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Symmetries of Quantum Spaces. Subgroups and Quotient Spaces of Quantum $SU(2)$ and $SO(3)$ Groups

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Abstract: We prove that each action of a compact matrix quantum group on a compact quantum space can be decomposed into irreducible representations of the group. We give the formula for the corresponding multiplicities in the case of the quotient quantum spaces. We describe the subgroups and the quotient spaces of quantum $SU(2)$ and $SO(3)$ groups.

0. Introduction

Quantum groups have been already applied in various areas of physics, like conformal field theory and exactly solvable models in statistical mechanics. It is especially interesting that they could possibly describe symmetries of (quantum) space-time in a future quantum gravity. In the same time, the nature and properties of quantum groups are still under investigation. The local description of quantum groups is given in terms of quantum universal enveloping algebras (cf. e.g. [Dr, J]). In the global description we investigate the functions on quantum groups (cf. e.g. [W2, RTF]). A deep insight in that global structure is given by the topological approach developed in the series of papers of S.L. Woronowicz [W1-W6]. We use that approach in the present paper.

The classical $SU(2)$ and $SO(3)$ groups play an important role in description of spherically symmetric, stationary problems in physics. Also their subgroups are important in description of various physical systems. The description of quantum $SU(2)$ groups was given in [W2]. Their quantum homogeneous spaces, quantum 2-spheres, were investigated in [P1, P2, P5] (cf. also [VS2]). However, the general theory of quantum subgroups and quantum homogeneous spaces was only touched there. In the present paper we want to treat that subject in more detail. We also provide more examples.

In Sect. 1 we investigate the general theory of the (right) actions of (compact matrix) quantum groups on (compact) quantum spaces. In Sect. 2 and 3 the theory is illustrated on the example of quantum $SU(2)$ and $SO(3)$ groups. We classify their subgroups and describe the corresponding quotient spaces. Provided examples of finite quantum groups can have an application in the theory of pseudogroups of

Ocneanu. In the course of the paper we substantiate some statements made in [P1] and [P5]. The results of the paper were partially contained in [P3] and partially announced in [P4].

Throughout the paper we use the terminology and results of [W2, W3]. All considered C^*-algebras and C^*-homomorphisms are unital. The symbol $\approx$ denotes a C^*-isomorphism. If M is a subset of a C^*-algebra A then $< M >$ denotes the closure of span M. Let us recall (cf. [W1]) that (compact) quantum spaces X are abstract objects which are in bijective correspondence with C^*-algebras $C(X)$. In particular, if X is a usual (compact Hausdorff) space then $C(X)$ has the usual meaning of C^*-algebra of continuous functions on X. Each commutative C^*-algebra can be obtained in that way (up to a C^*-isomorphism).

We use the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

We sum over repeated indices which are not taken in brackets (Einstein's convention). For $x \in \mathbf{R}$,$E(x)$ denotes the integer part of x.

1. Symmetries of Quantum Spaces

In this section we define the notion of subgroup of (compact matrix) quantum group. We also provide the basic notions concerning the actions of quantum groups on (compact) quantum spaces. We prove that each such action can be decomposed into irreducible representations of the quantum group. We give the formula for the corresponding multiplicities in the case of quotient quantum spaces.

Let us recall

Definition 1.1 ([W3, W7]). *$G = (A, u)$ is called a (compact matrix) quantum group if $A \neq \{0\}$ is a C^*-algebra, $u = (u_{ij})_{i,j=1}^N$ is an $N \times N$ matrix with entries belonging to A and*

1. *A is the smallest C^*-algebra containing all matrix elements of u.*
2. *There exists a C^*-algebra homomorphism $\Phi : A \longrightarrow A \otimes A$ such that*

$$\Phi(u_{kl}) = \sum_{r=1}^{N} u_{kr} \otimes u_{rl} \quad k, l = 1, 2, \ldots, N . \tag{1}$$

3. *u and $u^T = (u_{lk})_{k,l=1}^N$ are invertible.*

In particular, each compact group of matrices $G \subset GL(N, \mathbf{C})$ is a quantum group [W3]. Then $A = C(G)$ and u corresponds to the fundamental representation of $G : u_{ij}(g) = g_{ij} \in \mathbf{C},\ g \in G, i, j = 1, \ldots, N$. Each quantum group with commutative A is of that kind (up to a C^*-isomorphism). We use the notation $A = C(G)$ for any quantum group. We say [W3] that w is a (smooth nondegenerate) representation of G if w is an invertible $M \times M$ matrix with entries in A and

$$\Phi(w_{kl}) = \sum_{r=1}^{N} w_{kr} \otimes w_{rl}, \quad k, l = 1, 2, \ldots, M ,$$

for some $M \in \mathbf{N}$. We denote $M = \dim w$. It is easy to see (cf. [W3]) that w^T is also invertible and therefore the w-image of G is a quantum group:

Proposition 1.2. *Let w be a representation of a quantum group, $M = \dim w$. Then*

$$(C^*(\{w_{ij} : i,j = 1,\ldots,M\}), w)$$

is also a quantum group.

Note. Let $C^*(\{w_{ij} : i,j = 1,\ldots,M\}) = A$. Then quantum groups (A,u) and (A,w) have the same Φ and can be identified.

The unital $*$-algebra generated by all matrix elements of u is denoted by $\mathscr{A}$. Tensor product ($Ⓣ$), direct sum ($\oplus$), equivalence ($\simeq$) and irreducibility of representations of G are defined as for usual matrices (cf. [W3]). In particular, representations w, w' are equivalent if $\dim w = \dim w'$ and there exists $S \in GL(\dim w, \mathbf{C})$ such that $w = Sw'S^{-1}$. Each representation is equivalent to a representation which is unitary (as matrix). Let $\{u^\tau\}_{\tau \in \hat{G}}$ be the set of all nonequivalent irreducible unitary representations of G. We denote by u^0 the trivial representation ($0 \in \hat{G}$, $\dim u^0 = 1$ and $u^0_{11} = I$). Set $d_\tau = \dim u^\tau$. Due to [W3, Prop. 4.7], the matrix elements of all $u^\tau, \tau \in \hat{G}$ give a linear basis of $\mathscr{A}$. The Haar measure h is the state on $C(G)$ which is equal to 1 on I and 0 on other matrix elements of $u^\tau, \tau \in \hat{G}$. It is invariant, i.e. $(id \otimes h)\Phi(x) = (h \otimes id)\Phi(x) = h(x)I, x \in C(G)$ [W3, Th. 4.2]. According to (5.10) and (5.15) of [W3], there exist matrices F_α, $\alpha \in \hat{G}$, such that

$$h(u^\alpha_{km}{}^* u^\beta_{ln}) = (TrF_{(\alpha)})^{-1}\delta_{\alpha\beta}\delta_{mn}(F^{-1}_{(\alpha)})_{lk} \ .$$

We set $x^\alpha_{sm} = (TrF_{(\alpha)})\ u^\alpha_{km}{}^*(F_{(\alpha)})_{ks}, \rho^\alpha_{sm}(x) = h(x^\alpha_{sm}x)$ for $x \in C(G), \alpha \in \hat{G}$, $s,m = 1,\ldots,d_\alpha$. Then ρ^α_{sm} are continuous linear functionals on $C(G)$ and $\rho^\alpha_{sm}(u^\beta_{ln}) = \delta_{\alpha\beta}\delta_{sl}\delta_{mn}$, $\beta \in \hat{G}, l,n = 1,2,\ldots,d_\beta$. Hence,

$$(\rho^\alpha_{sm} \otimes \rho^\beta_{ij})\Phi_G = \delta_{mi}\delta_{\alpha(\beta)}\rho^\beta_{sj} \tag{2}$$

(the action of both sides on all u^τ_{ab} is the same). We put $P^\alpha_{sm} = (id \otimes \rho^\alpha_{sm})\Phi_G, \rho^\alpha = \rho^\alpha_{ss} \in C(G)'$ (Einstein's convention!). Then

$$P^\alpha_{sm}C(G) \subset \operatorname{span}\{u^\alpha_{is} : i = 1,2,\ldots,d_\alpha\} \ , \tag{3}$$

$$\rho^\alpha(u^\beta_{ln}) = \delta_{\alpha\beta}\delta_{ln} \tag{4}$$

(cf. [W3, eq. 5.37]). In particular $\rho^0 = h$. The basic notion of this section is given by

Definition 1.3. *We say that a quantum group $H = (B,v)$ is a (compact) subgroup of a quantum group $G = (A,u)$ if $\dim v = \dim u$ and there exists a C^*-homomorphism $\theta_{HG} : A \longrightarrow B$ such that $\theta_{HG}(u_{ij}) = v_{ij}, i,j = 1,2,\ldots,\dim u$.*

Notice that θ_{HG} must be a C^*-epimorphism.

Let $H \subset G$ be two compact groups of matrices. The conditions of Def. 1.3 are then satisfied by $\theta_{HG} = i^*$, where $i : H \longrightarrow G$ is the natural embedding. Conversely, let G be a compact group of matrices. Then each subgroup in the sense of Def. 1.3 is also a compact subgroup in the usual sense (up to a C^*-isomorphism).

According to Def. 1.3, $S_qU(N), q \in (0,1]$ (see [W4]) is a subgroup of $S_qU(N + 1)$ (we use the identification of the note after Prop. 1.2 for the representation $w = u \oplus u^0$ of $S_qU(N)$, cf. Eq. (1.7) of [NYM]).

The second main notion of the paper is introduced as follows.

Definition 1.4. *Let X be a quantum space and G be a quantum group. We say that a C^*-homomorphism $\Gamma : C(X) \longrightarrow C(X) \otimes C(G)$ is an action of G on X if*
a) $(\Gamma \otimes id)\Gamma = (id \otimes \Phi_G)\Gamma$,
b) $\langle (I \otimes y)\Gamma x : x \in C(X), y \in C(G) \rangle = C(X) \otimes C(G)$.

Remark 1. This definition is more restrictive than that used in [P1]. Nevertheless, Thm. 1 and Thm. 2 of [P1] remain true if we use instead Def. 1.4 (cf. Corollary 1.6).

Remark 2. In the classical case (i.e. if X is a usual compact Hausdorff space and G a compact group of matrices), Def. 1.4 means that $\Gamma = \sigma^*$, where $\sigma : X \times G \longrightarrow X$ is a right continuous action of G on X in the usual sense (including the condition $\sigma(x, e) = x$ for $x \in X$).

