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$SU(2)$ WZW Theory at Higher Genera

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Abstract: We compute, by free field techniques, the scalar product of the $SU(2)$ Chern-Simons states on genus > 1 surfaces. The result is a finite-dimensional integral over positions of "screening charges" and one complex modular parameter. It uses an effective description of the CS states closely related to the one worked out by Bertram [1]. The scalar product formula allows to express the higher genus partition functions of the WZW conformal field theory by finite-dimensional integrals. It should provide the hermitian metric preserved by the Knizhnik-Zamolodchikov-Bernard connection describing the variations of the CS states under the change of the complex structure of the surface.

1. Introduction

As noted in [2], there exists a close relation between the Chern-Simons (CS) topological gauge theory in 3D and the Wess-Zumino-Witten (WZW) model of conformal field theory in 2D. The (fixed time) quantum states of the CS theory on a Riemann surface Σ without boundary are solutions of the current algebra Ward identities of the WZW theory. The states of the CS theory are holomorphic functionals Ψ on the space $\mathscr{A}^{01}$ of (smooth) 0,1-forms A^{01} with values in the complexified Lie algebra $\mathbf{g}^{\mathbb{C}}$ of a compact Lie group G. The functionals Ψ are required to be invariant under the complex (chiral) gauge transformations $\Psi \mapsto {}^h\Psi$, where

$$ {}^h\Psi(A^{01}) = \mathrm{e}^{-kS(h,A^{01})}\,\Psi\big({}^{h^{-1}}\!A^{01}\big) \tag{1.1}$$

for $h : \Sigma \to G^{\mathbb{C}}$. In the above formula, $S(h,A^{01})$ denotes the action of the WZW model [3] in the external gauge field A^{01}. For a general gauge field, it takes the form

$$\begin{aligned} S(h,A^{10}+A^{01}) = &-\frac{i}{4\pi}\int_\Sigma \mathrm{tr}\,(h^{-1}\partial h)\wedge(h^{-1}\bar\partial h) - \frac{i}{12\pi}\int_\Sigma d^{-1}\mathrm{tr}\,(h^{-1}dh)^{\wedge 3} \\ &+ \frac{i}{2\pi}\int_\Sigma \mathrm{tr}\big[(h\partial h^{-1})\wedge A^{01} + A^{10}\wedge(h^{-1}\bar\partial h) \\ &+ hA^{10}h^{-1}\wedge A^{01} - A^{10}\wedge A^{01}\big]\,. \end{aligned} \tag{1.2}$$

The non-negative integer k is called the level of the model. The invariance $\Psi = {}^h\Psi$ is exactly the chiral gauge symmetry Ward identity for the WZW partition function. Moreover, adding static Wilson lines in the CS theory, one obtains the chiral Ward identities for the Green functions of the primary fields of the WZW theory. For the sake of simplicity, we shall concentrate here on the case of the WZW partition function and we shall take $G = SU(2)$.

The space of CS states has finite dimension. The CS states Ψ may be viewed as sections of a complex line bundle over the orbit space $\mathscr{A}^{01}/\mathscr{G}^{\mathbb{C}}$ of the group $\mathscr{G}^{\mathbb{C}}$ of complex gauge transformations. The orbits $\mathscr{G}^{\mathbb{C}}A^{01}$ are in one to one correspondence with the isomorphism classes of holomorphic vector bundles (h.v.b.) E of rank 2, with trivial determinant, given by the $SL(2,\mathbb{C})$-valued holomorphic 1-cocycles $(g_{\alpha\beta})$,

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}\,, \tag{1.3}$$

where $A^{01} = g_\alpha^{-1}\bar{\partial}g_\alpha$ locally and $g_{\alpha\beta} = g_\alpha g_\beta^{-1}$. If, for the genus g of $\Sigma > 1$, one limits oneself to the open dense (in the C^∞ topology) subset $\mathscr{A}_s^{01} \subset \mathscr{A}^{01}$ corresponding to the stable bundles, then the orbit space $\mathscr{A}_s^{01}/\mathscr{G}^{\mathbb{C}}$ becomes a complex variety $\mathscr{N}_s$ of dimension $3g-3 \equiv N$. Besides, $\mathscr{N}_s$ possesses a natural compactification $\mathscr{N}_{ss}$, the Seshadri moduli space of semi-stable bundles [4, 5]. The CS states coincide with the holomorphic sections of the k^{th}power of the natural determinant bundle $\mathscr{D}$ over $\mathscr{N}_{ss}$. The spaces $H^0(\mathscr{D}^k)$ of such sections have dimensions given by the Verlinde formula [6]. They form a holomorphic vector bundle $\mathscr{W}_k$ over the moduli space $\mathscr{M}$ of complex curves. This bundle may be equipped with a projectively flat "heat kernel" connection first described by Bernard [7], see also [2, 8, 9, 10], which generalizes the Knizhnik-Zamolodchikov connection [11] to the higher genus situation.

The partition function of the WZW model is formally given by the functional integral

$$Z(A^{10}+A^{01}) = \int e^{kS(g,A^{10}+A^{01})}Dg\,, \tag{1.4}$$

where Dg stands for the formal product $\prod_{x\in\Sigma} dg(x)$ of the Haar measures on G. It has been argued in [13] that the solution of (1.4) is given by

$$Z(A^{10}+A^{01}) = \sum_r \overline{\Psi_r(-(A^{01})^\dagger)}\Psi_r(A^{01})\, e^{\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}A^{10}\wedge A^{01}}\,, \tag{1.5}$$

for an arbitrary basis (Ψ_r) of the CS states orthonormal with respect to the scalar product corresponding to the norm

$$\|\Psi\|^2 = \int |\Psi(A^{01})|^2\, e^{\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\,A^{10}\wedge A^{01}} DA\,. \tag{1.6}$$

The functional integral in (1.6) is over the unitary gauge fields $A \equiv A^{10}+A^{01}$ with $A^{10} = -(A^{01})^\dagger$. This way the calculation of the partition functions in the WZW theory is reduced to that of the scalar product of the CS states.

Formal arguments show that the scalar product (1.6) should supply the bundle $\mathscr{W}_k$ of CS states with a hermitian structure preserved by the Knizhnik-Zamolodchikov-Bernard (KZB) connection, see [9]. Proving rigorously the metricity of the KZB connection is an important mathematical problem in conformal field theory still left open. The purpose of this work is to provide a candidate for its (constructive) solution by computing exactly the functional integral at Eq. (1.6). The result will have a form of an explicit finite-dimensional integral of a positive measure. Similar work has been done in the case of genus zero with insertions in [12, 13] for $G = SU(2)$ and in [14] for a general simple G. The integral formulae for the scalar product are the dual versions of the expressions for conformal blocks of the WZW theory in terms of the generalized hypergeometric integrals [15, 16, 17], at the core of the relations between the WZW models and the quantum groups and of the recently discovered relation between the WZW model and the Bethe ansatz [18, 19, 20, 21]. The extension to higher genera has required a nontrivial generalization of the low genus arguments and has taken some time.

The paper is organized as follows. In Sect. 2 we describe a slice in the space of gauge fields, transversal to generic $\mathscr{G}^{\mathbb{C}}$-orbits. It corresponds to realizing generic rank 2 determinant 1 holomorphic vector bundles as extensions of a one-parameter family of degree $g-1$ line bundles. In Sect. 3, we examine the restrictions of CS states to the slice. They become sections of the k^{th} power of the determinant bundle of the family of extensions. This picture of the higher genus CS states is closely related to the one worked out in [1], see also [22], based on considering the extensions of fixed degree g line bundle[1]. The relation between the two presentations is the subject of Sect. 4. Section 5 describes a projective version of the scalar product formula, from which surface dependent constants were omitted. It has been extracted from the full-fledged formula discussed later for the sake of a moderately interested reader who would not like to dwell into the details of the functional integration which occupies most of the rest of the paper. And so, in Sect. 6, using the slice of the space $\mathscr{A}^{01}$, we decompose the functional integral (1.6) to the one over $\mathscr{G}^{\mathbb{C}}$ and over the orbit space. The Jacobian of the relevant change of variables is computed in Sect. 6 by free field functional integration. The crucial Sect. 7 performs the integration over $\mathscr{G}^{\mathbb{C}}$ by reducing it to an iterative Gaussian integral. Finally, Sect. 8 assembles the complete formula for the scalar product. In Appendices, besides treating a number of technical points, we work out the details of the relation of our description of CS states to that of [1] (Appendix C) and submit the scalar product formula to simple consistency checks (Appendix F). What remains to be proven, however, is that the finite-dimensional integrals appearing in the formula actually always converge resulting in a hermitian metric on the bundle $\mathscr{W}_k$ of state spaces which is preserved by the KZB connection. What is also missing is an interpretation of the formula in terms of modular geometry, providing a counterpart of the analysis of the KZB connection carried out in [8]. As the first step in this direction one could try to simplify the formal arguments given below. Numerous cancellations occurring in intermediate steps of the calculation suggest that such simplification should be possible.

This is a non-rigorous work in its manipulation of formal functional integrals which lead, in the end, to a chain of Gaussian integrations. Handling these integrals required, nevertheless, careful treatment. As a result, the paper employs relatively sophisticated mathematical tools. It may be viewed as a piece of "theoretical mathematics" in the sense of [23]: it uses formal functional integral to extract an interesting mathematical structure which should be submitted now to rigorous analysis. Care was taken to clearly mark the non-rigorous steps in the discussion. The main result of the calculation performed here was announced in the note [24]. The case with insertion points will be treated in a separate publication.

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[1] We thank B. Feigin and S. Ramanan for attracting our attention to [1] and [22].

2. Space of Orbits

We shall need below an effective description of generic orbits $\mathscr{G}^{\mathbb{C}}A^{01}$. It will be based on the fact that every h.v.b. E has a line subbundle $L^{-1} \subset E$ or, equivalently, that the cocycle $(g_{\alpha\beta})$ of E may be always chosen in the triangular form

$$g_{\alpha\beta} = \begin{pmatrix} a_{\alpha\beta}^{-1} & b_{\alpha\beta} \\ 0 & a_{\alpha\beta} \end{pmatrix}, \tag{2.1}$$

where $(a_{\alpha\beta})$ is a 1-cocycle of a holomorphic line bundle (h.l.b.) L s.t. the dual bundle $L^{-1} \subset E$. $(b_{\alpha\beta})$ satisfy the twisted cocycle condition

$$a_{\alpha\beta}^{-1} b_{\beta\gamma} + b_{\alpha\beta} a_{\beta\gamma} = b_{\alpha\gamma} \tag{2.2}$$

which means that they define a holomorphic 1-cocycle with values in the h.l.b. L^{-2} (this may be better seen by rewriting Eq. (2.2) as $a_{\alpha\beta}^{-2} b'_{\beta\gamma} + b'_{\alpha\beta} = b'_{\alpha\gamma}$, where $b'_{\alpha\beta} \equiv a_{\alpha\beta}^{-1} b_{\alpha\beta}$). The corresponding cohomology class $[(b_{\alpha\beta})]$ in $H^1(L^{-2})$ describes E as an extension

$$0 \to L^{-1} \to E \to L \to 0 \tag{2.3}$$

of the line bundle L by L^{-1}. Proportional $[(b_{\alpha\beta})]$ give rise to isomorphic bundles (but the converse may be not true). By the Riemann-Roch Theorem,

$$\dim(H^1(L^{-2})) = g - 1 + 2\deg(L) \qquad \text{for} \qquad \deg(L) > 0\,. \tag{2.4}$$

Let $L(\pm x)$ denote the h.l.b. $L\mathscr{O}(\pm x)$ (omitting the sign of the tensor product between the bundles), where $\mathscr{O}(\pm x)$ is the degree ± 1 h.l.b. with divisor $\pm x$. We shall fix for the rest of the paper a h.l.b. L of degree g. For later convenience, we shall assume that this is done so that $L(-x)^2$ never coincides[2] with the canonical line bundle K of Σ. For any rank 2 h.v.b. bundle E with trivial determinant, there exists $x \in \Sigma$ and a non-trivial homomorphism

$$\phi : L(-x)^{-1} \to E\,, \tag{2.5}$$

see [5], Lemma 5.4. If ϕ has zeros (counted with multiplicities) at $x_1,\dots,x_r$, then ϕ induces an embedding of $L(-x - x_1 \cdots - x_r)^{-1}$ into E or, in other words, E is an extension of $L(-x - x_1 \cdots - x_r)$. Notice that $\deg(L(-x - x_1 \cdots - x_r)) = g - 1 - r$. If E is stable then it can have only negative degree subbundles so that, necessarily, $r < g - 1$ and, moreover, the extension has to be nontrivial.

The above discussion gives rise to the following description of the orbits $\mathscr{G}^{\mathbb{C}}A^{01}$, related to the picture of the moduli space $\mathscr{N}$ discussed in the papers [1, 22]. For $0 \leqq r < g - 1$, consider a holomorphic family $(L(-x - x_1 \cdots - x_r))$ of line bundles. By definition, it is a holomorphic line bundle $\mathscr{L}_r$ over[3] $\Sigma^{r+1} \times \Sigma$ whose restriction to the fiber $pr_1^{-1}(\{x,x_1,\dots,x_r\})$ of the projection on the first factor gives $L(-x - x_1 \cdots - x_r) \equiv L(-X_r)$. $\mathscr{L}_r$ is not unique: for each h.l.b. M over Σ^{r+1}, we may take $pr_1{}^*(M)\mathscr{L}_r$ instead of $\mathscr{L}_r$. Let W_r denote the first direct image $R^1 pr_{1*}(\mathscr{L}_r^{-2})$ of $\mathscr{L}_r^{-2}$ by pr_1 (W_r is a h.v.b. of dimension $N - 2r$ ($N \equiv 3g - 3$) over Σ^{r+1} with fibers $H^1(L(-X_r)^{-2})$). Let $\mathbb{P}W_r$ denote the corresponding holomorphic bundle of projective spaces $\mathbb{P}H^1(L(-X_r)^{-2})$. The total dimension of the

[2] as opposed to the choice of L employed in [24]

[3] We denote by Σ^n the symmetrized n-fold Cartesian product of Σ.

compact complex manifold $\mathbb{P}W_r$ is $N-r$. Now, each element $w \in \mathbb{P}W_r$, with the base point $\{x, x_1, \dots, x_r\} \equiv X_r$, defines (uniquely up to isomorphisms) a holomorphic bundle E of rank 2 and trivial determinant which is an extension of $L(-X_r)$. (Many w's may define the same E.) The dimensions imply that, generically, $r=0$ (recall that $\dim(\mathcal{N}) = N$). From this and the analysis of [1] and [22], the following picture of the orbit space emerges:

> an open dense subset of $\mathbb{P}W_0$ corresponding to stable bundles is a ramified ($2g$-fold) cover of a dense subset of the stable moduli space $\mathcal{N}_s$. The rest of $\mathcal{N}_s$ is in the image of subsets of $\mathbb{P}W_r$. In particular, the union of the $\mathcal{G}^{\mathbb{C}}$-orbits corresponding to h.v.b.'s E obtained from W_0 is dense in $\mathcal{A}^{01}$.

Other details of that geometry may be found in [1, 22].

We shall construct gauge fields corresponding to points in $\mathbb{P}W_0$ (a slice $\mathcal{S} : \mathbb{P}W_0 \to \mathcal{A}^{01}$). Let us start by an explicit construction of a family $(L(-x))$ of line bundles on Σ and of the corresponding bundle W_0. A more natural but less explicit construction will be discussed in the next section and, a somewhat pedantic, distinction between different realizations of the family $(L(-x))$ will later play an important role. In order to describe the first construction, fix a point $x_0 \in \Sigma$ and denote $L_0 \equiv L(-x_0)$. The family $(L(-x))$ will be obtained by twisting the $\bar\partial$-operator in L_0. Let $(\omega^i)_{i=1}^g$ be the basis of holomorphic 0,1-forms on Σ adapted to a marking of Σ, i.e. to a choice of a standard homology basis (a_i, b_j). $\int_{a_i} \omega^j = \delta^{ij}$ and $\int_{b_i} \omega^j = \tau^{ij}$, where τ is the period matrix. Define a 0,1-form

$$a = \pi \sum_{i,j=1}^{g} \left(\int_{x_0}^{x} \omega^i \right) \left(\frac{1}{\operatorname{Im}\, \tau} \right)_{ij} \bar\omega^j \equiv \pi \left(\int_{x_0}^{x} \omega \right) (\operatorname{Im} \tau)^{-1} \bar\omega \,. \tag{2.6}$$

Notice, that $a \equiv a_x$ depends on the lift $\mathbf{x}$ of the point x to the covering space $\tilde\Sigma$ of Σ (with the base point x_0). Denote by $L_{\mathbf{x}}$ the line bundle L_0 with $\bar\partial$ replaced by $\bar\partial_{L_{\mathbf{x}}} \equiv \bar\partial + a_{\mathbf{x}}$. It is a standard fact that all $L_{\mathbf{x}}$ corresponding to the same x are isomorphic to $L(-x)$. Consider the holomorphic bundle $\tilde\Sigma \times L_0$ over $\tilde\Sigma \times \Sigma$ with the antiholomorphic derivative $\bar\delta + \bar\partial$, where $\bar\delta$ differentiates in the trivial direction of $\tilde\Sigma$. We shall twist $\tilde\Sigma \times L_0$ by replacing its antiholomorphic derivative by $\bar\delta + \bar\partial + a$. Denote the resulting h.l.b. over $\tilde\Sigma \times \Sigma$ by $\tilde{\mathcal{L}}_0$. It gives a specific realization of a holomorphic family $(L_{\mathbf{x}})$. The action of the fundamental group $\Pi_1(\Sigma, x_0) \equiv \Pi_1$ on $\tilde\Sigma$ lifts to an action on $\tilde{\mathcal{L}}_0$ preserving the structure of the h.l.b. The lifted action of $p \in \Pi_1$ is

$$(\mathbf{x}, l_y) \mapsto (p\mathbf{x}, c_p(y)^{-1} l_y) \tag{2.7}$$

for l_y in the fiber of $L_{\mathbf{x}}$ over $y \in \Sigma$, where

$$c_p(y) = \mathrm{e}^{2\pi i \operatorname{Im}[(\int_p \omega)(\operatorname{Im}\tau)^{-1}(\int_{x_0}^{y} \bar\omega)]} \,. \tag{2.8}$$

Note that c_p is a function on Σ (it does not depend on the integration path from x_0 to y). Dividing $\tilde{\mathcal{L}}_0$ by the action of Π_1, we obtain a h.l.b. $\mathcal{L}_0$ over $\Sigma \times \Sigma$, the first explicit realization of the holomorphic family $(L(-x))$.

For a line bundle M, we shall denote by $\Gamma(M)$ the space of smooth sections of L and by $\wedge^{01}(M)$ the space of smooth 0,1-forms with values in M. The bundle $\widetilde{W}_0 = R^1 pr_{1*}(\tilde{\mathcal{L}}_0^{-2})$ (the first direct image of $\tilde{\mathcal{L}}_0^{-2}$ under pr_1) may be viewed as the quotient of the infinite-dimensional trivial bundle $\tilde{\Sigma} \times \wedge^{01}(L_0^{-2})$ by the subbundle whose fiber over $\mathbf{x}$ is the image by $\bar{\partial}_{L_{\mathbf{x}}^{-2}} \equiv \bar{\partial} - 2a_{\mathbf{x}}$ of $\Gamma(L_0^{-2})$. Indeed,

$$\wedge^{01}(L_0^{-2})/\bar{\partial}_{L_{\mathbf{x}}^{-2}}(\Gamma(L_0^{-2})) = \wedge^{01}(L_{\mathbf{x}}^{-2})/\bar{\partial}_{L_{\mathbf{x}}^{-2}}(\Gamma(L_{\mathbf{x}}^{-2})) \cong H^1(L_{\mathbf{x}}^{-2}), \tag{2.9}$$

which is the Dolbeault realization of $H^1(L_{\mathbf{x}}^{-2})$. Division by the action of Π_1 gives an explicit construction of the fiber bundle W_0.

Let us construct now a gauge field A^{01} whose $\mathscr{G}^{\mathbb{C}}$-orbit corresponds to a given point $w \in W_0$. To this end, we shall choose a smooth isomorphism U of rank 2 vector bundles over Σ with trivial determinants,

$$U : L_0^{-1} \oplus L_0 \to \Sigma \times \mathbb{C}^2 . \tag{2.10}$$

Let us twist the holomorphic structure of the vector bundle $E_0 \equiv L_0^{-1} \oplus L_0$ by replacing its $\bar{\partial}$-operator by

$$\bar{\partial} + \begin{pmatrix} -a_{\mathbf{x}} & b \\ 0 & a_{\mathbf{x}} \end{pmatrix} \equiv \bar{\partial} + B^{01}_{\mathbf{x},b} , \tag{2.11}$$

where $b \in \wedge^{01}(L_0^{-2})$. We shall denote the twisted bundle by E. Note that E is an extension of the line bundle $L_{\mathbf{x}}$ by $L_{\mathbf{x}}^{-1}$. We may use the smooth isomorphism U of (2.10) to transport the holomorphic structure from E to the trivial bundle where we get the $\bar{\partial}$-operator

$$\bar{\partial} + UB^{01}_{\mathbf{x},b}U^{-1} + U\bar{\partial}U^{-1} \equiv \bar{\partial} + A^{01}_{\mathbf{x},b} . \tag{2.12}$$

Let c be a constant $\neq 0$ or $c = c_p$, see Eq. (2.8), and let $v \in \Gamma(L_0^{-2})$.

$$g_{c,v} \equiv \begin{pmatrix} c^{-1} & cv \\ 0 & c \end{pmatrix} \tag{2.13}$$

is a smooth section of the bundle $\mathrm{Aut}(L_0^{-1} \oplus L_0)$ of automorphisms of $L_0^{-1} \oplus L_0$. The gauge transformation $B^{01}_{\mathbf{x},b} \mapsto {}^{g_{c,v}^{-1}}B^{01}_{\mathbf{x},b} \equiv g_{c,v}^{-1}B^{01}_{\mathbf{x},b}g_{c,v} + g_{c,v}^{-1}\bar{\partial}g_{c,v}$ preserves the form of the gauge field $B^{01}_{\mathbf{x},b}$ shifting

$$a_{\mathbf{x}} \mapsto a_{\mathbf{x}} + c^{-1}\bar{\partial}c \qquad \text{and} \qquad b \mapsto c^2(b + (\bar{\partial} - 2a_{\mathbf{x}})v) . \tag{2.14}$$

In particular, for $c = c_p$, $a_{\mathbf{x}} \mapsto a_{\mathbf{x}} + c_p^{-1}\bar{\partial}c_p = a_{p\mathbf{x}}$. The corresponding fields $A^{01}_{\mathbf{x},b}$ are gauge-related by

$$h_{c,v} \equiv Ug_{c,v}U^{-1} \in \mathscr{G}^{\mathbb{C}} , \tag{2.15}$$

so that they lie in the same $\mathscr{G}^{\mathbb{C}}$-orbit. Taking constant $c \neq 0$, we see that b leading to the same class in the projective space

$$\mathbb{P}(\wedge^{01}(L_0^{-2})/\bar{\partial}_{L_{\mathbf{x}}^{-2}}(\Gamma(L_0^{-2}))) \cong \mathbb{P}H^1(L_{\mathbf{x}}^{-2}) \tag{2.16}$$

give gauge fields $A^{01}_{\mathbf{x},b}$ in the same $\mathscr{G}^{\mathbb{C}}$-orbit. The class of b in $\mathbb{P}H^1(L_{\mathbf{x}}^{-2})$ is exactly the one describing the rank 2 h.v.b. E, an extension of $L_{\mathbf{x}}$, associated to the orbit $\mathscr{G}^{\mathbb{C}}A^{01}_{\mathbf{x},b}$. Choosing $\mathbf{x}$ in a fundamental domain in $\tilde{\Sigma}$ and one b in each class of

$\mathbb{P}H^1(L_{\mathbf{x}}^{-2})$, we obtain a slice $\mathscr{S} : \mathbb{P}W_0 \to \mathscr{A}^{01}$ which cuts a generic orbit a finite number ($= 2g$) of times. One may take $\mathscr{S}$ to be piecewise holomorphic.

