

Topological Quantum Field Theory and Invariants of Graphs for Quantum Groups

Anna Beliakova¹★, Bergfinnur Durhuus²

¹ Institut für Theoretische Physik, Freie Universität Berlin, Berlin, Germany

² Mathematics Institute, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark

Received: 8 September 1993/in revised form: 15 March 1994

Abstract: On the basis of generalized $6j$ -symbols we give a formulation of topological quantum field theories for 3-manifolds including observables in the form of coloured graphs. It is shown that the $6j$ -symbols associated with deformations of the classical groups at primitive even roots of unity provide examples of this construction. Computational methods are developed which, in particular, yield the dimensions of the state spaces as well as a rather simple proof of the relation, previously found by Turaev and Walker for the case of $U_q(sl(2, \mathbb{C}))$, between these models and corresponding ones based on the ribbon graph construction of Reshetikhin and Turaev.

1. Introduction

In ref. [TV] a novel combinatorial approach to 3-dimensional topological quantum field theory was proposed. Its basis is the observation that the $6j$ -symbols of $U_q(sl(2, \mathbb{C}))$ obey the symmetries of a tetrahedron and satisfy identities which may also be interpreted geometrically in terms of glued tetrahedra and which lead to the possibility of associating state sums (partition functions) with 3-dimensional triangulated manifolds which are independent of the triangulation, i.e. they are topological invariants.

This approach was generalized in [DJN, D] to a large class of algebras (replacing $U_q(sl(2, \mathbb{C}))$) with associated generalized $6j$ -symbols, thus leading to a class of (unitary) 3-dimensional topological quantum field theories satisfying all the standard properties (see [Wi, At]). Similar generalizations also appeared in [FG and T].

In the case of $U_q(sl(2, \mathbb{C}))$ a second generalization was introduced in [KS] by including observables in the form of coloured graphs on the boundary or the interior of the manifolds. This leads to effective computational methods which were used to calculate e.g. the dimensions of the state spaces in this model. Invariants of links and graphs in 3-manifolds had previously also been considered in [T2].

★ Supported by DAAD and DFG, SFB 288 “Differentialgeometrie und Quantenphysik”

In this paper we combine the approaches of [DJN, D] and [KS] by providing a simple geometric interpretation of the state sums introduced in [KS]. This leads to a simplification of the discussion in [KS] in a more general setting, including the case of 6j-symbols associated to quantum deformations of an arbitrary classical group.

As an application we calculate the dimensions of the state spaces in the general case in terms of the fusion matrices and we give a proof that the partition function $Z(M)$ associated to a closed 3-manifold M is related to the invariant $\tau(M)$ introduced by Reshetikhin and Turaev [RT2] in terms of ribbon graphs by

$$Z(M) = |\tau(M)|^2 \tag{1.1}$$

for quantum deformations of the classical groups at primitive even roots of unity. For the quantum group $U_q(sl(2, \mathbb{C}))$ alternative proofs of Eq. (1.1) were given in [T1 and Wa].

The paper is organized as follows. In Sect. 2 we formulate a general system of axioms for 6j-symbols appropriate for our construction. In Sect. 3 we construct the state sums $Z(M, G_{\underline{x}})$ where M is a 3-manifold and G is a graph on the boundary ∂M whose lines are coloured by labels indicated by \underline{x} , and we discuss the geometric meaning of $Z(M, G_{\underline{x}})$. Section 4 is devoted to an analysis of the properties of $Z(M, G_{\underline{x}})$, such as its behavior under cutting of handles or removal of tubes in M . This analysis yields the desired calculational tools which are applied in Sect. 5 to evaluate the dimensions of the state spaces and to prove Eq. (1.1). In Sect. 6 we establish the properties of 6j-symbols stated in Sect. 2 in the case of a quantum group by ribbon graph techniques. In addition, we prove that the state sum for a planar graph coincides with the corresponding ribbon graph invariant of [RT1].

2. Abstract 6j-Symbols

In this section we list the defining properties of the abstract 6j-symbols to be used in the subsequent construction.

Let I be a finite set with involution $*$: $I \rightarrow I (i \mapsto i^*)$ and a distinguished element $0 = 0^*$. The elements in I will be called colours. To each triple of colours $(i, j, k) \in I^3$ there is associated a finite dimensional complex vector space V_{ij}^k of dimension N_{ij}^k , and we assume there exist canonical isomorphisms

$$V_{ij}^k \simeq V_{jk}^{i^*}, \quad V_{j^*i^*}^{k^*} \simeq (V_{ij}^k)^*, \tag{2.1}$$

where $(V_{ij}^k)^*$ denotes the dual vector space of V_{ij}^k , and

$$V_{ij}^k \simeq V_{ji}^k. \tag{2.2}$$

In the following these isomorphisms will be used without further notice to identify these spaces. Moreover, we assume that

$$N_{ij^*}^0 = \delta_{ij}. \tag{2.3}$$

In [DJN] a rather general framework for the construction of Hilbert spaces V_{ij}^k fulfilling these properties was given, where I labels a set of irreducible representations of an algebra satisfying certain properties. The isomorphism (2.2) (see [DJN, Sect. 5]) is, however, not needed for the construction of a topological

quantum field theory for manifolds without graphs, but is crucial when graphs with vertices of order 4 are present, as will be seen below.

The first isomorphism in (2.1) allows us to associate a vector space with each coloured oriented 2-simplex (triangle) as follows. Let σ^2 be an oriented 2-simplex with boundary links $\sigma_1^1, \sigma_2^1, \sigma_3^1$ decorated by arrows as indicated in Fig. 1, where the orientation of σ^2 is supposed to be clockwise. A colouring of the links in σ^2 is a mapping $j: \sigma_i^1 \mapsto j(\sigma_i^1) \in I, i = 1, 2, 3$. Given a colouring j of σ^2 we associate with (σ^2, j) the vector space

$$V(\sigma^2, j) = V_{j(\sigma_1^1)j(\sigma_2^1)}^{j(\sigma_3^1)}. \tag{2.4}$$

If an arrow on a link σ_i^1 is reversed we replace in this definition $j(\sigma_i^1)$ by $j^*(\sigma_i^1)$. The first isomorphism in (2.1) shows that this definition is invariant under rotations of the triangle in R^2 and the second shows that the vector space associated with the same triangle with reversed orientation is the dual of the original one.

Next, we assume the existence of a mapping $i \mapsto \omega_i$ from I to $\mathbb{C} \setminus \{0\}$ such that $\omega_0 = 1, \omega_k^2 = \omega_{k^*}$ and

$$\sum_k \omega_k^2 N_{ij}^k = \omega_i^2 \omega_j^2, \quad \sum_i \omega_i^4 = \omega^2 \neq 0. \tag{2.5}$$

In [DJN] ω_i^2 was denoted by F_i^{-1} and in the case of a quantum group, to be discussed later, it equals the q -dimension of the representation i up to a sign. The assumption $\omega_k^2 = \omega_{k^*}$ then expresses the reality of the q -dimension while the first relation in (2.5) reflects the multiplicativity of the q -dimension w.r.t. tensor products.

Finally, to each ordered 6-tuple $(i, j, k, l, m, n) \in I^6$, is associated an abstract 6j-symbol,

$$\left| \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right| \in V_{ij}^k \otimes V_{kl}^m \otimes V_{i^*m}^n \otimes V_{j^*n}^l. \tag{2.6}$$

By the association of vector spaces with coloured 2-simplexes discussed above we see that the tensor product in (2.6) may be associated to the boundary of the coloured tetrahedron depicted in Fig. 2.

In order that the 6j-symbol in (2.6) define a unique vector associated to the tetrahedron it must be invariant under the tetrahedral symmetry group, i.e.

$$\left| \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right| = \left| \begin{matrix} k^* & i & j^* \\ n & l & m \end{matrix} \right| = \left| \begin{matrix} l & m^* & k^* \\ i & j^* & n^* \end{matrix} \right|. \tag{2.7}$$

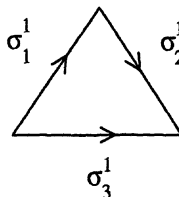


Fig. 1. An oriented 2-simplex

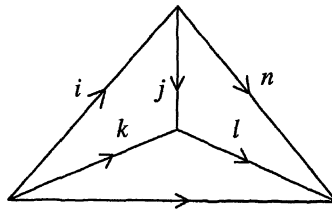


Fig. 2. An oriented, coloured tetrahedron

In addition, we shall assume that

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m^* & l^* & n^* \end{vmatrix}, \tag{2.8}$$

which implies that reversal of orientation of the tetrahedron is equivalent to applying the involution $*$ to all labels.

Besides the symmetry relations (2.7–8) the $6j$ -symbols are assumed to satisfy the following four relations, where products of $6j$ -symbols mean (unordered) tensor products together with contraction w.r.t. mutually dual pairs of vector spaces associated with certain pairs of factors:

1) Orthogonality:

$$\sum_k \omega_k^2 \begin{vmatrix} i & j & k \\ C & B & A \end{vmatrix} \begin{vmatrix} i & j & k \\ C & B & A' \end{vmatrix}^* = \omega_A^{-2} \delta_{AA'} 1_{V_{iA}^B \otimes V_{jC}^{A'}}, \tag{2.9}$$

where

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}^* = \begin{vmatrix} i^* & j^* & k^* \\ j^* & m^* & n^* \end{vmatrix}.$$

2) Biedenharn–Elliott relations:

$$\sum_n \omega_n^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} i & n & m \\ D & A & C \end{vmatrix} \begin{vmatrix} j & l & n \\ D & C & B \end{vmatrix} = \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix} \begin{vmatrix} k & l & m \\ D & A & B \end{vmatrix}. \tag{2.10}$$

3) Racah identities: There exists a mapping $k \mapsto q_k$ from I to $\mathbb{C} \setminus \{0\}$, such that $q_0 = 1, q_k = q_{k^*}$ and

$$\frac{q_k}{q_i q_j} \begin{vmatrix} i & j & k \\ A & B & C \end{vmatrix} = \sum_D \omega_D^2 \frac{q_A q_B}{q_C q_D} \begin{vmatrix} i & A & D \\ j & B & C \end{vmatrix} \begin{vmatrix} j & i & k \\ A & B & D \end{vmatrix}. \tag{2.11}$$

Moreover, we assume that the replacement of all q_i in (2.11) by q_i^{-1} leads to another identity.

4) Considering $1_{V_{iA}^B}$ as a vector in $V_{iA}^B \otimes (V_{iA}^B)^*$ and identifying $V_{i_0}^i, i \in I$, with C according to (2.3) we have

$$\begin{vmatrix} i' & A' & B \\ A^* & i & 0 \end{vmatrix} = \frac{\delta_{AA'} \delta_{ii'}}{\omega_i \omega_A} 1_{V_{iA}^B}. \tag{2.12}$$

The $6j$ -symbols defined in [DJN] with the notation

$$F_{pq} \begin{bmatrix} i & j \\ k & l \end{bmatrix} = \omega_p^2 \begin{vmatrix} i & j & q \\ l & k & p \end{vmatrix} \tag{2.13}$$

for the quantum groups obtained as deformations of the universal enveloping algebra $U_q\mathfrak{g}$ of the classical semisimple Lie algebra \mathfrak{g} , were shown to satisfy the assumptions (2.7), (2.9) and (2.10) if q is an even primitive root of unity and I is a certain set of irreducible representations of $U_q\mathfrak{g}$. These were shown to be sufficient for the construction of a topological quantum field theory for manifolds without graphs. The additional relations (2.8), (2.12) and Racah identities will turn out to be of importance for the inclusion of graphs with vertices of order 4. These relations were not discussed in [DJN], but they are relatively easy to establish in the case of quantum groups by the methods developed there, with

$$q_i^2 = i(c)$$

for each representation $i \in I$, where c is a certain central element in the quantum group (acting as multiplication by a scalar q_i^2 , since i is irreducible). In fact, it is straightforward to check (2.12) and it is possible to see that (2.8) follows from the Racah identities.

In order to establish the Racah identities one defines the braiding operator

$$B_{pq} \begin{bmatrix} i & j \\ k & l \end{bmatrix} : V_{ip}^k \otimes V_{jl}^p \rightarrow V_{jq}^k \otimes V_{il}^q$$

by setting its matrix element to be

$$\left\langle B_{pq} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \alpha \otimes \beta, \delta \otimes \gamma \right\rangle = \alpha \circ \beta \circ \check{R}_{12}^{ij} \circ \gamma^* \circ \delta^* \tag{2.14}$$

for $\alpha \in V_{ip}^k, \beta \in V_{jl}^p, \delta \in V_{jq}^k, \gamma \in V_{il}^q$, where the RHS is an intertwiner between k and k and hence a number, and, if V_i denotes the representation space of i , $\check{R}_{12}^{ij} : V_j \otimes V_i \rightarrow V_i \otimes V_j$ is given by ([Dr, DJN, D])

$$\check{R}_{12}^{ij} = \sigma \circ (j \otimes i)(R), \tag{2.15}$$

where $\sigma(x \otimes y) = y \otimes x$ for $x \in V_j, y \in V_i$ and $R \in U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ is the standard R -matrix.

