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The Galilean Group in 2 + 1 Space-Times and its Central Extension

S.K. Bose

Department of Physics, University of Notre Dame, Notre Dame, Indiana, 46556, U.S.A.
FAX: 219-631-5952. E-mail: Bosel @nd.edu

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Abstract: The problem of constructing the central extensions, by the circle group, of the group of Galilean transformations in two spatial dimensions; as well as that of its universal covering group, is solved. Also solved is the problem of the central extension of the corresponding Lie algebra. We find that the Lie algebra has a three parameter family of central extensions, as does the simply-connected group corresponding to the Lie algebra. The Galilean group itself has a two parameter family of central extensions. A corollary of our result is the impossibility of the appearance of non-integer-valued angular momentum for systems possessing Galilean invariance.

Ever since the pioneering work of Wigner [1] it has been appreciated that the representations of a symmetry group that are appropriate to quantum physics are the projective unitary (or anti-unitary) representations. That is, representations in a projective space P of a separable Hilbert space H that describes the state-space of a quantum-mechanical system. This idea also finds a reflection in the domain of classical mechanics [2]. Indeed, Wigner showed us how to understand the appearance of spin one-half particles in terms of the projective unitary irreducible representations of the Poincaré group. It also meanwhile became clear that the projective representations of a group are constructible from a knowledge of the ordinary (linear) representations of an associated group, which is the central extension of the original group by the circle group. Thus Bargmann [3] carried out his path-breaking analysis of the projective representations of continuous groups; in particular, of the Galilean group in $(3+1)$ space-times and showed how the concept of mass, with its associated superselection rule, arises via the central extension of the Galilean group. Later authors [4] provided further elaboration of the projective representations of the Galilean group as well as of the concept of a non-relativistic zero-mass system [4, 5].

The aim of the present paper is to solve the problem of finding central extensions of the proper Galilean group in $(2+1)$ space-time dimensions. There are several reasons for studying this problem. First, it is intrinsically interesting; the structure of the $(2+1)$ dimensional Galilean group is significantly different from that of the

$(3+1)$ dimensional Galilean group. For the latter, the subgroup of homogeneous transformations is perfect; not so for the former, where it is, instead, solvable. Secondly, it is felt that the problem may throw some light on the structure of non-relativistic systems that are effectively confined to two spatial dimensions (planar systems). In this context, let us recall that it has been claimed [6] that the angular momentum for a planar system need not be integer-valued and can have the spectrum integer $+\alpha$, where α is any value in the interval [0, 1]. This proposal, made by Wilczek [6], has been challenged by Divakaran [7], who notes that the planar rotation group $SO(2)$ does not admit a non-integer valued angular momentum since the Pontryagin dual of $SO(2)$ is Z – the additive group of integers (failure to appreciate this fact constitutes a widespread misconception among physicists, arising out of the situation that , *at the level of the Lie algebra*, the planar angular momentum is unrestricted. The restriction to integer quantisation follows from the representations of the *group*). So the only question that remains is whether non-integer-valued angular momentum could arise via the projective unitary representations. The answer is no; the group $SO(2)$ does not possess any central extension by the circle group [9], as Divakaran [7] notes. (This conclusion is also implicit in Theorem 7.2 of Bargmann's paper [3] which we now quote: "*Every continuous ray representation of a compact connected abelian Lie group G is induced by a representation of G.*" Thus although it is true that the universal cover of $SO(2)$ is the group R (additive group of reals) this fact is quite irrelevant, there being no connection between the universal cover of $SO(2)$ and its possible central extension, since $SO(2)$ is not semi-simple. To nail the situation down further, Divakaran [7] next studies the central extension of the Poincaré group in $2+1$ space-times. The homogeneous part of this group – the group $SO(2, 1)$ – is isomorphic with $PSL(2, R)$ and is semi-simple. Utilizing, amongst other things, a general theorem due to Raghunathan [8], Divakaran [7] next concludes that the universal central extension (for extensions by the circle group) of the $(2+1)$ dimensional Poincaré group is a semi-direct product of the subgroup of space-time translations with the universal covering group of $PSL(2, R)$. Unfortunately, this last object is a nasty beast. For instance, it cannot be realized as a sub-group of $GL(n, C)$, regardless of how large the positive integer n might be, even its *presentation* is fairly involved [10], and its representation theory, at the moment, unknown. Because of this reason, Divakaran [7] was unable to complete his investigation in the manner he had originally set out to do.

