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# $q$-Lorentz Group and Braided Coaddition on $q$-Minkowski Space

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**Abstract:** We present a new version of $q$-Minkowski space, which has both a coaddition law and an $SL_q(2,\mathbb{C})$-spinor decomposition. The additive structure forms a braided group rather than a quantum one. In the process, we obtain a $q$-Lorentz group which coacts covariantly on this $q$-Minkowski space.

## 1. Introduction

In recent years, there has been some speculation whether it could be possible to regularise singularities in quantum field theories by making spacetime slightly non-commutative. As well as the programme of A. Connes [3] based on the theory of operator algebras, there is also a more naive approach based on the idea of $q$-deformation. In this approach, which is the one we shall follow, non-commutativity is controlled by a parameter $q$ such that one recovers the commutative case for $q = 1$. This programme is motivated by examples of "Feynman-type" integrals over two-dimensional $q$-deformed planes which are of the form $\int(\ldots) = \frac{1}{q^2-1}(finite)$, i.e. are divergent only in the commutative case [7]. Moreover, one hopes in such a $q$-regularisation scheme to preserve all symmetries as $q$-symmetries, using the standard techniques for $q$-deforming Lie algebras, etc. One would then set $q = 1$ after intelligent renormalisation, although, to take account of Planck scale corrections to the geometry, one might even keep $q \neq 1$.

As an important element of such a $q$-regularisation scheme, many $q$-Lorentz groups and $q$-Minkowski spaces have been recently proposed [17, 2, 16, 15]. One of the points of view in these works, which will be our point of view also, is that $q$-Minkowski space should have a $q$-spinor decomposition. Mathematically, $q$-Minkowski space should be a $q$-deformed version of $2 \times 2$ Hermitean matrices and the $q$-Lorentz group should act on it by conjugation by two $q$-deformed $SL(2,\mathbb{C})$ transformations. The rôle of such a $q$-deformed $SL(2,\mathbb{C})$ can be provided by the quantum double [17], but $q$-Minkowski space and the $q$-Lorentz group itself are less well understood so far.

Naively, one might try to construct $q$-Minkowski space as quantum $2 \times 2$ matrices, but this algebra is not covariant under the coaction of the $q$-deformed $SL(2,\mathbb{C})$

[14]. The solution to this problem is to consider *braided* rather than quantum Hermitean matrices as $q$-Minkowski space [10, 15]. Braided matrices are an example of so-called braided bialgebras introduced by S. Majid in [9] as a generalization of bialgebras, for which the ordinary tensor product in the bialgebra axioms is replaced by a *braided tensor product*. Braided tensor products are like super tensor products encountered in the theory of superspaces, but with $\pm 1$ replaced by braid statistics. There is a general construction called *transmutation* [12] by means of which one can convert any suitable bialgebra, such as a usual quantum matrix algebra, into a braided bialgebra with better covariance properties. The algebra and coalgebra structure of such a braided bialgebra are covariant under the coaction of the quantum group. Thus braided $2 \times 2$ matrices as the transmutation of the well-known $2 \times 2$ quantum matrices are a natural candidate for the algebra of $q$-Minkowski space. It is covariantly coacted upon by the $q$-deformed $SL(2, \mathbb{C})$.

Braided $2 \times 2$ matrices have the same matrix coalgebra structure as quantum matrices, but a different multiplication [10]. Similar as for $2 \times 2$ quantum matrices, there is a *braided determinant* which is central and grouplike with respect to the braided coproduct [10] to play the rôle of a $q$-Minkowski norm. Furthermore, these braided matrices allow for a $q$-spinor decomposition [14] and can also be equipped with a $*$-structure appropriate for Hermitean matrices [15].

Considering braided Hermitean matrices seems to lead in the right direction, but a fundamental structure is still missing: so far there is no $q$-deformed analogue of the additive group structure of Minkowski space. In this paper we solve this problem and generalize the group structure on Minkowski space as a *braided coaddition* in the form of a new braided coalgebra structure for the algebra of braided matrices. The required braiding for the coaddition is a new one and gives rise to a $q$-Lorentz group which acts covariantly on $q$-Minkowski space.

An outline of the paper is as follows. In Sect. 2, we reformulate some classical considerations about the Lorentz group and Minkowski space suitable for later $q$-deformation. The $q$-Lorentz group of function algebra type is presented in Sect. 3. Section 4 discusses braided coaddition on $q$-Minkowski space. Finally, Sect. 5 presents a deformation of the universal enveloping algebra of the Lorentz group which is dual to the algebra discussed in Sect. 3.

## Preliminaries

When working with matrices, we use lower-case letters for indices which run from 1 to 2 or $n$, and upper-case letters for multi-indices, e.g. $A = (a_0 a_1) = (11), (12), \ldots, (n(n-1)), (nn)$.

For Hopf algebras, we use the notation and results from the standard textbooks [1, 21]. Recall that a complex *coalgebra* is a $\mathbb{C}$-vector space $A$ equipped with a $\mathbb{C}$-linear coassociative comultiplication $\Delta : A \to A \otimes A$ and a $\mathbb{C}$-linear counit $\varepsilon : A \to \mathbb{C}$ satisfying certain axioms. Elements $a$ in $A$ which obey $\Delta a = a \otimes a$ are called *grouplike*. We use the notation $\Delta a = a_{(1)} \otimes a_{(2)}$ for the coproduct and omit summation signs for brevity. We also use M. Sweedler's shorthand notation [21], where a suffix indicates the position in a matrix tensor product, e.g. $A_{12}B_{23}$ means $A^{ij}_{kl}B^{lm}_{xy}$, etc.

A complex *bialgebra* is an algebra and a coalgebra in a compatible way, such that both comultiplication and counit are algebra maps. If a bialgebra $H$ also allows for a $\mathbb{C}$-linear antipode $S : H \to H$ obeying $\cdot \circ (S \otimes id) \circ \Delta = \cdot \circ (id \otimes S) \circ \Delta = \eta \circ \varepsilon$, then $H$ is called *Hopf algebra*. Here $\eta$ denotes the injection of the identity. A $*$-*Hopf algebra* [22] is a Hopf algebra equipped with an antilinear involution "$*$" such that $(S \circ *)^2 = id$, $\Delta \circ * = (* \otimes *) \circ \Delta$, and $\varepsilon \circ * = * \circ \varepsilon$.

Two $*$-Hopf algebras $H$ and $H'$ are called *dually paired* if there exists a bilinear pairing $\langle \ , \ \rangle : H \otimes H' \to \mathbb{C}$ such that $\langle \alpha\beta, x\rangle = \langle \alpha \otimes \beta, \Delta x\rangle$, $\langle \alpha, xy\rangle = \langle \Delta\alpha, x \otimes y\rangle$, $\langle 1, x\rangle = \varepsilon(x)$, $\langle \alpha, 1\rangle = \varepsilon(\alpha)$, $\langle S\alpha, x\rangle = \langle \alpha, Sx\rangle$ and $\langle \alpha^*, x\rangle = \overline{\langle \alpha, (Sx)^*\rangle}$ for all $\alpha$, $\beta$ in $H$ and $x$, $y$ in $H'$.

We shall also need the notion of a *right comodule*, which is dual to the definition of a left module: a right comodule of a coalgebra $A$ is a pair $(C, \beta)$, where $C$ is a vector space and $\beta$ a linear map $\beta : C \to C \otimes A$ obeying $(id \otimes \Delta) \circ \beta = (\beta \otimes id) \circ \beta$ and $id = (id \otimes \varepsilon) \circ \beta$. If $\beta$ is also an algebra map, then the comodule is called *comodule algebra*.

Of particular interest to us are non-commutative bialgebras, for which the non-commutativity is controlled by a so-called *dual quasitriangular structure* [11], which is a convolution invertible map $\mathfrak{R} : A \otimes A \to \mathbb{C}$ such that $b_{(1)}a_{(1)}\mathfrak{R}(a_{(2)} \otimes b_{(2)}) = \mathfrak{R}(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)}$, $\mathfrak{R}(ab \otimes c) = \mathfrak{R}(a \otimes c_{(1)})\mathfrak{R}(b \otimes c_{(2)})$, and $\mathfrak{R}(a \otimes bc) = \mathfrak{R}(a_{(1)} \otimes c)\mathfrak{R}(a_{(2)} \otimes b)$ for all $a$, $b$, $c$ in $A$. In other words, $\mathfrak{R}$ is a bialgebra bicharacter. This notion is dual to the maybe more familiar concept of quasitriangularity due to Drinfel'd [4].

