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The Additivity of the η-Invariant. The Case of a Singular Tangential Operator

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Abstract: We prove the decomposition formula for the η-invariant of the compatible Dirac operator on a closed manifold M which is a sum of two submanifolds with common boundary.

0. Introduction

Let M be a compact odd-dimensional Riemannian manifold without boundary. Let $A : C^\infty(S) \to C^\infty(S)$ denote a compatible Dirac operator acting on sections of a bundle of Clifford modules S over M (see [6,8]). Then A is a self-adjoint elliptic operator. It has a discrete spectrum $\{\lambda_k\}_{k\in\mathbf{Z}}$. We define the eta function of the operator A as follows:

$$\eta(A;s) = \sum_{\lambda_k \neq 0} \text{sign } (\lambda_k)|\lambda_k|^{-s} . \tag{0.1}$$

Now $\eta(A;s)$ is a holomorphic function of s for $Re(s) > dim(M)$, and it has a meromorphic extension to $\mathbf{C}$, with isolated simple poles on the real axis and locally computable residue (see [1,8,13]). In particular, we know that if A is a compatible Dirac operator, then $\eta(A;s)$ is holomorphic for $Re(s) > -2$. The value of $\eta(A;s)$ at $s = 0$ is an important invariant of the operator, the bundle, and the manifold. We call $\eta(A;0)$ the eta invariant of A and denote it by η_A. We use the heat representation for the eta function and obtain the following formula for η_A:

$$\eta_A = \frac{1}{\sqrt{\pi}} \cdot \int_0^\infty \frac{1}{\sqrt{t}} \cdot Tr(Ae^{-tA^2})dt . \tag{0.2}$$

In this paper we study the decomposition of η_A into the contributions coming from different parts of the manifold M. The problem here is that η_A is not given by the local formula and it depends on the global geometry of the manifold and the operator (see [1,13]). Therefore it is somewhat surprising that we can present a satisfactory result.

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Assume that we are given a decomposition of M into $M_1 \cup M_2$, where M_1 and M_2 are compact manifold with boundary such that

$$M_1 \cap M_2 = Y = \partial M_1 = \partial M_2 \ . \tag{0.3}$$

We also assume that the Riemannian metric on M and Hermitian product on S are products in $N = [-1,1] \times Y$, the bicollar neighborhood of Y in $M(M_1 \cap N = [-1,0] \times Y)$. In this case A has the following form in N:

$$A = \Gamma(\partial_u + B) \ , \tag{0.4}$$

where $\Gamma : S|Y \to S|Y$ is a unitary bundle automorphism (Clifford multiplication by the unit normal vector) and $B : C^\infty(Y; S|Y) \to C^\infty(Y; S|Y)$ is the tangential part of A on Y. B is the corresponding Dirac operator on Y, hence it is a self-adjoint, elliptic operator of the first order. Furthermore, Γ and B do not depend on u and they satisfy the following identities:

$$\Gamma^2 = -Id \quad and \quad \Gamma B = -B\Gamma \ . \tag{0.5}$$

In particular, $S|Y$ decomposes into the direct sum $S^+ \bigoplus S^-$ of subbundles of eigenvectors of Γ corresponding to the eigenvalues $\pm i$. The operator B has the following representation with respect to this decomposition:

$$B = \begin{bmatrix} 0 & B^- = (B^+)^* \\ B^+ & 0 \end{bmatrix} \ . \tag{0.6}$$

We consider first the case $ker(B) = \{0\}$. Let $\Pi_>$ (respectively, $\Pi_<$) denote the spectral projection of B onto the subspace of $L^2(Y; S|Y)$ spanned by the eigenvectors corresponding to the positive (resp., negative) eigenvalues. It is well-known (see [6,12]) that $\Pi_>$ is a self-adjoint elliptic boundary condition for the operator $A|M_2$. This means that the operator $\mathcal{A}_2$ defined by

$$\begin{cases} \mathcal{A}_2 = A|M_2 \\ dom(\mathcal{A}_2) = \{s \in H^1(M_2; S|M_2); \Pi_>(s|Y) = 0\} \end{cases} \tag{0.7}$$

is an unbounded self-adjoint operator such that $\mathcal{A}_2 : dom(\mathcal{A}_2) \to L^2(M_2; S|M_2)$ is a Fredholm operator and the kernel of $\mathcal{A}_2$ consists of smooth sections of $S|M_2$. It turns out that the eta-function of $\mathcal{A}_2$ is well-defined and enjoys all the properties of the eta-function of the Dirac operator defined on a closed manifold (see [12]). In particular $\eta_{\mathcal{A}_2}$, the eta-invariant of $\mathcal{A}_2$, is well-defined. Likewise, $\Pi_<$ is a self-adjoint boundary condition for the operator $A|M_1$, and we define the operator $\mathcal{A}_1$ using the formula which is the obvious modification of (0.7). We proved the following result in [20]:

Theorem 0.1.

$$\eta_A = \eta_{\mathcal{A}_1} + \eta_{\mathcal{A}_2} \, mod \, \mathbf{Z} \ . \tag{0.8}$$

It was explained in [20] that the integer jumps in formula (0.8) are due to the presence of "*small*" eigenvalues of the operator A.

