Commun. Math. Phys. 167, 373–393 (1995)

Communications in Mathematical Physics

S^7 and $\widehat{S^7}$

Martin Cederwall★**, Christian R. Preitschopf**★★

Institute for Theoretical Physics, Chalmers University of Technology and University of Göteborg, S-412 96 Göteborg, Sweden

Received: 8 September 1993

Abstract: We investigate the seven-sphere as a group-like manifold and its extension to a Kac–Moody-like algebra. Covariance properties and tensorial composition of spinors under S^7 are defined. The relation to Malcev algebras is established. The consequences for octonionic projective spaces are examined. Current algebras are formulated and their anomalies are derived, and shown to be unique (even regarding numerical coefficients) up to redefinitions of the currents. Nilpotency of the BRST operator is consistent with one particular expression in the class of (field-dependent) anomalies. A Sugawara construction is given.

1. Preliminaries

This paper is devoted to an investigation of the seven-sphere as a manifold equipped with group-like multiplication, and to its extension to a Kac–Moody-like algebra. As is well known, the seven-sphere is not a group manifold, but shares a great number of properties with the group manifolds. It is the parallelizability property that enables us to consider transformations generated by vectors tangent to the seven-sphere. There are essentially two routes to take when trying to generalize the Lie algebra concept. One, which has been extensively explored in the mathematical literature, is based on abandoning the Jacobi identities in favour of a weaker structure, which leads to Malcev algebras. The other is to maintain the Jacobi identities and give up the invariance of the structure constants. It is this latter option that will be pursued in this paper, for the simple reason that the multiplication rules defined by Poisson brackets and commutators used in physics automatically obey Jacobi identities. We will also comment on the exact relation to Malcev algebras.

The paper is organized as follows. Section 1 gives a brief summary of division algebras, specifically aiming at octonions, which are an almost indispensable tool

★ e-mail tfemc@fy.chalmers.se
★★ e-mail tfecp@fy.chalmers.se

for investigating the structure of the seven-sphere. Here we also deal with the parallelizability properties of unit spheres related to the division algebras. Section 2 uses the parallelizability to define seven-sphere transformations and defines covariance properties under these transformations. The current algebra is formulated and its relation to Malcev algebras is established. We discuss the implications for octonionic projective spaces, which are naturally defined in our framework, and give a set of homogeneous coordinates for $\mathbb{O}P^2$. Applications are exemplified by ten-dimensional twistor theory. In Sect. 3 we consider the Kac–Moody-like structure arising from the map $S^1 \to S^7$. We calculate the Schwinger terms, derive conditions for quantum-mechanical nilpotency of the BRST operator and give a Sugawara construction for an energy-momentum tensor. Sections 2 and 3 contain the results of this paper. Section 4, finally, is devoted to a brief discussion of the results.

1.1. Division Algebras. The class of algebras of interest in this paper are the alternative division algebras, especially the algebra $\mathbb{O}$ of octonions or Cayley numbers [Cay, Zorn]. As we will see, the properties of these algebras are directly related to the corresponding algebras of transformations as (properly defined) multiplication by an element of unit norm.

An algebra $\mathfrak{A}$ (not necessarily associative) is called a *division algebra* (see *e.g.* [Scha, Po], which together with [Zorn] are the main sources of this section) if left and right multiplication

$$L_a x \equiv ax, \quad R_a x \equiv xa \tag{1.1}$$

have inverses (for $0 \neq a \in \mathfrak{A}$). We will only consider division algebras over the field $\mathbb{R}$ of real numbers. The existence of inverses implies that there are no divisors of zero: $x \neq 0$ and $y \neq 0$ in $\mathfrak{A}$ gives $xy \neq 0$.

An *alternative algebra* is algebra where the associator

$$[a, b, c] \equiv (ab)c - a(bc) \tag{1.2}$$

obeys the relation

$$[a, a, b] = 0 \ . \tag{1.3}$$

This implies (consider $[a + b, a + b, c]$) that the associator alternates, *i.e.* changes sign under any odd permutation of the entries. The alternativity implies a number of useful relations, among which are the Moufang identities [Zorn]

$$\begin{aligned} (axa)y &= a(x(ay)) \ , \\ (ax)(ya) &= a(xy)a \ . \end{aligned} \tag{1.4}$$

The first of these equations is equivalent to

$$[a, xa, y] = -a[x, a, y] \ . \tag{1.5}$$

One can prove that any alternative algebra $\mathfrak{A}$ with a unit element 1 where any nonzero element x has an inverse x^{-1} ($xx^{-1} = 1 = x^{-1}x$) is a division algebra. Namely, it follows from the Moufang identity (1.5) that

$$[a^{-1}, a, x] = 0 \ , \tag{1.6}$$

which means that left multiplication is invertible (cf. (1.1)), and analogously for right multiplication.

Conjugation of an element in $\mathfrak{A}$ is defined as an anti-automorphism $x \to x^*$: $(x^*)^* = x$, $(xy)^* = y^* x^*$, with

$$x + x^* \in \mathbb{R} \ni x^* x . \tag{1.7}$$

For convenience, we introduce the notation

$$[x] \equiv \tfrac{1}{2}(x + x^*), \quad \{x\} \equiv \tfrac{1}{2}(x - x^*), \quad |x| = (x^* x)^{1/2} . \tag{1.8}$$

If $xx^* \neq 0$ we can obviously express the inverse in terms of the conjugate element as $x^{-1} = (xx^*)^{-1} x^*$. Thus, if there are no zero divisors, an alternative $\mathfrak{A}$ with conjugation as above is a division algebra.

