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Symplectic Structure of the Moduli Space of Flat Connection on a Riemann Surface

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Abstract: We consider the canonical symplectic structure on the moduli space of flat $\mathscr{G}$-connections on a Riemann surface of genus g with n marked points. For $\mathscr{G}$ being a semisimple Lie algebra we obtain an explicit efficient formula for this symplectic form and prove that it may be represented as a sum of n copies of Kirillov symplectic form on the orbit of dressing transformations in the Poisson–Lie group G^* and g copies of the symplectic structure on the Heisenberg double of the Poisson–Lie group G (the pair (G, G^*) corresponds to the Lie algebra $\mathscr{G}$).

1. Introduction

Being an interesting object of investigations, the moduli space of flat connections on a Riemann surface attracted the attention of many physicists and mathematicians when its relation to the Chern–Simons theory had been discovered [12]. By definition the moduli space (we shall often refer to the moduli space of flat connections in this way) is a quotient of the infinite dimensional space of flat connections over the infinite dimensional gauge group. It is remarkable that this quotient appears to be finite dimensional.

The moduli space $\mathscr{M}$ carries a nondegenerate symplectic structure [3]. It implies the existence of a nondegenerate Poisson bracket on $\mathscr{M}$. Recently the combinatorial description of the moduli space has been suggested [5]. The main idea is to represent the same space $\mathscr{M}$ as a quotient of the finite dimensional space $\mathscr{P}$ over the finite dimensional group action. The Poisson structure has been defined on $\mathscr{P}$ and proved to reproduce the canonical Poisson structure on the moduli space after reduction.

In the first part of this paper we give a combinatorial description of the canonical symplectic structure on $\mathscr{M}$ (see Theorem 1, Sect. 3). This is a bit more natural

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object to consider because the symplectic form may be canonically mapped from $\mathcal{M}$ to $\mathcal{P}$ by means of the pull-back, whereas the Poisson bracket may be defined on $\mathcal{P}$ in many ways.

The nonabelian 3-dimensional Chern–Simons theory has been solved because it is related to the 2-dimensional Wess–Zumino model and to the Quantum Groups. In particular, let us consider the Hilbert space $\mathcal{H}$ of the CS theory associated to the simple Lie algebra $\mathcal{G}$ on an equal time Riemann surface Σ of genus g with n marked points. By construction, there is a representation I_i assigned to each marked point. Then the Hilbert space $\mathcal{H}$ is isomorphic to the space of invariants

$$\mathcal{H} = Inv_q(I_1 \otimes \ldots \otimes I_n \otimes \mathfrak{R}^{\otimes g}) \tag{1.1}$$

in the tensor product of the corresponding representations of the quantum group $U_q(\mathcal{G})$. In formula (1.1), we denote by $\mathfrak{R}$ the regular representation of $U_q(\mathcal{G})$ corresponding to a handle. In this paper we prove a quasi-classical analogue of this statement (see Theorem 2, Sect. 4).

The first attempt in this direction had been made in [4]. There the cases of torus and a disc with one marked point had been considered. However, the key object which will enter into the answer appeared quite recently [6,1]. This is the set of symplectic forms associated to Poisson–Lie groups which replace quantum groups in the quasi-classical limit. More precisely, there is a family of symplectic forms ϑ on the orbits of dressing transformations [10]. They are naturally assigned to the marked points. Besides we have a symplectic form θ on the so-called Heisenberg double (analogue of the cotangent bundle) which is responsible for a handle. So, we prove that the symplectic structure on the moduli space of flat connections on a Riemann surface may be represented as a direct sum of n copies of ϑ and g copies of θ:

$$\Omega = \sum_{i=1}^{n} \vartheta_i + \sum_{i=1}^{g} \theta_i \,. \tag{1.2}$$

The paper is organized as follows. For the convenience of the reader we collect in Sect. 2 the information about the gauge field approach to the moduli space of flat connections, symplectic structures associated to Poisson–Lie groups, dual pairs and Hamiltonian reduction. Each of these subjects is considered in a separate subsection. Section 3 is devoted to a combinatorial description of the symplectic structure on the moduli space of flat connections. A new efficient formula is obtained for the case of a surface with marked points. In our approach one gets a simple proof of the fact that the symplectic form defined via finite dimensional construction is closed. The main result of the paper is presented in Sect. 4. This is the relation of the symplectic structure on the moduli space and Poisson–Lie symplectic structures. Formula (1.2) is proved for any complex simple Lie group or its split real form. With some minor modifications the same is true for compact forms of complex simple Lie groups.

2. Preliminaries

This section includes a collection of facts which we shall use throughout the paper.

2.1. Definition of the Symplectic Structure on the Moduli Space

Let Σ be a Riemann surface of genus g with n marked points. Consider a connection A on Σ taking values in a simple Lie algebra $\mathscr{G}$. We denote the Killing form on $\mathscr{G}$ by Tr. In order to make the construction of this subsection mathematically precise, one should consider the compact form of $\mathscr{G}$ so that the Killing form defines a Euclidean metric on the space of connections. However, we usually ignore subtleties of this kind and pay more attention to the algebraic aspects of the construction.

There is a canonical symplectic structure on the space $\mathscr{A}$ of all smooth connections [3]:

$$\Omega_{\mathscr{A}} = \frac{k}{4\pi} Tr \int_{\Sigma} \delta A \wedge \delta A \, . \tag{2.1}$$

Here we have introduced a coefficient $\frac{k}{4\pi}$ in order to make our notations closer to the ones accepted in the physical literature.

The form (2.1) is obviously nondegenerate and invariant with respect to the action of the gauge group G_Σ:

$$A^g = g^{-1}Ag + g^{-1}dg \, . \tag{2.2}$$

We denote the exterior derivative on the Riemann surface by d, whereas the exterior derivative on the space of connections, moduli space or elsewhere is always δ. The action (2.2) is actually Hamiltonian and the corresponding moment mapping is given (up to a multiplier) by the curvature:

$$\mu(A) = \frac{k}{2\pi} F \, ;$$
$$F = dA + A^2 \, . \tag{2.3}$$

Let us start with a case when there is no marked points.

Definition 1. *The space of flat connections $\Im_g$ on a Riemann surface of genus g is defined as a zero level surface of the moment mapping* (2.3) :

$$F(z) = 0 \, . \tag{2.4}$$

Definition 2. *The moduli space of flat connections is a quotient of the space of flat connections $\Im_g$ over the gauge group action* (2.2) :

$$\mathscr{M}_g = \Im_g / G_\Sigma . \tag{2.5}$$

The curvature being the moment mapping for the gauge group, the moduli space may be obtained by Hamiltonian reduction from the space of smooth connections. General theory of Hamiltonian reduction [2,13] ensures that the moduli space carries a canonical nondegenerate symplectic structure induced from the symplectic structure (2.1) on $\mathscr{A}$.

Now we turn to the more sophisticated case of the Riemann surface with marked points. Among several possible approaches we choose the one which is more convenient for further consideration.