3. Determinant Bundle

Let us fix a hermitian structure on the h.l.b. L_0. It induces a metric connection on L_0 whose covariant derivative in the antiholomorphic direction coincides with the $\bar{\partial}$ operator. Let F_0 denote the curvature form of this connection (normalized so that $\int_\Sigma \frac{i}{2\pi} F_0 = \deg(L_0) = g - 1$). The hermitian metric on L_0 induces a hermitian structure and a connection on L_0^{-1} and, putting both together, a hermitian structure and a connection on $L_0^{-1} \oplus L_0$. Let us denote by ∇ the holomorphic covariant derivative in $L_0^{-1} \oplus L_0$. The complete covariant derivative is $\nabla + \bar{\partial}$. Clearly, its curvature form

$$\operatorname{curv}(\nabla + \bar{\partial}) = \begin{pmatrix} -F_0 & 0 \\ 0 & F_0 \end{pmatrix} . \tag{3.1}$$

Let us assume that the smooth isomorphism $U : L_0^{-1} \oplus L_0 \to \Sigma \times \mathbb{C}^2$ maps the hermitian metric of $L_0^{-1} \oplus L_0$ into the one coming from the standard scalar product of $\mathbb{C}^2$. Using U, we may transport the connection on $L_0^{-1} \oplus L_0$ to the trivial bundle $\Sigma \times \mathbb{C}^2$:

$$U(\nabla + \bar{\partial})U^{-1} = d + U\nabla U^{-1} + U\bar{\partial}U^{-1} \equiv d + A_0^{10} + A_0^{01} \equiv d + A_0 . \tag{3.2}$$

The right-hand side gives a unitary connection, so that $A_0^{10} = -(A_0^{01})^\dagger$. The curvature forms are related by

$$\operatorname{curv}(U(\nabla + \bar{\partial})U^{-1}) = F(A_0) \equiv dA_0 + A_0 \wedge A_0 = U \begin{pmatrix} -F_0 & 0 \\ 0 & F_0 \end{pmatrix} U^{-1} . \tag{3.3}$$

We shall represent the CS states by holomorphic sections of a line bundle over $\mathbb{P}W_0$. To this end, let us define, for a CS state Ψ, a holomorphic function

$$\psi(\mathbf{x}, b) \equiv \exp\left[\frac{ik}{2\pi} \int_\Sigma \operatorname{tr} A_0^{10} \wedge A_{\mathbf{x},b}^{01} \right] \Psi(A_{\mathbf{x},b}^{01}) \tag{3.4}$$

of $\mathbf{x} \in \tilde{\Sigma}$ and $b \in \wedge^{01}(L_0^{-2})$. As is shown in Appendix F, only the normalization of ψ depends on the choice of the hermitian structure on L_0 and of the isomorphism U. Since the $\mathscr{G}^{\mathbb{C}}$-orbits of the chiral gauge fields $A_{\mathbf{x},b}^{01}$ form a dense subset of $\mathscr{A}^{01}$, Ψ is uniquely determined by the function ψ. The gauge relations between the forms $A_{\mathbf{x},b}^{01}$ induce constraints for functions ψ, due to the gauge invariance of CS states Ψ. In particular,

$$\psi(\mathbf{x}, b + (\bar{\partial} - 2a_{\mathbf{x}})v) = \exp[kS(h_v, A_0^{10} + A_{\mathbf{x},b}^{01})]\ \psi(\mathbf{x}, b) , \tag{3.5}$$

$$\psi(\mathbf{x}, c^2 b) = \exp[kS(h_c, A_0^{10} + A_{\mathbf{x},b}^{01})]\ \psi(\mathbf{x}, b) , \tag{3.6}$$

$$\psi(p\mathbf{x}, c_p^2 b) = \exp[kS(h_{c_p}, A_0^{10} + A_{\mathbf{x},b}^{01})]\ \psi(\mathbf{x}, b) , \tag{3.7}$$

where h_v is given by Eq. (2.15) with $c = 1$, h_c by the same formula with $v = 0$ and $c \in \mathbb{C}^\times \equiv \mathbb{C} \backslash \{0\}$ and h_{c_p} by setting $v = 0$ and $c = c_p$, see (2.8).

Let us study these transformation properties in greater detail. The first two equations become much more transparent if we rewrite them in the infinitesimal form. Since $\frac{\delta}{\delta h}\big|_{h\equiv 1} S(h,A) = \frac{i}{2\pi}F(A)$, we obtain

$$\begin{aligned}\frac{\delta}{\delta v}\Big|_{v\equiv 0} S(h_v, A_0^{10} + A_{\mathbf{x},b}^{01}) &= \frac{i}{2\pi}\,\mathrm{tr}\,\sigma^+ U^{-1} F(A_0^{10} + A_{\mathbf{x},b}^{01})U \\ &= \frac{i}{2\pi}\,\mathrm{tr}\,\sigma^+(\mathrm{curv}(\nabla + \bar\partial) + \nabla(B_{\mathbf{x},b}^{01})) \\ &= \frac{i}{2\pi}\,\mathrm{tr}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\left(\begin{pmatrix}-F_0 & 0\\ 0 & F_0\end{pmatrix}\right. \\ &\quad \left. + \begin{pmatrix}-\partial a_{\mathbf{x}} & \nabla(b)\\ 0 & \partial a_{\mathbf{x}}\end{pmatrix}\right) = 0\,, \end{aligned} \tag{3.8}$$

where we have used the identity: $\mathrm{curv}(\nabla + \bar\partial + B_{\mathbf{x},b}^{01}) = \mathrm{curv}(\nabla + \bar\partial) + \nabla(B_{\mathbf{x},b}^{01})$. Above, $\nabla(b)$ stands for the holomorphic covariant derivative of the L_0^{-2}-valued 0,1-form b. Similarly,

$$\begin{aligned}\frac{\delta}{\delta c}\Big|_{c=1} S(h_c, A_0^{10} + A_{\mathbf{x},b}^{01}) &= \frac{1}{2\pi i}\int_\Sigma \mathrm{tr}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(\begin{pmatrix}-F_0 & 0\\ 0 & F_0\end{pmatrix} + \begin{pmatrix}-\partial a_{\mathbf{x}} & \nabla(b)\\ 0 & \partial a_{\mathbf{x}}\end{pmatrix}\right) \\ &= \frac{i}{\pi}\int_\Sigma F_0 = 2(g-1)\,. \end{aligned} \tag{3.9}$$

As for the relation (3.7), first notice that we may reduce the calculation to the case $b = 0$ since for $h = h_{c_p}$,

$$\begin{aligned} S(h, A_0^{10} + A_{\mathbf{x},b}^{01}) =& S(h, A_0^{10} + A_{\mathbf{x},o}^{01}) \\ &+ \frac{i}{2\pi}\int_\Sigma \mathrm{tr}\,[(h\partial h^{-1} + hA_0^{10}h^{-1} - A_o^{10}) \wedge U\sigma^+ bU^{-1}] \\ =& S(h, A_0^{10} + A_{\mathbf{x},0}^{01}) + \frac{i}{2\pi}\int_\Sigma \mathrm{tr}\,[\sigma^3\sigma^+ c_p^{-1}\partial c_p \wedge b] \\ =& S(h, A_0^{10} + A_{\mathbf{x},0}^{01})\,. \end{aligned}$$

We shall show in Appendix A that for c a non-vanishing function on Σ, for $g_c = \begin{pmatrix}c^{-1} & 0\\ 0 & c\end{pmatrix}$ and for $h_c = Ug_cU^{-1}$,

$$\exp[S(h_c, A_0^{10} + A_{\mathbf{x},0}^{01})] = \mathrm{e}^{\frac{1}{2\pi i}\int_\Sigma c^{-1}\partial c \wedge (c^{-1}\bar\partial c + 2a_{\mathbf{x}})} v(c)\,. \tag{3.10}$$

Note that, except for the last factor, the expression on the right-hand side is $\mathrm{e}^{S(g_c, -\sigma^3 a_{\mathbf{x}})}$, where g_c is viewed as a standard $SL(2,\mathbb{C})$-valued field. The correction term is

$$v(c) = \mathrm{e}^{\frac{i}{\pi}\int_\Sigma F_0 \ln c} \prod_{j=1}^{g}\left(W_{a_j}^{-\frac{i}{\pi}\int_{b_j} c^{-1}dc}\, W_{bj}^{\frac{i}{\pi}\int_{a_j} c^{-1}dc}\right)\,, \tag{3.11}$$

where $W_{a_j}(W_{b_j})$ stand for the holonomy of the metric connection on L_0 along the $a_j(b_j)$ cycle. $\ln c(x) = \ln c(x_0) + \int_{x_0}^{x} c^{-1}dc$, where the integration path is taken inside a fundamental domain of Σ obtained by cutting the surface along the cycles a_j, b_j starting at x_0. Altogether,

$$\exp[S(h_{c_p}, A_0^{10} + A_{\mathbf{x},b}^{01})] = \mathrm{e}^{\pi(\int_p \bar{\omega})(\mathrm{Im}\,\tau)^{-1}(\int_p \omega) + 2\pi(\int_p \bar{\omega})(\mathrm{Im}\,\tau)^{-1}(\int_{x_0}^{\mathbf{x}} \omega)} \nu(c_p)$$
$$\equiv \mu(p, \mathbf{x})\, \nu(c_p)\,. \tag{3.12}$$

Gathering Eqs. (3.5) and (3.8), (3.6) and (3.9), (3.7) and (3.12), we obtain

Proposition. *Holomorphic functions ψ possess the following transformation properties:*

$$\psi(\mathbf{x}, \lambda b + (\bar{\partial} - 2a_{\mathbf{x}})v) = \lambda^{k(g-1)}\psi(\mathbf{x}, b)\,, \tag{3.13}$$

$$\psi(p\mathbf{x}, c_p^2 b) = \mu(p, \mathbf{x}^k)\nu(c_p)^k\, \psi(\mathbf{x}, b)\,. \tag{3.14}$$

In particular, for fixed $\mathbf{x}, \psi(\mathbf{x},\ \cdot\ )$ is a homogeneous polynomial of degree $k(g-1)$ on $H^1(L_{\mathbf{x}}^{-2})$. Note that the factor $\mu(p, \mathbf{x})$ appears in the transformation property for the square of Riemann's theta function:

$$\mathrm{e}^{\pi(\int_{x_0}^{p\mathbf{x}} \omega)\frac{1}{\mathrm{Im}\,\tau}(\int_{x_0}^{p\mathbf{x}} \omega)} \vartheta\left(\int_{x_0}^{p\mathbf{x}} \omega | \tau\right)^2 = \mu(p, \mathbf{x})\ \mathrm{e}^{\pi(\int_{x_0}^{\mathbf{x}} \omega)\frac{1}{\mathrm{Im}\,\tau}(\int_{x_0}^{\mathbf{x}} \omega)} \vartheta\left(\int_{x_0}^{\mathbf{x}} \omega | \tau\right)^2\,. \tag{3.15}$$

The map $\mathbf{x} \mapsto \vartheta(\int_{x_0}^{\mathbf{x}} \omega|\tau)^2$ defines a holomorphic section of the bundle $K(2x_0)$. The map $\Pi_1 \ni p \mapsto \nu(c_p)$ is a character of the fundamental group Π_1. We show in Appendix A that it defines the flat bundle $L(-gx_0)^2$. Hence a holomorphic function $\phi(\mathbf{x})$ s.t.

$$\phi(p\mathbf{x}) = \mu(p, \mathbf{x})\, \nu(c_p)\, \phi(\mathbf{x}) \tag{3.16}$$

defines (upon multiplication by $\mathrm{e}^{-\pi(\int_{x_0}^{\mathbf{x}} \omega)(\mathrm{Im}\,\tau)^{-1}(\int_{x_0}^{\mathbf{x}} \omega)}$) a section of the line bundle $L^2K((2-2g)x_0)$. The transformation properties (3.13) and (3.14) may be rephrased by saying that ψ is a holomorphic section of the k^{th} power of a h.l.b., which we shall denote DET, over the total space of the fiber bundle $\mathbb{P}W_0$. Explicitly,

$$\mathrm{DET} = \varpi^*(L^2K((2-2g)x_0))\ \mathrm{Hf}(W_0)^{(1-g)}\,, \tag{3.17}$$

where ϖ is the bundle projection of $\mathbb{P}W_0$ and $\mathrm{Hf}(W_0)$ is the tautological bundle over $\mathbb{P}W_0$. In particular, the restriction of the h.l.b. DET to the fiber $\mathbb{P}H^1(L(-x)^{-2})$ of $\mathbb{P}W_0$ over $x \in \Sigma$ is the $(1-g)^{\mathrm{th}}$ power of the tautological bundle over $\mathbb{P}H^1(L(-x)^{-2})$. As we shall see in the next section, the h.l.b. $L^2K((2-2g)x_0)$ is isomorphic to (the Π_1 quotient of) the determinant bundle of the family $(\bar{\partial} - \sigma^3 a_{\mathbf{x}})$ of $\bar{\partial}$-operators in $L_0^{-1} \oplus L_0$. In turn, the line bundle DET is isomorphic to (the Π_1 quotient of) the determinant bundle for the family $(\bar{\partial} + B_{\mathbf{x},b}^{01})$ of $\bar{\partial}$-operators in $L_0^{-1} \oplus L_0$. Note the simple way by which the addition of the upper-diagonal gauge field b in the $\bar{\partial}$-operator manifests itself in the determinant bundle. The map $\Psi \mapsto \psi$ embeds the space of CS states onto a subspace $\mathscr{H} \subset H^0(\mathrm{DET}^k)$. The space $H^0(\mathrm{DET}^k)$ of the holomorphic sections of a line bundle over the compact space W_0 is finite dimensional so that the finite-dimensionality of the space of CS states follows. Given $\psi \in H^0(\mathrm{DET}^k)$, the question whether it comes from a CS state should be determined by its behavior at the codimension one subvariety defined by $\mathbb{P}W_1$

in the moduli space $\mathcal{N}_s$ of stable bundles. We shall return below to this question which requires some refinement of the analysis of [1, 22].

For the later use, it will be convenient to write the homogeneous polynomial $\psi(\mathbf{x}, \cdot)$ on $H^1(L_{\mathbf{x}}^{-2})$ in an integral form following from the Serre duality:

$$\psi(\mathbf{x}, b) = \int_{\Sigma^{k(g-1)}} \chi(\mathbf{x}; x_1, \ldots, x_{k(g-1)})\; b(x_1) \ldots b(x_{k(g-1)}) \tag{3.18}$$

for $b \in \wedge^{01}(L_0)$ and $\chi(\mathbf{x}; \cdot) \in H^0(S^{k(g-1)}(L_{\mathbf{x}}^2 K))$, where, for a h.l.b. M over Σ, $S^n M$ stands for the n-fold (symmetrized) tensor product with the base space Σ^n, the n-fold symmetric Cartesian product of Σ. The $\mathbf{x}$-dependence of χ again gives rise to a section of $L^2K((2-2g)x_0)$ so that we may view $\chi(\cdot\,;\cdot)$ as a holomorphic, pr_2-horizontal $k(g-1)$,0-form on $\Sigma \times \Sigma^{k(g-1)}$ with values in the h.l.b.

$$pr_1{}^*(L^2K((2-2g)x_0))^k \; S^{k(g-1)}(\mathscr{L}_0{}^2) \equiv B_k\ . \tag{3.19}$$

4. Relation to Bertram's Picture

Paper [1], see also [22], describes a somewhat different, simpler, construction of the space of CS states. It is based on the realization of a generic rank 2 determinant 0 bundle as an extension of a fixed degree g line bundle L. Taking

$$B_{b'}^{\prime 01} \equiv \begin{pmatrix} 0 & b' \\ 0 & 0 \end{pmatrix}, \tag{4.1}$$

where $b' \in \wedge^{01}(L^{-2})$, we shall put

$$A_{b'}^{\prime 01} = U' A_{b'}^{\prime 01} U'^{-1} + U' \bar{\partial} U'^{-1}\ , \tag{4.2}$$

where $U' : L^{-1} \oplus L \to \Sigma \times \mathbb{C}^2$ is a fixed smooth isometric isomorphism. Defining

$$\psi'(b') = \exp\left[\frac{ik}{2\pi}\int_\Sigma \mathrm{tr} A_0^{\prime 10} \wedge A_{b'}^{\prime 01}\right] \Psi(A_{b'}^{\prime 01})\ , \tag{4.3}$$

we infer in the same way as above that

$$\psi'(\lambda(b' + \bar{\partial} v')) = \lambda^{kg} \psi'(b')\ , \tag{4.4}$$

compare (3.13). Hence each ψ' is a degree kg homogeneous polynomial on $H^1(L^{-2})$. The latter space has dimension $3g-1$ and the number of independent homogeneous polynomials of degree kg on it is $\binom{kg+3g-2}{3g-2}$. By the Serre duality, we may write

$$\psi'(b') = \int_{\Sigma^{kg}} \chi'(x_1, \ldots, x_{kg})\, b'(x_1) \ldots b'(x_{kg})\ , \tag{4.5}$$

where $\chi' \in H^0(S^{kg}(L^2K))$.

Theorem 2a of [1][4]. *The CS states correspond exactly to the polynomials ψ' s. t. $\chi \in H^0(S^{kg}(L^2K))$ vanish whenever $k+1$ of their arguments x_n coincide.*

[4] We have learned this reformulation of the result of [1] from B. Feigin who discussed in [25] its generalization to the case with insertions and with arbitrary simple groups.

The dimension of the space of such polynomials was shown in [22] to be

$$\left(\frac{k+2}{2}\right)^{g-1} \sum_{j=0,\frac{1}{2},\dots,\frac{k}{2}} \left(\sin\,\frac{\pi(2j+1)}{k+2}\right)^{2-2g} , \tag{4.6}$$

in agreement with the Verlinde formula [6] for the dimensions of the spaces of CS states. Let us set

$$\chi(x;x_1,\dots,x_{k(g-1)}) \equiv \chi'(\underbrace{x,\dots,x}_{k\ \text{times}},x_1,\dots,x_{k(g-1)}) \,. \tag{4.7}$$

Note that

$$\chi(x;x,x_2,\dots,x_{k(g-1)}) = 0 \quad \text{and} \quad \frac{\partial}{\partial x_1}\bigg|_{x_1=x} \chi(x;x_1,\dots,x_{k(g-1)}) = 0 \,, \tag{4.8}$$

where the second equality is obtained by differentiating the first one over x. In particular, for fixed $x, \chi(x;\ \cdot\ ) \in (L^2K)^k|_x \otimes H^0(S^{k(g-1)}(L^2K(-2x)))$. We may also interpret $\chi(x;\ \cdot\ )$ as a holomorphic $k(g-1),0$-form on $\Sigma^{k(g-1)}$ with values in the line bundle $(L^2K)^k|_x \otimes S^{k(g-1)}(L^2(-2x))$.

We shall show that giving χ is equivalent to giving ψ in the description of the CS states of the previous section. For this purpose, let us consider a line bundle $\mathcal{O}(-\Delta)$ over $\Sigma \times \Sigma$, where Δ denotes the diagonal. $\mathscr{L}'_0 \equiv pr_2^*(L)\mathcal{O}(-\Delta) \equiv pr_2{}^*(L)(-\Delta)$ is another realization of a family $(L(-x))$ of the h.l.b.'s, a different one from $\mathscr{L}_0$ described before. It is not difficult to see that

$$\mathscr{L}'_0 \cong pr_1^*(\mathcal{O}(-x_0))\mathscr{L}_0 \,. \tag{4.9}$$

An explicit isomorphism is given in Appendix B. If $W'_0 \equiv R^1 pr_{1*}(\mathscr{L}_0'^{-2})$, then

$$W'_0 \cong \mathcal{O}(2x_0)W_0, \quad \mathbb{P}W'_0 \cong \mathbb{P}W_0 \quad \text{and} \quad \mathrm{Hf}(W'_0) \cong \varpi^*(\mathcal{O}(2x_0))\ \mathrm{Hf}(W_0) \,. \tag{4.10}$$

Now χ, with its x-dependence taken into account, may be viewed as a holomorphic (pr_2-horizontal) $k(g-1)$, 0-form on $\Sigma \times \Sigma^{k(g-1)}$ with the values in the h.l.b.

$$B'_k \equiv pr_1{}^*(L^2K)^k\ S^{k(g-1)}(\mathscr{L}_0'^2) \,. \tag{4.11}$$

Since the h.l.b.'s B'_k of Eq. (4.11) and B_k of (3.19) are isomorphic due to (4.9), χ introduced by formula (4.7) is the same type of object as χ considered in Sect. 3, see Eq. (3.18). Indeed, in Appendix C we show that the two χ's coincide completely under an explicit isomorphism. This will establish the precise relation between the two descriptions of the CS states: by functions ψ which we shall employ in this work and by polynomials ψ'. Let us briefly sketch here the geometric picture due to [1, 22] which is at the core of the detailed analysis of Appendix C. One may embed the curve Σ into $\mathbb{P}H^1(L^{-2})$ by

$$\Sigma \ni x \mapsto [b'_x] \in \mathbb{P}H^1(L^{-2}) \,, \tag{4.12}$$

where, for $\eta' \in H^0(L^2K)$, $\int_\Sigma \eta' b'_x = \eta'(x)$ in some trivialization of L^2K around x. We shall see in Appendix C that $[b'_x]$ corresponds to an extension of L which, as a rank two bundle, is isomorphic to the split bundle $L(-x)^{-1} \oplus L(-x)$. The condition of Theorem 2a of [1] means[5] that ψ' vanishes to order $k(g-1)-1$ on (the image of)

[5] this was the original formulation of Theorem 2a of [1]

Σ. In this description, the CS states are thus realized as homogeneous polynomials ψ' on $(3g-1)$-dimensional vector space which vanish to some order on the explicit embedding of Σ into the space. ψ's are given by the $k(g-1)^{\text{th}}$ Taylor coefficients (the first non-trivial order) of ψ' on Σ.

Similarly, one may embed Σ into $\mathbb{P}H^1(L(-x)^{-2})$ by mapping y to $[b_y]$ s.t. $\int_\Sigma \eta b_y = \eta(y)$ for each $\eta \in H^0(L(-x)^2K)$. $[b_y]$ defines an extension of $L(-x)$ which, as a h.v.b., is isomorphic to $L(-x-y)^{-1} \oplus L(-x-y)$. Changing also x, we get an embedding of Σ^2 into the bundle $\mathbb{P}W_0$ considered in Sect. 2. Replacement of the image of Σ^2 in $\mathbb{P}W_0$ by $\mathbb{P}W_1$ is the second blow-up step of [1, 22]. Their analysis shows that, again, $\chi(x; y_1, \ldots, y_{k(g-1)})$ vanishes whenever $k+1$ of y_n's coincide. We may then equate k of y_n's and continue the process including higher and higher $\mathbb{P}W_r$'s into the game. It is interesting to know whether the vanishing of $\chi(\mathbf{x}; x_1, \ldots, x_{k(g-1)})$ at $k+1$ coincident points characterizes completely the sections $\psi \in H^0(\text{DET}^k)$ coming from the CS states.

Let us remark, in the end, that the h.l.b.'s isomorphisms $\det Rpr_{1*}(\mathscr{L}_0') \cong L^{-1}$ and $\det Rpr_{1*}(\mathscr{L}_0'^{-1}) \cong (LK)^{-1}$ following from the exact sequence

$$0 \longrightarrow pr_2{}^*(L)(-\Delta) \longrightarrow pr_2{}^*(L) \longrightarrow pr_2{}^*(L)|_\Delta \longrightarrow 0$$

and the relation (4.9) imply that the determinant bundle $\det^{-1} Rpr_{1*}(\mathscr{L}_0^{-1} \oplus \mathscr{L}_0)$ of the family $(\bar\partial - \sigma^3 a_{\mathbf{x}})$ of $\bar\partial$-operators in $L_0^{-1} \oplus L_0$ is isomorphic to the h.l.b. $L^2K((2-2g)x_0)$. This provides an interpretation of the first factor on the right-hand side of Eq. (3.17).