Let X be a quantum space and G be a quantum group. Let us fix a C^*-homomorphism $\Gamma : C(X) \to C(X) \otimes C(G)$. We say that a vector subspace $W \subset C(X)$ corresponds to a representation v of G if there exists a basis $e_1, \ldots, e_d$ in W such that $\dim v = d$ and $\Gamma e_k = e_m \otimes v_{mk}, k = 1, 2, \ldots d$. It occurs that if Γ is an action of G on X then $C(X)$ can be decomposed into vector subspaces corresponding to irreducible representations of G:

Theorem 1.5. *Let Γ be an action of a quantum group G on a quantum space X. We denote $E^\alpha = (id \otimes \rho^\alpha)\Gamma, W_\alpha = E^\alpha C(X) \subset C(X)$ for $\alpha \in \hat{G}$ (see (4)). Then*
1) $C(X) = \overline{\oplus_{\alpha \in \hat{G}} W_\alpha}$.
2) *For each $\alpha \in \hat{G}$ there exists a set I_α and vector subspaces $W_{\alpha i}$, $i \in I_\alpha$, such that*
a) $W_\alpha = \oplus_{i \in I_\alpha} W_{\alpha i}$.
b) *$W_{\alpha i}$ corresponds to u^α for each $i \in I_\alpha$.*
3) *Each vector subspace $V \subset C(X)$ corresponding to u^α is contained in W_α .*
4) *The cardinal number of I_α doesn't depend on the choice of $\{W_{\alpha i}\}_{i \in I_\alpha}$. It is denoted by c_α and called the multiplicity of u^α in the spectrum of Γ.*

Proof. 1)2) Set $E^\alpha_{sm} = (id \otimes \rho^\alpha_{sm})\Gamma : C(X) \longrightarrow C(X), \alpha \in \hat{G},\ s, m = 1, 2, \ldots, d_\alpha$. Using condition a) of Def. 1.4 and (2), we get

$$E^\alpha_{sm} E^\beta_{ij} = [id \otimes (\rho^\alpha_{sm} \otimes \rho^\beta_{ij})\Phi_G]\Gamma = \delta_{mi}\delta_{\alpha(\beta)} E^\beta_{sj} . \tag{5}$$

By virtue of $\langle x^\alpha_{sm} : \alpha \in \hat{G}, s, m = 1, \ldots, d_\alpha \rangle = C(G)$ and condition b) of Def. 1.4, we obtain

$$\left.\begin{array}{l} \langle E^\alpha_{sm} x : \alpha \in \hat{G}, s, m = 1, \ldots, d_\alpha, x \in C(X) \rangle \\ = \langle (id \otimes h)(I \otimes y)\Gamma x : y \in C(G), x \in C(X) \rangle = C(X) \end{array}\right\} . \tag{6}$$

Let $W^{\alpha s} = E^\alpha_{(s)(s)} C(X)$. Using (5) and (6), we get

$$C(X) = \overline{\bigoplus_{\alpha, s} W^{\alpha s}} . \tag{7}$$

But $E^\alpha = E^\alpha_{ss}$, hence

$$W_\alpha = \bigoplus_{s=1}^{d_\alpha} W^{\alpha s} , \tag{8}$$

which proves 1). Let $\{e_{\alpha i1}\}_{i\in I_\alpha}$ be a basis of $W^{\alpha 1}$. We set $e_{\alpha is} = E^{(\alpha)}_{s1} e_{\alpha i1}, s = 1,\ldots,d_\alpha,\ i \in I_\alpha$. In virtue of (5), $\{e_{\alpha is}\}_{i\in I_\alpha}$ is a basis of $W^{\alpha s}$. Putting $W_{\alpha i} = \operatorname{span}\{e_{\alpha is} : s = 1,\ldots,d_\alpha\}$ and using (8), we get 2a). Using condition a) of Def. 1.4 and (3), we get

$$\begin{aligned}\Gamma e_{\alpha is} &= \Gamma E^{\alpha}_{(s)(s)} e_{\alpha is} = \Gamma(id \otimes \rho^{\alpha}_{(s)(s)})\Gamma e_{\alpha is}\\ &= (id \otimes (id \otimes \rho^{\alpha}_{(s)(s)})\Phi_G)\Gamma e_{\alpha is}\\ &= [id \otimes P^{\alpha}_{(s)(s)}]\Gamma e_{\alpha is} \in C(X) \otimes \operatorname{span}\{u^{\alpha}_{js} : j = 1,\ldots,d_\alpha\}\,.\end{aligned}$$

Therefore $\Gamma e_{\alpha is} = x_{(\alpha)i(s)j} \otimes u^{\alpha}_{js}$ for some $x_{\alpha isj} \in C(G), j = 1,2,\ldots,d_\alpha$. Acting on both sides by $(id \otimes \rho^{(\alpha)}_{k(s)}), k = 1,2,\ldots,d_\alpha$, we obtain $E^{(\alpha)}_{k(s)} e_{\alpha is} = x_{\alpha isk}$, hence $x_{\alpha isk} = e_{\alpha ik},\ \Gamma e_{\alpha is} = e_{(\alpha)ij} \otimes u^{\alpha}_{js},\ s = 1,2,\ldots,d_\alpha, i \in I_\alpha$. It proves b).

3)4) Let $e_1, e_2, \ldots, e_{d_\alpha}$ form a basis of $V \subset C(X)$ such that $\Gamma e_s = e_j \otimes u^{\alpha}_{js},\ s = 1,\ldots,d_\alpha$. We get $E^{\alpha}_{r1} e_1 = e_r, r = 1,\ldots,d_\alpha$. Thus $e_r = E^{\alpha}_{(r)(r)} e_r \in W^{\alpha r} \subset W_\alpha$, which proves 3). Moreover, we see that each decomposition of type 2a) can be obtained as in proof of 2). Therefore the cardinal number of I_α is equal to $\dim W^{\alpha 1}$ for each choice of $\{W_{\alpha i}\}_{i\in I_\alpha}$. □.

Corollary 1.6. *Let X be a quantum space, G be a quantum group and $\Gamma : C(X) \longrightarrow C(X) \otimes C(G)$ be a C^*-homomorphism. Then Γ is an action of G on X iff there exist sets J_α, $\alpha \in \hat{G}$, and linearly independent and linearly dense elements $e_{\alpha mj}, \alpha \in \hat{G}, m \in J_\alpha,\ j = 1,\ldots,d_\alpha$, in $C(X)$ such that $\Gamma e_{\alpha mj} = e_{\alpha ms} \otimes u^{(\alpha)}_{sj}$. In that case $\#J_\alpha = c_\alpha$ if one of these values is finite.*

Proof. " $\Rightarrow$ " is contained in Theorem 1.5. Conversely, let such elements $e_{\alpha mj}$ be given. Then condition a) of Def. 1.4 is satisfied (it suffices to check it on $e_{\alpha mj}$), while the condition b) follows from $e_{\alpha mk} \otimes w = \{I \otimes [w(u^{\alpha T})^{-1}_{kj}]\}\Gamma e_{(\alpha)mj}$, where $w \in C(G), \alpha \in \hat{G}, m \in J_\alpha, k = 1,2,\ldots,d_\alpha$. Moreover (see the proof of Th. 1.5), $W^{\alpha s} = \langle e_{\alpha ms} : m \in J_\alpha\rangle, c_\alpha = \dim\langle e_{\alpha m1} : m \in J_\alpha\rangle$, which proves the last statement. □.

Now we shall find the numbers c_α for the quotient spaces. Let H be a subgroup of a quantum group G. The quotient space $H\backslash G$ is defined by

$$C(H\backslash G) = \{x \in C(G) : (\theta_{HG} \otimes id)\Phi_G x = I \otimes x\}$$

(cf. [P1, Sect. 6]). Similarly as in [P1, Sect. 6] we get that $E_{H\backslash G} = (h_H \otimes id)(\theta_{HG} \otimes id)\Phi_G$ is a completely bounded projection from $C(G)$ onto $C(H\backslash G)$. Moreover, $(E_{H\backslash G} \otimes id)\Phi_G = \Phi_G E_{H\backslash G}$. Thus we can define

$$\Gamma_{H\backslash G} = \Phi_G|_{C(H\backslash G)} : C(H\backslash G) \longrightarrow C(H\backslash G) \otimes C(G)\,.$$

Let $\alpha \in \hat{G}$. The representation $\theta_{HG}(u^\alpha)$ of the group H can be decomposed into a direct sum of irreducible representations among which the trivial one appears with a multiplicity which we denote by n_α. Taking a suitable form of u^α we get

$$h_H(\theta_{HG}(u^{\alpha}_{ij})) = \begin{cases} \delta_{ij} & \text{for } i = j,\ i = 1,2,\ldots,n_\alpha,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore $E_{H\backslash G}u^{\alpha}_{mj} = u^{\alpha}_{mj}$ for $1 \leqq m \leqq n_{\alpha}$ and $E_{H\backslash G}u^{\alpha}_{mj} = 0$ for $n_{\alpha} < m \leqq d_{\alpha}, j = 1,\dots,d_{\alpha}, \alpha \in \hat{G}$. Hence, $e_{\alpha mj} = u^{\alpha}_{mj}, m = 1,\dots,n_{\alpha}, j = 1,\dots,d_{\alpha}, \alpha \in \hat{G}$, have the same properties as in Corollary 1.6 with $J_{\alpha} = \{1,\dots,n_{\alpha}\}$. We obtain

Theorem 1.7. *Let H be a subgroup of a quantum group G. Then $\Gamma_{H\backslash G} = \Phi_{G|_{C(H\backslash G)}} : C(H\backslash G) \longrightarrow C(H\backslash G) \otimes C(G)$ is an action of G on $H\backslash G$. Moreover, $c_{\alpha} = n_{\alpha}$ (the multiplicity of the trivial representation (I) in the decomposition of $\theta_{HG}(u^{\alpha})$ into irreducible components).*

Definition 1.8. *All the pairs $(H\backslash G, \Gamma_{H\backslash G})$ obtained in the above way (and the pairs isomorphic to them) are called quotient. Let Γ be an action of a quantum group G on a quantum space X. We say that a pair (X,Γ) is embeddable if $C(X) \neq \{0\}$ and there exists a faithful C^*-homomorphism $\psi : C(X) \longrightarrow C(G)$ such that $\Phi_G\psi = (\psi \otimes id)\Gamma$ (cf. [VS2]). We say that (X,Γ) is homogeneous if $c_0 = 1$.*

Remark 3. In the classical case $C(H\backslash G)$ is the commutative C^*-algebra of functions which are constant on the orbits $Hg (g \in G)$ of the subgroup H of G. Let π be the continuous projection $\pi : G \longrightarrow H\backslash G$. Then π^* identifies that C^*-algebra with the C^*-algebra of continuous functions on the usual quotient space $H\backslash G$. Then Γ is identified with σ^*, where σ is the usual right continuous action of G on $H\backslash G$.