It occurred to the author that questions concerning planar angular momentum could also be addressed within the framework of the Galilean group in $(2+1)$ space-times. After all, this is what Bargmann [3] did to understand non-relativistic particle with definite mass and spin in three spatial dimensions. Moreover, the whole controversy has arisen in the context of physical systems that are entirely non-relativistic.

This paper is organized as follows. In the next section, the central extension problem is studied at the level of the Lie algebra, that is, central extensions of the Lie algebra by R—— the one-dimensional real vector space. A three parameter set of central extensions is found. In Sect. 3, assorted known facts concerning the Galilean group (denoted by G hereafter) and its universal covering group (denoted $\overline{G}$ in the sequel) are collected under one roof. In Chapter 4 the problem of central extension of G is solved. It is found that of the three (families of) central extensions of the Lie algebra (denoted Lie(G)) only two "exponentiate" to the group G. The central extension of $\overline{G}$ is considered next, in Chapter 4A. It is found, as expected on general grounds, that all three extensions of Lie(G) get elevated to those of the

group $\overline{G}$. The mutual relationship between the central extensions of G and $\overline{G}$ is explored further in this chapter. The final conclusions concerning our structure of the extended groups are summarized in Sect. 4B. In Chapter 5, we make concluding remarks.

A corollary of our results for the central extension of G is this. The rotation subgroup of the extended groups continue to be $SO(2)$. Thus in the linear representation of the extended groups (and hence in the projective representation of the original group G), the spectrum of angular momentum must be integral. Anyons cannot arise via the projective unitary representations of G.

2. The Lie Algebra of the 2 + 1 Dimensional Galilean Group and its Central Extension

Let G denote the Galilean group in $(2+1)$ space-times and Lie(G) its Lie algebra. We choose a basis for Lie(G) in which the infinitesimal generators of rotation, the boosts along the two spatial directions, that of time translation and those of spatial translation are denoted respectively as M, N_i, H and P_i $(i = 1,2)$. The commutation relations for these operators are

$$\begin{aligned} &[M,N_i] = \varepsilon_{ij}\, N_j, && [N_i,N_j] = 0\,, \\ &[P_i,H] = 0, && [P_i,P_j] = 0\,, \\ &[M,P_i] = \varepsilon_{ij}\, P_j, && [N_i,P_j] = 0\,, \\ &[M,H] = 0, && [N_i,H] = P_i\,. \end{aligned} \tag{2.1}$$

In the above, ε_{ij} is the antisymmetric symbol with $\varepsilon_{12} = -\varepsilon_{21} = 1$. Summation convention for a repeated index is implied. The physical significance of the generators are well-known. M corresponds to the angular momentum in the plane, H the Hamiltonian and P_i the components of linear momentum.

To carry out the central extension of Lie(G), we have found it convenient to relabel the generators. The six generators of Lie(G) are now denoted L_i, with i running from 1 to 6, with the following identification:

$$L_1 = M, L_2 = H, L_3 = N_1, L_4 = N_2, L_5 = P_1, L_6 = P_2\,. \tag{2.2}$$

Equation (1) may now be written in the form

$$[L_i,L_j] = C_{ij}^k L_k\,, \quad i,j,k = 1,\ldots 6\,, \tag{2.3}$$

where the non-vanishing structure constants are

$$C_{41}^3 = C_{13}^4 = C_{32}^5 = C_{61}^5 = C_{15}^6 = C_{42}^6 = 1\,, \tag{2.4}$$

and those that follow from the antisymmetry of C_{ij}^k in the two lower indices.