Now we will discuss the situation in which $ker(B) \neq \{0\}$. In this case, the unbounded Fredholm operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are no longer self-adjoint. We have to modify the boundary conditions in order to take care of the kernel of the operator

B. We follow Appendix A of [12]. Formulas (0.4)–(0.6) show that Γ defines a symplectic structure in $ker(B)$. We use the *Cobordism Theorem for Dirac Operators* (see [6, 18]) which implies that *index* $B^+ = 0$. In particular, it gives the equality

$$dim\,\ker(B^+) = dim\,\ker(B^-) \ . \tag{0.9}$$

The last fact implies the existence of Lagrangian subspaces in $ker(B)$. We choose two such subspaces W_1 and W_2. Let σ_i denote the orthogonal projection of $L^2(Y;S|Y)$ onto W_i. The operator $\mathscr{A}_{2,\sigma_2}$ is defined by

$$\begin{cases} \mathscr{A}_{2,\sigma_2} = A|M_2 \\ dom(\mathscr{A}_{2,\sigma_2}) = \{s \in H^1(M_2;S|M_2);(|\Pi_> + \sigma_2)(s|Y) = 0\} \end{cases} . \tag{0.10}$$

The operator $\mathscr{A}_{1,\sigma_1}$ is determined analogously by the condition $(\Pi_< + \sigma_1)$. Note that $\mathscr{A}_i, \sigma_i$ are self-adjoint operators. We refer the reader to [4 and 12] for a discussion of the space of self-adjoint generalized Atiyah-Patodi-Singer boundary conditions. In general $\Pi_> + \sigma_2 \neq Id - (\Pi_< + \sigma_1)$ and this is the reason that we have a correction term in our additivity formula. The correction term is the η-invariant of the boundary problem on the cylinder. We consider the operator $A = \Gamma(\partial_u + B)$ on the manifold $[0,1] \times Y$ subject to the boundary condition $(\Pi_> + \sigma_1)$ at $u = 0$ and $(\Pi_< + \sigma_2)$ at $u = 1$. Let $\eta(\sigma_1, \sigma_2)$ denote the η-invariant of this operator. In this paper we prove that

Theorem 0.2.

$$\eta_A = \eta_{\mathscr{A}_{1,\sigma_1}} + \eta_{\mathscr{A}_{2,\sigma_2}} + \eta(\sigma_1, \sigma_2)\, mod\, \mathbf{Z} \ . \tag{0.11}$$

Remark 0.3.

(a) Theorem 0.2 shows that partial localization of the η-invariant can be achieved when the manifold is a sum of two submanifolds joined by the cylinder. In this case the η-invariant is the sum of the contributions coming from different parts of the manifold, plus the error term due to the cylinder.

(b) We also have a corresponding result in case M is a manifold with boundary, which leads to a formula for the variation of the η-invariant under *cutting and pasting* of the operator. This subject is discussed in Sect. 4 (see Theorem 4.3 and Theorem 4.4).

(c) In this paper, for simplicity, we discuss only compatible Dirac operator. The result, however, holds for any operator of Dirac type (see [6 and 8]. We reduce general case to the compatible. Details will be presented elsewhere.

The main step in the proof is the reduction to the case in which the tangential operator is invertible. Fix a Lagrangian subspace W of $ker(B)$. Let σ denote the corresponding orthogonal projection of $L^2(Y;S|Y)$ onto W. We define the operator $\gamma : L^2(Y;S|Y) \to L^2(Y;S|Y)$ by

$$\gamma = \begin{cases} 2\sigma - Id_{ker(B)} & on \quad ker(B) \\ \quad 0 & on \quad ker(B)^\perp \end{cases} . \tag{0.12}$$

We define $\{A_r\}_{\{0 \leqq r \leqq 1\}}$, a family of modified Dirac operators, where the operator A_r is given by the formula

$$A_r = A + rf(u)\Gamma\gamma \ , \tag{0.13}$$

and where $f : [-1, +1] \to [0, 1]$ is a smooth function equal to 0 for $|u| > \frac{1}{2}$ and equal to 1 for $|u| \leqq \frac{1}{4}$, which is extended by 0 to the whole manifold M. The tangential part of A_r, $B + r\gamma$, is invertible for $r \neq 0$. The technical problem we face here is that the operators A_r are not pseudodifferential. Therefore we have to be careful when dealing with *heat kernel formulas* giving the values of η_{A_r}. We employ *Duhamel's Principle*, which allows us to show that $Tr(A_r e^{-tA_r^2})$ and $Tr(\dot{A}_r e^{-tA_r^2})$ have the same asymptotic expansion as $t \to 0$, as in the case of compatible Dirac operators. We use the standard notation here - $\dot{A}_r$ denotes the derivative of A_r with respect to a parameter r. Once we have established the necessary asymptotic behavior of $Tr(\dot{A}_r e^{-tA_r^2})$, we show that $\dot{\eta} A_r$ is 0. Now we can directly apply the argument from [20] in which we treated the "*invertible*" case. This gives the following result:

$$\eta_A = \eta_{\mathscr{A}_1,\ Id_{ker(B)} - \sigma} + \eta_{\mathscr{A}_{2,\sigma}}\ mod\ \mathbf{Z}\ . \tag{0.14}$$

The general case now follows from the results of [14], in which we studied the variation of $\eta_{\mathscr{A}_{2,\sigma}}$ under perturbation of the boundary condition.

Remark 0.4. Theorem 0.2 was announced in [5]. We have a corresponding decomposition formula for the index (see [5,6,7]) and for the spectral flow. The last was obtained by L. Nicolaescu (see [17]; see also [5]).

Remark 0.5.

(a) It should be mentioned that Jeff Cheeger was the first who considered the localization problem for the η-invariant in his papers [10 and 11]. He suggested to blow down Y to a cone in order to separate pieces M_i. He also pointed out that this procedure should correspond to the choice of the specific boundary conditions on M_i, of the type considered here.