One of the interesting properties of dual quasitriangular bialgebras is that the right comodules of such a bialgebra $A$ form a quasi-tensor or *braided* category denoted by $\mathcal{M}^A$. This means that $\mathcal{M}^A$ can be equipped with a bifunctor $\underline{\otimes} : \mathcal{M}^A \times \mathcal{M}^A \to \mathcal{M}^A$, which was called "braided tensor product" in the introduction and which satisfies some associativity conditions. Furthermore, for any two objects $X$, $Y$ of $\mathcal{M}^A$ (i.e. for any two comodules) there is a natural isomorphism $\Psi_{X,Y} : X\underline{\otimes}Y \cong Y\underline{\otimes}X$, called *braiding*. For $\mathcal{M}^A$, this braiding is given in terms of the dual quasitriangular structure $\mathfrak{R}$ and the coactions of the respective comodules as [12] $\Psi_{B,B'} = (\tau_{B,B'} \otimes \mathfrak{R}) \circ \tau_{A,B'} \circ (\beta \otimes \beta')$, where $\tau$ denotes the twist map. If we are now given two $A$-comodule *algebras* $B$ and $B'$, we can use $\Psi$ to define their *braided tensor product* $B\underline{\otimes}B'$ as $B \otimes B'$ equipped with the new multiplication $(a \otimes b)(c \otimes d) = a\Psi(b \otimes c)d$. Due to the properties of $\underline{\otimes}$ and $\Psi$, the braided tensor product of two comodule algebras turns out to be a comodule algebra again, i.e. the braided tensor product provides a covariant way of combining two covariant systems. Recall further that a *braided bialgebra* [12] is an algebra $B$ living in a braided category equipped with a braided coproduct $\underline{\Delta} : B \to B\underline{\otimes}B$ obeying axioms similar to the bialgebra axioms, but with $\underline{\Delta}$ a homomorphism to the braided tensor product $B\underline{\otimes}B$. All maps are morphisms, i.e. covariant under the coaction of the background quantum group $A$. The braided matrices mentioned in the introduction and which form our $q$-Minkowski space are of this type.

## 2. The Classical Case

In this section, we reformulate some classical considerations about the Lorentz group and Minkowski space in an algebraic language suitable for later generalization. As familiar from other appications, we present Minkowski space as Hermitean $2 \times 2$ matrices. One usually chooses this description in order to give a simple exposition of the covering of the subgroup of proper orthochronous Lorentz transformations by $SL(2, \mathbb{C})$. This map also enables one to construct the well-known spinor decomposition of Lorentz tensors. If Minkowski space is given as Hermitean $2 \times 2$ matrices, then the Minkowski metric can be expressed in terms of the $SL(2,\mathbb{C})$ spinor metric

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as $g^{AB} = \epsilon^{a_0 b_0}\epsilon_{a_1 b_1}$, and the Lorentz group $L$ is given as the subspace of real $4 \times 4$ matrices $\Lambda \in M(4,\mathbb{R})$ which satisfy $\Lambda^A_C \Lambda^B_D g^{CD} = g^{AB}$. We use the convention $\epsilon_{bc}\epsilon^{ac} = \delta^a_b$ for the definition of the inverse spinor metric.

In principle, we are interested in $\mathscr{C}(X)$, the algebra of continuous $\mathbb{C}$-valued functions on a subset $X$ of $\mathbb{R}^n$, such as the Lorentz group $L$ or Minkowski space $M$. However, in order to avoid the discussion of convergence problems and other complications due to the non-compact nature of these spaces, we consider only $\mathscr{P}(X)$, the algebra of polynomial functions, which is almost the same as $\mathscr{C}(X)$, since on arbitrarily large compact subsets $X'$ of $X$, the algebra $\mathscr{P}(X')$ is dense in $\mathscr{C}(X')$.

For the application to the Lorentz group, we are particularly interested in the case where $X$ is a subset of real or complex $n \times n$ matrices $M(n,\mathbb{R}/\mathbb{C})$. For sake of clarity, we recall some result of this special case: $\mathscr{P}(M(n,\mathbb{C}))$ is a commutative and associative $\mathbb{C}$-algebra generated by 1 and the linear coordinate functionals $t^a_b$ and their complex conjugates $\bar{t}^a_b$. It has the structure of a bialgebra with pointwise multiplication, comultipication $\Delta t^a_b = t^a_c \otimes t^c_b$ and counit $\varepsilon t^a_b = \delta^a_b$. A coalgebra structure of this type is called of *matrix multipication type*. If we are given a matrix group $G \subset M(n,\mathbb{C})$ then $\mathscr{P}(G)$ is a Hopf algebra.

For the special case of $SL(2,\mathbb{C})$ one finds that $\mathscr{P}(SL(2,\mathbb{C}))$ is generated by 1, $t^a_b$ and $t^{\dagger a}{}_b$ with relations $t^a_c t^b_d \epsilon^{cd} = \epsilon^{ab}$. It is a Hopf algebra with coalgebra structure of matrix multiplication type, antipode given by $St^a_b = \epsilon_{bc} t^c_d \epsilon^{ad}$ and $*$-structure $t^{a*}_b = t^{\dagger b}{}_a$. One obtains $\mathscr{P}(SU(2))$ as a "real form" of $\mathscr{P}(SL(2,\mathbb{C}))$. Using these results, we find for the algebra of polynomial functions on the Lorentz group:

**Proposition 2.1.** *The algebra of polynomial functions on the Lorentz group $\mathscr{P}(L)$ is generated by the linear coordinate functional $\lambda^A_B$ on $M(4,\mathbb{R})$ with relations $\lambda^A_C \lambda^B_D g^{CD} = g^{AB}$. It forms a commutative $*$-Hopf algebra with pointwise multiplication and*

$$\Delta\lambda^A_B = \lambda^A_C \otimes \lambda^C_B, \quad S\lambda^A_B = g_{BC}\lambda^C_D g^{AD}, \quad \lambda^{*A}_B = \lambda^{\bar{A}}_{\bar{B}}\,.$$

*$\bar{A}$ is shorthand for the twisted multi-index $\bar{A} = (a_1 a_0)$. Furthermore, there is a $*$-Hopf algebra homomorphism $\varphi : \mathscr{P}(L) \to \mathscr{P}(SL(2,\mathbb{C}))$ given by $\varphi(\lambda^A_B) = t^{\dagger b_0}{}_{a_0} t^{a_1}_{b_1}$. Its image is $\mathscr{P}(SL(2,\mathbb{C}))^{\mathbb{Z}_2}$, the fixed-point set of the $\mathbb{Z}_2$-action $\sigma$ given by $\sigma(t) = -t$ and $\sigma(t^\dagger) = -t^\dagger$.*

Composition with the map $\varphi$ defines a *push forward* of comodules, i.e. a covariant monoidal functor $\Phi : \mathscr{M}^{\mathscr{P}(L)} \to \mathscr{M}^{\mathscr{P}(SL(2,\mathbb{C}))}$. This is the spinor decomposition of Lorentz tensors on the level of polynomial function algebras.

Next, we come to Minkowski space $M$ in this algebraic form. Minkowski space has an additive group structure and not a multiplicative one as the matrix groups discussed so far. This additive group structure of spacetime is recovered as a *coaddition* on $\mathscr{P}(M)$:

**Proposition 2.2.** *The polynomial functions on Minkowski space form a commutative associative $\mathbb{C}$-algebra generated by* 1 *and* 4 *linear coordinate functionals* $x_A$. $\mathscr{P}(M)$ *is a* $*$-*Hopf algebra with*

$$\Delta x_A = x_A \otimes 1 + 1 \otimes x_A, \quad S x_A = -x_A, \quad \varepsilon x_A = 0, \quad x_A^* = x_{\bar{A}} .$$

*Furthermore, we have* $x_A x_B g^{AB} = 2 \det x = 2(x_{11}x_{22} - x_{12}x_{21})$, *i.e. the norm is given by the determinant.*

The $*$-Hopf algebra $\mathscr{P}(M)$ is covariantly coacted upon by $\mathscr{P}(L)$ with right coaction $\beta_{\mathscr{P}(M)} : \mathscr{P}(M) \to \mathscr{P}(L) \otimes \mathscr{P}(M)$ given by $x_A \mapsto x_B \otimes \lambda^B_A$. In particular, one finds that the "norm" $x_A x_B g^{AB}$ is invariant under this coaction. Applying the functor $\Phi$ establishes that $\mathscr{P}(M)$ is also a right $\mathscr{P}(SL(2,\mathbb{C}))$-comodule algebra with coaction $(id \otimes \varphi) \circ \beta_{\mathscr{P}(M)}$.