Again, the middle Latin indices take up values 1 and 2 in Eq. (2.1), whereas in Eq. (2.3) they take up values 1 to 6. This should cause no confusion.

We proceed to carry out the desired central extension of Lie(G). Let $\widehat{L}_i$ together with a set d_{ij} of central operators, generate the extended Lie algebra. We have

$$[\widehat{L}_i,\widehat{L}_j] = C_{ij}^k \widehat{L}_k + d_{ij}\,, \tag{2.5}$$

where the central generators d_{ij} commute with $\widehat{L}_i$ and with each other and possess the antisymmetry property $d_{ij} = -d_{ji}$. Further, they satisfy the closure condition

$$C^m_{ij} d_{mk} + C^m_{jk} d_{mi} + C^m_{ki} d_{mj} = 0 \tag{2.6}$$

that follows from the Jacobi identity for the extended algebra (2.5). Certain solution of Eq. (2.6) are, for our purpose, trivial. These correspond to exact forms on the Lie algebra, which can be set equal to zero by a suitable redefinition of the $\widehat{L}_i$ generators [2]. Thus we are representing each equivalence class of closed modulo exact forms on Lie(G) by means of a typical representative from that class. To solve Eq. (2.6) with the aid of Eq. (2.4), along the lines indicated, we proceed in the following eight steps.

(1) By suitable redefinitions of the $\widehat{L}_i$ generators we can set

$$d_{13} = d_{14} = d_{15} = d_{16} = 0 \,. \tag{2.7}$$

The result just stated becomes intuitively obvious when we recall that the sets (L_1, L_3, L_4) and (L_1, L_5, L_6) of operators generate two isomorphic copies of the Lie algebra of the Euclidean group in a plane.

(2) Take $i = 1$, $j = 2$ and $k = 5$ to derive from Eq. (2.6) that

$$d_{26} = 0 \,. \tag{2.8}$$

(3) Take $i = 1$, $j = 2$ and $k = 6$ to conclude from Eq. (2.6) that

$$d_{25} = 0 \,. \tag{2.9}$$

(4) Take $i = 3$, $j = 6$ and $k = 2$ in Eq. (2.6) to derive

$$d_{56} = 0 \,. \tag{2.10}$$

(5) Take $i = 1$, $j = 2$ and $k = 3$ in Eq. (2.6) and use Eq. (2.7) to derive

$$d_{24} = 0 \,. \tag{2.11}$$

(6) Take $i = 1$, $j = 3$ and $k = 4$ use Eqs. (2.6) and (2.7) to obtain

$$d_{23} = 0 \,. \tag{2.12}$$

(7) Take the combinations $i = 1$, $j = 3$, $k = 5$ and $i = 1$, $j = 5$, $k = 3$ to derive two equations that together lead to

$$d_{45} = d_{36} = 0 \,. \tag{2.13}$$

(8) Take $i = 1$, $j = 3$ and $k = 6$ to derive

$$d_{35} = d_{46} \,. \tag{2.14}$$

It is now a matter of straightforward, if somewhat tedious, calculation to verify that all the remaining components of Eq. (2.6) are now identically satisfied. Thus we are left with three independent central generators d_{12}, d_{34} and $d_{35}(= d_{46})$. We have thus proved the

Theorem. *The vector space of central extensions of Lie* (G) *is three-dimensional.*

Finally, we may note that in terms of the notation of Eq. (2.1) the structure of our centrally extended algebra is given by

$$\begin{aligned}
&[M,N_i] = \varepsilon_{ij}\ N_j, && [N_i,N_j] = \varepsilon_{ij}\ d\ ,\\
&[H,P_i] = O, && [P_i,P_j] = O\ ,\\
&[M,P_i] = \varepsilon_{ij}\ P_j, && [N_i,P_j] = \delta_{ij}\ m\ ,\\
&[M,H] = \ D, && [N_i,H] = P_i
\end{aligned} \tag{2.15}$$

where we have set $\mathrm{d}_{12} = D$, $\mathrm{d}_{34} = d$ and $d_{35} = m$.