5. Projective Formula

The outcome of the calculation performed in this paper is much simpler than the calculation itself. We shall start by describing a softened version of its result. It will give the scalar product of CS states up to a Σ-dependent constant. Such data are enough to generate a projective connection on the bundle $\mathscr{W}_k$ of state spaces. This should coincide with the projective class of the KZB connection and hence be flat. Section 9 contains a more detailed scalar product formula with the normalization fixed, up to an overall constant depending only on the level k and genus g (which could be traced through the calculation). The detailed formula gives also the dependence of the scalar product on the metric of the surface (i.e., in particular, on its complex structure).

In the simplified formula, we shall use, as the geometric input, the representation of the CS states by the holomorphic $k(g-1), 0$-forms χ with values in the line bundle B_k', see (4.11), as well as hermitian structures on these bundles. It will be convenient to choose the latter in a specific way. Following [31], we shall call a hermitian metric on a h.l.b. M on Σ admissible, if the curvature form of the induced connection is proportional to the 2-form $\alpha \equiv \frac{i}{2g}\omega(\operatorname{Im}\tau)^{-1} \wedge \bar\omega$. Admissible hermitian structures exist and are unique up to normalization. We shall call a Riemannian metric on Σ admissible if it induces an admissible hermitian structure on the holomorphic tangent bundle of Σ. Let $G(x, y)$ denote the Green function of the scalar Laplacian Δ on Σ chosen so that $\int_\Sigma G(\,\cdot\,,y)\,\alpha(y)=0$. $G(x,y)$ has a logarithmic singularity at coinciding points. Define

$$:G(x,x):=\lim_{\varepsilon\to 0}\left(G(x,x')-\frac{1}{2\pi}\ln\varepsilon\right), \tag{5.1}$$

with the distance $d(x,x')=\varepsilon$. Choose an admissible Riemannian metric on Σ normalized so that $:G(x,x):\equiv 0$, called the Arakelov metric [32]. It will also be convenient to fix the dependence of the hermitian structures on the parameter for families of h.l.b.'s on Σ. In particular, the line bundle $\mathcal{O}(\Delta)$ over $\Sigma\times\Sigma$ may be provided with a hermitian metric by setting

$$|1(x,y)|^2=\mathrm{e}^{4\pi\,G(x,y)} \tag{5.2}$$

for its canonical section 1. Fixing also an admissible hermitian structure on the h.l.b. L, we obtain this way a hermitian metric on $\mathscr{L}'=pr_2{}^*(L)(-\Delta)$ which may be viewed as a family of admissible hermitian structures on the family $(L(-x))$ of h.l.b.'s realized as $\mathscr{L}'_0$. The above choices determine a hermitian metric on the h.l.b. B'_k of (4.11) in which forms χ of Eq. (4.7) take values.

Another geometric input in the scalar product formula comes from the linear map

$$H^0(K)\ni v\overset{l(x,b'')}{\longmapsto}v[b'']\in H^1(L(-x)^{-2}K)\cong H^0(L(-x)^2)^* \tag{5.3}$$

defined for each $b''\in\wedge^{01}(L(-x)^2)$ and depending only on the class of b'' in $H^1(L(-x)^{-2})$. As we shall see in Sect. 6.3, the rank of this map controls the (local) regularity of the projection from $\mathscr{A}^{01}$ into the orbit space $\mathscr{A}^{01}/\mathscr{G}^{\mathbb{C}}$. $l(x,b'')$ may be viewed as an element of the vector space $H^0(K)^*\otimes H^0(L(-x)^2)^*$. Since

$$\wedge^{g-1}H^0(K)^*\cong H^0(K)\otimes\ \det{}^{-1}H^0(K)\,, \tag{5.4}$$

$\wedge^{g-1}l(\,\cdot\,,b'')$ induces a holomorphic 1,0-form on Σ with values in the bundle $\det^{-1}H^0(K)\otimes\det^{-1}R^0pr_{1*}\mathscr{L}_0'^2$. Since it depends homogeneously on $[b'']\in H^1(L(-x)^{-2})$, we may write

$$\wedge^{g-1}l(x,b'')=\int\limits_{\Sigma^{g-1}}\phi(x;x_1,\dots,x_{g-1})\,b''(x_1)\dots b''(x_{g-1})\,, \tag{5.5}$$

where ϕ is a holomorphic $g,0$-form on $\Sigma\times\Sigma^{g-1}$ with values in

$$\det{}^{-1}H^0(K)\otimes pr_{1*}(\det{}^{-1}R^0pr_{1*}\mathscr{L}_0'^2)\,S^{g-1}(\mathscr{L}_0'^2)\,. \tag{5.6}$$

Note that the choices of the metric on Σ and of the hermitian structure on $\mathscr{L}'_0$ described above induce a hermitian metric on the h.l.b. (5.6).

The functional integral calculation which we describe in this paper implies the following scalar product formula for the CS states:

$$\begin{aligned}\|\Psi\|^2=\ &\text{const.}\ i^{-1-M}\int\det{}'(\bar\partial^\dagger_{L(-x)^2}\bar\partial_{L(-x)^2})\\ &\cdot|\mathscr{S}_M(\phi(x;x_1,\dots,x_{g-1})\,\chi(x;x_g,\dots,x_M))|^{\wedge 2}\prod_{m_1\neq m_2}\mathrm{e}^{-\frac{4\pi}{k+2}G(x_{m_1},x_{m_2})}\end{aligned} \tag{5.7}$$

Above, $M\equiv(k+1)(g-1)$, $\mathscr{S}_M$ stands for the symmetrizer of $(x_m)_{m=1}^M$, $|\cdot|^{\wedge 2}$ denotes the $(1+M),(1+M)$-form obtained by pairing a $(1+M),0$-form with values in a h.l.b. with itself using the hermitian structures described above. The prefactor i^{-1-M} assures the positivity of the integrated form. The determinant of the operator $\bar{\partial}^\dagger_{L(-x)^2}\bar{\partial}_{L(-x)^2}$ restricted to the subspace orthogonal to the zero modes should be zeta-function regularized. Notice that the product on the right-hand side of Eq. (5.7) has a form of the Boltzmann factor for a gas of two-dimensional particles interacting with attractive Coulomb forces. Appearance of such "Coulomb (or, more properly, Newton) gas representation" was a characteristic feature of the genus zero scalar product formulae, see [13, 14]. As mentioned above, we have not proven that the above equation defines the scalar product of CS states which induces (projectivized) KZB connection. The first thing which remains to be shown is that the integral on the right-hand side of (5.7) (over the modular parameter x and over M positions of "screening charges" at points x_m) converges for χ corresponding to the CS states. Although not proven in general, the convergence seems very plausible in view of the analysis of Sect. 9 below. In particular, it is evident for genus 2. Appendix F discusses other consistency checks of the complete scalar product formula worked out in Sect. 9.

6. Change of Variables

As we have mentioned, the main idea of this work is a brute-force calculation of the functional integral (1.6) giving the formal scalar product of the CS states. This will be a long process in which the first step is the change of variables

$$A^{01} = {}^{h^{-1}}\!A^{01}(n)\,, \tag{6.1}$$

where $n \mapsto A^{01}(n)$ parametrizes holomorphically a $(3g-3) \equiv N$-dimensional slice of $\mathscr{A}^{01}$ (generically) transversal to the chiral gauge orbits. The reparametrization (6.1) permits to transform the formal scalar product formula into

$$\|\Psi\|^2 = \int |\Psi({}^{h^{-1}}\!A^{01}(n))|^2 e^{-\frac{ik}{2\pi}\int_\Sigma \operatorname{tr}\,({}^{h^{-1}}\!A^{01}(n))^\dagger \wedge {}^{h^{-1}}\!A^{01}(n)} \left|\frac{\partial({}^{h^{-1}}\!A^{01}(n))}{\partial(h,n)}\right|^2 Dh \prod_\alpha d^2 n_\alpha\,, \tag{6.2}$$

where $Dh = \prod_x dh(x)$ is a formal local product of the Haar measures on $SL(2,\mathbb{C})$. We have to compute the Jacobian $|\frac{\partial({}^{h^{-1}}\!A^{01}(n))}{\partial(h,n)}|^2$ of the change of variables. Notice that under an infinitesimal variation of h and n,

$$\delta({}^{h^{-1}}\!A^{01}(n)) = {}^{h^{-1}}\!\bar{D}_n(h^{-1}\delta h) + h^{-1}\frac{\partial A^{01}(n)}{\partial n_\alpha}\delta n_\alpha\, h\,, \tag{6.3}$$

where $\bar{D}_n \equiv \bar{\partial} + [A^{01}(n),\ \cdot\ ]$ and ${}^{h^{-1}}\!\bar{D}_n \equiv \mathrm{Ad}_{h^{-1}}\bar{D}_n\,\mathrm{Ad}_h$. Assume that $A^{01}(n_0)$ defines a stable vector bundle. Then, for n close to n_0 one may choose a basis $(\omega^\alpha(n))_{\alpha=1}^N$ of 1,0-forms with values in $sl(2,\mathbb{C})$ such that $\bar{D}_n\omega^\alpha(n) = 0$ and $\omega^\alpha(n)$ depends holomorphically on n. Notice that the relation $\bar{D}_n\omega^\alpha(n) = 0$ holds if and only if

$$\int_\Sigma \operatorname{tr}\,\omega^\alpha(n) \wedge \bar{D}_n \Lambda = 0 \tag{6.4}$$

for all (smooth) $sl(2,\mathbb{C})$-valued function A so that $\omega^\alpha(n)$ may be viewed as co-vectors tangent to the orbit space $\mathscr{A}^{01}/\mathscr{G}^{\mathbb{C}}$ since they vanish on the variations $\bar{D}_n A$ tangent to the orbit of $\mathscr{G}^{\mathbb{C}}$ through $A^{01}(n)$. The space $\mathscr{A}^{01}$ has a natural scalar product corresponding to the norm

$$\|A^{01}\|^2 = i\int_\Sigma \mathrm{tr}\,(A^{01})^\dagger \wedge A^{01} \,. \tag{6.5}$$

Using this scalar product, we may decompose

$$\mathscr{A}^{01} = \mathrm{im}({}^{h^{-1}}\bar{D}_n) \oplus (\mathrm{im}({}^{h^{-1}}\bar{D}_n))^\perp \,. \tag{6.6}$$

Notice that the subspace $(\mathrm{im}({}^{h^{-1}}\bar{D}_n))^\perp$ orthogonal to the image of ${}^{h^{-1}}\bar{D}_n$ is spanned by the forms $(h^{-1}\omega^\alpha(n)h)^\dagger$. The (holomorphic) derivative of the change of variables (6.1) is

$$\frac{\delta({}^{h^{-1}}A^{01}(n))}{\delta(h,n)} = \begin{pmatrix} {}^{h^{-1}}\bar{D}_n & \\ 0 & h^{-1}\frac{\partial A^{01}(n)}{\partial n_\gamma}h \end{pmatrix} = \begin{pmatrix} {}^{h^{-1}}\bar{D}_n & \cdots \\ 0 & (h^{-1}\frac{\partial A^{01}(n)}{\partial n_\alpha}h)^\perp \end{pmatrix} , \tag{6.7}$$

where

$$\left(h^{-1}\frac{\partial A^{01}(n)}{\partial n_\alpha}h\right)^\perp = i(h^{-1}\omega^\gamma(n)h)^\dagger \;\; (\Omega(hh^\dagger,n)^{-1})_{\gamma\beta}\int_\Sigma \mathrm{tr}\;\omega^\beta(n)\wedge\frac{\partial A^{01}(n)}{\partial n_\alpha} \tag{6.8}$$

with $\Omega(hh^\dagger,n)^{\beta\gamma} = i\int_\Sigma \mathrm{tr}\; h^{-1}\omega^\beta(n)h\wedge(h^{-1}\omega^\gamma(n)h)^\dagger$. It follows that the Jacobian of the change of variables (6.1) is

$$\begin{aligned} \left|\frac{\partial({}^{h^{-1}}A^{01}(n))}{\partial(h,n)}\right|^2 &= \det\,(({}^{h^{-1}}\bar{D}_n)^\dagger\,{}^{h^{-1}}\bar{D}_n)\;\det\,(\Omega(hh^\dagger,n))^{-1} \\ &\cdot\left|\det\left(\int_\Sigma \mathrm{tr}\;\omega^\beta(n)\wedge\frac{\partial A^{01}(n)}{\partial n_\alpha}\right)\right|^2 \,. \end{aligned} \tag{6.9}$$

Of course, $\det\,(({}^{h^{-1}}\bar{D}_n)^\dagger\,{}^{h^{-1}}\bar{D}_n)$ has to be regularized, e.g. by the zeta-function prescription. The chiral anomaly permits to compute the h-dependence of the regularized Jacobian:

$$\begin{aligned} &\det\,(({}^{h^{-1}}\bar{D}_n)^\dagger\,{}^{h^{-1}}\bar{D}_n)\;\det\,(\Omega(hh^\dagger,n))^{-1} \\ &\quad = \mathrm{e}^{4\,S(hh^\dagger,\,A(n))}\;\det\,(\bar{D}_n^\dagger\bar{D}_n)\;\det\,(\Omega(1,n))^{-1}\,, \end{aligned} \tag{6.10}$$

where $A(n) \equiv -(A^{01}(n))^\dagger + A^{01}(n)$. Also a short calculation using the transformation properties (1.1) shows that

$$\begin{aligned} |\Psi({}^{h^{-1}}A^{01}(n))|^2\,\mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}({}^{h^{-1}}A^{01}(n))^\dagger\wedge{}^{h^{-1}}A^{01}(n)} &= |\Psi(A^{01}(n))|^2\,\mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}(A^{01}(n))^\dagger\wedge A^{01}(n)} \\ &\cdot\,\mathrm{e}^{kS(hh^\dagger,\,A(n))}\,. \end{aligned} \tag{6.11}$$

Inserting Eqs. (6.9),(6.10) and (6.11) into the functional integral (6.2), we obtain

$$\|\Psi\|^2 = \int |\Psi(A^{01}(n)|^2 \, \mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\,(A^{01}(n))^\dagger \wedge A^{01}(n)} \, \mathrm{e}^{(k+4)S(hh^\dagger, A(n))} \det(\bar{D}_n^\dagger \bar{D}_n)$$
$$\cdot \det(\Omega(1,n))^{-1} \left| \det \left(\int_\Sigma \mathrm{tr}\,\omega^\beta(n) \wedge \frac{\partial A^{01}(n)}{\partial n_\alpha} \right) \right|^2 D(hh^\dagger) \prod_\alpha d^2 n_\alpha \,, \quad (6.12)$$

where we have used the fact that the integrand depends on h only through $hh^\dagger$, related to the gauge invariance of the original integral (1.6), to reduce the h-integration to that over the $hh^\dagger$ fields effectively taking values in the hyperbolic space $SL(2,\mathbb{C})/SU(2)$. $D(hh^\dagger)$ should then be interpreted as the local formal product $\prod_x d(hh^\dagger)(x)$ of $SL(2,\mathbb{C})$-invariant measures on $SL(2,\mathbb{C})/SU(2)$.

Formula (6.12) decomposes the original functional integral (1.6) over $\mathscr{A}^{01}$ into the one along the orbits of the chiral gauge transformations, which has the form of the partition function of an $SL(2,\mathbb{C})/SU(2)$-valued WZW model [26, 27]

$$\int \mathrm{e}^{(k+4)S(hh^\dagger, A(n))} \, D(hh^\dagger)\,, \quad (6.13)$$

and the integral along a slice $n \mapsto A^{01}(n)$ of $\mathscr{A}^{01}$ which we shall parametrize by the complex bundle $\mathbb{P}W_0$. More exactly, as discussed in Sect. 2, we shall consider the map

$$(\mathbf{x}, b) \overset{s}{\longmapsto} A^{01}_{\mathbf{x},b} \quad (6.14)$$

with $\mathbf{x}$ running through a fundamental domain of Π_1 in $\tilde{\Sigma}$ and one b in each class of $\mathbb{P}H^1(L_\mathbf{x}^{-2})$. Such a map gives a multiply parametrized slice of $\mathscr{A}^{01}$ (as we have seen in Sect. 2, the induced map from $\mathbb{P}W_0$ is essentially a multiple covering of $\mathscr{A}^{01}/\mathscr{G}^{\mathbb{C}}$; one may show that multiplicity is equal of $2g$). Let $(\eta_\mathbf{x}^\alpha)_{\alpha=1}^N$ be a basis of $H^0(L_\mathbf{x}^2 K)$. $\eta_\mathbf{x}^\alpha$ may be chosen locally as depending holomorphically on $\mathbf{x}$. The integrals

$$z_\mathbf{x}^\alpha = \int_\Sigma \eta_\mathbf{x}^\alpha \wedge b \quad (6.15)$$

provide coordinates on $H^1(L_\mathbf{x}^{-2})$ (homogeneous coordinates on $\mathbb{P}H^1(L_\mathbf{x}^{-2})$) and a local (holomorphic) trivialization of W_0. We shall have to find explicit expressions for various terms under the integral (6.12).

6.1. Term $|\Psi(A^{01}(n)|^2 \, \mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\,(A^{01}(n))^\dagger \wedge A^{01}(n)}$.
Using Eq. (3.4), we obtain

$$|\Psi(A^{01}_{\mathbf{x},b})|^2 \, \mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\,(A^{01}_{\mathbf{x},b})^\dagger \wedge A^{01}_{\mathbf{x},b}} = |\psi(\mathbf{x}, b)|^2$$
$$\cdot\, \mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\,(A_0^{10} + (A^{01}_{\mathbf{x},b})^\dagger) \wedge (-A_0^{01} + A^{01}_{\mathbf{x},b}) - \frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\, A_0^{10} \wedge A_0^{01}} \,. \quad (6.16)$$

Recall that $A^{01}_{\mathbf{x},b} = UB^{01}_{\mathbf{x},b}U^{-1} + U\bar{\partial}U^{-1}$, $A_0^{10} = U\nabla U^{-1} = -(U\bar{\partial}U^{-1})^\dagger = -(A_0^{01})^\dagger$ with $B^{01}_{\mathbf{x},b} = \begin{pmatrix} -a_\mathbf{x} & b \\ 0 & a_\mathbf{x} \end{pmatrix}$. Hence $-A_0^{01} + A^{01}_{\mathbf{x},b} = UB^{01}_{\mathbf{x},b}U^{-1}$ and $A_0^{10} + (A^{01}_{\mathbf{x},b})^\dagger = U(B^{01}_{\mathbf{x},b})^\dagger U^{-1}$, where $(B^{01}_{\mathbf{x},b})^\dagger = \begin{pmatrix} \overline{a_\mathbf{x}} & 0 \\ b & -\overline{a_\mathbf{x}} \end{pmatrix}$ with the L_0^{-2}-valued 1,0-form $b^\dagger$ being the conjugate of b with respect to the hermitian metric $\langle \cdot\,, \cdot \rangle$ of L_0^{-2}. It follows that $\int_\Sigma \mathrm{tr}(A_0^{10} + (A^{01}_{\mathbf{x},b})^\dagger) \wedge (-A_0^{01} + A^{01}_{\mathbf{x},b}) = 2\int_\Sigma \overline{a_{\mathbf{x}}} \wedge a_{\mathbf{x}} + \int_\Sigma \langle b, \wedge b\rangle$. Consequently,

$$|\Psi(A^{01}_{\mathbf{x},b})|^2\, \mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}(A^{01}_{\mathbf{x},b})^\dagger \wedge A^{01}_{\mathbf{x},b}} = \mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr} A_0^{10}\wedge A_0^{01} - 2\pi k(\int_{x_0}^{\mathbf{x}}\bar\omega)\frac{1}{\mathrm{Im}\,\tau}(\int_{x_0}^{\mathbf{x}}\omega)} |\psi(\mathbf{x},b)|^2\,. \tag{6.17}$$

6.2. Term $\mathrm{e}^{(k+4)S(hh^\dagger, A(n))}$.

Consider the field $U^{-1}hh^\dagger U$. It is a smooth section of the bundle $\mathrm{End}(L_0^{-1} \oplus L_0)$ taking values in the positive endomorphisms. It is easy to see that, necessarily,

$$U^{-1}hh^\dagger U = \begin{pmatrix} \mathrm{e}^{\varphi} + \mathrm{e}^{-\varphi}\langle v, v\rangle & \mathrm{e}^{-\varphi} v \\ \mathrm{e}^{-\varphi} v^\dagger & \mathrm{e}^{-\varphi}\end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\varphi/2} & \mathrm{e}^{-\varphi/2} v \\ 0 & \mathrm{e}^{-\varphi/2}\end{pmatrix}\begin{pmatrix} \mathrm{e}^{\varphi/2} & \mathrm{e}^{-\varphi/2} v \\ 0 & \mathrm{e}^{-\varphi/2}\end{pmatrix}^\dagger \equiv gg^\dagger \tag{6.18}$$

for unique real function φ on Σ and $v \in \Gamma(L_0^{-2})$. We shall prove that

$$\begin{aligned} S(hh^\dagger, A_{\mathbf{x},b}) = {} & \frac{i}{2\pi}\int_\Sigma \varphi(\partial\bar\partial\varphi - 2F_0) \\ & - \frac{i}{2\pi}\int_\Sigma \mathrm{e}^{-2\varphi}\langle b + (\bar\partial - 2a_{\mathbf{x}})v), \wedge(b + (\bar\partial - 2a_{\mathbf{x}})v)\rangle \\ & + \frac{i}{2\pi}\int_\Sigma \langle b, \wedge b\rangle\,. \end{aligned} \tag{6.19}$$

Using the formula (A.2) of Appendix A, we see that

$$\begin{aligned} \delta S(hh^\dagger, A_{\mathbf{x},b}) &= \frac{i}{2\pi}\int_\Sigma \mathrm{tr}(hh^\dagger)^{-1}\delta(hh^\dagger)\; F(-(A^{01}_{\mathbf{x},b})^\dagger + {}^{(hh^\dagger)^{-1}}A^{01}_{\mathbf{x},b}) \\ &= \frac{i}{2\pi}\int_\Sigma \mathrm{tr}(gg^\dagger)^{-1}\delta(gg^\dagger)\;\mathrm{curv}(\nabla - (B^{01}_{\mathbf{x},b})^\dagger \\ &\quad + (gg^\dagger)^{-1}(\bar\partial + B^{01})(gg^\dagger))\,. \end{aligned} \tag{6.20}$$

By a straightforward computation, under holomorphic variations of v

$$\delta S(hh^\dagger, A^{01}_{\mathbf{x},b}) = -\frac{i}{2\pi}\int_\Sigma \delta v\,(\bar\partial + 2a_{\mathbf{x}})\left(\mathbf{e}^{-2\varphi}(b + (\bar\partial - 2a_{\mathbf{x}})v)\right)^\dagger\,, \tag{6.21}$$

and under antiholomorphic ones

$$\delta S(hh^\dagger, A^{01}_{\mathbf{x},b}) = \frac{i}{2\pi}\int_\Sigma \delta v^\dagger\,(\nabla + 2\overline{a_{\mathbf{x}}})\left(\mathbf{e}^{-2\varphi}(b + (\bar\partial - 2a_{\mathbf{x}})v)\right) \tag{6.22}$$

which coincides with the v-variations of the right-hand side of (6.19). Thus we may assume that $v = 0$. Then the variation of S with respect to φ becomes

$$\delta S(hh^\dagger, A^{01}_{\mathbf{x},b}) = \frac{i}{\pi}\int_\Sigma \delta\varphi\left(\partial\bar\partial\varphi - F_0 + \mathbf{e}^{-2\varphi} b^\dagger \wedge b\right) \tag{6.23}$$

(recall that $\bar\partial a_{\mathbf{x}} = 0$) which is also the variation of the right-hand side of Eq. (6.19) for $v = 0$. This ends the proof of (6.19) since $S(hh^\dagger, A^{01}_{\mathbf{x},b}) = 0$ for $hh^\dagger = 1$.

