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Generalized Bethe Ansatz Equations for Hofstadter Problem

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Abstract: The problem of diagonalization of the quantum mechanical Hamiltonian, governing dynamics of an electron on a two-dimensional triangular or square lattice in external uniform magnetic field, applied perpendicularly to the lattice plane, the flux through lattice cell, divided by the elementary quantum flux, being a rational number, is reduced to the generalized Bethe ansatz like equations on the high genus algebraic curve. Our formulae for the trigonometric case, where the genus of the curve vanishes, contain as a particular case a recent result of Wiegmann and Zabrodin.

1. Introduction

In this paper we consider the diagonalization problem of the following Hamilton operator:

$$\mathscr{H} = \mu(\alpha S + \alpha^{-1}S^{-1}) + \nu(\beta T + \beta^{-1}T^{-1}) + \rho(\gamma U + \gamma^{-1}U^{-1}), \tag{1.1}$$

where unitary operators S, T, and U satisfy Weyl commutation relations

$$ST = \omega TS, \quad TU = \omega UT, \quad US = \omega SU \tag{1.2}$$

with ω being a primitive N^{th} root of unity:

$$\omega = \exp(2\pi i M/N), \quad (M,N) = 1 \tag{1.3}$$

for some mutually prime integers $N > M \geqq 1$, the complex parameters α, β, γ have unit absolute values, while μ, ν and ρ are real parameters. Phases of α, β, γ are constrained to lay between 0 and $2\pi/N$:

$$0 \leqq \arg(\alpha), \quad \arg(\beta), \quad \arg(\gamma) \leqq 2\pi/N\,. \tag{1.4}$$

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The Hamiltonian (1.1) appears in various problems of solid state physics. With parameter ρ, taken to be zero, it governs the dynamics of an electron on a two-dimensional square lattice in the external uniform magnetic field, applied perpendicularly to the lattice plane, phases of α and β playing the role of quasi-momentum, and the phase of ω, the role of magnetic flux through lattice cell, divided by the elementary quantum flux. The same problem on triangular lattice is described by (1.1) with non-zero parameter ρ. Parameters μ, ν and ρ, called hopping parameters (or amplitudes), serve as length scales along corresponding directions on the lattice.

N^{th} powers of operators S, T, U are central elements and we fix them to be unity:

$$S^N = T^N = U^N = 1 . \tag{1.5}$$

From (1.2) and (1.5) one can deduce that operator

$$C = STU \tag{1.6}$$

is also a central element, and its N^{th} power is 1 (-1) for odd (even) N:

$$C^N = (-1)^{N-1} . \tag{1.7}$$

If we fix C by some number, satisfying (1.7), then operators S, T, U have a unique N-dimensional irreducible representation. For the problem under consideration only algebraic properties of these operators, given by (1.2), (1.5) are important. Different reducible realizations can affect degeneracies of eigenvalues, but not the spectrum itself. We will use this fact, working with an N^3-dimensional reducible representation.

Exploration of the spectrum of (1.1), or its various particular cases, is associated with many names, and especially Hofstadter, who at $\rho = 0$ made a detailed numerical analysis of dependence of this spectrum on integers M and N for $N \leqq 50$ [H]. A shortened list of physical and mathematical papers, concerning this problem, contains also [W, HK, BKS, CEY].

Last years the idea about possible relevance to above problem of methods, used in quantization of integrable models, attracted some attention. This idea has been materialized in recent paper [WZ], where Wiegmann and Zabrodin showed that in the case, equivalent to $\rho = 0$, $\mu = -\nu = 1$, and $\alpha^N = \beta^N = 1$, the spectrum of (1.1) can be represented in terms of solutions of Bethe ansatz (BA) type algebraic equations.

