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Group Actions on C*-Algebras, 3-Cocycles and Quantum Field Theory

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Abstract: We study group extensions $\Delta \to \Gamma \to \Omega$, where Γ acts on a C*-algebra $\mathscr{A}$. Given a twisted covariant representation π, V of the pair $\mathscr{A}$, Δ we construct 3-cocycles on Ω with values in the centre of the group generated by $V(\Delta)$. These 3-cocycles are obstructions to the existence of an extension of Ω by $V(\Delta)$ which acts on $\mathscr{A}$ compatibly with Γ. The main theorems of the paper introduce a subsidiary invariant Λ which classifies actions of Γ on $V(\Delta)$ and in terms of which a necessary and sufficient condition for the the cohomology class of the 3-cocycle to be non-trivial may be formulated. Examples are provided which show how non-trivial 3-cocycles may be realised. The framework we choose to exhibit these essentially mathematical results is influenced by anomalous gauge field theories. We show how to interpret our results in that setting in two ways, one motivated by an algebraic approach to constrained dynamics and the other by the descent equation approach to constructing cocycles on gauge groups. In order to make comparisons with the usual approach to cohomology in gauge theory we conclude with a Lie algebra version of the invariant Λ and the 3-cocycle.

1. Introduction

Group three cocycles arise from the descent equation approach to the study of the cohomology of gauge groups. This early work was motivated by the need to understand anomalies in gauge theories [2, 4, 10, 26]. From the viewpoint of Dirac's constrained dynamical systems these are models with second class constraints. There have been many attempts to explain and interpret these 3-cocycles [2, 3, 4, 10, 25, 26, 30]. For example Dirac's quantisation condition for the charge of a magnetic monopole [3, 10, 30] has been interpreted as the vanishing of a 3-cocycle whilst non-vanishing 3-cocycles have been interpreted as nonassociative algebra multiplication [4, 10, 26]. Neither of these approaches has yielded to mathematical analysis. In [4] the conventional mathematical interpretation of 3-cocycles as obstructions was described, but the question remained: obstructions to what? In [1] one of us pointed out an interpretation of a 3-cocycle on a group of symmetries of a quantum system as the obstruction to implementing an extension of the symmetry group of the system by unitary operators on the Hilbert space of states of the theory. Mathematical questions were ignored in that discussion in favour of presenting a simple account of this interpretation and in particular no conditions for non-triviality of the 3-cocycle were given.

It eventuated that further development of the mathematical framework in which 3-cocycles are "obstructions" was needed to encompass the motivating examples. Our particular concern is with conditions for non-triviality. Although the basic constructions may be found in the mathematical literature [22 and references therein] they need considerable adapting in order to handle the situation we consider. The framework we derive is capable of a variety of different interpretations and we will present several of these. The basic data is a short exact sequence of groups

$$\Delta \to \Gamma \to \Omega = \Gamma/\Delta$$

and a twisted or μ-representation $V\colon \Delta \to U(\mathscr{H})$ by unitaries on a Hilbert space $\mathscr{H}$ with associated 2-cocycle $\mu \in C^2(\Delta, \mathscr{W})$, where $\mathscr{W} \subset U(\mathscr{H})$ is some coefficient group. (Thus $V_d V_{d'} = \mu(d,d')\, V_{dd'}$.) This is sufficient to determine a mechanism for producing 3-cocycles $\kappa \in C^3(\Omega, \mathscr{W})$. We present the main results of the paper as theorems about group actions on C^*-algebras. However the motivation for proving them depends on an insight into the underlying physics. In one of the interpretations we consider here (which is slightly different from that in [1]) Δ is the gauge group, the group Γ contains all the automorphisms of the algebra of the system in question which we wish to consider while the symmetries of the observables will be represented by the quotient group Ω.

The more usual interpretation [1] is to take Γ/Δ to be the gauge group acting by automorphisms of the field algebra for a coupled Yang–Mills-fermion system and Γ to be the Mickelsson–Faddeev extension [2, 14] of Γ/Δ by the abelian group Δ. The latter is identified with a group of $U(1)$ valued functions on the space of connections (on which the gauge group is acting). Then 3-cocycles would then arise as obstructions to the existence of a representation of this extension by unitary operators on a Hilbert space. More details may be found in Sect. 5. Within this second interpretation our aim in this paper is to lay the groundwork for an investigation of a cohomological version of Pickrell's theorem [17] which suggests that the Mickelsson–Faddeev extension has no separable unitary representations.

To avoid technical problems in Sects. 1 to 3 we will assume all groups are equipped with the discrete topology.

1.1. Constrained Dynamical Systems. To interpret our results in the context of gauge theories we need to choose a suitable framework. The first of these is the C^*-algebraic theory of constrained dynamical systems described in [6, 7]. One may read the main theorems and examples of our paper without reference to this physical framework.

In the terminology of [6, 7] we suppose we are given a unital C^*-algebra $\mathscr{A}$ (called the field algebra), a set $\mathscr{U} \subset \mathscr{A}$ of unitaries (called the constraint set) which is first class in the sense that the identity operator $\mathbf{1}$ is not an element of the C^*-subalgebra of $\mathscr{A}$ generated by $\{U - \mathbf{1} \mid U \in \mathscr{U}\}$. Denote this latter algebra by $C^*(\mathscr{U} - \mathbf{1})$. This guarantees the existence of a distinguished class of states of $\mathscr{A}$ called the Dirac states in [6, 7]:

$$\mathscr{S}_D := \{\omega \in \mathscr{S}(\mathscr{A}) \mid \omega(U) = 1 \quad \text{for all } U \in \mathscr{U}\}\ ,$$

where $\mathscr{S}(\mathscr{A})$ denotes the state space of $\mathscr{A}$. We use the notation: $[.]$, for the closed linear span of a set. Then the algebra

$$\mathscr{O} := \{A \in \mathscr{A} \mid UAU^{-1} - A \in \mathscr{D} \quad \text{for all } U \in \mathscr{U}\}$$

is called the observable algebra, where

$$\mathscr{D} := [\mathscr{A}C^*(\mathscr{U} - 1)] \cap [C^*(\mathscr{U} - 1)\mathscr{A}]$$

is the unique maximal C^*-algebra contained in $\cap\{\ker\omega \mid \omega \in \mathscr{S}_D\}$ (it is a closed 2-sided ideal of $\mathscr{O}$). The algebra $\mathscr{R} = \mathscr{O}/\mathscr{D}$ is called the physical algebra and the physical transformations of $\mathscr{A}$ are those automorphisms which descend to $\mathscr{R}$:

$$\Gamma := \{\alpha \in \operatorname{Aut}\mathscr{A} \mid \alpha(\mathscr{D}) = \mathscr{D}\} .$$

As $\mathscr{O}$ is the relative multiplier algebra of $\mathscr{D}$ in $\mathscr{A}$, (written $M(\mathscr{D})$), if α preserves $\mathscr{D}$ it also preserves $\mathscr{O}$ and the descent to $\mathscr{R}$ gives a homomorphism $\gamma : \Gamma \to \operatorname{Aut}\mathscr{R}$ (i.e. $\gamma_\alpha(\theta(A)) := \theta(\alpha(A))$ for all $A \in \mathscr{O}$, where $\theta : \mathscr{O} \to \mathscr{R}$ is the canonical quotient map). We interpret the automorphisms of $\mathscr{A}$ which lie in the kernel of γ as the guage group and denote it by Δ so that the quotient group $\Gamma/\Delta = \Omega$ is the group of automorphisms of the physical algebra which descend from $\mathscr{A}$. Notice that the group $\operatorname{Ad}\mathscr{U}$ of automorphisms defined by conjugation by elements of $\mathscr{U}$ are in Δ and that Δ may be strictly larger than this group. It is possible that there are non-trivial cocycles on Δ which are trivial on $\operatorname{Ad}\mathscr{U}$. In any case we may suppose that $V: \Delta \to U(\mathscr{H})$ is a μ-representation of Δ on some Hilbert space which carries a representation of $\mathscr{A}$. This shows how the basic data leading to a 3-cocycle can occur in a situation in which there are only first class constraints.

In the physics literature a different interpretation is often used (and is described above and in Sect. 5) in which one treats the elements of the Lie algebra of the gauge group in a projective representation as imposing constraints (in the sense of Dirac) which are necessarily second class. We do not go into the operator algebra approach to this alternative picture in detail here. We note only that the framework of Sects. 3 and 4 applies. We discuss the implications of the results of this paper for this picture in Sect. 5.

1.2. Summary. The plan of the paper is as follows. In Sect. 2 we establish notation and prove the basic results, that is give the mathematical framework for the construction of non-trivial 3-cocycles. It involves the definition of a cohomological invariant Λ which classifies actions of the group Γ on the unitaries $\{V_d \mid d \in \Delta\}$. In Sect. 3 we show how to construct a 3-cocycle given the data of Sect. 2, interpret it and explore its dependence on choices made in the construction. We then prove the main result (Theorem 3.4) which gives a condition for non-triviality. Section 4 contains an example of some mathematical interest in which we use continuous trace C*-algebras to realise some 3-cocycles explicitly. Whereas the physical interpretation we have given above depends for its realisation on representations which have not been shown to exist, the examples in Sect. 4 are complete in all details and indeed are of independent interest. In Sect. 5 we consider the Lie algebra versions of the constructions of Sect. 2. In this and the final section we make contact with the physics literature and Pickrells theorem in particular.

The exact sequence in Theorem 3.4 has already appeared in the algebra literature [8, 12, 24], but the proofs and constructions are expressed in terms of crossed extensions rather than cocycles. Since we are primarily interested in the cocycles (and indeed these are essential to the applications), we have given cocycle-theoretic constructions of the homomorphisms and self-contained proofs of the crucial arguments. For some of the less informative calculations and other side issues we refer to [22], where the relationship between group cohomology and C*-dynamical systems is investigated in the context of locally compact groups and the Borel cochain theory of Moore [15].

2. Preliminaries

Assume as in the introduction a short exact sequence of groups

$$1 \to \Delta \to \Gamma \to \Omega \to 1$$

and that Γ is given as a subgroup of the automorphism group of a unital C^*-Algebra $\mathscr{A}$. We suppose we are given a faithful representation π of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ and a group $\mathscr{V}$ of unitaries with the property that for each $d \in \Delta$ there is a $U_d \in \mathscr{V}$ with

$$(\mathrm{Ad}\, U_d)(\pi(A)) = U_d \pi(A) U_d^{-1} = \pi(d(A))$$

for all $d \in \Delta$, $A \in \mathscr{A}$. Conversely we suppose each $V \in \mathscr{V}$ defines an automorphism of $\mathscr{A}$ via $\mathrm{Ad}: \mathscr{V} \to \Delta$, i.e. we require $(\mathrm{Ad}\, V_d)\pi(A) = \pi(d(A))$ for some $d \in \Delta$ which is uniquely determined by V.

Remark. The choice of $\mathscr{V}$ will effect the cohomology theory that we are considering. Enlarging $\mathscr{V}$ can result in non-trivial cocycles becoming trivial (an example is given in [27]). In practice $\mathscr{V}$ is usually specified by other considerations for example, in the continuous trace case of Sect. 4 the automorphisms they generate are inner or locally inner with respect to $\mathscr{A}$.

Now introduce the group $\mathscr{W} = \ker(\mathrm{Ad}: \mathscr{V} \to \Delta)$ that is:

$$\mathscr{W} = \{V \in \mathscr{V} \mid V\pi(A) = \pi(A)V \quad \text{for all } A \in \mathscr{A}\}\,.$$

Note that $\mathscr{W}$ need not be abelian. In examples there may be additional constraints on π which restrict $\mathscr{W}$, for example, irreducibility of π forces $\mathscr{W}$ to be the circle group. However as noted in [27] nonabelian $\mathscr{W}$ must also be considered

Now we have the following diagram of exact sequences, where $V: \Delta \to \mathscr{V}$ and $\omega: \Omega \to \Gamma$ are sections chosen such that $V_e = 1$, $\omega_e = e$.

$$\begin{tikzcd}
 & 1 \arrow[d] & & & \\
 & \mathscr{W} \arrow[d] & & & \\
 & \mathscr{V} \arrow[d, "Ad|_{\mathscr{A}}"'] & & & \\
1 \arrow[r] & \Delta \arrow[r] \arrow[d] \arrow[u, bend right, "V"'] & \Gamma \arrow[r, "\gamma"'] \arrow[d, phantom, "\cap"] & \Omega \arrow[r] \arrow[l, bend right, "\omega"'] & 1 \\
 & 1 & \mathrm{Aut}\,\mathscr{A} & &
\end{tikzcd}$$

For each of these sections, there is a noncommutative 2-cocycle. The first is $\mu: \Delta \times \Delta \to \mathscr{W}$ defined by $V_d V_k = \mu(d,k) V_{dk}$ for all $d, k \in \Delta$. Then

$$\mu(d,k)\mu(dk,l) = (V_d \mu(k,l) V_d^{-1})\mu(d,kl) \quad (d,k,l \in \Delta) \tag{2.1}$$

by associativity. A second 2-cocycle $\sigma: \Omega \times \Omega \to \Delta$ is defined by $\omega_g \cdot \omega_h = \sigma(g,h)\omega_{gh}$ for all $g, h \in \Omega$. In this case

$$\sigma(g,h)\sigma(gh,f) = (\omega_g \sigma(h,f)\omega_g^{-1})\sigma(g,hf) \quad (g,h,f \in \Omega)\,. \tag{2.2}$$

Now the 3-cocycle we will construct depends for its definition on the existence of an action $\delta: \Gamma \to \operatorname{Aut} \mathscr{V}$ with the property that on Δ we have $\delta_d(v) = V_d v V_d^{-1}$ for all $v \in \mathscr{V}$, $d \in \Delta$ and on $\mathscr{A}$:

$$Ad(\delta_g(V_d)) = gdg^{-1}$$

for all $g \in \Gamma$, $d \in \Delta$. That is $\delta_g(V_d) \in \mathscr{W} V_{gdg^{-1}}$ for all g, d, and thus δ defines a map $\lambda: \Gamma \times \Delta \to \mathscr{W}$ by

$$\lambda(g,d) := \delta_g(V_d) V_{gdg^{-1}}^{-1} \quad (d \in \Delta, g \in \Gamma)\,. \tag{2.3}$$

The main question we need to address is the existence of such a δ. This is handled by our first result.

Theorem 2.1. *Given the exact sequences as above, fix a section $V: \Delta \to \mathscr{V}$, with $V_e = 1$. Let $\delta: \Gamma \to \operatorname{Aut} \mathscr{W}$ be a given action on the coefficient group $\mathscr{W}$ satisfying $\delta_d(w) = V_d w V_d^{-1}$ for all $d \in \Delta$, $w \in \mathscr{W}$. Then δ extends to an action $\delta: \Gamma \to \operatorname{Aut} \mathscr{V}$ such that $\delta_g(V_d) \in \mathscr{W} V_{gdg^{-1}}$ and $\delta_d = \operatorname{Ad}|_{\mathscr{V}} V_d$ for all $d \in \Delta$, $g \in \Gamma$ if and only if*

$$\mu(d,k) = V_d V_k V_{dk}^{-1} \in Z(\mathscr{V}) \cap \mathscr{W}$$

for all $d, k \in \Delta$ and there is a map $\lambda: \Gamma \times \Delta \to Z(\mathscr{V}) \cap \mathscr{W}$ satisfying, for all $g, h \in \Gamma$, $d, k, \in \Delta$:

(i) $\lambda(e,d) = \lambda(g,e) = 1$,
(ii) $\delta_g(\mu(d,k)) \cdot \lambda(g,dk) = \lambda(g,d)\lambda(g,k)\mu(gdg^{-1}, gkg^{-1})$
(iii) $\lambda(gh,d) = \delta_g(\lambda(h,d)) \cdot \lambda(g, hdh^{-1})$
(iv) $\lambda(d,k) = \mu(d,k)\mu(dkd^{-1},d)^{-1}$.