Remark 4. In the classical case (X,Γ) is homogeneous iff $X \neq \emptyset$ is homogeneous w.r.t. the action σ of the group G (see Remark 2).

Proof. Let $x \in C(X)$. Then $x \in W_0$ iff $\Gamma x = x \otimes I$ iff $x(pg) = x(p), p \in X, g \in G$ iff x is constant on the orbits pG of G.

$\Leftarrow$: If $X \neq \emptyset$ is homogeneous then $pG = G, W_0 = \mathbf{C}I, c_0 = 1$.

$\Rightarrow$: $X \neq \emptyset$ since $c_0 = 1 > 0$. Assume ad absurdum that X is not homogeneous. Then there exist $p, p' \in X$ such that $p' \notin pG$. By the Urysohn lemma there exists $f \in C(X)$ such that $0 \leqq f \leqq 1, f_{|pG} = 0, f(p') = 1$. Let $k = (id \otimes h)\Gamma f$. Then $k = E^0 f \in W^0 = \mathbf{C}I$. But $k(p) = \int_G f(pg)dg = 0,\ k(p') = \int_G f(p'g)dg > 0$. This contradiction proves the homogeneity of X. □.

A relation among the above notions is given by

Proposition 1.9. *Let Γ be an action of a quantum group G on a quantum space X. Then*

a) *(X,Γ) is quotient $\Rightarrow$ (X,Γ) is embeddable $\Rightarrow$ (X,Γ) is homogeneous.*
b) *In the classical case (X,Γ) is quotient $\Longleftrightarrow$ (X,Γ) is embeddable $\Longleftrightarrow$ (X,Γ) is homogeneous.*

Proof. a) The first implication holds for $\psi = id : C(H\backslash G) \longrightarrow C(G)$. Let now (X,Γ) be embeddable, $x \in C(X), \Gamma x = x \otimes I$. Then $\Phi_G\psi(x) = \psi(x) \otimes I$. Acting on both sides by $id \otimes h_G$ we get $\psi(x) = h_G(\psi(x))I \in \mathbf{C}I, x \in \mathbf{C}I$. Thus $W_0 = \mathbf{C}I$, $c_0 = 1$.
b) In this case each homogeneous space is (up to a homeomorphism) quotient, which proves the implications opposite to that of a). □.

Remark 5. Examples of non-compact quantum homogeneous spaces are given by [W8].

2. Subgroups and Quotient Spaces of Quantum $SU(2)$ Groups

In this section we classify the subgroups of quantum groups $SU_q(2), q \in [-1,1]\backslash\{0\}$. The corresponding quotient spaces are described (for $q \in (-1,1)\backslash\{0\}$).

First, let us recall that compact subgroups of $SO(3)$ are given by

a) $SO(3)$,
b) $SO(2)_{\mathbf{n}}$ (all rotations around the axis given by $\mathbf{n}$),
c) $DO(2)_{\mathbf{n}}$ (the elements of $SO(2)_{\mathbf{n}}$ and all rotations through angle π around axes perpendicular to $\mathbf{n}$),
d) $C_{m,\mathbf{n}}$ (rotations through angles $\frac{2\pi}{m}k, k = 0,1,\ldots,m-1$, around the axis given by $\mathbf{n}$), $m = 1,2,\ldots$,
e) $D_{m,\mathbf{n},\phi}$ (the elements of $C_{m,\mathbf{n}}$ and rotations through angle π around m axes in plane $\sigma_{\mathbf{n}}$ perpendicular to $\mathbf{n}$, with equal angles between neighbouring axes, where ϕ denotes the angle in $\sigma_{\mathbf{n}}$ between the projection of $\mathbf{e}_3$ on $\sigma_{\mathbf{n}}$ (we take $\mathbf{e}_1$ instead of $\mathbf{e}_3$ if $\mathbf{n} = \pm\mathbf{e}_3$) and the first axis in the anti-clockwise direction), $m = 2,3,\ldots, 0 \leqq \phi < \frac{\pi}{m}$,
f) $T_{\mathbf{n},\phi}$ (the symmetries of regular tetrahedron with one of vertices in the direction of $\mathbf{n}$, where ϕ is now measured towards a projection of an edge starting in this vertex), $0 \leqq \phi < \frac{2\pi}{3}$,
g) $O_{\mathbf{n},\phi}$ (the symmetries of regular octahedron with $\mathbf{n},\phi$ defined as in f)), $0 \leqq \phi < \frac{\pi}{2}$,
h) $I_{\mathbf{n},\phi}$ (the symmetries of regular icosahedron with $\mathbf{n},\phi$ defined as in f)), $0 \leqq \phi < \frac{2\pi}{5}$,

where $\mathbf{n}$ is a unit vector. We have two opposite choices of $\mathbf{n}$ for any subgroup in b)-f) (in the case of f) the change of sign of $\mathbf{n}$ corresponds to the inversion of the tetrahedron) and many choices of $\mathbf{n}$ corresponding to the vertices of the solid for any subgroup in f)-h) (thus we have 8 choices for f); ϕ is unique for a given $\mathbf{n}$, but can depend on its choice); moreover C_1 does not depend on $\mathbf{n}$, D_2 depends only on the set of three perpendicular axes; other subgroups are distinct.

Let $\beta : SU(2) \to SO(3)$ be the standard continuous two–folded covering: $[\beta(g)]w = gwg^{-1}$, where $g \in SU(2)$, $w = x\sigma_x + y\sigma_y + z\sigma_z \simeq (x,y,z) \in \mathbf{R}^3$. The compact subgroups $H \subset SU(2)$ fall into two classes:

1) $-I \notin H$. Then $\beta(H)$ can not contain $C_{m,\mathbf{n}}$ for any even m. We must have $\beta(H) = C_{m,\mathbf{n}}$ for some odd m. Then we have exactly one such subgroup $H = (\mathbf{Z}_m)_{\mathbf{n}} \equiv \{R^{2k} : k = 0,1,\ldots,m-1\}$, where R is any generator of the cyclic group $\beta^{-1}(C_{m,\mathbf{n}})$ (the choice of R is irrelevant, opposite choices of $\mathbf{n}$ give the same subgroup, for $m = 1$ $\mathbf{n}$ is irrelevant, other subgroups are distinct).
2) $-I \in H$. Then all distinct possibilities are given by $H = \beta^{-1}(W)$, where W is any compact subgroup of $SO(3)$.

Quantum $SU(2)$ groups [W2] are defined as $SU_q(2) = (A,u)$, $q \in [-1,1]\backslash\{0\}$, where A is the universal C^*-algebra generated by two elements α,γ satisfying

$$\left.\begin{array}{l} \alpha^*\alpha + \gamma^*\gamma = I, \alpha\alpha^* + q^2\gamma^*\gamma = I, \gamma^*\gamma = \gamma\gamma^*, \\ \alpha\gamma = q\gamma\alpha, \alpha\gamma^* = q\gamma^*\alpha \end{array}\right\} \tag{9}$$

and

$$u = \begin{pmatrix} \alpha, & -q\gamma^* \\ \gamma, & \alpha^* \end{pmatrix} .$$

For $q=1$ we get the usual $SU(2)$ group. According to [W2, Sect. 5], all nonequivalent irreducible representations of $SU_q(2)$ can be chosen as $\{d_\alpha\}_{\alpha\in\mathbf{N}/2}$, $\dim d_\alpha = 2\alpha+1$, $d_\alpha \textcircled{T} d_\beta \simeq d_{|\alpha-\beta|} \oplus d_{|\alpha-\beta|+1} \oplus \ldots \oplus d_{\alpha+\beta}$. We can put $d_{1/2}=u$. The classification of subgroups of $SU_q(2), q\in(-1,1)\backslash\{0\}$, is given by

Theorem 2.1. $SU_q(2), q \in [-1,1]\backslash\{0\}$, *has the following subgroups*:
a) $SU_q(2)=(A,u)$.
b) $U(1)=\left(C(S^1), \begin{bmatrix} z, & 0 \\ 0, & \bar{z}\end{bmatrix}\right)$ *where* $S^1=\{e^{i\phi}:\phi\in\mathbf{R}\}$ *and* $z\in C(S^1)$ *is given by* $z(e^{i\phi})=e^{i\phi}, \phi\in\mathbf{R}$.
c) $\mathbf{Z}_n=\left(C(Z_n), \begin{bmatrix} z_{(n)}, & 0 \\ 0, & \overline{z_{(n)}}\end{bmatrix}\right)$ *where* $Z_n=\{e^{2\pi ik/n}: k=0,1,\ldots,n-1\}$ *and* $z_{(n)}\in C(Z_n)$ *is defined by* $z_{(n)}(e^{2\pi ik/n})=e^{2\pi ik/n}$, $n=1,2,\ldots$.

For $q\in(-1,1)\backslash\{0\}$ the above list contains all subgroups of $SU_q(2)$ (up to C^*-isomorphisms, without repetitions).

Proof. a) is obvious.

b) Compact group of matrices S^1 corresponds to the quantum group $(C(S^1),z)$. By Prop. 1.2 (for $w=z\oplus\bar{z}$), $U(1)$ is also a quantum group. The elements $\tilde{\alpha}=z$ and $\tilde{\gamma}=0$ satisfy (9), hence $\theta_{U(1)SU_q(2)}$ exists, $U(1)$ is a subgroup of $SU_q(2)$.

c) can be proved analogously.

We shall prove the last statement. Let $H=(B,v)$ be a subgroup of $SU_q(2)$, $q\in(-1,1)\backslash\{0\}$. Then

$$v=\begin{bmatrix}\tilde{\alpha}, & -q\tilde{\gamma}^* \\ \tilde{\gamma}, & \tilde{\alpha}^*\end{bmatrix},$$

where $\tilde{\alpha},\tilde{\gamma}$ satisfy (9). Moreover, $B=C^*(\tilde{\alpha},\tilde{\gamma})$. A detailed analysis of relations (9) shows (cf. [W2, VS1]) that (up to a C^*-isomorphism of the C^*-algebra B)

1) $\tilde{\alpha}=\alpha_0\otimes I_{C(\Delta)}$, $\tilde{\gamma}=\gamma_0\otimes U$, or
2) $\tilde{\alpha}=U, \tilde{\gamma}=0$,

where $\alpha_0,\gamma_0\in B(H_\infty)$, H_∞ is a Hilbert space with an orthonormal basis $f_0, f_1,\ldots$,

$$\alpha_0 f_m=(1-q^{2m})^{1/2} f_{m-1},\ \gamma_0 f_m = q^m f_m\ (f_{-1}=0),\ m=0,1,\ldots,$$

$U\in C(\Delta)$ is given by $U(e^{i\phi})=e^{i\phi}$ for $e^{i\phi}\in\Delta$ and Δ is a nonempty compact subset of S^1.