In this paper, using more sophisticated methods of integrable models, we show that BA like equations can be obtained also for general Hamiltonian (1.1). They are similar to the equations, found in [BS] for the case of the chiral Potts model [AMPTY, MPTS, BPA]. Rapidity variables, satisfying generalized BA equations, live on high genus algebraic curve. The rational limit, where the genus vanishes, contains also the case, considered in [WZ]. We hope, that our result will be useful for further investigation of Hamiltonian (1.1), especially in the limit of large N.

In Sect. 2 we represent Hamiltonian (1.1) as a part of three-site transfer matrix, constructed through elementary L-operator, intertwined by the six-vertex R-matrix. In Sect. 3 we calculate Baxter's vector, which enables to derive functional equation, determining in principle the spectrum of Hamiltonian (1.1) in terms of solutions of BA like equations on high genus algebraic curve. Section 4 contains specialization to the case with $\rho = 0$ in (1.1). In Sect. 5 a detailed investigation of the trigonometric case, corresponding to genus zero curve, is performed.

2. Transfer Matrix

In this section we introduce a simple L-operator, intertwined by the six-vertex R-matrix, and define transfer matrix, containing Hamiltonian (1.1).

Consider two operators, X and Y, satisfying Weyl permutation relation, and with unit N^{th} powers:

$$XY = \omega YX, \quad X^N = Y^N = 1\,, \tag{2.1}$$

with ω from (1.3). These operators can be realized as N-by-N matrices with the following matrix elements:

$$\langle m|X|n\rangle = \omega^m \delta_{m,n}, \quad \langle m|Y|n\rangle = \delta_{m,n+1}\,, \tag{2.2}$$

where indices m and n run over N values $0,1,\ldots,N-1$ and are considered $(\mathrm{mod}\, N)$, so they are elements of Z_N.

Introduce 2-by-2 matrix, $L(x,h)$, with operator valued matrix elements:

$$L(x,h) = \begin{pmatrix} aXY & xbY \\ xcX & d \end{pmatrix}, \tag{2.3}$$

where x and $h = (a,b,c,d)$ are some complex parameters. The N-dimensional linear space, where operators X and Y act, will be referred to as a "quantum" space while the two-dimensional space, where $L(x,h)$ acts as 2-by-2 matrix, as an "auxiliary" one.

Matrix (2.3), called an L-operator, is intertwined by the six-vertex R-matrix:

$$R(x/y)L(x,h) \otimes L(y,h) = (1 \otimes L(y,h))(L(x,h) \otimes 1)R(x/y)\,, \tag{2.4}$$

where $L(x,h)$ and $L(y,h)$ act independently in two different auxiliary spaces and in one and the same quantum space, while $R(x/y)$ is a matrix in the tensor product of the auxiliary spaces with numerical matrix elements:

$$R(x) = \begin{pmatrix} x\omega - x^{-1} & 0 & 0 & 0 \\ 0 & \omega(x - x^{-1}) & \omega - 1 & 0 \\ 0 & \omega - 1 & x - x^{-1} & 0 \\ 0 & 0 & 0 & x\omega - x^{-1} \end{pmatrix}. \tag{2.5}$$

Note that this R-matrix differs from the usual one in two diagonal elements. Our choice enables us to escape square roots of ω for the time being and to consider both odd and even N simultaneously. Matrix (2.3) has been used in [BBP] for the derivation of functional equations for the chiral Potts model. It can be extracted also by specialization of parameters in L-operators, written in [BKMS] and [T].

Now introduce the following transfer matrix, acting in the tensor product of three quantum spaces:

$$\mathcal{T}(x) = \mathrm{tr}(L(x,h_0) \otimes L(x,h_1) \otimes L(x,h_2))\,, \tag{2.6}$$

where the matrix products and trace are performed in the auxiliary space, and the indexed h's mean that three different sets of parameters $h_i = (a_i, b_i, c_i, d_i)$ are taken. To write down explicitly $\mathcal{T}(x)$, we introduce special notations for certain combinations of parameters h_i:

$$e_i = b_{i-1}d_i c_{i+1}, \quad f_i = c_{i-1}a_i b_{i+1}, \quad i = 0,1,2\,(\mathrm{mod}\,3)\,, \tag{2.7}$$

where index i is considered as an element of Z_3. Then, for $\mathscr{T}(x)$ we have the following explicit expression:

$$\mathscr{T}(x) = \mathscr{T}_0 + x^2 \mathscr{T}_2 , \tag{2.8}$$

where

$$\mathscr{T}_0 = \omega a_0 a_1 a_2 C + d_0 d_1 d_2 , \tag{2.9}$$

and

$$\mathscr{T}_2 = e_2 S + f_2 C S^{-1} + e_1 T + f_1 C T^{-1} + e_0 U + f_0 C U^{-1} , \tag{2.10}$$

with

$$S = X \otimes Y \otimes 1, \quad T = Y \otimes 1 \otimes X, \quad U = 1 \otimes X \otimes Y , \tag{2.11}$$

$$C = \omega^{-1} XY \otimes XY \otimes XY . \tag{2.12}$$

Operators S, T, U and C, defined in (2.11) and (2.12), satisfy algebraic relations (1.2), (1.5), and (1.6), operator C being commutative with others (so it can be considered as a number). Thus, operators (1.1) and (2.10) can be identified:

$$\mathscr{T}_2 = \mathscr{H} , \tag{2.13a}$$

where

$$\alpha = e_2^{1/2} (f_2 C)^{-1/2}, \quad \beta = e_1^{1/2} (f_1 C)^{-1/2}, \quad \gamma = e_0^{1/2} (f_0 C)^{-1/2} , \tag{2.13b}$$

$$\mu = (e_2 f_2 C)^{1/2}, \quad \nu = (e_1 f_1 C)^{1/2}, \quad \rho = (e_0 f_0 C)^{1/2} . \tag{2.13c}$$

To conclude the section note, that one and the same transfer matrix $\mathscr{T}(x)$ can be represented by (2.6) with different L's:

$$\mathscr{T}(x) = \mathrm{tr}(\tilde{L}_0(x, h_0) \otimes \tilde{L}_1(x, h_1) \otimes \tilde{L}_2(x, h_2)) , \tag{2.14}$$

where $\tilde{L}$'s differ from L's by a gauge transformation:

$$\tilde{L}_i(x, h_i) = A_i L(x, h_i) A_{i+1}^{-1}, \quad i \in Z_3 \tag{2.15}$$

with some invertible numerical 2-by-2 matrices A_i. In the next section we will use this freedom to construct Baxter's vector.

3. Baxter's Vector

Following [B1, FT, BS], let us turn to the calculation of Baxter's vector for transfer matrix (2.6). For this let us use its representation by (2.14) with matrices A_i, chosen as

$$A_i = \begin{pmatrix} 1 & \eta_i \\ 1 & \xi_i \end{pmatrix}, \quad \eta_i = \xi_i - 1, \quad i \in Z_3 , \tag{3.1}$$

where ξ_i are some complex parameters to be fixed later on. Transformed L-operators $\tilde{L}_i(x, h_i)$ have the following explicit form:

$$\tilde{L}_i(x, h_i) = \begin{pmatrix} F(x, h_i; \eta_i, \xi_{i+1}) & -F(x, h_i; \eta_i, \eta_{i+1}) \\ F(x, h_i; \xi_i, \xi_{i+1}) & -F(x, h_i; \xi_i, \eta_{i+1}) \end{pmatrix}, \quad i \in Z_3 , \tag{3.2}$$

where operator valued function F is defined by

$$F(x,h;\xi,\xi') = (\xi' aX - xb)Y + \xi(x\xi' cX - d)\,. \tag{3.3}$$

Now let us try to fix parameters ξ_i by demanding that equations

$$F(x,h_i;\xi_i,\xi_{i+1})|\phi_i\rangle = 0, \quad i \in Z_3 \tag{3.4}$$

have non-zero solutions. Writing them in the basis (2.2), we get the following recurrence relations for matrix elements of vectors $|\phi_i\rangle$:

$$\frac{\langle m|\phi_i\rangle}{\langle m-1|\phi_i\rangle} = -\xi_i^{-1}\frac{\xi_{i+1}a_i\omega^m - xb_i}{x\xi_{i+1}c_i\omega^m - d_i}, \quad i \in Z_3\,. \tag{3.5}$$