(b) This paper presents results of the research, which was started long ago with a joint project of the author and Ron Douglas. The fundamental analytical tools were developed in the joint work of author and Ron Douglas (see [12]).

(c) Mazzeo and Melrose obtained a formula which corresponds to formula (0.8) in Theorem 0.1, in the framework of the b-calculus of pseudodifferential operators (see [15]).

(d) Theorem 0.2 was also announced by U. Bunke (see [9]). His proof was based on the *finite propagation speed method.*

(e) The η-invariant of boundary value problems of the type considered in this paper were also studied in a recent work of Werner Müller (see [16]).

In Sect. 1 we use Duhamel's Principle in order to show that η_{A_r} enjoys all properties of the η-invariant of a compatible Dirac operator. In particular we show that we can use formula (0.2) in order to represent η_{A_r}.

In Sect. 2 we study the variation of η_{A_r} with respect to a parameter r. Results of Sect. 1 allows us to prove that

$$\eta_A = \eta_{A_r}\ mod\ \mathbf{Z}\ .$$

In Sect. 3 we explain how Theorem 0.2 follows from the results of Sect. 1 and 2, and of [14].

In Sect. 4 we discuss cutting and pasting of the η-invariant. The corresponding problem for the index was settled and solved in the early Eighties (see [19]; see also [3, 6]). In this paper we present a simple corollary from a generalization of

Theorem 0.2 for the case of manifolds with boundary (Theorem 4.4). We will analyze the general situation in future work.

1. The η-Invariant of the Operator A_r

In this section we discuss the η-invariant of the operator A_r given by formula (0.13). It is not difficult to show that $\eta(A_r;s)$ is well-defined for $Re(s)$ large. The question is about the meromorphic continuation of $\eta(A_r;s)$. We show that $Tr(A_r e^{-tA_r^2})$ has the same asymptotic expansion as $t \to 0$ as the corresponding trace for the compatible Dirac operator A. We have the following result:

Theorem 1.1. *There exist positive constants c_1, c_2 such that for any $0 < t < 1$,*

$$|Tr(A_r e^{-tA_r^2}) - Tr(Ae^{-tA^2})| < c_1 \cdot e^{-\frac{c_2}{t}} . \tag{1.1}$$

In particular, we have a positive constant c_3 such that for any $0 < t < 1$,

$$|Tr(A_r e^{-tA_r^2})| < c_3 \cdot \sqrt{t} . \tag{1.2}$$

Proof. We apply *Duhamel's Principle* (see for instance [6,10]). Let us observe that

$$A_r^2 = A^2 - rf'(u)\gamma + r^2 f^2(u)\mathscr{P} , \tag{1.3}$$

where $\mathscr{P}$ denotes orthogonal projection of $L^2(Y;S|Y)$ onto kernel of the operator B. We also have

$$Tr(A_r e^{-tA_r^2}) - Tr(Ae^{-tA^2}) = Tr(A_r - A)e^{-tA_r^2} + Tr(A\{e^{-tA_r^2} - e^{-tA^2}\}) . \tag{1.4}$$

We discuss $Tr(A_r - A)e^{-tA_r^2} = Tr(rf(u)\Gamma\gamma e^{-tA_r^2}$ first. We are interested in the *small time asymptotics* only. Therefore we can replace the operator $e^{-tA_r^2}$ by a suitable *parametrix*. Let $\mathscr{F}_r(t;x,z)$ denote the kernel of $e^{-tA_r^2}$ and let $\mathscr{E}_1(t;x,z)$ denote the kernel of the operator e^{-tA^2} on the manifold M. We also introduce $\mathscr{E}_2(t;x,z)$, the kernel of the operator $e^{-t(-\partial_u^2+B^2-rf'(u)\gamma+r^2f^2(u)\mathscr{P})}$ on the infinite cylinder $(-\infty,+\infty) \times Y$. We define an operator $Q(t)$ by defining its kernel to be

$$Q(t;x,z) = \sum_{k=1}^{2} \phi_k(x)\mathscr{E}_i(t;x,z)\psi_i(z) , \tag{1.5}$$

where $\{\psi_i\}_{i=1}^2$ is a partition of unity on M such that $\psi_2(z)$ is equal to 0 for $z \notin N$ and ψ_2 is a function of the normal coordinate u in N satisfying

$$\psi_2(u) = \begin{cases} 1 & for \quad |u| \leqq 1/2 \\ 0 & for \quad |u| \geqq 5/8 \end{cases} . \tag{1.6}$$

The corresponding function $\phi_2(x)$ is 0 outside N and is a function of the normal variable u in N such that

$$\phi_2(u) = \begin{cases} 1 & for \quad |u| \leqq 3/4 \\ 0 & for \quad |u| \geqq 7/8 \end{cases} . \tag{1.7}$$

Similarly, $\phi_1(x)$ is equal to 1 outside of N. Inside N, $\phi_1(x)$ is given by the formula

$$\phi_1(u) = \begin{cases} 1 & for \quad |u| \leqq 3/8 \\ 0 & for \quad |u| \geqq 1/8 \end{cases} . \tag{1.8}$$

The choice of the cut-off functions implies that

$$dist(supp\left(\frac{\partial \phi_i}{\partial u}\right); supp(\psi_i)) \geqq \frac{1}{8} . \tag{1.9}$$

We have the following result (see, for instance, [12]; Sect. 2).