Defining the isomorphism $V_{ij}^k \simeq V_{ji}^k$ by

$$\beta \mapsto \hat{\beta} = \frac{q_k}{q_i q_j} \beta \circ \check{R}_{12}^{ij}, \tag{2.16}$$

we have $\beta = \hat{\hat{\beta}}$ and one finds that

$$\omega_p^{-2} B_{pq} \begin{bmatrix} i & j \\ k & l \end{bmatrix} = \frac{q_p q_q}{q_k q_l} \begin{bmatrix} i & l & q \\ j & k & p \end{bmatrix}, \tag{2.17}$$

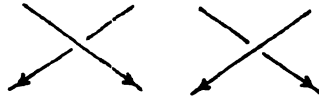
which is the vector that will be associated to a 4-vertex (or rectangle) (see Fig. 6 below).

Using the definition (2.14) it is quite easy to verify Eq. (2.11) by using complete reducibility of tensor products of representations in I and the properties of R (Eq. (5.4) in [DJN]). Replacing in (2.16) $q_i \rightarrow q_i^{-1}$ and $R \rightarrow R^{-1}$ we obtain in a similar way the second Racah identity.

In Sect. 6 we describe an alternative method of establishing the properties of the 6j-symbols.

3. State Sum for Manifolds with Coloured Graphs

Let M be an oriented compact 3-manifold with triangulation X whose 1-simplices (links) are assumed to be decorated by arrows in some arbitrary way, and let $j: X \ni \sigma^1 \mapsto j(\sigma^1) \in I$ be a colouring of the 1-simplices in X . Furthermore, let G be an oriented graph on the boundary ∂M , i.e. a finite 1-dimensional simplicial complex without boundary which is compatible with X , i.e. each edge (or line) of G consists of a sequence of 1-simplices of ∂X . We assume that G has only 2-, 3- and 4-vertices. The 4-vertices can be of two types:



which we shall call inverse to each other. Orientations of links on opposite sides of 2- and 4-vertices are assumed to coincide. A colouring of G is a mapping \underline{x} from its lines (maximal connected sets of links joined by vertices of order 2) such that the colours on opposite sides of 4-vertices are identical. The graph G coloured by \underline{x} will be denoted by $G_{\underline{x}}$.

We associate now with (M, X, G) a new triangulated pseudo-manifold (M_G, X_G) as follows: Let $\mathcal{C} = \{c_1, \dots, c_n\}$ denote the set of connected components of $\partial M \setminus G$, which we shall identify with new additional vertices. For each triangle $\sigma^2 \in c_i$ we glue on to ∂M a new tetrahedron, which has base σ^2 and an opposite vertex c_i . Furthermore, we glue these tetrahedra together along triangles, which they share, i.e. for two triangles $\sigma_1^2 \in c_i$ and $\sigma_2^2 \in c_i$ with $\sigma_1^2 \cap \sigma_2^2 = [AB]$ the corresponding tetrahedra are glued along the triangle (ABc_i) (see Fig. 3).

Next, for each link $\sigma^1 \in \partial X$, contained in a line of G , and which is contained in the boundaries of the components, c_i, c_j in \mathcal{C} (Fig. 4), we glue on a tetrahedron along the two triangles containing σ^1 and c_i , respectively c_j , and continually glue along common triangles of the added tetrahedra. Moreover, we use the orientations of M and G to fix, by a suitable convention, an arrow on the link (c_i, c_j) (see Fig. 4 where, say, the tetrahedra are assumed to be oriented such that positive normals point outwards). If, in addition, the line in G is x -coloured, the link (c_i, c_j) is likewise coloured by x .

To the simplicial complex obtained in this way we glue for each 3-vertex u of G a new tetrahedron which contains the vertices (u, c_i, c_j, c_k) , where c_i, c_j, c_k are the components in \mathcal{C} sharing u (Fig. 5).

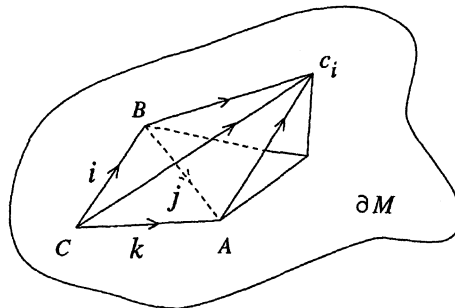


Fig. 3. Gluing tetrahedra onto the boundary triangles of M in c_i

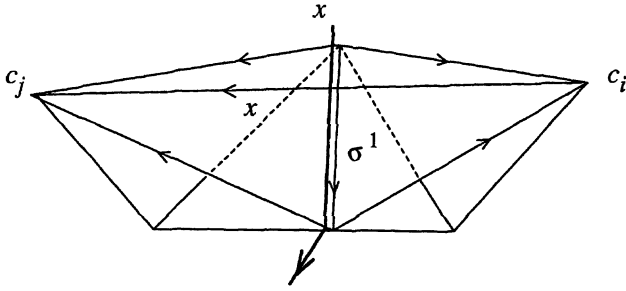


Fig. 4. Gluing a tetrahedron along the boundary of two connected components c_i and c_j

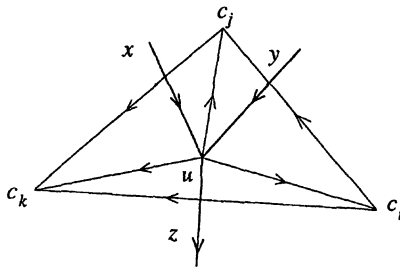


Fig. 5. Gluing a tetrahedron at a 3-vertex u of G

This finishes the construction of the triangulated pseudo-manifold (M_G, X_G) . Its principal feature is that it contains (M, X) in its interior except for the original 4-vertices in G , which are now 4-vertices in ∂X_G . It is convenient to view the four triangles in ∂X_G sharing such a 4-vertex v as making up a rectangle with v as center. Then ∂X_G consists of a set of triangles and rectangles, namely one triangle for each 3-vertex in G and one rectangle for each 4-vertex in G , and two of these share a link if and only if the corresponding vertices in G are connected by a line. In this sense the links and vertices of ∂X_G constitute a graph that equals the dual of $G \subseteq \partial M$.

As noted above a colouring \underline{x} of G induces a colouring of the links in ∂X_G and the orientation of G (and M) induces a decoration of those links by arrows. We denote (M_G, X_G) with this extra structure by $(M_{G_{\underline{x}}}, X_{G_{\underline{x}}})$. Thus by colouring the interior links of $X_{G_{\underline{x}}}$ one obtains a full colouring of X_G . We then associate with $(M, G_{\underline{x}})$ the state sum

$$Z(M, G_{\underline{x}}) = Z(M_{G_{\underline{x}}}), \tag{3.1}$$

where $Z(M_{G_{\underline{x}}})$ is defined in analogy with [TV, DJN] as follows: Given a decoration by arrows as well as a colouring \underline{j} of the interior links of $X_{G_{\underline{x}}}$, a factor ω^{-2} is attached to each interior vertex $\sigma^0 \in \text{int} X_{G_{\underline{x}}}$, a factor ω_i^2 to each i -coloured interior link $\sigma^1 \in \text{int} X_{G_{\underline{x}}}$ and a $6j$ -symbol to each tetrahedron $\sigma^3 \in X_{G_{\underline{x}}}$ as described above. Finally, to each coloured rectangle in $\partial X_{G_{\underline{x}}}$, as depicted in Fig. 6, whose orientation is assumed to be clockwise, we attach the braiding operator

$$\frac{q_A q_B}{q_C q_D} \begin{vmatrix} y & A & D \\ x & B & C \end{vmatrix} \in V_{Ay}^D \otimes V_{Dx}^B \otimes V_{By}^C \otimes V_{Cx}^A. \tag{3.2}$$

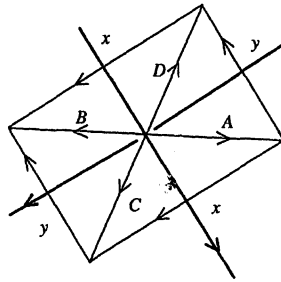


Fig. 6. Coloured rectangle on the boundary of X_G

For the inverse 4-vertex the braiding operator can be obtained from (3.2) by replacement of all q_i by q_i^{-1} . We then form the product of all factors so associated to X_{G_x} , contract w.r.t. all interior triangles as well as the triangles inside the rectangles in ∂X_{G_x} and sum over all colourings \underline{j} of the interior links of X_{G_x} , thus obtaining the desired vector

$$Z(M_{G_x}) \in V(\partial M, G_x), \tag{3.3}$$

where $V(\partial M, G_x)$ is the tensor product of vector spaces associated to the coloured triangles in ∂M_{G_x} , which are dual to 3-vertices in G_x . Note that $Z(M_{G_x})$ is independent of the initial distribution of arrows on the interior links in X_{G_x} , since $i \mapsto i^*$ is a bijective mapping on I .

This definition of $Z(M_{G_x})$ is identical to the one in [DJN] except for the issues concerning 4-vertices in G and that no factors ω^{-1} are associated to vertices in ∂X_{G_x} . By the same arguments as in [DJN] it then follows that $Z(M_{G_x})$ is independent of the interior of the triangulation X_{G_x} as a consequence of the identities (2.9–10). In particular, it is independent of the triangulation X of M and is invariant under isotopy changes of G_x , since the latter clearly only affect the interior of X_{G_x} .

In general M_G is not a manifold. For example, for empty G one point c_i will be added to X for each connected component of ∂M and the points corresponding to components not homeomorphic to S^2 will be singular points of M_G . On the other hand, if G is sufficiently “large” then M_G will be homeomorphic to M . A particular graph of this type on a connected surface Σ of genus $g \geq 1$ may be obtained by first choosing a homology basis $a_1, \dots, a_g, b_1, \dots, b_g$ whose representatives have one point P in common, and then deforming it such that P is separated into $4g - 2$ vertices of order 3. We shall adopt the notation of [KS] and call this graph G^Σ . The dual graph of G^Σ on Σ is easily seen to yield a proper triangulation¹ of Σ and M_{G^Σ} is obtained from M , $\partial M = \Sigma$, by gluing on a cylinder $\Sigma \times [0, 1]$, as may be easily verified.

In general, for a graph G without vertices of order 4 such that M_G is homeomorphic to M , the partition function $Z'(M)$ of [DJN] is given by

$$Z'(M) = \omega^{-N_G} \otimes_x Z(M, G_x), \tag{3.4}$$

¹ We use here a more general notion of triangulation than the standard one, in that we allow identifications of subsimplices of co-dimension > 1 in a simplex of maximal dimension.

where N_G is the number of connected components of $\partial M \setminus G$ and the triangulation of ∂M is given by the dual graph of G . In particular,

$$Z'(M) \in V(\partial M, G), \tag{3.5}$$

where

$$V(\partial M, G) \equiv \bigoplus_x V(\partial M, G_x), \tag{3.6}$$

is the vector space associated to the surface ∂M triangulated by the dual graph of G .

Given a closed, compact, oriented surface Σ , a graph G in Σ and a compact oriented 3-manifold M with $\partial M = \Sigma$, we call G a *canonical graph* in Σ if M_G is obtained from M by gluing on a cylinder $\Sigma \times [0, 1]$ along Σ . Clearly, this property only depends on Σ and G but not on M . In particular, a graph G on Σ with no vertices of order 4 is canonical if and only if its dual graph in Σ yields a triangulation of Σ .

The correspondence (3.4) allows us to rephrase results in [DJN] in terms of manifolds with graphs. Of particular interest is the gluing property (axiom (3) of [At]) whose triangulated version holds by construction for $Z'(M)$ (see [DJN]) and may be reformulated as follows.

Theorem 3.1. *Let M be a (connected or disconnected) oriented compact 3-manifold with $\partial M = \Sigma_1 \cup \Sigma_2 \cup \Sigma \cup \Sigma^*$ (disjoint union), where $\Sigma_1, \Sigma_2, \Sigma, \Sigma^*$ are closed oriented surfaces such that there exists an orientation reversing diffeomorphism $F: \Sigma \rightarrow \Sigma^*$. Moreover, let $G_{\Sigma_1}^1 \subseteq \Sigma_1$ and $G_{\Sigma_2}^2 \subseteq \Sigma_2$ be arbitrary coloured graphs and let $G \subseteq \Sigma$ be a canonical graph without vertices of order 4 and G^F its image in Σ^* by F . Then, if M_Σ denotes the 3-manifold obtained by gluing M along Σ and Σ^* by F , we have*

$$Z(M_\Sigma, G_{\Sigma_1}^1 \cup G_{\Sigma_2}^2) = \omega^{-2N_G} \sum_x \prod_{l \subseteq G} \omega_{x_l}^2 (Z(M, G_{\Sigma_1}^1 \cup G_{\Sigma_2}^2 \cup G_x \cup G_x^F))_{G_x}, \tag{3.7}$$

where the product on the RHS is over lines l in G , and $(\cdot)_{G_x}$ indicates a contraction w.r.t. all pairs of vector spaces associated to coloured 3-vertices in G_x and G_x^F .

This theorem generalizes Theorem 7.1 of [KS] and is identical to Eq. (4.6) in [DJN] in the case that G^1 and G^2 are canonical graphs, but its validity for arbitrary G^1 and G^2 follows, as in [DJN], immediately from the construction of $Z(M_{G_x})$, and its independence of the interior of the triangulation X_G of M_G .