## 3. $q$-Lorentz Group of Function Algebra Type

In this section, we give a non-commutative generalisation of $\mathscr{P}(L)$, making use of the standard technique of deforming the commutative bialgebra of polynomial functions on a matrix group as a non-commutative dual quasitriangular bialgebra [18]. The resulting algebraic objects are called *quantum matrix groups*. The basic idea is to make the linear coordinate functionals $t^a_b$ commutative only up to conjugation by an invertible solution $R = \sum R^{(1)} \otimes R^{(2)} \in GL(\mathbb{C}^n \otimes \mathbb{C}^n)$ of the quantum Yang–Baxter equation (QYBE) $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. Explicitly, one defines $A(R)$ to be the free associative $\mathbb{C}$-algebra generated by 1 and $n^2$ symbols $t^a_b$; $a,b = 1,\dots,n$ divided by the ideal generated by the relations $R_{12}t_1t_2 = t_2t_1R_{12}$ (i.e. $R^{ab}_{cd}t^c_e t^d_f = t^b_d t^a_c R^{cd}_{ef}$). It is known that $A(R)$ is a dual quasitriangular bialgebra with coproduct of matrix multiplication type and a dual quasitriangular structure $\mathfrak{R} : A(R) \otimes A(R) \to \mathbb{C}$ given by $\mathfrak{R}(t \otimes 1) = \mathfrak{R}(1 \otimes t) = id$ and $\mathfrak{R}(t_1 \otimes t_2) = R_{12}$ extended as a bialgebra bicharacter [8]. The dual quasitriangularity of $A(R)$ follows from the fact that the so-called *fundamental matrix representations* $\rho_\pm : A(R) \to M(4,\mathbb{C})$ defined by $\rho_+(t^a_c)^b_d = R^{ab}_{cd}$ and $\rho_-(t^a_c)^b_d = R^{-1ba}_{dc}$ respect the relations in $A(R)$ and indeed extend to algebra maps [8]. This means that if we divide $A(R)$ by some further relations in order to obtain a generalisation of the Hopf algebra of polynomial functions on a matrix group, then it is sufficient to show that these additional relations are respected by $\rho_\pm$ in order to establish dual quasitriangularity of the quotient. Note, however, that these additional relations usually fix the normalisation of the dual quasitriangular structure.

We shall now give a non-commutative version of $\mathscr{P}(L)$ as a dual quasitriangular $*$-Hopf algebra of the form

$$\mathscr{L}_q = A(R_L)/(q\text{-deformed metric relation}) ,$$

where $R_L$ is an invertible solution of the four-dimensional QYBE which we introduce. This algebra $\mathscr{L}_q$ should generalise all features of $\mathscr{P}(L)$ from Proposition 2.1. In order to obtain such a matrix $R_L$ and a $q$-deformed metric, we make use of the important rôle which $\mathscr{P}(SL(2,\mathbb{C}))$ plays as a building block of $\mathscr{P}(L)$.

A non-commutative version of $\mathscr{P}(SL(2,\mathbb{C}))$ can be constructed as a "complexification" of the standard $q$-deformation of $\mathscr{P}(SU(2))$ from [18]: Let

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad q \in \mathbb{R} \tag{1}$$

be the well-known invertible solution of the two-dimensional QYBE. This matrix is of *real type*, i.e. obeys $R^{ab}_{cd} = R^{dc}_{ba}$. The algebra $SU_q(2)$ is then defined as $A(R)$ with generators $\tau^a_b$ and relations $\tau^a_c \tau^b_d \epsilon^{ab}$, where

$$\epsilon^{ab} = \begin{pmatrix} 0 & \sqrt{q} \\ -1/\sqrt{q} & 0 \end{pmatrix}$$

is the $q$-deformed spinor metric with inverse defined by $\epsilon_{bc}\epsilon^{ac} = \delta^a_b$. The algebra $SU_q(2)$ is a dual quasitriangular $*$-Hopf algebra with antipode $S\tau^a_b = \epsilon_{bc}\tau^c_d\epsilon^{ad}$, $*$-structure $\tau^{a*}_b = S\tau^b_a$ and standard dual quasitriangular structure defined in terms of the rescaled $R$-matrix $q^{-1/2}R$.

We deform $\mathscr{P}(SL(2,\mathbb{C})) \cong \mathscr{P}(SU(2)) \otimes \mathscr{P}(SU(2))$ as $SU_q(2) \bowtie SU_q(2)$, the *double cross product* Hopf algebra [6] of two copies of $SU_q(2)$ acting on each other in a compatible way. This double cross product coincides with $SU_q(2) \otimes SU_q(2)$ as a coalgebra, but has a different algebra structure given in terms of the compatible actions. By applying the general construction from [13, Sect. 4] one obtains cross relations $R^{ab}_{cd}(1 \otimes \tau^c_e)(\tau^d_g \otimes 1) = (\tau^b_d \otimes 1)(1 \otimes \tau^a_c)R^{cd}_{eg}$. The double cross product has a $*$-structure given in terms of the new generators $t^a_b = 1 \otimes \tau^a_b$ and $t^{\dagger a}{}_b = S\tau^a_b \otimes 1$ as $t^{\dagger a}{}_b = t^{*b}{}_a$. One obtains $SU_q(2)$ as a real form of $SL_q(2,\mathbb{C})$.

**Definition 3.1.** *The q-Lorentz group $\mathscr{L}_q$ is defined as the algebra $A(R_L)$ with generators $\lambda^A_B$ divided by the ideal generated by the relations $\lambda^A_C\lambda^B_D g^{CD} = g^{AB}$. In terms of the $SU_q(2)$ R-matrix and the q-deformed spinor metric the R-matrix $R_L$ and the preserved metric $g^{AB}$ are given by*

$$R_{L\,CD}^{AB} = R^{c_0\alpha}_{\beta b_0} R^{b_1\beta}_{\gamma a_0} R^{a_1\gamma}_{\delta d_1} \tilde{R}^{\delta d_0}_{c_1\alpha} .$$

*and $g^{AB} = q\epsilon_{a_0\alpha}R^{a_1\alpha}_{\beta b_0}\epsilon^{\beta b_1}$. Here we used the notation $\tilde{R} = ((R^{t_2})^{-1})^{t_2}$, where $t_2$ denotes transposition in the second tensor component. Explicitly, one finds*

$$g^{AB} = \begin{pmatrix} 1/q-q & 0 & 0 & q \\ 0 & 0 & -q & 0 \\ 0 & -q^3 & 0 & 0 \\ q & 0 & 0 & 0 \end{pmatrix}$$

*for the q-deformed metric. Its inverse is defined by $g_{BC}g^{AC} = \delta^A_B$.*

For this algebra to be a quantum group, we need to show that $R_L$ is an invertible solution of the QYBE, which can easily be established by explicit calculation. However, by making use of a result from [10], one can show more generally that any composed R-matrix of this form is a solution of the QYBE provided $R$ is a *Hecke type* solution of the QYBE, i.e. obeys $0 = (PR - q)(PR + q^{-1})$, where $P$ is the permutation matrix.

**Lemma 3.2.** *Let $R$ be an invertible Hecke type solution of the n-dimensional matrix QYBE. Then*

$$\mathbf{R}^{AB}_{CD} = R^{c_0\alpha}_{\beta b_0} R^{b_1\beta}_{\gamma a_0} R^{a_1\gamma}_{\delta d_1} \tilde{R}^{\delta d_0}_{c_1\alpha}$$

*satisfies the $n^2$-dimensional QYBE.*

*Proof.* It is known from [10, Lemma 3.1] that the matrix $\Psi^{DC}_{BA} = R^{c_0\alpha}_{\beta b_0} R^{-1\beta b_1}_{a_0\gamma} R^{a_1\gamma}_{\delta d_1} \tilde{R}^{\delta d_0}_{c_1\alpha}$ obeys the QYBE. (We use the notation from [10]: $\Psi$ is a matrix and not the braiding!) This means that also $\Theta^{AB}_{CD} = \Psi^{DC}_{BA}$ satisfies the QYBE. However, using the Hecke property of $R$, one finds

$$\mathbf{R}^{AB}_{CD} = R^{c_0\alpha}_{\beta b_0} R^{b_1\beta}_{\gamma a_0} R^{a_1\gamma}_{\delta d_1} \tilde{R}^{\delta d_0}_{c_1\alpha} = R^{c_0\alpha}_{\beta b_0} (PR^{-1}P + (q-q^{-1})P)^{b_1\beta}_{\gamma a_0} R^{a_1\gamma}_{\delta d_1} \tilde{R}^{\delta d_0}_{c_1\alpha} ,$$

i.e. $\mathbf{R}$ looks like $\Theta$ but with $R^{-1} + (q-q^{-1})P$ substituted for $R^{-1}$. It is easy to see that $R^{-1} + (q-q^{-1})P$ obeys the QYBE and acts like $R^{-1}$ in mixed QYBEs with $R$ and $\tilde{R}$. Thus the proof in [10] for $\Psi$ implies that also $\mathbf{R}$ obeys the QYBE. □