6.3. Term $\det(\Omega(1,n))^{-1}\left|\det\left(\int_\Sigma \mathrm{tr}\,\omega^\beta(n)\wedge\frac{\partial A^{01}(n)}{\partial n_\alpha}\right)\right|^2\prod_\alpha d^2n_\alpha$.

We have to look for $sl(2,\mathbb{C})$-valued 1,0-forms $\omega(\mathbf{x},b)\equiv\omega$ such that $\bar\partial\omega+A^{01}_{\mathbf{x},b}\omega+\omega A^{01}_{\mathbf{x},b}=0$. Such forms represent vectors cotangent to the orbit space $\mathscr{A}^{01}/\mathscr{G}^{\mathbb{C}}$. Writing

$$\omega=U\begin{pmatrix}-\mu & \lambda\\ \eta & \mu\end{pmatrix}U^{-1}\equiv U\rho U^{-1}\,, \tag{6.24}$$

where $\eta\in\wedge^{10}(L_0^2)$, $\mu\in\wedge^{10}$ and $\lambda\in\wedge^{10}(L_0^{-2})$, the condition for ω becomes $\bar\partial\rho+B^{01}_{\mathbf{x},b}\rho+\rho B^{01}_{\mathbf{x},b}=0$ or, in components,

$$(\bar\partial+2a_{\mathbf{x}})\eta=0,\qquad \bar\partial\mu=-\eta\wedge b,\qquad (\bar\partial-2a_{\mathbf{x}})\lambda=2\mu\wedge b\,. \tag{6.25}$$

The first of these equations requires that $\eta\in H^0(L^2_{\mathbf{x}}K)$ which has dimension N. The second one has a solution if and only if $\int_\Sigma\eta\wedge b=0$ which, for b corresponding to a non-zero element in $H^1(L_{\mathbf{x}}^{-2})$, defines a $N-1$ dimensional subspace in $H^0(L^2_{\mathbf{x}}K)$. For η in this subspace,

$$\mu=2i\partial\int G(\,\cdot\,,y)(\eta\wedge b)(y)+\nu\equiv\mu^0(\eta)+\nu\,, \tag{6.26}$$

where $G(x,y)$ is a Green function of the Laplacian on Σ and ν is an arbitrary holomorphic 1,0-form on Σ. Finally, let $(\kappa^r)_{r=1}^{g-1}$ be a basis of $H^0(L^2_{\mathbf{x}})$. κ^r may be chosen locally depending holomorphically on $\mathbf{x}$. The third of the equations (6.25) has a solution for λ if and only if

$$\int_\Sigma\kappa^r\mu\wedge b=\int_\Sigma\kappa^r\mu^0(\eta)\wedge b+\int_\Sigma\kappa^r\nu\wedge b=0 \tag{6.27}$$

for each r. Then the solution for λ is unique since $H^0(L_{\mathbf{x}}^{-2}K)=\{0\}$, since we have chosen L so that $L^2_{\mathbf{x}}$ is never isomorphic to K. Let us consider more carefully the condition (6.27). Notice that the exterior multiplication by b induces a linear map

$$l(\mathbf{x},b):H^0(K)\longrightarrow H^1(L_{\mathbf{x}}^{-2}K)\,. \tag{6.28}$$

$l(\mathbf{x},b)$ depends only on the class $[b]$ of b in $H^1(L_{\mathbf{x}}^{-2})$. The dimensions of the spaces are $\dim(H^0(K))=g$ and $\dim(H^1(L_{\mathbf{x}}^{-2}K))=\dim(H^0(L^2_{\mathbf{x}}))=g-1$. If $l(\mathbf{x},b)$ maps onto then, for fixed $\mu^0(\eta)$, there exists ν solving (6.27) and it is unique up to the addition of ν from the one-dimensional kernel of $l(\mathbf{x},b)$. Altogether, the space of solutions of Eqs. (6.25) is then N-dimensional: $N-1$ dimensions of the freedom to choose η and 1 dimension in the choice of ν. Let us examine the condition of surjectivity of $l(\mathbf{x},b)$ which guarantees that the space tangent to the $\mathscr{G}^{\mathbb{C}}$-orbit through $A_{\mathbf{x},b}$ is of maximal codimension ($=N$). Taking the standard basis $(\omega^i)_{i=1}^g$ of $H^0(K)$, this condition means that the matrix

$$\left(\int_\Sigma\kappa^r\omega^i\wedge b\right) \tag{6.29}$$

has rank $=g-1$. The $[b]$'s in $H^1(L_{\mathbf{x}}^{-2})$ for which this fails are common zeros of g homogeneous polynomials giving the $(g-1)\times(g-1)$ minors of the matrix (6.29). If these equations are non-trivial, it follows that $l(\mathbf{x},b)$ is surjective except for a subvariety of positive codimension. To see their non-triviality notice[6] that $l(\mathbf{x},b)$

[6] We thank J.-B. Bost for this argument.

fails to be surjective if and only if for some $0 \neq \kappa \in H^0(L_{\mathbf{x}}^2)$,

$$[b] \in B_\kappa \equiv \{ \ [b] \ | \ \int_\Sigma \kappa v \wedge b = 0 \text{ for all } v \in H^0(K) \ \} \,.$$

But $\cup_\kappa B_\kappa$ is at most $3g-5$ dimensional ($\dim(B_\kappa) = 2g-3$ and B_κ depends only on the class of κ in the $(g-2)$-dimensional projective space $\mathbb{P}H^0(L_{\mathbf{x}}^2)$).

For given $b \in \wedge^{01}(L_{\mathbf{x}}^{-2})$ corresponding to a non-trivial element in $H^1(L_{\mathbf{x}}^{-2})$, we may choose the basis $(\eta^\alpha)_{n=1}^N$ of $H^0(L_{\mathbf{x}}^2 K)$ so that $z^1 (\equiv \int_\Sigma \eta^1 \wedge b) \neq 0$ and $z^\alpha = 0$ for $\alpha > 1$. Suppose also that the non-zero $(g-1)\times(g-1)$ minor of the matrix (6.29) corresponds to $i < g$. Then, we may take

$$\eta = 0, \quad \mu^1 \equiv \omega^g - \omega^i M_{ir} \left(\int_\Sigma \kappa^r \omega^g \wedge b \right), \quad \lambda^1 \equiv 2\bar{\partial}^{-1}(\mu^1 \wedge b)\,, \tag{6.30}$$

where (M_{ir}) is the matrix inverse to $(\int_\Sigma \kappa^r \omega^i \wedge b)_{i<g}$, and, for $\alpha > 1$,

$$\eta^\alpha, \ \mu^\alpha \equiv \mu^0(\eta^\alpha) - \omega^i M_{ir} \left(\int_\Sigma \kappa^r (\mu^0(\eta^\alpha)) \wedge b \right), \quad \lambda^\alpha \equiv 2\bar{\partial}^{-1}(\mu^\alpha \wedge b) \tag{6.31}$$

as giving a basis of solutions of Eqs. (6.25) and, consequently, a basis $(\omega^\alpha(\mathbf{x}, b))$ of the $sl(2,\mathbb{C})$-valued 1,0-forms representing covectors tangent to the orbit space $\mathscr{A}^{01}/\mathscr{G}^{\mathbb{C}}$ at the orbit passing through $A^{01}_{\mathbf{x},b}$. Above $\bar{\partial}^{-1} \equiv (\bar{\partial}_{L_{\mathbf{x}}^{-2}K})^{-1}$. With this choice,

$$\det(\Omega(1,n)) \equiv \det(\Omega(1,\mathbf{x},b)) = \det\left(\int_\Sigma \frac{1}{i}(2\overline{\mu^\alpha} \wedge \mu^\beta + \langle \eta^\alpha, \wedge \eta^\beta \rangle + \langle \lambda^\alpha, \wedge \lambda^\beta \rangle) \right) \tag{6.32}$$

(η^1 should be replaced by zero). Moreover, since $\int_\Sigma \operatorname{tr} \omega^\beta(\mathbf{x}, b) \delta A^{01}_{\mathbf{x},b} = \int_\Sigma (2\mu^\beta \delta a_{\mathbf{x}} + \eta^\beta \delta b)$,

$$\begin{aligned} &\left| \det \left(\int_\Sigma \operatorname{tr} \omega^\beta(n) \wedge \frac{\partial A^{01}(n)}{\partial n_\alpha} \right) \right|^2 \prod_\alpha d^2 n_\alpha \\ &= \frac{8\pi^2}{i} |\omega^g(x) - \omega^\iota(x) M_{ir} \left(\int_\Sigma \kappa^r \omega^g \wedge b \right)|^{\wedge 2} \prod_{\alpha=2}^N d^2 z^\alpha, \end{aligned} \tag{6.33}$$

where, for a form ω, $|\omega|^{\wedge 2}$ denotes the form $\bar{\omega} \wedge \omega$. A simple algebra shows that

$$\begin{aligned} \omega^g(x) - \omega^\iota(x) M_{ir} \left(\int_\Sigma \kappa^r \omega^g \wedge b \right) &= \det \left(\int_\Sigma \kappa^r \omega^i \wedge b \right)^{-1}_{i<g} \sum_{j=1}^g (-1)^{g-j} \\ &\times \det \left(\int_\Sigma \kappa^r \omega^\iota \wedge b \right)_{\iota \neq j} \omega^j(x)\,. \end{aligned} \tag{6.34}$$

It will be convenient to represent $\det(\Omega(1,\mathbf{x},b))$ as a finite dimensional integral. To this end, consider a linear map $B : H^0(L_{\mathbf{x}}^2 K) \oplus H^0(K) \to \mathbb{C}^g$ given by

$$B(\eta, v) = \left(\int_\Sigma \eta \wedge b, \int_\Sigma \kappa^r(\mu^0(\eta) + v) \wedge b \right) \tag{6.35}$$

and another linear map $C:\ \ker(B)\to\wedge^{10}(\mathrm{End}(L_0^{-1}\oplus L_0))$ s.t.

$$C(\eta,v)=\begin{pmatrix}-\mu^0(\eta)-v & 2\bar\partial^{-1}((\mu^0(\eta)+v)\wedge b)\\ \eta & \mu^0(\eta)+v\end{pmatrix}, \tag{6.36}$$

(with $\bar\partial^{-1}\equiv(\bar\partial_{L_{\mathbf{x}}^{-2}K})^{-1}$). A straightforward calculation shows that with $V\equiv(\eta,v)$ and DV standing for the volume element of $H^0(L_{\mathbf{x}}^2K)\oplus H^0(K)$ coming from the scalar product induced by the hermitian metric of L_0 and the metric of Σ,

$$\int\delta(BV)\,\mathrm{e}^{-\|CV\|^2}\,DV=|z^1|^{-2}\left|\det\left(\int_\Sigma\kappa^r\omega^i\wedge b\right)_{i<g}\right|^{-2}\det(H_0)$$
$$\cdot\det(\mathrm{Im}\,\tau)\,\det(\Omega(1,\mathbf{x},b))^{-1}, \tag{6.37}$$

where

$$(H_0)^{\alpha\beta}\equiv\frac{1}{i}\int_\Sigma\langle\eta^\alpha,\wedge\eta^\beta\rangle\,. \tag{6.38}$$

Putting together Eqs. (6.33), (6.34) and (6.37), we obtain

$$\det(\Omega(1,n))^{-1}\left|\det\left(\int_\Sigma\mathrm{tr}\,\omega^\beta(n)\wedge\frac{\partial A^{01}(n)}{\partial n_\alpha}\right)\right|^2\prod_\alpha d^2n_\alpha$$
$$=\text{const.}\ i^{-N}\det(H_0)^{-1}\,\det(\mathrm{Im}\,\tau)^{-1}\left(\int\delta(BV)\,\mathrm{e}^{-\|CV\|^2}DV\right)$$
$$\cdot\,|\varepsilon_{\alpha_1,\dots,\alpha_N}z^{\alpha_1}dz^{\alpha_2}\wedge\ldots\wedge dz^{\alpha_N}|^{\wedge2}\left|\sum_{j=1}^g(-1)^j\det\left(\int_\Sigma\kappa^r\omega^i\wedge b\right)_{i\neq j}\omega^j(x)\right|^{\wedge2} \tag{6.39}$$

(with a numerical, easy to trace, g-dependent positive constant in front; the power of i makes the right-hand side a positive measure). We have given the term $|z^1dz^2\wedge\ldots\wedge dz^N|^2$ a form independent of the assumed relations $z^1\neq0$, $z^\alpha=0$ for $\alpha>1$.

7. Calculation of $\det(\bar D_n^\dagger\bar D_n)$

Let Λ be an $sl(2,\mathbb{C})$ valued function on Σ. Writing

$$\Lambda=U\begin{pmatrix}-X & Y\\ Z & X\end{pmatrix}U^{-1}, \tag{7.1}$$

where X is a function, $Y\in\Gamma(L_0^{-2})$ and $Z\in\Gamma(L_0^2)$, we obtain

$$\bar D_n\Lambda=U\begin{pmatrix}-\bar\partial X+Zb & (\bar\partial-2a_{\mathbf{x}})Y+2Xb\\ (\bar\partial+2a_{\mathbf{x}})Z & \bar\partial X-Zb\end{pmatrix}U^{-1}. \tag{7.2}$$

It follows that

$$(\Lambda,\bar D_n^\dagger\bar D_n\Lambda)\equiv i\int_\Sigma\mathrm{tr}(\bar D_n\Lambda)^\dagger\wedge\bar D_n\Lambda=i\int_\Sigma(2(\overline{\bar\partial X-Zb})\wedge(\bar\partial X-Zb)$$
$$+\langle\bar\partial Y+2Xb,\wedge(\bar\partial Y+2Xb)\rangle+\langle\bar\partial Z,\wedge\bar\partial Z\rangle), \tag{7.3}$$

where in the last line $\bar{\partial} \equiv \bar{\partial}_{L_{\mathbf{X}}^{-2}}(\bar{\partial}_{L_{\mathbf{X}}^{2}})$ when acting on $Y(Z)$. Formally,

$$\det(\bar{D}_n^\dagger \bar{D}_n)^{-1} = \int e^{-(\Lambda, \bar{D}_n^\dagger \bar{D}_n \Lambda)} D\Lambda$$
$$= \int e^{-i\int_\Sigma (2\overline{(\bar{\partial}X - Zb)} \wedge (\bar{\partial}X - Zb) + \langle \bar{\partial}Y + 2Xb, \wedge(\bar{\partial}Y + 2Xb)\rangle + \langle \bar{\partial}Z, \wedge \bar{\partial}Z\rangle)} DY\, DX\, DZ \qquad (7.4)$$

and we shall compute the latter Gaussian integral iteratively, first integrating over Y then over X and in the end over Z. As we shall see, this procedure requires a correction if we want to assure that the final result gives the zeta-function regularized determinant of $\bar{D}_n^\dagger \bar{D}_n$. It would be more natural to consider Λ as anticommuting ghost field rather than the commuting one. Indeed, this is $\det(\bar{D}_n^\dagger \bar{D}_n)$ and not its inverse which appears in the expression for the scalar product of the CS states. The choice of commuting fields Λ in this calculation is purely a matter of convenience.

7.1. Integral over Y.

$$I_Y \equiv \int e^{-i\int_\Sigma \langle \bar{\partial}Y + 2Xb, \wedge(\bar{\partial}Y + 2Xb)\rangle} DY = \det(\bar{\partial}_{L_{\mathbf{X}}^{-2}}^\dagger \bar{\partial}_{L_{\mathbf{X}}^{-2}})^{-1} e^{-4i\int_\Sigma \langle (Xb)^\perp, (Xb)^\perp\rangle}, \qquad (7.5)$$

where $(Xb)^\perp$ is the component of Xb orthogonal to $\bar{\partial}(\Gamma(L_{\mathbf{X}}^{-2})) \subset \wedge^{01}(L_{\mathbf{X}}^{-2})$. Recall that the scalar product in the spaces of sections is induced by the fixed hermitian structure of L_0 and the metric on Σ. Explicitly,

$$(Xb)^\perp = i(\eta^\alpha)^\dagger (H_0^{-1})_{\beta\alpha} \int_\Sigma \eta^\beta \wedge Xb\,, \qquad (7.6)$$

where $(H_0)^{\alpha\beta}$ is given by Eq. (6.38). We have

$$i\int_\Sigma \langle (Xb)^\perp, (Xb)^\perp\rangle = \int_\Sigma \overline{\eta^\alpha \wedge Xb}\, (H_0^{-1})_{\beta\alpha} \int_\Sigma \eta^\beta \wedge Xb\,. \qquad (7.7)$$

It will be convenient to express $e^{-4i\int_\Sigma \langle (Xb)^\perp, (Xb)^\perp\rangle}$ as a $6(g-1)$-dimensional Gaussian integral. Namely

$$e^{-4i\int_\Sigma \langle (Xb)^\perp, (Xb)^\perp\rangle} = \pi^{-N} \det(H_0) \int e^{2i\, c_\alpha \int_\Sigma \eta^\alpha \wedge Xb + 2i\, \bar{c}_\alpha \int_\Sigma \overline{\eta^\alpha \wedge Xb} - \bar{c}_\alpha (H_0)^{\alpha\beta} c_\beta} \cdot \prod_\alpha d^2 c_\alpha\,. \qquad (7.8)$$

7.2. Integral over X.
We have to calculate

$$I_X \equiv \int e^{-i\int_\Sigma 2\overline{(\bar{\partial}X - Zb)} \wedge (\bar{\partial}X - Zb) + 2i\, c_\alpha \int_\Sigma \eta^\alpha \wedge Xb + 2i\, \bar{c}_\alpha \int_\Sigma \overline{\eta^\alpha \wedge Xb}} DX\,. \qquad (7.9)$$

We have to fix the constant mode $X_0 \equiv (\int X \text{ vol})/(\text{area})^{1/2}$ which corresponds to the flat direction in the above Gaussian integral. vol is the Riemannian volume form of Σ and area $\equiv \int_\Sigma$ vol. Let us multiply I_X by $1 = \text{area} \cdot \int \delta(X_0 - a(\text{area})^{1/2}) d^2a$. Changing the order of integration and shifting X to $X + a$, we obtain

$$I_X = \text{area} \cdot \int \delta(X_0 - a(\text{area})^{1/2})\, d^2a\, I_X$$
$$= \text{area} \cdot \int e^{-i\int_\Sigma 2\overline{(\bar{\partial}X - Zb)} \wedge (\bar{\partial}X - Zb)}$$
$$\cdot\, e^{2i\, c_\alpha \int_\Sigma \eta^\alpha \wedge Xb + 2i\, \bar{c}_\alpha \int_\Sigma \overline{\eta^\alpha \wedge Xb} + 2i\, a\, c_\alpha \int_\Sigma \eta^\alpha \wedge b + 2i\, \bar{a}\, \bar{c}_\alpha \int_\Sigma \overline{\eta^\alpha \wedge b}}\, \delta(X_0)\, DX\, d^2a\,. \qquad (7.10)$$

Performing the a-integral first, we obtain

$$I_X = \frac{\pi^2}{4} \cdot \text{ area } \cdot \delta\Big(c_\alpha \int_\Sigma \eta^\alpha \wedge b\Big)$$

$$\cdot \int \mathrm{e}^{-i\int_\Sigma 2(\overline{\bar\partial X - Zb})\wedge(\bar\partial X - Zb) + 2i\, c_\alpha \int_\Sigma \eta^\alpha \wedge Xb + 2i\, \bar c_\alpha \int_\Sigma \overline{\eta^\alpha \wedge Xb}}\, \delta(X_0)\ DX$$

$$= \frac{\pi^2}{4} \cdot \text{ area } \cdot \delta\Big(c_\alpha \int_\Sigma \eta^\alpha \wedge b\Big)\, \mathrm{e}^{-2i\int_\Sigma \overline{Zb}\wedge Zb}$$

$$\cdot \int \mathrm{e}^{-\int_\Sigma \bar X(-\Delta X)\mathrm{vol} + 2i\int_\Sigma X(c_\alpha \eta^\alpha \wedge b + \bar\partial(\overline{Zb})) + 2i\int_\Sigma \bar X(\overline{c_\alpha \eta^\alpha \wedge b} - \partial(Zb))}\, \delta(X_0)\ DX\, d^2a$$

$$= \text{const. } \left(\frac{\det'(-\Delta)}{\text{area}}\right)^{-1} \mathrm{e}^{-2i\int_\Sigma \overline{Zb}\wedge Zb}\, \mathrm{e}^{4\int_\Sigma\int_\Sigma(\overline{c_\alpha \eta^\alpha \wedge b} - \partial(Zb))(x)\ G(x,y)\ (c_\gamma \eta^\alpha \wedge b + \bar\partial(\overline{Zb}))(y)}$$

$$\cdot\, \delta\Big(c_\alpha \int_\Sigma \eta^\alpha \wedge b\Big)$$

$$= \text{const. } \left(\frac{\det'(-\Delta)}{\text{area}}\right)^{-1} \mathrm{e}^{-2i\int_\Sigma \overline{Zb}\wedge Zb - 4\int_\Sigma\int_\Sigma(\partial(Zb))(x)\, G(x,y)\ (\bar\partial(\overline{Zb}))(y)}$$

$$\cdot\ \mathrm{e}^{-2i\int_\Sigma \mu^0(c_\alpha\eta^\alpha)\wedge Zb - 2i\int_\Sigma \overline{\mu^0(c_\alpha\eta^\alpha)\wedge Zb} + 2i\int_\Sigma \overline{\mu^0(c_\alpha\eta^\alpha)}\wedge \mu^0(c_\alpha\eta^\alpha)}\, \delta\Big(c_\alpha \int_\Sigma \eta^\alpha \wedge b\Big)\,, \qquad (7.11)$$

where $\det'$ denotes the determinant of the operator restricted to the subspace orthogonal to its kernel, $G(x, y)$ is a Green function of the Laplacian Δ on Σ and $\mu^0(\eta) \equiv 2i\partial \int G(\,\cdot\,, y)(\eta \wedge b)(y)$, as in Eq. (6.26). It is easy to see that

$$\mathrm{e}^{-2i\int_\Sigma \overline{Zb}\wedge Zb - 4\int_\Sigma\int_\Sigma(\partial(Zb))(x)\ G(x,y)\ (\bar\partial(\overline{Zb}))(y)} = \mathrm{e}^{-2i\int_\Sigma \overline{(Zb)^\perp}\wedge(Zb)^\perp}\,, \qquad (7.12)$$

where $(Zb)^\perp$ is the component of Zb orthogonal to the image of $\bar\partial$ acting on functions on Σ. Explicitly,

$$(Zb)^\perp = \frac{i}{2}\bar\omega^i \left(\frac{1}{\operatorname{Im}\tau}\right)_{ij} \int_\Sigma \omega^j \wedge Zb\,,$$

$$2i\int_\Sigma \overline{(Zb)^\perp} \wedge (Zb)^\perp = \left(\int_\Sigma \overline{\omega^i \wedge Zb}\right)\left(\frac{1}{\operatorname{Im}\tau}\right)_{ij}\left(\int_\Sigma \omega^j \wedge Zb\right). \qquad (7.13)$$

We shall rewrite the exponential of the latter expression as a $2g$-dimensional Gaussian integral:

$$\mathrm{e}^{-2i\int_\Sigma \overline{(Zb)^\perp}\wedge(Zb)^\perp} = \pi^{-g}\det(\operatorname{Im}\tau)$$

$$\cdot \int \mathrm{e}^{-4\bar e_i\ (\operatorname{Im}\tau)^{ij} e_j - 2i\ e_i\int_\Sigma \omega^i \wedge Zb - 2i\ \bar e_i \int_\Sigma \overline{\omega^i \wedge Zb}} \prod_i d^2 e_i\,. \qquad (7.14)$$