It should be noted that we have not exhibited the dependence of $Z'(M) \equiv Z'(M, G)$ on G in Eq. (3.4) for the following reason [TV]. Let G be a canonical graph in $\partial M \equiv \Sigma$ without 4-vertices as in Eq. (3.4), and consider the cylindrical 3-manifold $\Sigma \times [0, 1]$ oriented such that we may identify Σ^* with $\Sigma \times \{0\}$ and Σ with $\Sigma \times \{1\}$. Then

$$\begin{aligned} P_{(\Sigma, G)} &\equiv Z'(\Sigma \times [0, 1], G \times \{0\}, G \times \{1\}) \in V(\Sigma, G)^* \otimes V(\Sigma, G) \\ &\simeq \text{Hom}(V(\Sigma, G), V(\Sigma, G)). \end{aligned}$$

Since $M \cup_\Sigma (\Sigma \times [0, 1])$ is homeomorphic to M it follows from Eq. (3.7) that

$$Z'(M, G) = P_{(\Sigma, G)} Z'(M, G) \in P_{(\Sigma, G)} V(\Sigma, G) \equiv V'(\Sigma, G).$$

Here, we note that factors ω_x^2 in Eq. (3.7) associated to the coloured lines in G_x are included in the definition of the bilinear pairing of $V(\Sigma^*, G)$ and $V(\Sigma, G)$ (see Eq. (3.30) in [DJN]).

It follows, moreover, from Eq. (3.7) (see Sect. 4 of [DJN] for details) that $P_{(\Sigma, G)}$ is a projection and that $V'(\Sigma, G_1)$ and $V'(\Sigma, G_2)$ for two arbitrary canonical graphs in Σ may be consistently identified by the mappings

$$P_{(\Sigma, G_1), (\Sigma, G_2)} \equiv Z'(\Sigma \times [0, 1], G_1 \times \{0\}, G_2 \times \{1\}) \in V(\Sigma, G_1)^* \otimes V(\Sigma, G_2) \\ \simeq \text{Hom}(V(\Sigma, G_1), V(\Sigma, G_2))$$

by which the vectors $Z'(M, G_1)$ and $Z'(M, G_2)$ are also identified.

In force of these identifications we thus obtain the vector space

$$V_\Sigma \simeq V'(\Sigma, G) \subseteq V(\Sigma, G)$$

containing the vector

$$Z'(M) \equiv Z'(M, G)$$

for all 3-manifolds M with $\partial M = \Sigma$.

By Eq. (3.7) and the fact that $P_{(\Sigma, G)}$ is a projection we have

$$\dim V_\Sigma = \text{tr} P_{(\Sigma, G)} = Z'(\Sigma \times S^1) = Z(\Sigma \times S^1). \tag{3.8}$$

For later use we note the following simple lemma, valid for any compact oriented 3-manifolds M, M_1, M_2 .

Lemma 3.2. (i) *If $D^3 \subseteq \text{int}M$ is diffeomorphic to the 3-ball, then for any coloured graph $G_{\underline{x}} \subseteq \partial M$, we have*

$$(i) \quad Z(M, G_{\underline{x}}) = \omega^{-2} Z(M \setminus D^3, G_{\underline{x}}). \tag{3.9}$$

$$(ii) \quad Z(S^3) = \omega^{-2}, \quad Z(D^3) = 1. \tag{3.10}$$

(iii) *If M^* denotes M with the opposite orientation, then*

$$Z(M) = Z(M^*). \tag{3.11}$$

(iv) *If M is the connected sum of M_1 and M_2 , i.e. $M = (M_1 \setminus D_1^3) \cup_{S^2} (M_2 \setminus D_2^3)$, where $D_1^3 \subseteq \text{int}M_1$ and $D_2^3 \subseteq \text{int}M_2$ are diffeomorphic to the 3-ball and their boundaries S^2 are identified, then*

$$Z(M, G_{\underline{x}_1}^1 \cup G_{\underline{x}_2}^2) = \omega^2 Z(M_1, G_{\underline{x}_1}^1) \otimes Z(M_2, G_{\underline{x}_2}^2) \tag{3.12}$$

for arbitrary coloured graphs $G_{\underline{x}_1}^1 \subseteq \partial M_1$ and $G_{\underline{x}_2}^2 \subseteq \partial M_2$.

Proof.

i) follows since $M_G = (M \setminus D^3)_G$, but no factor ω^{-2} is associated to the vertex c corresponding to the component $\partial D^3 \subseteq \partial(M \setminus D^3)$ in the definition of $Z(M \setminus D^3, G_{\underline{x}})$.

ii) By triangulating S^3 by two tetrahedra the first relation follows from (2.9) and the definition of ω^2 . The second relation then follows from i) applied to $M = S^3$.

iii) follows from (2.8) since all colours are summed over and $i \mapsto i^*$ is a bijective mapping from I to I .

iv) We first note that M is diffeomorphic to

$$(M_1 \setminus D_1^3) \bigcup_{S^2 \times \{0\}} (S^2 \times [0, 1]) \bigcup_{S^2 \times \{1\}} (M_2 \setminus D_2^3). \tag{3.13}$$

Next we note that the graph \bar{G} in S^2 consisting of two 3-vertices connected by three lines is canonical since its dual yields a triangulation of S^2 by two triangles. We can obtain a triangulation of $S^2 \times [0, 1]$ such that $S^2 \times \{0\}$ and $S^2 \times \{1\}$ are both triangulated by two triangles by taking two prisms, each triangulated in a standard fashion by three tetrahedra, and gluing them together along their sides. Using this triangulation a straightforward calculation using (2.9) and (2.5) yields

$$Z(S^2 \times [0, 1], \bar{G}_y \times \{0\}, \bar{G}_z \times \{1\}) = \omega^2 1_{V_{y_1, y_2}^3} \otimes 1_{V_{z_1, z_2}^3}, \tag{3.14}$$

where $y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ are the colours of the lines (suitably oriented) in $\bar{G} \times \{0\}$ and $\bar{G} \times \{1\}$, respectively. Applying Eq. (3.7) to (3.13) the result follows from (3.14). \square

We close this section with some remarks on a few concepts that will be of importance in the following (see also [KS]).

An *interior graph* \mathcal{G} in a 3-manifold M consists of 1) a core $c(\mathcal{G})$ which is a finite unoriented graph embedded in $\text{int}M$, 2) a tubular neighborhood $\mathcal{T}_{\mathcal{G}}$ of $c(\mathcal{G})$ in $\text{int}M$ and 3) an oriented graph G in $\partial\mathcal{T}_{\mathcal{G}}$ as introduced at the beginning of this section. If G is coloured by \underline{x} we say that \mathcal{G} is coloured by \underline{x} and denote the coloured interior graph by $\mathcal{G}_{\underline{x}}$. A priori there need not be any connection between $c(\mathcal{G})$ and G , although this will be the case in the applications below. As a particular example we mention the case in which each component k_i of $c(\mathcal{G})$ is homeomorphic to the circle S^1 and the corresponding component \mathcal{T}_i of $\mathcal{T}_{\mathcal{G}}$ contains exactly one component l_i of G and l_i is homotopic to k_i in \mathcal{T}_i . In this case \mathcal{G} is a framed link in M , and it will be denoted by \mathcal{L} .

For a compact oriented 3-manifold M with an interior coloured graph $\mathcal{G}_{\underline{x}}$ as above we define

$$Z(M, \mathcal{G}_{\underline{x}}) \equiv Z(M \setminus \mathcal{T}_{\mathcal{G}}, G_{\underline{x}}). \tag{3.15}$$

Of course, this definition can be generalized to the case where in addition a graph is present in ∂M .

Of particular relevance for the following is the concept of a meridian and of left- and righthanded lines. Assume that the 3-manifold M contains an empty tube $T = S^1 \times [0, 1] \subseteq \partial M$, i.e. M can be obtained from a 3-manifold \tilde{M} by removing a cylinder C diffeomorphic to $D^2 \times [0, 1]$, where D^2 is the two-dimensional unit disk. A *meridian* on T is an oriented circle $m = S^1 \times \{p\}$, $p \in]0, 1[$.

A 3-manifold M containing a set of empty tubes $T_m, T_{m'}, \dots$ with meridians m, m', \dots will be denoted by $M(T_m, T_{m'}, \dots)$. Given a graph G on ∂M the lines in G may over- or undercross the meridians m, m', \dots , such that these together with G constitute a graph $G \cup m \cup m' \cup \dots$. For each colouring \underline{x} of G we may thus define ([KS, T2])

$$Z(M(T_m, T_{m'}, \dots), G_{\underline{x}}) = \sum_{a, a', \dots} \frac{\omega_a^2 \omega_{a'}^2}{\omega^2 \omega^2} \dots Z(M, G_{\underline{x}} \cup m_a \cup m_{a'} \cup \dots), \tag{3.16}$$

where the sum is over colourings of the meridians m, m', \dots .

A line l in G which intersects a meridian m exactly once is called left-, respectively righthanded w.r.t. m if l over-, respectively undercrosses m . For the particular case of a framed link \mathcal{L} as defined above the corresponding *left-, respectively righthanded link* \mathcal{L}_L , resp. \mathcal{L}_R , is obtained by introducing a meridian m_i on each of the tubes \mathcal{T}_i such that each component l_i is left-, respectively righthanded w.r.t. m_i . The state sum associated to this left-, respectively righthanded

framed link with colouring \underline{x} is then by definition given by (3.16) with the \mathcal{T}_i replacing T_m, T'_m, \dots and will be denoted by $Z(M, (\mathcal{L}_{\underline{x}})_L)$, resp. $Z(M, (\mathcal{L}_{\underline{x}})_R)$.

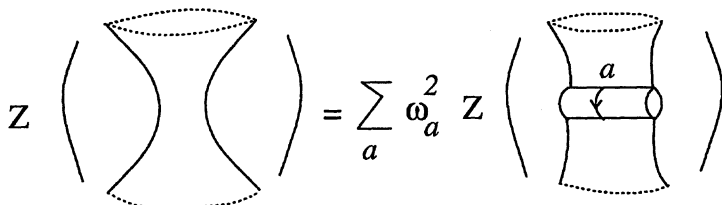
The main reason for introducing the definition (3.16) is illustrated by the following lemma, which asserts that filling an empty tube is equivalent (up to the factor ω^2) to introducing a meridian on the tube.

Lemma 3.3. *Let \tilde{M} be a compact oriented 3-manifold that contains a cylinder $C = D^2 \times [0, 1]$, with $D^2 \times \{0, 1\} \subseteq \partial\tilde{M}$, and let M denote the manifold obtained by removing C from \tilde{M} . Then*

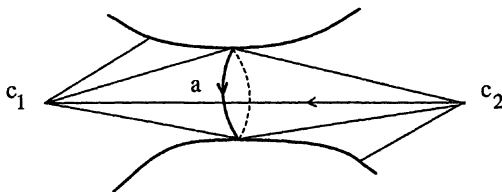
$$Z(\tilde{M}, G_{\underline{x}}) = \omega^2 Z(M(T_m), G_{\underline{x}}) \tag{3.17}$$

for any coloured graph $G_{\underline{x}} \subseteq \partial\tilde{M}$ such that $G \cap (D^2 \times \{0, 1\}) = \emptyset$ and where T_m denotes the tube $\partial D^2 \times [0, 1] \subseteq \partial M$ with meridian m .

Pictorially (3.17) may be written as



Proof. This follows simply by realizing that the cylinder C is reinserted automatically when $M_{G \cup m}$ is constructed. More explicitly, for a given triangulation X of M , $M_{G_{\underline{x}} \cup m_a}$ equals $\tilde{M}_{G_{\underline{x}}}$ with a triangulation in which the (interior) link connecting the vertices c_1 and c_2 associated with the components in $\partial M \setminus (G \cup m)$ on either side of m has a fixed colour a , as indicated on the following figure.



Thus inserting the factor ω_a^2 and summing over a one obtains $Z(\tilde{M}_{G_{\underline{x}}})$ according to its definition. □

On the other hand the following result shows that a handle in M can be cut without changing the state sum.

Lemma 3.4. *If the 3-manifold M contains a cylinder $C = D^2 \times [0, 1]$ with $\partial D^2 \times [0, 1] \subseteq \partial M$, and \tilde{M} denotes the manifold obtained by removing C from M , then*

$$Z(M, G_{\underline{x}}) = Z(\tilde{M}, G_{\underline{x}}) \tag{3.18}$$

for any coloured graph $G_{\underline{x}} \subseteq \partial M$ such that $G \cap (\partial D^2 \times [0, 1]) = \emptyset$.

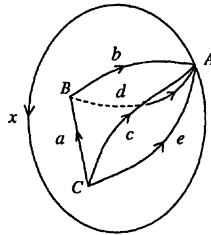
Proof. By choosing a triangulation X of M such that the handle C can be cut along a single triangle t (with $\partial t \subseteq \partial D^2 \times]0, 1[$), the result follows from Eq. (3.14) by an argument essentially identical to the one used in the proof of Lemma 3.2. \square

Whereas Lemma 3.4 deals with the cutting of handles disjoint from G the following lemma can be applied to handles traversed by a single line in G .