The class of finite-dimensional real division algebras is quite restricted, they can be of dimensions 1, 2, 4 or 8 only [Hur, BoMi, Ker, Ad]. If one also demands that they be alternative, there are only four algebras left: $\mathbb{R}$, the reals (dimension 1), $\mathbb{C}$, the complex numbers (dimension 2), $\mathbb{H}$, the quaternions (dimension 4) and $\mathbb{O}$, the octonions or Cayley numbers (dimension 8). The algebra of octonions is unique in that it is the only non-associative alternative division algebra. We will refer to the above algebras as $\mathbb{K}_\nu$, where ν is the dimension.

There is a number of equivalent ways to represent the multiplication table of the octonions. The simplest one, in our opinion, is given as follows. We chose an orthonormal basis

$$\mathbb{O} \ni x = \sum_{a=0}^{7} x_a e_a = [x] + \sum_{i=1}^{7} x_i e_i \quad (e_0 = 1) \tag{1.9}$$

and state the multiplication rule

$$e_i e_j = -\delta_{ij} + \sigma_{ijk} e_k , \tag{1.10}$$

where the structure constants σ are completely antisymmetric and equal to one for the combinations

$$(ijk) = (124), (235), (346), (457), (561), 672) \text{ and } (713) , \tag{1.11}$$

i.e. we have

$$e_i e_{i+1} = e_{i+3}, \tag{1.12}$$

where $e_{i+7} = e_i$. Is is then easy to verify that the structure constants for commutators and associators are given by

$$[e_i, e_j] = 2\sigma_{ijk} e_k, \quad [e_i, e_j, e_k] = 2\rho_{ijkl} e_l,$$

$$\rho_{ijkl} = -(^*\sigma)_{ijkl} = -\tfrac{1}{6} \varepsilon_{ijklmnp} \sigma_{mnp} . \tag{1.13}$$

1.2. The Seven-Sphere. Parallelizability. The seven-sphere can be trivially represented as the set of unit octonions:

$$S^7 = \{X \in \mathbb{O} \mid X^* X = 1\} . \tag{1.14}$$

It is the unique compact and simply connected non-group manifold[1] to share with the group manifolds the property of *global parallelizability* [CaSch, BoMi, Ker, Wolf, Hus]. Of the spheres, only $S^{\nu-1}$, where ν is the dimension of one of the alternative division algebras, are globally parallelizable. The lower-dimensional spheres are constructed analogously to above. There one has the isomorphisms $S^1 \approx U^1$ and $S^3 \approx SO(3)$, which leaves S^7 as the only non-group example.

Global parallelizability of a manifold $\mathfrak{M}$ (in the following referred to as just "parallelizability") means that there exist m linearly independent globally defined and nowhere vanishing vectorfields on $\mathfrak{M}$, where m is the dimension of $\mathfrak{M}$. Then the m vectorfields can be linearly combined to constitute an orthonormal basis of the tangent space $\mathscr{M}(X)$ at any point X in $\mathfrak{M}$. Letting the orientation of this basis define parallel transport on $\mathfrak{M}$, one immediately obtains, since parallel transport is independent of path,

$$0 = [\tilde{\mathscr{D}}_\mu, \tilde{\mathscr{D}}_\nu] = \tilde{R}_{\mu\nu} , \tag{1.15}$$

where $\tilde{\mathscr{D}}$ and $\tilde{R}$ are the covariant derivative and the curvature tensor defined with respect to this parallel transport. If we write

$$\tilde{\mathscr{D}} = \partial + \tilde{\Gamma} = \mathscr{D} - T , \tag{1.16}$$

where $\mathscr{D} = \partial + \Gamma$, Γ being the metric connection, we have the parallelizing connection $\tilde{\Gamma} = \Gamma - T$. T is the *parallelizing torsion.* If one considers the covariant derivative of the vielbein e^a_μ (roman letters $a, b, \ldots,$ denote tangent space indices), one finds, since $\mathscr{D}_\mu e^a_\nu = 0$,

$$\tilde{\mathscr{D}}_\mu e^a_\nu = -T^{\;a}_{\mu\nu} , \tag{1.17}$$

or equivalently

$$\tilde{\mathscr{D}}_\mu e^\nu_a = T^{\;\nu}_{\mu a} , \tag{1.18}$$

which can be taken as the definition of torsion.

2. The Seven-Sphere as a Transformational Manifold

2.1. From Parallelizability to Algebra. Suppose we have a manifold $\mathfrak{M}$ of parallelizable type [Wolf], *i.e.* a direct product of group manifolds and seven-spheres (including the complexified and non-compact versions). Then define infinitesimal "translations" generated by the tangent space covariant derivatives. The parallellizability property assures that the translations form a closed algebra:

$$[\mathscr{D}_a, \mathscr{D}_b] = [e^\mu_a \mathscr{D}_\mu, e^\nu_b \mathscr{D}_\nu] = 2e^\mu_a [\mathscr{D}_\mu, e^\nu_b] \mathscr{D}_\nu = 2e^\mu_a T^{\;\nu}_{\mu b} = 2T^{\;c}_{ab} \mathscr{D}_c , \tag{2.1}$$

where we have utilized (1.15) and the antisymmetry of the torsion tensor [Wolf].