3. The Galilean Group in 2 + 1 Dimensions

Let x denote the coordinate of a space with two dimensions and t that of time. The Galilean transformations are

$$\begin{aligned}
x' &= wx + tv + u\,,\\
t' &= t + \eta\,.
\end{aligned} \tag{3.1}$$

Here u and v are two-dimensional vectors signifying space translation and boost respectively and η a real number (time translation). w represents rotation in the plane. The set of all transformations (3.1) form the Galilean group G, under composition. If we write the element r of G in the fashion (w,η,v,u), then the multiplication law is

$$(w,\eta,v,u)\circ(w',\eta',v',u') = (ww',\eta+\eta',v+wv',u+wu'+\eta'v)\,. \tag{3.2}$$

Notice that $r \to M(r)$ is a faithful representation, where $\mathrm{M}(r)$ is the 3×3 matrix

$$M(r) = \begin{pmatrix} w & v & u \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{3.3}$$

The unit element of G is $(1,0,0,0)$ and the inverse of r is

$$r^{-1} = \left(w^{-1}, -\eta, -w^{-1}v, -w^{-1}(u-\eta v)\right)\,. \tag{3.4}$$

Let us look at the subgroups of G. The subgroup A of all translations in space-time

$$A = \{(1,\eta,0,u)\} \tag{3.5}$$

is normal and closed in G. The closed subgroup G_0

$$G_0 = \{(w,0,v,0)\} \tag{3.6}$$

is the homogeneous Galilean group. It is a solvable Lie group with degree of solvability equal to 2. Every $r \in G$ admits the decomposition

$$(w,\eta,v,u) = (1,\eta,0,u)\circ(w,0,v,0)$$

and further

$$(w,0,v,0)(1,\eta,0,u)(w,0,v,0)^{-1} = (1,\eta,0,wu+\eta v)\,, \tag{3.7}$$

which displays explicitly the fact that G is semi-direct product of A with G_0. The subgroup G_0 is a (closed) Lie subgroup of G (under the quotient topology). The subgroup A is naturally isomorphic with R^3. We can analyze G_0 further. Let M and H denote the subgroups of G_0,

$$M = \{(1,0,v,0)\}\,,$$
$$H = \{(w,0,0,0,)\}\,. \tag{3.8}$$

M is closed, normal subgroup of G_0 and

$$(w,0,0,0,)\circ(1,0,v,0)(w,0,0,0)^{-1} = (1,0,wv,0)\,. \tag{3.9}$$

Thus G_0 is a semi-direct product of M with H, with respect to the above action. M is naturally isomorphic with R^2 and H with the group $SO(2)$. Finally, note the subgroup $E(2)$ of G

$$E(2) = \{(w,0,0,u)\}\,, \tag{3.10}$$

which is the Euclidean group on the plane. It is a semi-direct product of $U = \{(1,0,0,u)\}$ – the subgroup of space translations – with H, given explicitly by

$$(w,0,0,0)(1,0,0,u)(w,0,0,0)^{-1} = (1,0,0,wu)\,. \tag{3.11}$$

The action wu and wv that appears in the foregoing expressions (3.2)–(3.11) can be made explicit by choosing suitable coordinates in G (canonical coordinates). We display H – the subgroup of rotations – as the multiplicative group of complex numbers of unit magnitude. The element r of G is now written as

$$r = (z,\eta,v,u)\,, \tag{3.12}$$

where η, v, u are as before and z a complex number with $|z| = 1$. The action, wu, wv are now given explicitly by

$$wv = \rho(\theta)v\,,$$

where θ is the argument of $z\Big(z = \exp(i\theta)\Big)$ and $\rho(\theta)$ is the 2×2 matrix

$$\rho(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3.13}$$

that act on the two component vectors u and v by the matrix rule.