Proof. ($\Rightarrow$) Let $\delta: \Gamma \to \operatorname{Aut} \mathscr{V}$ be an action preserving $\mathscr{W}$ and such that

$$\lambda(g,d) := \delta_g(V_d) V_{gdg^{-1}}^{-1} \in \mathscr{W} \quad \text{for all } d \in \Delta,\ g \in \Gamma \text{ and } \delta_d = \operatorname{Ad} V_d,\ d \in \Delta\,. \tag{2.4}$$

Then

$$\begin{aligned}\delta_{gd}(v) &= \delta_g(V_d v V_d^{-1}) = \delta_g(V_d)\delta_g(v)\delta_g(V_d)^{-1}\\ &= \lambda(g,d) V_{gdg^{-1}} \delta_g(v) V_{gdg^{-1}}^{-1} \lambda(g,d)^{-1}\\ &= \lambda(g,d)\delta_{gdg^{-1}}(\delta_g(v))\lambda(g,d)^{-1} \quad \text{since } gdg^{-1} \in \Delta\\ &= \lambda(g,d)\delta_{gd}(v)\lambda(g,d)^{-1} \quad \text{for all } d \in \Delta,\ g \in \Gamma,\ v \in \mathscr{V}\,,\end{aligned}$$

and so $\lambda(g,d) \in \mathscr{W} \cap Z(\mathscr{V})$. Similarly that $\mu(d,k) \in \mathscr{W} \cap Z(\mathscr{V})$ follows from $\delta_d(v) = V_d v V_d^{-1}$ for all $d \in \Delta$, $v \in \mathscr{V}$ and the fact that δ is an action. This implies that $\delta_d(w) = w$ for all $d \in \Delta$, $w \in \mathscr{K}$.

Requirement (i) is clear, using $V_e = 1$. As for (ii):

$$\begin{aligned}\delta_g(V_d V_k) &= \delta_g(\mu(d,k))\delta_g(V_{dk}) = \delta_g(\mu(d,k))\lambda(g,dk)V_{gdkg^{-1}} \\ &= \delta_g(V_d)\delta_g(V_k) = \lambda(g,d)V_{gdg^{-1}}\lambda(g,k)V_{gkg^{-1}} \\ &= \lambda(g,d)(V_{gdg^{-1}}\lambda(g,k)V_{gdg^{-1}}^{-1})\mu(gdg^{-1},gkg^{-1})V_{gdkg^{-1}} \\ &= \lambda(g,d)\lambda(g,k)\mu(gdg^{-1},gkg^{-1})V_{gdkg^{-1}} ,\end{aligned}$$

which proves (ii). For (iii), use $\delta_{gh} = \delta_g\delta_h$ as follows:

$$\begin{aligned}\delta_{gh}(V_d) &= \lambda(gh,d)V_{ghdh^{-1}g^{-1}} = \delta_g\delta_h(V_d) = \delta_g(\lambda(h,d)V_{hdh^{-1}}) \\ &= \delta_g(\lambda(h,d))\lambda(g,hdh^{-1})V_{ghdh^{-1}g^{-1}} ,\end{aligned}$$

which proves (iii). For (iv) we use $\delta_d = \mathrm{Ad}\, V_d$, $d \in \Delta$ then:

$$\begin{aligned}\delta_d(V_{d^{-1}kd}) &= \lambda(d,d^{-1}kd)V_k = V_d V_{d^{-1}kd}V_d^{-1} = \mu(d,d^{-1}kd)V_{kd}V_d^{-1} \\ &= \mu(d,d^{-1}kd)\mu(k,d)^{-1}V_k \quad \text{for all } d,k \in \Delta ,\end{aligned}$$

so that replacing k by dkd^{-1} we obtain (iv).

($\Leftarrow$) Conversely, assume we have an action $\delta:\Gamma \to \mathrm{Aut}\,\mathscr{W}$ and a $\lambda:\Gamma\times\Delta \to Z(\mathscr{V})\cap\mathscr{W}$ satisfying (i) to (iv) with V and hence μ given. For $d \in \Delta$ we define $\delta_g(V_d) := \lambda(g,d)V_{gdg^{-1}}$. Since $V:\Delta \to \mathscr{V}$ is a section, each $v \in \mathscr{V}$ has an expression $v = wV_d$ for some $d \in \Delta$, $w \in \mathscr{W}$, so using this, define $\delta_g(v) := \delta_g(w)\delta_g(V_d)$. As a map $\delta_g:\mathscr{V} \to \mathscr{V}$ this is well-defined because for all $v \in \mathscr{V}$ its expression $v = wV_d$ is unique. First we show that $\delta_g \in \mathrm{Aut}\,\mathscr{V}$. Given two elements $v = wV_d$, $v' = w'V_{d'} \in \mathscr{V}$:

$$\begin{aligned}\delta_g(vv') &= \delta_g(wV_d w'V_{d'}) = \delta_g(w(V_d w'V_d^{-1})V_d V_{d'}) \\ &= \delta_g(w(V_d w'V_d^{-1})\mu(d,d'))\lambda(g,dd')V_{gdd'g^{-1}} \\ &= \delta_g(w)\delta_g(V_d w'V_d^{-1})\delta_g(\mu(d,d'))\lambda(g,dd')V_{gdd'g^{-1}} \\ &= \delta_g(w)\delta_{gd}(w')\lambda(g,d)\delta_{gdg^{-1}}(\lambda(g,d'))\mu(gdg^{-1},gd'g^{-1})V_{gdd'g^{-1}} \quad \text{(using (ii))} \\ &= \delta_g(w)\lambda(g,d)\delta_{gd}(w')\delta_{gdg^{-1}}(\lambda(g,d'))V_{gdg^{-1}}V_{gd'g^{-1}} \quad \text{(by (2.1))} \\ &= \delta_g(w)\lambda(g,d)\delta_{gd}(w')V_{gdg^{-1}}\lambda(g,d')V_{gd'g^{-1}} \quad (\delta_d = \mathrm{Ad}\, V_d) \\ &= (\delta_g(w)\lambda(g,d)V_{gdg^{-1}})\delta_{gdg^{-1}}^{-1}(\delta_{gd}(w'))\lambda(g,d')V_{gd'g^{-1}} \\ &= \delta_g(v)\delta_g(w')\lambda(g,d')V_{gd'g^{-1}} = \delta_g(v)\delta_g(v')\end{aligned}$$

so δ_g is a homomorphism.

Clearly $\delta_g(v) = \delta_g(w)\delta_g(V_d) = \mathbb{1}$ iff $\delta_g(w)\lambda(g,d) = \mathbb{1}$ and $V_{gdg^{-1}} = \mathbb{1}$, which is the case iff $d = e$ and $w = \mathbb{1}$, i.e. $v = \mathbb{1}$. Thus δ_g is one to one, so $\delta_g \in \mathrm{Aut}\,\mathscr{V}$. To see that it is an action, i.e. $\delta_{gh} = \delta_g\delta_h$, let $v = wV_d \in \mathscr{V}$, then for all $g,h \in \Gamma$:

$$\delta_{gh}(v) = \delta_{gh}(w)\lambda(gh,d)V_{ghdh^{-1}g^{-1}} = \delta_{gh}(w)\delta_g(\lambda(h,d))\lambda(g,hdh^{-1})V_{ghdh^{-1}g^{-1}} ,$$

making use of (iii). Furthermore

$$\begin{aligned}\delta_g\circ\delta_h(v) &= \delta_g(\delta_h(w)\lambda(h,d)V_{hdh^{-1}}) = \delta_g(\delta_h(w)\lambda(h,d))\lambda(g,hdh^{-1})V_{ghdh^{-1}g^{-1}} \\ &= \delta_{gh}(w)\delta_g(\lambda(h,d))\lambda(g,hdh^{-1})V_{ghdh^{-1}g^{-1}} = \delta_{gh}(v) .\end{aligned}$$

By definition we have $\delta_g(V_d) \in \mathscr{W} V_{gdg^{-1}}$, so to prove that

$$\delta_d(v) = V_d v V_d^{-1} \, (v \in \mathscr{V}, d \in \Delta) ,$$

set $v = wV_{d'}$, then

$$\begin{aligned} V_d v V_d^{-1} &= V_d w V_{d'} V_d^{-1} = (V_d w V_d^{-1}) V_d V_{d'} V_d^{-1} \\ &= \delta_d(w) \mu(d,d') V_{dd'} V_{d^{-1}} \mu(d,d^{-1})^{-1} \\ &= \delta_d(w) \mu(d,d') \mu(dd',d^{-1}) (V_{dd'd^{-1}} \mu(d,d^{-1})^{-1} V_{dd'd^{-1}}^{-1}) V_{dd'd^{-1}} \\ &= \delta_d(w) \mu(d,d') \mu(dd'd^{-1},d)^{-1} V_{dd'd^{-1}} \quad \text{by (2.1)} \\ &= \delta_d(w) \lambda(d,d') V_{dd'd^{-1}} = \delta_d(v) \quad \text{by (iv)} . \end{aligned}$$

□

Remarks. (1) It is essential to start with some action on the coefficient group $\mathscr{W}$ because it occurs in (ii) and (iii). The 2-cocycle relation (2.1) holds automatically for μ by its definition.

(2) The fact that μ and λ take their values in $\mathscr{K} = \mathscr{W} \cap Z(\mathscr{V})$ means that the cohomology is commutative. This gives the simplifications that V disappears from (2.1), $\delta_d(w) = w$ for all $d \in \Delta$, $w \in \mathscr{K}$. Thus for twisted representations with cocycles μ not commuting with $\mathscr{V}$, no action exists as in the theorem. If we are given a situation where $\mathscr{K} \neq \mathscr{W}$ and we want actions as in Theorem 2.1, we might as well factor the redundant information in $\mathscr{W} \backslash \mathscr{K}$ out of the theory, to prohibit the occurrence of cohomology over $\mathscr{W}$. Usually we will assume $\mathscr{W} = \mathscr{K} \subset Z(\mathscr{V})$. Consider 2.1 (ii):

$$\delta_g(\mu(d,k)) = [\lambda(g,d)\lambda(g,k)\lambda(g,dk)^{-1}] \mu(gdg^{-1}, gkg^{-1}) ,$$

so if we write $\mu^g(d,k) := \delta_g(\mu(g^{-1}dg, g^{-1}kg))$, then $\mu \sim \mu^g \in Z^2(\Delta, \mathscr{K})$. Thus actions δ as in 2.1 will only exist for $\mu \in Z^2(\Delta, \mathscr{K})$ which are cohomologically Γ-invariant, i.e. $\mu \sim \mu^g$ for all $g \in \Gamma$. This is also a necessary (but not sufficient) condition for a 2-cocycle μ on Δ to extend to a 2-cocycle on Γ with the same coefficient group.

(3) Though the problem is essentially a group theoretical one ($\mathscr{A}$ only contributes the homomorphism of the vertical exact sequence) we prefer to retain C^*-algebras in the picture for the interpretation of our results in the context of gauge groups. When $\mathscr{W}$ is isomorphic to the unitaries in the centre of some C^*-algebra $\mathscr{B}$, denoted $UZ(\mathscr{B})$ we can remove the Hilbert space $\mathscr{H}$, using twisted crossed products as in [16]. Assume we are given a faithful action $\alpha : \Gamma \to \operatorname{Aut} \mathscr{A}$ and a Borel cocycle $\mu \in Z^2(\Delta, \mathscr{W})$. The twisted crossed product [16], $\mathscr{C} = M(\Delta \times_{\alpha,\mu} \mathscr{A} \otimes \mathscr{B})$ contains a copy of Δ given by $u : \Delta \to U(\mathscr{C})$ (the unitaries in $\mathscr{C}$) and a copy of $\mathscr{A} \otimes \mathscr{B}$ given by a homomorphism $k : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ related by

$$k(\alpha_d(a) \otimes 1) = u_d(k(a) \otimes 1) u_d^* \quad (a \in \mathscr{A}, d \in \Delta)$$

$$\text{and} \quad u_d u_f = k(1 \otimes \mu(d,f)) u_{df} \quad (d, f \in \Delta) .$$

The map u is Borel measurable and there is a bijection between representations of $\Delta \times_{\alpha,\mu} \mathscr{A} \otimes \mathscr{B}$ and μ-covariant representations of the action $\alpha : \Delta \to \operatorname{Aut} \mathscr{A}$. In this context we choose $\mathscr{V}$ be the group generated by $\{u_d \mid d \in \Delta\}$.

(4) As a final remark, observe that there is a special case of Theorem 2.1. where it is clear where to look for the actions $\delta : \Gamma \to \operatorname{Aut} \mathscr{W}$ (and which seems natural from a physical perspective). As Γ is meant to be a group of symmetries of a physical system, it is plausible to assume there is a covariant representation for the action $\alpha : \Gamma \to \operatorname{Aut} \mathscr{A}$, which may be twisted on Δ. Thus we have a group of

unitary operators $\mathscr{Y}$ on some Hilbert space such that for each $g \in \Gamma$ there is a $Y \in \mathscr{Y}$ such that $\mathrm{Ad}|_{\mathscr{A}} Y = g$. Clearly then $\mathscr{V} \subset \mathscr{Y}$ and we have an extra exact sequence defined as follows. Let $\mathscr{X} = \{Y \in \mathscr{Y} \mid [Y, A] = 0 \text{ for all } A \in \mathscr{A}\}$, $\mathscr{V} = \{u \in \mathscr{Y} \mid \mathrm{Ad}|_{\mathscr{A}} u \in \Delta\}$ and $\mathscr{W} = \mathscr{V} \cap \mathscr{X}$. Observe that $\mathscr{V}$ is a normal subgroup of $\mathscr{Y}$ and hence we have the diagram:

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & & \\
 & \mathscr{W} \arrow[d] & \subseteq & \mathscr{X} \arrow[d] & & \\
 & \mathscr{V} \arrow[d, "\mathrm{Ad}|_{\mathscr{A}}"'] & \subset & \mathscr{Y} \arrow[d, "Ad|_{\mathscr{A}}"'] & & \\
1 \arrow[r] & \Delta \arrow[u, bend right, "V"'] \arrow[d] \arrow[rr] & & \Gamma \arrow[u, bend right, "V'"'] \arrow[d] \arrow[r] & \Omega \arrow[l, bend left, "\gamma"', "\omega"] \arrow[r] & 1 \\
 & 1 & & 1 & &
\end{tikzcd}$$

In this diagram $V' : \Gamma \to \mathscr{Y}$ is a section, $V'(e) = 1$, then using the fact that $\mathscr{V}$ is normal in $\mathscr{Y}$, we can define an "action" of Γ on $\mathscr{V}$ by $\delta_g(v) := V'_g v V'^{-1}_g$ for all $g \in \Gamma$, $v \in \mathscr{V}$.

Now since $Ad|_{\mathscr{A}}(\delta_g(V_d)) = gdg^{-1}$ for all $g \in \Gamma$, $d \in \Delta$, we have

$$\delta_g(V_d) = V'_g V_d V'^{-1}_g \in \mathscr{W} V_{gdg^{-1}} \,.$$

On $\mathscr{V}$ we note that $V'_d V_d^{-1} \in \mathscr{W}$ for all $d \in \Delta$, so $\varphi : \Delta \to \mathscr{W}$ defined by $\varphi(d) := V'_d V_d^{-1}$ measures the difference between V' and V on Δ.

With the data introduced in remark (4) we can prove our second preliminary result.