Gathering Eqs. (7.11), (7.12) and (7.14), we obtain

$$I_X = \text{const. } \det(\operatorname{Im}\tau)\left(\frac{\det'(-\Delta)}{\text{area}}\right)^{-1} e^{-2i\int_\Sigma \mu^0(c_\gamma\eta^\gamma)\wedge Zb - 2i\int_\Sigma \overline{\mu^0(c_\alpha\eta^\alpha)\wedge Zb}}$$
$$\cdot\, e^{2i\int_\Sigma \overline{\mu^0(c_\alpha\eta^\alpha)}\wedge\mu^0(c_\gamma\eta^\alpha)}\int e^{-4\bar{e}_i\,(\operatorname{Im}\tau)^{ij}e_j - 2i\,e_i\int_\Sigma\omega^i\wedge Zb - 2i\,\bar{e}_i\int_\Sigma\overline{\omega^i\wedge Zb}}\prod_i d^2e_i$$
$$\cdot\,\delta\Big(c_\alpha\int_\Sigma\eta^\alpha\wedge b\Big)\,. \tag{7.15}$$

7.3. Integral over Z.
The integral to calculate is

$$I_Z \equiv \int e^{-i\int_\Sigma\langle\bar\partial Z,\wedge\bar\partial Z\rangle - 2i\int_\Sigma Z(\mu^0(c_\gamma\eta^\alpha)+e_i\omega^i)\wedge b - 2i\int_\Sigma\overline{Z(\mu^0(c_\gamma\eta^\gamma)+e_i\omega^i)\wedge b}}DZ\,. \tag{7.16}$$

Let us decompose Z into the part Z_0 in the kernel of $\bar\partial$ (i.e. in $H^0(L^2_{\mathbf{x}})$) and the part Z' orthogonal to $H^0(L^2_{\mathbf{x}})$. Writing $Z_0 = f_r\kappa^r$, where $(\kappa^r)_{r=1}^{g-1}$ is a basis of $H^0(L^2_{\mathbf{x}})$, we obtain

$$I_Z = \int e^{-i\int_\Sigma\langle\bar\partial Z',\wedge\bar\partial Z'\rangle - 2i\int_\Sigma Z'(\mu^0(c_\alpha\eta^\alpha)+e_i\omega^i)\wedge b - 2i\int_\Sigma\bar{Z}'\overline{(\mu^0(c_\alpha\eta^\gamma)+e_i\omega^i)\wedge b}}DZ'$$
$$\cdot\,\text{const. } \det(K_0)\int e^{-2if_r\int_\Sigma\kappa^r(\mu^0(c_\gamma\eta^\gamma)+e_i\omega^i)\wedge b - 2i\bar{f}_r\int_\Sigma\bar\kappa^r\overline{(\mu^0(c_\alpha\eta^\alpha)+e_i\omega^i)\wedge b}}\prod_r d^2f_r$$
$$= \text{const. } \det(K_0)\ \det{}'(\bar\partial^\dagger_{L^2_{\mathbf{x}}}\bar\partial_{L^2_{\mathbf{x}}})^{-1}\prod_r\delta\Big(\int_\Sigma\kappa^r(\mu^0(c_\alpha\eta^\alpha)+e_i\omega^i)\wedge b\Big)$$
$$\cdot\, e^{4i\int_\Sigma\langle\bar\partial^{-1}((\mu^0(c_\alpha\eta^\gamma)+e_i\omega^i)\wedge b),\,\bar\partial^{-1}((\mu^0(c_\gamma\eta^\gamma)+e_i\omega^i)\wedge b)\rangle}\,, \tag{7.17}$$

where

$$(K_0)^{rs} \equiv \int_\Sigma\langle\kappa^r,\kappa^s\rangle\text{vol}\,, \tag{7.18}$$

and, in the last line of (7.17), $\bar\partial^{-1}$ stands for the inverse of $\bar\partial_{L^{-2}_{\mathbf{x}}K}$. We may finally collect Eqs. (7.5), (7.8), (7.9), (7.15), (7.16) and (7.17):

$$I_{XYZ} \equiv e^{-i\int_\Sigma(2(\overline{\bar\partial X - Zb})\wedge(\bar\partial X - Zb) + \langle\bar\partial Y + 2Xb,\,\wedge(\bar\partial Y + 2Xb)\rangle + \langle\bar\partial Z,\wedge\bar\partial Z\rangle)}DY\,DX\,DZ$$
$$= \text{const. } \det(H_0)\ \det(\operatorname{Im}\tau)\det(K_0)\det(\bar\partial^\dagger_{L^{-2}_{\mathbf{x}}}\bar\partial_{L^{-2}_{\mathbf{x}}})^{-1}\left(\frac{\det'(-\Delta)}{\text{area}}\right)^{-1}$$
$$\times\det{}'(\bar\partial^\dagger_{L^2_{\mathbf{x}}}\bar\partial_{L^2_{\mathbf{x}}})^{-1}$$
$$\cdot\int\delta\Big(\int_\Sigma c_\alpha\eta^\alpha\wedge b\Big)\prod_r\delta\Big(\int_\Sigma\kappa^r(\mu^0(c_\alpha\eta^\alpha)+e_i\omega^i)\wedge b\Big)$$
$$\times\, e^{-\bar{c}_\gamma(H_0)^{\alpha\beta}c_\beta + 2i\int_\Sigma\overline{\mu^0(c_\alpha\eta^\gamma)}\wedge\mu^0(c_\gamma\eta^\alpha)}$$
$$\cdot\, e^{-4\bar{e}_i(\operatorname{Im}\tau)^{ij}e_j + 4i\int_\Sigma\langle\bar\partial^{-1}((\mu^0(c_\gamma\eta^\gamma)+e_i\omega^i)\wedge b),\,\bar\partial^{-1}((\mu^0(c_\alpha\eta^\gamma)+e_i\omega^i)\wedge b)\rangle}\prod_\alpha d^2c_\alpha\prod_i d^2e_i$$
$$= \text{const. } \det(K_0)\det\Big(\bar\partial^\dagger_{L^{-2}_{\mathbf{x}}}\bar\partial_{L^{-2}_{\mathbf{x}}}\Big)^{-1}\left(\frac{\det'(-\Delta)}{\text{area}}\right)^{-1}\det{}'(\bar\partial^\dagger_{L^2_{\mathbf{x}}}\bar\partial_{L^2_{\mathbf{x}}})^{-1}$$
$$\cdot\,\Big(\int\delta(BV)e^{-\|CV\|^2}DV\Big)\,, \tag{7.19}$$

where the last integral is the same as the one introduced in Sect. 4.3. Notice that, with the use of Eq. (6.37), one obtains then

$$I_{XYZ}\det(\Omega(1,n)) = \text{const.}\,|z^1|^{-2}\left|\det\Big(\int_\Sigma \kappa^r\omega^i\wedge b\Big)_{i<g}\right|^{-2}\det(H_0)\det(\operatorname{Im}\tau)\det(K_0)$$
$$\times\det(\bar\partial^\dagger_{L^{-2}_{\mathbf{X}}}\bar\partial_{L^{-2}_{\mathbf{X}}})^{-1}\left(\frac{\det'(-\Delta)}{\text{area}}\right)^{-1}\det{}'(\bar\partial^\dagger_{L^2_{\mathbf{X}}}\bar\partial_{L^2_{\mathbf{X}}})^{-1}\,, \qquad (7.20)$$

i.e. the V-integrals cancel. This ends the formal calculation of $\det(\bar D_n^\dagger \bar D_n)$.

Clearly, the determinants appearing on the right-hand side of the expression (7.19) need regularization. If we use the zeta-function procedure to give sense to them, it is not guaranteed that the result will coincide with $\det(\bar D_n^\dagger\bar D_n)$ regularized by the zeta-function prescription. Indeed, the latter should satisfy the chiral anomaly relation (6.10) but for $h = U\left(\begin{smallmatrix}1 & v\\ 0 & 1\end{smallmatrix}\right)U^{-1}$, we obtain

$$\det(({}^{h^{-1}}\bar D_n)^{\dagger}\,{}^{h^{-1}}\bar D_n)\,\det(\Omega(hh^\dagger,n))^{-1} = \det(\bar D_n^\dagger\bar D_n)\,\det(\Omega(1,n))^{-1}\,, \qquad (7.21)$$

if we use for $\det(\bar D_n^\dagger\bar D_n)^{-1}$ the expression on the right-hand side of Eq. (7.19), instead of

$$\det(({}^{h^{-1}}\bar D_n)^{\dagger}\,{}^{h^{-1}}\bar D_n)\det(\Omega(hh^\dagger,n))^{-1}$$
$$= e^{-\frac{2i}{\pi}\int_\Sigma\langle b+\bar\partial v,\wedge(b+\bar\partial v)\rangle+\frac{2i}{\pi}\int_\Sigma\langle b,\wedge b\rangle}\det(\bar D_n^\dagger\bar D_n)\det(\Omega(1,n))^{-1}\,, \qquad (7.22)$$

given by (6.10) (and Eqs. (6.18, (6.19)). It is easy to guess that we should correct the formal result for $\det(\bar D_n^\dagger\bar D_n)$ by taking

$$\det(\bar D_n^\dagger\bar D_n) = \text{const.}\ \det(K_0)^{-1}\det(\bar\partial^\dagger_{L^{-2}_{\mathbf{X}}}\bar\partial_{L^{-2}_{\mathbf{X}}})\left(\frac{\det'(-\Delta)}{\text{area}}\right)\det{}'(\bar\partial^\dagger_{L^2_{\mathbf{X}}}\bar\partial_{L^2_{\mathbf{X}}})$$
$$\cdot\, e^{-\frac{2i}{\pi}\int_\Sigma\langle b,\wedge b\rangle}\left(\int\delta(BV)\,e^{-\|CV\|^2}DV\right)^{-1}. \qquad (7.23)$$

Indeed, Eq. (7.20) is replaced then by the relation

$$\det(\bar D_n^\dagger\bar D_n)\det(\Omega(1,n))^{-1} = \text{const.}|z^1|^2\left|\det\Big(\int_\Sigma\kappa^r\omega^i\wedge b\Big)_{i<g}\right|^2 e^{-\frac{2i}{\pi}\int_\Sigma\langle b,\wedge b\rangle}$$
$$\cdot\,\det(H_0)^{-1}\det(\operatorname{Im}\tau)^{-1}\det(K_0)^{-1}\det(\bar\partial^\dagger_{L^{-2}_{\mathbf{X}}}\bar\partial_{L^{-2}_{\mathbf{X}}})\left(\frac{\det'(-\Delta)}{\text{area}}\right)\det{}'(\bar\partial^\dagger_{L^2_{\mathbf{X}}}\bar\partial_{L^2_{\mathbf{X}}})\,, \qquad (7.24)$$

and (7.22) follows. We shall show in Appendix D that formula (7.23) is, indeed, the right expression for the zeta-function regularized determinant. Putting it together with the Eq. (6.39) from the end of Sect. 4.3, we obtain the following explicit expression :

$$\det(\bar{D}_n^\dagger \bar{D}_n)\det(\Omega(1,n))^{-1}\left|\det\left(\int_\Sigma \mathrm{tr}\,\omega^\beta(n)\wedge\frac{\partial A^{01}(n)}{\partial n_\alpha}\right)\right|^2 \prod_\alpha d^2 n_\alpha = \text{const.}\, i^{-N}$$
$$\cdot \det(H_0)^{-1}\det(\mathrm{Im}\,\tau)^{-1}\det(K_0)^{-1}\det(\bar\partial^\dagger_{L_{\mathbf{x}}^{-2}}\bar\partial_{L_{\mathbf{x}}^{-2}})\left(\frac{\det'(-\Delta)}{\text{area}}\right)\det{}'(\bar\partial^\dagger_{L^2_{\mathbf{x}}}\bar\partial_{L^2_{\mathbf{x}}})$$
$$\cdot\, \mathrm{e}^{-\frac{2i}{\pi}\int_\Sigma\langle b,\wedge b\rangle}|\epsilon_{\alpha_1,\dots,\alpha_N} z^{\alpha_1} dz^{\alpha_2}\wedge\dots\wedge dz^{\alpha_N}|^{\wedge 2}$$
$$\cdot\, \frac{1}{i}\left|\sum_{j=1}^{g}(-1)^j \det\left(\int_\Sigma \kappa^r\omega^i\wedge b\right)_{i\neq j}\omega^j(x)\right|^{\wedge 2}. \tag{7.25}$$

8. Functional Integral over $hh^\dagger$

We shall attempt now a direct calculation of the functional integral (6.12). Only the $\mathrm{e}^{(k+4)S(hh^\dagger,A_{\mathbf{x},b})}$ term under it depends on $hh^\dagger$ and, as noticed before, its $hh^\dagger$-integral should give the partition function of the $SL(2,\mathbb{C})/SU(2)$-valued WZW model. Below, we shall find its form somewhat surprising. The action $S(hh^\dagger,A_{\mathbf{x},b})$ is explicitly given by Eq. (6.19) in the parametrization (6.18) of $hh^\dagger$ by real functions φ and $v\in\Gamma(L_0^{-2})$. It will be more convenient to use $w\equiv e^{-\varphi}v$ instead of v. The formal measure $D(hh^\dagger)$ becomes then the product of the formal Lebesgue measures Dw and $D\varphi$ determined by the L^2 scalar products $\int_\Sigma\langle w,w\rangle\mathrm{vol}$ and $\int_\Sigma|\varphi|^2\mathrm{vol}$. Thus we have:

$$\int \mathrm{e}^{(k+4)S(hh^\dagger,A_{\mathbf{x},b})}D(hh^\dagger) = \mathrm{e}^{-\frac{k+4}{2\pi i}\int_\Sigma\langle b,\wedge b\rangle}$$
$$\cdot \int \mathrm{e}^{\frac{k+4}{2\pi i}\int_\Sigma[-\varphi(\partial\bar\partial\varphi-2F_0)+\langle \mathrm{e}^{-\varphi}b+(\bar\partial+(\bar\partial\varphi))w,\wedge(\mathrm{e}^{-\varphi}b+(\bar\partial+(\bar\partial\varphi))\omega)\rangle]}Dw\, D\varphi\,, \tag{8.1}$$

where $\bar\partial\equiv\bar\partial_{L_{\mathbf{x}}^{-2}}$ when acting on w and F_0 is the curvature form of the holomorphic connection of L_0 preserving the fixed hermitian metric. Note that the field w enters quadratically in the action so that the w-integral is Gaussian and may be easily performed:

$$\int \mathrm{e}^{\frac{k+4}{2\pi i}\int_\Sigma\langle \mathrm{e}^{-\varphi}b+(\bar\partial+(\bar\partial\varphi))w,\wedge(\mathrm{e}^{-\varphi}b+(\bar\partial+\bar\partial(\varphi))\omega)\rangle}Dw$$
$$= \mathrm{e}^{\frac{k+4}{2\pi i}\int_\Sigma\langle P_\varphi(\mathrm{e}^{-\varphi}b),\wedge P_\varphi(\mathrm{e}^{-\varphi}b)\rangle}\det\left((\bar\partial+(\bar\partial\varphi))^\dagger(\bar\partial+(\bar\partial\varphi))\right)^{-1}, \tag{8.2}$$

where P_φ denotes the orthogonal projector on the kernel of $(\bar\partial+(\bar\partial\varphi))^\dagger$. Explicitly,

$$P_\varphi(\mathrm{e}^{-\varphi}b) = i\mathrm{e}^{\varphi}(\eta^\alpha)^\dagger(H_\varphi^{-1})_{\beta\alpha}\int_\Sigma\eta^\beta\wedge b\,,$$
$$i\int_\Sigma\langle P_\varphi(\mathrm{e}^{-\varphi}b),\wedge P_\varphi(\mathrm{e}^{-\varphi}b)\rangle = \left(\int_\Sigma\overline{\eta^\alpha\wedge b}\right)(H_\varphi^{-1})_{\beta\alpha}\left(\int_\Sigma\eta^\alpha\wedge b\right)\equiv\bar z^\alpha(H_\varphi^{-1})_{\beta\alpha}z^\beta\,, \tag{8.3}$$

with the matrix of the modified scalar products of the vectors of the basis (η^α) of $H^0(L^2_{\mathbf{x}}K)$

$$(H_\varphi)^{\alpha\beta}\equiv\frac{1}{i}\int_\Sigma\mathrm{e}^{2\varphi}\langle\eta^\alpha,\wedge\eta^\beta\rangle\,, \tag{8.4}$$

compare Eq. (6.38). For convenience, we shall rewrite the exponential on the right-hand side of Eq. (8.2) as a finite-dimensional Gaussian integral:

$$\mathrm{e}^{\frac{k+4}{2\pi i}\int_\Sigma\langle P_\varphi(\mathrm{e}^{-\varphi}b),\wedge P_\varphi(\mathrm{e}^{-\varphi}b)\rangle} = \mathrm{e}^{-\frac{k+4}{2\pi}\bar z^\alpha(H_\varphi^{-1})_{\beta\alpha}z^\beta}$$
$$= \left(\frac{2}{k+4}\right)^N \det(H_\varphi)\int \mathrm{e}^{-\frac{2\pi}{k+4}\bar c_\alpha(H_\varphi)^{\alpha\beta}c_\beta+ic_\alpha z^\alpha+i\bar c_\gamma\bar z^\alpha}\prod_\alpha d^2c_\alpha\,. \tag{8.5}$$

The φ-dependence of the product of determinants $\det(H_\varphi)\det\big((\bar\partial+(\bar\partial\varphi))^\dagger(\bar\partial+(\bar\partial\varphi))\big)^{-1}$ with the second one regularized by the zeta-function prescription (or any other gauge invariant procedure) is given by the chiral anomaly:

$$\delta\ln\left(\det(H_\varphi)\det\big((\bar\partial_{L_{\mathbf{x}}^{-2}}+(\bar\partial\varphi))^\dagger(\bar\partial_{L_{\mathbf{x}}^{-2}}+(\bar\partial_\varphi))\big)^{-1}\right)=\frac{2}{\pi i}\int_\Sigma(\delta\varphi)(\partial\bar\partial\varphi-F_0)$$
$$+\frac{1}{2\pi i}\int_\Sigma(\delta\varphi)R\,, \tag{8.6}$$

where R is the metric curvature form of (the holomorphic tangent bundle of) Σ normalized so that $\int_\Sigma R=4\pi i(g-1)$. The global form of the formula (8.6) is

$$\det(H_\varphi)\det\left((\bar\partial_{L_{\mathbf{x}}^{-2}}+(\bar\partial\varphi))^\dagger(\bar\partial_{L_{\mathbf{x}}^{-2}}+(\bar\partial\varphi))\right)^{-1}$$
$$=\mathrm{e}^{\frac{1}{\pi i}\int_\Sigma\varphi(\partial\bar\partial\varphi-2F_0)+\frac{1}{2\pi i}\int_\Sigma\varphi R}\,\det(H_0)\,\det(\bar\partial^\dagger_{L_{\mathbf{x}}^{-2}}\bar\partial_{L_{\mathbf{x}}^{-2}})^{-1}\,. \tag{8.7}$$

Gathering Eq. (8.2), (8.5) and (8.7), we obtain

$$\int\mathrm{e}^{\frac{k+4}{2\pi i}\int_\Sigma\langle e^{-\varphi}b+(\bar\partial+(\bar\partial\varphi))w,\wedge(e^{-\varphi}b+(\bar\partial+\bar\partial(\varphi))w)\rangle}Dw$$
$$=\text{const.}\det(H_0)\det(\bar\partial^\dagger_{L_{\mathbf{x}}^{-2}}\bar\partial_{L_{\mathbf{x}}^{-2}})^{-1}\mathrm{e}^{\frac{1}{\pi i}\int_\Sigma\varphi(\partial\bar\partial\varphi-2F_0)+\frac{1}{2\pi i}\int_\Sigma\varphi R}$$
$$\cdot\int\mathrm{e}^{-\frac{2\pi}{k+4}\bar c_\alpha(H_\varphi)^{\gamma\beta}c_\beta+\imath c_\gamma z^\gamma+\imath\bar c_\alpha\bar z^\gamma}\prod_\alpha d^2c_\alpha\,. \tag{8.8}$$

Here appears a new difficulty in the calculation of the scalar product of CS states, as compared to the genus zero and one cases studied in [13, 14] and [27], respectively. There, the $hh^\dagger$ integral for the partition function of the $SL(2,\mathbb{C})/SU(2)$-valued WZW theory led, after parametrization of $hh^\dagger$ by φ and w, to an iterative Gaussian integral: after calculation of the Gaussian w integral, the remaining φ integral was, miraculously, also becoming Gaussian. This does not seem to be the case here. The right-hand side of Eq. (8.8) includes the term

$$\mathrm{e}^{-\frac{2\pi}{k+4}\bar c_\alpha(H_\varphi)^{\alpha\beta}c_\beta}\equiv\mathrm{e}^{\frac{2\pi\imath}{k+4}\int_\Sigma e^{2\varphi}\langle c_\alpha\eta^\alpha,\wedge c_\beta\eta^\beta\rangle} \tag{8.9}$$

with the Liouville type terms containing $\mathrm{e}^{2\varphi}$ in the exponential. So the φ-integral obtained after integrating out w seems to be of a non-Gaussian type, in contrast to the low genera situation.