To each marked point z_i we assign a coadjoint orbit in the space $\mathscr{G}^*$ dual to the Lie algebra $\mathscr{G}$. Having the nondegenerate Killing form on $\mathscr{G}$, we can actually identify $\mathscr{G}$ and $\mathscr{G}^*$. In this case the coadjoint orbit may be viewed just as a conjugacy class in $\mathscr{G}$. Using a matrix realization of the Lie algebra we get

$$T \in \mathscr{O}_D \Leftrightarrow T = v^{-1}Dv, \quad v \in G \, . \tag{2.6}$$

Here D is any element of $\mathcal{G}$ which belongs to the orbit $\mathcal{O}$. For example, we can choose it in such a way that it will be represented by a diagonal matrix. Any coadjoint orbit $\mathcal{O}_D$ carries a nondegenerate symplectic form [8] which is often called the Kirillov form. Linear coordinates T on the orbit are not appropriate for description of the Kirillov form. It is much more efficient to use v and D instead of T. Then one can represent the pull-back of the Kirillov form to the group G parametrized by the variable v as

$$\varpi_D = Tr\, D(\delta v v^{-1})^2 . \tag{2.7}$$

It is easy to check that formula (2.7) indeed defines the nondegenerate closed two-form on the orbit $\mathcal{O}_D$ invariant with respect to conjugations. It is worth mentioning that T is a moment mapping for the group action

$$T^g = g^{-1} T g, \quad v^g = vg . \tag{2.8}$$

Definition 3. *A decorated Riemann surface with n marked points is a Riemann surface and a set of coadjoint orbits $\mathcal{O}_1, \dots, \mathcal{O}_n$ assigned to the marked points $z_1, \dots, z_n$.*

One can use the notion of decoration in order to describe possible singularities which may be developed by connections at marked points. Let us introduce the local coordinate ϕ_i in the small neighborhood of the marked point z_i so that

$$\oint_{S_i} d\phi_i = 2\pi . \tag{2.9}$$

Here S_i is a closed contour which surrounds the marked point. Apparently, the coordinate ϕ_i measures the angle in the neighborhood of z_i. On the surface with marked points we shall admit connections which have singularities of the form

$$A(z)_{z\sim z_i} = A_i d(\frac{\phi_i}{2\pi}) + \tilde{A}(z) , \tag{2.10}$$

where A_i are constant coefficients and $\tilde{A}(z)$ is a smooth connection. We call the coefficients A_i singular parts of A.

Definition 4. *The space of connections $\mathcal{A}_{g,n}$ on a decorated Riemann surface with marked points is defined by the requirement that the singular parts of the connection belong to the coadjoint orbits assigned to the corresponding marked points:*

$$\frac{2\pi}{k} A_i \in \mathcal{O}_i . \tag{2.11}$$

It is remarkable that the symplectic structure (2.1) may be used for the space $\mathcal{A}_{g,n}$ as well. It is convenient to introduce one more symplectic space which is the direct product of $\mathcal{A}_{g,n}$ and its collection of coadjoint orbits:

$$\mathcal{A}^{tot}_{g,n} = \mathcal{A}_{g,n} \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_n . \tag{2.12}$$

It carries the symplectic structure

$$\Omega^{tot}_{\mathcal{A}} = \Omega_{\mathcal{A}} + \sum_i^n \varpi_i , \tag{2.13}$$

The action of the gauge group may be defined on the space $\mathcal{A}^{tot}_{g,n}$ as follows:

$$A^g = g^{-1} A g + g^{-1} dg :$$
$$T_i^g = g(z_i)^{-1} T_i g(z_i), \quad v_i^g = v_i g(z_i). \tag{2.14}$$

As we see, the modified gauge transformations are combined from the standard gauge transformations (2.2) and orbit conjugations (2.8). The moment mapping for the gauge group action (2.14) looks very similar to (2.3):

$$\mu(z) = \sum_i^n T_i \delta(z - z_i) + \frac{k}{2\pi} F(z) . \tag{2.15}$$

It is easy to see that the definition of $\mathscr{A}_{g,n}$ ensures that there is a lot of solutions of the zero level conditions.

Definition 5. *The space of flat connections on a decorated Riemann surface $\Im_{g,n}$ is defined as a space of solutions of the following equation which replaces the zero curvature condition:*

$$\mu(z) = 0 . \tag{2.16}$$

Let us choose a loop S_i surrounding the marked point z_i. One can define the monodromy matrix (or parallel transport) M_i along this way. It is easy to check that if A and $\{T_i\}$ satisfy (2.16), the monodromy matrix M_i belongs to the conjugacy class of the exponent of D_i,

$$M_i = u_i^{-1} \exp\left(\frac{2\pi}{k} D_i\right) u_i . \tag{2.17}$$

Definition 6. *The moduli space of flat connections on a Riemann surface of genus g with n marked points $\mathscr{M}_{g,n}$ is defined as a quotient of the space of the flat connection on a decorated Riemann surface over the gauge group action* (2.14) :

$$\mathscr{M}_{g,n} = \Im_{g,n} / G_\Sigma . \tag{2.18}$$

It is important that the moduli space $\mathscr{M}_{g,n}$ is obtained by Hamiltonian reduction from the symplectic space $\mathscr{A}_{g,n}^{tot}$. This procedure provides the nondegenerate symplectic form on $\mathscr{M}_{g,n}$ which is the main object of this paper.

Let us finish this subsection by the remark that symplectic spaces $\mathscr{M}_{g,n}$ naturally appear as phase spaces in the Hamiltonian Chern–Simons theory (see for example [4]). So, the results concerning the corresponding symplectic forms may be always reinterpreted on the language of the Chern–Simons theory.

2.2 Poisson–Lie Groups

Let us consider the Lie group G associated to the Lie algebra $\mathscr{G}$. We shall introduce a Poisson bracket on G such that the multiplication:

$$G \times G \to G \tag{2.19}$$

is a Poisson mapping. A Lie group endowed with such a Poisson bracket is called a Poisson–Lie group. To give an expression for this bracket we need some notations. Let r_+ and r_- be classical r-matrices corresponding to the Lie algebra $\mathscr{G}$:

$$r_+ = \frac{1}{2} \sum h_i \otimes h^i + \sum_{\alpha \in \Delta_+} e_\alpha \otimes e_{-\alpha} , \tag{2.20}$$

$$r_- = -\frac{1}{2} \sum h_i \otimes h^i - \sum_{\alpha \in \Delta_+} e_{-\alpha} \otimes e_\alpha . \tag{2.21}$$

Then the Poisson bracket on the matrix elements of the group G is the following [11]:

$$\{g^1, g^2\} = [r_+, g^1 g^2] = [r_-, g^1 g^2] \,. \tag{2.22}$$

Here we use the tensor notation $g^1 = g \otimes I, g^2 = I \otimes g$. A simple Lie group G equipped with brackets (2.22) is a Poisson–Lie group. Another Poisson–Lie group which we need is called G^*. An element of G^* is a pair (L_+, L_-), where $L_+(L_-)$ is an element of the Borel subgroup generated by positive (negative) roots of $\mathscr{g}$. The Cartan part of L_+ is inverse to the one of L_-. In the simplest case of $G = SL(N), L_+$ is represented by an upper-triangular matrix and L_- by a lower-triangular one. The multiplication on the group G^* is component-wise:

$$(L_+, L_-)(L'_+, L'_-) = (L_+ L'_+, L_- L'_-) \,. \tag{2.23}$$

The Poisson bracket on G^* looks as follows [10]:

$$\begin{aligned} \{L_+^1, L_+^2\} &= [r_+, L_+^1 L_+^2] \,, \\ \{L_-^1, L_-^2\} &= [r_+, L_-^1 L_-^2] \,, \\ \{L_+^1, L_-^2\} &= [r_+, L_+^1 L_-^2] \,. \end{aligned} \tag{2.24}$$

The formulae for classical r-matrices and Poisson brackets in G^* make sense either for complex simple groups or for their split real forms which means that they admit a $*$-invariant root basis. The example of such a real form is provided by the group $SL(N, R)$.