The cases of group manifolds are trivial; there are parallelizing torsion contains simply the structure constants, and is independent of the location in $\mathfrak{M}$. The Lie

[1] The non-compact spaces $SO(4,4)/SO(3,4)$ (topology $S^3 \times \mathbb{R}^4$), obtained from the split octonions [Jac] and $SO(8;\mathbb{C})/SO(7;\mathbb{C})$ (topology $S^7 \times \mathbb{R}^7$) also alrise in the classification [Wolf].

algebra of the group $\mathfrak{M}$ is obtained. The only "non-trivial" case is the seven-sphere, and it is the only case where the torsion tensor varies over the manifold. This statement contains exactly the same amount of information as the statement that $\mathbb{O}$ is the only non-associative alternative division algebra. We shall soon see the connection.

2.2. Infinitesimal Transformations. S^7 Spinors. Let $\xi \in \mathbb{O}$ and let the seven-sphere be parametrized by direction of ξ: $S^7 = \{\frac{\xi}{|\xi|} = X \in \mathbb{O}\}$. The tangent space at X is spanned by the units $\{Xe_i\}_{i=1}^7$. Considering the tangent space basis at two infinitesimally separated points, we see that the parallel transport of this basis is defined by an infinitesimal transformation

$$\delta_\alpha \xi = \xi\alpha, \quad [\alpha] = 0 . \tag{2.2}$$

The commutator of two such transformations can be calculated explicitly:

$$\begin{aligned}[\delta_\alpha, \delta_\beta]\xi &\equiv \delta_\alpha(\delta_\beta \xi) - \delta_\beta(\delta_\alpha \xi) = (\xi\alpha)\beta - (\xi\beta)\alpha \\ &= \xi(X^*((X\alpha)\beta) - X^*((X\beta)\alpha)) = \delta_{X^*((X\alpha)\beta) - X^*((X\beta)\alpha)}\xi .\end{aligned} \tag{2.3}$$

Here alternativity in the form (1.6) has been used. The parameter $X^*((X\alpha)\beta) - X^*((X\beta)\alpha)$ of the transformation on the right-hand side is twice the parallelizing torsion [GuTze, Roo]. In component notation,

$$T_{ijk}(X) = [(e_i^* X^*)(Xe_j)e_k] , \tag{2.4}$$

which is completely antisymmetric in the three indices, and (2.3) can be written as

$$[\delta_i, \delta_j] = 2T_{ijk}(X)\delta_k . \tag{2.5}$$

The variation δ is indeed the parallelizing covariant derivative of (2.1). It should be mentioned that the specific parallelizing torsion used here is only one in a big family, parametrized by the choice of left or right multiplication in (2.2) and by the choice of the north pole [Roo][2]. To our knowledge, the algebra (2.5) was first considered in reference [ESTPS].

Now the question arises how to transform other fields than ξ. One can not simultaneously interpret two fields as parametrizing the seven-sphere. We want to introduce another boson η, $\frac{\eta}{|\eta|} = Y$ with some S^7 transformation rule, maintaining (2.5). This excludes the simplest candidate $\delta_\alpha Y = Y\alpha$. The two fields are bound to transform differently. The correct transformation rule turns out to be

$$\delta_\alpha \eta = X^*((Xn)\alpha) = (\eta X^*)(X\alpha) = (\eta(\alpha X^*))X \equiv \eta \underset{X}{\circ} \alpha , \tag{2.6}$$

where the equalities are derived from (1.5). By comparing with the tangent space basis introduced above, one sees that the product (2.6) indeed can be interpreted as multiplication in the basis $(X, \{Xe_i\})$ at the point $X \in S^7$. This multiplication fulfills the same conditions as the ordinary multiplication at the northpole $X = 1$, and differs from it only by an associator. We see that it is the non-associativity of $\mathbb{O}$ that

[2] Here we have deduced the parallelizing torsion in an indirect way, using (2.3) and (2.1). In reference [LuMi], it is shown how the torsionless and "flat" seven-spheres arise naturally as quotient spaces.

is responsible for the non-constancy of the torsion tensor (while the non-commutativity accounts for its non-vanishing) and for the necessity of utilizing inequivalent products associated with different points $X \in S^7$. We call this field-dependent multiplication the *X-product*.

One should note that the transformation (2.6) relies on the transformation of the *parameter field* X (2.2), while for group manifolds (and thus for the lower-dimensional spheres S^1 and S^3 associated with $\mathbb{C}$ and $\mathbb{H}$) ξ and η transform independently. A consequence is that fermions cannot transform without the presence of a parameter field, since a fermionic octonion is not invertible.

We call a field (bosonic or fermionic) transforming according to (2.6) a *spinor under* S^7. Note that also the transformation of X can be written $\delta_\alpha X = X \underset{X}{\circ} \alpha$, and that the commutator of variations is

$$[\delta_\alpha, \delta] = \delta \underset{X}{\circ} \alpha - \alpha \underset{X}{\circ} \delta \,, \tag{2.7}$$

where δ is thought of as an imaginary octonion: $\delta_\alpha \equiv [\alpha^* \delta]$.