Lemma 1.2. *There exist positive constants* c_1, c_2 *such that for any* $0 < t < 1$,

$$\|\mathscr{F}(t;x,y) - Q(t;x,y)\| < c_1 \cdot e^{-c_2 \cdot \frac{d^2(x,y)}{t}} . \tag{1.10}$$

It follows from the choice of functions f, ϕ_i, and ψ_i, that, up to exponentially decaying summand, we can replace $tr\, r f \Gamma \gamma \mathscr{F}(t;x,x)$ by

$$tr\, r f(u) \Gamma(y) \phi_2(u)(\gamma \mathscr{E}_2)(t;(u,y),(u,y)\psi_2(u) . \tag{1.11}$$

The trace (1.11) is equal to 0. The point is that, up to the cut-off functions, we have trace of the kernel

$$r\Gamma\gamma e^{-t(-\partial_u^2 + B^2 - rf'(u)\gamma + r^2 f^2(u)\mathscr{P})} = r\Gamma\gamma e^{-tB^2} e^{-t(-\partial_u^2 - rf'(u)\gamma + r^2 f^2(u)\mathscr{P})} .$$

The operator γ commutes with $(-\partial_u^2 + B^2 - rf'(u)\gamma + r^2 f^2(u)\mathscr{P})$ and we have

$$r\Gamma\gamma e^{-t(-\partial_u^2 + B^2 - rf'(u)\gamma + r^2 f^2(u)\mathscr{P})} = -r\gamma\Gamma e^{-t(-\partial_u^2 + B^2 - rf'(u)\gamma + r^2 f^2(u)\mathscr{P})} .$$

This shows that the trace of (1.11) has to be equal to 0 and we have just finished the proof of the estimate

$$|Tr(A_r - A)e^{-tA_r^2}| \leqq c_1 \cdot e^{-\frac{c_2}{t}} . \tag{1.12}$$

This takes care of the first summand in (1.4). We estimate the second summand

$$Tr(A\{e^{-tA_r^2} - e^{-tA^2}\})$$

in the same way. We use *Duhamel's Principle* to show that up to an exponentially decaying summand this is equal to the trace of

$$\Gamma(\partial_u + B)\phi(u)e^{-tB^2}(t;y,z)(e^{-t(-\partial_u^2 + W)} - e^{t\partial_u^2})(t;u,v)\psi(v) , \tag{1.13}$$

where $\phi(u)$ and $\psi(v)$ are the suitable cut-off functions and W denotes the operator $-rf'(u)\gamma + r^2 f^2(u)\mathscr{P}$. It is easy to observe that the trace in the y-direction in formula (1.13) is 0. This ends the proof of (1.1). Then (1.2) follows from the corresponding estimate for the compatible Dirac operators (see [2, 8]).
We have an immediate Corollary:

Corollary 1.3. *The* η*-invariant of the operator* A_r *is given by the formula* (0.2):

$$\eta_{A_r} = \frac{1}{\sqrt{\pi}} \cdot \int_0^\infty \frac{1}{\sqrt{t}} \cdot Tr(A_r e^{-tA_r^2})dt .$$

2. Derivative of η_{A_r} with Respect to the Parameter

In this section we follow the argument given in Sect. 4 of [14]. We pick a small real number $c > 0$ which is not an eigenvalue of $A = A_0$. By the continuity of the eigenvalues, there exists $\epsilon \geqq 0$ such that c is not an eigenvalue of A_r for $0 \leqq r < \epsilon$. Let P_c denote the orthogonal projection of $L^2(M;S)$ onto the subspace spanned by the eigenvectors of A corresponding to the eigenvalues λ with $|\lambda| < c$. Put

$$A'_r = A_r(Id - P_c) + P_c \ .$$

Then A'_r is an invertible operator for $0 \leqq r < \epsilon$, and depends smoothly on r. Further, P_c has finite rank and therefore the η-function of A'_r is defined and

$$\eta(A_r;s) = \eta(A'_r;s) + \sum_{|\lambda_j|<c} sign(\lambda_j)|\lambda_j|^{-s} - Tr(P_c) \ . \tag{2.1}$$

Formula (2.1) shows that the difference $\eta(A_r;s) - \eta(A'_r;s)$ is a holomorphic function on the whole complex plane. In particular, we have

$$\eta_{A_r} = \eta_{A'_r} \, mod\, \mathbf{Z} \ , \tag{2.2}$$

and we can use formula (0.2) to evaluate $\eta_{A'_r}$. We differentiate

$$\begin{aligned}
\frac{d}{dr}\eta_{A_r} &= \frac{d}{dr}\eta_{A'_r} = \frac{d}{dr}\left\{\frac{1}{\sqrt{\pi}} \cdot \int_0^\infty \frac{1}{\sqrt{t}} \cdot Tr(A'_r e^{-t(A'_r)^2})dt\right\} \\
&= \frac{1}{\sqrt{\pi}} \cdot \int_0^\infty \frac{1}{\sqrt{t}}\left(1 + 2t\frac{\partial}{\partial t}\right) Tr(\dot{A}'_r e^{-t(A'_r)^2})\,dt \\
&= \frac{1}{\sqrt{\pi}} \cdot \int_0^\infty \frac{1}{\sqrt{t}} Tr(\dot{A}'_r e^{-t(A'_r)^2})\,dt + \frac{2}{\sqrt{\pi}} \cdot \int_0^\infty \sqrt{t}\frac{\partial}{\partial t} Tr(\dot{A}'_r e^{-t(A'_r)^2})\,dt \\
&= \frac{2}{\sqrt{\pi}} \cdot \lim_{\epsilon\to 0} \sqrt{\epsilon} \cdot Tr(\dot{A}'_r e^{-t(A'_r)^2})|_{\epsilon}^{\frac{1}{\epsilon}} = -\frac{2}{\sqrt{\pi}} \cdot \lim_{\epsilon\to 0} \sqrt{\epsilon} \cdot Tr(\dot{A}_r e^{-\epsilon A_r^2}) + \\
&\quad - \frac{2}{\sqrt{\pi}} \cdot \lim_{\epsilon\to 0} \sqrt{\epsilon} \cdot Tr((\dot{A}'_r - \dot{A}_r)e^{-\epsilon A_r^2}) \\
&\quad - \frac{2}{\sqrt{\pi}} \cdot \lim_{\epsilon\to 0} \sqrt{\epsilon} \cdot Tr(\dot{A}'_r(e^{-\epsilon(A'_r)^2} - e^{-\epsilon A_r^2})) \ .
\end{aligned}$$