It is useful to introduce a mapping α from G^* to G

$$\alpha : (L_+, L_-) \to L = L_+ L_-^{-1} \,. \tag{2.25}$$

The group structures of G and G^* are different and the mapping α is not a group homomorphism. However, we shall see in Sect. 4 that it may be useful if we replace the requirements of the group homomorphism by some weaker conditions.

The matrix elements of the resulting element L have the following Poisson bracket:

$$\{L^1, L^2\} = r_+ L^1 L^2 + L^1 L^2 r_- - L^1 r_+ L^2 - L^2 r_- L^1 \,. \tag{2.26}$$

The Poisson bracket (2.26) is degenerate. So, one can describe its symplectic leaves. To this end we consider the action of G on G^* by means of dressing transformations [10]:

$$L \to g^{-1} L g, \quad L \in \alpha(G^*), \quad g \in G \,. \tag{2.27}$$

This action is a Poisson one. It means that the mapping

$$G \times G^* \to G^* \tag{2.28}$$

is consistent with Poisson structures on G and G^*.

Dressing transformations are useful when one describes symplectic structures associated to G^*. The result has been obtained in two steps. First, it was proved [10] that symplectic leaves of the Poisson bracket (2.26) are orbits of dressing transformations (2.27) and then the expression for the symplectic forms was found in [6,1]. To write down the answer we choose a particular orbit of dressing transformations:

$$\mathscr{D}_C = \{L = g^{-1} C g, \quad L \in \alpha(G^*), \quad g \in G\}, \tag{2.29}$$

where C is an element of the Cartan subgroup which parametrizes the orbit. So we have the mapping $\pi : G \to G^*$ given by (2.29). It is convenient to use coordinates L_+, L_- and g on the orbit simultaneously. We have the following formula for the pull-back of the symplectic form on the orbit (2.29) along the projection π:

$$\vartheta(g, C) = \frac{1}{2} Tr\{C\delta g g^{-1} \wedge C^{-1}\delta g g^{-1} + L_+^{-1}\delta L_+ \wedge L_-^{-1}\delta L_-\} . \tag{2.30}$$

We shall see in Sect. 4 that the orbit of dressing transformations may be naturally associated to each marked point on the Riemann surface.

Now we have a full analogue of the classical theory of coadjoint orbits of the group G for the Poisson–Lie case. The dressing transformations replace the coadjoint action and form (2.30) replaces the Kirillov form (2.6). To complete the program we should find an object which corresponds to the cotangent bundle T^*G. Actually, it has been introduced in [10] and called the Heisenberg double D_+. In the case at hand (simple Lie group with Poisson brackets (2.22)) D_+ is isomorphic to the Cartesian product of two copies of G:

$$D_+ \simeq G \times G . \tag{2.31}$$

So D_+ is a Lie group with component-wise multiplication. There exists a Poisson structure on D_+ such that the following embeddings of G and G^* into D_+ are Poisson mappings:

$$G \to D_+ : \ \ h \to (h, h), \tag{2.32}$$

$$G^* \to D_+ : \ \ L \to (L_+, L_-). \tag{2.33}$$

We do not write this Poisson bracket (see for example [10]), but make two remarks about it. First, D_+ is not a Poisson–Lie group (*i.e.* this bracket is not consistent with multiplication). Second, the Poisson structure on D_+ is degenerate, but there is the symplectic leaf

$$\mathcal{L} = GG^* \cap G^*G \tag{2.34}$$

which is open and dense in D_+. In formula (2.34) G, G^* are embedded into D_+ by means of the mappings (2.33). To write down the symplectic form on this leaf let us consider a set of coordinates

$$L = L_+L_-^{-1} = g^{-1}Cg, \quad L' = L'_+(L'_-)^{-1} = g'^{-1}C^{-1}g'. \tag{2.35}$$

One can express h as

$$h = g^{-1}g'. \tag{2.36}$$

The pull-back of the symplectic form on $\mathcal{L}$ is the following [1]:

$$\begin{aligned}\theta(g, g', C) = &\frac{1}{2} Tr\{C\delta g g^{-1} \wedge C^{-1}\delta g g^{-1} + L_+^{-1}\delta L_+ \wedge L_-^{-1}\delta L_-\}+\\ &+ \frac{1}{2} Tr\{C^{-1}\delta g' g'^{-1} \wedge C\delta g' g'^{-1} + L_+'^{-1}\delta L'_+ \wedge L_+'^{-1}\delta L'_-\}+\\ &+ Tr\, \delta C C^{-1} \wedge (\delta g g^{-1} - \delta g' g'^{-1}).\end{aligned} \tag{2.37}$$

As one can see the symplectic form on D_+ consists of two terms similar to the symplectic forms on the orbits (2.30). So we have two orbit systems (their dynamical variables are denoted by letters (g,L) and (g',L') which contain points C and C^{-1}. The last term in (2.37) is designed to take into account the fact that now C is a dynamical variable as well. The form (2.37) will appear in Sect. 4. It will correspond to the contribution of one handle into a symplectic form on the moduli space.

2.3. Dual Pairs

One of the powerful tools in Hamiltonian mechanics is the language of dual pairs. Let X be a symplectic space. Obviously, it carries a nondegenerate Poisson structure.

Definition 7. *A pair of Poisson mappings*

$$\begin{aligned} \mu &: X \to Y, \\ v &: X \to Z \end{aligned} \tag{2.38}$$

is called a dual pair if

$$\{\{f,h\} = 0, \forall f = \tilde{f} \circ \mu, \tilde{f} : Y \to \mathbb{C}\} \Leftrightarrow \{\exists \tilde{h} : Z \to \mathbb{C}, h = \tilde{h} \circ v\}\,. \tag{2.39}$$

In other words, any function lifted from Y is in involution with any function lifted from Z and moreover, if some function commutes with any function lifted from Y it means that it is lifted from Z.

The standard source of dual pairs is Hamiltonian reduction. If we have a Hamiltonian action of a group G on a symplectic manifold X, the following pair of projections is dual:

$$\begin{aligned} \mu &: X \to \mathcal{G}^*, \\ v &: X \to X/G. \end{aligned} \tag{2.40}$$

Here the mapping μ is the moment mapping from the manifold X to the space dual to the Lie algebra $\mathcal{G}$.

Dual pairs provide the method to classify symplectic leaves in the Poisson spaces Y and Z. For any point $y \in Y$ the subspace $v(\mu^{-1}(y))$ is a symplectic leaf in Z. It carries a nondegenerate symplectic structure. The same is true in the other direction. Take any point $z \in Z$, then the subspace $\mu(v^{-1}(z))$ is a symplectic leaf in Y. Actually, in this paper we don't need the full machinery of dual pairs. Only one simple fact will be of importance for us.

Lemma 1. *Let the pair of mappings (μ, v) (2.38) be a dual pair. Suppose that the Poisson bracket on Y is equal to zero at the point y. Under these conditions the restriction of the symplectic form Ω on X to the subspace $\mu^{-1}(y)$ coincides with the pull back of the symplectic form ω_y on the symplectic leave $v(\mu^{-1}(y))$ along the projection v:*

$$\Omega|_{\mu^{-1}(y)} = v^* \omega_y\,. \tag{2.41}$$

This lemma relates the symplectic structure of the reduced phase space with the symplectic structure of the global space X which is usually much simpler.

A particular example of the conditions of Lemma 1 is provided by the Hamiltonian reduction over the origin of the moment mapping. Indeed, the

Poisson structure of the space $\mathcal{G}^*$ is described by Kirillov–Kostant–Sourieu bracket:

$$\{y^a, y^b\} = f^{ab}_c y^c, \tag{2.42}$$

where f^{ab}_c are structure constants of the Lie algebra $\mathcal{G}$. At the origin of $\mathcal{G}^*$ coordinates y^c are equal to zero and the Poisson bracket is obviously equal to zero for any functions on $\mathcal{G}^*$. It means that Lemma 1 is applicable for the moduli space of flat connections on a Riemann surface with marked points. The symplectic structure in question may be investigated using the relatively simple symplectic form (2.13) on the space $\mathcal{A}^{tot}_{g,n}$. The subject of the next section is how to make this description indeed efficient.