With this lemma, we can now prove a generalisation of Proposition 2.1 for the algebra $\mathscr{L}_q$:

**Proposition 3.3.** (i) *The q-Lorentz group $\mathscr{L}_q$ is a dual quasitriangular $*$-Hopf algebra with matrix coalgebra structure, and antipode and $*$-structure*

$$S\lambda^A_B = g_{BC}\lambda^C_D g^{AD}, \quad \lambda^{*A}_B = \lambda^{\bar{A}}_{\bar{B}} .$$

*The standard dual quasitriangular structure $\mathfrak{R}$ on $A(\kappa R_L)$ descends to a dual quasitriangular structure on $\mathscr{L}_q$, where we have to choose a normalization factor $\kappa = 1/q$.*

(ii) *There exists a $*$-Hopf algebra map $\varphi : \mathscr{L}_q \to SL_q(2,\mathbb{C})$ given by $\varphi(\lambda^A_B) = t^{\dagger b_0}{}_{a_0} t^{a_1}_{b_1}$. As in the commuting case, $Im(\varphi) = SL_q(2,\mathbb{C})^{\mathbb{Z}_2}$ is the fixed-point set of a $\mathbb{Z}_2$-action.*

The Hopf algebra map $\varphi$ induces a push forward of comodules, i.e. a $q$-spinor decomposition of $q$-Lorentz tensors.

*Proof.* (i) We know from Lemma 3.2 that $R_L$ satisfies the QYBE since the $SU_q(2)$ R-matrix (1) is of Hecke type. Thus, apart from a simple check of Hopf algebra axioms, there remain two non-trivial statements to be shown: firstly that the operation "$*$" is a $*$-structure on a Hopf algebra, and secondly, that $\mathfrak{R}$ defines a dual quasitriangular structure, i.e. that it is compatible with the $q$-metric relation. In order to show that "$*$" respects the algebra relations in $\mathscr{L}_q$, note that using the relations $R^{ab}_{cd} = R^{dc}_{ba}$ and $\epsilon_{ab} = -\epsilon^{ba}$, one can show: $R^{AB}_{L\,CD} = R^{c_0\alpha}_{\beta b_0} R^{b_1\beta}_{\gamma a_0} R^{a_1\gamma}_{\delta d_1} \tilde{R}^{\delta d_0}_{c_1\alpha}$ $= R^{b_0\beta}_{\alpha c_0} R^{a_0\gamma}_{\beta b_1} R^{d_1\delta}_{\gamma a_1} \tilde{R}^{\alpha c_1}_{d_0\delta} = R^{\bar{B}\bar{A}}_{L\,\bar{D}\bar{C}}$, and $g^{AB} = q\epsilon_{a_0\alpha} R^{a_1\alpha}_{\beta b_0} \epsilon^{\beta b_1} = q\epsilon^{\alpha a_0} R^{b_0\beta}_{\alpha a_1} \epsilon_{b_1\beta} = g^{\bar{B}\bar{A}}$.

Therefore,

$$
\begin{aligned}
(R_{L\,CD}^{AB}\lambda_E^C\lambda_F^D)^* &= R_{L\,CD}^{AB}\lambda_{\bar F}^{\bar D}\lambda_{\bar E}^{\bar C}\\
&= R_{L\,\bar D\bar C}^{\bar B\bar A}\lambda_{\bar F}^{\bar D}\lambda_{\bar E}^{\bar C}\\
&= \lambda_{\bar C}^{\bar A}\lambda_{\bar D}^{\bar B}R_{L\,\bar F\bar E}^{\bar D\bar C}\\
&= \lambda_{\bar C}^{\bar A}\lambda_{\bar D}^{\bar B}R_{L\,EF}^{CD}\\
&= (\lambda_D^B\lambda_C^A R_{L\,EF}^{CD})^*\,,
\end{aligned}
$$

and also $(\lambda_C^A\lambda_D^B g^{CD})^* = \lambda_{\bar D}^{\bar B}\lambda_{\bar C}^{\bar A}g^{\bar D\bar C} = g^{\bar B\bar A} = (g^{AB})^*$. Thus, "$*$" can be extended as an anti-algebra map. The other axioms can be easily checked, e.g. $(S\circ *)^2(\lambda_B^A) = S\circ *(g_{\bar B C}\lambda_D^C g^{\bar A D}) = g_{\bar B C}g_{\bar D E}\lambda_F^E g^{\bar C F}g^{\bar A D} = \lambda_B^A$.

In order to show that $\mathfrak{R}$ descends to a dual quasitriangular structure on $\mathscr{L}_q$, we only have to prove that the fundamental representations defined above respect the metric relation. By explicit calculation one can show $q^{-2}R_{L\,CM}^{AE}R_{L\,DF}^{BM}g^{CD} = g^{AB}\delta_F^E$ and $q^2R_{L\,MC}^{-1\,EA}R_{L\,FD}^{-1\,MB}g^{CD} = g^{AB}\delta_F^E$, and hence

$$
\begin{aligned}
\rho_+(\lambda_C^A\lambda_D^B g^{CD})_F^E &= \rho_+(\lambda_C^A)_M^E\rho_+(\lambda_D^B)_F^M g^{CD}\\
&= q^{-2}R_{L\,CM}^{AE}R_{L\,DF}^{BM}g^{CD}\\
&= g^{AB}\delta_F^E\\
&= \rho_+(g^{AB})_F^E\,,\\
\rho_-(\lambda_C^A\lambda_D^B g^{CD})_F^E &= \rho_-(\lambda_C^A)_M^E\rho_-(\lambda_D^B)_F^M g^{CD}\\
&= q^2R_{L\,MC}^{-1\,EA}R_{L\,FD}^{-1\,MB}g^{CD}\\
&= g^{AB}\delta_F^E\\
&= \rho_-(g^{AB})_F^E\,.
\end{aligned}
$$

This is the place where one needs the normalisation factor $\kappa$.

(ii) We have to prove that $\varphi$ can be extended as a $*$-Hopf algebra map. Using the algebra structure on $SL_q(2,\mathbb{C})$, which can also be written as $t^{\dagger b}{}_d R_{cg}^{ad}t_e^c = t_c^a R_{ed}^{cb}t^{\dagger d}{}_g$ or $t^{\dagger d}{}_g\tilde R_{ed}^{cb}t_c^a = t_e^c\tilde R_{cg}^{ad}t^{\dagger b}{}_d$, one can show:

$$
\begin{aligned}
\varphi(\lambda_A^E\lambda_B^F R_{L\,DC}^{BA}) &= \varphi(\lambda_A^E)\varphi(\lambda_B^F)R_{g_0a_0}^{d_0h_0}R_{h_1b_0}^{a_1g_0}R_{g_1c_1}^{b_1h_1}\tilde R_{d_1h_0}^{g_1c_0}\\
&= t^{\dagger a_0}{}_{e_0}t_{a_1}^{e_1}R_{h_1b_0}^{a_1g_0}t^{\dagger b_0}{}_{f_0}t_{b_1}^{f_1}R_{g_0a_0}^{d_0h_0}R_{g_1c_1}^{b_1h_1}\tilde R_{d_1h_0}^{g_1c_0}\\
&= R_{g_0a_0}^{d_0h_0}t^{\dagger a_0}{}_{e_0}t^{\dagger g_0}{}_{b_0}R_{a_1f_0}^{e_1b_0}t_{h_1}^{a_1}t_{b_1}^{f_1}R_{g_1c_1}^{b_1h_1}\tilde R_{d_1h_0}^{g_1c_0}\\
&= R_{b_0e_0}^{g_0a_0}R_{a_1f_0}^{e_1b_0}R_{b_1h_1}^{f_1a_1}t^{\dagger d_0}{}_{g_0}t^{\dagger h_0}{}_{a_0}\tilde R_{d_1h_0}^{g_1c_0}t_{g_1}^{b_1}t_{c_1}^{h_1}\\
&= R_{b_0e_0}^{g_0a_0}R_{a_1f_0}^{e_1b_0}R_{b_1h_1}^{f_1a_1}\tilde R_{g_1a_0}^{b_1h_0}t^{\dagger d_0}{}_{g_0}t_{d_1}^{g_1}t^{\dagger c_0}{}_{h_0}t_{c_1}^{h_1}\\
&= \varphi(R_{L\,GH}^{FE}\lambda_D^G\lambda_C^H)\,,
\end{aligned}
$$