We shall show however, that this difficulty may be solved by a trick used in the Liouville theory [28, 29]. The functional integral over φ which we are left with has the form:

$$I_\varphi\equiv\int\mathrm{e}^{-\frac{k+2}{2\pi i}\int_\Sigma\varphi(\partial\bar\partial\varphi-2F_0)+\frac{1}{2\pi i}\int_\Sigma\varphi R-\frac{2\pi}{k+4}\bar c_\alpha(H_\varphi)^{\alpha\beta}c_\beta}D\varphi\,. \tag{8.10}$$

We shall integrate first over the zero mode of $\varphi_0 \equiv (\int_\Sigma \varphi \text{vol})/(\text{area})^{1/2}$. For this purpose, let us multiply I_φ by $1 = (\text{area})^{1/2} \cdot \int \delta(\varphi_0 - a(\text{area})^{1/2}) da$. Changing the order of integration, shifting φ to $\varphi + a$ and setting $M \equiv (k+1)(g-1)$, we obtain

$$
\begin{aligned}
&(\text{area})^{-1/2} I_\varphi = \\
&\quad \int e^{-2aM} e^{-\frac{k+2}{2\pi i}\int_\Sigma \varphi(\partial\bar{\partial}\varphi - 2F_0) + \frac{1}{2\pi i}\int_\Sigma \varphi R - \frac{2\pi}{k+4} e^{2\gamma} \bar{c}_\gamma (H_\varphi)^{\gamma\beta} c_\beta} da\, \delta(\varphi_0)\, D\varphi \\
&\quad = \frac{1}{2}\Gamma(-M)\left(\frac{2\pi}{k+4}\right)^M \int e^{-\frac{k+2}{2\pi i}\int_\Sigma \varphi(\partial\bar{\partial}\varphi - 2F_0) + \frac{1}{2\pi i}\int_\Sigma \varphi R} (\bar{c}_\alpha (H_\varphi)^{\alpha\beta} c_\beta)^M \delta(\varphi_0)\, D\varphi\,,
\end{aligned}
\tag{8.11}
$$

where we have used the relations $\frac{i}{2\pi}\int_\Sigma F_0 = \deg(L_0) = g - 1$ and $\frac{i}{2\pi}\int_\Sigma R = \deg(K^{-1}) = 2(1-g)$. As we see, the integration over the zero mode of φ diverges but may be easily (multiplicatively) regularized by removing the overall divergent factor $(-1)^M \Gamma(-M)$. Now, the c-integral is easy to perform:

$$
\int (-\bar{c}_\alpha (H_\varphi)^{\alpha\beta} c_\beta)^M e^{i\, c_\gamma z^\alpha + i\, \bar{c}_\gamma \bar{z}^\alpha} \prod_\alpha d^2 c_\alpha = (2\pi)^{(2N)} ((H_\varphi)^{\alpha\beta} \partial_{\bar{z}^\gamma} \partial_{z^\beta})^M \prod_\alpha \delta(z^\alpha)\,. \tag{8.12}
$$

Gathering the above results, we obtain the following "Coulomb gas representation" for the higher genus partition function of the $SL(2,\mathbb{C})/SU(2)$-valued WZW model:

$$
\begin{aligned}
&\int e^{(k+4)S(hh^\dagger, A_{\mathbf{x}}, b)} D(hh^\dagger) = \text{const.}\ (\text{area})^{1/2} \det(H_0) \det(\bar{\partial}^\dagger_{L_{\mathbf{x}}^{-2}} \bar{\partial}_{L_{\mathbf{x}}^{-2}})^{-1} e^{-\frac{k+4}{2\pi i}\int_\Sigma \langle b, \wedge b\rangle} \\
&\quad \cdot \left(\int e^{-\frac{k+2}{2\pi i}\int_\Sigma \varphi(\partial\bar{\partial}\varphi - 2F_0) + \frac{1}{2\pi i}\int_\Sigma \varphi R} ((H_\varphi)^{\alpha\beta} \partial_{\bar{z}^\alpha} \partial_{z^\beta})^M \delta(\varphi_0) D\varphi\right) \prod_\alpha \delta(z^\alpha) \\
&= \text{const.}\ (\text{area})^{1/2} \det(H_0) \det(\bar{\partial}^\dagger_{L_{\mathbf{x}}^{-2}} \bar{\partial}_{L_{\mathbf{x}}^{-2}})^{-1} e^{-\frac{k+4}{2\pi i}\int_\Sigma \langle b, \wedge b\rangle} \left(\prod_m \partial_{\bar{z}^\alpha_m} \partial_{z^\beta_m} \prod_\alpha \delta(z^\alpha)\right) \\
&\quad \cdot \int \left(\int e^{-\frac{k+2}{2\pi i}\int_\Sigma \varphi(\partial\bar{\partial}\varphi - 2F_0) + \frac{1}{2\pi i}\int_\Sigma \varphi R + 2\sum_m \varphi(x_m)} \delta(\varphi_0) D\varphi\right) \prod_m \frac{1}{i} \langle \eta^{\alpha_m}, \wedge \eta^{\beta_m}\rangle(x_m)\,,
\end{aligned}
\tag{8.13}
$$

where m runs from 1 to M. Observe, that the integrand in the functional integral over φ is now invariant under constant shifts of φ, except for the term $\delta(\varphi_0)$ (the neutrality of the Coulomb gas). The φ-integral is of the Gaussian form

$$
\int e^{\frac{k+2}{2\pi i}\int_\Sigma \partial\varphi \wedge \bar{\partial}\varphi - i\int_\Sigma \varphi\sigma} \delta(\varphi_0) D\varphi = \text{const.}\ \det{}'(-\Delta)^{-1/2} e^{\frac{\pi}{k+2}\int_\Sigma\int_\Sigma \sigma(x) G(x,y) \sigma(y)}\,, \tag{8.14}
$$

where $G(\,\cdot\,,\,\cdot\,)$ is a Green function of the Laplacian on Σ and $\sigma = \frac{k+2}{\pi} F_0 + \frac{1}{2\pi} R + 2i \sum_m \delta_{x_m}$. Since $G(x,y) \sim \frac{1}{2\pi} \ln d(x,y)$, where $d(\,\cdot\,,\,\cdot\,)$ is the metric distance, the terms $e^{-\frac{4\pi}{k+2} G(x_m, x_m)}$ on the right-hand side of (8.14) are divergent. They may be easily multiplicatively renormalized by replacing them by their normal ordered version $e^{-\frac{4\pi}{k+2} :G(x_m, x_m):}$, see Eq. (5.1) for the definition. This way, choosing for simplicity the Green function satisfying $\int_\Sigma G(\,\cdot\,, y)((k+2)F_0(y) + \frac{1}{2} R(y)) = 0$, we obtain

$$
\begin{aligned}
\int e^{\frac{k+2}{2\pi i}\int_\Sigma \partial\varphi \wedge \bar{\partial}\varphi - i\int_\Sigma \varphi\sigma} \delta(\varphi_0) D\varphi &= \text{const.}\ \det{}'(-\Delta)^{-1/2} \left(\prod_{m_1 \neq m_2} e^{-\frac{4\pi}{k+2} G(x_{m_1}, x_{m_2})}\right) \\
&\quad \cdot \left(\prod_m e^{-\frac{4\pi}{k+2} :G(x_m, x_m):}\right).
\end{aligned}
\tag{8.15}
$$

The substitution of this result into Eq. (8.13) results in the relation

$$\int e^{(k+4)S(hh^\dagger, A_{\mathbf{x},b})} D(hh^\dagger)$$
$$= \text{const.} \cdot \det(H_0) \det(\bar\partial^\dagger_{L_{\mathbf{x}}^{-2}} \bar\partial_{L_{\mathbf{x}}^{-2}})^{-1} \left(\frac{\det'(-\Delta)}{\text{area}}\right)^{-1/2}$$
$$\times e^{-\frac{k+4}{2\pi i}\int_\Sigma \langle b, \wedge b\rangle} \left(\prod_m \partial_{\bar z^{\alpha}_m} \partial_{z^{\beta}_m} \prod_\alpha \delta(z^\alpha)\right)$$
$$\cdot \int \left(\prod_{m_1 \neq m_2} e^{-\frac{4\pi}{k+2} G(x_{m_1}, x_{m_2})}\right) \left(\prod_m e^{-\frac{4\pi}{k+2} :G(x_m, x_m):} \frac{1}{i} \langle \eta^{\alpha_m}, \wedge \eta^{\beta_m}\rangle(x_m)\right), \tag{8.16}$$

which is the final formula for the higher genus partition function of the $SL(2,\mathbb{C})/SU(2)$-valued WZW model.

Equation (8.16) reduces the functional integral over $hh^\dagger$ to a finite dimensional integral over M copies of Σ. The integrand is a smooth function except for $\mathcal{O}(d(x_{m_1}, x_{m_2})^{-\frac{4}{k+2}})$ singularities at coinciding points. Power counting shows that the integral converges for $g = 2$ but for higher genera it diverges unless special combinations

$$\gamma_{(\alpha_m),(\beta_m)} \prod_m \langle \eta^{\alpha_m}, \wedge \eta^{\beta_m}\rangle(x_m) \tag{8.17}$$

of forms are integrated. We shall return to this issue below. Another feature of the right-hand side of Eq. (8.16) may look even more surprising in a candidate for the partition function: its dependence of the external field $A_{\mathbf{x},b}$ is not functional but distributional! The entire dependence on $b \in \wedge^{01}(L_0)$ resides in the term $\prod_m \partial_{\bar z^{\alpha}_m} \partial_{z^{\beta}_m} \prod_\alpha \delta(z^\alpha)$ (recall that $z^\alpha \equiv \int_\Sigma \eta^\alpha \wedge b$). This is not so astonishing in view of the fact that the partition function of the $SL(2,\mathbb{C})/SU(2)$ WZW may be expected, by formal arguments similar to the ones used in [30], to be the hermitian square of a holomorphic section of a negative power of the determinant bundle. But there are no such sections. There exist, however distributional solutions of the corresponding Ward identities and the right-hand side of (8.16) is one of them.

9. Assembling the Final Formula

The main results (7.25) and (8.16) of the calculations of the last two sections permit to reduce the formal scalar product formula (6.12) to the following finite-dimensional integral:

$$\|\Psi\|^2 = \text{const.}\, i^{-N} \det(\operatorname{Im}\tau)^{-1} \left(\frac{\det'(-\Delta)}{\text{area}}\right)^{1/2} e^{-\frac{ik}{2\pi}\int_\Sigma \operatorname{tr} A_0^{10} \wedge A_0^{01}}$$
$$\cdot \int \det\left(\int_\Sigma \langle \kappa^r, \kappa^s\rangle \text{vol}\right)^{-1} \det{}'(\bar\partial^\dagger_{L^2_{\mathbf{x}}} \bar\partial_{L^2_{\mathbf{x}}}) e^{-2\pi k (\int_{x_0}^{\mathbf{x}} \bar\omega)(\operatorname{Im}\tau)^{-1}(\int_{x_0}^{\mathbf{x}} \omega)}$$
$$\cdot \left|\sum_{j=1}^{g} (-1)^j \det\Big(\int_\Sigma \kappa^r \omega^\imath \wedge b\Big)_{\imath \neq j} \omega^j(x) \psi(\mathbf{x}, b)\right|^{\wedge 2} \left(\prod_m \partial_{\bar z^{\alpha}_m} \partial_{z^{\beta}_m} \prod_\alpha \delta(z^\alpha)\right)$$
$$\cdot \left|\epsilon_{\alpha_1,\ldots,\alpha_N} z^{\alpha_1} dz^{\alpha_2} \wedge \ldots \wedge dz^{\alpha_N}\right|^{\wedge 2} \left(\prod_{m_1 \neq m_2} e^{-\frac{4\pi}{k+2} G(x_{m_1}, x_{m_2})}\right)$$
$$\cdot \left(\prod_m e^{-\frac{4\pi}{k+2} :G(x_m, x_m):} \frac{1}{i} \langle \eta^{\alpha_m}, \wedge \eta^{\beta_m}\rangle(x_m)\right). \tag{9.1}$$

The integral is, for fixed $\mathbf{x}$, over the $(N-1) \equiv (3g-4)$-dimensional projective space with homogeneous coordinates (z^α), over the Cartesian product of $M \equiv (k+1)(g-1)$ copies of Σ (variables x_m) and, finally, over the projection x of $\mathbf{x} \in \tilde{\Sigma}$ to Σ. Let us discuss first the z^α integral. It has the form

$$I_z \equiv \int\limits_{\mathbb{P}H^1(L_{\mathbf{x}}^{-2})} |P(z)|^2 \Big(\prod_m \partial_{\bar{z}_m^\gamma} \partial_{z_m^\beta} \prod_\alpha \delta(z^\alpha) \Big) |\epsilon_{\alpha_1,\ldots,\alpha_N} z^{\alpha_1} dz^{\alpha_2} \wedge \ldots \wedge dz^{\alpha_N}|^{\wedge 2} , \tag{9.2}$$

where

$$P(z) = \sum_{j=1}^{g} (-1)^j \det \big(\int_\Sigma \kappa^r \omega^i \wedge b \big)_{i \neq j} \omega^j(x) \, \psi(\mathbf{x}, b) \tag{9.3}$$

is a homogeneous polynomial in (z^α) of degree M (with values in $K|_x$). The integrand is a distributional $2(N-1)$-form on $H^1(L_{\mathbf{x}}^{-2})$ invariant under complex rescalings of z . Note that formally

$$\int\limits_{H^1(L_{\mathbf{x}}^{-2})} |P(z)|^2 \Big(\prod_m \partial_{\bar{z}_m^\gamma} \partial_{z_m^\beta} \prod_\alpha \delta(z^\alpha) \Big) |dz^1 \wedge \ldots \wedge dz^N|^2 = \frac{1}{(N-1)!} \big(\int_{\mathbf{C}} |\lambda^{-1} d\lambda|^2 \big) I_z , \tag{9.4}$$

where the divergent integral on the right-hand side is over the fibers of the projection of $H^1(L_{\mathbf{x}}^{-2})$ onto $\mathbb{P}H^1(L_{\mathbf{x}}^{-2})$. Of course, the left-hand side is perfectly well defined and we shall take it as a definition of the right-hand side. One may expect to reabsorb this way the infinite constant $(-1)^M \Gamma(-M)$ produced by the integration of the zero mode of the field φ, see Eq. (8.11). This is more than formal gymnastics. In Appendix E, we show that changing the order of integration in the above arguments by computing the integral over $\mathbb{P}H^1(L_{\mathbf{x}}^{-2})$ modular degrees of freedom just after the w functional integration and the one over the scalar field φ only afterwards, one obtains the same final result but no infinite constants, apart from those of the Wick ordering, appear in the intermediate steps. This way, it is rather the convergent integration over the (part of) the modular degrees of freedom than the divergent Gupta–Trivedi–Wise–Goulian–Li trick which removes the cumbersome Liouville-type terms from the effective action for φ and renders the φ integral calculable. It is an interesting question whether similar arguments may be used to substantiate the Goulian-Li trick in the gravity case.

With the above interpretation of I_z, we obtain the following expression for the scalar product of genus g CS states:

$$\begin{aligned} \|\Psi\|^2 =& \text{const.} \det(\operatorname{Im}\tau)^{-1} \left(\frac{\det'(-\Delta)}{\text{area}} \right)^{1/2} e^{-\frac{ik}{2\pi} \int_\Sigma \operatorname{tr} A_0^{10} \wedge A_0^{01}} \\ & \cdot \int \det \Big(\int_\Sigma \langle \kappa_{\mathbf{x}}^r, \kappa_{\mathbf{x}}^s \rangle \text{vol} \Big)^{-1} \det'(\bar{\partial}^\dagger_{L_{\mathbf{x}}^2} \bar{\partial}_{L_{\mathbf{x}}^2}) \, e^{-2\pi k (\int_{x_0}^{\mathbf{x}} \bar{\omega})(\operatorname{Im}\tau)^{-1}(\int_{x_0}^{\mathbf{x}} \omega)} \\ & \cdot \prod_m \partial_{\bar{z}_{\mathbf{x}}^{\gamma_m}} \partial_{z_{\mathbf{x}}^{\beta_m}} \Big|_{b=0} \left(\frac{1}{i} \left| \sum_{j=1}^{g} (-1)^j \det \Big(\int_\Sigma \kappa_{\mathbf{x}}^r \omega^i \wedge b \Big)_{i \neq j} \omega^j(x) \, \psi(\mathbf{x}, b) \right|^{\wedge 2} \right) \\ & \cdot \Big(\prod_{m_1 \neq m_2} e^{-\frac{4\pi}{k+2} G(x_{m_1}, x_{m_2})} \Big) \Big(\prod_m e^{-\frac{4\pi}{k+2} :G(x_m, x_m):} \frac{1}{i} \langle \eta_{\mathbf{x}}^{\alpha_m}, \wedge \eta_{\mathbf{x}}^{\beta_m} \rangle (x_m) \Big) , \end{aligned} \tag{9.5}$$

with the $(M+1)$-fold integration over Σ (over the projection x of $\mathbf{x}$ to Σ and over the positions x_m of M "screening charges"). We have restored the $\mathbf{x}$ subscript to stress the $\mathbf{x}$ dependence of various entries in the integrated function. In fact, it is easy to see that the latter depends only on x. In Appendix F, we submit formula (9.5) to few consistency checks showing that $\|\Psi\|^2$ does not depend on the choices of the bases $(\kappa^r_{\mathbf{x}})$ of $H^0(L^2_{\mathbf{x}})$ and $(\eta^\alpha_{\mathbf{x}})$ of $H^0(L^2_{\mathbf{x}}K)$, and of the choice of a hermitian structure of L_0. We also show that upon multiplication of the Riemannian metric of Σ by a function e^σ, $\|\Psi\|^2$ picks up the factor

$$\exp\left[-\frac{i}{24\pi}\frac{3k}{k+2}\int_\Sigma\left(\frac{1}{2}\partial\sigma\wedge\bar{\partial}\sigma+\sigma R\right)\right] \tag{9.6}$$

which guarantees the right value $c=\frac{3k}{k+2}$ of the Virasoro central charge of the theory with partition function given by Eq. (1.5).

We shall rewrite the scalar product formula putting it into a form both more geometric and closer to the spirit of discussion of Sect. 4. To this end consider

$$\chi(\mathbf{x};x_1,\dots,x_{k(g-1)})=\sum_{(\alpha_m)}\prod_{m=1}^{k(g-1)}\left(\eta^{\alpha_m}_{\mathbf{x}}(x_m)\partial_{z^{\alpha_m}_{\mathbf{x}}}\right)\psi(\mathbf{x},b)\,. \tag{9.7}$$

Clearly, the relation (3.18) holds so that χ is the holomorphic $k(g-1)$, 0-form on $\Sigma\times\Sigma^{k(g-1)}$ with values in the h.l.b. B_k of Eq. (3.19) discussed at the end of Sect. 3. It is essentially the same object as χ introduced by Eq. (4.7) in Sect. 4, directly related to Bertram's picture [1] of CS states. The precise relation between the two χ's is given by Eq. (C.18) of Appendix C. With its use, one obtains from Eq. (9.5) a fully normalized formula for the scalar product which uses the description of CS states by polynomials ψ' discussed in Sect. 4.

The expression

$$e^{-2\pi k(\int_{x_0}^{\mathbf{x}}\bar{\omega})(\mathrm{Im}\,\tau)^{-1}(\int_{x_0}^{\mathbf{x}}\omega)}|\Phi(\mathbf{x})|^2 \tag{9.8}$$

with Φ as in Eq. (3.16) defines an admissible hermitian structure on $L^2K((2-2g)x_0)$, see the beginning of Sect. 5: it induces a connection with the curvature $2\pi\omega(\mathrm{Im}\,\tau)^{-1}\wedge\bar{\omega}=-4\pi ig\alpha$. In order to find the geometric interpretation, of the other terms in the scalar product formula (9.5), let us return to the linear map (6.28),

$$H^0(K)\ni v\overset{l(\mathbf{x},b)}{\mapsto}v[b]\in H^1(L^{-2}_{\mathbf{x}}K)\cong H^0(L^2_{\mathbf{x}})^*\,. \tag{9.9}$$

Recall from Sect. 6.3 that surjectivity of $l(\mathbf{x},b)$ assured the local regularity of the projection from $\mathscr{A}^{01}$ into the orbit space $\mathscr{A}^{01}/\mathscr{G}^C$. We may view $\wedge^{g-1}l(\cdot,b)$ as a holomorphic 1,0-form on Σ with values in the bundle $\det^{-1}H^0(K)\otimes\det^{-1}R^0pr_{1*}\mathscr{L}^2_0$ with the representation

$$\wedge^{g-1}l(\mathbf{x},b)=\sum_{j=1}^{g}(-1)^j\det\left(\int_\Sigma\kappa^r_{\mathbf{x}}\omega^i\wedge b\right)_{i\neq j}\omega^j(x)\otimes\left(\wedge_i\omega^i\right)^{-1}\left(\wedge_r\kappa^r_{\mathbf{x}}\right)^{-1}, \tag{9.10}$$

compare the discussion after (5.3) in Sect. 5. Setting

$$\begin{aligned}\phi(\mathbf{x}:x_1,\dots,x_{g-1})&\equiv\sum_{(\alpha_r)}\prod_{r=1}^{g-1}\eta^{\alpha_r}_{\mathbf{x}}(x_r)\partial_{z^{\alpha_r}_{\mathbf{x}}}\wedge^{g-1}l(\mathbf{x},b)\\&=(g-1)!\mathscr{S}_{g-1}\Big(\sum_{j=1}^{g}(-1)^j\\&\quad\times\det(\kappa^r_{\mathbf{x}}(x_r)\omega^i(x_r))_{i\neq j}\,\omega^j(x)\Big)\otimes\left(\wedge_i\omega^i\right)^{-1}\left(\wedge_r\kappa^r_{\mathbf{x}}\right)^{-1},\end{aligned} \tag{9.11}$$

with $\mathscr{S}_{g-1}$ symmetrizing the variables x_r, we obtain a holomorphic g,0-form on $\Sigma \times \Sigma^{g-1}$ with values in

$$C \equiv \det{}^{-1} H^0(K) \otimes pr_{1*}(\det{}^{-1} R^0 pr_{1*} \mathscr{L}_0^2) S^{g-1}(\mathscr{L}_0^2)$$
$$\cong \det{}^{-1} H^0(K) \otimes pr_{1*}(\det{}^{-1} R^0 pr_{1*} \mathscr{L}_0'^2) S^{g-1}(\mathscr{L}_0'^2) \equiv C' , \qquad (9.12)$$

where the isomorphism of the h.l.b.'s on $\Sigma \times \Sigma^{g-1}$ C' and C is induced by the isomorphism (4.9). In the right-hand realization, ϕ coincides with the one introduced by Eq. (5.5) in Sect. 5. Now, notice that

$$\sum_{(\alpha_m)} \prod_{m=1}^{M} (\eta_{\mathbf{x}}^{\alpha_m}(x_m) \partial_{z_{\mathbf{x}}^{\alpha_m}}) \wedge^{g-1} l(\mathbf{x}, b) \psi(\mathbf{x}, b)$$
$$= \binom{M}{g-1} \mathscr{S}_M(\phi(\mathbf{x}; x_1, \ldots, x_{g-1}) \chi(\mathbf{x}; x_g, \ldots, x_M)) . \qquad (9.13)$$

Clearly, the right-hand side is a holomorphic $(1+M)$,0-form on $\Sigma \times \Sigma^M$ with values in the h.l.b.

$$\det{}^{-1} H^0(K) \otimes pr_1{}^*((\det{}^{-1} R^0 pr_{1*} \mathscr{L}_0^2)(L^2 K((2-2g)x_0))^k\ S^M(\mathscr{L}_0^2)$$
$$\equiv \det{}^{-1} H^0(K) \otimes pr_1{}^*((\det{}^{-1} R^0 pr_{1*} \mathscr{L}_0'^2)(L^2 K))^k\ S^M(\mathscr{L}_0'^2) . \qquad (9.14)$$

It is easy to see that

$$\det(\operatorname{Im} \tau)^{-1} e^{-2\pi k (\int_{x_0}^{\mathbf{x}} \bar{\omega}) \frac{1}{\operatorname{Im} \tau} (\int_{x_0}^{\mathbf{x}} \omega)} \Bigg| \sum_{(\alpha_m)} \prod_{m=1}^{M} (\eta_{\mathbf{x}}^{\alpha_m}(x_m) \partial_{z_{\mathbf{x}}^{\alpha_m}}) \sum_{j=1}^{g} (-1)^j \det \left(\int_\Sigma \kappa_{\mathbf{x}}^r \omega^l \wedge b \right)_{i+j}$$
$$\cdot\, \omega^j(x) \psi(\mathbf{x}, b) \Bigg|^{\wedge 2} = \binom{M}{g-1}^2 |\mathscr{S}_M(\phi(\mathbf{x}; x_1, \ldots, x_{g-1}) \chi(\mathbf{x}; x_g, \ldots, x_M))|^{\wedge 2} , \qquad (9.15)$$

where, if we interpret $\mathscr{S}_M(\phi\chi)$ as a $(1+M)$,0-form on $\Sigma \times \Sigma^M$ with values in the bundle on the left-hand side of (9.14), we should use on the latter the hermitian metric induced by the Riemannian metric of Σ, the hermitian structure of L_0 and an admissible hermitian structure of $L^2 K((2-2g)x_0)$. It will then be simpler to work only with the admissible hermitian metrics on all occurring line bundles, including the holomorphic tangent bundle whose hermitian structure is given by the Riemannian metric. With such choices, we may rewrite

$$\|\Psi\|^2 = \text{const. } i^{-M-1} \int \det{}'(\bar{\partial}^\dagger_{L(-x)^2} \bar{\partial}_{L(-x)^2})\ |\mathscr{S}_M(\phi(x; x_1, \ldots, x_{g-1})$$
$$\cdot\, \chi(x; x_g, \ldots, x_M))|^{\wedge 2} \prod_{m_1 \neq m_2} e^{-\frac{4\pi}{k+2} G(x_{m_1}, x_{m_2})} \prod_m e^{-\frac{4\pi}{k+2} :G(x_m, x_m):} , \qquad (9.16)$$

including the prefactors into the normalization of the hermitian metric and replacing $\mathbf{x}$ by x in accordance with the interpretation of the $\mathbf{x}$-dependence as giving rise to geometric objects on Σ. The Green function $G(x,y)$ in (9.16) should be orthogonal to the 2-form $\alpha \equiv \frac{i}{2g} \omega (\operatorname{Im} \tau)^{-1} \wedge \bar{\omega}$. We have rewritten $\det{}'(\bar{\partial}^\dagger_{L^2_{\mathbf{x}}} \bar{\partial}_{L^2_{\mathbf{x}}})$ as $\det{}'(\bar{\partial}^\dagger_{L(-x)^2} \bar{\partial}_{L(-x)^2})$ using the fact that the latter determinant is independent of the normalization of the hermitian structure on $L(-x)$ so that it takes the same value for any admissible metric on $L(-x)$. When specified to the case of Arakelov metric on Σ, this is exactly the expression (5.7) of Sect. 5, if we reinterpret $\mathscr{S}_M(\phi\chi)$ according to the right-hand side of (9.14) and use the relation between the hermitian structures induced by isomorphism (4.9), see the discussion in Appendix B.