3. Combinatorial Description of the Symplectic Structure on the Moduli Space

As it was pointed in Subsect. 2.3, the pull-back of the canonical symplectic structure on the moduli space to the space of flat connections on the decorated Riemann surface is easy to describe because it coincides with the restriction of the canonical symplectic structure on the space $\mathcal{A}^{tot}_{g,n}$. The drawback of this description is that we have to use flat connections as coordinates on the moduli space. The space of flat connections is infinite dimensional, whereas the moduli space is finite dimensional for finite g and n. So, we should look for more efficient coordinate mappings. The simplest example of such a mapping may be constructed in the following way. Let us choose a point P on the Riemann surface which does not coincide with marked points z_i. One can define a subgroup of the gauge group $G_\Sigma(P)$ by the requirement:

$$G_\Sigma(P) = \{g \in G_\Sigma,\ \ g(P) = I\}\ . \tag{3.1}$$

The quotient space

$$\mathcal{H}_{g,n} = \Im_{g,n}/G_\Sigma(P) \subset \mathrm{Hom}(\pi(\Sigma_{g,n}), G) \tag{3.2}$$

is already finite dimensional and admits efficient parametrization.

Let us draw a bunch of circles on the Riemann surface so that there is only one intersection point P. In this bunch we have two circles for each handle (corresponding to a- and b- cycles) and one circle for each marked point. We shall denote the circles corresponding to the i's handle by a_i and $b_i (i = 1, \ldots, g)$ and we shall use symbols $m_i (i = 1, \ldots, n)$ for the circles surrounding marked points. We assume that the circles on Σ are chosen in such a way that the only defining relation in $\pi_1(\Sigma_{g,n})$ looks like

$$m_1 \cdots m_n (a_1 b_1^{-1} a_1^{-1} b_1) \cdots (a_g b_g^{-1} a_g^{-1} b_g) = id\ . \tag{3.3}$$

To each circle we assign the corresponding monodromy matrix defined by the flat connection A. Let us denote these matrices by A_i, B_i and M_i for a-, b- and m-circles. The set of monodromy matrices provides coordinates on $\mathcal{M}_{g,n}$ and a representation of the fundamental group $\pi_1(\Sigma_{g,n})$. It implies the relation

$$M_1 \ldots M_n (A_1 B_1^{-1} A_1^{-1} B_1) \ldots (A_g B_g^{-1} A_g^{-1} B_g) = I \tag{3.4}$$

imposed on the values of A_i, B_i and M_i.

For a Riemann surface with marked points $\mathscr{H}_{g,n}$ is only a subset in the space of representations of the fundamental group $\mathrm{Hom}(\pi(\Sigma_{g,n}),G)$ defined as follows:

$$\mathscr{H}_{g,n} = \{\rho \in \mathrm{Hom}(\pi(\Sigma_{g,n}),G), M_i = \rho(m_i) \in \mathscr{C}_i\} , \tag{3.5}$$

where $\mathscr{C}_i$ are certain conjugacy classes in the group G defined by the decoration:

$$\mathscr{C}_i = \left\{M = u_i^{-1}C_iu_i, u_i \in G, C_i = \exp\left(\frac{2\pi}{k}D_i\right)\right\} . \tag{3.6}$$

So the space $\mathscr{H}_{g,n}$ is a subspace in

$$\mathscr{F}_{g,n} = G^{2g} \times \prod_{i=1}^{n}\mathscr{C}_i , \tag{3.7}$$

defined by the relation (3.4).

The original moduli space may be represented as a quotient of $\mathscr{M}_{g,n}$ over the residual gauge group which is isomorphic to the group G:

$$\mathscr{M}_{g,n} = \mathscr{H}_{g,n}/G. \tag{3.8}$$

It is convenient to define some additional coordinates K_i on $\mathscr{F}_{g,n}$:

$$\begin{aligned} K_0 &= I, \\ K_i &= M_1 \dots M_i, 1 \leqq i \leqq n, \\ K_{n+2i-1} &= K_{n+2i-2}A_i, \\ K_{n+2i} &= K_{n+2i-1}B_i^{-1}A_i^{-1}B_i. \end{aligned} \tag{3.9}$$

It follows from Eq. (3.4) that

$$K_{n+2g} = K_0 = I . \tag{3.10}$$

The canonical symplectic form on the moduli space without marked points may be efficiently described in terms of A,B and K. Since marked points are admitted, we have to make one more step and introduce a new space $\tilde{\mathscr{F}}$:

$$\tilde{\mathscr{F}} = G^{n+2g} \times H^{n+g} . \tag{3.11}$$

Here H is a Cartan subgroup of G. $\tilde{\mathscr{F}}$ may be parametrized by matrices $u_i, i = 1,\dots,n+2g$ from the group G and by Cartan elements $C_i, i = 1,\dots,n+2g, C_{n+2i} = C_{n+2i-1}^{-1}$. We define a projection from $\tilde{\mathscr{F}}$ to $\mathscr{F}$ by the formulae:

$$\begin{aligned} M_i &= u_i^{-1}C_iu_i, \\ A_i &= u_{n+2i-1}^{-1}C_{n+2i-1}u_{n+2i-1}, \\ B_i &= u_{n+2i-1}^{-1}u_{n+2i}. \end{aligned} \tag{3.12}$$

Let us recall that the Kirillov symplectic form on a coadjoint orbit can't be efficiently described in terms of linear coordinates on the orbit. We get a nice formula making a pull-back to the group (see Subsect. 2.1). Replacing the space $\mathscr{F}$ by the space $\tilde{\mathscr{F}}$ we do essentially the same work for the moduli space. The symplectic structure assigned to marked points resembles the Kirillov symplectic structure and it needs a pull-back to the group for an efficient description. The new variables u furnish this task.

Let us call $\tilde{\mathscr{H}}_{g,n}$ the preimage of $\mathscr{H}_{g,n}$ in $\tilde{\mathscr{F}}$. We collect all information about the projections and embeddings in the following diagram:

$$\begin{array}{ccccc} \tilde{\mathscr{F}} & \longrightarrow & \mathscr{F} & & \\ \cup & & \cup & & \\ \tilde{\mathscr{H}}_{g,n} & \longrightarrow & \mathscr{H}_{g,n} & \longrightarrow & \mathscr{M}_{g,n} \end{array} . \tag{3.13}$$

The spaces $\mathscr{H}_{g,n}$ and $\mathscr{F}$ are parametrized by A, B, M and K. The coordinates on $\tilde{\mathscr{H}}_{g,n}$ and $\tilde{\mathscr{F}}$ are given by u and C. Actually, all these spaces are designed to provide efficient parametrization of $\mathscr{M}_{g,n}$ which does not admit good coordinates itself.

After these lengthy preparations we are ready to formulate the main result of this section.