Periodicity conditions impose algebraic relations of high order on ξ's:

$$\xi_i^N = (-1)^N\frac{a_i^N\xi_{i+1}^N - b_i^N x^N}{c_i^N(x\xi_{i+1})^N - d_i^N}, \quad i \in Z_3\,. \tag{3.6}$$

Considering ξ's and x as variables, while keeping all h's fixed, we get three equations on four variables, which define an algebraic curve Γ. Let p be some point of this curve, specified by a particular set of coordinates (x,ξ_i), satisfying (3.6). Then vectors $|\phi_i\rangle$ depend on this point, so from now on we will use another notation for them:

$$|\phi_i\rangle = |p\rangle_i, \quad p \in \Gamma\,. \tag{3.7a}$$

Besides, choose the following normalization:

$$\langle 0|p\rangle_i = 1, \quad i \in Z_3\,. \tag{3.7b}$$

To proceed further, define two automorphisms of our curve, $\tau_\pm$, which act on coordinates x and ξ_i as follows:

$$\tau_\pm\colon \Gamma \to \Gamma, \quad \tau_\pm^* x = \omega^{\pm 1/2}x, \quad \tau_\pm^*\xi_i = \omega^{-1/2}\xi_i\,. \tag{3.8}$$

Using this definition, one can easily show that

$$F(x,h_i;\xi_i - 1,\xi_{i+1})|p\rangle_i = -|\tau_- p\rangle_i(x\xi_{i+1}c_i - d_i)\,, \tag{3.9a}$$

and

$$F(x,h_i;\xi_i,\xi_{i+1} - 1)|p\rangle_i = -|\tau_+ p\rangle_i\xi_i(a_id_i - x^2b_ic_i)/(\xi_{i+1}a_i - xb_i)\,. \tag{3.9b}$$

Denoting

$$|p\rangle = |p\rangle_0 \otimes |p\rangle_1 \otimes |p\rangle_2\,, \tag{3.10}$$

and using (2.14), (3.2), (3.4), (3.9) together with multiplication properties of triangular matrices, we come to the relation:

$$\mathcal{T}(x)|p\rangle = |\tau_- p\rangle\Delta_-(p) + |\tau_+ p\rangle\Delta_+(p)\,, \tag{3.11}$$

where

$$\Delta_-(p) = \prod_{i\in Z_3}(d_i - x\xi_{i+1}c_i)\,, \tag{3.12a}$$

$$\Delta_+(p) = \prod_{i\in Z_3}\xi_i(a_id_i - x^2b_ic_i)/(\xi_{i+1}a_i - xb_i)\,. \tag{3.12b}$$

To get the functional equation from (3.11) we just multiply it from the left by the eigenvector $\langle\varphi|$ of $\mathscr{T}(x)$, corresponding to some eigenvalue $\Lambda(x)$. The vector $\langle\varphi|$ evidently does not depend on p, so we obtain eventually the scalar relation

$$\Lambda(x)Q(p) = Q(\tau_- p)\Delta_-(p) + Q(\tau_+ p)\Delta_+(p)\,, \tag{3.13}$$

where

$$Q(p) = \langle\varphi|p\rangle \tag{3.14}$$

is a function on our algebraic curve with known poles (they can be extracted from recurrence relations (3.5)), while its zeros, the number of them being equal to that of poles, are determined from the generalized Bethe ansatz equations, obtained from (3.13) by taking p as various zeros of $Q(p)$:

$$\frac{Q(\tau_- p_k)}{Q(\tau_+ p_k)} = -\frac{\Delta_+(p_k)}{\Delta_-(p_k)}, \quad Q(p_k) = 0, \quad k = 1,\ldots,\#(\text{poles})\,. \tag{3.15}$$