3A. The Universal Covering Group of G. For the sake of completeness of discussions and also because of the fact that we propose to discuss this concept in the next section, we consider $\overline{G}$ —— the universal covering group of the Galilean group in $(2+1)$ dimensional space-times. The group $\overline{G}$ is the semi-direct product of A (subgroup of space-time translations) with $\overline{G}_0$, which is the universal covering group of the homogeneous Galilean group G_0. The subgroup $\overline{G}_0$ is the semi-direct product of M (subgroup of boosts) with the universal cover $\overline{H}$ of H; $\overline{H}(=\overline{SO(2)})$ is naturally isomorphic with R —— the additive group of reals.

An explicit coordinatisation of $\overline{G}$ is afforded by

$$\overline{G} = \{(z,\eta,v,u)\}\,, \tag{3.14}$$

where η, v, u are as before and z *is a real number*. The multiplication rule is

$$(z,\eta,v,u)\circ(z',\eta',v',u') = (z+z',\eta+\eta',v+zv',u+zu'+\eta' v)\,, \tag{3.15}$$

and the actions $zu \equiv \rho(z)u, zv \equiv \rho(z)v$ with

$$\rho(z) = \begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix}\,. \tag{3.16}$$

In summary, the description of the two groups G and $\overline{G}$ run parallel. The only difference is that the group parameter z for G in (3.12) is a complex number of unit magnitude, whereas z is a real number for $\overline{G}$ in (3.14).

4. Central Extensions of the Galilean Group in (2 + 1) Dimensions

Let G denote the Galilean group in $(2+1)$ space-times. We consider central extensions of G by T —— the circle group. Recall that a central extension is a triple $(\widehat{G}, i, \pi)$,where i and π are (continuous) homomorphisms such that the sequence

$$1 \to T \xrightarrow{i} \widehat{G} \xrightarrow{\pi} G \to 1$$

is exact, and $i(T)$ is central in $\widehat{G}$, which is the centrally extended group. It is clear from above that i is injective (exactness at T) and π is surjective (exactness at G). We recall now the basic result in the theory of group extensions: *Central extensions of G are numbered by the elements of $H^2(G,T)$ – the second cohomology group of G with coefficients in T.* In other words, $H^2(G,T)$ is the group of group extensions. Let $Z^2(G,T)$ denote the group of 2-cocycles, its elements $\gamma(x,y)$ are T-valued (continuous) functions on $G\times G$ that satisfy the cocycle identities:

$$\gamma(xy,z)\gamma(x,y) = \gamma(x,yz)\gamma(y,z)\,, \tag{4.1a}$$

$$\gamma(1,x) = \gamma(x,1) = 1 \tag{4.1b}$$

for $x,y,z \in G$, that follow from the associativity of multiplication in $\widehat{G}$. Let $B^2(G,T)$ denote the subgroup of 2-coboundaries consisting of those functions $\gamma : G\times G \to T$ for which there exists $\beta : G \to T$ such that

$$\gamma(x,y) = \beta(xy)\beta(y)^{-1}\beta(x)^{-1} \tag{4.2}$$

for all $(x,y) \in G\times G$. The second cohomology group $H^2(G,T)$ is the factor group $Z^2(G,T)$ modulo $B^2(G,T)$. Its elements are equivalence classes in which two cocycles that differ by a coboundary

$$\gamma'(x,y) = \gamma(x,y)\beta(xy)\beta(y)^{-1}\beta(x)^{-1} \tag{4.3}$$

are identified.