Theorem 2.2. *Given the three sequences above, fix a section $V : \Delta \to \mathscr{V}$, $V_e = 1$ as before, then a section $V' : \Gamma \to \mathscr{Y}$, $V'_e = 1$ defines an action $\delta : \Gamma \to \mathrm{Aut}\,\mathscr{V}$ by $\delta_g := \mathrm{Ad}\, V'_g$ such that $\delta_g(V_d) \in \mathscr{W} V_{gdg^{-1}}$ and $\delta_d = \mathrm{Ad}\, V_d$ for all $d \in \Delta$ iff*

$$\varphi(d) := V'_d V_d^{-1} \in Z(\mathscr{V}) \cap \mathscr{W}$$

$$\textit{and } \tilde{\mu}(g,h) := V'_g V'_h V'^{-1}_{gh} \in Z(\mathscr{X}, \mathscr{V}) := \{u \in \mathscr{X} \mid uv = vu, \ \textit{for all } v \in \mathscr{V}\} \,.$$

In this case the map λ associated to δ by Theorem 2.1 satisfies

$$\lambda(g,d) := \delta_g(V_d) V^{-1}_{gdg^{-1}} = \varphi(gdg^{-1}) V'_g \varphi(d)^{-1} V'^{-1}_g \tilde{\mu}(g,d)(\tilde{\mu}(gdg^{-1}, g))^{-1} \,.$$

Proof. Write $V'_d = \varphi(d) V_d$, then we have

$$(\mathrm{Ad}\, V'_d)(v) = \varphi(d) V_d v V_d^{-1} \varphi(d)^{-1} = (\mathrm{Ad}\, V_d)(v) \quad (v \in \mathscr{V}, d \in \Delta)$$

iff $\varphi(d)$ commutes with $\mathscr{V}$, that is $\varphi(d) \in Z(\mathscr{V}) \cap \mathscr{W}$. For $\delta_g = \mathrm{Ad}\, V'_g$ to be an action, i.e. $\delta_{gh} = \delta_g \cdot \delta_h$:

$$\delta_g(\delta_h(v)) = V'_g V'_h v V'^{-1}_h V'^{-1}_g = \tilde{\mu}(g,h) V'_{gh} v V'^{-1}_{gh} \tilde{\mu}(g,h)^{-1} \,,$$

which is equal to $\delta_{gh}(v) = V'_{gh} v V'^{-1}_{gh}$ for all $v \in \mathscr{V}$ if and only if $\tilde{\mu}(g,h)$ commutes with $\mathscr{V}$, i.e. $\tilde{\mu}(g,h) \in Z(\mathscr{X}, \mathscr{V})$. For the condition on λ:

$$\begin{aligned}\lambda(g,d) &= \delta_g(V_d) V^{-1}_{gdg^{-1}} = V'_g V_d V'^{-1}_g V^{-1}_{gdg^{-1}} = V'_g \varphi(d)^{-1} V'_d V'^{-1}_g V'^{-1}_{gdg^{-1}} \varphi(gdg^{-1}) \\ &= V'_g \varphi(d)^{-1} V'_d (\tilde{\mu}(gdg^{-1}, g) V'_{gd})^{-1} \varphi(gdg^{-1}) \\ &= V'_g \varphi(d)^{-1} V'^{-1}_g \tilde{\mu}(g,d) \tilde{\mu}(gdg^{-1}, g)^{-1} \varphi(gdg^{-1}) .\end{aligned}$$

□

Clearly if $V'|_{\Delta} = V$, then $\lambda(g,d) = \tilde{\mu}(g,d)\tilde{\mu}(gdg^{-1},g)^{-1}$, thus extending (iv) of Theorem 2.1. Moreover since the range of δ is in $Ad|_{\mathscr{V}} \mathscr{Y} \subset \operatorname{Aut} \mathscr{V}$, this is only a subclass of possible actions $\Delta : \Gamma \to \operatorname{Aut} \mathscr{V}$ as in Theorem 2.1. Observe that the existence of a section $V' : \Gamma \to \mathscr{Y}$ as above, shows that μ is cohomologous to a 2-cocycle $\tilde{\mu}$ on Δ which extends to a 2-cocycle on Γ, but with a possibly different coefficient group $\mathscr{X}$. So this scenario can occur even when μ has no extensions to Γ over the group $\mathscr{W}$.

Next we wish to define a cohomology for the pairs (λ, μ) introduced in Theorem 2.1, so that the classes can classify the actions δ. Start with a section $V : \Delta \to \mathscr{V}$ as in Theorem 2.1 and choose another section $\bar{V} : \Delta \to \mathscr{V}$ in such a way that it leaves the action $\delta_d = Ad\, V_d = Ad\, \bar{V}_d$, $d \in \Delta$, invariant on Δ. Keep the action $\delta : \Gamma \to \operatorname{Aut} \mathscr{V}$ fixed. Then there is a map $\psi : \Delta \to \mathscr{W}$ such that $\bar{V}_d = \psi(d) V_d$ for all $d \in \Delta$, and so (as usual) the cocycle $\bar{\mu}$ associated with $\bar{V}$ is related to μ by:

$$\begin{aligned}\bar{\mu}(d,k) &= \psi(d)(V_d \psi(k) V_d^{-1}) \mu(d,k) \psi(dk)^{-1} \\ &= \psi(d)\psi(k)\psi(dk)^{-1}\mu(d,k) =: (\partial\psi)(d,k)\mu(d,k) , \qquad (2.5)\end{aligned}$$

where we used the fact that the requirement $\operatorname{Ad} V_d = \operatorname{Ad} \bar{V}_d$ forces $\psi : \Delta \to \mathscr{W} \cap Z(\mathscr{V})$ (in case we had not chosen $\mathscr{W} \subset Z(\mathscr{V})$). Moreover

$$\begin{aligned}\bar{\lambda}(g,d) &:= \delta_g(\bar{V}_d) \bar{V}^{-1}_{gdg^{-1}} = \delta_g(\psi(d) V_d) \psi(gdg^{-1})^{-1} V^{-1}_{gdg^{-1}} \\ &= \delta_g(\psi(d)) \lambda(g,d) V_{gdg^{-1}} \psi(gdg^{-1})^{-1} V^{-1}_{gdg^{-1}} \\ &= \psi(gdg^{-1})^{-1} \delta_g(\psi(d)) \lambda(g,d) . \qquad (2.6)\end{aligned}$$

Denote by $D\psi$ the pair (λ_0, μ_0), where $\lambda_0(g,d) := \psi(gdg^{-1})^{-1}\delta_g(\psi(d))$ for all $g \in \Gamma$, $d \in \Delta$, and $\mu_0 = \partial\psi$, then, denoting componentwise multiplication of pairs by '$\cdot$', $(\bar{\lambda}, \bar{\mu}) = D\psi \cdot (\lambda, \mu)$.

Definition 2.3. *On the coefficient group $\mathscr{W}$ fix an action*

$$\delta : \Gamma \to \operatorname{Aut} \mathscr{W}, \quad \delta_\Delta \subset \operatorname{Ad}|_{\mathscr{W}} \mathscr{V} .$$

Let $Z(\Gamma, \Delta, \mathscr{K})$ be the set of all pairs (λ, μ) satisfying the second part of Theorem 2.1 *where $\mu : \Delta \times \Delta \to Z(\mathscr{V}) \cap \mathscr{W} =: \mathscr{K}$ is a normalised 2-cocycle, $\mu(d,e) = \mu(e,d) = 1$ for all $d \in \Delta$. Then $Z(\Gamma, \Delta, \mathscr{K})$ is an abelian group under pointwise multiplication. Introduce the subgroup*

$$B(\Gamma, \Delta, \mathscr{K}) := \{D\psi \mid \psi : \Delta \to \mathscr{K}\} \subset Z(\Gamma, \Delta, \mathscr{K})$$

and define:

$$\Lambda(\Gamma, \Delta, \mathscr{K}) := Z(\Gamma, \Delta, \mathscr{K}) / B(\Gamma, \Delta, \mathscr{K}) .$$

Clearly $\Lambda(\Gamma, \Delta, \mathscr{K})$ classifies the actions $\delta : \Gamma \to \operatorname{Aut} \mathscr{V}$ which extend the action on $\mathscr{W}$ and are compatible with some section V. Now note that in Theorem 2.1 we characterised the actions $\delta : \Gamma \to \operatorname{Aut} \mathscr{V}$ in terms of a section $V : \Delta \to \mathscr{V}$ and above we measured with the Λ invariant how δ depends on the choice of V.

Remarks. This discussion left open two existence questions:

(i) given a section V does there exist an appropriate action δ (i.e. a λ), and
(ii) given an action $\delta : \Gamma \to \operatorname{Aut} \mathscr{V}$ such that for all $v \in \mathscr{V}$, $g \in \Gamma$, $\operatorname{Ad}|_{\mathscr{A}} \delta_g(v)) = g(\operatorname{Ad}|_{\mathscr{A}} v)g^{-1}$ in $\operatorname{Aut} \mathscr{A}$ and $\delta_\Delta \subset \operatorname{Inn} \mathscr{V}$, is there a section $V : \Delta \to \mathscr{V}$ such that $\operatorname{Ad}|_{\mathscr{A}} \delta_g(V_d) = gdg^{-1} \in \operatorname{Aut} \mathscr{A}$ and $\delta_d = \operatorname{Ad} V_d$ for all $d \in \Delta$?

We first consider (i). The section V determines the cocycle μ with coefficients in the module $\mathscr{K} = Z(\mathscr{V}) \cap \mathscr{W}$, and the problem is to construct a compatible λ. There are two obstructions. First, the class $[\mu]$ of μ in $H^2(\Delta, \mathscr{K})$ must be Γ-invariant (this is the content of equation (ii) of Theorem 2.1). If so then we have a 1-cocycle $g \to [\mu^g][\mu]^{-1}$ with values in the coboundaries $B^2(\Delta, \mathscr{K})$ and our second obstruction is the one arising from the short exact sequence

$$0 \to \operatorname{Hom}(\Delta, \mathscr{K}) \to C^1(\Delta, \mathscr{K}) \xrightarrow{\partial} B^2(\Gamma/\Delta, \mathscr{K}) \to 0$$

when we try to lift this cocycle to $\phi : \Gamma \to C^1(\Delta, \mathscr{K})$. If we can lift it, taking

$$\lambda(g, d) = \phi(g)(g^{-1}dg) \quad g \in \Gamma, d \in \Delta \tag{2.7}$$

defines the required λ. The details of this argument are supplied in [22].

For (ii) we need to assume the compatibility of $\delta : \Delta \to \operatorname{Inn} \mathscr{V}$ and the action of Δ by implemented automorphisms of $\mathscr{A}$. This is covered by assuming that for each $d \in \Delta$ there is a $w \in \mathscr{V}$ such that $d = Ad|_{\mathscr{A}} w$ and $\delta_d = Ad\, w$ so that:

$$\delta_{Ad|_{\mathscr{A}}(w)}(v) = wvw^* \ . \tag{2.8}$$

Then the condition $Ad|_{\mathscr{A}} \delta_g(v) = g(Ad|_{\mathscr{A}} v)g^{-1}$ says that this w will satisfy $d = Ad|_{\mathscr{A}} w$ and

$$gdg^{-1} = g(Ad|_{\mathscr{A}} w)g^{-1} = Ad|_{\mathscr{A}} \delta_g(w) \ .$$

We can now take V to be any section for $Ad|_{\mathscr{A}} : \mathscr{V} \to \Delta$ (by this we mean $\delta_d = Ad|_{\mathscr{V}} V_d$).

Definition 2.4. *Let Σ be the set of all such actions δ satisfying* (2.8).

Theorem 2.5. *For all actions $\delta \in \Sigma$ and section V for $Ad|_{\mathscr{A}} : \mathscr{V} \to \Delta$, which define a pair $(\lambda, \mu) \in Z(\Gamma, \Delta, \mathscr{K})$ as in Theorem 2.1, the class $d(\delta) := [\lambda, \mu]$ of (λ, μ) in $\Lambda(\Gamma, \Delta, \mathscr{K})$ is independent of the choice of V, and there is a "Green twisting map"* [18] *that is, a section V for $Ad|_{\mathscr{A}} : \mathscr{V} \to \Delta$ such that $\delta_g(V_d) = V_{gdg^{-1}}$, $g \in \Gamma$, $d \in \Delta$, iff $[\lambda, \mu]$ is trivial.*

Proof. Only the last statement needs proof. If a Green twisting map exists, then $\lambda = 1 = \mu$, so $[\lambda, \mu] = 1$. Conversely, let $[\lambda, \mu] = 1$, i.e. there is a map $\psi : \Delta \to \mathscr{K}$ such that

$$\lambda(g, d) = \psi(gdg^{-1})^{-1} \delta_g(\psi(d)) \quad \text{and } \mu(d, k) = \psi(d)\psi(k)\psi(dk)^{-1} = V_d V_k V_{dk}^{-1}$$

for some section V. Define $\bar{V} : \Delta \to \mathscr{V}$ by $\bar{V}_d = \psi(d)^{-1} V_d$, then $\delta_d = Ad\ V_d = Ad\ \bar{V}_d$ for all $d \in \Delta$ and

$$\begin{aligned} \delta_g(\bar{V}_d) &= \delta_g(\psi(d))^{-1} \delta_g(V_d) = \delta_g(\psi(d))^{-1} \lambda(g, d)\, V_{gdg^{-1}} \\ &= \delta_g(\psi(d))^{-1} \psi(gdg^{-1}) \delta_g(\psi(d))\, V_{gdg^{-1}} \\ &= \psi(gdg^{-1})^{-1} V_{gdg^{-1}} = \bar{V}_{gdg^{-1}} \ , \end{aligned}$$

i.e. $\bar{V}$ is a Green twisting map. □

Remark. In the particular situation of Theorem 2.2, for actions of the form $\delta_g = Ad|_{\mathscr{V}} V'_g$, where $V' : \Gamma \to \mathscr{U}$ is a section with $V'_e = 1$, we have $\delta \in \Sigma$ and the pair $(\tilde{\lambda}, \tilde{\mu})$ defined by the triple V', Γ, Δ must be cohomologous to (λ, μ) for any other choice of section V.

3. The Three Cocycle

In this section we will produce a 3-cocycle $K : \Omega^3 \to \mathscr{W} \cap Z(\mathscr{V})$ from the framework of the preceding section, interpret it and work out conditions for nontriviality. Start with the section $\omega : \Omega \to \Gamma$ with corresponding cocycle $\sigma : \Omega^2 \to \Delta$ as in (2.2):

$$\sigma(g,h)\sigma(gh,f) = (\omega_g \sigma(h,f)\omega_g^{-1})\sigma(g,hf) \quad \text{for all } f, g, h \in \Omega$$

and take the image of this relation in Δ under a chosen section V for $\mathrm{Ad}|_{\mathscr{A}} : \mathscr{V} \to \Delta$ associated with an action $\delta \in \Sigma$ (and thus a pair $(\sigma, \lambda) \in Z(\Gamma, \Delta, \mathscr{K})$). So:

$$\begin{aligned}
V(\sigma(g,h)\sigma(gh,f)) &= \mu(\sigma(g,h), \sigma(gh,f))^{-1} V(\sigma(g,h)) V(\sigma(gh,f)) \\
&= V((\omega_g \sigma(h,f)\omega_g^{-1})\sigma(g,hf)) \\
&= \mu(\omega_g \sigma(h,f)\omega_g^{-1}, \sigma(g,hf))^{-1} V(\omega_g \sigma(h,f)\omega_g^{-1}) V(\sigma(g,hf)) \\
&= \mu(\omega_g \sigma(h,f)\omega_g^{-1}, \sigma(g,hf))^{-1} \\
&\quad \cdot \lambda(\omega_g, \sigma(h,f))^{-1} \delta_{\omega_g}(V(\sigma(h,f))) V(\sigma(g,hf)) \,,
\end{aligned}$$

so that on abbreviating the notation $V(g,h) := V(\sigma(g,h))$ we get

$$K(g,h,f) V(g,h) V(gh,f) = \delta_{\omega_g}(V(h,f)) V(g,hf) \tag{3.1}$$

for all $g, h, f \in \Omega$, where

$$K(g,h,f) := \mu(\sigma(g,h), \sigma(gh,f))^{-1} \mu(\omega_g \sigma(h,f)\omega_g^{-1}, \sigma(g,hf)) \lambda(\omega_g, \sigma(h,f)) \,. \tag{3.2}$$

Then by definition $K : \Omega^3 \to \mathscr{K} = Z(\mathscr{V}) \cap \mathscr{W}$.