It follows from the fact that P_c is an operator of finite rank that the last two summands are 0 and we have the following result:

Lemma 2.1. *$\frac{d}{dr}\eta_{A_r}$ is given by the following formula:*

$$\frac{d}{dr}\eta_{A_r} = -\frac{2}{\sqrt{\pi}} \cdot \lim_{\epsilon\to 0} \sqrt{\epsilon} \cdot Tr(\dot{A}_r e^{-\epsilon A_r^2}) \ . \tag{2.3}$$

Now we are ready to apply *Duhamel's Principle*. We have

$$\dot{A}_r e^{-\epsilon A_r^2} = f(u)\Gamma\gamma e^{-\epsilon A_r^2} \ ,$$

and we are in exactly the same situation as we were in the first part of the proof of Theorem 1.1 when we discussed $Tr((A_r - A)e^{-tA_r^2}) = rf(u)\Gamma\gamma e^{-tA_r^2}$. We repeat the argument and obtain

Lemma 2.2. *There exist positive constants* c_1 , c_2 *such that for any* $0 < t < 1$,

$$|Tr(\dot{A}_r e^{-tA_r^2})| < c_1 e^{-\frac{c_2}{t}} . \tag{2.4}$$

Equation (2.3) combined with (2.4) gives the main result of this section.

Theorem 2.3. *The derivative of* η_{A_r} *with respect to the parameter* r *is equal to* 0 *and*

$$\eta_A = \eta_{A_r}\ mod\, \mathbf{Z} . \tag{2.5}$$

3. The Additivity Formula

In the previous section we have shown that $\eta_A = \eta_{A_1}\quad mod\, \mathbf{Z}$. Now A_1 is an operator with invertible tangential part. The fact that A_1 is not a pseudodifferential operator does not have any influence on the proof of the additivity formula offered in [20]. Let $\mathscr{A}^1_{2;\sigma}$ denote the operator:

$$\begin{cases} \mathscr{A}^1_{2;\sigma} = A_1|M_2 \\ dom(\mathscr{A}^1_{2;\sigma}) = \{s \in H^1(M_2; S|M_2); (\Pi_> + \sigma)(s|Y) = 0\} \end{cases} . \tag{3.1}$$

We define the operator $\mathscr{A}^1_{1;\sigma\perp}$ on M_1, using the boundary condition $(\Pi_< + \sigma^\perp)$, where $\sigma^\perp$ denotes the projection of the kernel of B onto the subspace orthogonal to the range of σ. We follow Sect. 4 and Appendix A in [12] to show that the η-invariants of both operators are well-defined and are given by the formula (0.2). The next result follows from the argument given in [20].

Theorem 3.1. *The* η*-invariant of* A *is given by the following formula*

$$\eta_A = \eta_{\mathscr{A}_{1,\sigma\perp}} + \eta_{\mathscr{A}_{2;\sigma}}\ mod\, \mathbf{Z} . \tag{3.2}$$

Proof. First we repeat the proof of Theorem 0.1 from [20] for the operator A_1. The fact that A_1 is not a differential operator does not change the argument. The most important point here is that A_1 has the form (0.4) in N, which allows us to use the specific spectral decomposition on the cylinder and the fact that A_1 is a differential operator outside of the cylinder N. This gives us

$$\eta_A = \eta_{A_1} = \eta_{\mathscr{A}^1_{1;\sigma\perp}} + \eta_{\mathscr{A}^1_{2;\sigma}}\ mod\, \mathbf{Z} . \tag{3.3}$$

In the second part of the proof we have to show that, at least $mod\, \mathbf{Z}, \eta_{\mathscr{A}^1_{1;\sigma\perp}}$ is equal to $\eta_{\mathscr{A}_{1,\sigma\perp}}$ and $\eta_{\mathscr{A}^1_{2,\sigma}} = \eta_{\mathscr{A}_{2,\sigma}}$. Let us focus on $\eta_{\mathscr{A}^1_{2,\sigma\perp}}$. The argument for $\eta_{\mathscr{A}^1_{1;\sigma\perp}}$ goes exactly the same way. We do not have any problem with showing the formula

$$\frac{d}{dr}\eta_{\mathscr{A}^r_{2,\sigma}} = -\frac{2}{\sqrt{\pi}} \cdot \lim_{\epsilon \to 0} \sqrt{\epsilon} \cdot Tr\left(\dot{\mathscr{A}}^r_{2;\sigma} e^{-\epsilon(\mathscr{A}^r_{2;\sigma})^2}\right) . \tag{3.4}$$