Relations (3.13)–(3.15) at this stage only in principle solve the diagonalization problem of (1.1), since Eqs. (3.15) are too complicated to work with. The main problem, of course, is that of the suitable choice of coordinates on Γ. Unfortunately, uniformization of algebraic curves is still unsolved in the general, long standing problem of mathematics. Our hope, however, is that Γ has a very special structure, which we believe should admit a particular approach to it. Taking into account successful analysis of the curves in the chiral Potts model, [AMP1, AMP2, B2, MR], we think that the analytic structure of Γ deserves a further study.

4. Hofstadter Hamiltonian

Here we specify results of Sect. 3 to the case with $\rho = 0$ in (1.1), which corresponds to

$$a_0 = d_0 = 0, \quad b_0 = c_0 = 1\,. \tag{4.1}$$

The algebraic curve Γ, defined by (3.6), reduces to a disjoint set of N copies of another curve, Γ_0, defined by two high order equations on three variables ξ_0, ξ_2 and x:

$$\xi_0^{-N} = -\frac{a_1^N \xi_2^N - b_1^N x^N}{c_1^N (x\xi_2)^N - d_1^N}, \quad \xi_2^N = (-1)^N \frac{a_2^N \xi_0^N - b_2^N x^N}{c_2^N (x\xi_0)^N - d_2^N}\,, \tag{4.2}$$

while the s^{th} copy in the set is specified by the value of ξ_1:

$$\xi_1 = \omega^{s-1/2}/\xi_0, \quad s \in Z_N \tag{4.3}$$

with $-\omega^{1/2}$ being chosen as the N^{th} root of minus unity:

$$(-\omega^{1/2})^N = -1\,. \tag{4.4}$$

Now Baxter's vector $|p\rangle$ as well as the functions $\Delta_\pm(p)$ acquire the index s:

$$|p\rangle \to |p,s\rangle, \quad \Delta_\pm(p) \to \Delta_\pm(p,s), \quad s \in Z_N\,, \tag{4.5}$$

so relation (3.11) now reads

$$\mathscr{T}(x)|p,s\rangle = |\tau_- p, s-1\rangle\Delta_-(p,s) + |\tau_+ p, s-1\rangle\Delta_+(p,s)\,, \tag{4.6}$$

where automorphisms $\tau_\pm$ of Γ_0 are defined by (3.8) with index i being restricted to only two values, 0 and 2. Taking into account (4.3) together with (3.12), we have explicitly:

$$\Delta_\pm(p,s) = \omega^s \Delta_\pm(p,0)\,. \tag{4.7}$$

Noting that

$$\omega^s = \Phi(s)\Phi(1)/\Phi(s-1), \quad \Phi(s) = \omega^{s(s+N)/2}\,, \tag{4.8}$$

then multiplying (4.6) by $\omega^{st}/\Phi(s)$ and summing over s, we get Baxter's relation for the case of Hofstadter Hamiltonian:

$$\mathscr{T}(x)|p,t\rangle' = |\tau_- p,t\rangle'\Delta_-(p,t)\Phi(1) + |\tau_+ p,t\rangle'\Delta_+(p,t)\Phi(1), \quad t \in Z_N\,, \tag{4.9}$$

where

$$|p,t\rangle' = \sum_{s\in Z_N} |p,s\rangle \omega^{st}/\Phi(s)\,, \tag{4.10}$$

and we used (4.7). Repeating arguments from the end of Sect. 3, one can get direct counterparts of relations (3.13)–(3.15).

5. Rational Limit

In this section we consider the rational limit of formulae from Sect. 3.