Theorem 1. *The pull-back of the canonical symplectic form on $\mathscr{M}_{g,n}$ to $\tilde{\mathscr{H}}_{g,n}$ coincides with the restriction of the following two-form defined on $\tilde{\mathscr{F}}$:*

$$\Omega_{\mathscr{F}} = \frac{k}{4\pi} Tr \left[\sum_{i=1}^{n+2g} C_i \delta u_i u_i^{-1} C_i^{-1} \wedge \delta u_i u_i^{-1} - \sum_{i=1}^{n+2g} \delta K_i K_i^{-1} \wedge \delta K_{i-1} K_{i-1}^{-1} + \right.$$
$$\left. + 2 \sum_{i=n+1}^{n+2g} \delta C_i C_i^{-1} \wedge \delta u_i u_i^{-1} \right] . \tag{3.14}$$

Remark. The combinatorial description of the symplectic structure on the moduli space of flat connections on a Riemann surface without marked points was given by W.Goldman [7]. An elegant formulation of this result in terms of equivariant cohomology of a certain bicomplex was suggested by A.Weinstein [14]. Recently the moduli space for a surface with marked points was considered in [15]. They use a generalization of the cohomological technique of Goldman. This is a way to get a nice presentation for complicated expressions of the symplectic structure. By contrast, our technique of pull-backs provides a bit lengthy but more concrete formulae which will be used in Sect. 4.

The rest of the section is devoted to proof of Theorem 1.

Proof of Theorem 1 Let us cut the surface along every circle a_i, b_i, m_i. We get $n+1$ disconnected parts. The first n are similar. Each of them is a neighborhood of the marked point with the cycle m_i as a boundary. We denote these disjoint parts by P_i. The last one is a polygon. There are no marked points inside and the boundary is composed of a-,b- and m-cycles as it is prescribed by formula (3.3). We denote the polygon by P_0.

Being restricted to P_0 a flat connection A becomes trivial:

$$A|_{P_0} = g_0^{-1} dg_0 . \tag{3.15}$$

For any other part P_i we get a bit more complicated expression:

$$A|_{P_i} = -\frac{1}{k} g_i^{-1} D_i g_i d\phi_i + g_i^{-1} dg_i . \tag{3.16}$$

We recall that D_i is a diagonal matrix which characterizes the orbit attached to the marked point z_i. There is a set of consistency conditions which tells that the connection described by formulae (3.15,3.16) is actually smooth on the Riemann surface everywhere except the marked points. It means that when one approaches the cuts from two sides, one always gets the same value of A. To be explicit, let us consider the m-cycle which surrounds the marked point z_i. Comparison of Eq.(3.15,3.16) gives:

$$g_0^{-1}dg_0\Big|_{m_i} = \Big(-\frac{1}{k} g_i^{-1} D_i g_i d\phi_i + g_i^{-1} dg_i \Big)\Big|_{m_i} . \tag{3.17}$$

This equation may be easily solved:

$$g_0|_{m_i} = NMg_i|_{m_i} , \tag{3.18}$$

where N is an arbitrary constant matrix and M is equal to

$$M(\phi_i) = \exp\Big(-\frac{1}{k} D_i \phi_i \Big) . \tag{3.19}$$

Now we turn to consistency conditions which arise when one considers a- or b-cycles. In this case both sides of the cut belong to the polygon P_0. Let us denote the restrictions of g_0 on the cut sides by g' and g''. So we have:

$$g'^{-1}dg' = g''^{-1}dg'' . \tag{3.20}$$

We conclude that the matrices g' and g'' may differ only by a constant left multiplier:

$$g'' = Ng' . \tag{3.21}$$

By now we considered connection A in the region of the surface where it is flat. However, it is not true at the marked points. We calculate the curvature in the region P_i and get a δ-function singularity:

$$F(z)|_{P_i} = -\frac{2\pi}{k} g_i^{-1} D_i g_i \delta(z - z_i) . \tag{3.22}$$

Equations (3.22,2.15,2.16) imply that the value $g_i(z_i)$ coincides with the matrix v_i:

$$g_i(z_i) = v_i \tag{3.23}$$

Let us recall that v_i diagonalizes the matrix T_i attached to the marked point z_i by definition of the decorated Riemann surface.

Now we are prepared to consider the symplectic structure on the space of flat connections. First, let us rewrite the definition (2.13) in the following way:

$$\Omega^{tot} = \omega_0 + \sum_{i=1}^{n} \omega_i, \tag{3.24}$$

where the summands correspond to different parts of the Riemann surface:

$$\omega_0 = \frac{k}{4\pi} Tr \int_{P_0} \delta A \wedge \delta A,$$

$$\omega_i = \frac{k}{4\pi} Tr \int_{P_i} \delta A \wedge \delta A + \varpi_i . \tag{3.25}$$

The next step must be to substitute (3.15,3.16) into formulae (3.25). The following lemma provides an appropriate technical tool for this operation.

Lemma 2. *Let A be a $\mathscr{G}$-valued connection defined in the region P of the Riemann surface Σ. Suppose that*

$$A = g^{-1}Bg + g^{-1}dg\,. \tag{3.26}$$

Then the canonical symplectic form

$$\omega_P = Tr\int_P \delta A \wedge \delta A \tag{3.27}$$

may be rewritten as

$$\omega_P = Tr\int_P \{\delta B \wedge \delta B + 2\delta[F_B\delta g g^{-1}]\} + Tr\int_{\partial P} \{\delta g g^{-1} d(\delta g g^{-1}) - 2\delta[B\delta g g^{-1}]\}\,, \tag{3.28}$$

where F_B is a curvature of the connection B

$$F_B = dB + B^2\,. \tag{3.29}$$

One can prove Lemma 2 by straightforward calculation.

Let us apply Lemma 2 to the polygon P_0. In this case $B = 0$ and the answer reduces to

$$\omega_0 = \frac{k}{4\pi} Tr \int_{\partial P_0} \delta g_0 g_0^{-1} d(\delta g_0 g_0^{-1})\,. \tag{3.30}$$

The boundary of the polygon ∂P_0 consists of $n + 4g$ cycles (3.3). So actually we have $n + 4g$ contour integrals in the r.h.s. of (3.30).

Now we use formula (3.28) to rewrite symplectic structures ω_i:

$$\begin{aligned}\omega_i = &\frac{k}{4\pi} Tr \int_{\partial P_i} \{\delta g_i g_i^{-1} d(\delta g_i g_i^{-1}) + \frac{2}{k}\delta[D_i \delta g_i g_i^{-1}] d\phi_i\} - \\ &- Tr\int_{P_i} \delta\{D_i \delta g_i g_i^{-1}\}\delta(z - z_i) + Tr\, D_i(\delta v_i v_i^{-1})^2.\end{aligned} \tag{3.31}$$

The last term in (3.31) represents the Kirillov form attached to the marked point z_i. Taking into account relation (3.23) we discover that this term together with the third term in (3.31) cancel each other.

At this point it is convenient to denote the values of g_0 at the corners of the polygon. We enumerate the corners by the index $i = 0, \ldots, n + 4g - 1$ so that the end-points of the cycle m_i are labeled by $i - 1$ and i. One can easily read from formula (3.3) the enumeration of the ends of a- and b-cycles (see Fig. 1). For example, the end-points of a_i are labeled by $n + 4(i - 1)$ and $n + 4(i - 1) + 1$, whereas the end-points of a_i^{-1} entering in the same word are labelled by $n + 4(i - 1) + 2$ and $n + 4(i - 1) + 3$. We denote the value of g_0 at the i's corner by h_i.