2.3. S^7 Tensor Algebra. In order to examine covariance properties and tensorial composition of spinors we will first examine the X-product a little closer. We introduce the related commutators and associators:

$$\begin{aligned} [a, b]_X &= a \underset{X}{\circ} b - b \underset{X}{\circ} a \,, \\ [a, b, c]_X &= (a \underset{X}{\circ} b) \underset{X}{\circ} c - a \underset{X}{\circ} (b \underset{X}{\circ} c) \,. \end{aligned} \tag{2.8}$$

Ordinary $*$ conjugation is still an anti-automorphism with respect to the X-product. One also has

$$[a \underset{X}{\circ} b] = [ab], \quad [a(b \underset{X}{\circ} c)] = [(a \underset{X}{\circ} b)c] \,. \tag{2.9}$$

The Moufang identities (1.4) or (1.5) may be used to express the X-associator in a number of ways. The inverse a^{-1} is also the inverse with respect to the X-product, and alternativity, and thus also (1.6) holds.

Let $\mathfrak{r}, \mathfrak{s}, \ldots$ be S^7 spinors, *i.e.* $\delta_\alpha \mathfrak{r} = \mathfrak{r} \underset{X}{\circ} \alpha$, etc. The generators δ should be thought of as transforming in an adjoint representation according to (2.7). Can this representation be formed as a tensor product of spinor representations? Due to the non-linearity, the answer is no. The current $\mathfrak{J}$ (see the following section) is the unique object to transform this way. The only reasonable candidate for a spinor bilinear in the adjoint $\{\mathfrak{r}^* \underset{X}{\circ} \mathfrak{s}\}$ which does not have good transformation properties.[3]

On the other hand, consider a bilinear

$$K = \mathfrak{r} \underset{X}{\circ} \mathfrak{s}^* \,. \tag{2.10}$$

Had one used the ordinary multiplication ($X = 1$), K would not have sensible transformation properties, but now also the product itself transforms. We obtain

$$\delta_\alpha K = (\mathfrak{r} \underset{X}{\circ} \alpha) \underset{X}{\circ} \mathfrak{s}^* - [\mathfrak{r}, \alpha, \mathfrak{s}^*]_X - \mathfrak{r} \underset{X}{\circ} (\alpha \underset{X}{\circ} \mathfrak{s}^*) = 0 \,, \tag{2.11}$$

[3] See also Sect. 3.2 on BRST analysis. We want to emphasize that we do not yet have a full representation theory.

where the middle term comes from transformation of the product, which effectively *cancels the non-associativity*. Let K be scalar. The same cancellation of non-associativity occurs in

$$\mathfrak{r}' = K \underset{X}{\circ} \mathfrak{r} : \delta_\alpha \mathfrak{r}' = \mathfrak{r}' \underset{X}{\circ} \alpha \,, \tag{2.12}$$

so that $\mathfrak{r}'$ is a spinor. As a consequence, we can form spinors as trilinears of spinors as

$$\mathfrak{u} = (\mathfrak{r} \underset{X}{\circ} \mathfrak{s}^*) \underset{X}{\circ} t \,, \tag{2.13}$$

and in this way only.

2.4. Current Algebra. As mentioned in Sect. 1, one of the main motivations for the introduction of the field-dependent transformation rules is that the Jacobi identities are fulfilled. Either this can be proven explicitly by calculation, or it suffices to find a current that generates the transformations (2.2), (2.6) by commutators or Poisson brackets.

Let $\pi^{(\xi)}$ and $\pi^{(n)}$ be the conjugate momenta of ξ and η, *i.e.*

$$\{\xi_a, \pi_b^{(\xi)}\} = \delta_{ab} = \{\eta_a, \pi_b^{(\eta)}\} \,, \tag{2.14}$$

the curly brackets denoting Poisson brackets or commutators. The transformations of ξ (2.2) is then easily derived from a generator $\mathfrak{J}^{(\xi)} = \{\pi^{(\xi)*}\xi\}$ and that of η (2.6) from $\mathfrak{J}^{(\eta)} = \{\pi^{(\eta)*} \underset{X}{\circ} \eta\}$. $\mathfrak{J}_\alpha^{(\xi)} \equiv [\alpha^* \mathfrak{J}^{(\xi)}]$ fulfills the algebra (2.3) with respect to the $\{.\,,.\}$ product, but $\mathfrak{J}_\alpha^{(\eta)}$ does so only when X transforms, *i.e.* only in combination with with $\mathfrak{J}_\alpha^{(\xi)}$. The generator of the simultaneous transformations (2.2), (2.6) is thus

$$\mathfrak{J} = \{\pi^{(\xi)*}\xi\} + \{\pi^{(\eta)*} \underset{X}{\circ} \eta\} \,. \tag{2.15}$$

In this expression, ξ is necessarily a boson, and η may be bosonic or fermionic. Any number of fields can be introduced in $\mathfrak{J}$ in the same way as η. Self-conjugated fermions ($\{S_a, S_b\} = \delta_{ab}$) give a contribution $\frac{1}{2} S^* \underset{X}{\circ} S$ to $\mathfrak{J}$.