Our aim is to prove that K is a 3-cocycle, but before doing so we make one comment on its definition. Observe that $\bar{\delta}_q(v) := \delta_{\omega_q}(v)$, $v \in \mathscr{K}$, $q \in \Omega$ defines an action of Ω on $\mathscr{K}$, because δ_{ω_q} preserves $Z(\mathscr{V}) \cap \mathscr{W} = \mathscr{K}$ and $\delta_{\omega_q \cdot \omega_r} = \delta_{\sigma(q,r)\omega_{qr}} = \delta_{\sigma(q,r)} \delta_{qr}$, however since $\sigma(q,r) \in \Delta$, $\delta \in \Sigma$, so $\delta_{\sigma(q,r)} \in \mathrm{Inn}\, \mathscr{V}$ and thus $\delta_{\sigma(q,r)}|_{Z(\mathscr{V}) \cap \mathscr{W}} = id$, so $\bar{\delta}_q \bar{\delta}_r = \bar{\delta}_{qr}$. Moreover, this action $\bar{\delta}$ is independent of ω, for if ω' is another section, there is a map $\psi : \Omega \to \Delta$ such that $\omega'_q = \psi(q)\omega(q)$ and so

$$\delta_{\omega'_q}|_{\mathscr{K}} = \delta_{\psi(q)} \delta_{\omega_q}|_{\mathscr{K}} = \delta_{\omega_q}|_{\mathscr{K}}$$

using $\delta_{\psi(q)}|_{\mathscr{K}} = id$.

Theorem 3.1. *With the preceding notation and $\Sigma \neq \emptyset$, we have for $\delta \in \Sigma$, and section V for $\mathrm{Ad}|_{\mathscr{A}} : \mathscr{V} \to \Delta$ and a section $\omega : \Omega \to \Gamma$, the expression* (3.2) *defines a* 3*-cocycle, $K \in Z^3(\Omega, \mathscr{K})$ that is for all $q, r, s, t \in \Omega$:*

$$\bar{\delta}_q(K(r,s,t)) K(q,rs,t) K(q,r,s) = K(qr,s,t) K(q,r,st) \,. \tag{3.3}$$

Proof. We use Maclane's argument ([13], p. 126, Lemma 8.4) and evaluate

$$L := \delta_{\omega_q}[\delta_{\omega_r}(V(s,t)) V(r,st)] V(q,rst)$$

in two different ways:

(i) $$\begin{aligned} L &= \delta_{\omega_q}[K(r,s,t)V(r,s)V(rs,t)]V(q,rst) \quad \text{by (3.1)} \\ &= \delta_{\omega_q}[K(r,s,t)\delta_{\omega_q}(V(r,s))\delta_{\omega_q}(V(rs,t))V(q,rst) \\ &= \delta_{\omega_q}(K(r,s,t))K(q,r,s)V(q,r)V(qr,s)V(q,rs)^{-1} \\ &\quad \cdot K(q,rs,t)V(q,rs)V(qrs,t)V(q,rst)^{-1}V(q,rst) \quad \text{by (3.1)} \\ &= \delta_{\omega_q}(K(r,s,t))K(q,r,s)K(q,rs,t)V(q,r)V(qr,s)V(qrs,t)\,. \end{aligned}$$

(ii) $$\begin{aligned} L &= \delta_{\omega_q\omega_r}(V(s,t))\delta_{\omega_q}(V(r,st))V(q,rst) \\ &= \delta_{\sigma(q,r)}(\delta_{\omega_{qr}}(V(s,t))K(q,r,st))V(q,r)V(qr,st) \quad \text{by (3.1)} \\ &= \delta_{\sigma(q,r)}[K(qr,s,t)V(q,r,s)V(qrs,t)V(qr,st)^{-1}]\cdot K(q,r,st)V(q,r)V(qr,st)\,. \end{aligned}$$

So relation (3.3) will hold iff

$$V(q,r)V(qr,s)V(qrs,t) = \delta_{\sigma(q,r)}[V(qr,s)V(qrs,t)V(qr,st)^{-1}]V(q,r)V(qr,st)\,,$$

where we made use of $\delta_{\sigma(q,r)}(\mathrm{K}(qr,s,t)) = K(qr,s,t)$ since $\sigma(q,r) \in \Delta$. That is

$$(\mathrm{Ad}\, V(q,r))(U(q,r,s,t)) = \delta_{\sigma(q,r)}(U(q,r,s,t))\,,$$

Where $U(q,r,s,t) = V(qr,s)V(qrs,t)V(qr,st)^{-1}$. Thus since it is a property of $\delta \in \Sigma$ and a section V for $Ad|_{\mathscr{A}}: \mathscr{V} \to \Delta$ that $\delta_d = Ad\, V_d$ for all $d \in \Delta$, we have

$$\delta_{\sigma(q,r)} = \mathrm{Ad}\, V(\sigma(q,r)) = \mathrm{Ad}\, V(q,r)\,,$$

which completes the proof of (3.3). □

From Eq. (3.1) we see that K depends on the choices of δ, V and ω. By analogy with the 3-cocycles in Maclane [13], we expect the class $[K]$ of K to be independent of V and ω.

Theorem 3.2. *Given the hypotheses of Theorem* 3.1,

(i) *the cohomology class of* $K \in Z^3(\Omega, \mathscr{K})$ *is independent of the choice of section* V *for* $Ad|_{\mathscr{A}}: \mathscr{V} \to \Delta$

(ii) *the cohomology class of* K *is independent of the choice of section* $\omega: \Omega \to \Gamma$.

Proof. (i) Using Theorem 2.5 we show that if $(\lambda,\mu) \sim (\lambda',\mu')$ in $Z(\Gamma,\Delta,\mathscr{K})$, then $K \sim K'$ for the corresponding 3-cocycles. Now $(\lambda,\mu) \sim (\lambda',\mu')$ means there is a map $\psi: \Delta \to \mathscr{K}$ such that

$$\begin{aligned} \lambda'(g,d) &= \psi(gdg^{-1})^{-1}\delta_g(\psi(d))\lambda(g,d) \quad \forall g \in \Gamma,\ d \in \Delta\,, \\ \mu'(d,k) &= \psi(d)\psi(k)\psi(dk)^{-1}\mu(d,k) \quad \forall d,k \in \Delta\,. \end{aligned}$$

So by (3.2):

$$\begin{aligned} K'(g,h,f) &= \mu'(\sigma(g,h),\sigma(gh,f))^{-1}\mu'(\omega_g\sigma(h,f)\omega_g^{-1},\sigma(g,hf))\lambda'(\omega_g,\sigma(h,f)) \\ &= \psi(\sigma(g,h))^{-1}\psi(\sigma(gh,f))^{-1}\psi(\sigma(g,h)\sigma(gh,f))\mu(\sigma(g,h),\sigma(gh,f))^{-1} \end{aligned}$$

$$\cdot\,\psi(\omega_g\sigma(h,f)\omega_g^{-1})\psi(\sigma(g,hf))\psi(\omega_g\sigma(h,f)\omega_g^{-1}\sigma(g,hf))^{-1}$$

$$\cdot\,\mu(\omega_g\sigma(h,f)\omega_g^{-1},\sigma(g,hf))\psi(\omega_g\sigma(h,f)\omega_g^{-1})^{-1}$$

$$\cdot\,\delta_{\omega_g}(\psi(\sigma(h,f))\lambda(\omega_g,\sigma(h,f))$$

$$=\psi(\sigma(g,h))^{-1}\psi(\sigma(gh,f))^{-1}\psi(\sigma(g,hf))$$

$$\cdot\,\delta_{\omega_g}(\psi(\sigma(h,f)))K(g,h,f)\quad \text{using (2.2)}$$

$$=\rho(g,h)^{-1}\rho(gh,f)^{-1}\rho(g,hf)\delta_{\omega_g}(\rho(h,f))K(g,h,f)\,,$$

where $\rho(g,h):=\psi(\sigma(g,h))$, and so $K\sim K'$.
(ii) The proof is complicated and unenlightening but routine. We refer the reader to [22] for details noting only that the method is to choose a second section $\omega':\Omega\to\Gamma$ with consequent 3-cocycle K' and map $n:\Omega\to\Gamma$ such that $\omega'_s=n_s\omega_s$ for each $s\in\Omega$ and prove that $K'=\delta(\xi)K$, where

$$\xi(r,s)=\lambda(\omega_r,\omega_r n_s\omega_r^{-1})\mu(n_r\omega_r n_s\omega_r^{-1}\sigma(r,s),n_{rs}^{-1})^{-1}\mu(n_{rs}^{-1},n_{rs})$$

$$\cdot\,\mu(n_r\omega_r n_s\omega_r^{-1},\sigma(r,s))^{-1}\mu(n_r,\omega_r n_s\omega_r^{-1})^{-1}\,.$$

□

Our interpretation of the 3-cocycle K will rest on the next theorem. Observe that if K is trivial, that is there is a 2-cochain $\rho:\Omega^2\to\mathscr{K}$ such that

$$K(g,h,f)=\rho(g,h)^{-1}\rho(gh,f)^{-1}\rho(g,hf)\bar{\delta}_g(\rho(h,f))\,,$$

then substitution in (3.1) produces:

$$(\rho(g,h)^{-1}V(g,h))(\rho(gh,f)^{-1}V(gh,f))=\delta_{\omega_g}(\rho(h,f)^{-1}V(h,f))(\rho(g,hf)^{-1}V(g,hf))\,,\tag{3.4}$$

that is

$$W(g,h):=\rho(g,h)^{-1}V(g,h)\tag{3.5}$$

is a non-commutative 2-cocycle.

Proposition 3.3. *Given the hypotheses of Theorem* 3.1 *and hence a* 3*-cocycle* K *associated with* $\delta\in\Sigma$, *assume that* K *is trivial. Then*

(i) *the extension defined by*

$\mathscr{E}:=\Omega\times\mathscr{V}$ *with multiplication* $(g_1,u_1)(g_2,u_2)=(g_1g_2,u_1\delta_{\omega_{g_1}}(u_2)W(g_1,g_2))\,,$

is a group (*where* W *is given by* (3.5)).
(ii) *The map* $v:\mathscr{E}\to\mathrm{Aut}\,\mathscr{A}$ *defined by*

$$v(g,u)(A)=u\omega_g(A)u^{-1},\quad A\in\mathscr{A},g\in\Omega,u\in\mathscr{V}$$

is an action.

Proof. The proof is standard, but we include it for completeness.
(i) Since $\omega_e = e$, $W(e,g) = \mathbb{1}$, the identity of $\mathscr{E}$ is $(e, \mathbb{1})$. The inverse of (g,u) is $(g^{-1}, W(g^{-1},g)^{-1}\delta_{\omega_{g^{-1}}}(u^{-1}))$ and so it is only necessary to verify associativity:

$$\begin{aligned}(g_1,u_1)[(g_2,u_2)(g_3,u_3)] &= (g_1,u_1)(g_2g_3, u_2\delta_{\omega_{g_2}}(u_3)W(g_2,g_3)) \\ &= (g_1g_2g_3, u_1\delta_{\omega_{g_1}}[u_2\delta_{\omega_{g_2}}(u_3)W(g_2,g_3)]W(g_1,g_2g_3)) \\ &= [(g_1,u_1)(g_2,u_2)](g_3,u_3) \\ &= (g_1g_2, u_1\delta_{\omega_{g_1}}(u_2)W(g_1,g_2))(g_3,u_3) \\ &= (g_1g_2g_3, u_1\delta_{\omega_{g_1}}(u_2)W(g_1,g_2)\delta_{\omega_{g_1g_2}}(u_3)W(g_1g_2,g_3)),\end{aligned}$$

that is, we need to prove that

$$\delta_{\omega_{g_1}\omega_{g_2}}(u_3)\delta_{\omega_{g_1}}(W(g_2,g_3))W(g_1,g_2g_3) = W(g_1,g_2)\delta_{\omega_{g_1g_2}}(u_3)W(g_1g_2,g_3). \quad (*)$$

Now

$$\begin{aligned}\delta_{\omega_{g_1}\omega_{g_2}} &= \delta_{\sigma(g_1,g_2)}\delta_{\omega_{g_1g_2}} = (\mathrm{Ad}|_{\mathscr{V}}\, V(g_1,g_2))\delta_{\omega_{g_1g_2}} \\ &= (\mathrm{Ad}|_{\mathscr{V}}\, W(g_1,g_2))\delta_{\omega_{g_1g_2}},\end{aligned}$$

and thus (also using (3.4)), both sides of $(*)$ are equal to

$$W(g_1,g_2)\delta_{\omega_{g_1g_2}}(u_3)W(g_1,g_2)^{-1}\delta_{\omega_{g_1}}(W(g_2,g_3))W(g_1,g_2g_3).$$

(ii) To see that v is in action:

$$\begin{aligned}v((g_1,u_1)(g_2,u_2))(A) &= v(g_1g_2, u_1\delta_{\omega_{g_1}}(u_2)W(g_1,g_2))(A) \\ &= (\mathrm{Ad}|_{\mathscr{A}}(u_1\delta_{\omega_{g_1}}(u_2)W(g_1,g_2)))\delta_{\omega_{g_1g_2}}(A) \\ &= (\mathrm{Ad}|_{\mathscr{A}}\,u_1)(Ad|_{\mathscr{A}}\,\delta_{\omega_{g_1}}(u_2))\cdot\sigma(g_1,g_2)\omega_{g_1g_2}(A) \\ &= (\mathrm{Ad}|_{\mathscr{A}}\,u_1)\omega_{g_1}((Ad|_{\mathscr{A}}\,u_2)\omega_{g_2})(A) \\ &= v(g_1,u_1)v(g_2,u_2)(A), \quad A\in\mathscr{A}, g_i\in\Omega,\ u_i\in\mathscr{V}. \quad \square\end{aligned}$$

As a result of this last proposition we conclude that if K is trivial there is an action v of an extension $\mathscr{E}$ of Ω by $\mathscr{V}$ on $\mathscr{A}$. Thus for a given action $\delta\in\Sigma$ (and section $\omega:\Omega\to\Gamma$) we interpret $[K]$ in the terminology of Sect. 1.1 as the obstruction to the construction of this extension $\mathscr{E}$ of the physical transformation group Ω by the group of implementers of gauge transformations $\mathscr{V}$.

Next we wish to find conditions for the nontriviality of K. Such triviality conditions are best expressed in terms of an exact sequence

$$H^2(\Gamma,\mathscr{K}) \xrightarrow{\zeta} \Lambda(\Gamma,\Delta,\mathscr{K}) \xrightarrow{\chi} H^3(\Omega,\mathscr{K})$$

whose existence is proven below. The homomorphism $\chi:\Lambda(\Gamma,\Delta,\mathscr{K})\to H^3(\Omega,\mathscr{K})$ is given by the formula $\bar{\chi}(\lambda,\mu) = K$, where K is given by Eq. (3.2) for a particular choice of ω. Then from Theorem 3.2, $\bar{\chi}$ respects cohomology classes so lifts to produce the desired homomorphism χ. The homomorphism $\zeta:H^2(\Gamma,\mathscr{K})\to\Lambda(\Gamma,\Delta,\mathscr{K})$ is given by

$$\bar{\zeta}(\tilde{\mu}) = (\lambda,\mu)\in Z(\Gamma,\Delta,\mathscr{K})$$

where for $\tilde{\mu}\in Z^2(\Gamma,\mathscr{K})$ we have

$$\lambda(g,d) := \tilde{\mu}(g,d)\tilde{\mu}(gdg^{-1},g)^{-1}$$

and $\mu = \tilde{\mu}|_{\Delta\times\Delta}$ (cf. remark below Theorem 2.2 for motivation). Then $\bar{\zeta}$ respects cohomology classes so lifts to the desired homomorphism ζ. The next result is the main theorem of the paper.