Monodromies A_i, B_i and M_i may be expressed in terms of h_i as

$$M_i = h_{i-1}^{-1} h_i\,, \tag{3.32}$$

$$A_i = h_{n+4(i-1)}^{-1} h_{n+4(i-1)+1} = h_{n+4(i-1)+3}^{-1} h_{n+4(i-1)+2}\,, \tag{3.33}$$

$$B_i = h_{n+4(i-1)+2}^{-1} h_{n+4(i-1)+1} = h_{n+4(i-1)+3}^{-1} h_{n+4i}\,. \tag{3.34}$$

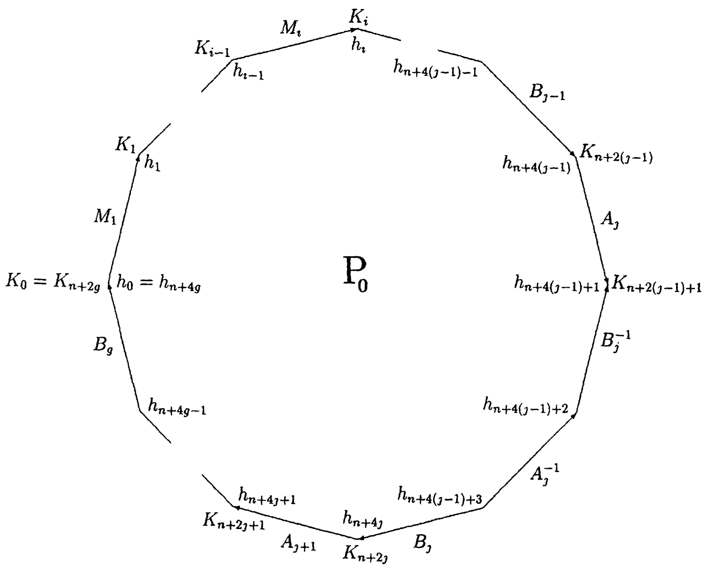

Fig. 1.

Let us remark that without loss of generality we can choose g_0 in such a way that its value h_0 is equal to the unit element in G. After that some of the corner values h_i may be identified with K_i;

$$K_i = \begin{cases} h_i & \text{for } 1 \leqq i \leqq n \\ h_{2i-n-1} & \text{for } (i-n) \text{ odd}, \ n < i \\ h_{2i-n} & \text{for } (i-n) \text{ even}, \ n < i \end{cases} . \tag{3.35}$$

Our strategy is to adjust notations to the description of Poisson–Lie symplectic forms (see Subsect. 2.2). Using formula (3.18) one can diagonalize M_i,

$$M_i = u_i^{-1} C_i u_i . \tag{3.36}$$

Here u_i is the value of the variable g_i at the point P.

Let us rewrite formula (3.30) in the following way:

$$\omega_0 = \sum_{i=1}^{n} \varphi_i + \sum_{i=1}^{g} \psi_i . \tag{3.37}$$

Here φ_i is a contribution corresponding to the marked point:

$$\varphi_i = \frac{k}{4\pi} Tr \int_{m_i} \delta g_0 g_0^{-1} d(\delta g_0 g_0^{-1}) , \tag{3.38}$$

and ψ_i is a contribution of the handle:

$$\psi_i = \frac{k}{4\pi} Tr \int_{a_i b_i^{-1} a_i^{-1} b_i} \delta g_0 g_0^{-1} d(\delta g_0 g_0^{-1}). \tag{3.39}$$

First, we are going to evaluate the total contribution of the given M-cycle which is equal to a sum of two terms:

$$\Omega_i = \omega_i + \varphi_i \,. \tag{3.40}$$

Actually, each summand in (3.40) includes an integral over the m-cycle. However, this sum of integrals is an integral of exact form and it depends only on some finite number of boundary values. This situation is typical and will repeat when we consider a contribution of a handle.

Lemma 3. *The form ω_i depends only on a finite number of parameters and may be written as*

$$\omega_i = \frac{k}{4\pi} Tr[C_i \delta u_i u_i^{-1} \wedge C_i^{-1} \delta u_i u_i^{-1} - \delta K_i K_i^{-1} \wedge \delta K_{i-1} K_{i-1}^{-1}]. \tag{3.41}$$

To prove Lemma 3 one should substitute formula (3.18) into the expression for φ_i, integrate by parts and compare the result with the expression for ω_i. The integrals in φ_i and ω_i cancel each other and after rearrangements the boundary terms reproduce formula (3.41).

Now we turn to the contribution of a handle ψ_i into the symplectic form on the moduli space. One can see that each a-cycle and each b-cycle enter twice into expression (3.37). These two contributions correspond to two sides of the cut. As usual, the result simplifies if we combine the contributions of two cut sides together.

Lemma 4. *Let g', g'' be two mappings from the segment $[x_1, x_2]$ into the group G with boundary values $g'_{1,2}, g''_{1,2}$. Suppose that these mappings differ by the x-independent left multiplier*

$$g'' = Ng'. \tag{3.42}$$

Then the following equality holds:

$$\begin{aligned} \Omega_{[x_1,x_2]} &= Tr\int_{x_1}^{x_2} \delta g'' g''^{-1} d(\delta g'' g''^{-1}) - Tr\int_{x_1}^{x_2} \delta g' g'^{-1} d(\delta g' g'^{-1}) = \\ &= Tr(g_1'^{-1}\delta g_1' \wedge g_1''^{-1}\delta g_1'' - g_2'^{-1}\delta g_2' \wedge g_2''^{-1}\delta g_2''). \end{aligned} \tag{3.43}$$

Proof is straightforward.

Let us parametrize A_i and B_i as in (3.12):

$$A_i = u_{n+2i-1}^{-1} C_{n+2i-1} u_{n+2i-1}, \quad u_{n+2i} = u_{n+2i-1} B_i. \tag{3.44}$$

One of the motivations for such notations is the following identity:

$$B_i^{-1} A_i^{-1} B_i = u_{n+2i}^{-1} C_{n+2i-1}^{-1} u_{n+2i} \,. \tag{3.45}$$

In principle, one can introduce the following uniformal variables:

$$\begin{aligned} M_{n+2i-1} &= A_i = u_{n+2i-1}^{-1} C_{n+2i-1} u_{n+2i-1}, \\ M_{n+2i} &= B_i^{-1} A_i^{-1} B_i = u_{n+2i}^{-1} C_{n+2i} u_{n+2i}, \end{aligned} \tag{3.46}$$

so that the defining relation (3.4) looks like

$$M_1 \dots M_n M_{n+1} \dots M_{n+2g} = I \,. \tag{3.47}$$

In these variables we treat handles and marked points in the same way. Roughly speaking, one handle produces two marked points which have the inverse values of $C: C_1 = C_{n+2i-1}, C_2 = C_{n+2i-1}^{-1}$. It resembles the relation between the double D_+ and two orbits of dressing transformations (see Subsect. 2.2). Using the definition of M (3.46) we can clarify the definition of K_i:

$$K_i = M_1 \dots M_i \,. \tag{3.48}$$

Now we turn to the contribution ψ_i of a handle into symplectic form (3.37).

Lemma 5. *The handle contribution into symplectic form depends only on the values of g_0 at the end-points of the corresponding a- and b-cycles and may be written as*

$$\begin{aligned}\psi_i =& \frac{k}{4\pi} Tr[C_{n+2i-1}\delta u_{n+2i-1}u_{n+2i-1}^{-1} \wedge C_{n+2i-1}^{-1}\delta u_{n+2i-1}u_{n+2i-1}^{-1} - \\ &- \delta K_{n+2i-1}K_{n+2i-1}^{-1} \wedge \delta K_{n+2(i-1)}K_{n+2(i-1)}^{-1} + \\ &+ C_{n+2i-1}^{-1}\delta u_{n+2i}u_{n+2i}^{-1} \wedge C_{n+2i-1}\delta u_{n+2i}u_{n+2i}^{-1} - \delta K_{n+2i}K_{n+2i}^{-1} \wedge \delta K_{n+2i-1}K_{n+2i-1}^{-1} + \\ &+ 2\delta C_{n+2i-1}C_{n+2i-1}^{-1} \wedge (\delta u_{n+2i-1}u_{n+2i-1}^{-1} - \delta u_{n+2i}u_{n+2i}^{-1})].\end{aligned} \tag{3.49}$$

If we take into account Lemma 4, the proof of Lemma 5 becomes a straightforward but long calculation. Let us remark that the terrible formula (3.49) contains two copies of the marked point contribution (3.41) with parameters C_{n+i} and C_{n+i}^{-1}. The last term includes $\delta C_{n+2i-1}C_{n+2i-1}^{-1}$ and coincides with the corresponding additional term in formula (2.37) for the symplectic form on the double D_+.