Let us restrict parameters h_i, $i \in Z_3$, by

$$a_i = q^{-1}d_i, \quad b_i = q^{-1}c_i, \quad i \in Z_3\,. \tag{5.1}$$

Everywhere in this section $q = \omega^{1/2}$ satisfies (4.4). Then, relations (3.6) become very simple:

$$\xi_i^N = \xi^N, \quad i \in Z_3, \quad \xi^{2N} = 1\,. \tag{5.2}$$

Here the variable x does not enter, so, we have just several copies of genus zero curve, spanned by the coordinate x, while all ξ's are fixed up to N^{th} roots of unity. Let us put $\xi = q^l$ for some $l = 0,\ldots,2N-1$ and choose

$$\xi_i = \xi, \quad i \in Z_3\,, \tag{5.3}$$

then point p of Γ can be identified with the pair (x,l), and (3.11) becomes

$$\mathscr{T}(x)|x,l\rangle = |xq^{-1},l-1\rangle\Delta_-(x,l) + |xq,l-1\rangle\Delta_+(x,l), \quad l = 0,\ldots,2N-1\,, \tag{5.4}$$

where

$$\Delta_-(x,l) = \prod_{i\in Z_3}(d_i - c_i x q^l)\,, \tag{5.5a}$$

$$\Delta_+(x,l) = \prod_{i\in Z_3}(d_i^2 - c_i^2 x^2)/(d_i - c_i x q^{-l})\,. \tag{5.5b}$$

Define function

$$f(x,l) = \prod_{i\in Z_3}\prod_{j=0}^{[l/2]}(d_i - x c_i q^{-l}\omega^j)/(d_i - x c_i q^l \omega^{-j}), \quad l = 0,\ldots,2N-1\,, \tag{5.6a}$$

where

$$[l/2] = \begin{cases} l/2, & l = 0 \pmod 2 ; \\ (l-1)/2, & l = 1 \pmod 2 , \end{cases} \tag{5.6b}$$

and for any $m \in Z_N$ introduce new vectors

$$|x,m\rangle_e = \sum_{n=0}^{N-1} |x,2n\rangle\, f(x,2n)\omega^{mn} ,$$
$$|x,m\rangle_o = \sum_{n=0}^{N-1} |x,2n+1\rangle\, f(x,2n+1)\omega^{mn} . \tag{5.7}$$

Combining them into 2-component row vectors

$$|x,m\rangle\rangle = (|x,m\rangle_e \;\; |x,m\rangle_o) , \tag{5.8}$$

one can rewrite (5.4) in a matrix form:

$$\mathscr{T}(x)|x,m\rangle\rangle = |xq^{-1},m\rangle\rangle D_-(x,m) + |xq,m\rangle\rangle D_+(x,m) , \tag{5.9}$$

where

$$D_\pm(x,m) = \begin{pmatrix} 0 & \Delta_\pm(x,-1) \\ \omega^m \Delta_\pm(x,0) & 0 \end{pmatrix} \tag{5.10}$$

Relations (5.9) do not change their form under gauge transformations:

$$|x,m\rangle\rangle \to |x,m\rangle\rangle U(x,m) ,$$
$$D_\pm(x,m) \to U(xq^{\pm 1},m)^{-1} D_\pm(x,m) U(x,m) \tag{5.11}$$

for any invertible 2-by-2 matrix $U(x,m)$. In the case of arbitrary N and without further restrictions on parameters, there is no gauge, where matrices $D_\pm(x,m)$ would be diagonal, and rational in x, so, from now on let us work with only odd N:

$$N = 2P+1, \quad P \geqq 1 . \tag{5.12}$$

In this case $q = \omega^{1/2}$, satisfying (4.4), can be represented as an integer power of ω:

$$q = \omega^{1/2} = \omega^{P+1} . \tag{5.13}$$

Define the function

$$u(x) = \prod_{\iota \in Z_3} \prod_{j=0}^{P} (d_\iota - x c_\iota \omega^j) , \tag{5.14}$$

and choose the gauge transformation matrix in (5.11) of the form

$$U(x,m) = \begin{pmatrix} u(qx) & q^{-m}u(x) \\ -q^m u(qx) & u(x) \end{pmatrix} . \tag{5.15}$$