Theorem 3.4. *Assume that $\Sigma \neq \emptyset$. Then*

(1) *there is an exact sequence*

$$H^2(\Gamma, \mathcal{K}) \xrightarrow{\zeta} \Lambda(\Gamma, \Delta, \mathcal{K}) \xrightarrow{\chi} H^3(\Omega, \mathcal{K}),$$

where the homomorphisms ζ, χ are as defined above,
(2) *the pair (λ, μ) produces a trivial $[K]$ if and only if $[\lambda, \mu] \in \operatorname{Ran}\zeta$, or equivalently if and only if there is a pair $(\lambda', \mu') \sim (\lambda, \mu)$ for which μ' extends to Γ and*

$$\lambda'(g,d) = \mu'(g,d)\mu'(gdg^{-1},g)^{-1} \quad (g \in \Gamma, d \in \Delta).$$

Proof. Only (1) needs proof of which we adapt the arguments in [22].
Given that for a choice of section ω:

$$\bar{\chi}(\lambda,\mu)(g,h,f) = \mu(\sigma(g,h),\sigma(gh,f))^{-1}\cdot\mu(\omega_g\sigma(h,f)\omega_g^{-1},\sigma(g,hf))\cdot\lambda(\omega_g,\sigma(h,f))$$

we see immediately that

$$\bar{\chi}((\lambda_1,\mu_1)(\lambda_2,\mu_2)) = \bar{\chi}(\lambda_1\lambda_2,\mu_1\mu_2) = \bar{\chi}(\lambda_1,\mu_1)\bar{\chi}(\lambda_2,\mu_2),$$

so using Theorem 3.2, $\bar{\chi}$ lifts to a well-defined homomorphism $\chi: \Lambda(\Gamma,\Delta,\mathcal{K}) \to H^3(\Omega,\mathcal{K})$. For ζ, we first show that

$$\bar{\zeta}(\tilde{\mu}) \in Z(\Gamma,\Delta,\mathcal{K}) \quad \forall \tilde{\mu} \in H^2(\Gamma,\mathcal{K}).$$

Now condition (i) of Theorem 2.1 is clear since $\lambda(e,d) = \tilde{\mu}(e,d)\tilde{\mu}(d,e)^{-1} = 1 = \lambda(g,e)$.
For condition (ii) of Theorem 2.1:

$$\begin{aligned}
&\lambda(g,d)\lambda(g,k)\mu(gdg^{-1},gkg^{-1})\\
&\quad= \tilde{\mu}(g,d)\tilde{\mu}(gdg^{-1},g)^{-1}\tilde{\mu}(g,k)\tilde{\mu}(gkg^{-1},g)^{-1}\tilde{\mu}(gdg^{-1},gkg^{-1})\\
&\quad= \tilde{\mu}(g,d)\tilde{\mu}(gd,k)\tilde{\mu}(gdg^{-1},gk)^{-1}\tilde{\mu}(gdg^{-1},gk)\tilde{\mu}(gdkg^{-1},g)^{-1}\\
&\quad= \tilde{\mu}(g,d)\tilde{\mu}(gd,k)\tilde{\mu}(gdkg^{-1},g^1)^{-1}\\
&\quad= \delta_g(\tilde{\mu}(d,k))\tilde{\mu}(g,dk)\tilde{\mu}(gdkg^{-1},g)^{-1}\\
&\quad= \delta_g(\mu(d,k))\lambda(g,dk) \quad \text{for all } g \in \Gamma, \quad d,k \in \Delta,
\end{aligned}$$

using $\delta_d(w) = w\ \forall d \in \Delta,\ w \in \mathcal{K}$.
For condition (iii) of Theorem 2.1:

$$\begin{aligned}
\delta_g(\lambda(h,d))\lambda(g,hdh^{-1}) &= \delta_g(\tilde{\mu}(h,d)\tilde{\mu}(hdh^{-1},h)^{-1})\tilde{\mu}(g,hdh^{-1})\tilde{\mu}(ghdh^{-1}g^{-1},g)^{-1}\\
&= \tilde{\mu}(g,h)\tilde{\mu}(gh,d)\tilde{\mu}(g,hd)^{-1}\tilde{\mu}(g,hdh^{-1})^{-1}\tilde{\mu}(ghdh^{-1},h)^{-1}\\
&\qquad\times\tilde{\mu}(g,hd)\tilde{\mu}(g,hdh^{-1})\tilde{\mu}(ghdh^{-1}g^{-1},g)^{-1}\\
&= \tilde{\mu}(gh,d)\tilde{\mu}(ghdh^{-1}g^{-1},gh)^{-1}\tilde{\mu}(ghdh^{-1}g^{-1},gh)\\
&\qquad\times\tilde{\mu}(ghdh^{-1}g^{-1},g)^{-1}\tilde{\mu}(g,h)\tilde{\mu}(ghdh^{-1},h)^{-1}\\
&= \lambda(gh,d).
\end{aligned}$$

Then (iv) of Theorem 2.1 is true by definition of $\bar{\zeta}$ and so $\bar{\zeta}(\tilde{\mu}) \in Z(\Gamma,\Delta,\mathcal{K})$.

Next we show that $\bar{\zeta}$ respects cohomology: let $\tilde{\mu}_1 \sim \tilde{\mu}_2$, which means there is a $\psi : \Gamma \to \mathscr{K}$ such that $\tilde{\mu}_1 = (\partial\psi) \cdot \tilde{\mu}_2$. Hence we have $\mu_1 \sim \mu_2$, where μ_i are the restrictions onto $\Delta \times \Delta$. Moreover

$$\begin{aligned}\lambda_1(g,d) &= \tilde{\mu}_1(g,d)\tilde{\mu}_1(gdg^{-1},g)^{-1} \\ &= \psi_g \delta_g(\psi_d)\psi_{gd}^{-1} \cdot \psi_{gdg^{-1}}^{-1}\delta_{gdg^{-1}}(\psi_g^{-1})\psi_{gd}\lambda_2(g,d) \\ &= \delta_g(\psi_d)\psi_{gdg^{-1}}^{-1}\lambda_2(g,d)\,,\end{aligned}$$

that is, $(\lambda_1, \mu_1) = D\psi \cdot (\lambda_2, \mu_2)$. Thus $\bar{\zeta}$ lifts to a map $\zeta : H^2(\Gamma, \mathscr{K}) \to \Lambda(\Gamma, \Delta, \mathscr{K})$.

For the homomorphism property: $\bar{\zeta}(\tilde{\mu}_1 \cdot \tilde{\mu}_2) = (\lambda_1 \cdot \lambda_2, \mu_1 \cdot \mu_2)$ we use

$$\lambda_i(g,d) = \tilde{\mu}_i(g,d)\,\tilde{\mu}_i(gdg^{-1},g)^{-1}\,,$$

so that

$$\bar{\zeta}(\tilde{\mu}_1 \cdot \tilde{\mu}_2) = (\lambda_1, \mu_1) \cdot (\lambda_2, \mu_2) = \bar{\zeta}(\tilde{\mu}_1) \cdot \bar{\zeta}(\tilde{\mu}_2)\,.$$

Next we need to establish exactness of the given sequence. First we show $\chi \circ \zeta = 0$, that is, $\operatorname{ran}\zeta \subseteq \operatorname{Ker}\chi$. Let $(\lambda, \mu) \in \operatorname{ran}\bar{\zeta}$ so there is a $\mu \in Z^2(\Gamma, \mathscr{K})$ such that

$$\lambda(g,d) = \mu(g,d)\mu(gdg^{-1},g)^{-1}\,.$$

Then for all $g, h, f \in \Omega$ and a choice of section ω,

$$\begin{aligned}\bar{\chi}(\lambda,\mu)(g,h,f) &= K(g,h,f) \\ &= \mu(\sigma(g,h),\sigma(gh,f))^{-1}\mu(\omega_g\sigma(h,f)\omega_g^{-1},\sigma(g,hf))\lambda(\omega_g,\sigma(h,f)) \\ &= \mu(\omega_g\omega_h\omega_{gh}^{-1},\omega_{gh}\omega_f\omega_{ghf}^{-1})^{-1}\mu(\omega_g\omega_h\omega_f\omega_{hf}^{-1}\omega_g^{-1},\omega_g\omega_{hf}\omega_{ghf}^{-1}) \\ &\quad \cdot \mu(\omega_g,\omega_h\omega_f\omega_{hf}^{-1})\mu(\omega_g\omega_h\omega_f\omega_{hf}^{-1}\omega_g^{-1},\omega_g)^{-1} \\ &= \mu(\omega_g\omega_h\omega_{gh}^{-1},\omega_{gh}\omega_f\omega_{ghf}^{-1})^{-1}\mu(\omega_g,\omega_h\omega_f\omega_{hf}^{-1}) \\ &\quad \cdot \mu(\omega_g\omega_h\omega_f\omega_{hf}^{-1},\omega_{hf}\omega_{ghf}^{-1}) \\ &\quad \cdot \mu(\omega_g,\omega_{hf}\omega_{ghf}^{-1})^{-1} \quad \text{(using the 2-cocycle relation (2.1) on the} \\ &\qquad\qquad\qquad\qquad\qquad \text{2}^{\text{nd}} \text{ and } 4^{\text{th}} \text{ terms)} \\ &= \mu(\omega_g\omega_h\omega_{gh}^{-1},\omega_{gh})^{-1}\mu(\omega_g\omega_h,\omega_f\omega_{ghf}^{-1})^{-1}\mu(\omega_{gh},\omega_f\omega_{ghf}^{-1}) \\ &\quad \cdot \mu(\omega_g,\omega_h\omega_f\omega_{hf}^{-1}) \\ &\quad \cdot \mu(\omega_g,\omega_{hf}\omega_{ghf}^{-1})^{-1}\mu(\omega_g\omega_h\omega_f\omega_{hf}^{-1},\omega_{hf}) \\ &\quad \cdot \mu(\omega_g\omega_h\omega_f,\omega_{ghf}^{-1})\delta_{\omega_g}(\mu(\omega_{hf},\omega_{ghf}^{-1})^{-1}) \\ &\quad (\text{using } \mu(ab^{-1},bc) = \mu(ab^{-1},b)\mu(a,c)\delta_{ab^{-1}}(\mu(b,c))^{-1} \\ &\quad\ \text{on the } 1^{\text{st}} \text{ and } 3^{\text{rd}}\ \mu) \\ &= \mu(\omega_g\omega_h\omega_{gh}^{-1},\omega_{gh})^{-1}\mu(\omega_{gh},\omega_f\omega_{ghf}^{-1})\delta_{\omega_g}(\mu(\omega_h\omega_f\omega_{hf}^{-1},\omega_{hf})) \\ &\quad \cdot \mu(\omega_g,\omega_h\omega_f) \\ &\quad \cdot \mu(\omega_g,\omega_{hf})^{-1}\mu(\omega_g\omega_{hf},\omega_{ghf}^{-1})^{-1}\mu(\omega_g\omega_h,\omega_f)^{-1}\delta_{\omega_g\omega_h}(\mu(\omega_f,\omega_{ghf}^{-1}))\end{aligned}$$

where we used (2.1) on the 4^{th} and 6^{th} terms, then on the 5^{th} and 8^{th} terms, and using (2.1) on $\mu(\omega_g\omega_h,\omega_f\omega_{ghf}^{-1})^{-1}\mu(\omega_g\omega_h\omega_f,\omega_{ghf}^{-1})$.

Now apply (2.1) to the 4th and 7th μ's, regroup and set

$$\beta(g,h) := \mu(\omega_g\omega_h\omega_{gh}^{-1}, \omega_{gh})\mu(\omega_g, \omega_h)^{-1}$$

to get

$$\begin{aligned} K(g,h,f) &= \beta(g,h)^{-1}\delta_{\omega_g}(\beta(h,f))\mu(\omega_g,\omega_{hf})^{-1}\mu(\omega_{gh},\omega_f\omega_{ghf}^{-1}) \\ &\quad\cdot\mu(\omega_g\omega_{hf},\omega_{ghf}^{-1})^{-1}\delta_{\omega_{gh}}(\mu(\omega_f,\omega_{ghf}^{-1})) \quad (\text{recall } \delta_{\omega_g\omega_h} = \delta_{\omega_{gh}} \text{ on } \mathscr{K}) \\ &= \beta(g,h)^{-1}\delta_{\omega_g}(\beta(h,f))\mu(\omega_g,\omega_{hf})^{-1}\mu(\omega_{gh},\omega_f)\mu(\omega_{gh}\omega_f,\omega_{ghf}^{-1}) \\ &\quad\cdot\mu(\omega_g\omega_{hf},\omega_{ghf}^{-1})^{-1} \quad (\text{using (2.1) on the } 2^{\text{nd}} \text{ and } 4^{\text{th}}\ \mu\text{'s}) . \end{aligned}$$

Now $\mu(ab^{-1},b) = \mu(ab^{-1},bc)\mu(a,c)^{-1}\delta_{ab^{-1}}(\mu(b,c))$. So with $a = \omega_g\omega_{hf}$, $b = \omega_{ghf}$, $c = \omega_{ghf}^{-1}$ we see that

$$\mu(\omega_g\omega_{hf},\omega_{ghf}^{-1})^{-1} = \mu(\omega_g\omega_{hf}\omega_{ghf}^{-1},\omega_{ghf})\mu(\omega_{ghf},\omega_{ghf}^{-1})^{-1} ,$$

and when $a = \omega_{gh}\omega_f$ instead, we have

$$\mu(\omega_{gh}\omega_f,\omega_{ghf}^{-1}) = \mu(\omega_{gh}\omega_f\omega_{ghf}^{-1},\omega_{ghf})^{-1}\mu(\omega_{ghf},\omega_{ghf}^{-1}) .$$

So on substituting these into the expression for K

$$K(g,h,f) = \beta(g,h)^{-1}\delta_{\omega_g}(\beta(h,f))\beta(g,hf)\beta(gh,f)^{-1} = (\partial\beta)(g,h,f) ,$$

that is, $\chi\circ\zeta = 0$.

For the more difficult part of proving that if $[\lambda,\mu]\in\operatorname{Ker}\chi$, then $[\lambda,\mu]\in\operatorname{ran}\zeta$, we first need the following lemma, stating that K is the boundary of a cochain ρ over Γ (but not necessarily over Ω).