Now, once again, we replace the kernel of the operator $\dot{\mathscr{A}}_{2;\sigma^r} e^{-\epsilon(\mathscr{A}^r_{2;\sigma})^2}$ by the corresponding parametrix. Let $\mathscr{T}(t;x,z)$ denote the kernel of the operator

$$e^{-t((\Gamma(\partial_u + B) + rf(u)\Gamma\gamma)_\sigma)^2} ,$$

where $(\Gamma(\partial_u + B) + rf(u)\Gamma\gamma)_\sigma$ denotes the operator $\Gamma(\partial_u + B) + rf(u)\Gamma\gamma$ on the infinite cylinder $[0, +\infty) \times Y$ subject to the boundary condition $\Pi_> + \sigma$ at $u = 0$. Let $\mathscr{T}_\gamma(t; x, z)$ denote the kernel of the operator $\Gamma\gamma e^{-t((\Gamma(\partial_u+B)+rf(u)\Gamma\gamma)_\sigma)^2}$. We have an explicit representation of $\mathscr{T}(t; x, z)$ (see [12], Sect. 3; see also [16]) in terms of the spectral decomposition of the operator B. This gives us the representation of $\mathscr{T}_\gamma(t; x, z)$ as well. In particular,

$$\int_Y f(u) \cdot tr(\mathscr{T}_\gamma(t; (u, y), (u, y)))dy = 0 \ . \tag{3.5}$$

Equation (3.5) implies the desired equality of the η-invariants. We have

$$\begin{aligned} \frac{d}{dr}\eta_{\mathscr{A}^r_{2;\sigma}} &= -\frac{2}{\sqrt{\pi}} \cdot \lim_{\epsilon \to 0} \sqrt{\epsilon} \cdot Tr\left(\dot{\mathscr{A}}^r_{2;\sigma} e^{-\epsilon(\mathscr{A}^r_{2,\sigma})^2}\right) \\ &= -\frac{2}{\sqrt{\pi}} \cdot \lim_{\epsilon \to 0} \sqrt{\epsilon} \cdot \int_0^{+\infty} f(u)du \int_Y tr(\mathscr{T}_\gamma(t; (u, y), (u, y)))dy = 0 \ . \end{aligned} \tag{3.6}$$

This ends the proof of Theorem 3.1.

Now the general additivity formula follows from the results of [14]. The main result of [14] can be formulated in our context as follows:

Theorem 3.2. *Let σ_1 and σ_2 denote the projection of the kernel of B onto two different Lagrangian subspaces and let $\mathscr{A}_{2,\sigma_i}$ denote the corresponding boundary problems on M_2. We have the following formula for the difference of the η-invariants*

$$\eta_{\mathscr{A}_{2,\sigma_2}} - \eta_{\mathscr{A}_{2,\sigma_1}} = \eta(\sigma_2, \sigma_1) \, mod \, \mathbf{Z} \ , \tag{3.7}$$

where $\eta(\sigma_2, \sigma_1)$ is the η-invariant on the cylinder defined in the Introduction (see Theorem 0.2 and (0.9)–(0.11)).

We apply this result. We have the following sequence of equalities *mod* $\mathbf{Z}$

$$\begin{aligned} \eta_A &= \eta_{\mathscr{A}_{1,\sigma_1}} + \eta_{\mathscr{A}_{2;\sigma_1^\perp}} = \eta_{\mathscr{A}_{1,\sigma_1}} + \eta_{\mathscr{A}_{2,\sigma_2}} - (\eta_{\mathscr{A}_{2,\sigma_2}} - \eta_{\mathscr{A}_{2;\sigma_1^\perp}}) \\ &= \eta_{\mathscr{A}_{1,\sigma_1}} + \eta_{\mathscr{A}_{2,\sigma_2}} - \eta(\sigma_2, \sigma_1^\perp) \ . \end{aligned}$$

We also use the following equalities, which holds *mod* $\mathbf{Z}$ for any σ, σ_1, and σ_2:

$$\eta(\sigma, \sigma^\perp) = 0 \ , \ \eta(\sigma_1, \sigma_2) = -\eta(\sigma_2, \sigma_1) \ , \ \eta(\sigma_2, \sigma) + \eta(\sigma, \sigma_1) = \eta(\sigma_2, \sigma_1) \ . \tag{3.8}$$

Equation (3.8) follows from the formula for $\eta(\cdot, \cdot)$ given in Theorem 2.1. of [14]. Now we finish the proof of Theorem 0.2.

End of the proof of Theorem 0.2.

$$\begin{aligned} \eta_{\mathscr{A}} &= \eta_{\mathscr{A}_{1,\sigma^\perp}} + \eta_{\mathscr{A}_{2,\sigma}} = \eta_{\mathscr{A}} = \eta_{\mathscr{A}_{1,\sigma_1}} + \eta_{\mathscr{A}_{2;\sigma_1^\perp}} \\ &= \eta_{\mathscr{A}_{1,\sigma_1}} + \eta_{\mathscr{A}_{2,\sigma_2}} - (\eta(\sigma_2, \sigma_1^\perp) + \eta(\sigma_1^\perp, \sigma_1)) = \eta_{\mathscr{A}_{1,\sigma_1}} + \eta_{\mathscr{A}_{2,\sigma_2}} - \eta(\sigma_2, \sigma_1) \\ &= \eta_{\mathscr{A}_{1,\sigma_1}} + \eta_{\mathscr{A}_{2,\sigma_2}} + \eta(\sigma_1, \sigma_2) \, mod \, \mathbf{Z} \ . \end{aligned} \tag{3.9}$$