In the case of 1) we define the unitary operator $D_c\in B(H_\infty)$ by $D_c f_k = e^{ick} f_k, k=0,1,2,\ldots, c\in\mathbf{R}$. Using $\Phi_H\tilde{\gamma}=\tilde{\gamma}\otimes\tilde{\alpha}+\tilde{\alpha}^*\otimes\tilde{\gamma}$ we get

$$(D_c^*\otimes id\otimes D_c\otimes id)\Phi_H\tilde{\gamma}(D_c^*\otimes id\otimes D_c\otimes id)^*=e^{-ic}\Phi_H\tilde{\gamma}.$$

Therefore

$$Sp(\Phi_H\tilde{\gamma})=e^{-ic}Sp(\Phi_H\tilde{\gamma}). \tag{10}$$

But $\Phi_H\tilde{\alpha}$ and $\Phi_H\tilde{\gamma}\neq 0$ also satisfy (9), hence they are (up to a C^*-isomorphism) such as in 1) for some Δ'. By virtue of (10), $\Delta'=e^{-ic}\Delta'$ for all $c\in\mathbf{R}$, $\Delta'=S^1$. Since $Sp(\Phi_H\tilde{\gamma})\subset Sp\tilde{\gamma}$, $\Delta'\subset\Delta$, $\Delta=S^1$. We can identify H with $SU_q(2)$.

In the case of 2) $\Phi_H\tilde{\alpha}=\tilde{\alpha}\otimes\tilde{\alpha}$. Therefore

$$\Delta\cdot\Delta=Sp(\tilde{\alpha}\otimes\tilde{\alpha})=Sp\Phi_H\tilde{\alpha}\subset Sp\tilde{\alpha}=\Delta.$$

We get $\Delta=S^1$ or $\Delta=Z_n$, $n=1,2,\ldots$. Hence, H is such as in b) or c).

These subgroups are distinct since the corresponding C^*-algebras are non-isomorphic.

Remark 1. In the case of $q = 1, SU_q(2) = SU(2) = \beta^{-1}(SO(3))$,

$$U(1) = \left\{ \begin{bmatrix} e^{i\phi}, & 0 \\ 0, & e^{-i\phi} \end{bmatrix} : \phi \in \mathbf{R} \right\} = \beta^{-1}(SO(2)_{e_3}),$$

$$\mathbf{Z}_n = \left\{ \begin{bmatrix} e^{2\pi ik/n}, & 0 \\ 0, & e^{-2\pi ik/n} \end{bmatrix} : k = 0, 1, \ldots, n-1 \right\},$$

which is $(\mathbf{Z}_n)_{e_3}$ for odd n and $\beta^{-1}(C_{n/2,e_3})$ for even n.

Now we shall classify the subgroups of $G = SU_{-1}(2)$. Some related facts were already given in [Z]. Here we proceed in a little bit more complete way. First, analysing the set $Sp(A)$ of unitary equivalence classes of nondegenerate irreducible representations of the C^*-algebra A (cf. [W2, Remark after Th. A2.3]) we get

Proposition 2.2. *Let $q = -1$. There exists the surjection $\tau : SU(2) \longrightarrow Sp(A)$ such that*

1) $[\tau(w)](\alpha) = a\sigma_x, [\tau(w)](\gamma) = c\sigma_y$ *for* $ac \neq 0$,
2) $[\tau(w)](\alpha) = a$, $[\tau(w)](\gamma) = 0$ *for* $c = 0$,
3) $[\tau(w)](\alpha) = 0$, $[\tau(w)](\gamma) = c$ *for* $a = 0$,

where

$$w = \begin{pmatrix} a, & -\bar{c} \\ c, & \bar{a} \end{pmatrix} \in SU(2). \tag{11}$$

We denote $\pi_w = \tau(w)$ and $\pi_Z = \oplus_{w \in Z} \pi_w$ for any $Z \subset SU(2)$.

Remark 2. Let $\pi \in Sp(A)$. If $\dim \pi = 1$ then $\tau^{-1}(\pi)$ consists of 1 element (case 2) or 3) above). If $\dim \pi = 2$ then $\tau^{-1}(\pi)$ consists of four elements of $SU(2)$ which give equivalent irreducible representations. These elements can be obtained one from another by change of sign of a or/and c.

Let $S \subset Sp(A)$. We set

$$\tilde{S} = \left\{ \pi \in Sp(A) : \| \pi(x) \| \leqq \| \bigoplus_{\rho \in S} \rho(x) \| \text{ for all } x \in A \right\}$$

(cf. [Dix, Sect. 3.1]). Clearly $S \subset \tilde{S}$, $\tilde{\tilde{S}} = \tilde{S}$. Denoting $Z_S = \tau^{-1}(S)$ and by $\overline{Z}$ the closure of a subset $Z \subset SU(2)$ we have

Proposition 2.3.

$$Z_{\tilde{S}} = \overline{Z_S}.$$

Proof. "$\supset$": If $w \in \overline{Z_S}$ then there exists a sequence $Z_S \ni w_n \longrightarrow w$. It is easy to check that

$$\| \pi_w(x) \| \leqq \sup_n \| \pi_{w_n}(x) \| \leqq \| \bigoplus_{\rho \in S} \rho(x) \|, \quad x \in A,$$

hence $\pi_w \in \tilde{S}$, $w \in Z_{\tilde{S}}$.

"$\subset$": If $w \in Z_{\tilde{S}}$ then $\pi_w \in \tilde{S}, \| \pi_w(x) \| \leqq \sup_{z \in Z_S} \| \pi_z(x) \|$ for all $x \in A$. Let us first consider the case where $ac \neq 0$ (see (11)). Setting

$$x = 8I - \left[(\alpha^2 - a^2)^*(\alpha^2 - a^2) + (\gamma^2 - c^2)^*(\gamma^2 - c^2) \right],$$

we get $\sup_{z\in Z_S} \| \pi_z(x) \| \leqq 8 = \| \pi_w(x) \|$. Hence, there exists a sequence $z_n \in Z_S$ such that $\| \pi_{z_n}(x) \| \longrightarrow 8$. One can prove that (maybe replacing z_n by z_n' with $\tau(z_n) = \tau(z_n')$) $z_n \longrightarrow w$, $w \in \overline{Z_S}$. The other cases can be dealt with in the same manner (for $c = 0$ we set $x = 4I - (\alpha - a)^*(\alpha - a) - \gamma^*\gamma$, for $a = 0$ we set $x = 4I - (\gamma - c)^*(\gamma - c) - \alpha^*\alpha$). □.

Let us introduce a new (non-associative) product $*$ in $SU(2)$. We set $x * y = x \cdot y$ for all $x, y \in SU(2)$ except of the case

$$x = \begin{pmatrix} 0, & -\bar{c} \\ c, & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0, & -\bar{c}' \\ c', & 0 \end{pmatrix}, \quad | c | = | c' | = 1 ,$$

when $x * y = -x \cdot y$. We say that $Z \subset SU(2)$ is *τ-conformable* if $Z = Z_S$ for some $S \subset SpA$. The subsets and subgroups of $S_{-1}U(2)$ are characterized by

Proposition 2.4. (cf. Sect. 2.3 of [Z])

1. *Let Z be compact, τ-conformable subset of $SU(2)$. Then $\pi_Z : A \longrightarrow \pi_Z(A)$ is a C^*-epimorphism. In that way we obtain all C^*-epimorphisms from A (up to C^*-isomorphisms of the image, without repetitions.)*

2. *Let Z be a compact, τ-conformable subset of $SU(2)$ such that $Z * Z \subset Z$. Then $G_Z = (\pi_Z(A), (\pi_Z(u_{ij}))_{i,j=1}^2)$ is a subgroup of $S_{-1}U(2)$. In that way we obtain all subgroups of $S_{-1}U(2)$ (up to C^*-isomorphisms, without repetitions).*

Proof. 1. Each C^*-epimorphism from A has (up to a C^*-isomorphism of the image) the form $\oplus_{\rho\in S}\rho$ for some $S \subset Sp(A)$ (A is separable). It is clear that $\| \bigoplus_{\rho\in S} \rho(a) \| = \| \bigoplus_{\rho\in\tilde{S}} \rho(a) \|$ for any $S \subset Sp(A)$ and $a \in A$. Moreover, if $\| \bigoplus_{\rho\in S} \rho(a) \| = \| \bigoplus_{\rho\in T} \rho(a) \|$ for some $S = \tilde{S}$, $T = \tilde{T}$ and any $a \in A$ then $S = T$ (since $S \subset \tilde{T}$, $T \subset \tilde{S}$). Passing to the subsets of $SU(2)$ and using Prop. 2.3, we get our statement.

2. Let $\rho', \rho'' \in Sp(A)$. It is easy to check that $(\rho' \otimes \rho'')\Phi_G$ is unitarily equivalent to $\bigoplus_{\rho\in S} \rho$, where $Z_S = Z_{\{\rho'\}} * Z_{\{\rho''\}}$. Hence (for Z as in 1.), $Z * Z \subset Z$ iff $\|(\pi_Z \otimes \pi_Z)\Phi_G(a) \| \leqq \| \pi_Z(a) \|$ (for all $a \in A$) iff there exists a C^*-homomorphism $\Psi : \pi_Z(A) \longrightarrow \pi_Z(A) \otimes \pi_Z(A)$ such that $(\pi_Z \otimes \pi_Z)\Phi_G = \Psi\pi_Z$ iff G_Z is a subgroup of $S_{-1}U(2)$ (the conditions 1,3 of Def. 1.1 for G imply the same properties for G_Z). □.

In the following we assume that Z is a compact, τ-conformable subset of $SU(2)$ such that $Z * Z \subset Z$. We shall find all such subsets. Let us set

$$L = \left\{ \begin{pmatrix} a, & -\bar{c} \\ c, & \bar{a} \end{pmatrix} \in SU(2) : ac = 0 \right\} .$$

We have two cases.