Lemma 3.5. *Let M be a compact oriented 3-manifold and G_x a coloured graph in ∂M . If an x -coloured line L in G_x does not separate two different components in $\partial M \setminus G$ and there exists in ∂M a contractible loop in M intersecting L (transversally) only once, then*

$$Z(M, G_x) = 0 \quad \text{when } x \neq 0. \tag{3.19}$$

Proof. Let X be a triangulation of M and denote by A the vertex in X_G corresponding to the component of $\partial M \setminus G$ containing L in its boundary. Then there is an x -coloured link l in X_G (dual to L) whose end-points both equal A . The tetrahedron in X_{G_x} containing l and an a -coloured link in L as opposite links then looks as follows:



By the assumption the two triangles (ABC) may be connected by a thickened disk. By retriangulation of this thickened disk one may obtain a (singular) triangulation in which the two triangles (ABC) are identical. The link (BC) is then only contained in one tetrahedron. Summing over its colour a one obtains a contribution

$$\sum_a \omega_a^2 \begin{vmatrix} a & b & c \\ x & c & b \end{vmatrix},$$

contracted w.r.t. the dual pair of spaces $V_{ab}^c, V_{a^*c}^b$. But this expression equals 0 for $x \neq 0$, since it may be rewritten by Eqs. (2.12) and (2.9) as:

$$\sum_a \omega_a^2 \begin{vmatrix} a & b & c \\ x & c & b \end{vmatrix} \begin{vmatrix} a & b & c \\ 0 & c & b \end{vmatrix} = \delta_{x0} 1_{V_{c_0}^c \otimes V_{b_0}^b}. \quad \square$$

Remark 3.6. In Lemma 4.2 we shall see that a line with colour 0 can be deleted from G without changing $Z(M, G_x)$. Together with Lemmas 3.4–5 this implies that if a single x -coloured line traverses a handle, then the state sum is either zero (if $x \neq 0$) or (if $x = 0$) the line can be removed and the handle cut.

4. Computational Methods

In this section we establish a number of invariance properties or transformation rules for the state sums $Z(M, G_x)$ under local changes of the graph G . These are

generalizations of the results derived in [KS], but we give different and hopefully more transparent proofs.

In the following let M be a compact oriented 3-manifold and G_x a coloured graph in ∂M .

Lemma 4.1. *If $G_x = G_{x_1}^1 \cup G_{x_2}^2$, where G^1 is contained in a disk $D^2 \subseteq \partial M$ such that $G^2 \cap D^2 = \emptyset$, i.e. G contains an isolated planar subgraph G^1 , then*

$$Z(M, G_x) = Z(G_{x_1}^1) \otimes Z(M, G_{x_2}^2), \tag{4.1}$$

where

$$Z(G_x) \equiv Z(D^3, G_x) \tag{4.2}$$

and D^3 denotes a 3-ball.

Proof. By choosing a suitable triangulation X of M we can represent (M, G_x) as two manifolds $(M', G_{x_1}^1)$ and $(D^3, G_{x_2}^2)$, which are glued together along the triangle $t \subseteq \partial M' \cap \partial D^3$ with $\partial t \subseteq \partial M$, and such that M' is homeomorphic to M . Then M_G equals the connected sum of $D_{G^1}^3$ and M'_{G^2} , obtained by first removing from $D_{G^1}^3$, resp. from M'_{G^2} , the tetrahedron with base t and opposite vertex c , resp. c' , corresponding to the connected component of $\partial D^3 \setminus G^1$, resp. $\partial M' \setminus G^2$, containing t , and subsequently gluing the two resulting manifolds together along the two copies of S^2 (the boundaries of the removed tetrahedra), such that c and c' are identified. Equation (4.1) then follows from Eq. (3.14) as in the proof of Lemma 3.2 iv), taking into account that factors ω^{-2} are associated neither to $c \in D_{G^1}^3$ and $c' \in M'_{G^2}$, nor to the vertex resulting from their identification in M_G . \square

Examples. 1) For a coloured circle S_x^1 we have

$$Z(S_x^1) = Z\left(\begin{array}{c} \curvearrowright \\ (x) \end{array}\right) = Z\left(\begin{array}{c} \curvearrowleft \\ (x) \end{array}\right) \equiv Z\left(\begin{array}{c} \circ \\ (x) \end{array}\right) = \omega_x^2. \tag{4.3}$$

Observe that, for a given triangulation X of D^3 , $(D_{S^1}^3, X_{S^1})$ equals (S^3, X_{S^1}) , where X_{S^1} contains two distinguished vertices c and c' connected by a link with colour x . Triangulating S^3 by two tetrahedra Eq. (4.3) follows from Eq. (2.9).

2)

$$Z\left(\begin{array}{c} i \quad j \\ \circ \\ k \end{array}\right) = 1_{V_{ij}^k}. \tag{4.4}$$

Note that D_G^3 , for the graph in (4.4), equals a 3-ball with boundary triangulated by two triangles and thus can be obtained by gluing two tetrahedra along three common triangles. Hence, Eq. (4.4) is again a consequence of Eq. (2.9).

3)

$$Z\left(\begin{array}{c} n \\ \circ \\ i \quad j \\ \circ \\ k \quad l \end{array}\right) = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}. \tag{4.5}$$

For the graph in (4.5) D_G^3 equals the 3-ball whose boundary is triangulated by the dual graph to G and hence can be obtained as a single tetrahedron. Equation (4.5) then follows from the definition of $Z(M, G_x)$.

4)

$$Z \left(\begin{array}{c} \text{circle with vertices } i, j, k, l, m, n \text{ and internal edges} \end{array} \right) = \frac{g_l g_m}{g_k g_n} \begin{vmatrix} j & l & n \\ i & m & k \end{vmatrix}. \quad (4.6)$$

We leave the verification of this identity to the reader.

Lemma 4.2. *The state sum $Z(M, G_x)$ is invariant under the following local changes of G_x :*

i)

$$\begin{array}{c} \text{circle with vertices } i, j \text{ and external edges } k, k' \end{array} \longleftrightarrow \frac{\delta_{kk'}}{\omega_k^2} I V_{ij}^k \begin{array}{c} \text{vertical line with edge } k \end{array}, \quad (4.7)$$

ii)

$$\sum_k \omega_k^2 \begin{array}{c} \text{X-shaped graph with vertices } i, j \text{ and edge } k \end{array} \longleftrightarrow \begin{array}{c} \text{two parallel vertical lines with vertices } i, j \end{array}, \quad (4.8)$$

where the curly line indicates the contraction with respect to the dual pair of spaces associated to the 3-vertices it connects.

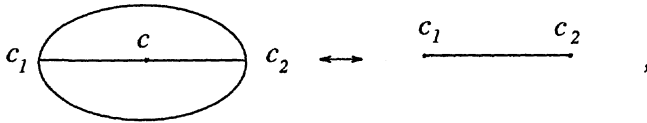
iii)

$$\begin{array}{c} \text{graph with vertices } x, x', y, y' \text{ and edge } 0 \end{array} \longleftrightarrow \frac{\delta_{x^*x'} \delta_{y^*y'}}{\omega_x \omega_y} \begin{array}{c} \text{two curved lines with vertices } x, y \end{array}, \quad (4.9)$$

where the vector spaces $V_{xx^*}^0$ and $V_{yy^*}^0$ have been identified with \mathbb{C} .

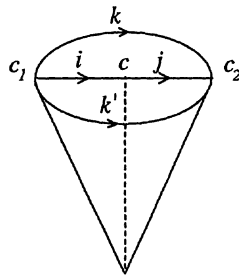
Proof.

i) The substitution (4.7) corresponds to the following local transformation of the triangulation on the boundary of M_G :

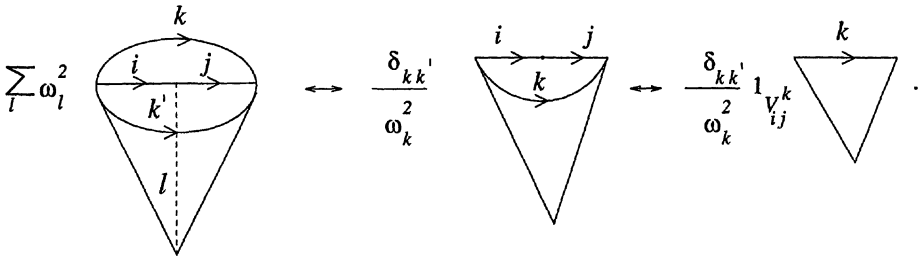


where c_1, c and c_2 are the vertices associated to the left, middle and the right components of $\partial M \setminus G$, respectively, on the lefthand side of (4.7), and similarly for the righthand side.

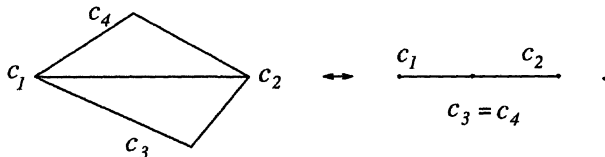
By choosing a triangulation of M_{G_x} which in the neighborhood of c_1, c, c_2 looks like



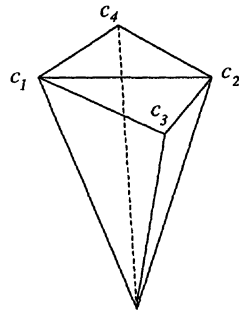
the result follows from the orthogonality relations, which can be interpreted in terms of the collapsing of two tetrahedra glued along two common triangles [DJN]:



ii) The substitution (4.8) corresponds to the following local transformation of the triangulation on the boundary ∂M_G :



Choosing a triangulation of M_G which in the neighborhood of c_1, c_2, c_3, c_4 looks like



we see that the contraction on the lefthand side of (4.8) corresponds to gluing the two triangles $(c_1c_2c_4)$ and $(c_1c_2c_3)$ together. Now we may again apply Eq. (2.9) as above and obtain:

$$\omega_l^2 \omega_n^2 \sum_k \omega_k^2 \begin{array}{c} i \quad j \\ \nearrow \quad \searrow \\ \text{---} k \text{---} \\ \searrow \quad \nearrow \\ l \quad n \end{array} \longleftrightarrow \delta_{ln} \omega_l^2 \begin{array}{c} i \quad j \\ \nearrow \quad \searrow \\ \text{---} l \text{---} \\ \searrow \quad \nearrow \end{array} ,$$

which yields the desired result.

iii) The substitution (4.9) corresponds to the following local transformation of the triangulation of the boundary of M_G :

$$\begin{array}{c} c_3 \\ \nearrow \quad \searrow \\ x' \quad y' \\ \nearrow \quad \searrow \\ c_1 \quad 0 \quad c_2 \\ \searrow \quad \nearrow \\ x \quad y \\ c_4 \end{array} \longleftrightarrow \begin{array}{c} x = x' \quad y = y' \\ \text{---} \text{---} \\ c_1 \quad c_2 \\ c_3 = c_4 \end{array} .$$

By choosing a suitable triangulation of M_G one finds that (4.9) is a simple consequence of Eq. (2.12). □

Next we note that the Racah identity (2.11) implies invariance of $Z(M, G_x)$ under the following local substitution in G_x :

$$\begin{array}{c} x \quad y \\ \searrow \quad \nearrow \\ \text{---} \\ \searrow \quad \nearrow \\ z \end{array} \longleftrightarrow \frac{q_z}{q_x q_y} \begin{array}{c} x \quad y \\ \searrow \quad \nearrow \\ \text{---} \\ \searrow \quad \nearrow \\ z \end{array} \tag{4.10}$$

which is easily seen in terms of a suitable local choice of triangulation of M_G . Similarly, the second Racah identity yields invariance under the substitution.

$$\begin{array}{c} x \quad y \\ \curvearrowright \\ \downarrow z \end{array} \iff \frac{q_x q_y}{q_z} \begin{array}{c} x \quad y \\ \searrow \quad \swarrow \\ \downarrow z \end{array} \cdot \tag{4.11}$$

By combining these two identities with (4.8) one obtains invariance of $Z(M, G_x)$ under the following local change of G_x :

$$\begin{array}{c} i \quad j \\ \curvearrowright \\ \downarrow \end{array} \iff \begin{array}{c} i \\ \downarrow \end{array} \begin{array}{c} j \\ \downarrow \end{array} \cdot \tag{4.12}$$

The Biedenharn–Elliott relations give rise to invariance under the substitution

$$\begin{array}{c} i \quad j \\ \swarrow \quad \searrow \\ \downarrow k \\ a \rightarrow \end{array} \iff \begin{array}{c} i \quad j \\ \swarrow \quad \searrow \\ \downarrow k \\ a \rightarrow \end{array} \tag{4.13}$$

which follows in a similar way as above by a convenient choice of a triangulation of M_G .

Combining (4.13) with (4.8) and (4.10) we get in addition invariance under the substitution

$$\begin{array}{c} x \quad y \\ \swarrow \quad \searrow \\ \downarrow z \end{array} \iff \begin{array}{c} x \quad y \\ \swarrow \quad \searrow \\ \downarrow z \end{array} \cdot \tag{4.14}$$

Of course, invariance of $Z(M, G_x)$ also holds under the substitution analogous to (4.12) (resp. (4.13)), where the j -line (resp. a -line) overcrosses the i -line (resp. i - and j -lines) on the lefthand side.