4. Cutting and Pasting of the η-Invariant

In the last section of the paper we discuss the variation of the η-invariant under cutting and pasting of the manifold, the bundle and the operator. We avoid the discussion of certain technicalities, and we present here the simplest possible case of the cutting and pasting operation. Let $\Phi : S|Y \to S|Y$ denote a bundle isomorphism covering $f : Y \to Y$, a diffeomorphism of the manifold Y. This means that for any $y \in Y$, the map

$$\Phi(y) : S_y \to S_{f(y)}$$

is a linear isomorphism. Assume that

$$\Phi(y)\Gamma(y) = \Gamma(f(y))\Phi(y), \quad \Phi(y)b(y;\zeta) = b(f(y);(f^{-1})^*(\zeta))\Phi(y) \tag{4.1}$$

for any $y \in Y$, and for any $\zeta \in T^*_y Y$, where $b(y;\zeta) : S_y \to S_y$ denotes the principal symbol of the tangential operator B. Moreover, let us assume that f is an isometry and that Φ is a unitary isomorphism. We obtain a manifold M^f by taking M_1 and M_2, and pasting them along Y using f. We identify

$$M_1 \supset Y \ni y \quad \sim \quad f(y) \in Y \subset M_2 \ . \tag{4.2}$$

Similarly we define a bundle S^Φ using the isomorphism Φ.

Now we define the operator A^Φ to be equal to the operator A outside the cylinder $[-1,0] \times Y \in M_1$, and equal to the operator $\Gamma(\partial_u + B_u)$ inside this cylinder. The family $\{B_u\}_{u\in[-1,0]}$ is defined by the formula

$$B_u = B + h(u)(\Phi^{-1}B\Phi - B) \ , \tag{4.3}$$

where $h(u)$ is a smooth function equal to 0 for $-1 \leqq u \leqq -\frac{1}{2}$ and equal to 1 for $-\frac{3}{4} \leqq u \leqq 0$. Then A^Φ is a compatible Dirac operator with respect to the introduced structures. We want to find a formula for the difference $\eta_{A^\Phi} - \eta_A$.

Remark 4.1. The corresponding problem for index was stated and solved a long time ago (see [3, 19]; see also [6]).

We have

$$\begin{aligned}\eta_{A^\Phi} &= \eta_{\mathscr{A}^\Phi_{1,\Phi^{-1}\sigma^\Phi}} + \eta_{\mathscr{A}^\Phi_{2,\sigma\perp}} = \eta_{\mathscr{A}^\Phi_{1,\Phi^{-1}\sigma\Phi}} + \eta_{\mathscr{A}_{2;\sigma\perp}} \\ &= (\eta_{\mathscr{A}^\Phi_{1,\Phi^{-1}\sigma\Phi}} - \eta_{\mathscr{A}_{1,\sigma}}) + \eta_{\mathscr{A}_{1,\sigma}} + \eta_{\mathscr{A}_{2;\sigma\perp}} \\ &= (\eta_{\mathscr{A}^\Phi_{1,\Phi^{-1}\sigma\Phi}} - \eta_{\mathscr{A}_{1,\sigma}}) + \eta_A \, mod\, \mathbf{Z} \ .\end{aligned}$$

In the formula above, $\mathscr{A}^\Phi_{1,\Phi^{-1}\sigma^\Phi}$ denotes the operator $A^\Phi|M_1$, subject to the boundary condition $\Phi^{-1}(\Pi_< + \sigma)\Phi$ (the boundary condition is determined by the spectral projection of the operator $\Phi^{-1}B\Phi$). We rewrite the equality given above as follows.

$$\eta_{A^\Phi} - \eta_A = \eta_{\mathscr{A}^\Phi_{1,\Phi^{-1}\sigma\Phi}} - \eta_{\mathscr{A}_{1,\sigma}} \, mod\, \mathbf{Z} \ . \tag{4.4}$$

The simplest way to interpret this formula is to introduce the manifold $\tilde{M}^f_1 = M_1 \cup_f M_1$ and the bundle $\tilde{S}^\Phi_1 = (S|M_1) \cup_\Phi (S|M_1)$. Now we define the operator $\tilde{A}^\Phi_1 = (A^\Phi|M_1) \cup (-A|M_1)$ equal to $A^\Phi|M_1$ on one copy of M_1 and equal to $-(A|M_1)$ on the other copy (see [6] for the details of the construction).

Theorem 4.2. *$\eta_{A^\Phi} - \eta_A$ is given by the following formula:*

$$\eta_{A^\Phi} - \eta_A = \eta_{\tilde{A}_1^\Phi} \, mod \, \mathbf{Z} \ . \tag{4.5}$$

Proof. We apply Theorem 0.2. to the operator $\tilde{A}_1^\Phi$, and obtain

$$\eta_{\tilde{A}_1^\Phi} = \eta_{\mathscr{A}^\Phi_{1,\Phi^{-1}\sigma\Phi}} - \eta_{\mathscr{A}_{1,\sigma}} \, mod \, \mathbf{Z} \ . \tag{4.6}$$

Theorem 4.2, however, is not the result we want to have, as we would like to achieve at least partial localization. We introduce an operator on the mapping torus. We introduce the manifold $Y^f = [0,1] \times Y/\sim$, which is obtained from $[0,1] \times Y$ by an obvious identification, $(1,y) \sim (0,f(y))$. In the same way we define a bundle of spinors $(S|Y)^\Phi$ and then we introduce a Dirac operator $D^\Phi : C^\infty(Y^f;(S|Y)^\Phi) \to C^\infty(Y^f;(S|Y)^\Phi)$,

$$D^\Phi = \Gamma(\partial_u + B + h(u)\Phi^{-1}[B,\Phi]) \ . \tag{4.7}$$

The main result of this section is the following theorem:

Theorem 4.3. *$\eta_{A^\Phi} - \eta_A$ is equal to the η-invariant of the operator D^Φ*

$$\eta_{A^\Phi} - \eta_A = \eta_{D^\Phi} \, mod \, \mathbf{Z} \ . \tag{4.8}$$

We introduce a generalization of Theorem 0.2 in order to prove this result. Let Z denote a cobordism between Y and another closed manifold W. We paste M_1 and Z along Y and obtain a compact manifold X

$$X = M_1 \cup Z \ ,$$

with boundary W. Let E denote a bundle of Clifford modules over X such that $E|M_1 = S|M_1$. Once again we assume that all metric structures are products in the collar neighborhoods of Y and W. Let $A_X : C^\infty(X;E) \to C^\infty(X;E)$ denote a compatible Dirac operator on X such that $A_X|M_1 = A|M_1$. Let σ_1 and σ_2 denote the orthogonal projections onto a Lagrangian subspaces of $ker(B)$, and let σ_3 denote the orthogonal projection onto Lagrangian subspace of the kernel of the tangential part of the operator A_X restricted to W. We define an operator $\mathscr{A}_{X,\sigma_3}$ as the operator A_X subject to a boundary condition σ_3 on W. We denote by $\eta(A_Z;\sigma_2,\sigma_3)$ the eta invariant of the operator $A_Z = A_X|Z$, subject to the boundary condition defined by σ_2 on Y and σ_3 on W.

Theorem 4.4. *The η-invariant of the operator $\mathscr{A}_{X,\sigma_3}$ is given by the following formula:*

$$\eta_{\mathscr{A}_{X,\sigma_3}} = \eta_{\mathscr{A}_{1,\sigma_1}} + \eta(\sigma_1,\sigma_2) + \eta(A_Z;\sigma_2,\sigma_3) \, mod \, \mathbf{Z} \ . \tag{4.9}$$

The proof of Theorem 4.4 is almost the same as the proof of Theorem 0.2. First, we use *Duhamel's Principle* and *adiabatic argument*, as in [20], to prove the result in the case of invertible tangential operators. Then we follow Sect. 1, 2, and 3 of this paper. We leave details to the reader.

Proof of Theorem 4.3. We have already shown (see (4.4))

$$\eta_{A^\Phi} - \eta_A = \eta_{\mathscr{A}^\Phi_{1,\Phi^{-1}\sigma\Phi}} - \eta_{\mathscr{A}_{1,\sigma}} \, mod \, \mathbf{Z} \ .$$

Now we apply Theorem 4.4. in the case $Z = [-\frac{1}{2}, 0] \times Y$, and we obtain

$$\eta_{\mathscr{A}^{\Phi}_{1,\Phi^{-1}\sigma\Phi}} = \eta_{\mathscr{A}'_{1,\sigma}} + \eta(A_Z; \sigma^{\perp}, \Phi^{-1}\sigma\Phi) \ . \tag{4.10}$$

In this formula, $A_Z = \Gamma(\partial_u + B + h(u)\Phi^{-1}[\Phi, B])$ and the summand $\eta(\sigma_1, \sigma_2) = \eta(\Gamma(\partial_u + B); \sigma; \sigma^{\perp})$, which appears in the formula (4.9), is equal to 0. Now $\mathscr{A}'_{1,\sigma}$ is equal to the operator A on the manifold $M'_1 = M_1 \backslash [-\frac{1}{2}, 0] \times Y \cong M_1$ subject to a boundary condition, defined by σ on $\{-\frac{1}{2}\} \times Y$. It was shown by Werner Müller that the η-invariant of the boundary problem, of the type considered here, does not depend on the length of the cylinder (see [16]; Proposition 2.16) and we have

$$\eta_{\mathscr{A}'_{1,\sigma}} = \eta_{\mathscr{A}_{1,\sigma}} \ . \tag{4.11}$$

The only thing left is to apply Theorem 0.2 to the operator D^{Φ} on the mapping torus. We can assume

$$Y^f = \left(\left[-1, -\frac{1}{2}\right] \times Y\right) \cup_{f,Id} \left(\left[-\frac{1}{2}, 0\right] \times Y\right) ,$$

where we use f (and Φ) to make a pasting at $u = -\frac{1}{2}$, and we use Id_Y (and $Id_{S|Y}$) to paste the manifold (and bundle) at $u = 0$ to the manifold at $u = -1$. D^{Φ} is equal to $\Gamma(\partial_u + B)$ on $[-1, -\frac{1}{2}] \times Y$ and it is equal to $A_Z = \Gamma(\partial_u + B + h(u)\Phi^{-1}[\Phi, B])$ on $[-\frac{1}{2}, 0] \times Y$. Theorem 0.2 gives the equality

$$\eta_{D^{\Phi}} = \eta(A_Z; \sigma^{\perp}, \Phi^{-1}\sigma\Phi) \, mod \, \mathbf{Z} \ . \tag{4.12}$$

This allows us to finish the proof

$$\begin{aligned} \eta_{A^{\Phi}} - \eta_A &= \eta_{\mathscr{A}'_{1,\sigma}} + \eta(A_Z; \sigma^{\perp}, \Phi^{-1}\sigma\Phi) - \eta_{\mathscr{A}_{1,\sigma}} \\ &= \eta(A_Z; \sigma^{\perp}, \Phi^{-1}\sigma\Phi) = \eta_{D^{\Phi}} \, mod \, \mathbf{Z} \ . \end{aligned}$$

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