1. $Z \subset L$. For $\phi \in \mathbf{R}$ we put

$$O_\phi = \begin{pmatrix} e^{i\phi}, & 0 \\ 0, & e^{-i\phi} \end{pmatrix}, K_\phi = \begin{pmatrix} 0, & -e^{-i\phi} \\ e^{i\phi}, & 0 \end{pmatrix} .$$

Using the definition of $*$ we find that Z has any of the following distinct forms:

a) $Z = L = \beta^{-1}(DO(2)_{e_3})$,
b) $Z = \{O_\phi : \phi \in \mathbf{R}\} = \beta^{-1}(SO(2)_{e_3})$ (then $G_Z \approx U(1)$),

c) $Z = \{O_{2\pi k/n}, K_{2\pi k/n+\phi_0-\pi/2} : k = 0, 1, \ldots, n-1\}$, where $n = 1, 2, \ldots, 0 \leqq \phi_0 < \frac{2\pi}{n}$ ($Z = \beta^{-1}(D_{n/2,\mathbf{e}_3,\phi_0})$ for n even, Z is not a group for n odd),
d) $Z = \{O_{2\pi k/n} : k = 0, 1, \ldots n-1\}$, where $n = 1, 2, \ldots$, (then $G_Z \approx \mathbf{Z}_n$; $Z = \beta^{-1}(C_{n/2,\mathbf{e}_3})$ for n even, $Z = (\mathbf{Z}_n)_{\mathbf{e}_3}$ for n odd).

II. $Z \not\subset L$. Then $-I \in Z$. Indeed, let $w \in Z$ be such that $ac \neq 0$. Hence, $w^k = w * \cdots * w$ (k times) belongs to Z for any $k = 1, 2, \ldots$. Therefore $w^{-1} \in Z$ (cf. the argument in [W3, proof of Th. 1.5]). But (τ-conformability) $-w \in Z$. We get $-I = -w * w^{-1} \in Z$. Consequently, the considered set Z must have a form $Z = \beta^{-1}(W)$, where W is any compact subgroup of $SO(3)$ such that $g_0 W g_0^{-1} \subset W$ (where g_0 is the rotation through the angle π around the axis x_3) and $W \not\subset DO(2)_{\mathbf{e}_3}$ (omitting the last condition is equivalent with adding the cases Ia)-Ib) and the cases Ic)-Id) with even n).

We get the following W:

a) $SO(3)$ (then $G_Z \approx SU_{-1}(2)$),
b) $SO(2)_{\mathbf{n}}$ with $\mathbf{n}$ perpendicular to $\mathbf{e}_3$,
c) $DO(2)_{\mathbf{n}}$ with $\mathbf{n}$ perpendicular to $\mathbf{e}_3$,
d) $C_{m,\mathbf{n}}$ with $\mathbf{n}$ perpendicular to $\mathbf{e}_3, m = 3, 4, \ldots$,
e) $D_{m,\mathbf{n},0}$ with $\mathbf{n}$ perpendicular to $\mathbf{e}_3, m = 3, 4, \ldots$,
e′) $D_{m,\mathbf{n},\pi/2m}$ with $\mathbf{n}$ perpendicular to $\mathbf{e}_3, m = 2, 3, \ldots$,
f) T_ϕ (the group of symmetries of regular tetrahedron freely hanging on a horizontal edge, ϕ gives the angle between $\mathbf{e}_1$ and this edge in anti-clockwise direction), $0 \leqq \phi < \pi/2$,
f′) T'_ϕ (the group of symmetries of a regular tetrahedron with one edge in vertical position, the opposite one in horizontal position, ϕ gives the angle between $\mathbf{e}_1$ and this edge in anti-clockwise direction), $0 \leqq \phi < \pi$,
g) O_ϕ (the group of symmetries of regular octahedron defined as in f)), $0 \leqq \phi < \pi$,
g′) $O_{\mathbf{n},\phi}$ with $\mathbf{n} = \mathbf{e}_3$, $0 \leqq \phi < \pi/2$,
h) I_ϕ (the group of symmetries of regular icosahedron defined as in f)), $0 \leqq \phi < \pi$.

Change of sign of $\mathbf{n}$ does not change W in any of the cases b)-e′). Besides, we have distinct W and distinct $Z = \beta^{-1}(W)$. Combining the above results with Prop. 2.4, we get all non-identical subgroups of $S_{-1}U(2)$.

Remark 3. Let Z be a τ-conformable subset of $SU(2)$. Then (using an argument similar to that in [W3, proof of Lemma 4.8]) we get $\dim \pi_Z(A) = \#Z$ (whether $\#Z \in \mathbf{N}$ or $\#Z = \infty$). In particular, under assumptions of Prop. 2.4.2, $\dim C(G_Z) = \#Z$. Therefore the above classification gives us many examples of finite-dimensional $*$-Hopf algebras.

Let us consider the quotient spaces $H\backslash G$ w.r.t. the above subgroups H of $G = SU_q(2)$. Our aim is to compute the multiplicity c_α of d_α in the spectrum of the action of G on $H\backslash G, \alpha \in \mathbf{N}/2$. According to Thm 1.7, c_α are equal to the multiplicities n_α of the trivial representation in the decomposition of $\theta_{HG}(d_\alpha)$ into irreducible components. We start with

Proposition 2.5. *Let $G = SU_q(2), q \in [-1, 1]\backslash\{0\}$, and H be a subgroup of G described in Thm 2.1. Consider the quotient space $H\backslash G$. Then c_α are equal*

a) $c_0 = 1, c_\alpha = 0$ *for* $\alpha > 0$ *in the case of* $H = SU_q(2)$,
b) $c_k = 1, c_{k+1/2} = 0, k = 0, 1, 2, \ldots$, *in the case of* $H = U(1)$,

c) $c_k = 2E(\frac{2k}{m}) + 1, c_{k+1/2} = 0$, $k = 0, 1, 2, \ldots$, *in the case of* $H = \mathbf{Z}_m$, *m even;*

$c_k = 2E(\frac{k}{m}) + 1$, $c_{k+1/2} = 2E(\frac{2k+1}{m}) - 2E(\frac{k}{m})$, $k = 0, 1, 2, \ldots$, *in the case of* $H = \mathbf{Z}_m$, *m-odd.*

Proof. a) $\theta_{HG}(d_\alpha) = d_\alpha, \alpha \in \mathbf{N}/2$, which gives the trivial representation only for $\alpha = 0$ with $c_0 = 1$.
b) $\theta_{HG}(d_\alpha) \simeq z^{-2\alpha} \oplus z^{-2\alpha+2} \oplus \ldots \oplus z^{2\alpha}$ (see [P1, eq. 10]). Thus the trivial subrepresentation occurs only for $\alpha \in \mathbf{N}$ (with multiplicity 1).
c) Similarly as in b), $\theta_{HG}(d_\alpha) \simeq z_{(m)}^{-2\alpha} \oplus z_{(m)}^{-2\alpha+2} \oplus \ldots \oplus z_{(m)}^{2\alpha}$, hence n_α is the number of elements in the set $\{-2\alpha, -2\alpha + 2, \ldots, 2\alpha\}$, which are divisible by m. The precise computation gives the results as in the formulation of the proposition. □

The remaining results concerning $q = \pm 1$ will be given in Sect. 3.

In the following we shall illustrate Prop. 1.9. using the results of [P1]. Let $q \in [-1, 1]\backslash\{0\}$. We say that (X, Γ) is a quantum sphere if X is a quantum space, Γ is an action of $SU_q(2)$ on X and

1) $c_k = 1, c_{k+1/2} = 0, k = 0, 1, 2, \ldots,$
2) W_1 generates $C(X)$ (as C^*-algebra with unity).

Note. The notion of the quantum sphere, which is used in the present paper is more restrictive (for $q \in (-1, 1)\backslash\{0\}$) than that in [P1]. The present notion coincides with the assumption (for general $q \in [-1, 1]\backslash\{0\}$) and the condition (i′) of [P1, Th. 2].

In the remaining part of the section $q \in (-1, 1)\backslash\{0\}$. According to [P1, Th. 2], (X, Γ) is a quantum sphere iff (X, Γ) is isomorphic to (S^2_{qc}, σ_{qc}) for $c \in [0, \infty]$. Moreover, the constant c is unique.

In [P1] we also considered $(S^2_{qc(n)}, \sigma_{qc(n)})$, where $c(n) = -q^{2n}/(1 + q^{2n})^2, n = 1, 2, \ldots$. They satisfy the definition of the quantum sphere with 1) replaced by $1)_n$ $c_k = 1$ for $k = 0, 1, \ldots, n-1$, all other c_k vanish (see [P1, Eq. 13b and Eq. 14]).

Proposition 2.6 (cf. [P1, Sect. 6]). *Let* $q \in (-1, 1)\backslash\{0\}, c \in \{c(1), c(2), \ldots\} \cup [0, \infty]$. *Then*

a) (S^2_{qc}, σ_{qc}) *is quotient* $\iff c = c(1)$ *or* $c = 0$,
b) (S^2_{qc}, σ_{qc}) *is embeddable* $\iff c = c(1)$ *or* $c \in [0, \infty]$,
c) (S^2_{qc}, σ_{qc}) *is homogeneous.*

Thus the implications in Prop. 1.9.a cannot be replaced by equivalences.

Proof. a) It follows from Thm. 2.1, Prop. 2.5 and the properties of quantum spheres
b) Due to [P1, Prop. 4.II] (cf. also [MNW, Corollary 3.8]), $C(S^2_{qc(n)}) \approx \pi_+(C(S^2_{qc(n)})) = B(\mathbf{C}^n), n = 2, 3, \ldots$, which has no characters. Therefore $(S^2_{qc(n)}, \sigma_{qc(n)})$ is not embeddable (otherwise $e\psi$ would be a character, where e is the counit of $SU_q(2)$). For $c = c(1)$ we set $\psi I = I$. For $c \in [0, \infty]$ we set $\psi(e_i) = s_k d_{1,ki}, i = -1, 0, 1$ where (s_{-1}, s_0, s_1) equals $(c^{1/2}, 1, c^{1/2})$ for $c \in [0, \infty)$ and $(1, 0, 1)$ for $c = \infty$ (due to $\overline{s_{-k}} = s_k$, $a_{lm} s_l s_m = \rho$, $b_{lm,k} s_l s_m = \lambda s_k, k = -1, 0, 1$, and [P2, Eq. 5], $\psi(e_i)$ satisfy [P1, Eq. 2]). It is easy to check the equation $\Phi\psi = (\psi \otimes id)\sigma_{qc}$ on the

generators e_i. The faithfulness of ψ follows from an argument similar as in the proof of [P1, Th. 1(i)⇒(ii)].

c) is obvious.

3. Quantum $SO(3)$ Groups

In this section we describe subgroups and quotient spaces of quantum groups $SO_q(3)$, $q \in [-1, 1]\backslash\{0\}$. We treat as well the quotient spaces of $SU_q(2), q = \pm 1$.