In order to formulate the following lemma, which will be of importance in the next section, we define the matrix S by (see [Wi2] and [KS])

$$S_{ab} = \omega^{-1} Z \left(\begin{array}{c} \text{graph with two overlapping circles } a \text{ and } b \end{array} \right) = \omega^{-1} \sum_c \frac{\omega_c^2 q_c^2}{q_a^2 q_b^2} N_{ab}^c \tag{4.15}$$

and set

$$\omega_b^2(a) = \frac{\omega}{\omega_a^2} S_{ab} . \tag{4.16}$$

The last equality in Eq. (4.15) follows by first applying (4.8) to the double line in the middle of the graph in (4.15), then applying (4.10) and finally (4.4). We note that

$$S_{ab} = S_{ba} = S_{a^*b^*} , \tag{4.17}$$

since $q_{a^*} = q_a$, $\omega_{c^*}^2 = \omega_c^2$ and $N_{ab}^c = N_{ba}^c = N_{a^*b^*}^{c^*}$, and that

$$\omega_b^2(0) = \omega S_{0b} = \omega_b^2 , \tag{4.18}$$

since $q_0 = \omega_0 = 1$ and $N_{0b}^c = \delta_{cb}$.

Lemma 4.3. i) *The state sum $Z(M, G_{\mathbb{X}})$ is invariant under the following local substitution in $G_{\mathbb{X}}$:*

$$\tag{4.19}$$

ii) *We have*

$$\sum_b N_{cd}^b \omega_b^2(a) = \omega_c^2(a) \omega_d^2(a) , \tag{4.20}$$

i.e. $\omega_c^2(a)$, $a \in I$, is an eigenvalue of the matrix N_c defined by

$$(N_c)_d^b = N_{cd}^b, \quad b, d \in I , \tag{4.21}$$

for each $c \in I$, with eigenvector $(\omega_d^2(a))_{d \in I}$ (provided the latter is non-zero).

iii) *If $q_a^2 \neq 1$ for all $a \in I \setminus \{0\}$ and*

$$\Delta = \sum_c q_c^2 \omega_c^4 \neq 0 , \tag{4.22}$$

then the following formulae hold:

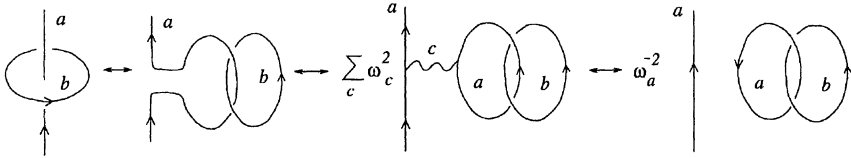
$$\sum_b \omega_b^2 \omega_b^2(a) = \omega^2 \delta_{a0} , \tag{4.23}$$

$$\sum_b S_{ab} S_{bc^*} = \delta_{ac} , \tag{4.24}$$

$$N_{ab}^c = \sum_d \frac{S_{ad} S_{bd} S_{c^*d}}{S_{0d}} . \tag{4.25}$$

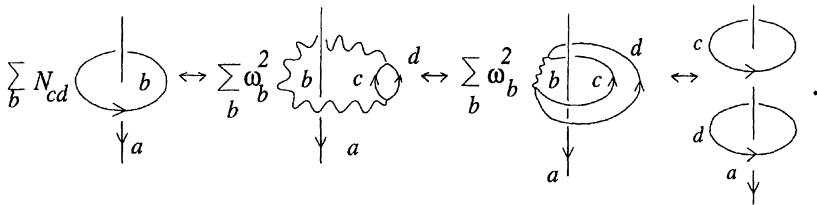
Proof. The statements follow by rather obvious modifications of the arguments in [KS], Appendix A. For completeness we give some details:

i) Using (4.8), Lemma 3.5, (4.9) and Lemma 4.1 we obtain invariance under the following substitutions:



as desired.

ii) follows from i) and invariance under the following substitutions:



iii) It is easy to see (Lemma A.2 in [KS]) that the assumption $q_a^2 \neq 1$ and $\Delta \neq 0$ imply the existence of $b \in I$, such that $\omega_b^2 \neq \omega_b^2(a)$. Then (4.23) follows for $a \neq 0$ from

$$(\omega_b^2 - \omega_b^2(a)) \sum_c \omega_c^2 \omega_{c^*}^2(a) = \sum_{cd} (N_{bc}^d \omega_d^2 \omega_{c^*}^2(a) - N_{bc^*}^d \omega_d^2(a) \omega_c^2) = 0,$$

where we have used ii) and the fact that $N_{bc^*}^d = N_{bd^*}^c$ by (2.1-2) as well as $\omega_{c^*}^2 = \omega_c^2$. For $a = 0$ Eq. (4.23) follows from the definition of ω^2 . Equation (4.24) coincides with Eq. (4.23) for $a = 0$ or $c = 0$ in view of Eqs. (4.16-18). Multiplying by ω_b^2 and summing over $b \in I$ in (4.19) we get from Eq. (4.23) invariance under the substitution

$$\sum_b \omega_b^2 \left[\text{Diagram: vertical line } a \text{ with loop } b \text{ around it} \right] \longleftrightarrow \omega^2 \delta_{a0} \left[\text{Diagram: vertical line } a \text{ with a downward arrow} \right]. \tag{4.26}$$

Using this Eq. (4.24) follows easily for general $a, b \in I$ as in [KS], Lemma A.3.

Finally, Eq. (4.25) follows immediately from Eqs. (4.20) and (4.16) using (4.24). □

We note that, in view of Eq. (4.24), Eq. (4.25) states that the matrix S diagonalizes all the matrices N_a ; $a \in I$.

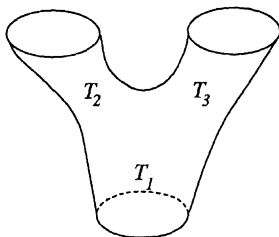
We end this section by discussing some useful properties of the state sums for manifolds with interior graphs, which can be derived easily from the rules developed in this and the previous sections.

1) Suppose that two meridians m and m' are introduced on an empty tube T in M disjoint from G . According to Lemma 3.3 this is equivalent, up to a factor ω^4 , to filling the tube by two cylinders in the neighborhood of m and m' . Between the cylinders there will then be an empty 3-ball. Filling this 3-ball is equivalent to multiplying the state sum by ω^{-2} by Lemma 3.2 i). Thus, it follows that the presence of two (or more) meridians on T is equivalent to the presence of just one, i.e.

$$(Z(M(T_m, T_{m'}), G_x) = Z(M(T_m), G_x) . \tag{4.27}$$

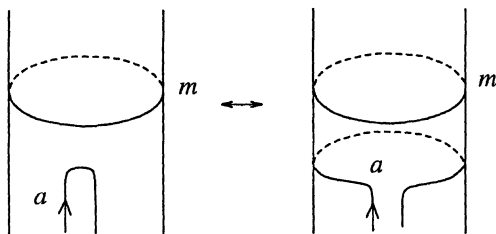
In fact, the purpose of the factors ω^{-2} in (3.16) is to ensure this projection property of the meridians.

2) Consider in M a branching of an empty tube T_1 into two empty tubes T_2 and T_3 , which are all disjoint from G (see the picture below).



An argument similar to the previous one implies equivalence of any two configurations of meridians on the tubes T_1, T_2, T_3 , as long as at least two of the tubes contain a meridian.

3) Similarly one sees that lines in G along a tube containing a meridian may be deformed non-trivially as follows:

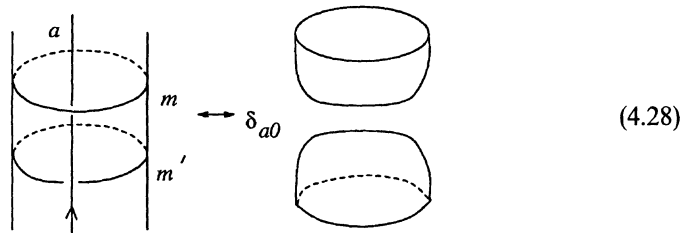


4) The statements in 1–3) may be generalized to the case where lines in G (in addition to the a -line in 3)) are traversing the tubes, as long as all these lines overcross (or all undercross) the meridians (and the a -line in 3)). This follows by repeated use of (4.8) and (4.12–4.14).

5) Combining the generalized versions of 2) and 3) with Lemma 3.3, it is easy to show that if T_m and $T_{m'}$ are two empty tubes in M such that T , resp T' , is traversed by a lefthanded, resp. righthanded, line then T_m and $T_{m'}$ have trivial braiding, i.e. they may be moved through each other.

6) Using (4.26) and 3) as well as Lemma 3.4 it follows that if a tube in M is traversed by an a -coloured line in G_x which overcrosses one meridian and undercrosses another then $Z(M, G_x)$ vanishes unless $a = 0$, in which case the line and the

meridians can be deleted and the tube filled:



We leave the details of these arguments to the reader. Alternatively, [KS] can be consulted.

Finally, we note the following lemma, which we shall need in the next section.

Lemma 4.4. *Let \mathcal{L}_a and \mathcal{L}'_b be arbitrary disjoint framed coloured links in S^3 . Then the following holds:*

i)
$$Z(S^3, (\mathcal{L}_a)_L \cup (\mathcal{L}'_b)_R) = \omega^2 Z(S^3, (\mathcal{L}_a)_L) Z(S^3, (\mathcal{L}'_b)_R); \tag{4.29}$$

ii)
$$Z(S^3, (\mathcal{L}_a)_L) = \omega^{-2} Z(G_L(\mathcal{L})_a), \tag{4.30}$$

where $G_L(\mathcal{L})$ is the planar graph (with no 3-vertices) naturally obtained from the framed link \mathcal{L} by projecting onto a plane (and specified more precisely below);

iii)
$$Z(S^3, (\mathcal{L}_a)_R) = \omega^{-2} Z(G_R(\mathcal{L})_a), \tag{4.31}$$

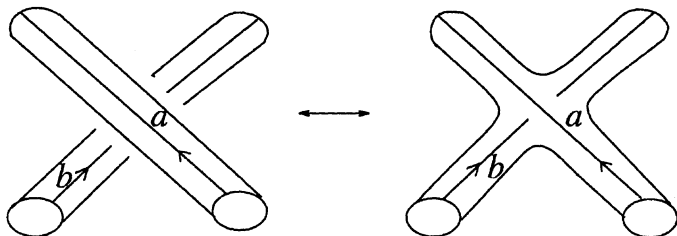
where $G_R(\mathcal{L})$ is obtained from $G_L(\mathcal{L})$ by replacing all 4-vertices by their inverses.

Proof.

i) follows immediately from 5) above and Lemma 3.2 iv).

ii) Consider \mathcal{L} as embedded into $\mathbf{R}^3 \subseteq S^3$ and choose a plane π such that \mathcal{L} lies on one side of π . We may then deform \mathcal{L} such that the circles on the tori constituting the boundary of the tubular neighborhood \mathcal{T}_φ can be obtained by translation from the corresponding cores in the direction orthogonal to and away from π . (In terms of ribbons given by the framing of \mathcal{L} this is equivalent to deforming the ribbons such that no twists are present (see [RT1]).) The coloured graph $G_L(\mathcal{L})_a$ is then obtained by projecting the circles on $\partial\mathcal{T}_\varphi$ onto π , taking into account over- and undercrossings as well as orientation and colouring \underline{a} of \mathcal{L} .

In order to prove (4.30), we consider the crossings of tubes in \mathcal{L} (see the LHS of the figure below) corresponding to 4-vertices in $G_L(\mathcal{L})$. By Lemma 3.3 and 2) above we may connect the two tubes at each crossing by a tube at the cost of a factor ω^2 . Using lefthandedness and the general version of 3) above (or (4.8)), it follows that the two tubes may be joined as indicated in the following figure.



Thus one obtains an interior graph in S^3 whose core is a copy of $G_L(\mathcal{L})$ and whose corresponding coloured graph on the boundary of its tubular neighborhood is a copy of $G_L(\mathcal{L})_a$, and all of whose lines are lefthanded.

Now it is not difficult to see that all meridians can be eliminated, partly by use of 2) above and partly by using Lemmas 3.5 and 4.2 at the cost of factors ω^{-2} . Finally, handles (corresponding to the faces of the graph $G_L(\mathcal{L})$) may be cut by Lemma 3.4, thus obtaining D^3 . Keeping track of the ω -factors one arrives at (4.30).

iii) Follows from ii) by observing that in the case of righthanded lines the 4-vertex on the RHS of the figure above must be replaced by its inverse. \square

5. Dimension of the State Spaces and Factorization of State Sums

This section is devoted to deriving a formula related to Verlinde’s formula [V] for the dimension of the state space V_Σ associated to a closed, compact, and oriented surface Σ , and to showing that the state sum $Z(M)$ for a closed, compact, oriented 3-manifold M factorizes into a lefthanded and a righthanded contribution, which in the quantum group case equal $\tau(M)$ and $\tau(M^*)$, respectively, where $\tau(M)$ is the invariant introduced in [RT2].

Let us first note that by (3.8), (3.14) and (2.5) we have, for Σ connected of genus 0,

$$\dim V_{S^2} = Z(S^2 \times S^1) = \omega^{-4} \sum_{a,b,c} N_{ab}^2 \omega_a^2 \omega_b^2 \omega_c^2 = 1 . \tag{5.1}$$

Let the matrix $|\bar{N}|^2$ be defined by

$$|\bar{N}|^2 = \sum_a N_a^t N_a = \sum_a N_{a^*} N_a , \tag{5.2}$$

where the upper index t denotes transposition and the last equality follows from (2.1–2).