Lemma 3.5. *For each* $(\lambda,\mu)\in Z(\Gamma,,\Delta,\mathscr{K})$, *define* $\rho\in C^2(\Gamma,\mathscr{K})$ *by*:

$$\rho(g,h) := \lambda(\omega_g, h\omega_h^{-1})\mu(g\omega_g^{-1},\omega_g h\omega_h^{-1}\omega_g^{-1})\mu(gh\omega_h^{-1}\omega_g^{-1},\sigma(g,h)) \quad \forall g,h\in\Gamma$$

(*where we use notation* $\omega_g = \omega_{g\Delta}$ *and* $\sigma(g,h) = \sigma(g\Delta,d\Delta)$). *Then*

$$\begin{aligned} K(g\Delta,h\Delta,f\Delta) &= \rho(g,h)^{-1}\rho(gh,f)^{-1}\rho(gh,f)\delta_{\omega_g}(\rho(h,f)) \\ &= (\partial\rho)(g,h,f) \quad \forall g,h,f\in\Gamma . \end{aligned}$$

Proof. Note that on $\mathscr{K}$, $\delta_{\omega_{g_\Delta}} = \delta_g$, so

$$\begin{aligned} (\partial\rho)(g,h,f) &= \delta_g[\lambda(\omega_h,f\omega_f^{-1})\mu(h\omega_h^{-1},\omega_h f\omega_f^{-1}\omega_h^{-1})\mu(hf\omega_f^{-1}\omega_h^{-1},\sigma(h,f))] \\ &\quad\cdot\lambda(\omega_g,hf\omega_{hf}^{-1})\mu(g\omega_g^{-1},\omega_g hf\omega_{hf}^{-1}\omega_g^{-1})\mu(ghf\omega_{hf}^{-1}\omega_g^{-1},\sigma(g,hf)) \\ &\quad\cdot\lambda(\omega_g,h\omega_h^{-1})^{-1}\mu(g\omega_g^{-1},\omega_g h\omega_h^{-1}\omega_g^{-1})^{-1}\mu(gh\omega_h^{-1}\omega_g^{-1},\sigma(g,h))^{-1} \\ &\quad\cdot\lambda(\omega_{gh},f\omega_f^{-1})^{-1}\mu(gh\omega_{gh}^{-1},\omega_{gh}f\omega_f^{-1}\omega_{gh}^{-1})^{-1} \\ &\quad\cdot\mu(ghf\omega_f^{-1}\omega_{gh}^{-1},\sigma(gh,f))^{-1} . \end{aligned} \tag{3.6}$$

Now note that by the Theorem 2.1 (iii), the first λ-term is

$$\begin{aligned} \delta_g[\lambda(\omega_h,f\omega_f^{-1})] &= \lambda(\omega_g\omega_h,f\omega_f^{-1})\lambda(\omega_g,\omega_h f\omega_f^{-1}\omega_h^{-1})^{-1} \\ &= \lambda(\omega_{gh},f\omega_f^{-1})\lambda(\sigma(g,h),\omega_{gh}f\omega_f^{-1}\omega_{gh}^{-1}) \\ &\quad\cdot\lambda(\omega_g,\omega_h f\omega_f^{-1}\omega_h^{-1})^{-1} \\ &= \lambda(\omega_{gh},f\omega_f^{-1})\lambda(\omega_g,\omega_h f\omega_f^{-1}\omega_h^{-1})^{-1}\mu(\sigma(g,h),\omega_{gh}f\omega_f^{-1}\omega_{gh}^{-1}) \\ &\quad\cdot\mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1},\sigma(g,h))^{-1} , \end{aligned} \tag{3.7}$$

and by Theorem 2.1 (ii), the next two λ-terms are

$$\begin{aligned}
&\lambda(\omega_g, hf\omega_{hf}^{-1})\lambda(\omega_g, h\omega_h^{-1})^{-1}\\
&\quad= \lambda(\omega_g, \omega_h f\omega_{hf}^{-1})\mu(\omega_g h\omega_h^{-1}\omega_g^{-1}, \omega_g\omega_h f\omega_{hf}^{-1}\omega_g^{-1})\delta_g(\mu(h\omega_h^{-1}, \omega_h f\omega_{hf}^{-1})^{-1})\\
&\quad= \lambda(\omega_g, \omega_h f\omega_f^{-1}\omega_h^{-1})\lambda(\omega_g, \omega_h\omega_f\omega_{hf}^{-1})\mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \omega_g\sigma(h,f)\omega_g^{-1})\\
&\qquad\cdot\delta_g[\mu(\omega_h f\omega_f^{-1}\omega_h^{-1}, \sigma(h,f))^{-1}\mu(h\omega_h^{-1}, \omega_h f\omega_{hf}^{-1})^{-1}]\\
&\qquad\cdot\mu(\omega_g h\omega_h^{-1}\omega_g^{-1}, \omega_g\omega_h f\omega_{hf}^{-1}\omega_g^{-1})\,. \qquad (3.8)
\end{aligned}$$

Thus on collecting all λ-terms in (3.6), substituting and cancelling we get:

$$\lambda(\omega_g, \sigma(h,f))\times \mu\text{-terms in (3.7)}\times \mu\text{-terms in (3.8)}\,.$$

Now the $\delta_g\cdot\mu$-terms in (3.6) cancel with the $\delta_g\mu^{-1}$-terms in (3.8) because

$$\begin{aligned}
&\mu(h\omega_h^{-1}, \omega_h f\omega_f^{-1}\omega_h^{-1})\mu(hf\omega_f^{-1}\omega_h^{-1}, \sigma(h,f))\\
&\quad= \mu(\omega_h f\omega_f^{-1}\omega_h^{-1}, \sigma(h,f))\mu(h\omega_h^{-1}, \omega_h f\omega_{hf}^{-1})\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&(\partial\rho)(g,h,f)\\
&\quad= \lambda(\omega_g, \sigma(h,f))\mu(\sigma(g,h), \omega_{gh} f\omega_f^{-1}\omega_{gh}^{-1})\\
&\qquad\cdot\mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \sigma(g,h))^{-1}\mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \omega_g\sigma(h,f)\omega_g^{-1})\\
&\qquad\cdot\mu(\omega_g h\omega_h^{-1}\omega_g^{-1}, \omega_g\omega_h f\omega_{hf}^{-1}\omega_g^{-1})\mu(g\omega_g^{-1}, \omega_g hf\omega_{hf}^{-1}\omega_g^{-1})\\
&\qquad\cdot\mu(ghf\omega_{hf}^{-1}\omega_g^{-1}, \sigma(g,hf))\mu(g\omega_g^{-1}, \omega_g h\omega_h^{-1}\omega_g^{-1})^{-1}\\
&\qquad\cdot\mu(gh\omega_h^{-1}\omega_g^{-1}, \sigma(g,h))^{-1}\mu(gh\omega_{gh}^{-1}, \omega_{gh} f\omega_f^{-1}\omega_{gh}^{-1})^{-1}\\
&\qquad\cdot\mu(ghf\omega_f^{-1}\omega_{gh}^{-1}, \sigma(gh,f))^{-1}\\
&\quad= \lambda(\omega_g, \sigma(h,f))\cdot\mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \sigma(g,h))^{-1}\\
&\qquad\cdot\mu(\sigma(g,h), \omega_{gh} f\omega_f^{-1}\omega_{gh}^{-1})\mu(gh\omega_h^{-1}\omega_g^{-1}, \sigma(g,h))^{-1}\mu(gh\omega_{gh}^{-1}, \omega_{gh} f\omega_f^{-1}\omega_{gh}^{-1})^{-1}\\
&\qquad\cdot\mu(\omega_g h\omega_h^{-1}\omega_g^{-1}, \omega_g\omega_h f\omega_{hf}^{-1}\omega_g^{-1})\mu(g\omega_g^{-1}, \omega_g hf\omega_{hf}^{-1}\omega_g^{-1})\\
&\qquad\cdot\mu(g\omega_g^{-1}, \omega_g h\omega_h^{-1}\omega_g^{-1})^{-1}\\
&\qquad\cdot\mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \omega_g\sigma(h,f)\omega_g^{-1})\mu(ghf\omega_{hf}^{-1}\omega_g^{-1}, \sigma(g,hf))\\
&\qquad\cdot\mu(ghf\omega_f^{-1}\omega_{gh}^{-1}, \sigma(gh,f)^{-1}\\
&\quad= \lambda(\omega_g, \sigma(h,f))\cdot\mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \sigma(g,h))^{-1}\\
&\qquad\cdot\{\mu(gh\omega_h^{-1}\omega_g^{-1}, \omega_g\omega_h f\omega_f^{-1}\omega_{gh}^{-1})\}^{-1}\{\mu(gh\omega_h^{-1}\omega_g^{-1}, \omega_g\omega_h f\omega_{hf}^{-1}\omega_g^{-1})\}\\
&\qquad\cdot\mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \omega_g\sigma(h,f)\omega_g^{-1})\mu(ghf\omega_{hf}^{-1}\omega_g^{-1}, \sigma(g,hf))\\
&\qquad\cdot\mu(ghf\omega_f^{-1}\omega_{gh}^{-1}, \sigma(gh,f))^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= \lambda(\omega_g, \sigma(h,f)) \cdot \mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \sigma(g,h))^{-1} \\
&\quad \cdot \mu(gh\omega_h^{-1}\omega_g^{-1}, \omega_g\omega_h f\omega_f^{-1}\omega_{gh}^{-1})^{-1}\mu(gh\omega_h^{-1}\omega_g^{-1}, \omega_g\omega_h f\omega_{hf}^{-1}\omega_g^{-1}) \\
&\quad \cdot \mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \omega_g\sigma(h,f)\omega_g^{-1}) \\
&\quad \cdot [\mu(ghf\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \omega_g\sigma(h,f)\omega_g^{-1})^{-1} \\
&\quad \cdot \mu(\omega_g\sigma(h,f)\omega_g^{-1}, \sigma(g,hf)) \cdot \mu(ghf\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \omega_g\sigma(h,f)\omega_g^{-1}\sigma(g,hf))] \\
&\qquad\qquad \cdot \mu(gh\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \sigma(g,h)) \\
&\quad \cdot \mu(\sigma(g,h), \sigma(gh,f))^{-1}\mu(ghf\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \sigma(g,h)\sigma\sigma(gh,f))^{-1}\,,
\end{aligned}
$$

so using (2.2) we obtain the cancellation. On regrouping:

$$
\begin{aligned}
&(\partial\rho)(g,h,f) \\
&\quad = \lambda(\omega_g, \sigma(h,f))\mu(\sigma(g,h), \sigma(gh,f))^{-1}\mu(\omega_g\sigma(h,f)\omega_g^{-1}, \sigma(g,hf)) \\
&\quad \cdot \mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \sigma(g,h))^{-1}\mu(gh\omega_h^{-1}\omega_g^{-1}, \omega_g\omega_h f\omega_f^{-1}\omega_{gh}^{-1})^{-1} \\
&\quad \cdot \mu(gh\omega_h^{-1}\omega_g^{-1}, \omega_g\omega_h f\omega_{hf}^{-1}\omega_g^{-1})\mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \omega_g\sigma(h,f)\omega_g^{-1}) \\
&\quad \cdot \mu(ghf\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \omega_g\sigma(h,f)\omega_g^{-1})^{-1}\mu(ghf\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \sigma(g,h)) \\
&= K(g,h,f) \cdot \mu(\omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \sigma(g,h))^{-1} \\
&\quad \cdot \mu(gh\omega_h^{-1}\omega_g^{-1}, \omega_g\omega_h f\omega_f^{-1}\omega_{gh}^{-1})^{-1}\mu(gh\omega_h^{-1}\omega_g^{-1}, \omega_g\omega_h f\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}) \\
&\quad \cdot \mu(ghf\omega_f^{-1}\omega_h^{-1}\omega_g^{-1}, \sigma(g,h)) \\
&= K(g,h,f)\,.
\end{aligned}
$$

□

The final step is to show, using this lemma, that if $[K] = \chi[\lambda,\mu] = 0$, then there is a $\theta \in Z^2(\Gamma, \mathscr{K})$ such that $(\lambda,\mu) = \bar{\zeta}(\theta)$. Equivalently $\mu = \theta|_{\Delta\times\Delta}$ and $\lambda(g,d) = \theta(g,d)\cdot\theta(gdg^{-1},g)^{-1}$, so that $\ker\chi \subseteq \operatorname{ran}\zeta$. Now since $[K] = 0$, there is some normalised $\tau \in C^2(\Omega, \mathscr{K})$ such that $K = \partial\tau$. Define $\tilde{\tau} \in C^2(\Gamma, \mathscr{K})$ by $\tilde{\tau}(g,h) := \tau(g\Delta, h\Delta)$. Then $\partial(\rho\cdot\tilde{\tau}^{-1}) = K\cdot K^{-1} = 1$, and so we set $\theta := \rho\cdot\tilde{\tau}^{-1} \in Z^2(\Gamma, \mathscr{K})$. Trivially

$$\theta|_{\Delta\times\Delta} = (\rho\cdot\tilde{\tau}^{-1})|_{\Delta\times\Delta} = \rho|_{\Delta\times\Delta} = \mu\,.$$

Moreover since $\tilde{\tau}(g,h) = 1$ if either g or $h \in \Delta$, we see that

$$
\begin{aligned}
\theta(g,d)\cdot\theta(gdg^{-1},g)^{-1} &= \rho(g,d)\rho(gdg^{-1},g)^{-1} \\
&= \lambda(\omega_g, d)\mu(g\omega_g^{-1}, \omega_g d\omega_g^{-1})\mu(gd\omega_g^{-1}, \sigma(g,d)) \\
&\quad \cdot \lambda(e,g)^{-1}\mu(gdg^{-1}, g\omega_g^{-1})\mu(gd\omega_g^{-1}, \sigma(gdg^{-1},g))^{-1} \\
&= \lambda(\omega_g, d)\mu(g\omega_g^{-1}, \omega_g d\omega_g^{-1})\mu(gdg^{-1}, g\omega_g^{-1})^{-1} \\
&= \lambda(\omega_g, d)\lambda(g\omega_g^{-1}, \omega_g d\omega_g^{-1}) \quad \text{by Theorem 2.1(iv)} \\
&= \lambda(g,d) \quad \text{by Theorem 2.1 (iii)}\,.
\end{aligned}
$$

Thus $K \in \operatorname{ran}\bar{\zeta}$. □

Summary: The circumstances in which this framework will produce a nontrivial 3-cocycle K are that $(\lambda, \mu) \notin \operatorname{ran} \bar{\zeta}$ and this will be the case when either:

(i) μ does not extend to Γ over $\mathscr{K}$ (but remember we must still have that $\mu \sim \mu^g \, \forall g \in \Gamma$ or else K will not exist), or

(ii) μ does extend to Γ over $\mathscr{K}$, but there is no extension $\tilde{\mu}$ which satisfies

$$\lambda(g,d) \sim \tilde{\mu}(g,d)\tilde{\mu}(gdg^{-1},g)^{-1} .$$

Note finally that in 3.4 we have used only part of the long exact sequence contained in [22].

4. Examples

In the absence of constructive approaches to gauge field theories which enable us to exhibit representations of gauge groups on Hilbert spaces which also carry a representation of the "field algebra" we turn to more conventional C^*-algebras for examples of the three cocycles of the previous section. The examples we construct here are for continuous trace C^* algebras which bear some formal similarities to the algebras one would expect to arise in quantum field theory. To keep the discussion brief we assume familiarity with [18, 23].

Example 4.1. For the first example consider the commutative diamond:

$$\begin{tikzcd} & E \arrow[dl, "q^*p"'] \arrow[dr, "p^*q"] & \\ S^3 \arrow[dr, "q"'] & & S^3 \arrow[dl, "p"] \\ & S^2 & \end{tikzcd}$$

of principal $\mathbf{T}$-bundles, in which both p and q are the Hopf vibration. Then there is a continuous trace algebra A with spectrum S^3 and an action $\alpha: R \to \operatorname{Aut} A$ inducing the given action of $T = R/Z$ on $S^3 = \hat{A}$ and such that the spectrum $(A \times_\alpha R)\hat{}$ is isomorphic as a principal circle bundle to $q: S^3 \to S^2 = \hat{A}/R$. (Noting that $H^4(S^2, Z) = 0$, this is a corollary of [21, Theorem 3.1]. It also follows by applying [23, 4.12] to any stable algebra whose Dixmier–Douady class $\delta(A)$ satisfies $p_!(\delta(A)) = [q]$, where $p_!$ is the map in the Gysin sequence

$$0 = H^3(S^2, Z) \xrightarrow{p^*} H^3(S^3, Z) \xrightarrow{p_!} H^2(S^2, Z) \xrightarrow{[p]\cup} H^4(S^2, Z) = 0$$

with $[p]\cup$ denoting the map which takes $[q]$ to the cup product $[p]\cup[q]$. Exactness of this sequence also tells us that there is just one class $\delta(A)$ mapping onto $[q]$ under $p_!$.)