We take d_1 in the form

$$d_1 = (d_{1,ij})_{i,j=-1,0,1} = \begin{pmatrix} \alpha^{*2}, & -(q^2+1)\alpha^*\gamma, & -q\gamma^2 \\ \gamma^*\alpha^*, & I-(q^2+1)\gamma^*\gamma, & \alpha\gamma \\ -q\gamma^{*2}, & -(q^2+1)\gamma^*\alpha, & \alpha^2 \end{pmatrix}$$

(see [P1, Sect. 2]). According to Prop. 1.2, $SO_q(3)=(C^*(\{d_{1,ij}, i, j = -1, 0, 1\}), d_1), q \in [-1, 1]\backslash\{0\}$, are quantum groups (cf. [P1, Remark 3 after Th. 2]). The set of their all nonequivalent irreducible representations can be chosen as $\{d_\alpha\}_{\alpha\in\mathbf{N}}$ (they are generated by d_1). Therefore

$$C(SO_q(3)) = \langle d_{\alpha,mn} : m, n = -\alpha, -\alpha+1, \ldots, \alpha, \alpha \in \mathbf{N}\rangle = C(\mathbf{Z}_2\backslash SU_q(2)) \,. \quad (12)$$

That is generated by the elements

$$K = \gamma^*\gamma,\ A = \alpha\gamma,\ C = \alpha\gamma^*,\ G = \gamma^2,\ L = \alpha^2 \,. \quad (13)$$

It is easy to check that

$$\left.\begin{aligned} &L^*L = (I-K)(I-q^{-2}K), LL^* = (I-q^2K)(I-q^4K)\,,\\ &G^*G = GG^* = K^2,\ A^*A = K-K^2,\ AA^* = q^2K - q^4K^2,\\ &C^*C = K-K^2,\ CC^* = q^2K-q^4K^2,\ LK = q^4KL\,,\\ &GK = KG,\ AK = q^2KA,\ CK = q^2KC, LG = q^4GL\,,\\ &LA = q^2AL,\ AG = q^2GA,\ CA = AC,\ LG^* = q^4G^*L\,,\\ &A^2 = q^{-1}LG,\ A^*L = q^{-1}(I-K)C,\ K^* = K\,. \end{aligned}\right\} \quad (14)$$

Proposition 3.1. *Let $q \in (-1, 1)\backslash\{0\}$. Then $C(SO_q(3))$ is the universal C^*-algebra generated by K, A, C, G, L satisfying* (14).

Proof. It follows from

Lemma 3.2. *Let $q \in (-1, 1)\backslash\{0\}$ and $\tilde{K}, \tilde{A}, \tilde{C}, \tilde{G}, \tilde{L}$ be bounded operators in a Hilbert space H, which satisfy* (14). *Then there exist bounded operators $\tilde{\alpha}, \tilde{\gamma}$ in H which satisfy* (9) *and* (13).

Proof. We analyse the representations of (14). □

Remark 1. Let $q \in (-1, 1)\backslash\{0\}$. Then [Ta] (using the language of Hopf algebras) denotes $SO_q(3)$ by $SO_{q^2}(3)$ and gives its relationship with $O_{q^2}(3)$ of [RTF], [Ta].

Proposition 3.3. (cf. [Ta], [P1, Remark 3 after Th. 2], [Z, Sect. 2.3]). *Let* $q \in [-1,1]\backslash\{0\}$. *Then* $SO_q(3)$ *is similar to* $SO_{-q}(3)$: *there exists a* C^**-isomorphism* $\rho_q : C(SO_q(3)) \longrightarrow C(SO_{-q}(3))$ *such that*

a) $\rho_q(K) = K,\ \rho_q(A) = iA,\ \rho_q(C) = iC,\ \rho_q(G) = G,\ \rho_q(L) = L,$
b) $\rho_q(d_1) = Qd_1Q^{-1}$, *where* $Q = \mathrm{diag}\,(1, -i, -1)$.

Proof. We define C^*-homomorphism $T_q : C(SU_q(2)) \longrightarrow C(SU_{-q}(2)) \otimes B(\mathbf{C}^2)$ by $T_q(\alpha) = \alpha \otimes \sigma_x,\ T_q(\gamma) = \gamma \otimes \sigma_y$ (cf. Sect. 1.1 of [Z] in the case of $q = -1$). Then $T_q : C(SO_q(3)) \longrightarrow C(SO_{-q}(3)) \otimes \mathrm{span}\{I, \sigma_z\}$. The eigenvalues $1, -1$ of σ_z correspond to C^*-homomorphisms $\rho_q, \tilde{\rho}_q : C(SO_q(3)) \longrightarrow C(SO_{-q}(3))$. Using (13), we get a) and an analogous formula for $\tilde{\rho}_q$, with i replaced by $-i$. Therefore $\tilde{\rho}_{-q} \circ \rho_q = id,\ \rho_q \circ \tilde{\rho}_{-q} = id,\ \rho_q$ is a C^*-isomorphism. The property b) follows from a). □.

Remark 2. (cf. [P1, Remark 3 after Th. 2]). Compact group of matrices $SO_1(3)$ is the image of $SU_1(2) = SU(2)$ under $p = d_1 : SU(2) \rightarrow SO_1(3)$. Since d_1 and β are both three-dimensional irreducible representations of $SU(2)$, there exists a matrix $M \in GL(3, \mathbf{C})$ such that $p(x) = M\beta(x)M^{-1}$, $x \in SU(2)$. Thus $SO_1(3) = MSO(3)M^{-1}$ is similar to $SO(3)$. We can take

$$M = \begin{bmatrix} -1, & -i, & 0 \\ 0, & 0, & 1 \\ -1, & i, & 0 \end{bmatrix}.$$

Due to Prop. 3.3, we have also $SO_{-1}(3) = Q^{-1}SO_1(3)Q$ is similar to $SO_1(3)$.

We want to describe subgroups and quotient spaces of $SO_q(3)$, $q \in [-1,1]\backslash\{0\}$. We start with

Proposition 3.4. *Let* H *be a subgroup of* $G = SU_q(2)$, $q \in [-1,1]\backslash\{0\}$, *such that* $\mathbf{Z}_2$ *is a subgroup of* H *(see Th. 2.1.c). We set* $\tilde{G} = SO_q(3)$, $\tilde{H} = (\theta_{HG}(C(\tilde{G})), (\theta_{HG}(d_{1,mn}))_{m,n=-1}^{1})$. *Then* $\tilde{H}$ *is a subgroup of* $\tilde{G}, C(\tilde{H}) = C(\mathbf{Z}_2\backslash H)$. *Moreover,* $C(\tilde{H}\backslash\tilde{G}) = C(H\backslash G) \subset C(\tilde{G}) \subset C(G)$, $\Gamma_{\tilde{H}\backslash\tilde{G}} = \Gamma_{H\backslash G}$ *and consequently* c_α, $\alpha \in \mathbf{N}$, *are the same for both quotient spaces,* c_α, $\alpha \in \mathbf{N} + 1/2$, *vanish.*

Proof. It is easy to check that $\tilde{H}$ is a quantum group and a subgroup of $\tilde{G}$ with $\Phi_{\tilde{H}} = \Phi_H|_{C(\tilde{H})}$,

$$\theta_{\tilde{H}\tilde{G}} = \theta_{HG}|_{C(\tilde{G})}\,. \tag{15}$$

Using

$$\theta_{\mathbf{Z}_2H}\theta_{HG} = \theta_{\mathbf{Z}_2G}\,, \tag{16}$$

we get

$$E_{\mathbf{Z}_2\backslash H}\theta_{HG}d_{\alpha,mn} = \begin{cases} \theta_{HG}d_{\alpha,mn} & \text{for } \alpha \in \mathbf{N} \\ 0 & \text{for } \alpha \in \mathbf{N} + 1/2\,. \end{cases}$$

Therefore $C(\mathbf{Z}_2\backslash H) = \theta_{HG}C(\tilde{G}) = C(\tilde{H})$. By virtue of (16), one can obtain $C(H\backslash G) \subset C(\mathbf{Z}_2\backslash G) = C(\tilde{G})$. Therefore, due to (15) and

$$\Phi_{\tilde{G}} = \Phi_G|_{C(\tilde{G})}\,, \tag{17}$$

$C(\tilde{H}\backslash\tilde{G}) = C(H\backslash G)$ as C^*-subalgebras of $C(\tilde{G})$. Using (17), we obtain $\Gamma_{H\backslash G} = \Gamma_{\tilde{H}\backslash\tilde{G}}$. Hence, Corollary 1.6 gives that c_α for $H\backslash G$ equals c_α for $\tilde{H}\backslash\tilde{G}$ if $\alpha \in \mathbf{N}$ and 0 if $\alpha \in \mathbf{N} + 1/2$. □.

Remark 3. Let us identify C^*-isomorphic objects. The assumptions of Prop. 3.4 are fulfilled for the following subgroups H of $SU_q(2)$:

1) $SU_q(2)$, $U(1)$, $\mathbf{Z}_{2m}(m \in \mathbf{N})$ for $q \in [-1,1]\backslash\{0\}$.
2) $\beta^{-1}(W)$, where W is any compact subgroup of $SO(3)$ for $q = 1$ (then $\tilde{H} = p(H) = MWM^{-1}$, i.e. $\tilde{H}$ is similar to W under M, hence c_α, $\alpha \in \mathbf{N}$, for $H\backslash SU(2)$ are the same as for $W\backslash SO(3)$, $c_\alpha, \alpha \in \mathbf{N} + 1/2$, vanish).
3) $G_{\beta^{-1}(W)}$, where W is any compact subgroup of $SO(3)$ such that $g_0 W g_0^{-1} \subset W$ for $q = -1$.

They are not fulfilled in the remaining cases:

1)′ $\mathbf{Z}_{2m+1}(m \in \mathbf{N})$ for $q \in [-1,1]\backslash\{0\}$,
2)′ $(\mathbf{Z}_{2m+1})_{\mathbf{n}}\,(m \in \mathbf{N})$ for $q = 1$ $((\mathbf{Z}_{2m+1})_{e_3} = \mathbf{Z}_{2m+1}$ for $q = 1)$,
3)′ the cases Ic)-Id) with odd n for $q = -1$.

It occurs that Prop. 3.4. gives all subgroups of $SO_q(3)$ for $q \in (-1,1)\backslash\{0\}$:

Theorem 3.5. $SO_q(3), q \in [-1,1]\backslash\{0\}$, *has the following subgroups*:

a) $SO_q(3) = (C(SO_q(3)), d_1) = SU_q^{\sim}(2)$,
b) $SO(2) = (C(S^1), \bar{z} \oplus I \oplus z) \approx U^{\sim}(1)$,
c) $C_n = (C(Z_n), \overline{z_{(n)}} \oplus I \oplus z_{(n)}) \approx \mathbf{Z}_{2n}^{\sim}, n = 1, 2, \ldots$.