Then matrices $D_\pm(x,m)$ become diagonal:

$$D_\pm(x,m) = q^m \psi(\pm x q^{\pm 1/2}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{5.16}$$

where

$$\psi(x) = \prod_{\iota \in Z_3} (d_\iota + x c_\iota q^{-1/2}) . \tag{5.17}$$

Now multiplying (5.9) from the left by the eigenvector $\langle\varphi|$ of $\mathscr{T}(x)$, corresponding to some eigenvalue $\Lambda(x)$, we get two scalar relations:

$$\pm q^{-m}\Lambda(x)Q_{\pm}(x) = \psi(-xq^{-1/2})Q_{\pm}(xq^{-1}) + \psi(xq^{1/2})Q_{\pm}(xq)\,, \tag{5.18}$$

where

$$(Q_{+}(x)\;\; Q_{-}(x)) = \langle\varphi|x,m\rangle\rangle\,. \tag{5.19}$$

From the structure of Baxter's vector, given by (3.5), it follows that $Q_{\pm}(x)$ is a polynomial in x, and sending x to zero and infinity, taking into account (2.8) and (2.9), we conclude, that

$$Q_{-}(x) = 0, \quad Q_{+}(x) = x^{P}Q(x), \quad \deg Q(x) = 2P, \quad m = P\,, \tag{5.20}$$

while the central element C in (1.6) equals unity. For convenience choose the following normalization for the parameters c_0, c_1, c_2:[3]

$$c_0 c_1 c_2 = \omega\,. \tag{5.21}$$

Then, the function $\psi(x)$ reads

$$\psi(x) = q^{1/2}(x+\mu)(x+\nu)(x+\rho)\,, \tag{5.22}$$

where μ, ν, ρ are the same as those in (1.1). As for the parameters α, β, γ, they are equal to $q^{1/2}$. Using (5.20), we rewrite (5.18) as

$$\Lambda(x)Q(x) = \psi(-xq^{-1/2})Q(xq^{-1}) + q^{-1}\psi(xq^{1/2})Q(xq), \quad \deg Q(x) = 2P\,, \tag{5.23}$$

$\Lambda(x)$ being

$$\Lambda(x) = \mu\nu\rho(q^{1/2} + q^{-1/2}) + x^2 E\,, \tag{5.24}$$

where E is an eigenvalue of Hamiltonian (1.1). Let $Q(x)$ have the following decomposition:

$$Q(x) = \prod_{m=1}^{2P}(x - 1/z_m) \tag{5.25}$$

for some z's, which should satisfy BA equations (3.15):

$$q^{-1/2}\frac{(\mu z_l + q^{1/2})(\nu z_l + q^{1/2})(\rho z_l + q^{1/2})}{(q^{1/2}\mu z_l - 1)(q^{1/2}\nu z_l - 1)(q^{1/2}\rho z_l - 1)} = \prod_{m=1, m\neq l}^{2P}\frac{qz_l - z_m}{z_l - qz_m}, \quad l = 1,\ldots,2P\,. \tag{5.26}$$

Differentiating (5.23) twice with respect to x at $x = 0$, we get the expression for E entirely in terms of z's:

$$\begin{aligned} E =& \mu\nu\rho(q - q^{-1})(q^{1/2} - q^{-1/2}) \sum_{1\leq m<n\leq 2P} z_m z_n \\ &- (q - q^{-1})(\mu\nu + \nu\rho + \rho\mu)\sum_{m=1}^{2P} z_m + (q^{1/2} + q^{-1/2})(\mu + \nu + \rho)\,. \end{aligned} \tag{5.27}$$

Formulae (5.26) and (5.27) at $\rho = 0$, $\mu = -\nu = 1$ reproduce correspondingly formulae (6) and (5) of paper [WZ], if we identify our z_m with their iz_m.[4]

[3] This can be done by rescaling variable x

[4] In fact our Eqs. (5.26) differ from (6) of [WZ] by a sign. We checked our equations by direct diagonalization at $N = 3$, so we think that there is a misprint in [WZ].