The facts that we have cited above are standard knowledge and can be found in books [11,12] or in the article of Raghunathan [8]. It should also perhaps be noted that in the theory of projective representations of groups [13] the 2-cocycles are called multipliers (and coboundaries the exact multipliers). With these preliminaries out of the way, we can now proceed to tackle the problem at hand.

We will adopt the following procedure. For each equivalence class of 2-cocycles, we shall select a typical representative. Then our task will amount to finding solutions to the cocycle identities (4.1a) and (4.1b) and making sure that the cocycle that we have found is not a coboundary. For the latter purpose, a very simple criterion has been given by Bargmann [3]. Let $\gamma(x,y)$ be a T-valued 2-cocycle and $\xi(x,y)$ the corresponding R-valued cocycle

$$\gamma(x,y) = \exp\{i\ a\ \xi(x,y)\} \quad a \in R^* , \tag{4.4}$$

where a is a non-zero real number. Then $\gamma(x,y)$ is trivial, $\xi(x,y)$ cohomologous to zero, provided

$$\xi(x,y) = \xi(y,x) \tag{4.5}$$

whenever the elements $x, y \in G$ commute with each other $xy = yx$. This follows easily from Eqs. (4.4) and (4.3). As far as construction of $\gamma(x,y)$ is concerned, we will again follow the procedure of Bargmann [3] which involves the construction of suitable homogeneous polynomials in the group elements.

We present our results. Let $r, r' \in G$ be as follows

$$r = (w, \eta, v, u), \quad r' = (w', \eta', v', u') . \tag{4.6}$$

Then there is a T-valued 2 cocycle $\gamma_1(r,r')$ given by

$$\gamma_1(r,r') = \exp\{i\ a\ m_1(r,r')\} \quad a \in R^* \tag{4.7}$$

with

$$m_1(r,r') = \langle u, wv' \rangle - \langle v, wu' \rangle + \eta' \langle v, wv' \rangle , \tag{4.7a}$$

where $\langle\ ,\ \rangle$ denotes the Euclidean inner product on 2-space.

Proof. By explicit verification that (4.7), (4.7a) satisfy the cocycle identities (4.1a), (4.1b), which is a matter of elementary calculations. □

The next task is to verify that the above $\gamma_1(r,r')$ is not trivial. Choose $r = (1,0,0,u)$ and $r' = (1,0,v',0)$, then $rr' = r'r = (1,0,v',u)$ but

$$m_1(r,r') - m_1(r',r) = 2\langle u, v' \rangle , \tag{4.8}$$

and the right-hand side of (4.8) need not, and in general, does not vanish. □

There is a second solution to the cocycle identities. This is

$$\gamma_2(r,r') = \exp\{ib\ m_2(r,r')\} \quad b \in R^* \tag{4.9}$$

with

$$m_2(r,r') = v \wedge wv' , \tag{4.9a}$$

where $\wedge$ connotes the determinant of the two vectors. That is, for any pair q and p of two-dimensional vectors $q \wedge p = q_1 p_2 - q_2 p_1$ in terms of the components of the vectors. Proof of the above assertion, again, is by explicit verification (of the cocycle identities) which is elementary. Next select $r = (1,0,v,u)$ and $r' = (1,0,v',u')$ so that $rr' = r'r = (1,0,v+v',u+u')$. Now, $m_2(r,r') = v \wedge v'$ and $m_2(r',r) = v' \wedge v$ and thus $m_2(r,r') \neq m_2(r',r)$ for non-parallel vectors v and v'. Thus our 2-cocycle is not trivial. □

There is no other non-trivial 2-cocycle for G.

The central extension of G that corresponds to the 2-cocycle $\gamma_1(r,r')$ is the global version of the central extension of Lie (G) that is heralded by the central generator m in Eq. (2.15). It is the analogue, in the present case, of that found in the case of the Galilean group of $(3+1)$ space times and has the same interpretation (the Bargmann superselection rule for mass). The additional central extension corresponding to the 2-cocycle $\gamma_2(r,r')$ is specific to the present case, it has no counterpart in the $(3+1)$ dimensional Galilean group. Furthermore, it is the global analogue of the Lie algebra extension that corresponds to the central generator d in Eq. (2.15). So far as the central generator D in Eq. (2.15) is concerned, it has no global extension from Lie (G) to G. However, it does elevate to a central extension of the universal covering group $\overline{G}$, as we show next.