$$\begin{aligned}\varphi(\lambda^A_C\lambda^B_D g^{CD}) &= q\epsilon_{c_0\alpha}t^{\dagger c_0}{}_{a_0}t^{a_1}_{c_1}R^{c_1\alpha}_{\beta d_0}t^{\dagger d_0}{}_{b_0}t^{b_1}_{d_1}\epsilon^{\beta d_1}\\ &= q\epsilon_{c_0\alpha}t^{\dagger c_0}{}_{a_0}t^{\dagger\alpha}{}_{d_0}R^{a_1d_0}_{c_1b_0}t^{c_1}_{\beta}t^{b_1}_{d_1}\epsilon^{\beta d_1}\\ &= q\epsilon_{a_0d_0}R^{a_1d_0}_{c_1b_0}\epsilon^{c_1b_1}\\ &= \varphi(g^{AB})\,.\end{aligned}$$

This also implies $\varphi(1)=1$ and $\varphi(S\lambda)=S(\varphi(\lambda))$. Similarly we find for the coproduct:

$$\Delta(\varphi(\lambda^A_B)) = \Delta(t^{\dagger b_0}{}_{a_0}t^{a_1}_{b_1}) = t^{\dagger c_0}{}_{a_0}t^{a_1}_{c_1}\otimes t^{\dagger b_0}{}_{c_0}t^{c_1}_{b_1} = \varphi(\lambda^A_C)\otimes\varphi(\lambda^C_B) = (\varphi\otimes\varphi)(\Delta\lambda^A_B)\,,$$

the counit $\varepsilon(\varphi(\lambda^A_B)) = \delta^A_B = \varphi(\varepsilon(\lambda^A_B))$, and the $*$-structure:

$$\varphi((\lambda^A_B)^*) = \varphi(\lambda^{\bar A}_{\bar B}) = t^{\dagger b_1}{}_{a_1}t^{a_0}_{b_0} = (t^{\dagger b_0}{}_{a_0}t^{a_1}_{b_1})^* = \varphi(\lambda^A_B)^*\,.$$

Thus $\varphi$ can be extended as a $*$-Hopf algebra map. □

## 4. $q$-Minkowski Space

We follow the general approach of [13] to construct a $q$-Minkowski space with braided coaddition: Let $R$ be an invertible solution of the $n$-dimensional QYBE and let $R'$ be a second matrix such that they satisfy the mixed QYBEs,

$$R'_{12}R_{13}R_{23} = R_{23}R_{13}R'_{12},\quad R_{12}R_{13}R'_{23} = R'_{23}R_{13}R_{12}\,. \tag{2}$$

Define an algebra of *quantum covectors* $V^*(R')$ as the free associative $\mathbb{C}$-algebra generated by 1 and $n$ generators $x_a$ with relations $x_ax_b = x_dx_cR'^{cd}_{ab}$. Similarly, the algebra of *quantum vectors* $V(R')$ is generated by 1 and $v^a$'s with relations $v^av^b = R'^{ab}_{cd}v^dv^c$. The algebra $V^*(R')$ is a right $A(R)$-comodule with coaction $x_a\mapsto x_b\otimes t^b_a$ and the braiding between two copies of $V(R')$ is given by $\Psi(x_a\otimes x_b) = x_d\otimes x_cR^{cd}_{ab}$. For $V^*(R')$ to be a braided Hopf algebra in with *braided coaddition* $\underline{\Delta}x_a = x_a\otimes 1+1\otimes x_a$, $\underline{\varepsilon}x_a=0$ and $\underline{S}x_a=-x_a$ in $\mathcal{M}^{A(R)}$, the two matrices $R$ and $R'$ have to satisfy the relation

$$0 = (PR+1)(PR'-1) = (PR'-1)(PR+1)\,, \tag{3}$$

where $P$ denotes the permutation matrix. This relation ensures that $\underline{\Delta}$ extends as an algebra map.

Thus we need to find a $q$-Minkowski R-matrix $R_M$ which satisfies (2) and (3) with $R_L$. We solve this problem more generally for any composed R-matrix $\mathbf{R}$ from Lemma 3.2.

**Lemma 4.1.** *Let $R$ be a Hecke type solution of the QYBE. Then the R-matrix*

$$\mathbf{R}'^{AB}_{CD} = R^{-1\,\delta c_0}_{b_0\alpha}R^{b_1\alpha}_{\beta a_0}R^{a_1\beta}_{\gamma d_1}\tilde{R}^{\gamma d_0}_{c_1\delta}\,, \tag{4}$$

*and the matrix $\mathbf{R}$ from Lemma 3.2 satisfy* (3) *and the mixed QYBEs* (2).

*Proof.* We have to show $P\mathbf{R}P\mathbf{R}' = P\mathbf{R}'P\mathbf{R} = P\mathbf{R} - P\mathbf{R}' + 1$. The first term can be rewritten as

$$\begin{aligned}(P\mathbf{R}P\mathbf{R}')^{AB}_{EF} &= \mathbf{R}^{BA}_{DC}\mathbf{R}'^{CD}_{EF}\\ &= (R^{a_1\beta}_{\gamma b_0}R^{b_1\gamma}_{\delta c_1}R^{d_0\alpha}_{\beta a_0}\tilde{R}^{\delta c_0}_{d_1\alpha})(R^{-1\,\varepsilon e_0}_{d_0\lambda}R^{d_1\lambda}_{\mu c_0}R^{c_1\mu}_{\nu f_1}\tilde{R}^{\nu f_0}_{e_1\varepsilon})\\ &= R^{a_1\beta}_{\gamma b_0}R^{b_1\gamma}_{\delta c_1}R^{d_0\alpha}_{\beta a_0}R^{-1\,\varepsilon e_0}_{d_0\alpha}R^{c_1\delta}_{\nu f_1}\tilde{R}^{\nu f_0}_{e_1\varepsilon}\\ &= \delta^{a_0}_{e_0}R^{a_1\beta}_{\gamma b_0}R^{b_1\gamma}_{\delta c_1}R^{c_1\delta}_{\nu f_1}\tilde{R}^{\nu f_0}_{e_1\beta}\,,\end{aligned}$$

and indeed equals the second term

$$\begin{aligned}(P\mathbf{R}'P\mathbf{R})^{AB}_{EF} &= \mathbf{R}'^{BA}_{DC}\mathbf{R}^{CD}_{EF}\\ &= R^{-1\,\alpha d_0}_{a_0\beta}R^{a_1\beta}_{\gamma b_0}R^{b_1\gamma}_{\delta c_1}\tilde{R}^{\delta c_0}_{d_1\alpha}R^{e_0\varepsilon}_{\lambda d_0}R^{d_1\lambda}_{\mu c_0}R^{c_1\mu}_{\nu f_1}\tilde{R}^{\nu f_0}_{e_1\varepsilon}\\ &= R^{-1\,\alpha d_0}_{a_0\beta}R^{a_1\beta}_{\gamma b_0}R^{b_1\gamma}_{\delta c_1}R^{e_0\varepsilon}_{\alpha d_0}R^{c_1\delta}_{\nu f_1}\tilde{R}^{\nu f_0}_{e_1\varepsilon}\\ &= \delta^{a_0}_{e_0}R^{a_1\beta}_{\gamma b_0}R^{b_1\gamma}_{\delta c_1}R^{c_1\delta}_{\nu f_1}\tilde{R}^{\nu f_0}_{e_1\beta}\,.\end{aligned}$$

Using the Hecke property $PRP = R^{-1} + (q - q^{-1})P$ of the matrix $R$, we can further simplify this expression as:

$$\begin{aligned}(P\mathbf{R}P\mathbf{R}')^{AB}_{EF} &= \delta^{a_0}_{e_0}R^{a_1\beta}_{\gamma b_0}(R^{-1\,\gamma b_1}_{c_1\delta} + (q - q^{-1})P^{\gamma b_1}_{c_1\delta})R^{c_1\delta}_{\nu f_1}\tilde{R}^{\nu f_0}_{e_1\beta}\\ &= \delta^{AB}_{EF} + (q - q^{-1})\delta^{a_0}_{e_0}R^{a_1\beta}_{c_1 b_0}R^{c_1 b_1}_{\nu f_1}\tilde{R}^{\nu f_0}_{e_1\beta}\,.\end{aligned}$$