Theorem 5.1. *Assume that $q_a^2 \neq 1$ for $a \neq 0$ and that $\Delta \neq 0$. Let Σ be a connected, closed, compact, oriented surface of genus $g \geq 1$. Then*

$$\dim V_\Sigma = (\text{tr}(|\bar{N}|^{2(g-1)}))^2 . \tag{5.3}$$

Proof. Given the computational rules developed in Sects. 3 and 4 the proof of (5.3) is identical to the proof of Thm. 7.4 (without punctures) in [KS] with only minor modifications. It suffices therefore here to indicate the main line of argument: According to Eq. (3.8) we have to calculate $Z(\Sigma \times S^1)$. Using Theorem 3.1 we may cut $\Sigma \times S^1$ along $\Sigma \times \{a\}$, for some $a \in S^1$, thus obtaining a manifold homeomorphic

to $\Sigma \times [0, 1]$ with a graph $(G \times \{0\}) \cup (G \times \{1\})$ on its boundary, where G is a canonical graph on Σ without 4-vertices, e.g. $G = G^\Sigma$ (see Sect. 3). By (4.28) we can then introduce an empty tube connecting $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ thus obtaining a handlebody with $2g$ handles. Using (4.8) the graph on its boundary can be reduced to one without 3-vertices and using in addition Lemmas 3.4–5 and (4.9) the handles can be cut yielding a 3-ball with a graph on its boundary. The corresponding state sum can then be evaluated and shown to equal the RHS of (5.3) by Lemma 4.3 iii). \square

We note here that, as a consequence of Eqs. (4.24–25), $|\vec{N}|^2$ on the RHS of (5.3) can be replaced by

$$\vec{N}^2 \equiv \sum_a (N^a)^2, \tag{5.4}$$

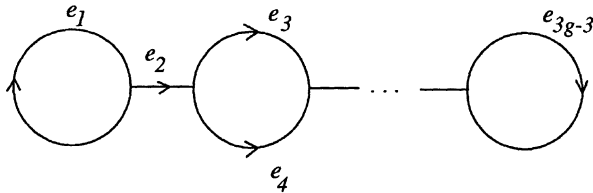
where the symmetric matrix N^a is defined by

$$(N^a)_{bc} = N^a_{bc}, \quad b, c \in I, \tag{5.5}$$

for each $a \in I$. Thus

$$\dim V_\Sigma = (\text{tr}(\vec{N}^{2(g-1)}))^2. \tag{5.6}$$

Let now G_g be the graph depicted below



with g circles and coloured by $\underline{e} = (e_1, \dots, e_{3g-3})$ for $g > 1$ and $\underline{e} = (e_1)$ for $g = 1$. Notice that the dimension of the vector space $V(\Sigma, G_g)$ associated to the graph G_g embedded (without self-intersections and such that the cyclic order of lines at each vertex are as indicated on the figure above) in a closed oriented surface Σ is given by

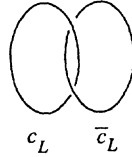
$$\dim V(\Sigma, G_g) = (\text{tr}(\vec{N}^{2(g-1)})). \tag{5.7}$$

as is easily seen by use of Lemma 4.3.

Let $M_{\Sigma^g} \subseteq R^3$ be a handlebody with g handles and $\partial M_{\Sigma^g} = \Sigma^g$, containing in its interior two disjoint (non-oriented) copies c_g^L and c_g^R of G_g as deformation retracts of Σ^g . Thus, in particular, the circles in c_g^L and c_g^R are non-contractible in M_{Σ^g} . We let \mathcal{G}_g^L and \mathcal{G}_g^R be two disjoint interior graphs in M_{Σ^g} whose cores are c_g^L and c_g^R , respectively, and whose associated graphs G_g^L and G_g^R are also copies of G_g on the boundaries of tubular neighborhoods \mathcal{T}^L and \mathcal{T}^R , respectively, which are equipped with $3g - 3$ meridians for $g > 1$, and one for $g = 1$, each such that all lines in G_g^L , resp. G_g^R , are lefthanded, resp. righthanded. Denote the vector spaces associated to $G_g^L \subseteq \partial \mathcal{T}^L$ and $G_g^R \subseteq \partial \mathcal{T}^R$ by V_g^L and V_g^R , respectively.

By introducing a canonical graph G without 4-vertices on Σ_g we may consider the partition function associated to M_{Σ_g} with interior graphs \mathcal{G}_g^L and \mathcal{G}_g^R and boundary graph G as a linear mapping from $(V_g^L \otimes V_g^R)^*$ into V_Σ . In fact it is not hard to show that this mapping is an isomorphism. We shall not need this general result in the following, but restrict our attention to the case $g = 1$.

In this case G_g is a circle and Σ_g is a torus T embedded in $R^3 \subseteq S^3$, and M_T is a solid torus whose complement \bar{M}_T in S^3 is also a solid torus with $\partial\bar{M}_T = T^*$. We thus consider copies c^L, c^R (resp. \bar{c}^L, \bar{c}^R) of the circle S^1 in M_T (resp. \bar{M}_T) and interior graphs \mathcal{G}^L and \mathcal{G}^R (resp. $\bar{\mathcal{G}}^L$ and $\bar{\mathcal{G}}^R$) whose associated tubular neighborhoods \mathcal{T}^L and \mathcal{T}^R (resp. $\bar{\mathcal{T}}^L$ and $\bar{\mathcal{T}}^R$) are solid tori, and whose associated graphs G^L, G^R (resp. \bar{G}^L, \bar{G}^R) are oriented circles, which we assume have zero linking number with c^L, c^R (\bar{c}^L, \bar{c}^R , respectively). Moreover, G^L and \bar{G}^L are lefthanded and G^R, \bar{G}^R are righthanded and we note that c^L and \bar{c}^L , resp. c^R and \bar{c}^R , are linked as indicated in the following picture:



As the graphs $G^L, \bar{G}^L, G^R, \bar{G}^R$ have no 3-vertices and only one line, the vector spaces associated to them may be identified with $\mathbf{C}^{|I|}$, where $|I|$ is the cardinality of I , (cf. Eq. (5.7)).

The partition function Z'_T (as defined in Eq. (3.4)) associated to M_T with interior graphs \mathcal{G}^L and \mathcal{G}^R and a canonical graph G without 4-vertices on T thus yields a linear mapping

$$K: \mathbf{C}^{|I|} \otimes \mathbf{C}^{|I|} \rightarrow V_T,$$

and similarly the partition function \bar{Z}'_T associated to \bar{M}_T with the interior graphs $\bar{\mathcal{G}}^L$ and $\bar{\mathcal{G}}^R$ and boundary graph G on T^* yields a linear mapping

$$L: V_T \rightarrow \mathbf{C}^{|I|} \otimes \mathbf{C}^{|I|}.$$

Gluing M_T and \bar{M}_T with interior graphs along T yields, of course, S^3 with interior graphs $\mathcal{G}^L, \mathcal{G}^R, \bar{\mathcal{G}}^L, \bar{\mathcal{G}}^R$. Thus the mapping $LK: \mathbf{C}^{|I|} \otimes \mathbf{C}^{|I|} \rightarrow \mathbf{C}^{|I|} \otimes \mathbf{C}^{|I|}$ is given by the matrix

$$(LK)_{(e,e')(f,f')} = Z(S^3, \mathcal{G}_e^L \cup \mathcal{G}_{f'}^R \cup \bar{\mathcal{G}}_e^L \cup \bar{\mathcal{G}}_{f'}^R) = \omega^2 Z(S^3, \mathcal{G}_e^L \cup \bar{\mathcal{G}}_e^L) Z(S^3, \mathcal{G}_{f'}^R \cup \bar{\mathcal{G}}_{f'}^R), \tag{5.8}$$

where we have used 5) at the end of Sect. 4 and Lemma 4.4 i). According to Lemma 4.4 ii)–iii) we have

$$Z(S^3, \mathcal{G}_e^L \cup \bar{\mathcal{G}}_e^L) = \omega^{-1} S_{e,e'}, \tag{5.9}$$

$$Z(S^3, \mathcal{G}_{f'}^R \cup \bar{\mathcal{G}}_{f'}^R) = \omega^{-1} S_{f',f'}^{-1}, \tag{5.10}$$

where $S_{e,e'}$ is defined by Eq. (4.15). Thus, Eq. (5.8) reads

$$LK = \omega^2 (\omega^{-1} S \otimes \omega^{-1} S^{-1}), \tag{5.11}$$

and we conclude now from the fact that $\dim V_T = \dim(\mathbf{C}^{|I|} \otimes \mathbf{C}^{|I|})$, that L and K are isomorphisms and that

$$K(S^{-1} \otimes S)L = 1_{V_T}, \tag{5.12}$$

provided the assumptions of Theorem 5.1 are fulfilled.

It was first observed by M. Karowski and R. Schrader [KS1] that the remarkable property of the vector spaces V_Σ being factorized into lefthanded and righthanded ones should lead to a rather simple proof of the relation between $Z(M)$ and the invariant $\tau(M)$ defined in [RT2]. This is the content of the following theorem where the context is assumed to be that of quantum deformations of a semisimple Lie algebra at an even root or unity, in which case the assumptions of Theorem 5.1 are known to be fulfilled, and in addition $|q_a| = 1$ for all $a \in I$.

Theorem 5.2. *Let M be a closed, compact, oriented 3-manifold. Then*

$$Z(M) = \tau(M)\tau(M^*) , \tag{5.13}$$

where $\tau(M)$ is an invariant defined by Eq. (5.17) below, which in fact equals the invariant introduced in [RT2], up to normalization.

Proof. As is well known M can be obtained (up to diffeomorphism) by surgery along a framed link in S^3 : Let $\mathcal{L}_1, \dots, \mathcal{L}_n$ be the components of \mathcal{L} , i.e. \mathcal{L}_i is a framed circle, and let \mathcal{T}_i be a tubular neighborhood of \mathcal{L}_i such that $\cap_i \mathcal{T}_i = \emptyset$. Then we obtain M as

$$(S^3 \setminus (\cup_i \mathcal{T}_i)) \cup_{F_1} \mathcal{T}_1 \cup_{F_2} \dots \cup_{F_i} \mathcal{T}_i \dots \cup_{F_n} \mathcal{T}_n ,$$

i.e. by first removing all \mathcal{T}_i from S^3 and then gluing them back with identification maps $F_i: \partial \mathcal{T}_i \rightarrow \partial(S^3 \setminus \cup_i \mathcal{T}_i)$, which are determined by the framing of \mathcal{L}_i . In other words if the framing of \mathcal{L}_i is given by the circle L_i on $\partial \mathcal{T}_i$ with linking number N w.r.t. the core of \mathcal{T}_i , then F_i is composed of an inversion (which interchanges the a- and b-cycles in a canonical homology basis for $\partial \mathcal{T}_i$) and an N -fold Dehn-twist. Equivalently, $L_i \subseteq \partial \mathcal{T}_i$ is identified with a meridian on $\partial \mathcal{T}_i \subseteq \partial(S^3 \setminus \cup_i \mathcal{T}_i)$.

After inserting the identity operator in the form of (5.12) for each $\partial \mathcal{T}_i$, a simple calculation yields

$$Z(M) = \sum_{e, f, e', f'} Z(S^3, \mathcal{L}_e)_L \cup (\mathcal{L}_f)_R \prod_{i=1}^n S_{e_i e'_i}^{-1} \delta_{e_i 0} S_{f_i f'_i} \delta_{f_i 0} , \tag{5.14}$$

where the factors $\delta_{e'_i 0} \delta_{f_i 0}$ arise because the inversions make the e'_i - and f'_i -lines traverse handles and hence their colours vanish by Lemma 4.2.

Using (4.18) and Lemma 4.4 i) in (5.14) we get

$$Z(M) = (\omega^2)^{-n+1} \sum_e \omega_e^2 Z(S^3, (\mathcal{L}_e)_L) \sum_f \omega_f^2 Z(S^3, (\mathcal{L}_f)_R) , \tag{5.15}$$

where $\omega_e^2 = \prod_{i=1}^n \omega_{e_i}^2$. Equation (5.15) may be rewritten as

$$Z(M) = \tau_L(M)\tau_R(M) , \tag{5.16}$$

if we define

$$\begin{aligned} \tau_L(M) &= \omega^{-n+1} (A_L \omega^{-1})^{\sigma(\mathcal{L})} \sum_e \omega_e^2 Z(S^3, (\mathcal{L}_e)_L) \\ &= \omega^{-n-1} (A_L \omega^{-1})^{\sigma(\mathcal{L})} \sum_e \omega_e^2 Z(G_L(\mathcal{L})_e) , \end{aligned} \tag{5.17}$$

and correspondingly τ_R by replacing L by R in (5.17), where we have used Lemma 4.4 ii) and iii). Here $\sigma(\mathcal{L})$ is the signature of a certain 4-manifold whose boundary is

M , $\Delta_L = \Delta$ is defined by Eq. (4.22) and Δ_R is defined by replacing q_c^2 by q_c^{-2} in Eq. (4.22). Since it is known (see [T, ch. 2]) that

$$\Delta_L \Delta_R = \omega^2,$$

it follows that (5.15) and (5.16) are equivalent.