4A. Central Extension of the Universal Covering Group. Let $\overline{G}$ be the universal covering group of the Galilean group in $(2+1)$ space times, as before. Bearing in mind our results in Sect. 2, we know that $\overline{G}$ possesses exactly three (families of) central extensions, in view of a general theorem [14] that we now quote: " *For a connected and simply-connected Lie group G, the central extensions of G (by T) are in bijective correspondence with those of Lie (G) (by R).*"

First of all, it is easy to check that the 2-cocycles $\gamma_1(r,r')$ and $\gamma_2(r,r')$ possess lifts from G to $\overline{G}$. Let $r,r' \in \overline{G}$, be as follows:

$$r = (z,\eta,v,u) \quad , \quad r' = (z',\eta',v',u') \, , \tag{4.10}$$

then the lifts of $\gamma_1(r,r')$ and $\gamma_2(r,r')$ are obtained by substituting (4.6) by (4.10) and (3.13) by (3.16) in the expressions (4.7a) and (4.9a). The remaining 2-cocycle is easily found. It is given by

$$\gamma_3(r,r') = \exp\{icf(r,r')\} \quad c \in R^* \tag{4.11}$$

with

$$f(r,r') = z\eta' - z'\eta \, . \tag{4.11a}$$

The proof is again by direct verification of the cocycle condition, Eqs. (4.1a), (4.1b). It is also easy to check that $f(r,r')$ is not equivalent to zero. Just take $r = (z,\eta,0,0)$, $r' = (z',\eta',0,0)$; then $rr' = r'r$ but $f(r,r') \neq f(r',r)$, in general.

We wish to understand in some detail as to why the cocycle $\gamma_3(r,r')$ does not survive the passage from $\overline{G}$ to G. Let h be the projection $h : \overline{G} \to G$, and let h_* be the induced homomorphism of cohomology groups, $h_* : H^2(\overline{G},T) \to H^2(G,T)$; and set $h_* f = f_*$. The homomorphism h is given explicitly by

$$h : (z,\eta,v,u) \to (e^{2\pi i z},\eta,v,u) \, . \tag{4.12}$$

The kernel of h consists of elements

$$\ker h = \{(n,0,0,0)\}, \quad n \in Z \, , \tag{4.13}$$

and thus $f_*(r,r')$ must vanish (see Eq. (4.1b)) whenever $r \in \overline{G}$ is of the form $r = (n,0,0,0)$, $n \in Z$ and r' arbitrary. From bilinearity of $f(r,r')$ it follows immediately that $f_*(r,r')$ vanishes also, for arbitrary $r,r' \in \overline{G}$. $\square$

The result that we have established by explicit construction above may also be seen on somewhat more abstract grounds. The subgroup of G, generated by elements of the form $(w,\eta,0,0)$, is isomorphic with $T \times R$. The corresponding subgroup of

$\overline{G}$ is $R \times R$. The result of the previous paragraph would imply that $H^2(T \times R, T)$ is trivial, whereas $H^2(R \times R, T)$ is not. Bearing in mind known facts [9] that $H^2(T,T)$ and $H^2(R,T)$ are trivial, an application of the Künneth formula to $T \times R$ yields the isomorphism $H^2(T \times R, T) = H^1(T,R)$. Now, the first cohomology group H^1 is the group of "crossed homomorphisms" [11]. In the present case, because T and R mutually commute (T action on R trivial), $H^1(T,R)$ becomes the group of ordinary homomorphisms. But there is no non-trivial homomorphism $T \to R$. Thus $H^1(T,R)$ and hence also $H^2(T \times R, T)$ is trivial. The same argument when applied to the subgroup $R \times R$ of $\overline{G}$ leads to the isomorphisms $H^2(R \times R, T) = H^1(R,R) = R$.