The transformation of any field ϕ is given by

$$\delta_\alpha \phi = \{\mathfrak{J}_\alpha, \phi\} \,. \tag{2.16}$$

Using (2.8), the torsion tensor can be written as

$$T_{\alpha\beta}(X) = \tfrac{1}{2}\,[\alpha, \beta]_X \,, \tag{2.17}$$

and the S^7 algebra (2.5) becomes

$$\{\mathfrak{J}_\alpha, \mathfrak{J}_\beta\} = \mathfrak{J}_{[\alpha,\beta]_X} \,. \tag{2.18}$$

Note the exact analogy to $S^3 \approx SO(3)$ obtained from $\mathbb{H}$, where one has $\{\mathfrak{J}_\alpha, \mathfrak{J}_\beta\} = \mathfrak{J}_{[\alpha,\beta]}$.

2.5. Finite Transformations. This section might be trivial, but the form of finite transformations may not be so obvious when field dependence is involved. A finite transformation is obtained by the limit procedure

$$\phi \to \phi' = \lim_{N\to\infty} \phi_N \,,$$

$$\phi_0 = \phi, \quad \phi_{n+1} = \frac{\theta}{N}\{\mathfrak{J}_\alpha, \phi_n\} + \phi_n \,, \tag{2.19}$$

where $\theta \in \mathbb{R}$. A straightforward calculation shows that

$$\xi \to \xi' = \xi \exp(\theta\alpha) = \xi\Omega , \tag{2.20}$$

where $\Omega \in S^7$. The corresponding calculation for a spinor $\mathfrak{r}$ other than ξ is a little more involved, but a careful analysis shows that all associators between X, α and $\mathfrak{r}$ cancel, and the finite transformation is

$$\mathfrak{r} \to \mathfrak{r}' = \mathfrak{r} \circ_X \Omega , \tag{2.21}$$

and likewise for the current,

$$\mathfrak{J} \to \mathfrak{J}' = \Omega^* \circ_X \mathfrak{J} \circ_X \Omega . \tag{2.22}$$

We will use this last equation later when looking at changes of parameter fields in connection to projective spaces.

2.6. Transformation of an Arbitrary Number of Fields. The characterization of spinors under S^7 made above is not complete. There are other ways for fields to transform than (2.6) that in a general treatment must be called spinorial. Suppose that at least two bosonic fields transform under $\mathfrak{J}$, and call two of them ξ and η as before. The choice of ξ as parameter field is arbitrary, one could as well have chosen η, and there is a way to move between the two forms of the current. Namely, if we define

$$\tilde{\mathfrak{J}} = ((\mathfrak{J}X^*)(XY^*))Y = (\mathfrak{J} \circ_X Y^*)Y = (\mathfrak{J}X^*) \circ_Y X , \tag{2.23}$$

we see that the roles of ξ and η have interchanged. We have

$$\tilde{\mathfrak{J}} = \{\pi^{(\eta)*}\eta\} + \{\pi^{(\xi)*} \circ_Y \xi\} + \dots . \tag{2.24}$$

Using only (2.23) and not its explicit form (2.24), one can show that $\tilde{\mathfrak{J}}$ fulfills the same algebra as $\mathfrak{J}$, but with the torsion tensor taken at the point Y instead of X:

$$\{\tilde{\mathfrak{J}}_\alpha, \tilde{\mathfrak{J}}_\beta\} = \tilde{\mathfrak{J}}_{[\alpha,\beta]_Y} . \tag{2.25}$$

Now, let there be yet another field ζ present, transforming the same way as η under $\mathfrak{J}$, and let $Z = \frac{\zeta}{|\xi|}$. The term $\mathfrak{J}^{(\zeta)} = \{\pi^{(\zeta)*} \circ_X \zeta\}$in $\mathfrak{J}$ does *not* change into $\{\pi^{(\zeta)*} \circ_Y \zeta\}$ under the transformation (2.23). Instead we obtain

$$\tilde{\mathfrak{J}} = \{\pi^{(\eta)*}\eta\} + \{\pi^{(\zeta)*} \circ_Y \xi\} + ((\pi^{(\zeta)})^*(\zeta X^*))(XY^*))Y . \tag{2.26}$$

Note that the objects ZX^* and XY^* occurring in this formula are S^7 scalars, according to (2.11), but that the remaining combination, YZ^* is not a scalar. We can visualize this in a linear diagram, where ξ is connected with η and ζ, but not η with ζ (Fig. 2, first diagram). The transformation rule of ζ derived from (2.26) is

$$\delta_\alpha \zeta = (\zeta X^*)((XY^*)(Y\alpha)) . \tag{2.27}$$

Then the more complicated product in (2.27) is thought of as a product defined by going from ζ to the (new) parameter field η along the connections in this diagram. This principle is completely generalizable to any connected "tree diagram" with arbitrary number of points (fields) and arbitrary number of branches. Closed loops are forbidden; due to non-associativity they lead to inconsistencies.