In Sect. 6 we show that

$$Z(G_L(\mathcal{L})_e) = F(\Gamma(\mathcal{L}, \underline{e})), \tag{5.18}$$

where F denotes the functor from the category of coloured ribbon graphs into the category of representations of the quantum group under consideration, which was introduced in [RT1]. Using the identity (5.18) in (5.17) one obtains the expression for the invariant $\tau(M)$ introduced in [RT2] (up to normalization) as given in [T, ch. 2]. Thus $\tau_L(M) = \tau(M)$. Since, moreover, it is known that $\tau(M^*) = \tau(M)^*$ and it is easy to verify that $\tau_R(M) = \tau_L(M)^*$ as a consequence of the properties of 6j-symbols, we have proven (5.13). \square

As a final topic in this section we shall briefly discuss how vector spaces can be associated to surfaces with punctures and the corresponding extension of Theorem 5.1.

Let Σ be a closed, compact, oriented surface with a set $p = (p_1, \dots, p_n)$ of n different, distinguished points, and consider a canonical graph G on Σ without 4-vertices, disjoint from $\{p_1, \dots, p_n\}$. Adjoining to G lines l_1, \dots, l_n such that l_i connects p_i to a point in G , different from the vertices in G and otherwise not intersecting G , we obtain a graph $G(p)$ on Σ whose vertices are all of order 2 or 3 except for p_1, \dots, p_n , which are of order 1. The dual graph of $G(p)$ on Σ , constructed in the standard way, then yields a proper triangulation of Σ except for the n disks D_1, \dots, D_n containing p_1, \dots, p_n , respectively, whose boundaries consist of a single link each, namely the dual links of l_1, \dots, l_n . Interpreting these disks as holes in Σ , we obtain a proper triangulation of $\Sigma \setminus \cup_i D_i$ such that the boundary of each hole consists of a single link (see Fig. 7)

We call graphs of the form $G(p)$ canonical graphs for the punctured surface $\Sigma(p)$ and we shall assume that the disks D_1, \dots, D_n around p_1, \dots, p_n , respectively, are fixed, and only consider graphs G disjoint from $\cup_i D_i$. Moreover, we assume that l_i intersects ∂D_i at a (unique) fixed point q_i .

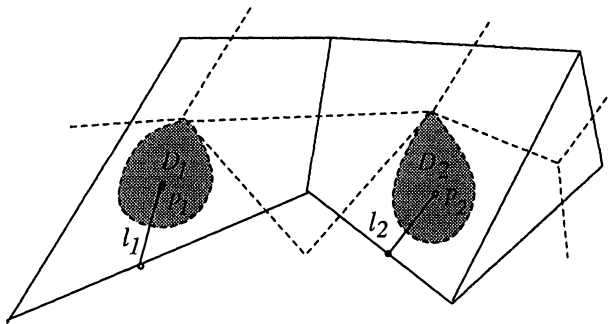


Fig. 7. A piece of the triangulated surface with punctures and a canonical graph on it

Fixing the colours of the boundary links in the triangulation, i.e. of the lines l_1, \dots, l_n to be $\underline{a} = (a_1, \dots, a_n)$ we define, in analogy with $V(\Sigma, G)$, the vector space $V_{\underline{a}}(\Sigma(\underline{p}), G(\underline{p}))$ to be the direct sum over colourings of the remaining lines in $G(\underline{p})$ of the tensor product of spaces associated to the coloured 3-vertices in $G(\underline{p})$.

Next we have to define a set of linear mappings between these spaces for different canonical graphs possessing properties corresponding to those of $P_{(\Sigma, G_1), (\Sigma, G_2)}$ discussed in Sect. 3. For this purpose let G_1 and G_2 be two canonical graphs on Σ and consider the graphs $G_1(\underline{p}) \times \{0\}$ and $G_2(\underline{p}) \times \{1\}$ on $\partial(\Sigma \times [0, 1])$. We construct the graph $G_1(\underline{p}) + G_2(\underline{p})$ on $\partial((\Sigma \setminus \cup_i D_i) \times [0, 1])$ by connecting the points $(q_i, 0)$ in $G_1(\underline{p}) \times \{0\}$ to $(q_i, 1)$ in $G_2(\underline{p}) \times \{1\}$ by the a -coloured line $q_i \times [0, 1]$, suitably oriented. Finally, we introduce meridians m_i on the tubes $T^i = \partial D_i \times [0, 1]$. It is then easy to see that $(G_1(\underline{p}) + G_2(\underline{p})) \cup \{m_i\}$ is a canonical graph on $\partial((\Sigma \setminus \cup_i D_i) \times [0, 1])$ and that the corresponding vector space with \underline{a} fixed is

$$V_{\underline{a}}(\Sigma(\underline{p}), G_1(\underline{p}))^* \otimes V_{\underline{a}}(\Sigma(\underline{p}), G_2(\underline{p})).$$

The partition function

$$Z'_{\underline{a}}(M(\{T^i_m\}), G_1(\underline{p}) + G_2(\underline{p}))$$

is defined in terms of (3.16) in analogy with (3.4) by summing over all colourings of $G_1(\underline{p}) + G_2(\underline{p})$ but keeping the colouring \underline{a} of lines traversing the tubes fixed. Hence, it may be considered as a mapping

$$P_{G_1, G_2}(\underline{a}): V_{\underline{a}}(\Sigma(\underline{p}), G_1(\underline{p})) \rightarrow V_{\underline{a}}(\Sigma(\underline{p}), G_2(\underline{p})).$$

Since this mapping depends on the chiralities of the \underline{a} -lines w.r.t. the meridians we change notation and denote by $\underline{a} = (a_1, \dots, a_r)$, resp. $\underline{b} = (b_1, \dots, b_s)$, the colours of the left-, resp. righthanded lines and correspondingly replace \underline{a} everywhere by $\underline{a}, \underline{b}$.

It is a simple consequence of Theorem 3.1 and the projection property of meridians that for any three canonical graphs G_1, G_2, G_3 without 4-vertices on Σ we have

$$P_{G_2, G_3}(\underline{a}, \underline{b}) P_{G_1, G_2}(\underline{a}, \underline{b}) = P_{G_1, G_3}(\underline{a}, \underline{b})$$

and, in particular, $P_{G, G}(\underline{a}, \underline{b})$ is a projection. Hence we let $V'_{\underline{a}, \underline{b}}(\Sigma(\underline{p}), G(\underline{p}))$ be the support of this projection and conclude, as in Sect. 3, that these spaces for different G may be consistently identified with a space $V_{\underline{a}, \underline{b}}(\Sigma(\underline{p}))$ by the mappings $P_{G_1, G_2}(\underline{a}, \underline{b})$ and that

$$\dim V_{\underline{a}, \underline{b}}(\Sigma(\underline{p})) = Z(\Sigma \times S^1, \mathcal{L}^L_{\underline{a}} \cup \mathcal{L}^R_{\underline{b}}), \tag{5.19}$$

where \mathcal{L} , respectively \mathcal{L} , consists of r , respectively s , framed circles of the form $p_i \times S^1$ with trivial framing, and $\mathcal{L}^L_{\underline{a}}$, resp. $\mathcal{L}^R_{\underline{b}}$, denote the corresponding coloured left-, resp. righthanded links.

On the basis of (5.19) a slight extension of the proof of Theorem 5.1 yields the following generalization (see Theorem 7.4 in [KS]).

Theorem 5.3. *Let $\Sigma(\underline{p})$ be a connected, closed, compact, oriented surface of genus $g \geq 1$ with n punctures $(p_1, \dots, p_n) \equiv \underline{p}$, r of which are lefthanded with colours $\underline{a} = (a_1, \dots, a_r)$ and s righthanded with colours $\underline{b} = (b_1, \dots, b_s)$. Then*

$$\dim V_{\underline{a}, \underline{b}}(\Sigma(\underline{p})) = \text{tr}(N_{a_1} \dots N_{a_r} |\bar{N}|^{2(g-1)}) \text{tr}(N_{b_1} \dots N_{b_s} |\bar{N}|^{2(g-1)}). \tag{5.20}$$

We remark that in (5.20) we have assumed a certain common orientation of the \underline{a} and \underline{b} coloured lines in the definition of $V_{\underline{a},\underline{b}}(\Sigma(p))$.

6. Ribbon Graphs and 6j-Symbols

The purpose of this section is partly to describe how the properties of the 6j-symbols of a quantum group can be demonstrated by the use of the notion of ribbon graphs and partly to verify Eq. (5.18).

As was proven in [A], q -deformations of the universal enveloping algebras of complex simple finite-dimensional Lie algebras $U_q\mathfrak{g}$ (quantum groups) with q -admissible modules $\{V_i\}$ constructed in [TW] for $q = \exp(i\pi/l)$ and l bigger then the Coxeter number of \mathfrak{g} give us natural examples of modular Hopf algebras. (For the Lie algebras of type \mathfrak{g}_2 l must also be odd and not divisible by 3.) For the reader's convenience we recall some of the basic notions that enter the definition of a modular Hopf algebra.

Definition [TW]. Let be (A, R, v) a ribbon Hopf algebra (see Def. 3.3 in [RT1]) over \mathbf{C} , where A is a quasitriangular Hopf algebra, R a universal R -matrix and $v \in A$ a central element with special properties (see Eq. (6.3) below). Assume the following data are given:

- i) a finite set I with involution $i \mapsto i^*: I \rightarrow I$ and a preferred element $0 = 0^*$,
- ii) a set of A -modules $\{V_i\}$ labeled by $i \in I$, where $V_0 = \mathbf{C}$ with the action of A determined by the counit $A \rightarrow \mathbf{C}$,
- iii) a set of A -linear isomorphisms

$$\{w_i: (V_i)^* \rightarrow V_{i^*}, i \in I\}, \quad w_0 = \text{id}_{\mathbf{C}}.$$

The triple (A, R, v) together with these data is called a modular Hopf algebra if the following axioms (1–5) are satisfied:

- 1) The modules $\{V_i, i \in I\}$ are mutually non-isomorphic, irreducible (i.e. do not contain proper non-trivial A -submodules), have a finite \mathbf{C} -dimension and all have non-zero quantum dimension (see def. below).
- 2) For each $i \in I$ the homomorphism

$$w_i^* \circ (w_{i^*}): V_i \rightarrow V_{i^{**}} = V_i$$

is the multiplication by $g = uv^{-1}$, where $u \in A$ is such that that ad_u equals the square of the antipode S of A (see also Eq. (6.3) below).

- 3) For any $k \geq 2$ and for any sequence $\Theta = (\lambda_1, \dots, \lambda_k) \in I^k$ there exists an A -linear decomposition:

$$V_{\lambda_1} \otimes V_{\lambda_2} \otimes \dots \otimes V_{\lambda_k} = Z_{\Theta} \oplus \bigoplus_{\lambda \in I} (V_{\lambda} \otimes \Omega_{\Theta}^{\lambda}),$$

where $\{\Omega_{\Theta}^{\lambda}\}$ are vector spaces over \mathbf{C} and Z_{Θ} is a certain A -module satisfying the next axiom (4).

- 4) For any $k \geq 2$, $\Theta \in I^k$ and any A -linear homomorphism $f: Z_{\Theta} \rightarrow Z_{\Theta}$ the q -trace of f is equal to 0, where q -trace of the operator $f: V \rightarrow V$ is defined as the trace of the operator

$$x \mapsto gf(x): V \rightarrow V,$$

and the quantum dimension $\text{dim}_q V$ equals $\text{tr}_q(\text{id}_V)$.

5) Let S_{ij} be the q -trace of the A -linear operator

$$a \mapsto R_{21}R_{12}a: V_i \otimes V_j \rightarrow V_i \otimes V_j.$$

Then the matrix S_{ij} must be invertible.

We shall henceforth use the standard notations for ribbon graphs and let F denote the functor introduced in [RT1] mentioned previously.

In order to prove Eq. (5.18) it is enough to verify that Z and F agree on the graphs depicted in (4.3–6), i.e. that the equations analogous to (4.3–6) with Z replaced by F hold. In fact, one uses repeatedly Lemma 3.2 and Lemma 4.2 ii)–iii) (most effectively in the form of the Wigner–Eckart type relations derived in [KS]) to decompose any planar graph G into pieces of the types in (4.3–6) and correspondingly obtains $Z(G)$ as a contraction of a linear combination of tensor products of partition functions of the pieces. Since the analogue of Lemma 4.2 also holds for the functor F (see below) the assertion follows.

The validity of the analogues of Eqs. (4.3–6) for F is, on the other hand, essentially obvious by inspection. In fact, Eq. (4.3) holds for F since ω_x^2 equals the quantum dimension of V_x , $x \in I$, up to a sign, and Eq. (4.4) for F can be obtained by a suitable choice of dual bases $\{\alpha\}$ and $\{\alpha^*\}$ in the mutually dual intertwiner spaces V_{ij}^k (from $V_i \otimes V_j$ to V_k) and V_k^{ij} (from V_k to $V_i \otimes V_j$), so that

$$\begin{array}{c} k \\ \downarrow \\ \alpha \\ \begin{array}{c} \curvearrowright \\ i \quad j \\ \curvearrowleft \end{array} \\ \beta^* \\ \downarrow \\ k' \end{array} \quad \longleftrightarrow \quad \frac{\delta_{kk'}}{\omega_k^2} \delta_{\alpha\beta} \quad \begin{array}{c} k \\ \downarrow \end{array} \quad (6.1)$$

which is the ribbon graph version of (4.7). We note that the dual basis $\{\alpha^*\}$ in V_k^{ij} to $\{\alpha\}$ in V_{ij}^k can be constructed w.r.t. a natural bilinear pairing or, alternatively by exploiting the natural inner product on $V_i \otimes V_j$ (see [D]).