Now we apply Theorem 2.2 of [23] to the above system which then identifies the diamond of bundles with the spectra of the algebras as indicated:

$$\begin{tikzcd}
 & (Z \times_\alpha A)\hat{} \arrow[dl, "\mathrm{Ind}"'] \arrow[dr, "\mathrm{Res}"] & \\
(R \times_\alpha A)\hat{} \arrow[dr, "q"'] & & A\hat{} \arrow[dl, "p"] \\
 & \hat{A}/R &
\end{tikzcd}$$

Here Ind denotes the induction of representations and Res takes a covariant pair (π, U) to π. Because $H^2(\hat{A}, Z) = H^2(S^3, Z) = 0$, the fibration Res is trivial, and $\alpha|_Z$ is inner by [18]. Hence we are essentially in the situation of Sect. 2 and 3. Now we know from [20] that q is the trivial fibration if and only if α is given by a Green twisting map on N, and hence if and only if the pair (λ, μ) (which is well defined because $\alpha|_Z$ consists of inner automorphisms) is also trivial. But we have constructed q to be non-trivial forcing the class of the pair (λ, μ) to be non-zero in $\Lambda(R, Z;\, C(X, T))$. From [23, Theorem 4.1] we have $H^2(R, C(X, T)) = 0 = H^3(R, C(X, T))$ and hence the exact sequence

$$H^2(R, C(X,T)) \to \Lambda(R, Z;\, C(X,T)) \xrightarrow{\Delta} H^3(T, C(X,T)) \to H^3(R, C(X,T))$$

implies that $[K] = \Delta([\lambda, \mu])$ is non-zero in $H^3(T, C(X, T))$.

Remark. Note that the two cocycle μ is zero as $\alpha|_Z$ is inner so that the class we have constructed is in the kernel of the map $\Lambda(R, Z;\, C(X, T)) \to H^2(Z, C(X, T))$-indeed the second cohomology of Z with any coefficient group is trivial.

Example 4.2. Let Γ be a discrete group, Δ a central subgroup and suppose we are given a continuous trace C^*-algebra B with $\hat{B} = X$ and a cocycle

$$\sigma \in Z^2(\Delta, UZM(B)) = Z^2(\Delta, C(X, T))\,.$$

We say that $\alpha\colon \Delta \to \mathrm{Inn}(B)$ has Mackey obstruction $\sigma = c(\alpha)$ if there is a map $u\colon \Delta \to UM(B)$ ($UM(B)$ denotes the unitaries in the multiplier algebra of B) such that $\alpha_d = \mathrm{Ad}\, u_d$ and $u_c u_d = \sigma(c, d) u_{cd}$. Let K denote the C^*-algebra of compact operators on a separable Hilbert space. If B is stable there is always such an action, for by [9, Theorem] we can find such an action β of Δ on $C_0(X, K)$ and take $\alpha = 1 \otimes \beta$ on

$$B \otimes_{C(X)} C_0(X, K) \cong B \otimes K \cong B\,.$$

Now let

$$A = \mathrm{Ind}_\Delta^\Gamma(B, \alpha) := \{ f \in \ell^\infty(\Gamma, B) \mid f(\gamma d) = \alpha_d^{-1}(f(\gamma))$$
$$\text{for all } d \in \Delta \text{ and the map } \gamma\Delta \to \|f(\gamma)\| \text{ vanishes at } \infty \text{ on } \Gamma/\Delta\}\,.$$

Because α fixes $X = \hat{B}$, A is a continuous trace C^*-algebra with spectrum $\Gamma/\Delta \times X$ and a natural action τ of Γ by left translation: $\tau_\gamma(f)(\chi) = f(\gamma^{-1}\chi)$. The homeomorphism which gives $\hat{A}$ is induced by the map $M\colon \Gamma \times \hat{B} \to \hat{A}$ defined by $M(\gamma, \pi)(f) = \pi(f(\gamma))$ [19].

Lemma 4.1. *The automorphism $\tau|_\Delta$ is inner and the Mackey obstruction $c(\tau|_\Delta)$ is given by*

$$c(\tau|_\Delta)(c, d)(\gamma\Delta, \chi) = (1 \otimes \sigma)(c, d)(\gamma\Delta, \chi) = \sigma(c, d)(\chi)\,.$$

Proof. If $d \in \Delta$, then because Δ is central

$$\tau_d(f)(\gamma) = f(d^{-1}\gamma) = f(\gamma d^{-1}) = \alpha_d(f(\gamma)) .$$

We define $v_d \in M(\mathrm{Ind}_\Delta^\Gamma B)$ by choosing a section $c: \Gamma/\Delta \to \Gamma$ (recall that Γ is discrete and so there is no topological problem in choosing a cross section) and setting $v_d(\gamma) = \alpha^{-1}_{c(\gamma\Delta)^{-1}\gamma}(u_d)$. It is easy to check that $v_d \in M(\mathrm{Ind}_\Delta^\Gamma B)$ and it is unitary because u_d is. Further, for any $c, d \in \Delta$ we have

$$\alpha_c(u_d) = u_c u_d u_c^* = \sigma(c,d)\sigma(d,c)^{-1} u_d u_c u_c^* = \sigma(c,d)\sigma(d,c)^{-1} u_d ,$$

so that $\alpha_c(u_d)$ implements $\alpha_d = Ad\, u_d$. Thus for each γ, $v_d(\gamma)$ implements α_d and

$$\mathrm{Ad} v_d(f)(\gamma) = v_d(\gamma) f(\gamma) v_d(\gamma)^* = \alpha_d(f(\gamma)) = \tau_d(f)(\gamma) .$$

Since

$$v_c v_d(\gamma) = \alpha^{-1}_{c(\gamma\Delta)^{-1}\gamma}(u_c u_d) = \alpha^{-1}_{c(\gamma\Delta)^{-1}\gamma}(\sigma(c,d) u_{cd})$$

$$= \sigma(c,d)\alpha^{-1}_{c(\gamma\Delta)^{-1}\gamma}(u_{cd}) = \sigma(c,d) v_{cd}(\gamma) .$$

If we identify $(\mathrm{Ind}_\Delta^\Gamma B)\hat{}$ with $\Gamma/\Delta \times X$ so that the unitary group of the centre of the multiplier algebra of B is just $C(\Gamma/\Delta \times X, T)$ then this says precisely that $v_c v_d = (1 \otimes \sigma(c,d)) v_{cd}$.

Now we may apply the analysis of Sects. 2 and 3 with $\mathscr{V}$ equal to the unitaries in the multiplier algebra of A, $\mathscr{W}$ the unitaries in the centre of the multiplier algebra of A and the section V_d given by v_d. The lemma implies that the cocycle μ of Theorem 2.1 is given by $\mu = 1 \otimes \sigma$.

Using Theorem 3.4 (namely the exactness of the long exact sequence at Δ) the 3-cocycle K which is the image of the pair (λ, μ) in $H^3(\Gamma/\Delta, \mathscr{W})$ vanishes if and only if (λ, μ) is the restriction of some $\nu \in H^2(\Gamma, \mathscr{W})$. In particular, only if μ extends to a cocycle $\nu: \Gamma \times \Gamma \leftarrow \mathscr{W}$.

Now if $\nu: \Gamma \times \Gamma \to C(\Gamma/\Delta \times X, T)$ is a cocycle extending $\mu = 1 \otimes \sigma$ and Γ/Δ is finite then

$$\rho(\gamma, \chi)(x) = \prod_{\gamma\Delta \in \Gamma/\Delta} \nu(\gamma, \chi)(\gamma\Delta, x)$$

will be a cocycle bounding $\mu^{|\Gamma/\Delta|}$. (Notice that for example $\rho|_{\{\Delta\} \times X}$ is not a cocycle because the action of Γ enters in the last variable.) On the other hand if we seek a cocycle $\nu: \Gamma \times \Gamma \to C(X, T)$ extending μ, then this amounts to making a different choice for $\mathscr{V}$ and hence a different choice for $\mathscr{W}$ (namely $C(X,T)$). Thus for example, with $\Gamma = Z \times Z$, $\Delta = 3Z \times Z$, $X = T$ and $\mu((3m_1, n_1), (3m_2, n_2)) = z^{n_1 m_2}$, then μ cannot possibly extend to a cocycle on $\Gamma \times \Gamma$. For if so the bicharacter $\tilde{\nu}$ obtained by antisymmetrising ν would satisfy

$$\tilde{\nu}((0,1),(3,0))(z) = \tilde{\nu}((0,1),(1,0))(z)^3 ,$$

which would only be consistent with $\tilde{\mu}((0,1),(3,0)) = z^2$ if we had a continuous cube root for the function $z \to z^2$ on T.

Remark. Note that this illustrates the sensitivity of the non-triviality question to the choice of coefficient group.

5. The 3-Cocycle a Lie Algebra Context

The literature on gauge group chomology actually mainly discusses Lie algebra cohomology (see [31]). It seems useful therefore to consider what our constructions yield in that context. Hence assume that the groups $\Delta, \Gamma, \Omega, \mathscr{V}, \mathscr{W}$ are all Lie groups with Lie algebras $\boldsymbol{\Delta}, \boldsymbol{\Gamma}, \boldsymbol{\Omega}, \mathscr{V}, \mathscr{W}$ and that the homomorphisms considered previously are locally C^∞ and hence we obtain the diagram below also for the Lie algebras. Also assume the sections $V: \Delta \to \mathscr{V}$ and $\omega: \Omega \to \Gamma$ are locally C^∞, and so define linear sections $\boldsymbol{v}: \boldsymbol{\Delta} \to \mathscr{V}$, $\boldsymbol{v}(0) = 0$, and $\boldsymbol{\omega}: \boldsymbol{\Omega} \to \boldsymbol{\Gamma}$, $\boldsymbol{\omega}(0) = 0$. We now have the diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & \mathscr{W} \arrow[d] & & & \\
 & \mathscr{V} \arrow[d] & & & \\
0 \arrow[r] & \boldsymbol{\Delta} \arrow[u, bend left, "\boldsymbol{v}"] \arrow[r] \arrow[d] & \boldsymbol{\Gamma} \arrow[r] & \boldsymbol{\Omega} \arrow[l, bend right, "\omega"'] \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}
$$

The relations $V_d V_k = \mu(d,k) V_{dk}$ and $\omega_g \omega_h = \sigma(g,h)\omega_{gh}$ have the Lie algebra versions:

$$[\boldsymbol{v}_d, \boldsymbol{v}_k] = \boldsymbol{v}_{[d,k]} + \boldsymbol{\mu}(d,k) \quad \text{for all } d,k \in \boldsymbol{\Delta}\ ,$$

$$\text{and} \quad [\boldsymbol{\omega}_g, \boldsymbol{\omega}_h] = \boldsymbol{\omega}_{[g,h]} + \boldsymbol{\sigma}(g,h) \quad \text{for all } g,h \in \boldsymbol{\Omega}\ ,$$

where $\boldsymbol{\mu}: \boldsymbol{\Delta} \times \boldsymbol{\Delta} \to \mathscr{W}$ and $\boldsymbol{\sigma}: \boldsymbol{\Omega} \times \boldsymbol{\Omega} \to \boldsymbol{\Delta}$ are Lie algebra 2-cocycles involving the infinitesimal versions of the actions

$$\delta_d(w) = V_d w V_d^{-1} \text{ (of } \Delta \text{ on } \mathscr{W}) \quad \text{and } \omega_g k \omega_g^{-1} \quad (\text{of } \Omega \text{ on } \Delta)\ ,$$

that is,

$$d \cdot w = [\boldsymbol{v}_d, w] \quad \text{and} \quad g \cdot k = [\boldsymbol{\omega}_g, k]\ .$$

Our first task is to obtain a Lie algebra version of $\Lambda(\Gamma, \Delta, \mathscr{K})$ and hence an analogue of Theorem 2.1. Introduce

$$\mathrm{Der}\,\mathscr{V} = \{\delta \in L(\mathscr{V}) \mid \delta([v_1, v_2]) = [\delta(v_1), v_2] + [v_1, \delta(v_2)]\}$$

and a Lie algebra action $\boldsymbol{\delta}: \boldsymbol{\Gamma} \to \mathrm{Der}\,\mathscr{W}$, as a map satisfying

$$\boldsymbol{\delta}_g + \boldsymbol{\delta}_h = \boldsymbol{\delta}_{g+h} \quad \text{and} \quad \boldsymbol{\delta}_{[g,h]} = [\boldsymbol{\delta}_g, \boldsymbol{\delta}_h] \in \mathrm{Der}\,\mathscr{W}\ .$$

(Observe that whilst Der $\mathscr{V}$ is not closed under composition, it is closed under the commutator, and is thus a Lie algebra.)

Theorem 5.1. *Given the exact sequences above and a Lie algebra action* $\boldsymbol{\delta}: \boldsymbol{\Gamma} \to \mathrm{Der}\,\mathscr{W}$ *on the coefficient Lie algebra* $\mathscr{W}$ *such that*

$$\boldsymbol{\delta}_d(w) = [\boldsymbol{v}_d, w] \quad \text{for all } d \in \boldsymbol{\Delta},\ w \in \mathscr{W}\ ,$$

then there is an extension to a Lie algebra action $\boldsymbol{\delta}:\boldsymbol{\Gamma}\to \operatorname{Der}\mathscr{V}$ such that

$$\boldsymbol{\delta}_g(\boldsymbol{v}_d)\in \mathscr{W}+\boldsymbol{v}_{[g,d]}$$
$$\text{and}\quad \boldsymbol{\delta}_d(v)=[\boldsymbol{v}_d,v]\quad \text{for all } g\in\boldsymbol{\Gamma},\ d\in\boldsymbol{\Delta},\ v\in\mathscr{V}\ .$$

if and only if

$$\boldsymbol{\mu}(d,k)\in Z(\mathscr{V})\cap\mathscr{W}=:\mathscr{K}\quad \text{for all } d,k\in\boldsymbol{\Delta}\ ,$$

and there exists a map $\boldsymbol{\lambda}:\boldsymbol{\Gamma}\times\boldsymbol{\Delta}\to\mathscr{K}$ such that for all $g,h\in\boldsymbol{\Gamma}$, $d,k,\in\boldsymbol{\Delta}$:

(i) $\boldsymbol{\lambda}(0,d)=0=\boldsymbol{\lambda}(g,0)$,
(ii) $\boldsymbol{\delta}_g(\boldsymbol{\mu}(d,k))+\boldsymbol{\lambda}(g,[d,k])=\boldsymbol{\mu}([g,d],k)+\boldsymbol{\mu}(d,[g,k])$
(iii) $\boldsymbol{\lambda}([g,h],d)=\boldsymbol{\delta}_g(\boldsymbol{\lambda}(h,d))-\boldsymbol{\delta}_h(\boldsymbol{\lambda}(g,d))+\boldsymbol{\lambda}(g,[h,d])-\boldsymbol{\lambda}(h,[g,d])$
(iv) $\boldsymbol{\lambda}(d,k)=\boldsymbol{\mu}(d,k)$.