The path $\mathfrak{P}[\mathfrak{r}_n]$ from any field $\mathfrak{r}_n$ to the parameter field $\mathfrak{r}_0$ is then uniquely determined, and letting it be the sequence

$$\mathfrak{P}[\mathfrak{r}_n] = \{\mathfrak{r}_{n-1}, \mathfrak{r}_{n-2}, \ldots, \mathfrak{r}_1, \mathfrak{r}_0\} \tag{2.28}$$

(we enumerate the points in a tree diagram by the label n, which is not an ordinary integer, but a set where subtraction by 1, *i.e.* stepping towards the parameter point is well-defined for $n \neq 0$, but addition is not, due to possible branching), we define the path-dependent product by

$$A \underset{\mathfrak{P}[\mathfrak{r}_n]}{\circ} B = (A\mathfrak{r}_{n-1}^{-1})((\mathfrak{r}_{n-1}\mathfrak{r}_{n-2}^{-1})(\ldots((\mathfrak{r}_2\mathfrak{r}_1^{-1})((\mathfrak{r}_1\mathfrak{r}_0^{-1})(\mathfrak{r}_0 B)))\ldots)), \tag{2.29}$$

or by induction as

$$A \underset{\mathfrak{P}[\mathfrak{r}_n]}{\circ} B = (A\mathfrak{r}_{n-1}^{-1})\, \mathfrak{r}_{n-1} \underset{\mathfrak{P}[\mathfrak{r}_{n-1}]}{\circ} B), \tag{2.30}$$

which is easily seen to reduce to the product defined in (2.6) for the case of a one-step and (2.27) for a two-step path. The generator of S^7 transformations is then

$$\mathfrak{J}_\alpha = -\sum_k [\pi_k^*(\mathfrak{r}_k \underset{\mathfrak{P}[\mathfrak{r}_k]}{\circ} \alpha)], \tag{2.31}$$

and it fulfills (2.18) with $X = \mathfrak{r}_0/|\mathfrak{r}_0|$. Transformations are given as

$$\delta_\alpha \mathfrak{r}_k = \mathfrak{r}_k \underset{\mathfrak{P}[\mathfrak{r}_n]}{\circ} \alpha\,. \tag{2.32}$$

One can also choose to view this as an ordinary multiplication by another (X-dependent) unit octonion α_k. In that case one uses the properties of the path product to rewrite (2.31) as

$$\mathfrak{J}_\alpha = -[\pi_0^* \mathfrak{r}_0 \alpha] - \sum_{k \neq 0} [(\pi_k^* \underset{\mathfrak{r}_{k-1}}{\circ} \mathfrak{r}_k)\alpha_k], \tag{2.33}$$

where $\alpha_k = 1 \underset{\mathfrak{P}[\mathfrak{r}_k]}{\circ} \alpha$, and obtains

$$\delta_\alpha \mathfrak{r}_{k-1} = \mathfrak{r}_{k-1}\alpha_k\,. \tag{2.34}$$

Fermions, due to non-invertibility, can be assigned to endpoints of the diagram only; no path may pass via a fermion.

Define the path from $\mathfrak{r}$ to $\mathfrak{s}$ as the composition

$$\mathfrak{P}[\mathfrak{r}, \mathfrak{s}] = \mathfrak{P}[\mathfrak{r}]\mathfrak{P}^{-1}[\mathfrak{s}] \tag{2.35}$$

of the path of $\mathfrak{r}$ followed by the reverse path of $\mathfrak{s}$. The invariance principle generalizing (2.10), (2.11) is

$$\delta_\alpha(\mathfrak{r} \underset{\mathfrak{P}[\mathfrak{r} \to \mathfrak{s}]}{\circ} \mathfrak{s}^*) = 0\,, \tag{2.36}$$

which contains the irreducible amount of information that $\delta_\alpha(\mathfrak{r}\mathfrak{s}^*) = 0$ whenever $\mathfrak{r}$ and $\mathfrak{s}$ are connected by a link in the diagram.

We conclude by the remark that change of parameter field along a link in the diagram is a finite S^7 transformation, however not with constant parameter. Consider $\mathfrak{J}' = X\mathfrak{J}X^*$, which has the property of generating *left* multiplication on X. One can prove that changing the parameter field from X to the neighboring Y amounts to a finite $\mathfrak{J}'$-transformation with (constant) parameter $\Omega' = YX^*$. By using a modified form of (2.22), generated by $\mathfrak{J}'$, we obtain exactly (2.23).

2.7. Octonionic Projective Spaces. This section will deal with the description of octonionic projective spaces [Po, Mou, Jo1, Fr, Bor, Jo2, JNW, GPR, Alb] in terms of sets of homogeneous coordinates modulo octonionic transformations with $\mathbb{O}\backslash\{0\} \approx S^7 \times \mathbb{R}^+$, and to establish relations with known coordinatizations. There is a number of "different" realizations of projective spaces: from homogeneous coordinates [Po], from explicit sewing together coordinate patches [Po, Mou], from Jordan algebras [Jo1, Fr], or as quotient spaces [Bor].

An important feature is that $\mathbb{K}_\nu P^n$ can be described topologically as the disjoint union of $\mathbb{K}_\nu^n$ and the space at infinity $\mathbb{K}_\nu P^{n-1}$ together with a map from the sphere at infinity $S^{n\nu-1}$ of $\mathbb{K}_\nu^n$ to $\mathbb{K}_\nu P^{n-1}$. For the trivial case $\mathbb{K}_\nu P^1$, the maps are just constant maps from $S^{\nu-1}$ to $\mathbb{K}_\nu P^0 = \{0\}$, that obviously can be taken as fibrations with the "group" $S^{\nu-1}$ in the sense of this paper. One has $\mathbb{K}_\nu P^1 = S^\nu$. For $\mathbb{K}_\nu P^2$, one has the maps from $S^{2\nu-1}$ to $\mathbb{K}_\nu P^1$, *i.e.* the Hopf maps, or Hopf fibrations [Hopf],

$$S^1 \xrightarrow[S^0]{} S^1 = \mathbb{R}P^1 \, ,$$

$$S^3 \xrightarrow[S^1]{} S^2 = \mathbb{C}P^1 \, ,$$

$$S^7 \xrightarrow[S^3]{} S^4 = \mathbb{H}P^1 \, ,$$

$$S^{15} \xrightarrow[S^7]{} S^8 = \mathbb{O}P^1 \, . \tag{2.37}$$

The map, together with scaling by positive real numbers, can be used to obtain $\mathbb{K}_\nu P^{n-1}$ from its homogeneous coordinates in $\mathbb{K}_\nu^n\backslash\{0\}$. This means that there is an equivalence between the existence of $\mathbb{K}_\nu P^n$ and homogeneous coordinates for $\mathbb{K}_\nu P^{n-1}$. This holds for $\nu \neq 8$, all n, and for $\nu = 8$, $n \leqq 2$.