This is equal to the third term:

$$\begin{aligned}\delta^{AB}_{EF} + P\mathbf{R}'^{AB}_{EF} - P\mathbf{R}^{AB}_{EF} &= \delta^{AB}_{EF} + R^{-1\,\alpha e_0}_{a_0\beta}R^{a_1\beta}_{\gamma b_0}R^{b_1\gamma}_{\delta f_1}\tilde{R}^{\delta f_0}_{e_1\alpha} - R^{e_0\alpha}_{\beta a_0}R^{a_1\beta}_{\gamma b_0}R^{b_1\gamma}_{\delta f_1}\tilde{R}^{\delta f_0}_{e_1\alpha}\\ &= \delta^{AB}_{EF} + (R^{-1\,\alpha e_0}_{a_0\beta} - R^{e_0\alpha}_{\beta a_0})R^{a_1\beta}_{\gamma b_0}R^{b_1\gamma}_{\delta f_1}\tilde{R}^{\delta f_0}_{e_1\alpha}\\ &= \delta^{AB}_{EF} + (q - q^{-1})\delta^{a_0}_{e_0}R^{a_1\beta}_{\gamma b_0}R^{b_1\gamma}_{\delta f_1}\tilde{R}^{\delta f_0}_{e_1\beta}\,.\end{aligned}$$

Hence, $\mathbf{R}$ and $\mathbf{R}'$ satisfy (3). In order to show that $\mathbf{R}$ and $\mathbf{R}'$ satisfy the mixed QYBEs, recall from [10] that $\Theta$ and $\mathbf{R}'$ satisfy (2) and use an argument similar to the one employed in the proof of Lemma 3.2. □

Thus for any Hecke type solution $R$ of the QYBE, the algebra of quantum covectors $V^*(\mathbf{R}')$ is a braided Hopf algebra with braided coaddition $\underline{\Delta}x_A = x_A\underline{\otimes}1 + 1\underline{\otimes}x_A$ in the braided category $\mathcal{M}^{A(\mathbf{R})}$. However, $V^*(\mathbf{R}')$ is nothing but the algebra of braided matrices $B(R)$ from [10]. Denoting $\mathbf{R}'$ from (4) in the case of the $SU_q(2)$ R-matrix by $R_M$, we define *q-Minkowski space* $M_q$ as $2 \times 2$ braided matrices $V^*(R_M) \cong B(R)$. It is necessary to introduce a new name for this algebra in order to avoid confusion: braided matrices $B(R)$ were constructed as the transmutation of the algebra of quantum matrices $A(R)$ and have the structure of a braided bialgebra with a braided coalgebra structure of matrix multiplication type. $q$-Minkowski space $M_q$ on the other hand is a braided Hopf algebra with braided coaddition and lives in a different category. It has the same algebra structure as $B(R)$ which is sufficient to ensure a *q-spinor decomposition* as an algebra map $M_q \to V(R)\underline{\otimes}V^*(R)$ [14].

The explicit algebra relations in $M_q$ were given in [10] as

$$\begin{aligned} ab &= q^{-2}ba, & ac &= q^2 ca\,, \\ ad &= da, & bc &= cb + (1-q^{-2})a(d-a)\,, \\ bd &= db - (1-q^{-2})ab, & cd &= dc + (1-q^{-2})bc\,, \end{aligned}$$

where $x = (a, b, c, d)$. Central elements in this algebra are the *braided trace* $qd + q^{-1}a$ and the *braided determinant* $\underline{\det} = ad - q^2 cb$. Reflecting the fact that ordinary determinants are multiplicative but not additive, the braided determinant is grouplike only in $B(R)$, but not in $M_q$.

The appropriate $*$-structure on $B(R)$ was discussed in [15]. Its axioms are slightly different from the ones recalled in the preliminaries. The main difference is that for braided bialgebras one requires $\underline{\Delta} \circ \underline{*} = \tau \circ (\underline{*} \otimes \underline{*}) \circ \underline{\Delta}$, where $\tau$ is the twist map. In [15] it was shown that $(x_A)^{\underline{*}} = x_{\bar{A}}$ defines such a $*$-structure on $B(R)$ and it is easy to see that this also defines a $*$-structure on $M_q$ as braided group with braided coaddition.

Similar to the commutative case, where the norm on Minkowski space is given by the determinant of the corresponding matrices, we find

**Proposition 4.2.** *The q-norm on q-Minkowski space is given by* $x_A x_B g^{AB} = (q^{-1} + q)\underline{\det}$, *is central and also real with respect to* $\underline{*}$.

Next, we have to address the question of the coaction of $\mathscr{L}_q$ on $q$-Minkowski space. The problem is that $M_q$ is an $\mathscr{L}_q$-comodule, but *not* a braided Hopf algebra in $\mathscr{M}^{\mathscr{L}_q}$ because we had to rescale $R_L$ in Proposition 3.3 with a normalization factor $\kappa = 1/q$ in order to obtain a dual quasitriangular structure on $\mathscr{L}_q$. This normalization is different from the one required in Proposition 4.1, which ensures that $M_q$ is a braided Hopf algebra in $\mathscr{M}^{A(R_L)}$. This problem of different normalisations was already encountered in [13]. It can be solved by extending $\mathscr{L}_q$ by a single invertible grouplike element $\varsigma$ which commutes with $\lambda$. Let $\mathbb{C}\mathbb{Z}$ denote the $\mathbb{C}$-vector space with monomials $\varsigma^a$, $a \in \mathbb{Z}$ as basis. It has the structure of a commutative and cocommutative Hopf algebra with $\Delta\varsigma = \varsigma \otimes \varsigma$, $\varepsilon\varsigma = 1$, $S\varsigma = \varsigma^{-1}$. We form the tensor product of $\mathscr{L}_q$ with $\mathbb{C}\mathbb{Z}$ and denote this extended $q$-Lorentz group by $\tilde{\mathscr{L}}_q$. The advantage of this construction is that one can define a dual quasitriangular structure on $\mathbb{C}\mathbb{Z}$ by $\mathfrak{R}(\varsigma^a \otimes \varsigma^b) = \kappa^{-ab}$, $a$, $b \in \mathbb{Z}$, which extends to a dual quasitriangular structure on $\tilde{\mathscr{L}}_q$ and "absorbs" the normalization factor $\kappa$. With the $q$-Lorentz group thus extended, one finds:

**Proposition 4.3.** *q-Minkowski space* $M_q$ *is a right* $\tilde{\mathscr{L}}_q$*-comodule* $*$*-algebra with coaction* $\tilde{\beta}_{M_q} : x_A \mapsto x_B \otimes \lambda^B_A \varsigma$. *Moreover, it is a braided Hopf algebra with braided coaddition in* $\mathscr{M}^{\tilde{\mathscr{L}}_q}$. *The braided coaddition is a right* $\tilde{\mathscr{L}}_q$*-comodule morphism between comodule algebras, i.e.* $\tilde{\mathscr{L}}_q$*-covariant.*

*Proof.* One can show by explicit calculation that the generators $\lambda$ of $\mathscr{L}_q$ obey $R_M \lambda_1 \lambda_2 = \lambda_2 \lambda_1 R_M$. This then implies that $M_q$ is a right $\mathscr{L}_q$-comodule algebra with coaction $\beta : x_A \mapsto x_B \otimes \lambda^B_A$. Since $\varsigma$ commutes with the generators of $\mathscr{L}_q$, we immediately find that $q$-Minkowski space is also a right $\tilde{\mathscr{L}}_q$-comodule $*$-algebra with coaction $\tilde{\beta}_{M_q} : x_A \mapsto x_B \otimes \lambda^B_A \varsigma$. The coaction by $\varsigma$ measures the degree (scaling dimension) of monomials in $x$, and is often called *dilation element* [20, 13].