4B. The Extended Groups. Corresponding to the two cocycles γ_1 and γ_2 that we have found for G, there are two centrally extended groups. We may assemble these into one big group $\widehat{G}$. The elements of $\widehat{G}$ are of the form $(r; t_1, t_2)$ where r is as in (4.6) and $(t_1, t_2) \in T \times T$ with the multiplication rule

$$(r; t_1, t_2)(r';\ t_1', t_2') = (rr';\ \gamma_1(r,r')t_1t_1', \gamma_2(r,r')t_2t_2') \tag{4.14}$$

and the inverse

$$(r;\ t_1, t_2)^{-1} = (r^{-1};\ t_1^{-1}, t_2^{-1})\,, \tag{4.15}$$

since $\gamma_1(r,r^{-1}) = \gamma_2(r,r^{-1}) = 1$, as follows from Eqs. (3.4), (4.7) and (4.9). Thus $\widehat{G}$ is the extension of G by the torus T^2.

As far as the universal covering group $\overline{G}$ is concerned, it will have three centrally extended groups. We can construct an universal central extension $U(\overline{G})$ in the sense of reference [8] as follows. The elements of $U(\overline{G})$ are of the form $(r; x_1, x_2, x_3)$, where r is as in (4.10) and $(x_1, x_2, x_3) \in R \times R \times R$. The multiplication law is

$$\begin{aligned}(r; x_1, x_2, x_3) \circ (r'; x_1', x_2', x_3') = (rr'; x_1 + x_1' + m_1(r,r'), x_2 + x_2' + m_2(r,r')\,,\\ x_3 + x_3' + f(r,r'))\,,\end{aligned} \tag{4.16}$$

where the real-valued cocycles m_1, m_2 and f have been described before, in Sect. 4A. Note $U(\overline{G})$ is simply-connected, and has, as its Lie algebra, the Lie algebra central extension of Lie (G), given by Eq. (2.15).

5. Concluding Remarks

We have carried out the central extensions of the $(2+1)$ dimensional Galilean group G, of its universal cover $\overline{G}$ and of their Lie algebra. The questions concerning the physical significance of the results is tied up with the problem of representations of the group extensions. We shall address this question on a separate occasion. However, one corollary of our results is quite obvious. The structure of the centrally extended group is such that its rotation subgroup continues to be the group $SO(2)$. Consequently, non-integral angular-momenta cannot arise as projective representations of G.

What moral should one draw from the above exercise concerning the question of the possible existence of anyons? We did not prove that anyons cannot exist; only that they cannot appear via representations of the Galilean group. To accommodate anyons, then, one possibility will be to dispense with group representations altogether. This presumably implies that anyons are not conventional *structureless* particles. Indeed, all known, models of anyons show internal structure; e.g., solitons

in 2 + 1 dimensional non-linear σ model [15], vortices of an abelian Higgs model with a Chern-Simons term [16], to quote two examples. Actually, the absence of group representation i.e., the lack of unitary implementability of the group operations is not, by itself, a new phenomenon. It has appeared before in the context of (compact) internal symmetry groups in the grab of "spontaneous breakdown" of symmetries. On the other hand, giving up group representations also means giving up a framework which provides us, unambiguously, with the concept of angular (and linear) momentum. A second possibility for anyons would be to discard the Galilean group and replace it with its universal covering group. Although the justification for such a procedure cannot validly be made from considerations of projective representations of the Galilean group, the effect itself could conceivably take place in any event; for instance, as a result of special dynamical properties of a model.

Our analysis, needless to say, does not throw any light on the question of exotic statistics [17–20], which is the other distinguishing characteristic of anyons.

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