We will show that the last of the Hopf fibrations (2.37) can be given a formulation as a fibration with fiber S^7 in the sense of this paper, *i.e.* as identification of points of S^{15} modulo infinitesimally generated S^7 orbits without fixed points. In view of the relation to homogeneous coordinates, the same holds for the map to $\mathbb{O}P^1$ from its homogeneous coordinates.

We start with $\mathbb{O}P^1$ ($\approx S^8$), whose standard atlas consists of the two charts

$$(1, y_1), \quad (x_2, 1) \tag{2.38}$$

with the overlap equation $x_2 = y_1^{-1}$ where both charts are valid. The standard homogeneous coordinates of $\mathbb{O}P^1$ are a pair of octonions (ξ, η) defined modulo (right) multiplication with the same octonion:

$$(\xi, \eta) \approx (\xi\Omega, \eta\Omega) \, . \tag{2.39}$$

The consistency with (2.38) is seen by choosing $\Omega = \xi^{-1}$ or $\Omega = \eta^{-1}$. One has to be careful, however. The transformations of (2.39) do not close to an algebra (see Sect. 2.2), so repeated use of them does not give an equivalence class of points corresponding to the same point $\mathbb{O}P^1$. A basepoint has to be chosen (preferably in one of the forms of (2.38)). A more natural way, at least from our point of view, would be

Fig. 1. Diagram for $\mathbb{O}P^1$

to define the homogeneous coordinates modulo S^7 transformations (and a real scale). We then have the transformations according to Fig. 1:

$$\begin{aligned}(\xi, \eta) &\approx (\xi\Omega, \eta \circ_X \Omega), \quad \xi \neq 0\,,\\ (\xi, \eta) &\approx (\xi \circ_Y \Omega', \eta\Omega'), \quad \eta \neq 0\,.\end{aligned} \tag{2.40}$$

The price being paid for the algebra structure for the variables parametrizing the S^7 fiber is that we cannot describe an equivalence class by only one of the transformations (2.40), since the X-(Y $-$) product is undefined at $\xi = 0(\eta = 0)$. Of course the transformations are equivalent for $\xi, \eta \neq 0$: $\Omega = Y^*(Y \circ_X \Omega')$, since the associated currents are related according to (2.23). The mapping from the homogeneous coordinates (modulo $\mathbb{R}^+$) to $\mathbb{O}P^1$ is a topologically equivalent modification of the Hopf map $S^{15} \to S^8$ [Hopf], that now has been turned into identification of points on *infinitesimally generated* S^7 orbits (see also Sect. (2.8) for a physical motivation).

The reason that the traditional homogeneous coordinates for $\mathbb{O}P^1$ exist, is that the specific Ω's taking (ξ, η) to (2.38) satisfy $[\Omega, \xi, \eta] = 0$. Trying the same procedure for $\mathbb{O}P^2$ is bound to fail – the atlas

$$(1, y_1, z_1), \quad (x_2, 1, z_2), \quad (x_3, y_3, 1) \tag{2.41}$$

with the overlap equations

$$\begin{aligned} x_2 &= y_1^{-1}, & z_2 &= z_1 y_1^{-1},\\ x_3 &= x_2 z_2^{-1}, & y_3 &= z_2^{-1},\\ y_1 &= y_3 x_3^{-1}, & z_1 &= x_3^{-1}\,. \end{aligned} \tag{2.42}$$

(consistency is easily checked) can not be reached from (ξ, η, ζ) by uniform right octonionic multiplication, due to non-associativity. We need the S^7 transformations, fulfilling Jacobi identities and thus effectively associative. Any set of coordinate patches of the generic type (2.41) resulting from identifying points on S^7 orbits in some coordinates *within one specific diagram* is automatically consistent in the regions where the overlap equations apply – this follows from the composition properties of the S^7 transformations.