It remains to show that $\underline{\Delta} : M_q \to M_q \underline{\otimes} M_q$ is a right $\tilde{\mathscr{L}}_q$-comodule morphism. On the generators of $M_q$, the coaction $\tilde{\beta}_{M_q}$ satisfies $(\underline{\Delta} \otimes id) \circ \tilde{\beta}_{M_q} = \tilde{\beta}_{M_q \underline{\otimes} M_q} \circ \underline{\Delta}$. Since both $\underline{\Delta}$ and the coactions $\tilde{\beta}_{M_q}$ and $\tilde{\beta}_{M_q \underline{\otimes} M_q}$ are algebra maps this extends to $M_q$. Thus the entire structure of $q$-Minkowski space is covariant under the coaction by the $q$-Lorentz group. □

Next note that the $q$-metric $g^{AB}$ does not only determine a $q$-norm in $M_q$, but it can also be used to raise and lower indices of $q$-Lorentz tensors in a covariant fashion:

**Proposition 4.4.** *There is a q-metric induced braided $*$-Hopf algebra isomorphism $G : V^*(R_M) \cong V(R_M)$ given by $x_A \mapsto v^A = x_B g^{AB}$, which is also an $\tilde{\mathscr{L}}_q$-comodule morphism.*

*Proof.* $V(R_M)$ is a braided Hopf algebra in $\mathscr{M}^{\tilde{\mathscr{L}}_q}$ with generators $v^A$, braided coaddition $\underline{\Delta} v^A = v^A \underline{\otimes} 1 + 1 \underline{\otimes} v^A$, $*$-structure $(v^A)^{\underline{*}} = v^{\bar{A}}$, and coaction $\beta : v^A \mapsto v^B \otimes S\lambda^A_B \varsigma$. In order to prove that $G$ extends as a $*$-Hopf algebra isomorphism, note that one can show by explicit calculation that $R^{KL}_{M\,EF} = g_{FP} g_{EQ} R^{QP}_{M\,AB} g^{KA} g^{LB}$. This implies $G(x_K)G(x_L) = x_A g^{KA} x_B g^{LB} = x_C x_D R^{DC}_{M\,AB} g^{KA} g^{LB} = x_C g^{FC} x_D g^{ED} g_{FP} g_{EQ} R^{QP}_{M\,AB} g^{KA} g^{LB} = R^{KL}_{M\,EF} G(u_F) G(u_E)$. Furthermore, $G$ is a $*$-homomorphism: $G(x_K)^{\underline{*}} = x_A g^{\bar{K}A} = G(x_{\bar{K}}) = G((x_K)^{\underline{*}})$. On the generators, we can also immediately verify that $G \circ \underline{\Delta} = \underline{\Delta} \circ G$ and $S \circ G = G \circ S$. Because of the algebra homomorphism properties of $G$, $\underline{\Delta}$ and $S$, this result extends to products. It remains to show that $G$ is a right $\tilde{\mathscr{L}}_q$-comodule morphism. On the generators, we have $\beta \circ G(x_K) = x_A \otimes \lambda^A_B g^{KB} \varsigma = u_A g^{BA} \otimes S\lambda^K_B \varsigma = (G \otimes id) \circ \tilde{\beta}_{M_q}(x_K)$, and since $\tilde{\beta}_{M_q}$, $\beta$ and $G$ are algebra maps, this extends to $M_q$. □

For sake of completeness, we also list the explicit form of the braiding between two copies of $M_q$, which is quite different from the braiding on $B(R)$ with its multiplicative braided coalgebra structure [10]:

$$
\begin{aligned}
\Psi(a \otimes a) &= q^2 a \otimes a\,, \\
\Psi(a \otimes b) &= b \otimes a\,, \\
\Psi(a \otimes c) &= q^2 c \otimes a + (q^2 - 1) a \otimes c\,, \\
\Psi(a \otimes d) &= d \otimes a + (q^2 - 1)(b \otimes c + (1 - q^{-2}) a \otimes a)\,, \\
\Psi(b \otimes a) &= q^2 a \otimes b + (q^2 - 1) b \otimes a\,, \\
\Psi(b \otimes b) &= q^2 b \otimes b\,, \\
\Psi(b \otimes c) &= c \otimes b + (1 - q^{-2})(d \otimes a + a \otimes d \\
&\qquad + (1 - q^{-2}) b \otimes c + (2 - q^{-2}) a \otimes a)\,, \\
\Psi(b \otimes d) &= d \otimes b + (q^2 - 1)(b \otimes (d - q^{-2} a) + (1 - q^{-2}) a \otimes b)\,,
\end{aligned}
$$

$$\begin{aligned}
\Psi(c\otimes a) &= a\otimes c\,,\\
\Psi(c\otimes b) &= b\otimes c+(1-q^{-2})a\otimes a\,,\\
\Psi(c\otimes c) &= q^2 c\otimes c\,,\\
\Psi(c\otimes d) &= q^2 d\otimes c+(q^2-1)c\otimes a\,,\\
\Psi(d\otimes a) &= a\otimes d+(q^2-1)(b\otimes c+(1-q^{-2})a\otimes a)\,,\\
\Psi(d\otimes b) &= q^2 b\otimes d+(q^2-1)a\otimes b\,,\\
\Psi(d\otimes c) &= c\otimes d+(q^2-1)((d-q^{-2}a)\otimes c-(1-q^{-2})c\otimes a)\,,\\
\Psi(d\otimes d) &= q^2 d\otimes d+(q^2-1)(c\otimes b-q^{-2}b\otimes c-q^{-2}(1-q^{-2})a\otimes a)\,.
\end{aligned}$$

Finally note that since $M_q$ is a braided group in the category of right $\tilde{\mathscr{L}}_q$-comodules, one can apply the results of [13] and construct a *q-Poincaré group* as a semidirect product $M_q \rtimes \tilde{\mathscr{L}}_q$. Similar to the double cross product, the algebra structure is given in terms of compatible actions. The resulting Hopf algebra structure is given explicitly in [13].

## 5. *q*-Lorentz Group of Enveloping Algebra Type

Instead of deforming the function algebra $\mathscr{P}(L)$, we would also deform the universal enveloping algebra of the Lorentz group. However, the standard procedure for $q$-deforming enveloping algebras is not applicable in this case, since the Lorentz group is not simple. We shall rather use the relation $U(so(3,1)) \cong U(su(2)) \otimes U(su(2))$ and the standard $q$-deformation $U_q(su(2))$ of the enveloping algebra of the simple Lie group $SU(2)$ to construct a $q$-deformation of $U(so(3,1))$. We also investigate how this algebra is related to the $q$-Lorentz group of function algebra type.

For sake of clarity, we first recall some standard constructions: For any algebra of quantum matrices $A(R)$, there is a *canonical dual* given by the bialgebra $U(R)$ [18, 8] with generators $l_b^{a\pm}$ and relations $Rl_2^{\pm}l_1^{\pm} = l_1^{\pm}l_2^{\pm}R$ and $Rl_2^{+}l_1^{-} = l_1^{-}l_2^{+}R$. The dual pairing is given by $\langle t_1, l_2^{\pm}\rangle = R^{\pm}$, where $R^+ = R_{12}$ and $R^- = R_{21}^{-1}$. This bialgebra is universal in the sense that there exists a bialgebra map to any other bialgebra dually paired with $A(R)$ [8].

In the special case of the $SU(2)$ R-matrix, $U(R)$ is known to be related to a deformation of the universal enveloping algebra of $su(2)$. The algebra $U_q(su(2))$ is defined as $U(R)$ with the further relations implied by the ansatz

$$l^+ = \begin{pmatrix} q^{\frac{H}{2}} & 0 \\ q^{-1/2}(q-q^{-1})X_+ & q^{-\frac{H}{2}} \end{pmatrix}, \quad l^- = \begin{pmatrix} q^{-\frac{H}{2}} & q^{1/2}(q-q^{-1})X_- \\ 0 & q^{\frac{H}{2}} \end{pmatrix}.$$

Using M. Jimbo's convention [5] and the usual $*$-structure for $U_q(su(2))$ one finds explicitly:

$$q^{\frac{H}{2}}X_{\pm}q^{-\frac{H}{2}} = q^{\pm}X_{\pm}, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q-q^{-1}}, \quad H^* = H,\ X_{\pm}^* = X_{\mp}\,.$$

This algebra is a quasitriangular $*$-Hopf algebra with coalgebra structure of matrix multiplication type and a well-known quasitriangular structure $\mathcal{R}$ and is dual to $SU_q(2)$ as a $*$-Hopf algebra (i.e. not only as bialgebras). The generators of $U(R)$ are given by $l^+ = \mathcal{R}^{(1)}\langle\tau, \mathcal{R}^{(2)}\rangle$ and $l^- = \langle\tau, \mathcal{R}^{-1(1)}\rangle\mathcal{R}^{-1(2)}$ in terms of the pairing and the dual quasitriangular structure and we also have $\langle\tau_1 \otimes \tau_2, \mathcal{R}\rangle = q^{-1/2}R_{12}$, where $\tau$ denotes the generators of $SU_q(2)$. We used the notation $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ and omitted summation signs.