Proof. ($\Rightarrow$) Assume an action $\boldsymbol{\delta}:\boldsymbol{\Gamma}\to\operatorname{Der}\mathscr{V}$ exists as above. Define

$$\boldsymbol{\lambda}(g,d):=\boldsymbol{\delta}_g(\boldsymbol{v}_f)-\boldsymbol{v}_{[g,d]}\in\mathscr{W},$$

for $g\in\boldsymbol{\Gamma}$, $d\in\boldsymbol{\Delta}$. Now

$$\begin{aligned}\boldsymbol{\delta}_{[d,k]}(v)&=[\boldsymbol{v}_{[d,k]},v]=[[\boldsymbol{v}_d,\boldsymbol{v}_k],v]-[\boldsymbol{\mu}(d,k),v]\\&=(\boldsymbol{\delta}_d\boldsymbol{\delta}_k-\boldsymbol{\delta}_k\boldsymbol{\delta}_d)(v)=[\boldsymbol{v}_d,[\boldsymbol{v}_k,v]]-[\boldsymbol{v}_k,[\boldsymbol{v}_d,v]]\quad\text{for all } v\in\mathscr{V},\ d,k\in\boldsymbol{\Delta}\ .\end{aligned}$$

Rearranging:

$$-[\boldsymbol{\mu}(d,k),v]=[v,[\boldsymbol{v}_d,\boldsymbol{v}_k]]+[\boldsymbol{v}_d,[\boldsymbol{v}_k,v]]+[\boldsymbol{v}_k[v,\boldsymbol{v}_d]]=0$$

by the Jacobi identity. Hence, $\boldsymbol{\mu}(d,k)\in Z(\mathscr{V})\cap(\mathscr{W})$ and we denote this latter algebra by $\mathscr{K}$. Similarly, let $g\in\boldsymbol{\Gamma}$, $d\in\boldsymbol{\Delta}$, then for all $v\in\mathscr{V}$:

$$\begin{aligned}(\boldsymbol{\delta}_g\boldsymbol{\delta}_d)(v)&=\boldsymbol{\delta}_g([\boldsymbol{v}_d,\boldsymbol{v}])=[\boldsymbol{\delta}_g(\boldsymbol{v}_d),v]+[\boldsymbol{v}_d,\boldsymbol{\delta}_g(v)]\\&=[\boldsymbol{\lambda}(g,d),v]+[\boldsymbol{v}_{[g,d]},v]+[\boldsymbol{v}_d,\boldsymbol{\delta}_g(v)]\\&=[\boldsymbol{\delta}_{[g,d]}+\boldsymbol{\delta}_d\boldsymbol{\delta}_g)(v)=[\boldsymbol{v}_{[g,d]},v]+[\boldsymbol{v}_d,\boldsymbol{\delta}_g(v)]\ .\end{aligned}$$

Hence $[\boldsymbol{\lambda}(g,d),v]=0$ or all $v\in\mathscr{V}$, that is, $\boldsymbol{\lambda}:\boldsymbol{\Gamma}\times\boldsymbol{\Delta}\to\mathscr{K}$. Using $\boldsymbol{v}(0)=0$, (i) now follows.

For (ii) consider $\boldsymbol{\delta}_g([\boldsymbol{v}_d,\boldsymbol{v}_k])=[\boldsymbol{\delta}_g(\boldsymbol{v}_d),\boldsymbol{v}_k]+[\boldsymbol{v}_d,\boldsymbol{\delta}_g(\boldsymbol{v}_k)]$. The left-hand side of this expression is

$$\boldsymbol{\delta}_g(\boldsymbol{v}_{[d,k]}+\boldsymbol{\mu}(d,k))=\boldsymbol{\delta}_g(\boldsymbol{\mu}(d,k))+\boldsymbol{v}_{[g,[d,k]]}+\boldsymbol{\lambda}(g,[d,k])\ ,$$

while the right-hand side becomes:

$$\begin{aligned}&[\boldsymbol{\lambda}(g,d)+\boldsymbol{v}_{[g,d]},\boldsymbol{v}_k]+[\boldsymbol{v}_d,\boldsymbol{\lambda}(g,k)+\boldsymbol{v}_{[g,k]}]\\&=[\boldsymbol{v}_{[g,d]},\boldsymbol{v}_k]+[\boldsymbol{v}_d,\boldsymbol{v}_{[g,k]}]\\&=\boldsymbol{v}_{[[g,d],k]}+\boldsymbol{\mu}([g,d],k)+\boldsymbol{v}_{[d,[g,k]]}+\boldsymbol{\mu}(d,[g,k])\ .\end{aligned}$$

Equating these and using the Jocobi identity and linearity of $\boldsymbol{v}$ we get

$$\boldsymbol{\delta}_g(\boldsymbol{\mu}(d,k))+\boldsymbol{\lambda}(g,[d,k])=\boldsymbol{\mu}([g,d],k)+\boldsymbol{\mu}(d,[g,k])\ .$$

For (iii):

$$\begin{aligned}\delta_{[g,h]}(v_d) &= \lambda([g,h],d) + v_{[[g,h],d]}\\ &= (\delta_g\delta_h - \delta_h\delta_g)(v_d) = \delta_g(\lambda(h,d) + v_{[h,d]}) - \delta_h(\lambda(g,d) + v_{[g,d]})\\ &= \delta_g(\lambda(h,d)) - \delta_h(\lambda(g,d)) + \lambda(g,[h,d]) + v_{[g,[h,d]]}\\ &\quad - \lambda(h,[g,d]) - v_{[h,[g,d]]}\,,\end{aligned}$$

so using linearity of v and the Jacobi identity, we get

$$\lambda([g,h],d) = \delta_g(\lambda(h,d)) - \delta_h(\lambda(g,d)) + \lambda(g,[h,d]) - \lambda(h,[g,d])\,.$$

For (iv): $\delta_d(v_k) = [v_d, v_k]$, so

$$\lambda(d,k) + v_{[d,k]} = v_{[d,k]} + \mu(d,k)\,.$$

($\Leftarrow$) The converse is a straightforward adaptation of the proof for the group case. Starting from an action $\delta: \Gamma \to \mathrm{Der}\,\mathscr{W}$ on the coefficient algebra and a map λ satisfying (i) to (iv), define a map $\delta: \Gamma \to \mathrm{Der}\,\mathscr{V}$ by

$$\delta_g(w + v_d) = \delta_g(w) + \lambda(g,d) + v_{[g,d]}, \quad \text{for all } w \in \mathscr{W},\ g \in \Gamma,\ d \in \Delta\,,$$

and verify that it is an action as claimed. □

Next we want to know how to obtain from a pair (λ, μ) a 3-cocycle $K: \Omega^3 \to \mathscr{K}$. As at the group level, we want to obtain K by finding the image of the 2-cocycle relation of σ under v, express this in terms of a coboundary of $v \circ \sigma$ and then the remainder is K.

The 2-cocycle relation for σ is: for all $g, h, f \in \Omega$:

$$\begin{aligned}&[\omega_g, \sigma(h,f)] - [\omega_h, \sigma(g,f)] + [\omega_f, \sigma(g,h)]\\ &\qquad - \sigma([g,h],f) + \sigma([g,f],h) - \sigma([h,f],g) = 0\end{aligned}$$

and its image under $v: \Delta \to \mathscr{V}$ is (using the notation $v(g,h) := v(\sigma(g,h))$):

$$\begin{aligned}&v([\omega_g, \sigma(h,f)]) - v([\omega_h, \sigma(g,f)]) + v([\omega_f, \sigma(g,h)])\\ &\quad - v([g,h],f) + v([g,f],h) - v([h,f],g) = 0\,,\end{aligned}$$

so that,

$$\begin{aligned}&\delta_{\omega_g}(v(h,f)) - \lambda(\omega_g, \sigma(h,f)) - \delta_{\omega_h}(v(g,f)) + \lambda(\omega_h, \sigma(g,f))\\ &\quad + \delta_{\omega_f}(v(g,h)) - \lambda(\omega_f, \sigma(g,h)) - v([g,h],f) + v([g,f],h) - v([h,f],g) = 0\,.\end{aligned}$$

So we set

$$\begin{aligned}K(g,h,f) &:= \lambda(\omega_g, \sigma(h,f)) - \lambda(\omega_h, \sigma(g,f)) + \lambda(\omega_f, \sigma(g,h)) = (\partial v)(g,h,f)\\ &= \delta_{\omega_g}(v(h,f)) - \delta_{\omega_h}(v(g,f)) + \delta_{\omega_f}(v(g,h)) - v([g,h],f)\\ &\quad + v([g,f],h) - v([h,f],g)\,.\end{aligned}$$

Theorem 5.2. *The expression*

$$K(g,h,f) = \lambda(\omega_g, \sigma(h,f)) - \lambda(\omega_h, \sigma(g,f)) + \lambda(\omega_f, \sigma(g,h))$$

defines a 3*-cocycle, that is,*

$$\begin{aligned}&\delta_{\omega_g}(K(h,f,s)) - \delta_{\omega_h}(K(g,f,s)) + \delta_{\omega_f}(K(g,h,s)) - \delta_{\omega_s}(K(g,h,f))\\ &\quad - K([g,h],f,s) + K([g,f],h,s) - K([g,s],h,f)\\ &\quad - K([h,f],g,s) + K([h,s],g,f) - K([f,s],g,h) = 0\,. \qquad (5.1)\end{aligned}$$

Proof. This is lengthy but straightforward verification of Eq. (5.1) making use of the listed identities in Theorem 5.1, the 2-cocycle identity for $\boldsymbol{\sigma}$ and linearity and anti-symmetry of the 2-cocycles.

Remarks. Observe that $\boldsymbol{\mu}$ has vanished from the Lie algebra version of K in contrast to the group situation. This is due to the fact that in constructing the Lie algebra version we are assuming additional regularity (locally smooth cocycles on a Lie group rather than cocycles on a discrete group). This translates in our Lie algebraic construction into linearity of the sections $\boldsymbol{v}, \boldsymbol{\omega}$. It means that the conditions in Theorem 5.1 are more restrictive than those in Theorem 2.1. In fact a change in $\boldsymbol{\mu}$ will definitely affect K but indirectly through $\boldsymbol{\lambda}$. The question of non-triviality at the Lie algebra level requires different techniques and will not be pursued here. It is relevant however to the discussion in the concluding section.

5.1. Interpretation of Lie Algebra Cocycles and Concluding Remarks. To connect up with the field theoretic viewpoint [2, 31] we need some further discussion.

Let M denote a three dimensional manifold (representing the spatial degrees of freedom) and we take $\boldsymbol{\Gamma}/\boldsymbol{\Delta}$ to be the Lie algebra of the gauge group which for the present purposes is taken to be the group of smooth maps from M into a compact Lie group G with Lie algebra $\boldsymbol{g}$. Then we take $\boldsymbol{\Gamma}$ to be the Faddeev–Mickelsson [2, 14] extension of $\boldsymbol{\Gamma}/\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}$ is a non-central abelian Lie algebra. For concreteness, if $\mathscr{A}$ denotes the affine space of connections on the trivial principal bundle P over the 3-dimensional manifold M and $\boldsymbol{g}$ is the Lie algebra of the compact group $SU(2)$, then $\boldsymbol{\Delta}$ may be taken to consist of $\mathbb{R}$-*valued functions* on $\mathscr{A}$ which are linear in the connections and their derivatives. This is because for this case the Faddeev–Mickelsson 2-cocycle $\boldsymbol{\sigma}$ on $\boldsymbol{\Gamma}/\boldsymbol{\Delta}$ which defines the extension $\boldsymbol{\Gamma}$ takes its values in the space of such linear functionals on $\mathscr{A}$.

One may construct a C^*-algebra $\mathscr{F}$ on which $\boldsymbol{\Gamma}$ acts as derivations [5]. We denote this action by $A \mapsto \eta_x(A)$, $x \in \boldsymbol{\Gamma}$, $A \in \mathscr{F}$ remarking however that for the purposes of the present discussion it is not essential to describe $\mathscr{F}$ explicitly. For the moment assume that $\mathscr{F}$ is represented irreducibly by bounded operators on a Hilbert space $\mathscr{H}$ say by $\pi : \mathscr{F} \to B(H)$. The usual procedure now would be to assume that the map $\boldsymbol{v}$ above is a projective Lie algebra algebra homomorphism from $\boldsymbol{\Delta}$ to the unbounded self-adjoint operators acting on a common dense invariant domain $\mathscr{S} \subseteq \mathscr{H}$ with cocycle $\boldsymbol{\mu}$ and that we have an (unbounded) self-adjoint operator $\boldsymbol{v}_x$ for each $x \in \boldsymbol{\Gamma}$ such that

$$[\boldsymbol{v}_x, \pi(A)] = \pi(\eta_x(A)), \quad A \in \mathscr{F} .$$

Now $\boldsymbol{v}_x$ for $x \in \boldsymbol{\Gamma}$ defines a derivation on $\boldsymbol{v}(\boldsymbol{\Delta})$ by

$$\boldsymbol{v}_D \mapsto [\boldsymbol{v}_x, \boldsymbol{v}_D] .$$

By our previous results, a scalar valued $\boldsymbol{\lambda}$ satisfying the conditions of Theorem 5.1 exists and hence a corresponding 3-cocycle. If this 3-cocycle is non-trivial then it is not difficult to check that $\{\boldsymbol{v}_x \mid x \in \boldsymbol{\Gamma}\}$ do not form a representation of $\boldsymbol{\Gamma}$. So one has an obstruction to the usual desiderata of quantum field theory.

In fact much of the preceding apparatus is redundant. The field algebra $\mathscr{F}$ is only required to force a scalar valued pair $\boldsymbol{\mu}, \boldsymbol{\lambda}$. All one really needs are the Lie algebras of the diagram at the beginning of this section.

One may now ask whether the 3-cocycles obtained by the descent equation [2, 31] could arise in this way. The answer is negative however because the descent equation method does not simultaneously construct 2-cocycles *and* 3-cocycles on the same group. This is because the Lie group $SU(2)$ has non-trivial cohomology in odd dimensions *only* and the descent equation method works by pulling back (using the evaluation map) forms representing these cohomology classes. It follows that the degrees of cocycles on a given algebra Γ which may be constructed via the descent equation method must all have even degree, ruling out the simultaneous construction of 2- and 3-cocycles by this method. However one might conjecture that our approach will shed some light on Pickrell's theorem [17] which suggests that the Lie group corresponding to $\boldsymbol{\Gamma}$ has no separable continuous unitary representations. As we noted in the previous paragraph, if the operators $\boldsymbol{v}_x$, $x \in \boldsymbol{\Gamma}$ can be chosen so as to provide a representation of $\boldsymbol{\Gamma}$, then $\boldsymbol{v} \circ \boldsymbol{\sigma}$, as a map from $\boldsymbol{\Gamma}/\boldsymbol{\Delta} \times \boldsymbol{\Gamma}/\boldsymbol{\Delta}$ to the unbounded self-adjoint operators on $\mathscr{S}$, must satisfy the 2-cocycle identity. A straightforward calculation then shows that if our three cocycle if non-trivial one has a contradiction. Thus one cannot have the desired representation of $\boldsymbol{\Gamma}$. To deduce a version of Pickrell's theorem from this, we would have to show that, for any map $\boldsymbol{\lambda}$ defined on $\boldsymbol{\Gamma}/\boldsymbol{\Delta} \times \boldsymbol{\Gamma}/\boldsymbol{\Delta}$ to the unbounded self-adjoint operators on $\mathscr{S}$, must satisfy the 2-cocycle identity. A straightforward calculation then shows that if our three cocycle if non-trivial one has a contradiction. Thus one cannot have the desired representation of $\boldsymbol{\Gamma}$. To deduce a version of Pickrell's theorem from this, we would have to show that, for any map $\boldsymbol{\lambda}$ defined on $\boldsymbol{\Gamma}/\boldsymbol{\Delta} \times \boldsymbol{\Delta}$ as above, the corresponding 3-cocycle is non-trivial. Considerable further work seems to be needed however to prove this.

On the other hand work in progress indicates that the 3-cocycle arising from the anomalous commutators in QCD [4] fits into the framework of this section.

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