Equation (4.5) can be written in terms of ribbon graphs as

$$F \left(\begin{array}{c} m \\ \uparrow \end{array} \right) \left(\begin{array}{c} \delta \\ \downarrow \\ \alpha \\ \downarrow \\ \beta^* \end{array} \right) \left(\begin{array}{c} k \\ \downarrow \\ j \\ \downarrow \\ n \end{array} \right) \left(\begin{array}{c} l \\ \downarrow \end{array} \right) \doteq \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}^{\alpha\beta^*\gamma^*\delta}, \quad (6.2)$$

and the analogue of Eq. (4.6) for F is given below in (6.6).

We next proceed to show that the 6j-symbols defined by (6.2) have the required properties (see Sect. 2). First we note that due to Lemma 5.1 of [RT2] ribbon graphs can be considered as two-sided objects, where the lower side of the ribbons and annuli have the dual colour and the opposite orientation compared to the upper side. We shall use an additional operation on the ribbons and annuli – so-called half-twists, as proposed by [N]. The right half-twist is illustrated in Fig. 8.

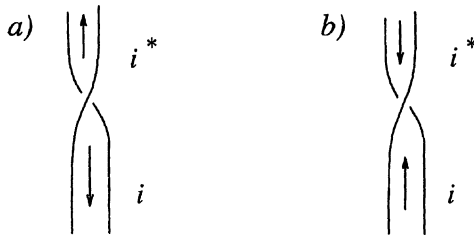


Fig. 8. Right half-twist on a ribbon

The transformation in Fig. 8 a) is given by the action of the operator $\pi_{V_i}(\tau): V_i \rightarrow (V_{i^*})^* = V_i$, and the one in Fig. 8 b) by the action of $\pi_{V_{i^*}}(\tau): V_{i^*} \rightarrow V_{i^*} = V_i^*$. We use here the standard notation:

$$\pi_{V_i^*}(a) = [\pi_{V_i}(S(a))]^t, \quad a \in A,$$

where $S(a)$ denotes the antipode of a , t the transposition w.r.t. the canonical pairing $V_i^* \otimes V_i \rightarrow \mathbb{C}$ and τ an invertible element of the extended Hopf algebra, which satisfies the following identities:

$$\tau^2 = v^{-1}, \quad u = ad_i S(u), \quad \varepsilon(\tau) = 1, \quad \Delta(\tau) = (\tau \otimes \tau)R, \quad S(\tau) = \tau^{-1}u^{-1}. \quad (6.3)$$

τ corresponds to the element w^{-1} in the [RT1] notation. The definition of the left half-twists is obtained from the above by replacing τ by τ^{-1} .

It turns out that the so defined half-twists have a nice geometrical property, namely they can be pulled off from left to right through all four annihilation and creation generators of the ribbon graph category:

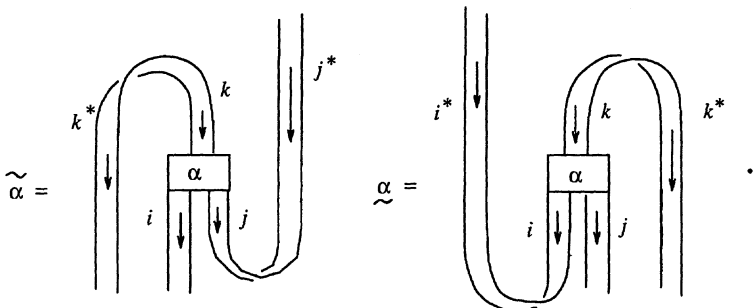


This follows directly from the properties of τ for non self-dual colours. In the case of $i = i^*$, there exists an intertwiner $T: V_i \rightarrow V_i^*$, and the above mentioned property can be reduced to the claim: $T = T^t$. This means that the two isomorphisms $V_i^* \otimes V_i \cong V_i \otimes V_i$ and $V_i \otimes V_i^* \cong V_i \otimes V_i$ give us the same invariant vector $\xi \in V_i \otimes V_i$, i.e.

$$\text{Perm}(\pi_{V_i}(g) \otimes 1)\xi = \xi. \quad (6.4)$$

This was proven for quantum groups in [DJN]. In the terminology of [T](6.4) distinguishes unimodular and modular Hopf algebras.

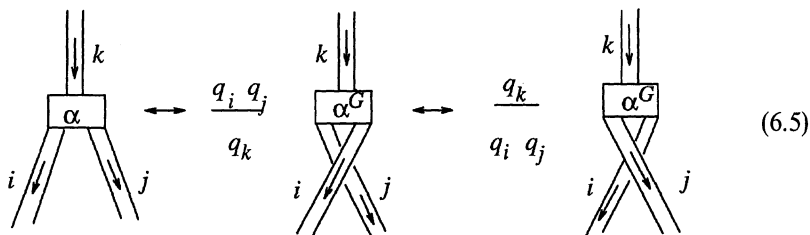
We may now construct isomorphisms $V_{ij}^k \simeq V_{k^*i}^{j^*} \simeq V_{jk^*}^{i^*}$ as follows (cf. also [DJN]). To each $\alpha \in V_{ij}^k$ we associate an $\tilde{\alpha} \in V_{k^*i}^{j^*}$ and an $\underline{\alpha} \in V_{jk^*}^{i^*}$, such that



It is easy to see from the well known identities for ribbon graphs [RT1], that

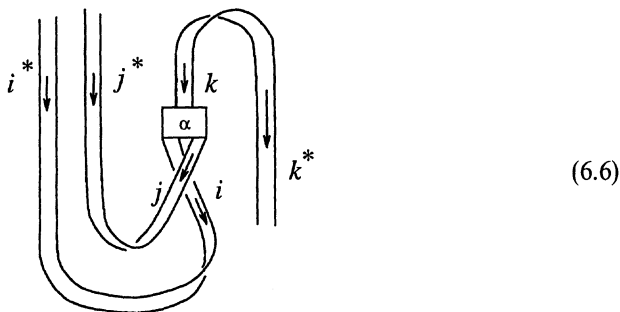
$$\tilde{\tilde{\alpha}} = \alpha, \tilde{\underline{\alpha}} = \alpha, \underline{\underline{\alpha}} = \alpha.$$

Due to the fact, that all admissible modules $\{V_i\}$ of a modular Hopf algebra have real q -dimension there exist isomorphisms $\alpha \rightarrow \alpha^G$ between V_{ij}^k and V_{ji}^k given by:



where $q_i = \sqrt{v_i^{-1}}$, $v_i = \pi_{v_i}(v)$, $v_i = v_{i^*}$. The modular property of ribbon Hopf algebras $R_{21}R_{12}(v^{-1} \otimes v^{-1})\Delta v = 1$ implies $\alpha^{GG} = \alpha$.

On the other hand the identity $\Delta(\tau) = (\tau \otimes \tau) R$ allows us to construct an isomorphism between V_{ij}^k and $V_{k^*i}^{j^*}$:



We denote by \mathcal{V}_{ij}^k the \mathbf{C} -module which is obtained by identifying $V_{ij}^k, V_{k^*i}^{j^*}, V_{jk^*}^{i^*}, V_{ji}^k$ and $V_{k^*i}^{j^*}$ by the isomorphisms constructed above. It is not difficult to see that the \mathbf{C} -module $\mathcal{V}_{i^*j^*}^{k^*}$ is dual to \mathcal{V}_{ij}^k . If we choose in (6.2) $\alpha \in \mathcal{V}_{ij}^k$, $\delta \in \mathcal{V}_{kl}^m, \gamma^* \in \mathcal{V}_n^{j^*l}, \beta^* \in \mathcal{V}_m^{i^*}$ the symmetry properties (2.7) follow directly from the isotopies of the corresponding ribbon graphs [RT1]. In addition, we can show using the isomorphisms (6.5) and (6.6), that

$$\left| \begin{array}{ccc} i^* & j^* & k^* \\ l^* & m^* & n^* \end{array} \right|^{\alpha^* \beta \gamma \delta^*} = F \left(\text{Diagram} \right) \quad (6.7)$$

Due to the properties 3) and 4) of the modular Hopf algebras, for closed graphs we always have the equivalence

$$\sum_{k \alpha} \omega_k^2 \left(\text{Diagram} \right) \leftrightarrow \left(\text{Diagram} \right), \quad (6.8)$$

which is the ribbon graph version of (4.8).

Taking (6.7), (6.1) and (6.8) into consideration one can quite easily prove the orthogonality, the Biedenharn–Elliott relations and the Racah identities for 6j-symbols defined by (6.2). Applying the isomorphisms (6.5) to each 3-vertex in (6.2) and using isotopies of ribbon graphs one obtains (2.8). The last condition (2.12) is a consequence of the normalization (6.1). In fact, to preserve (6.1) for $k = 0$ we must introduce the factor ω_i^{-1} after removing the 0-ribbon from $\alpha \in V_{ii^*}^0$.

Finally, one can analogously prove that

$$F \left(\text{Diagram} \right) = \frac{q_{AqB}}{q_C q_D} \left| \begin{array}{ccc} y & A & D \\ x & B & C \end{array} \right| \quad (6.9)$$

which is the ribbon graph version of Eq. (2.17).

7. Conclusion

We have in this paper developed calculational techniques applicable to a large class of 3-dimensional state sum models of topological quantum field theories, and extending those of [KS]. As a byproduct we have calculated the dimensions of the state spaces associated to surfaces and obtained a relatively simple proof of the relation between these models and those introduced in [RT1,2], and hence to continuum Chern–Simons theory with an arbitrary compact semisimple gauge group [Wi2]. Our arguments have been mainly based on simple geometrical considerations which we believe can be generalized rather straightforwardly to higher dimensions. Indeed, higher dimensional models of Turaev–Viro type have been suggested recently in the literature and it would be an obvious task to introduce observables into those models in the form of “higher dimensional graphs” by extending our construction in Sect. 3. In order to develop such a construction into an effective calculational tool it would be crucial to have at disposal analogues of the Racah identities. We leave these issues for future investigation.

Acknowledgements. The first author would like to thank M. Karowski and R. Schrader for many fruitful discussions, and the second author acknowledges the kind hospitality extended to him at the Yukawa Institute for Theoretical Physics, Kyoto, in the fall of 1992, where part of this work was carried out.

References

- [A] Andersen, H.H.: Tensor products of quantized tilting modules. *Commun. Math. Phys.* **149**, 149 (1992)
- [At] Atiyah, M.: Topological quantum field theories. *Publ. Math. I.H.E.S.* **68**, 175 (1989)
- [D] Durhuus, B.: A discrete approach to topological quantum field theories. *J. Geom. and Phys.* **11**, 155 (1993)
- [DJN] Durhuus, B., Jakobsen, H., Nest, R.: Topological quantum field theories from generalized 6j-symbols. *Rev. Math. Phys.* **5**, 1 (1993)
- [Dr] Drinfeld, V.G.: Quantum groups. *Proc. of ICM Berkeley 1986*. Providence, R.I., **1** (1987)
- [FG] Felder, G., Grandjean, O.: On combinatorial three-manifold invariant. Preprint (1992)
- [KMS] Karowski, M., Müller, W., Schrader, R.: State sum invariants of compact 3-manifolds with boundary and 6j-symbols. *J. Phys. A* **25**, 4847 (1992)
- [KS] Karowski, M., Schrader, R.: A combinatorial approach to topological quantum field theories and invariants of graphs. *Commun. Math. Phys.* **151**, 355 (1993)
- [KS1] Karowski, M., Schrader R.: Private communication
- [N] Nill, F.: Private communication
- [RT1] Reshetikhin, N., Turaev, V.: Ribbon graphs and their invariants derived from quantum groups. *Commun. Math. Phys.* **127**, 26 (1990)
- [RT2] Reshetikhin, N., Turaev, V.: Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.* **103**, 547 (1991)
- [T] Turaev, V.: Quantum invariants of 3-manifolds I. Preprint 509/p-295, Strasbourg, (1992) to be published in *Quantum invariants of 3-manifolds*, Berlin: Walter de Gruyter, 1994
- [T1] Turaev, V.: Topology of shadows. Preprint (1991)
- [T2] Turaev, V.: Quantum invariants of links and 3-valent graphs in 3-manifolds. *Publ. IHES* **77**, 121 (1993)
- [TV] Turaev, V., Viro, O.: State sum invariants of 3-manifolds and quantum 6j-symbols. *Topology* **31**, 865 (1992)

- [TW] Turaev, V., Wenzl H.: Quantum invariants of 3-manifolds associated with classical simple Lie algebras. *Int. J. Math.* **4**, 323 (1991)
- [V] Verlinde, E.: Fusion rules and modular transformations in 2d conformal field theory. *Nucl. Phys. B* **300**, 360 (1988)
- [Wa] Walker, K.: On Witten's 3-manifold invariants. Preprint (1991)
- [Wi1] Witten E.: Topological quantum field theory. *Commun. Math. Phys.* **117**, 353 (1988)
- [Wi2] Witten, E.: Quantum field theory and the Jones polynomial. *Commun. Math. Phys.* **121**, 351 (1989)

Communicated by G. Felder