Similarly as for the lower genus case, see [13], the natural conjecture is that the integral on the right-hand side of Eq. (9.16) converges if and only if the function ψ defines a globally non-singular CS state Ψ. The singularities under the integral in Eq. (9.16) come from the product

$$\prod_{m_1 \neq m_2} e^{-\frac{4\pi}{k+2} G(x_{m_1}, x_{m_2})} \sim \prod_{m_1 \neq m_2} d(x_{m_1}, x_{m_2})^{-\frac{2}{k+2}} . \tag{9.17}$$

The power counting when $Q+1$ of x_m converge shows that

$$\mathscr{S}_M(\phi\chi)(x; y, y+y_1, \dots, y+y_Q, x_{Q+2}, \dots, x_M) \tag{9.18}$$

has to have the vanishing Taylor expansion at zero in $y_1, \dots, y_Q$ up to order $\leqq Q(\frac{Q+1}{k+2} - 1)$. This also gives a set of sufficient conditions for the convergence of the integral in (9.16). Notice that for $g = 2$ when $M = k+1$ these conditions are always satisfied. For $g > 2$, taking $Q = k+1$, we infer that if the integral converges then $\mathscr{S}_M(\phi\chi)(x; x_1, \dots, x_M)$ has to vanish whenever $k+2$ of x_m's coincide. Let us see that this condition is, indeed, satisfied for ϕ corresponding to CS states. As we have explained at the end of Sect. 4, such states give sections $\chi(x; x_g, \dots, x_M)$ which vanish whenever $k+1$ of x_m's coincide. On the other hand,

$$\sum_{j=1}^{g} (-1)^j \det(\kappa_x^r(x_r)\omega^i(x_r))_{i \neq j}\, \omega^j(x) = -\det \begin{bmatrix} \kappa_x^1(x_1)\omega^i(x_1) \\ \cdot \\ \cdot \\ \kappa_x^{g-1}(x_{g-1})\omega^i(x_{g-1}) \\ \omega^i(x) \end{bmatrix} \tag{9.19}$$

vanishes whenever two x_r's coincide. Hence $\mathscr{S}_M(\phi\chi)(x; x_1, \dots, x_M)$ vanishes whenever $k+2$ of x_m's are equal. It is clear that a complete analysis of the convergence of the integral in Eq. (9.16) and of the related "fusion rule conditions" should be based on the geometry studied in [1, 22] and we shall postpone it to a future work.

4. Appendix A

Let us show that, for c a non-vanishing function on Σ, for $g_c = \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix}$ and for $h_c = U g_c U^{-1}$,

$$\exp[S(h_c, A_0^{10} + A_{\mathrm{x},0}^{01}] = e^{\frac{1}{2\pi i}\int_\Sigma c^{-1}\partial c \wedge (c^{-1}\bar\partial c + 2a_{\mathrm{x}})} \nu(c) , \tag{A.1}$$

where ν is given by Eq. (3.11). Recall that U is a smooth isomorphism of rank 2 vector bundles with trivial determinant, $U : L_0^{-1} \oplus L_0 \to \Sigma \times \mathbb{C}^2$. The gauge field $A_0^{10} + A_{\mathrm{x},0}^{01}$ represents the image under U of the diagonal connection $\nabla + \bar\partial - \sigma^3 a_{\mathrm{x}}$. First note that, by the gauge invariance of the WZW action, $S(h_c, A_0^{10} + A_{\mathrm{x},0}^{01})$ is independent of the choice of U. Moreover, since under an infinitesimal change of the field h,

$$\delta S(h, A) = \frac{i}{2\pi} \int_\Sigma \mathrm{tr}\, h^{-1}\delta h\, F(A^{10} + {}^{h^{-1}}\!A^{01}) , \tag{A.2}$$

which transforms covariantly, one infers that under small changes of the function c,

$$\delta S(h_c, A_0^{10} + A_{\mathbf{x},0}^{01} = \frac{i}{\pi} \int_\Sigma c^{-1} \delta c(\partial(c^{-1}\bar{\partial} c) + \partial a_{\mathbf{x}} + F_0)\,, \tag{A.3}$$

which is the variation of the right-hand side of Eq. (A.1). We may then assume that $c = 1$ on a small disc D. Using a trivialization of the bundle L_0 over D and over $\Sigma \backslash D$ with the transition function f defined around the boundary of D, we may take the isomorphism U equal to the identity on $\Sigma \backslash D$ and interpolating smoothly $\begin{pmatrix} f^{-1} & 0 \\ 0 & f \end{pmatrix}$ inside D. In any case, $h_c = \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix} = g_c$ everywhere. Let a_0 be the 1-form representing on $\Sigma \backslash D$ the metric connection of L_0. It follows easily, that

$$S(h_c, A_0^{10} + A_{\mathbf{x},0}^{01}) = S(g_c) - \frac{i}{\pi} \int_{\Sigma \backslash D} c^{-1} dc \wedge (a_0 + a_{\mathbf{x}}) \tag{A.4}$$

from which Eq. (A.1) follows by integration by parts on the cut surface.

Let us identify the flat bundle corresponding to the character $\Pi_1 \ni p \mapsto v(c_p) \in S^1$ of the fundamental group Π_1 of Σ. First note that $v(c_p)$ is independent of the choice of the metric on the h.l.b. L_0. Suppose that $L_0 = L(-x_0)$ has divisor $D \equiv \sum_{m=1}^{Q+g} y_m - \sum_{n=0}^{Q} x_n$, so that $L_0 \cong \mathcal{O}(D)$. Let us choose a hermitian metric on $\mathcal{O}(D)$ so that $|1(x)|^2 = \exp[4\pi(\sum_m G(x, y_m) - \sum_n G(x, x_n))]$, where 1 is the canonical section of $\mathcal{O}(D)$ with zeros at y_m and poles at x_n and $G(x, y)$ is a Green function of the Laplacian on Σ. 1 trivializes $\mathcal{O}(D)$ on $\Sigma \backslash \{y_m, x_n\}$ and the 1,0-form

$$a \equiv 4\pi\partial\left(\sum_{m=1}^{Q+g} G(\,\cdot\,, y_m) - \sum_{n=0}^{Q} G(\,\cdot\,, x_n)\right) \tag{A.5}$$

represents there the metric connection of $\mathcal{O}(D) \cong L_0$. In particular, its curvature F_0 is equal to $\bar{\partial}\partial a$. Cutting ϵ-balls around the points y_m and x_n (region B_ϵ) and integrating by parts, we obtain

$$\begin{aligned} v(c_p) &\equiv e^{\frac{i}{\pi}\int_\Sigma F_0 \ln c_p} \prod_{j=1}^{g} \left(W_{a_j}^{-\frac{i}{\pi}\int_{b_j} c_p^{-1} dc_p} W_{b_j}^{\frac{i}{\pi}\int_{a_j} c_p^{-1} dc_p} \right) \\ &= \exp\left[\frac{i}{\pi} \lim_{\varepsilon \to 0} \int_{\Sigma \backslash B_\epsilon} (\bar{\partial} a) \ln c_p \right] \prod_{j=1}^{g} \left(W_{a_j}^{-\frac{i}{\pi}\int_{b_j} c_p^{-1} dc_p} W_{b_j}^{\frac{i}{\pi}\int_{a_j} c_p^{-1} dc_p} \right) \\ &= \exp\left[\frac{i}{\pi} \lim_{\varepsilon \to 0} \int_{\Sigma \backslash B_\epsilon} \left(a\, d \ln c_p - \int_{\partial B_\epsilon} a \ln c_p \right) \right]. \end{aligned} \tag{A.6}$$

Since $\bar{\partial} G(x, y) = \frac{1}{4\pi(x-y)} +$ a smooth function, the boundary term contributes

$$\prod_{m=1}^{Q+g} c_p(y_m)^2 \prod_{n=0}^{Q} c_p(x_n)^{-2}\,, \tag{A.7}$$

whereas the volume term $\lim_{\varepsilon \to 0} \int_{\Sigma \backslash B_\epsilon} a\, d \ln c_p$ may be shown to vanish by using Eq. (A.5) and integrating once more by parts (c_p is harmonic). It is easy to see that the flat bundle corresponding to the character (A.7) of Π_1 is equivalent to the trivial bundle with the $\bar\partial$-operator

$$\bar\partial - 2\pi\Big(\sum_m \int_{x_0}^{y_m} \omega - \sum_n \int_{x_0}^{x_n} \omega\Big)\,(\mathrm{Im}\ \tau)^{-1}\bar\omega$$

and is isomorphic to $\mathcal{O}(D-(g-1)x_0)^2 \cong L(-gx_0)^2$.

Appendix B

Consider a function f_{x_0} on $\tilde\Sigma \times \Sigma$ given by

$$f_{x_0}(\mathbf{x},y) \equiv \mathrm{e}^{2\pi i(\int_{x_0}^{\mathbf{x}}\omega)\,(\mathrm{Im}\ \tau)^{-1}(\int_{x_0}^{y}\mathrm{Im}\ \omega)}\ \frac{\vartheta(-a+\int_{x_0}^{\mathbf{x}}\omega|\tau)\ \vartheta(a+\int_{x_0}^{y}\omega|\tau)}{\vartheta(a+\int_{\mathbf{x}}^{y}\omega|\tau)}\,, \tag{B.1}$$

where a is an odd characteristic. One has

$$f_{x_0}(p\mathbf{x},y) = c_p(y)^{-1}\, f_{x_0}(\mathbf{x},y), \qquad \Big(\bar\partial + \pi\Big(\int_{x_0}^{\mathbf{x}}\omega\Big)(\mathrm{Im}\,\tau)^{-1}\bar\omega(y)\Big)\ f_{x_0}(\mathbf{x},y) = 0\,.$$

Besides, f_{x_0} has first order zeros at $x = x_0$ and $y = x_0$ and a first order pole at $x = y$. It follows that multiplication by f_{x_0} establishes an isomorphism between the h.l.b.'s $\mathscr{L}_0'$ and $pr_1{}^*(\mathcal{O}(-x_0))\mathscr{L}_0$ over $\tilde\Sigma \times \Sigma$.

Note that the hermitian structure on the h.l.b. $\mathscr{L}_0$ coming from an admissible hermitian metric on the bundle L_0 (see the beginning of Sect. 5 for the definition of admissibility) induces the connection with curvature

$$\pi\, pr_1{}^*(\omega)\,(\mathrm{Im}\,\tau)^{-1}\, pr_2{}^*(\bar\omega) - \pi\, pr_1{}^*(\bar\omega)\,(\mathrm{Im}\,\tau)^{-1}\, pr_2{}^*(\omega) + (g-1)pr_2{}^*\alpha\,.$$

Taking also an admissible hermitian structure on the bundle $\mathcal{O}(-x_0)$ we obtain a hermitian metric on the h.l.b. $pr_1{}^*(\mathcal{O}(-x_0))\mathscr{L}^0$ corresponding to the curvature form

$$\begin{aligned} &- pr_1{}^*\alpha + \pi\, pr_1{}^*(\omega)(\mathrm{Im}\,\tau)^{-1} pr_2{}^*(\bar\omega) \\ &- \pi\, pr_1{}^*(\bar\omega)(\mathrm{Im}\,\tau)^{-1} pr_2{}^*(\omega) + (g-1)pr_2{}^*\alpha\,, \end{aligned} \tag{B.2}$$

the same as the curvature induced by the hermitian structure of $\mathscr{L}_0'$ described around Eq. (5.2) in Sect. 5. Hence multiplication by f_{x_0} must carry one hermitian structure into the other one, up to a constant factor.

Appendix C

Let us discuss in more detail the relation between the two descriptions of the CS states: the one discussed in Sect. 3 using functions $\psi(\mathbf{x},b)$ with $\mathbf{x} \in \tilde\Sigma$, $b \in \wedge^{01}(L_0^{-2})$ and the one of [1], discussed in Sect. 4, employing polynomials $\psi'(b')$, $b' \in \wedge^{01}(L^{-2})$. Let us fix $\mathbf{x}$ with $x \neq x_0$ and b. Viewing b as an element of $\wedge^{01}(L_{\mathbf{x}}^{-2})$, we may define a form $b'' \in \wedge^{01}(L(-x)^{-2})$ by setting

$$b'' \equiv f_{x_0}(\mathbf{x},\ \cdot\ )^2\, b\,, \tag{C.1}$$

since the multiplication by $f_{x_0}(\mathbf{x},\ \cdot\ )$ given by Eq. (B.1) establishes an isomorphism between $L(-x)$ and $L_{\mathbf{x}}$. We shall choose $\kappa \in H^0(L^2(-x))\backslash H^0(L(-x)^2)$ s.t.

$$\int_\Sigma \kappa v \wedge b'' = 0 \quad \text{for} \quad v \in H^0(K(-x))\,. \tag{C.2}$$

Except on a subset of b'''s of codimension at least 2 in $H^1(L(-x)^{-2})$, such κ exists and is unique up to normalization (compare the discussion around (6.28) in Sect. 6.3). Equation (C.2) guarantees that there exists a function $f_1 \in \Gamma(\mathcal{O}(x))$ s.t.

$$\bar\partial f_1 = b\kappa\,. \tag{C.3}$$

Let $f_t \equiv 1 + tf_1 \in \Gamma(\mathcal{O}(x))$ for $t \in \mathbb{C}$. Since, by our assumptions, $\kappa(x) \neq 0$ as an element of $L^2(-x)$ and, for small t, f_t may have zeros only close to x, it follows that, for such t, the map

$$L^{-1} \ni l \longmapsto (lf_t, l\kappa) \in L(-x)^{-1} \oplus L(-x) \tag{C.4}$$

is an embedding which, moreover, is holomorphic if we modify the $\bar\partial$ operator of $L(-x)^{-1} \oplus L(-x)$ by replacing it by $\bar\partial + \begin{pmatrix} 0 & tb'' \\ 0 & 0 \end{pmatrix}$. Choose now $\zeta \in \Gamma(\mathcal{O}(-x))$ and $s \in \Gamma(L^{-2}(x))$ s.t.

$$\zeta - s\kappa = 1\,. \tag{C.5}$$

Note that $s(x)$ has to be a non-vanishing element of $L^{-2}(x)$ because otherwise Eq. (C.5) could not be satisfied as ζ, viewed as a function on Σ, vanishes at x. Now, we shall perturb ζ and s by taking $\zeta_t \in \Gamma(\mathcal{O}(x))$ and $s_t \in \Gamma(L^{-2}(x))$ s.t.

$$\zeta_t f_t - s_t\kappa = 1 \tag{C.6}$$

and $\zeta_t = \zeta + t\zeta_1 + o(t)$, $s_t = s + ts_1 + o(t)$ are analytic in (small) t. This may be easily achieved by solving Eq. (C.6) for ζ_t with $s_t = s$ outside a small ball $B_\varepsilon(x)$ around x and for s_t with $\zeta_t = \zeta$ on $B_{2\varepsilon}(x)$ and by interpolating between the two solutions in $B_{2\varepsilon}(x)\backslash B_\varepsilon(x)$. Consider now the smooth isomorphism

$$V_t : L^{-1} \oplus L \to L(-x)^{-1} \oplus L(-x), \quad V_t = \begin{pmatrix} f_t & s_t \\ \kappa & \zeta_t \end{pmatrix} \tag{C.7}$$

depending analytically on (small) t. A straightforward computation shows that

$$V_t^{-1}\begin{pmatrix} 0 & tb'' \\ 0 & 0 \end{pmatrix} V_t + V_t^{-1}\bar\partial V_t = \begin{pmatrix} 0 & t\zeta_t^2 b'' + \zeta_t\bar\partial s_t - s_t\bar\partial\zeta_t \\ 0 & 0 \end{pmatrix}. \tag{C.8}$$

Notice that $t\zeta_t^2 b'' + \zeta_t\bar\partial s_t - s_t\bar\partial\zeta_t \equiv b'_t = b'_0 + tb'_1 + o(t) \in \wedge^{01}(L^{-2})$. In particular,

$$b'_0 = \zeta\bar\partial s - s\bar\partial\zeta = (1 + s\kappa)\bar\partial s - s\bar\partial(s\kappa) = \bar\partial s\,, \tag{C.9}$$

$$b'_1 = \zeta^2 b'' + \zeta\bar\partial s_1 + \zeta_1\bar\partial s - s\bar\partial\zeta_1 - s_1\bar\partial\zeta = b'' - \bar\partial(f_1 s) + \bar\partial s_1\,, \tag{C.10}$$

where we have used the relations $\zeta = 1 + s\kappa$ and $\zeta_1 = -\zeta f_1 + s_1\kappa$ following from Eqs. (C.5) and (C.6). Let us define two gauge fields

$$A^{01}_{\mathbf{x},tb} = U \begin{pmatrix} 0 & tb \\ 0 & 0 \end{pmatrix} U^{-1} + U\bar{\partial}U^{-1} ,$$

$$A'^{01}_{b'_t} = U' \begin{pmatrix} 0 & b'_t \\ 0 & 0 \end{pmatrix} U'^{-1} + U'\bar{\partial}U'^{-1} , \tag{C.11}$$

where $U : L_0^{-1} \oplus L_0 \to \Sigma \times \mathbb{C}^2$ and $U' : L^{-1} \oplus L \to \Sigma \times \mathbb{C}^2$ are smooth isometric isomorphisms, see Sects. 3 and 4. $A^{01}_{\mathbf{x},tb}$ and $A'^{01}_{b'_t}$ are gauge related:

$$A'^{01}_{b'_t} = h_t^{-1} A^{01}_{\mathbf{x},tb} h_t + h_t^{-1}\bar{\partial}h_t , \tag{C.12}$$

where

$$h_t = U \begin{pmatrix} f_{x_0}(\mathbf{x}, \cdot\,)^{-1} & 0 \\ 0 & f_{x_0}(\mathbf{x}, \cdot\,) \end{pmatrix} V_t U'^{-1} . \tag{C.13}$$

Expressing the same CS state Ψ in two descriptions corresponding to Eq. (4.3) and Eq. (3.4) and comparing them using the gauge invariance of Ψ, we obtain the relation

$$\psi'(b'_t) = \exp[kS(h_t, A^{01}_{\mathbf{x},tb}) + \frac{ik}{2\pi} \int_\Sigma \mathrm{tr}\,(A'^{10}_0 \wedge A'^{01}_{b'_t} - A^{01}_0 \wedge A^{01}_{\mathbf{x},tb})]\; \psi(\mathbf{x}, tb) . \tag{C.14}$$

Now, due to the homogeneity of $\psi(\mathbf{x}, \cdot\,)$,

$$\psi(\mathbf{x}, tb) = t^{k(g-1)}\, \psi(\mathbf{x}, b) . \tag{C.15}$$

On the other side, it is easy to see with the use of Eq. (C.9) that, for $\eta' \in H^0(L^2K)$,

$$\int_\Sigma \eta' \wedge b'_0 = \int_\Sigma \eta' \wedge \bar{\partial}s = \lim_{\varepsilon\to 0} \int_{\Sigma\setminus B_\varepsilon(x)} \eta' \wedge \bar{\partial}s = \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(x)} \eta' s = 2\pi i \frac{\eta'(x)}{d(s^{-1})(x)/dx} , \tag{C.16}$$

where s^{-1} is differentiated as a section of L^2 vanishing at x. Hence the class of b'_0 in $\mathbb{P}H^1(L^2)$ coincides with the image of $x \in \Sigma$ under the embedding (4.12) of Σ into $\mathbb{P}H^1(L^2)$. Specifying Eq. (C.8) to $t = 0$, we infer that the corresponding rank 2 holomorphic bundle is isomorphic by V_0 with the split bundle $L(-x)^{-1} \oplus L(-x)$. Using the integral presentation (4.5) of ψ' and Theorem 2a of [1], see Sect. 4, we obtain the relation

$$\psi'(b'_t) = \frac{(2\pi i)^k}{k!}\, t^{k(g-1)} \left(\frac{d(s^{-1})(x)}{dx}\right)^{-k} \int_{\Sigma^{k(g-1)}} \chi'(\underbrace{x,\ldots,x}_{k \text{ times}}; x_{k+1},\ldots,x_{kg})$$
$$\cdot\; b'_1(x_{k+1}) \ldots b'_1(x_{kg}) + o(t^{k(g-1)}) . \tag{C.17}$$

Besides, using Eq. (C.10), we may replace b'_1 by $b'' = f_{x_0}(\mathbf{x}, \cdot\,)^2 b$ on the right-hand side. Hence the relations (C.14), (C.15) and (C.17) imply that

$$\chi'(\underbrace{x,\ldots,x}_{k \text{ times}}; x_{k+1},\ldots,x_{kg}) = \mathscr{U}_{x_0}(\mathbf{x})\, \chi(\mathbf{x}; x_{k+1},\ldots,x_{kg})$$
$$\cdot\, f_{x_0}(\mathbf{x}, x_{k+1})^{-2} \ldots f_{x_0}(\mathbf{x}, x_{kg})^{-2} , \tag{C.18}$$

where

$$\mathscr{U}_{x_0}(\mathbf{x}) \equiv \frac{k!}{(2\pi i)^k}\left(\frac{d(s^{-1})(x)}{dx}\right)^k \exp[kS(h_0, A^{01}_{\mathbf{x},0}) + \frac{ik}{2\pi}\int_\Sigma \operatorname{tr}(A'^{10}_0 \wedge A'^{01}_{b'_0} - A^{01}_0 \wedge A^{01}_{\mathbf{x},0})]\,. \tag{C.19}$$

$\mathscr{U}_{x_0}(\mathbf{x})$ takes values in $(L^2K)^k$. It must be independent of the choice of κ, ζ and s since the other terms in Eq. (C.18) are. It gives an explicit realization of the isomorphism between functions $\Phi(\mathbf{x})$ on $\tilde{\Sigma}$ transforming by Eq. (3.16) and behaving like $(x-x_0)^{-2k(g-1)}$ around x_0 and sections of $(L^2K)^k$, see Appendix A. Such isomorphism is unique up to normalization and Eq. (C.19) fixes this normalization completely (in a way dependent on the isometric isomorphisms U and U'). Equation (C.18) establishes the precise relation between functions ψ used to represent the CS states in this paper and polynomials ψ' introduced in Sect. 4 and corresponding to the description of [1, 22].