A natural $q$-deformed generalization of $U(so(3,1))$ is given by the *twisted square* [19] $U_q(su(2)) \otimes_{\mathcal{R}} U_q(su(2))$ of two copies of the standard $q$-deformation of the universal enveloping algebra of $su(2)$ [5]. The twisted square has the tensor product algebra structure and a "twisted" coalgebra structure given in terms of the standard quasitriangular structure $\mathcal{R}$ on $U_q(su(2))$ as $\Delta_{\mathcal{R}}(x \otimes y) = \mathcal{R}_{23}^{-1}\Delta_{13}(x)\Delta_{24}(y)\mathcal{R}_{23}$ and corresponding antipode $S_{\mathcal{R}} = \mathcal{R}_{21}(S \otimes S)\mathcal{R}_{21}^{-1}$. It also has a $*$-structure defined by $(l_1^{\pm} \otimes l_2^{\pm})^* = \mathcal{R}_{21}(l_2^{\pm *} \otimes l_1^{\pm *})\mathcal{R}_{21}^{-1}$. The $*$-Hopf algebra pairing between $U_q(su(2))$ and $SU_q(2)$ then extends to a $*$-Hopf algebra pairing between the twisted square and the double cross product $SU_q(2) \bowtie SU_q(2)$ [13]. The twisted square is quasitriangular, but has more than one quasitriangular structure, one of which was given in [19]. For our purposes we need a different quasitriangular structure:

**Lemma 5.1.** *$\mathcal{R}_L = \mathcal{R}_{41}^{-1}\mathcal{R}_{24}\mathcal{R}_{13}\mathcal{R}_{23}$ defines a quasitriangular structure for $U_q(so(3,1))$.*

*Proof.* Using the fact that $\mathcal{R}$ obeys the axioms of a quasitriangular structure (as given e.g. in [8, Sect. 1.5]) we obtain:

$$\begin{aligned}
\mathcal{R}_L\Delta_{\mathcal{R}}(\cdot)\mathcal{R}_L^{-1} &= \mathcal{R}_{41}^{-1}\mathcal{R}_{24}\mathcal{R}_{13}\mathcal{R}_{23}(\mathcal{R}_{23}^{-1}\Delta_{13}\Delta_{24}\mathcal{R}_{23})\mathcal{R}_{23}^{-1}\mathcal{R}_{13}^{-1}\mathcal{R}_{24}^{-1}\mathcal{R}_{41} \\
&= \mathcal{R}_{41}^{-1}(\mathcal{R}_{24}\Delta_{24}\mathcal{R}_{24}^{-1})(\mathcal{R}_{13}\Delta_{13}\mathcal{R}_{13}^{-1})\mathcal{R}_{41} \\
&= \mathcal{R}_{41}^{-1}\Delta_{42}\Delta_{31}\mathcal{R}_{41} \\
&= \tau \circ \Delta_{\mathcal{R}}\,, \\
(id \otimes \Delta_{\mathcal{R}})(\mathcal{R}_L) &= \mathcal{R}_{45}^{-1}(id_{12} \otimes \Delta_{35}\Delta_{46})\mathcal{R}_{41}^{-1}\mathcal{R}_{24}\mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}_{45} \\
&= \mathcal{R}_{45}^{-1}(id \otimes \Delta_{46})(\mathcal{R}_{41}^{-1}\mathcal{R}_{24})(id \otimes \Delta_{35})(\mathcal{R}_{13}\mathcal{R}_{23})\mathcal{R}_{45} \\
&= \mathcal{R}_{45}^{-1}\mathcal{R}_{61}^{-1}\mathcal{R}_{41}^{-1}\mathcal{R}_{26}\mathcal{R}_{24}\mathcal{R}_{15}\mathcal{R}_{13}\mathcal{R}_{25}\mathcal{R}_{23}\mathcal{R}_{45} \\
&= (\mathcal{R}_{61}^{-1}\mathcal{R}_{26}\mathcal{R}_{15}\mathcal{R}_{25})(\mathcal{R}_{41}^{-1}\mathcal{R}_{24}\mathcal{R}_{13}\mathcal{R}_{23}) \\
&= \mathcal{R}_{L13}\mathcal{R}_{L12}\,.
\end{aligned}$$

The proof of $(\Delta \otimes id)(\mathcal{R}_L) = \mathcal{R}_{L13}\mathcal{R}_{L23}$ is similar. Thus $\mathcal{R}_L$ is a quasitriangular structure for the twisted square. □

By virtue of the $*$-Hopf algebra map $\varphi$, $\mathcal{L}_q$ is then also dually paired with the twisted square $U_q(so(3,1))$, and one finds

**Proposition 5.2.**

$$\langle\varphi(\lambda_C^A) \otimes \varphi(\lambda_D^B), \mathcal{R}_L\rangle = \kappa R_{L\,CD}^{AB}\,.$$

*Proof.*

$$\begin{aligned}\langle\varphi(\lambda^A_C)\otimes\varphi(\lambda^B_D),\mathcal{R}_L\rangle &=\langle S\tau^{c_0}_{a_0}\otimes\tau^{a_1}_{c_1}\otimes S\tau^{d_0}_{b_0}\otimes\tau^{b_1}_{d_1},\ \mathcal{R}^{-1}_{41}\mathcal{R}_{24}\mathcal{R}_{13}\mathcal{R}_{23}\rangle\\ &=\langle S\tau^{\beta}_{a_0},\mathcal{R}^{-1}_{(2)}\rangle\langle S\tau^{c_0}_{\beta},\mathcal{R}_{(1)}\rangle\langle\tau^{a_1}_{\delta},\mathcal{R}_{(1)}\rangle\langle\tau^{\delta}_{c_1},\mathcal{R}_{(1)}\rangle\\ &\quad\langle S\tau^{\alpha}_{b_0},\mathcal{R}_{(2)}\rangle\langle S\tau^{d_0}_{\alpha},\mathcal{R}_{(2)}\rangle\langle\tau^{b_1}_{\gamma},\mathcal{R}^{-1}_{(1)}\rangle\langle\tau^{\gamma}_{d_1},\mathcal{R}_{(2)}\rangle\\ &=\langle S\tau^{c_0}_{\beta}\otimes S\tau^{\alpha}_{b_0},\mathcal{R}\rangle\langle\tau^{b_1}_{\gamma}\otimes S\tau^{\beta}_{a_0},\mathcal{R}^{-1}\rangle\\ &\quad\langle\tau^{a_1}_{\delta}\otimes\tau^{\gamma}_{d_1},\mathcal{R}\rangle\langle\tau^{\delta}_{c_1}\otimes S\tau^{d_0}_{\alpha},\mathcal{R}\rangle\\ &=q^{-1}R^{c_0\alpha}{}_{\beta b_0}R^{b_1\beta}{}_{\gamma a_0}R^{a_1\gamma}{}_{\delta d_1}\tilde{R}^{\delta d_0}{}_{c_1\alpha}\\ &=q^{-1}R_L{}^{AB}_{CD}\,,\end{aligned}$$

using $\langle\tau_1\otimes\tau_2,\mathcal{R}\rangle=q^{-1/2}R_{12}$ and $\tilde{R}^{ab}_{cd}=q^{-1}\varepsilon_{dm}R^{am}_{cn}\varepsilon^{bn}$ and the properties of quasi-triangular structures as given in e.g. [8, Sect. 1.5]. □

On the other hand, however, there is the canonical dual of $\mathcal{L}_q$ given by the algebra $U(\kappa R_L)$. This bialgebra maps into the twisted square with a bialgebra map

$$\psi: U(\kappa R_L)\to U_q(so(3,1))$$

defined according to the general construction from [8] by $\psi(l^{+A}_B)=\mathcal{R}^{(1)}_L\langle\varphi(\lambda^A_B),\mathcal{R}^{(2)}_L\rangle$ and $\psi(l^{-A}_B)=\langle\varphi(\lambda^A_B),\mathcal{R}_L^{-1(1)}\rangle\mathcal{R}_L^{-1(2)}$. With Proposition 5.2 it follows that this map has the property $\langle\varphi(\lambda_1),\psi(L^{\pm}_2)\rangle=\kappa R^{\pm}_L$, i.e. the restriction of the pairing between the double cross product and the twisted square to the images of $\phi$ and $\psi$ recovers the standard pairing between $\mathcal{L}_q$ and its canonical dual.

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