For $q \in (-1,1)\backslash\{0\}$ the above list contains all subgroups of $SO_q(3)$ (up to C^*-isomorphisms, without repetitions).

Proof. We prove the first statement similarly as in Th. 2.1 (in order to prove the second relations we use the fact that C^*-homomorphisms $\psi : C(U(1)) \longrightarrow C(U(1))$ and $\psi_n : C(Z_n) \longrightarrow C(Z_{2n})$ defined by $\psi(z) = z^2, \psi_n(z_{(n)}) = z_{(2n)}^2, n = 1, 2, \ldots$, are faithful). Let now $H = (B, v)$ be a subgroup of $SO_q(3)$, $0 < |q| < 1$. Then

$$v = \begin{bmatrix} \tilde{L}^*, & -q^{-1}(q^2+1)\tilde{C}^*, & -q\tilde{G} \\ \tilde{A}^*, & I - (q^2+1)\tilde{K}, & \tilde{A} \\ -q\tilde{G}^*, & -q^{-1}(q^2+1)\tilde{C}, & \tilde{L} \end{bmatrix},$$

where the elements $\tilde{K}, \tilde{A}, \tilde{C}, \tilde{G}, \tilde{L}$ generate B and satisfy (14). Due to Lemma 3.2, there exist $\tilde{\alpha}, \tilde{\gamma}$, which satisfy (9) and (13). Let us replace $\tilde{\alpha}$ by $\tilde{\alpha} \oplus (-\tilde{\alpha})$ and $\tilde{\gamma}$ by $\tilde{\gamma} \oplus (-\tilde{\gamma})$ (it changes H only up to a C^*-isomorphism). Hence, $Sp\,\tilde{\gamma} = -Sp\,\tilde{\gamma}$. Moreover, we can assume that $\tilde{\alpha}, \tilde{\gamma}$ have one of two forms given in the proof of Th. 2.1. In the first case we use the equality

$$\Phi_H \tilde{G} = \tilde{L}^* \otimes \tilde{G} + (1 + q^{-2})\tilde{C}^* \otimes \tilde{A} + \tilde{G} \otimes \tilde{L} .$$

Similarly as for $SU_q(2)$ we get $Sp\,\Phi_H \tilde{G} = Sp\,\tilde{G} = Sp\,G$. Thus $Sp\,\tilde{\gamma} = Sp\,\gamma, \Delta = S^1$. We can identify H with $SO_q(3)$. In the second case $\tilde{K} = \tilde{A} = \tilde{C} = \tilde{G} = 0$, while $\tilde{L} = U^2$. But $\Phi_H \tilde{L} = \tilde{L} \otimes \tilde{L}$, hence $Sp\,\tilde{L} = S^1$ or $Sp\,\tilde{L} = Z_n, n = 1, 2, \ldots$. We get (up to a C^*-isomorphism) the cases b) or c). These subgroups are distinct since the corresponding C^*-algebras are nonisomorphic. □.

Combining Prop. 2.5 with Prop. 3.4, one gets the values of c_α, $\alpha \in \mathbf{N}$, for those subgroups.

Under similarities $SO_{-1}(3) \sim SO_1(3) \sim SO(3)$, compact subgroups of $SO_{-1}(3)$, $SO_1(3)$ and $SO(3)$ are in one to one correspondence, hence $c_\alpha, \alpha \in \mathbf{N}$, are the same for quotient spaces in all three cases. Those values are given by

Proposition 3.6. *Let W be a compact subgroup of $SO(3)$. Consider the quotient space $W\backslash SO(3)$. Then $c_k, k \in \mathbf{N}$, are equal*

a) $c_0 = 1, c_k = 0$ *for* $k > 0$ *in the case of* $W = SO(3)$,
b) $c_k = 1$ *for* $k \in \mathbf{N}$ *in the case of* $W = SO(2)_{\mathbf{n}}$,
c) $c_k = 1$ *for* $k \in 2\mathbf{N}, c_k = 0$ *for* $k \in 2\mathbf{N}+1$ *in the case of* $W = DO(2)_{\mathbf{n}}$,
d) $c_k = 2E(k/m) + 1$ *in the case of* $W = C_{m,\mathbf{n}}$,
e) $c_k = E(k/m) + \delta_k$ ($\delta_k = 1$ *for* $k \equiv 0$ (*mod.* 2), $\delta_k = 0$ *otherwise*) *in the case of* $W = D_{m,\mathbf{n},\phi}$,
f) $c_k = E(k/6) + \delta_k$ ($\delta_k = 1$ *for* $k \equiv 0,3,4$ (*mod.* 6), $\delta_k = 0$ *otherwise*) *in the case of* $T_{\mathbf{n},\phi}$,
g) $c_k = E(k/12) + \delta_k$ ($\delta_k = 1$ *for* $k \equiv 0,4,6,8,9,10$ (*mod.* 12), $\delta_k = 0$ *otherwise*) *in the case of* $O_{\mathbf{n},\phi}$,
h) $c_k = E(k/30) + \delta_k$ ($\delta_k = 1$ *for* $k \equiv$ 0, 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28 (*mod.* 30), $\delta_k = 0$ *otherwise*) *in the case of* $I_{\mathbf{n},\phi}$.

Proof. Due to Th. 1.7, c_k is equal to the multiplicity of the trivial representation of W in $\theta_{W\,SO(3)}d_k$. Let $\chi_k = Tr(\theta_{W\,SO(3)}d_k)$ and dw denotes the Haar measure on W. Then $c_k = (\chi_0|\chi_k) = \int_W \chi_k(w)dw$. Using the formula $\chi_k(w) = \sum_{s=-k,-k+1,\dots,k} e^{is\phi}$ (where ϕ is the angle of the rotation w), we get the above results. □.

Now we pass to the description of quotient spaces for $SU_q(2), q = \pm 1$. The cases 1), 1)′, 2) of Remark 3 are already solved. The case 3) is given by

Proposition 3.7. *Let W be a compact subgroup of $SO(3)$ such that $g_0 W g_0^{-1} \subset W$. Then*
1. (cf. [Z, Sect. 2.3.]) *Under similarities*

$$SO_{-1}(3) \sim^{\rho_1} SO_1(3) \sim SO(3)$$

we have

$$G_{\tilde{\beta^{-1}(W)}} \sim \tilde{\beta^{-1}(W)} \sim W$$

(*up to C^*-isomorphisms*).
2. *The quotient spaces* $G_{\beta^{-1}(W)}\backslash SU_{-1}(2), \beta^{-1}(W)\backslash SU(2), G_{\tilde{\beta^{-1}(W)}}\backslash SO_{-1}(3)$, $\tilde{\beta^{-1}(W)}\backslash SO_1(3), W\backslash SO(3)$ *have the same coefficients* $c_\alpha, \alpha \in \mathbf{N}$ ($c_\alpha, \alpha \in \mathbf{N} + 1/2$ *vanish for the first two quotient spaces*).

Proof. 1. We denote $Z = \beta^{-1}(W) \subset SU(2)$. We shall prove the first similarity. One has

$$\theta_{\tilde{G}_Z SO_{-1}(3)}\rho_1(d_1) = \theta_{G_Z SU_{-1}(2)}\rho_1 \begin{bmatrix} L^*, & -2C^*, & -G \\ A^*, & 1-2K, & A \\ -G^*, & -2C, & L \end{bmatrix}$$

$$= \bigoplus_{w\in Z} \pi_w \begin{bmatrix} L^*, & 2iC^*, & -G \\ -iA^*, & 1-2K, & iA \\ -G^*, & -2iC, & L \end{bmatrix} \approx \bigoplus_{w\in Z} \begin{bmatrix} \bar{a}^2, & 2\bar{a}c\sigma_z, & -c^2 \\ -\bar{a}\bar{c}\sigma_z, & 1-2\bar{c}c, & -ac\sigma_z \\ -\bar{c}^2, & 2a\bar{c}\sigma_z, & a^2 \end{bmatrix}$$

$$\approx \bigoplus_{w\in Z} \begin{bmatrix} \bar{a}^2, & -2\bar{a}c, & -c^2 \\ \bar{a}\bar{c}, & 1-2\bar{c}c, & ac \\ -\bar{c}^2, & -2a\bar{c}, & a^2 \end{bmatrix} \approx \theta_{Z\,SU(2)}d_1 = \theta_{\tilde{Z}\,SO_1(3)}d_1$$

(we used the fact $\begin{pmatrix} a, & -\bar{c} \\ c, & \bar{a} \end{pmatrix} \in Z \Rightarrow \begin{pmatrix} a, & \bar{c} \\ -c, & \bar{a} \end{pmatrix} \in Z$). Thus $\lambda\theta_{\tilde{Z}SO_1(3)} = \theta_{\tilde{G}_Z SO_{-1}(3)}\rho_1$ for some C^*-isomorphism $\lambda : C(\tilde{Z}) \longrightarrow C(\tilde{G}_Z)$. Therefore $G_{\beta^{-1}(W)}^{\sim} \sim \beta^{-1}\tilde{(W)}$. The second similarity is given in Remark 3.
2. It follows from 1. and Prop. 3.4. □.

In the case 2)′ the coefficients $c_\alpha, \alpha \in \mathbf{N}/2$, for $(\mathbf{Z}_{2m+1})_\mathbf{n}\backslash SU(2)$ are the same as in Prop. 2.5 in the case of $H = \mathbf{Z}_{2m+1}((\mathbf{Z}_{2m+1})_\mathbf{n}$ are similar one to another under unitary matrices; such similarities do not change $SU(2)$). It remains to consider the case 3)′ : the case Id) with odd n is covered by the case 1)′, while the case Ic) with odd n is solved by

Proposition 3.8. *Let* $Z = \{O_{2\pi k/n}, K_{2\pi k/n+\phi_0-\pi/2} : k = 0, 1, \ldots, n-1\}$, *where* $0 \leqq \phi_0 < 2\pi/n$, *n is odd. Consider the quotient space* $G_Z\backslash SU_{-1}(2)$. *Then* $c_l = E(l/n) + \delta_l, c_{l+1/2} = E\left(\frac{2l+1}{n}\right) - E\left(\frac{l}{n}\right)$, *where* $\delta_l = 1$ *for l even,* $\delta_l = 0$ *for l odd,* $l \in \mathbf{N}$.