Appendix D

We shall prove here the formula (7.23) for the zeta-function regularized determinant of the operator $\bar{D}_n^\dagger \bar{D}_n$. Let us consider the determinant line bundle $\mathscr{F}$ of the $\bar{\partial}$-family $(\bar{\partial} + [A^{01}_{\mathbf{x},b}, \cdot\,] \equiv \bar{D}_n)$ of operators acting in the trivial bundle $\Sigma \times sl(2,\mathbb{C})$. It is a holomorphic line bundle over the space of pairs $(\mathbf{x}, b)$ with the fibers

$$\det(\ker(\bar{D}_n))^{-1} \det(\operatorname{coker}(\bar{D}_n))\,.$$

Generically, $\ker(\bar{D}_n) = 0$ and the dual space to the $\operatorname{coker}(\bar{D}_n)$ is spanned by the $sl(2,\mathbb{C})$-valued 1,0-forms $\omega^\alpha(\mathbf{x},b) \equiv U\begin{pmatrix} -\mu^\alpha & \lambda^\alpha \\ \eta^\alpha & \mu^\alpha \end{pmatrix}U^{-1}$ constructed in Sect. 5.3, see formulae (6.30) and (6.31). The complex gauge transformations $h_{c,v} \equiv U g_{c,v} U^{-1}$, with $g_{c,v}$ as in Eq. (2.13), c a non-zero constant or $c = c_p$, act on $\mathscr{F}$. Division by their action gives the 4$^{\text{th}}$ power ($4 = 2\times$ the dual Coxeter number of $SU(2)$) of the h.l.b. DET over the compact space $\mathbb{P}W_0$ discussed in Sect. 3. The formula

$$\left\| 1 \otimes \bigwedge_\alpha \omega^\alpha(\mathbf{x},b)\right\|^2 = \det(\bar{D}_n^\dagger \bar{D}_n)\det(\Omega(1,n))^{-1} \tag{D.1}$$

defines Quillen's hermitian metric [33] on $\mathscr{F}$. Its curvature is easily calculable (from the Riemann-Roch-Grothendick Theorem, see e.g. [34]) to be

$$\frac{2}{\pi i}\int_\Sigma \operatorname{tr}(\delta A^{01}_{\mathbf{x},b})^\dagger \wedge \delta A^{01}_{\mathbf{x},b} = \frac{2}{\pi i}\int_\Sigma (2\overline{\delta a_{\mathbf{x}}} \wedge \delta a_{\mathbf{x}} + \langle \delta b, \wedge \delta b\rangle)\,. \tag{D.2}$$

The change of the Quillen metric on $\mathscr{F}$ under the complex gauge transformations $h_{c,v}$ may be inferred from the chiral anomaly formula (6.10) with $S(h_{c,v}h^\dagger_{c,v}, A(n))$ given by Eqs. (6.18) and (6.19). It is then easy to see that the modified metric

$$\|\cdot\|'^2 = \|\cdot\|^2\, e^{\frac{2i}{\pi}\int_\Sigma \langle b, \wedge b\rangle} \tag{D.3}$$

is invariant under the $h_{c,v}$ transformations and descends to DET^4. Its curvature is $\frac{4}{\pi i}\int_\Sigma \overline{\delta a_{\mathbf{x}}} \wedge \delta a_{\mathbf{x}}$. On the other hand, the right-hand side of Eq. (7.24) multiplied by $\exp[\frac{2i}{\pi}\int_\Sigma\langle b,\wedge b\rangle]$ also defines a hermitian structure on the h.l.b. DET^4 and the Riemann-Roch-Grothendick Theorem shows that the curvatures agree. Hence, the two metrics are proportional with the proportionality constant which might a *priory* depend on the metric (i.e. also on the complex structure) of Σ.

In order to see that Eq. (7.23) for $\det(\bar{D}_n^\dagger\bar{D}_n)$ represents properly also the dependence on the metric of Σ, it is enough to show that it produces the right behavior of the determinant in the limit when b is replaced by tb and $t\to 0$. This limit may be studied by the 2nd order perturbation theory. For $t=0$, $\bar{D}_n^\dagger\bar{D}_n$ has the g-dimensional kernel $U(H^0(L_{\mathbf{x}}^2)\sigma^- + \mathbb{C}\sigma^3)U^{-1}$. For $t\neq 0$, the operator is modified by a relatively compact perturbation [35]. If the rank of the matrix $(\int_\Sigma \kappa^r\omega^i\wedge b)$ is $g-1$, all the zero eigenvalues of $\bar{D}_n^\dagger\bar{D}_n$ move up and their product is easily calculated to be

$$Rt^{2g} \equiv 2^{-g}\, t^{2g}\, \mathrm{area}^{-1}\det(K_0)^{-1}|z^1|^2\,(H_0^{-1})_{11}$$
$$\times\det\left(\left(\int_\Sigma\overline{\kappa^r\omega^i\wedge b}\right)(\operatorname{Im}\tau)_{ij}\left(\int_\Sigma\kappa^s\omega^j\wedge b\right)\right) \tag{D.4}$$

in the leading nontrivial order (we have assumed that $z^\alpha(b)=0$ for $\alpha>1$). It follows that, when $t\to 0$ (and with the zeta-functions regularized determinants),

$$\det(\bar{D}_n^\dagger\bar{D}_n) = R\,t^{2g}\,\det{}'(\bar{D}_n^\dagger\bar{D}_n)|_{t=0} + o(t^{2g})\,. \tag{D.5}$$

On the other hand,

$$\lim_{t\to 0}\det(\Omega(1,\mathbf{x},tb)) = \frac{2}{i}\left(\int_\Sigma\bar{\mu}_1\wedge\mu_1\right)\det{}_{\alpha,\beta\neq 1}(H_0^{\alpha\beta}) \tag{D.6}$$

in the notation of Eq. (6.30). Using also the relation (6.34), we obtain

$$\lim_{t\to 0}\det(\Omega(1,\mathbf{x},tb)) = 4\left|\det\left(\int_\Sigma\kappa^r\omega^i\wedge b\right)_{i<g}\right|^2$$
$$\cdot\sum_{i',j'}(-1)^{i'+j'}\det\left(\int_\Sigma\overline{\kappa^r\omega^i\wedge b}\right)_{i\neq i'}(\operatorname{Im}\tau)_{i'j'}\det\left(\int_\Sigma\kappa^s\omega^j\wedge b\right)_{j\neq j'}\det(H_0)(H_0^{-1})_{11}$$
$$= 4\det(\operatorname{Im}\tau)\det(H_0)\,(H_0^{-1})_{11}\left|\det\left(\int_\Sigma\kappa^r\omega^i\wedge b\right)_{i<g}\right|^{-2}$$
$$\cdot\ \det\left(\left(\int_\Sigma\overline{\kappa^r\omega^i\wedge b}\right)(\operatorname{Im}\tau)^{-1}_{ij}\left(\int_\Sigma\kappa^s\omega^j\wedge b\right)\right), \tag{D.7}$$

where the last equality is a consequence of the identity $\det(\sum_j\overline{A^{rj}}A^{sj}) = \sum_j|\det(A^{ri})_{i\neq j}|^2$ for (A^{rj}) a $(g-1)\times g$ matrix which may be easily verified by taking (A^{rj}) with first $(g-1)$ columns forming a unit matrix. It follows now from Eq. (6.37) that

$$\lim_{t\to 0} t^{-2g}\left(\int\delta(BV)\,e^{-\|CV\|^2}DV\right)^{-1} = 4|z^1|^2(H_0^{-1})_{11}$$
$$\cdot\ \det\left(\left(\int_\Sigma\overline{\kappa^r\omega^i\wedge b}\right)(\operatorname{Im}\tau)^{-1}_{ij}\left(\int_\Sigma\kappa^s\omega^j\wedge b\right)\right) \tag{D.8}$$

and, consequently, the right-hand side of Eq. (7.23) behaves when $t \to 0$ in accordance with (D.5). This ends the proof of formula (7.23).

Appendix E

Let us denote

$$\begin{aligned} \mathscr{P}(\varphi) &\equiv \det(H_\varphi)^{-1}|z^1|^2 \int\limits_{\mathbb{C}^{N-1}} |P(z)|^2 \, \mathrm{e}^{-\frac{k+4}{2\pi}\bar{z}^\alpha (H_\varphi^{-1})_{\beta\alpha} z^\beta} \prod_{\alpha=2}^{N} d^2 z^\alpha \\ &= \left(\frac{2}{k+4}\right)^N |z^1|^2 \int\limits_{\mathbb{C}^{N-1}} |P(z)|^2 \left(\int \mathrm{e}^{-\frac{2\pi}{k+4}\bar{c}_\alpha (H_\varphi)^{\alpha\beta} c_\beta + i\, c_\alpha z^\alpha + i \bar{c}_\alpha \bar{z}^\alpha} \prod_\alpha d^2 c_\alpha \right) \prod_{\alpha=2}^{N} d^2 z^\alpha \,, \end{aligned} \tag{E.1}$$

where $P(z)$ is a homogeneous polynomial in variables (z^α) of degree $M \equiv (k+1)(g-1)$:

$$P(z) = \sum_{(\alpha_m)} \frac{1}{M!} \left(\prod_{m=1}^{M} \partial_{z^{\alpha_m}} P(z) \right) \prod_{m=1}^{M} z^{\alpha_m} \equiv \sum_{(\alpha_m)} P_{(\alpha_m)} \prod_{m=1}^{M} z^{\alpha_m} \,. \tag{E.2}$$

$\mathscr{P}(\varphi)$ with $P(z)$ given by Eq. (9.3) is the z-integral to be computed after the w-integration in Sect. 7 if we postpone the φ integral till after the one over $z^{\alpha\prime}s$, see Eq. (8.8). The integrals in (E.1) clearly converge. We shall show that the zero-mode integral

$$\int\limits_{\mathbb{R}} \mathrm{e}^{-2aM} \mathscr{P}(\varphi + a)\, da = \frac{1}{2}\pi^{M-1} \left(\frac{2}{k+4}\right)^{M+N} \sum_{(\alpha_m),(\beta_m)} \overline{P_{(\alpha_m)}} P_{(\beta_m)} \prod_{m=1}^{M} H_\varphi^{\alpha_m \beta_m} \,. \tag{E.3}$$

Consequently, integrating in our calculation of the right-hand side of (6.12) first over w then over $z^\alpha, \alpha > 1$ and at the end over φ one obtains the expression (9.5) without encountering other infinities than the standard ones removed by the zeta-function regularization of the determinants and the Wick ordering of the φ-field exponentials. In order to prove formula (E.3), let us rewrite

$$\begin{aligned} \int\limits_{\mathbb{R}} \mathrm{e}^{-2aM} \mathscr{P}(\varphi + a)\, da &= \frac{1}{2\pi} \int\limits_{\mathbb{C}} |t|^{2(M-1)} \mathscr{P}(\varphi - \ln|t|)\, d^2 t \\ &= \frac{1}{2\pi} \det(H_\varphi)^{-1} |z^1|^2 \int\limits_{\mathbb{C}^N} |t|^{2(M+N-1)} |P(z)|^2 \, \mathrm{e}^{-\frac{k+4}{2\pi}|t|^2 \bar{z}^\alpha (H_\varphi^{-1})_{\beta\alpha} z^\beta} \prod_{\alpha=2}^{N} d^2 z^\alpha \,. \end{aligned} \tag{E.4}$$

The integral clearly converges. By the change of variables

$$\zeta^1 = t z^1, \;\; \zeta^2 = t z^2, \ldots, \zeta^N = t z^N \,, \tag{E.5}$$

one obtains

$$\int\limits_{\mathbb{R}} \mathrm{e}^{-2aM} \mathscr{P}(\varphi + a)\, da = \frac{1}{2\pi} \det(H_\varphi)^{-1} \int\limits_{\mathbb{C}^N} |P(\zeta)|^2 \, \mathrm{e}^{-\frac{k+4}{2\pi}\bar{\zeta}^\gamma (H_\varphi^{-1})_{\beta\alpha} \zeta^\beta} \prod_{\alpha=1}^{N} d^2 \zeta^\alpha \,. \tag{E.6}$$

Now Eq. (E.3) follows by simple Gaussian integration.

Appendix F

We shall check the consistency of the formula (9.5) for the scalar product of the CS states. First of all, the integrand on the right-hand side is independent of the choice of the bases $(\kappa^r_{\mathbf{x}})$ of $H^0(L^2_{\mathbf{x}})$ and $(\eta^\alpha_{\mathbf{x}})$ of $H^0(L^2_{\mathbf{x}}K)$. Indeed, a change of $(\kappa^r_{\mathbf{x}})$ in $|\sum_{j=1}^{g}(-1)^j \det(\int_\Sigma \kappa^r_{\mathbf{x}}\omega^i \wedge b)_{i+j}\, \omega^j(x)|^{\wedge 2}$ is compensated by that in $\det(\int_\Sigma \langle \kappa^r_{\mathbf{x}}, \kappa^s_{\mathbf{x}}\rangle \,\mathrm{vol})^{-1}$ and $\eta^\alpha_{\mathbf{x}}\partial_{z^\alpha_{\mathbf{x}}}$ is independent of the choice of $(\eta^\alpha_{\mathbf{x}})$. Another consistency check is the independence of the integrand under change of the hermitian structure of L_0. Recall, that the Green function $G(x, y)$ of the Laplacian was chosen so that $\int_\Sigma G(\,\cdot\,, y)((k+2)F_0(y) + \frac{1}{2}R(y)) = 0$, where F_0 is the curvature form of L_0. The multiplication of the hermitian metric of L_0 by a positive function $\mathrm{e}^{\varphi/(k+2)}$ leads to the replacement $F_0 \mapsto F_0 + \frac{\bar\partial\partial\varphi}{k+2}$ and

$$
\begin{aligned}
G(x, y) \;\mapsto\; & G(x, y) + \frac{1}{4\pi M}(\varphi(x) + \varphi(y)) \\
& - \frac{i}{8\pi^2 M^2}\int_\Sigma (\partial\varphi \wedge \bar\partial\varphi + \varphi(2(k+2)F_0 + R))\,. \qquad \text{(F.1)}
\end{aligned}
$$

Since $\langle \eta^{\alpha_m}, \wedge\eta^{\beta_m}\rangle(x_m)$ picks up the factor $\mathrm{e}^{2\varphi(x_m)/(k+2)}$, the last line of Eq. (9.5) is multiplied by

$$
\exp\left[\frac{i}{2\pi(k+2)}\int_\Sigma (\partial\varphi \wedge \bar\partial\varphi + \varphi(2(k+2)F_0 + R))\right].
$$

By the chiral anomaly formula, $\det(\int_\Sigma \langle \kappa^r, \kappa^s\rangle \;\mathrm{vol})^{-1}\; \det{}'(\bar\partial^\dagger_{L^2_{\mathbf{x}}}\bar\partial_{L^2_{\mathbf{x}}})$ changes by the factor

$$
\exp\left[-\frac{i}{\pi(k+2)^2}\int_\Sigma \left(\partial\varphi \wedge \bar\partial\varphi + (k+2)\varphi\left(2F_0 + \frac{1}{2}R\right)\right)\right].
$$

Finally, we shall show that $\exp[-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\, A_0^{10} \wedge A_0^{01}]\,|\psi(\mathbf{x}, b)|^2$, which depends on the metric of L_0 through the smooth isometric isomorphism $U : L_0^{-1} \oplus L_0 \to \Sigma \times \mathbb{C}^2$, yields upon the multiplication of the metric by $\mathrm{e}^{\varphi/(k+2)}$ the factor

$$
\exp\left[-\frac{ik}{2\pi(k+2)^2}\int_\Sigma (\partial\varphi \wedge \bar\partial\varphi + 2(k+2)\varphi F_0)\right]
$$

cancelling the previous ones. First note that, by Eq. (6.17),

$$
\mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\, A_0^{10}\wedge A_0^{01}}\,|\psi(\mathbf{x}, b)|^2 = c(\mathbf{x})|\Psi(A^{01}_{\mathbf{x},b})|^2\, \mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\,(A^{01}_{\mathbf{x},b})^\dagger \wedge A^{01}_{\mathbf{x},b} + \frac{ik}{2\pi}\int_\Sigma \langle b, \wedge b\rangle} \qquad \text{(F.2)}
$$

with $c(\mathbf{x})$ independent of the choice of the metric on L_0. As follows from the transformation properties of Ψ and of the WZW action S,

$$
\begin{aligned}
& |\Psi({}^{h^{-1}}\!A^{01})|^2\, \mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\,({}^{h^{-1}}\!A^{01})^\dagger \wedge {}^{h^{-1}}\!A^{01}} \\
&= \mathrm{e}^{kS(hh^\dagger, -(A^{01})^\dagger + A^{01})}\; |\Psi(A^{01})|^2\, \mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\,(A^{01})^\dagger \wedge A^{01}}\,. \qquad \text{(F.3)}
\end{aligned}
$$

In particular, the expression does not change if h takes values in the compact group. It follows that, for a fixed metric on L_0, $\exp[-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr} A_0^{10}\wedge A_0^{01}]\,|\psi(\mathbf{x},b)|^2$ is independent of the choice of the isometric isomorphism $U : L_0^{-1}\oplus L_0 \to \Sigma\times\mathbb{C}^2$ since $A^{01}_{\mathbf{x},b}$ for two different choices are related by an $SU(2)$-valued gauge transformation.

After the multiplication of the hermitian structure on L_0 by e^{φ}, we may take $U\mathrm{e}^{-\varphi\sigma^3/2}$ as the new isometric isomorphism from $L_0^{-1}\oplus L_0$ to $\Sigma\oplus\mathbb{C}^2$. As a result of the change of U, $A^{01}_{\mathbf{x},b}$ changes to ${}^{h^{-1}}A^{01}_{\mathbf{x},b}$ with $h = U\mathrm{e}^{\varphi\sigma^3/2}U^{-1}$. By virtue of Eqs. (F.2) and (F.3), $\mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr} A_0^{10}\wedge A_0^{01}}\,|\psi(\mathbf{x},b)|^2$ is then multiplied by

$$\mathrm{e}^{kS(hh^\dagger,-(A^{01}_{\mathbf{x},b})^\dagger+A^{01}_{\mathbf{x},b})-\frac{ik}{2\pi}\int_\Sigma(\mathrm{e}^{-2\varphi}-1)\langle b,\wedge b\rangle}. \tag{F.4}$$

We are left with the calculation of $S(hh^\dagger,-(A^{01}_{\mathbf{x},b})^\dagger+A^{01}_{\mathbf{x},b})$. By Eq. (A.2), under the infinitesimal charge of φ,

$$\begin{aligned}
&\delta S(hh^\dagger,-(A^{01}_{\mathbf{x},b})^\dagger+A^{01}_{\mathbf{x},b})\\
&=\frac{i}{2\pi}\int_\Sigma \mathrm{tr}\, U\delta\varphi\sigma^3U^{-1}F(-(A^{01}_{\mathbf{x},b})^\dagger+{}^{(hh^\dagger)^{-1}}A^{01}_{\mathbf{x},b})\\
&=\frac{i}{2\pi}\int_\Sigma \mathrm{tr}\,\delta\varphi\sigma^3\mathrm{curv}\left(\nabla+\bar\partial+\begin{pmatrix}\bar a_{\mathbf{x}}-a_{\mathbf{x}}+\bar\partial\varphi & \mathrm{e}^{-2\varphi}b\\ b^\dagger & -\bar a_{\mathbf{x}}+a_{\mathbf{x}}-\bar\partial\varphi\end{pmatrix}\right)\\
&=\frac{i}{2\pi}\int_\Sigma \mathrm{tr}\,\delta\varphi\sigma^3\begin{pmatrix}-F_0+\partial\bar\partial\varphi-\mathrm{e}^{-2\varphi}\langle b,\wedge b\rangle & \nabla(\mathrm{e}^{-2\varphi}b)+2\mathrm{e}^{-2\varphi}\bar a_{\mathbf{x}}\wedge b\\ (\bar\partial+2a_{\mathbf{x}}-\bar\partial\varphi)b^\dagger & F_0-\partial\bar\partial\varphi+\mathrm{e}^{-2\varphi}\langle b,\wedge b\rangle\end{pmatrix}\\
&=\delta\left(-\frac{i}{2\pi}\int_\Sigma(\partial\varphi\wedge\bar\partial\varphi+2\varphi F_0-(\mathrm{e}^{-2\varphi}-1)\langle b,\wedge b\rangle)\right)
\end{aligned}\tag{F.5}$$

so that the factor (F.4) becomes $\mathrm{e}^{-\frac{ik}{2\pi}\int_\Sigma(\partial\varphi\wedge\bar\partial\varphi+2\varphi F_0)}$, as required.

As for the change of $\psi(\mathbf{x},b)$ itself, a straightforward calculation shows that if $h:\Sigma\to SL(2,\mathbb{C})$ then

$$\begin{aligned}
&\mathrm{e}^{\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\,{}^{h^\dagger}A_0^{10}\wedge{}^{h^{-1}}A^{01}_{\mathbf{x},b}}\,\psi({}^{h^{-1}}A^{01}_{\mathbf{x},b}) = \mathrm{e}^{kS(hh^\dagger,A_0^{10}+A^{01}_{\mathbf{x},b})-kS(hh^\dagger)+kS(h)-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr} A_0^{10}\wedge(h^\dagger)^{-1}\bar\partial h^\dagger}\\
&\qquad\cdot\,\mathrm{e}^{\frac{ik}{2\pi}\int_\Sigma \mathrm{tr} A_0^{10}\wedge A^{01}_{\mathbf{x},b}}\,\Psi(A^{01}_{\mathbf{x},b})\,.
\end{aligned}\tag{F.6}$$

It follows that if $U\mapsto U' = h^{-1}U$ with $SU(2)$-valued h then $\psi(\mathbf{x},b)$ changes only by a constant factor $\mathrm{e}^{-kS(h)-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr} A_0^{10}\wedge(h^\dagger)^{-1}\bar\partial h^\dagger}$. On the other hand, if the metric of L_0 is multiplied by e^{φ} and $U\mapsto U' = U\mathrm{e}^{-\varphi\sigma^3/2}\equiv h^{-1}U$, then $\psi(\mathbf{x},b)$ picks up the factor

$$\begin{aligned}
&\mathrm{e}^{kS(hh^\dagger,A_0^{10}+A^{01}_{\mathbf{x},b})-kS(hh^\dagger)+kS(h)-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr} A_0^{10}\wedge(h^\dagger)^{-1}\bar\partial h^\dagger}\\
&=\mathrm{e}^{kS(hh^\dagger,A_0)-kS(hh^\dagger)+kS(h)-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr} A_0^{10}\wedge(h^\dagger)^{-1}\bar\partial h^\dagger}\,,
\end{aligned}\tag{F.7}$$

where the last inequality may be checked by differentiating $S(hh^\dagger,A_0^{10}+A^{01}_{\mathbf{x},b})$ with respect to φ with the use of Eq. (A.2). Again $\psi(\mathbf{x},b)$ is multiplied by a constant independent of $\mathbf{x}$ and b.

In the next check, let us find the dependence of the right-hand side of Eq. (9.5) on the conformal factor of the Riemannian metric of Σ. If the metric is multiplied by a positive function e^{σ}, then

$$\begin{aligned} G(x,y) \ &\mapsto\ G(x,y) + \frac{1}{8\pi M}(\sigma(x)+\sigma(y)) \\ &\quad - \frac{i}{32\pi^2 M^2}\int_\Sigma (\partial\sigma \wedge \bar\partial\sigma + 2\sigma(2(k+2)F_0 + R))\,, \\ :G(x,x):\ &\mapsto\ :G(x,x): - \frac{M-1}{4\pi M}(\sigma(x)+\sigma(y)) \\ &\quad - \frac{i}{32\pi^2 M^2}\int_\Sigma (\partial\sigma \wedge \bar\partial\sigma + 2\sigma(2(k+2)F_0 + R))\,, \end{aligned} \tag{F.8}$$

and the last line of Eq. (9.5) is multiplied by

$$\exp\left[\frac{i}{4\pi(k+2)}\int_\Sigma\left(\frac{1}{2}\partial\sigma\wedge\bar\partial\sigma + \sigma(2(k+2)F_0+R)\right)\right].$$

By virtue of the conformal anomaly formula, $\det\left(\int_\Sigma \langle\kappa^r,\kappa^s\rangle\,\mathrm{vol}\right)^{-1}\det'(\bar\partial^\dagger_{L^2_{\mathbf{x}}}\bar\partial_{L^2_{\mathbf{x}}})$ and $\left(\frac{\det'(-\Delta)}{\text{area}}\right)^{1/2}$ change, respectively, by the factors

$$\exp\left[-\frac{i}{12\pi}\int_\Sigma\left(\frac{1}{2}\partial\sigma\wedge\bar\partial\sigma + \sigma\,(6F_0+R)\right)\right] \quad\text{and}\quad \exp\left[-\frac{i}{24\pi}\int_\Sigma\left(\frac{1}{2}\partial\sigma\wedge\bar\partial\sigma + \sigma R\right)\right].$$

Altogether, Eq. (9.5) picks the factor

$$\exp\left[-\frac{i}{24\pi}\,\frac{3k}{k+2}\int_\Sigma\left(\frac{1}{2}\partial\sigma\wedge\bar\partial\sigma + \sigma R\right)\right] \tag{F.9}$$

when the Riemannian metric is multiplied by e^{σ}. This guarantees the right value $c = \frac{3k}{k+2}$ of the Virasoro central charge of the theory with partition function given by the formula (1.5).

Another easy check of formula (9.5) shows that its right-hand side does not depend on the choice of x_0 used to fix the bundle $L_0 = L(-x_0)$. We leave it to the reader. A more involved problem which we have not addressed is the independence of the scalar product expression of the choice of the h.l.b. L of degree g.

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