Let us comment on the physical meaning of our result in the case of the Hofstadter Hamiltonian, i.e. the case with $\rho = 0$ in (1.1). Equations (5.26) and (5.27) at $\rho = 0$ describe particular points (one in each band) in the spectrum of an electron on the square lattice and in an external uniform magnetic field with rational flux through elementary cell. It appears we can in fact describe four points in each band, two of them being the band's end-points. Indeed, fix two positive numbers t_1 and t_2 and consider four different substitutions of the form $\mu = \varepsilon_1 t_1$, $\nu = \varepsilon_2 t_2$, $\rho = 0$ into (5.26), where ε_1, $\varepsilon_2 = \pm 1$. Then, all four cases describe the band's four different points of one and the same physical system with the hopping parameters being $\mu = t_1$, $\nu = t_2$, $\rho = 0$. Two cases with $\varepsilon_1 = \varepsilon_2 = \pm 1$ describe two end-points of a band, while the cases with $\varepsilon_1 = -\varepsilon_2 = \pm 1$ correspond to a pair of the band's interior points, if $t_1 \neq t_2$, and to a single band's mid-point, if $t_1 = t_2$.

It is worth noting that in the case of odd N one can start from the very beginning with another L-operator:

$$L(x,h) = \begin{pmatrix} aV & xbW \\ xcW^{-1} & dV^{-1} \end{pmatrix}, \tag{5.28}$$

where operators V and W satisfy

$$VW = qWV, \quad V^N = W^N = 1. \tag{5.29}$$

L-operators (5.28) and (2.3) are connected in a simple way. If we multiply (5.28) from the right by V, and replace the parameter a by qa, then we get (2.3) with

$$X = W^{-1}V, \quad Y = WV. \tag{5.30}$$

The L-operator (5.28) can be extracted as a particular case from the L-operator, considered by Bazhanov and Stroganov in [BS]. The case, where $a = d$ and $b = -c$, is related to the massless quantum Sine–Gordon model on the lattice [G, V, KT].

Using the L-operator (5.28), we can repeat the procedure developed in Sect. 2 and 3. Automorphisms $\tau_\pm$ in this case act on variables ξ_i and x in another way, namely, they are inverse to each other and transform only the variable x. So, in the rational limit we are not forced to use simultaneously several disjoint copies of the zero genus curve. Particularly, (5.23) and (5.26) arise as a direct specialization of general formulae, counterparts of (3.13) and (3.15), without intermediate transformations, which we made in this section. Nevertheless, the approach which we have presented in detail enables us to consider both even and odd N on equal footing. This explains why we prefer to work with the L-operator (2.3).

Summary

The main result of this paper consists in establishing the relation between the generalized Hofstadter Hamiltonian (1.1) and inhomogeneous XXZ chain of three sites. The available methods in lattice integrable models lead to the generalized Baxter relation (3.11), as well as Bethe ansatz (BA) type equations (3.15) on the high genus algebraic curve. In particular, in the rational case we have got BA-like equations (5.26) together with formula (5.27) for eigenvalues of the Hamiltonian (1.1).

To use Baxter's relation in the general case, one has to develop analysis on the high genus algebraic curve. Results in the chiral Potts model, [AMP1, AMP2,

B2, MR], can be useful here. Apparently, one needs also more information about automorphic functions, connected with the curve.

There is an opportunity to try to apply the known simplification properties of BA equations when the number of variables, satisfying these equations, tends to infinity. In our case this corresponds to $N \to \infty$. However, at the moment it is not clear how to utilize this possibility.

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