Proof. Since $Z \subset L$, $C(G_Z)$ is commutative. Using Th. 1.5 of [W3], we can identify G_Z with $\{O_{2\pi k/n}, S_{2\pi k/n+\phi_0-\pi/2} : k = 0, 1, \ldots, n-1\} \subset U(2)$, where

$$S_\phi = \begin{pmatrix} 0, & e^{-i\phi} \\ e^{i\phi}, & 0 \end{pmatrix} .$$

Then $c_l = \int_{G_Z} \chi_l(w)dw$, $l \in \mathbf{N}/2$, where $\chi_l = Tr(\theta_{G_Z SU_{-1}(2)} d_l)$ and dw denotes the Haar measure on G_Z. Using the formula $\chi_l\chi_{1/2} = \chi_{l-1/2} + \chi_{l+1/2}$, $l \geqq 1/2$ (it follows from $d_l \textcircled{T} d_{1/2} \simeq d_{l-1/2} \oplus d_{l+1/2}$), we get $\chi_l(O_\phi) = \sum_{s=-2l,-2l+2,\ldots,2l} e^{is\phi}$, $l \in \mathbf{N}/2, \chi_l(S_\phi) = (-1)^l$ for $l \in \mathbf{N}, \chi_l(S_\phi) = 0$ for $l \in \mathbf{N} + 1/2$. An easy computation gives the result. □.

Remark 4. (cf. [P1, Remark 3 after Th. 2]). Let $(X_{q\lambda\rho}, \sigma_{q\lambda\rho})$ be as in [P1, Sect. 3], $q \in [-1, 1]\backslash\{0\}$, and $e_k, k = -1, 0, 1$, be the corresponding generators. Then

$$\sigma_{q\lambda\rho} : C(X_{q\lambda\rho}) \longrightarrow C(X_{q\lambda\rho}) \otimes C(SO_q(3)) \subset C(X_{q\lambda\rho}) \otimes C(SU_q(2)) .$$

We set

$$\sigma'_{q\lambda\rho} = (id \otimes \rho_q)\sigma_{q\lambda\rho} : C(X_{q\lambda\rho}) \longrightarrow C(X_{q\lambda\rho}) \otimes C(SO_{-q}(3))$$
$$\subset C(X_{q\lambda\rho}) \otimes C(SU_{-q}(2)) .$$

The C^*-isomorphism $\Lambda : C(X_{q\lambda\rho}) \longrightarrow C(X_{-q\lambda\rho})$ given by $\Lambda(e_{-1}) = -ie_{-1}, \Lambda(e_0) = e_0, \Lambda(e_1) = ie_1$ identifies $(X_{q\lambda\rho}, \sigma'_{q\lambda\rho})$ with $(X_{-q\lambda\rho}, \sigma_{-q\lambda\rho})$. Therefore, using [P1, Remark 2 after Th. 2], we get that

a) $(S^2_{\pm 10}, \sigma_{\pm 10}) = (X_{\pm 101}, \sigma_{\pm 101})$ is a unique (up to an isomorphism) quantum sphere for $q = \pm 1$.
b) The above object and $(X_{\pm 1,1,(l^2-1)/4}, \sigma_{\pm 1,1,(l^2-1)/4}), l = 2, 3, \ldots$, are unique (up to an isomorphism) objects which satisfy the assumptions and the condition (i) of [P1, Th. 1] for $q = \pm 1$.

Using Prop. 3.4, Th. 3.5 and [P1, Sect. 6], we get the following realisations for the quotient quantum sphere (with the corresponding action of the group):

$$(X_{q,1-q^2,1}, \sigma_{q,1-q^2,1}) = (S^2_{q0}, \sigma_{q0}) \approx U(1)\backslash SU_q(2) = SO(2)\backslash SO_q(3) ,$$

$q \in [-1, 1]\backslash\{0\}$.

Remark 5. Here we use the terminology of [P5]. Let $q \in [-1,1]\backslash\{0\}, c \in [0,\infty]$ ($c = 0$ for $q = \pm 1$). Then A of Remark 4 identifies $\mathscr{A}_c$ for q with $\mathscr{A}_c$ for $-q$. Using that identification one can check that $(S^\wedge, \sigma^\wedge, d, *)$ is a $\binom{2}{\cdot}$-dimensional exterior algebra on S^2_{qc}, invariant w.r.t. σ_{qc} iff $(S^\wedge, (id \otimes \rho_q)\sigma^\wedge, d, *)$ is a $\binom{2}{\cdot}$-dimensional exterior algebra on S^2_{-qc}, invariant w.r.t. σ_{-qc}. The same holds if we restrict ourselves to $S^{\wedge 0} \oplus \dots S^{\wedge k}$ for some $k = 1, 2, \dots$, (with suitable restrictions of all structures in $\mathbf{S}^\wedge$, without $*$ or with $*$) instead of $\mathbf{S}^\wedge$. That and [P5, Theorem] for $q = 1$ prove that [P5, Theorem] holds also for $q = -1$ (cf. the remarks at the beginning of [P5, Sect. 2]).

Remark 6. The results of the paper give a proof that the Haar measure is faithful for all $SU_q(2)$, $SO_q(3)$, $q \in [-1,1]\backslash\{0\}$ (the proof in [P1, Sect. 4] is valid only for $SU_q(2)$, $q \in (-1,1)\backslash\{0\}$, the case of $SU_{-1}(2)$ follows from [Z, Sect. 2.3]). Indeed, let $G = (A, u)$ be a quantum group, h be the Haar measure, $J = \{b \in A : h(b^*b) = 0\}$ be the corresponding closed two-sided ideal, $\pi : A \longrightarrow A/J$ be the canonical projection and $u_r = (id \otimes \pi)u$. Then, according to [W3, p.656], $G_r = (A/J, u_r)$ is a quantum subgroup of $G, \{\pi(u^\alpha)\}_{\alpha \in \hat{G}}$ is the set of all nonequivalent irreducible unitary representations of G_r. Thus $c_\alpha = \delta_{0\alpha}$ for the quotient space $G_r\backslash G$. Therefore, for $G = SU_q(2)$ or $SO_q(3)$, our results show that $G_r = G$ (up to a C^*-isomorphism), $J = \{0\}, h$ is faithful.

Remark 7. Throughout the paper we dealt with the right actions. We say that a C^*-homomorphism $\Gamma : C(X) \longrightarrow C(G) \otimes C(X)$ is a left action of a quantum group G on a quantum space X if

a) $(id \otimes \Gamma)\Gamma = (\Phi_G \otimes id)\Gamma$,
b) $\langle (y \otimes I)\Gamma x : x \in C(X), y \in C(G) \rangle = C(G) \otimes C(X)$.

In that case a vector subspace $W \subset C(X)$ corresponds to a representation v of G if there exists a basis $e_1, \dots, e_d$ in W such that $d = \dim v$ and $\Gamma e_k = v_{km} \otimes e_m, k = 1, 2, \dots d$. It occurs that for $G = SU_q(2)$ or $G = SO_q(3), q \in [-1,1]\backslash\{0\}$, there exists a bijective correspondence between the right and the left actions on any quantum space X : if Γ is a right action then $\Gamma' = (\lambda \otimes id)\sigma\Gamma$ is a left action (C^*-isomorphisms $\lambda : C(G) \longrightarrow C(G)$ and $\sigma : C(X) \otimes C(G) \rightarrow C(G) \otimes C(X)$ are given by $\lambda(\alpha) = \alpha$, $\lambda(\gamma) = \gamma^*$, $\sigma(x \otimes y) = y \otimes x, x \in C(X)$, $y \in C(G)$; one has $\Phi_G\lambda = (\lambda \otimes \lambda)\sigma\Phi_G$, $e_G\lambda = e_G$). That correspondence gives a bijective correspondence between right and left quantum spheres (defined analogously as in the present paper).

There exist matrices $S_l \in M_{2l+1}(\mathbf{C})$, $l = 0, 1, \dots$, such that $\lambda(d_l)^T = (S_l^T)^{-1} d_l S_l^T$. We can take

$$S \equiv S_1 = (S_{1,ij})_{i,j=-1,0,1} = \begin{pmatrix} 1, & 0, & 0 \\ 0, & -q(q^2+1)^{-1}, & 0 \\ 0, & 0, & 1 \end{pmatrix} .$$

Let us consider the right quantum spheres $(S^2_{qc}, \sigma_{qc}), q \in [-1,1]\backslash\{0\}, c \in [0,\infty]$ (for $q = \pm 1$ we have $c = 0$), and their generators e_k, $k = -1, 0, 1$. We denote the corresponding left quantum spheres by $(S^{2'}_{qc}, \sigma'_{qc})$. Then $S^{2'}_{qc} = S^2_{qc}$. Setting $e'_k = S_{ik} e_i$, one has $\sigma'_{qc} e'_k = d_{1,kj} \otimes e'_j$, $k = -1, 0, 1$. We put $a' = (S^{-1} \otimes S^{-1})a \in M_{9 \times 1}(\mathbf{C})$, $b' = (S^{-1} \otimes S^{-1})bS$, $c' = (S^{-1} \otimes S^{-1})cS_2$. Then

$$a'^T (b'^T, c'^T, \text{resp.}) \text{ intertwines } d_1 \textcircled{T} d_1 \text{ with } d_0 (d_1, d_2, \text{resp.}).$$

One can use $X'_{q\lambda\rho} = X_{q\lambda\rho}$, $\sigma'_{q\lambda\rho}$, e'_k, a', b', c', instead of $X_{q\lambda\rho}$, $\sigma_{q\lambda\rho}$, e_k, a, b, c. Cf. also [NM].

All the results of the present paper preceding this remark, [P1, Thm. 1 and Thm. 2], [P2, Thm. 1] and [P5, Thm.] remain true if we use the left actions and the left quantum spheres (after appropriate modifications, e.g. Eq. (5) takes form $E^{\alpha}_{sm}E^{\beta}_{ij} = [(\rho^{\beta}_{ij} \otimes \rho^{\alpha}_{sm})\Phi_G \otimes id]\Gamma = \delta_{js}\delta_{\alpha(\beta)}E^{\beta}_{im}, e_{\alpha is} = E^{(\alpha)}_{1s}e_{\alpha i1}, E_{G/H} = (id \otimes h_H)(id \otimes \theta_{HG})\Phi_G$, for $c \in [0, \infty]$ we use the embedding $\psi' = \lambda\psi : \psi'(e'_i) = d_{1,ik}s'_k$ with $s'_k = S_{jk}s_j, \psi'$ gives the isomorphism $(S^{2'}_{q0}, \sigma'_{q0}) \approx SU_q(2)/U(1), q \in [-1, 1]\backslash\{0\}, c = 0)$.

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