Summarizing Lemma 3 and Lemma 5 we get the proof of Theorem 1 completed.

Let us remark that in our approach one can easily prove that the form $\Omega_{\mathcal{F}}$ is closed. It is assured by the construction, but we give here a sketch of the combinatorial proof. Let us define M_i for $i = 1, \dots, n+2g$ as

$$M_i = u_i^{-1}C_i u_i = K_{i-1}^{-1}K_i \,. \tag{3.50}$$

and introduce a three-form η by the formula

$$\eta = \frac{k}{4\pi}\sum_i^{n+2g} Tr(\delta M_i M_i^{-1})^3 \,. \tag{3.51}$$

This is a sum of standard $H^3(G)$ generators for $n+2g$ copies of G represented by M_i. One can split the two-form $\Omega_{\mathcal{F}}$ into two pieces:

$$\begin{aligned}\Omega_{\mathcal{F}} &= \Omega_1 - \Omega_2, \\ \Omega_1 &= \frac{k}{4\pi} Tr\left[\sum_{i=1}^{n+2g} C_i\delta u_i u_i^{-1}C_i^{-1} \wedge \delta u_i u_i^{-1} + \right. \\ &\qquad \left. + 2\sum_{i=n+1}^{n+2g} \delta C_i C_i^{-1} \wedge \delta u_i u_i^{-1}\right] \\ \Omega_2 &= \frac{k}{4\pi} Tr\left[\sum_{i=1}^{n+2g} \delta K_i K_i^{-1} \wedge \delta K_{i-1}K_{i-1}^{-1}\right] .\end{aligned} \tag{3.52}$$

There are two different representations of M_i in terms of u_i, C_i and in terms of K_i. Both these substitutions actually kill the nontrivial cohomology class of η and one can prove that

$$\delta\Omega_1 = \delta\Omega_2 = -\frac{1}{3}\eta \,. \tag{3.53}$$

Equation (3.53) assures that the symplectic form on the moduli space is closed.

4. Equivalence to Poisson–Lie Symplectic Structure

Formula (3.14) contains cross-terms with different indices i. In this section we represent the canonical symplectic structure as a direct sum of several terms. Using Subsect. 2.2, each term may be identified with either the Kirillov form for the Poisson–Lie group G^* or the symplectic form on the Heisenberg double D_+ of the Poisson–Lie group G. To achieve this result we have to make a change of variables. The new set of variables is designed to "decouple" contributions of different handles and marked points.

The following remark is important for understanding of the construction of decoupled variables. Monodromy matrices M_i, A_i and B_i are elements of the group G. In accordance with this fact we use G-multiplication to define the variable K_i (3.48) and to constraint monodromies (3.4). On the other hand, natural variables for description of orbits of dressing transformations of double D_+ must belong to G^*. In Sect. 2 we defined the mapping $\alpha : G^* \to G$. Unfortunately, α is not a group homomorphism. So, we would face difficulties applying α to identities (3.48,3.4). This is a motivation to introduce a notion of a weak group homomorphism.

Definition 8. *Let G and G' be two groups. A set of mappings*

$$\alpha^{(n)} : G^n \to G'^n \tag{4.1}$$

is called a weak homomorphism if the following diagram is commutative for any i:

$$\begin{array}{ccc} G^n & \stackrel{\alpha^{(n)}}{\longrightarrow} & G'^n \\ \mathbf{m}_i \downarrow & & \mathbf{m}'_i \downarrow \\ G^{n-1} & \stackrel{\alpha^{(n-1)}}{\longrightarrow} & G'^{n-1} . \end{array} \tag{4.2}$$

Here $\mathbf{m}_i$ and $\mathbf{m}'_i$ are multiplication mappings in G and G' correspondingly which map the product of n copies of the group into the product of $n-1$ copies:

$$\begin{aligned} \mathbf{m}_i &: (g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) \to (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) :, \\ \mathbf{m}'_i &: (g'_1, \ldots, g'_i, g'_{i+1}, \ldots, g'_n) \to (g'_1, \ldots, g'_i g'_{i+1}, \ldots, g'_n) : . \end{aligned} \tag{4.3}$$

The mapping α (2.25) may be considered as a first mapping of a weak homomorphism from G^* to G. To define the other mappings $\alpha^{(n)}$ we introduce the products

$$K_\pm(i) = L_\pm(1) \ldots L_\pm(i). \tag{4.4}$$

The action of $\alpha^{(n)}$ looks as follows. A tuple $(L_+(i), L_-(i)) \in G^*, i = 1, \ldots n$ is mapped into the tuple $M_i \in G, i = 1, \ldots n$:

$$M_i = K_-(i-1) L_i K_-(i-1)^{-1} . \tag{4.5}$$

Here L_i is the image of the pair $(L_+(i), L_-(i))$ under the action of α:

$$L_i = L_+(i) L_-(i)^{-1} . \tag{4.6}$$

One can easily check that the set of mappings (4.5) satisfies the requirements of a weak homomorphism.

The next step is to implement the definition (4.5) to the space $\tilde{\mathscr{F}}$. Let us introduce a set of variables on $\tilde{\mathscr{F}}$ which consists of $v_i, i = 1, \ldots, n+2g$ taking values in G and $C'_i, i = 1, \ldots, n+2g, C'_{n+2i} = C'^{-1}_{n+2i-1}$ taking values in H. In addition we introduce the elements of G:

$$\begin{aligned} L_i &= v_i^{-1} C'_i v_i \quad for 1 \leqq i \leqq n; \\ L_{n+2i-1} &= v_{n+2i-1} C'_{n+2i-1} v_{n+2i-1} \quad for 1 \leqq i \leqq g; \\ L_{n+2i} &= v_{n+2i} C'_{n+2i} v_{n+2i} \quad for 1 \leqq i \leqq g, \end{aligned} \tag{4.7}$$

together with their preimages in G^* with respect to α^{-1}. Technically that means that we assume the Bruhat components $L_\pm(i)$ (2.25) of $L(i)$ to exist. We have introduced the space $\tilde{\mathscr{F}}$ in order to parametrize the moduli space of flat connections. Now Poisson–Lie groups come into the game for the first time. One can reinterpret the space $\tilde{\mathscr{F}}$ as a space of parameters on the direct product of n orbits of dressing transformations and g copies of doubles D_+. Coordinate projection

$$q : \tilde{\mathscr{F}} \rightarrow \mathscr{D}_1 \times \ldots \times \mathscr{D}_n \times \mathscr{D}_+^g \tag{4.8}$$

is given by formulae (4.7). The space $\mathscr{D}_1 \times \ldots \times \mathscr{D}_n \times \mathscr{D}_+^g$ carries a nondegenerate symplectic form which is equal to the direct sum of symplectic forms on the orbits of dressing transformations (2.30) and on g copies of the double D_+ (2.37) (see Subsect. 2.2). The pull-back of this symplectic form to the space $\tilde{\mathscr{F}}$ is equal to

$$\Omega_{PL} = \sum_{i=1}^{n} \vartheta(v_i, C'_i) + \sum_{i=1}^{g} \theta(v_{n+2i-1}, v_{n+2i}, C'_{n+2i-1}) . \tag{4.9}$$

Let us compare the forms (3.14) and (4.9). Motivated by the definition (4.5) we introduce a mapping $\sigma : \tilde{\mathscr{F}} \rightarrow \tilde{\mathscr{F}}$ defined by the relations:

$$u_i = v_i K_-^{-1}(i-1), \quad C_i = C'_i . \tag{4.10}$$

Here $K_-(i)$ are defined as in (4.4). It is easy to see that the mapping σ induces the mapping $\alpha^{(n+2g)}$ from the set of pairs $(L_+(i), L_-(i))$ into the set of monodromies M_i. It is guaranteed by the definition of weak homomorphism that the G-product in the relation (3.4) is now replaced by a G^*-product:

$$K_\pm(n+2g) = L_\pm(1) \ldots L_\pm(n+2g) = I. \tag{4.11}$$

Equation (4.11) defines the preimage of $\tilde{\mathscr{H}}_{g,n}$ in $\tilde{\mathscr{F}}$ with respect to the mapping σ. It is worth mentioning that the matrices K_i from the previous section may be represented as

$$K_i = K_+(i)K_-(i)^{-1} \,. \tag{4.12}$$

This is also a consequence of the definition of weak homomorphism. Indeed, K_i has been defined as a product in G of the first i monodromies. Formula (4.4) defines a product in G^* of i first elements $(L_+(i), L_-(i))$. Using the basic property of weak homomorphism $(i-1)$ times we check (4.12).

Let us arrange the introduced spaces and mappings in the commutative diagram:

$$\begin{array}{lllll}
 & & & & G^* \\
 & & & \nearrow & \\
\tilde{\mathscr{F}} \rightarrow & \mathscr{D}_1 \times \ldots \times \mathscr{D}_n \times \mathscr{D}_+^g & \rightarrow \mathscr{D}_1 \times \ldots \times \mathscr{D}_n \times G^{*2g} & & \\
\uparrow & \uparrow & \qquad\qquad\qquad\uparrow & & G \\
|\sigma^{-1} & |\sigma^{-1} & \quad (\alpha^{n+2g})^{-1}| & \nearrow & \\
\tilde{\tilde{\mathscr{F}}} \quad \rightarrow & \mathscr{F} \quad \rightarrow & \mathscr{C}_1 \times \ldots \times \mathscr{C}_n \times G^{2g} & & \\
\cup & \cup & & & \\
\tilde{\mathscr{H}}_{g,n} \rightarrow & \mathscr{H}_{g,n} \quad \rightarrow & \mathscr{M}_{g,n} & &
\end{array} \tag{4.13}$$

We should make several comments concerning the structure and properties of the diagram (4.13).

Projection $\mathscr{D}_1 \times \ldots \times \mathscr{D}_n \times \mathscr{D}_+^g \rightarrow \mathscr{D}_1 \times \ldots \times \mathscr{D}_n \times G^{*2g}$ is defined by formulae (4.7) and prepares tuples of $n+2g$ elements of G^*. The next projection $\mathscr{D}_1 \times \ldots \times \mathscr{D}_n \times G^{*2g} \rightarrow G^*$ is a multiplication mapping in the group G^*. Projection $\mathscr{C}_1 \times \ldots \times \mathscr{C}_n \times G^{2g} \rightarrow G$ is a multiplication mapping in the group G. These two mappings are consistent with the mapping $\alpha : G \rightarrow G^*$ and represent the generalized moment mapping for Poisson–Lie groups in the sense of [9]. Actually, the image of $\mathscr{H}_{g,n}$ in the symplectic space $\mathscr{D}_1 \times \ldots \times \mathscr{D}_n \times \mathscr{D}_+^g$ coincides with the preimage of the unit element in G^* and the moduli space $\mathscr{M}_{g,n}$ may be obtained by the Poisson reduction from $\mathscr{D}_1 \times \ldots \times \mathscr{D}_n \times \mathscr{D}_+^g$. As a result of Poisson reduction we get some symplectic form on the moduli space $\mathscr{M}_{g,n}$. Our task is to compare this new form with the canonical symplectic structure of the moduli space. Given Theorem 1 the only necessary step is to compare the forms $\Omega_{\mathscr{F}}$ and Ω_{PL} on $\tilde{\mathscr{F}}$.

Lemma 6. *The two-form $\Omega_{\mathscr{F}}$ is proportional to the pull-back of the form Ω_{PL} along the mapping σ:*

$$\Omega_F = \frac{k}{2\pi}\sigma^*(\Omega_{PL}). \tag{4.14}$$

Lemma 6 may be proved by straightforward calculation.

Now we are ready to formulate the main result of this paper.

Theorem 2. *The pull-back of the direct sum of n copies of the Kirillov symplectic form on the orbit of dressing transformations in G^* and g copies of the canonical form on the Heisenberg double of the group G to the space $\tilde{\mathscr{H}}_{g,n}$ coincides up to a scalar multiplier with the pull-back of the canonical symplectic form on the moduli space of flat connections on the Riemann surface of genus g with n marked points.*

Proof of Theorem 2. The proof consists of three steps. First, in Subsect. 2.2 we established that the pull-back of the canonical symplectic form on $\mathscr{D}_1 \times \ldots \times \mathscr{D}_n \times D_+^g$ is equal to Ω_{PL}. Next, Lemma 6 provides the identification of the pull-back of Ω_{PL} with Ω_F up to a multiplier. Finally, Theorem 1 proves that the pull-back of Ω_F

to $\tilde{\mathscr{H}}_{g,n}$ coincides with the pull-back of the canonical form of $\mathscr{M}_{g,n}$ to the same space. Thus, Theorem 2 is proved.

Remark. In this formulation Theorem 2 is correct for G being a complex simple Lie group or its split real form (compare to Subsect. 2.2). In order to apply this result to a compact form, let us remark that the space $\mathscr{H}_{g,n}$ may be embedded into the complexified space $\mathscr{H}^{\mathbb{C}}_{g,n}$. This embedding is induced by the embedding $G \to G^{\mathbb{C}}$. Consider the following diagram:

$$\mathscr{D}_1 \times \ldots \times \mathscr{D}_n \times D_+^{\times g} \leftarrow \mathscr{H}^{\mathbb{C}}_{g,n} \leftarrow \mathscr{H}_{g,n} \to \mathscr{M}_{g,n}. \tag{4.15}$$

For G being a compact Lie group, the result of Theorem 2 may be generalized as follows. The pull-back of the Poisson–Lie symplectic form defined on the complex space $\mathscr{D}_1 \times \ldots \times \mathscr{D}_n \times D_+^{\times g}$ to the space $\mathscr{H}_{g,n}$ coincides with the pull-back of the canonical symplectic form on the moduli space along the projection $\mathscr{H}_{g,n} \to \mathscr{M}_{g,n}$.

5. Conclusions

As we promised in the Introduction, the symplectic form on the moduli space of flat connections may be split into n pieces corresponding to the orbits of dressing transformations and g pieces corresponding to the copies of the Heisenberg double. By the principle of orbit-representation correspondence [8] one should assign some irreducible representations I_i of the quantum group $U_q(\mathscr{G})$ to the orbits and the regular representation $\mathfrak{R}$ to each copy of the Heisenberg double. Taking into account the constraint (4.11), which means that the representation of the total spin is trivial, we have a complete quasi-classical analogue of formula (1.1).

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