Let us now try to construct homogeneous coordinates. We choose a linear diagram of the variables (ξ, η, ζ), with ξ as parameter field in the middle (Fig. 2, first diagram). Points on $S^7 \times \mathbb{R}^+$ orbits are identified as

$$(\xi, \eta, \zeta) \approx (\xi\Omega, \eta \circ_X \Omega, \zeta \circ_X \Omega) \tag{2.43}$$

(it is easily checked that (2.41) with (2.42) holds). This map has a problem for $\xi = 0$. In the twentythree-sphere $|\xi|^2 + |\eta|^2 + |\zeta|^2 = 1$, approaching the fifteen-sphere $\xi = 0$ from the seven-sphere direction X gives the $\mathbb{O}P^1$ charts

$$(0, 1, \zeta \circ_X \eta^{-1}), \quad (0, \eta \circ_X \zeta^{-1}, 1)\,, \tag{2.44}$$

and the orbits are not well defined on $\xi = 0$ unless we explictly specify the value of X there. This choice has to be consistent with the transformations, and we note that, whatever prescription is used,

$$\lim_{\xi \to 0} \pi(\xi, \eta, \zeta) \neq \pi(0, \eta, \zeta) \tag{2.45}$$

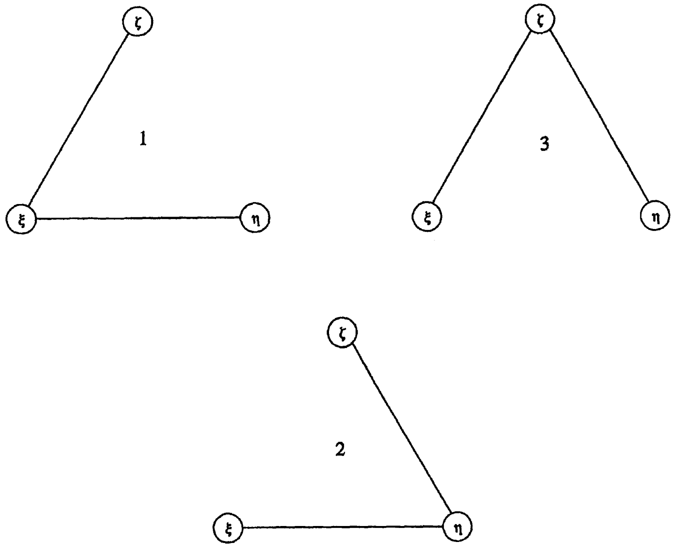

Fig. 2. Diagrams for $\mathbb{O}P^2$

for some directions, where π is the map $S^{23} \to \mathbb{O}P^2$. We arrive at a discontinuous fibration as a generalization of the Hopf map. We make the choice of prescription in (2.43):

$$X = \begin{cases} \xi/|\xi|, & \text{if } \xi \neq 0 \,; \\ \eta/|\eta|, & \text{if } \xi = 0,\ \eta \neq 0 \,; \\ \zeta/|\zeta| \ \ (\text{or any } X' \in S^7), & \text{if } \xi = \eta = 0 \,. \end{cases} \tag{2.46}$$

Using the map (2.43) with (2.46) at the infinity of $\mathbb{O}^3$, a space $\mathbb{O}P^3$ may be defined as $\mathbb{O}^3 \cup \mathbb{O}P^2$. One may show that also higher-dimensional octonionic spaces may be constructed once the requirement that the map $S^{8n-1} \to \mathbb{O}P^{n-1}$ be continuous fibrations is relaxed.

There is another solution to the problem of finding homogeneous coordinates (for $\mathbb{O}P^2$ only) that lies in the fact that in contrast to the case of $\mathbb{O}P^1$ (two transforming fields) we now have three inequivalent linear diagrams (Fig. 2) for the S^7 transformations. We define the associated transformations and hence the associated partial equivalence classes patchwise, with one patch for each diagram, and the complete class arises only after identifying points in different patches via transition functions. The set of homogeneous coordinates for $\mathbb{O}P^2$ is defined as follows:

$$(\xi_1, \eta_1, \zeta_1) \approx (\xi_1 \Omega, \eta_1 \circ_X \Omega, \zeta_1 \circ_X \Omega), \quad X = \frac{\xi_1}{|\xi_1|},$$

$$(\xi_2, \eta_2, \zeta_2) \approx (\xi_2 \circ_Y \Omega', \eta_2 \Omega', \zeta_2 \circ_Y \Omega'), \quad Y = \frac{\eta_2}{|\eta_2|},$$

$$(\xi_3, \eta_3, \zeta_3) \approx (\xi_3 \circ_Z \Omega'', \eta_3 \circ_Z \Omega'', \zeta_3 \Omega''), \quad z = \frac{\zeta_3}{|\zeta_3|}, \tag{2.47}$$

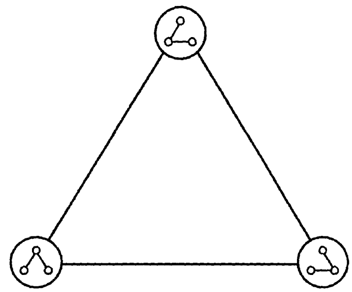

Fig. 3. Diagram of diagrams for $\mathbb{O}P^2$

valid for $\xi \neq 0, \eta \neq 0$ and $\zeta \neq 0$ respectively (the statements $\xi_n = 0$ *etc.* are independent of the subscripts). The overlap relations between the three patches are the transformations needed to go from one diagram to another, e.g.

$$\xi_2 = \xi_1, \quad \eta_2 = \eta_1, \quad \zeta_2 = ((\zeta_1 X_1^*)(X_1 Y_1^*))Y_1\,, \tag{2.48}$$

applying for $\xi \neq 0, \eta \neq 0$ (i.e. in the overlap of the regions where the charts 1 and 2 are valid), creating a link between ζ and η instead of the one between ζ and ξ. In this way, any of the three diagrams is related to any other in the region were both apply. Equation (2.48) is defined such that partial equivalence classes are transformed into each other. A consistency check is given by going around the closed loop in the "diagram of diagrams" (Fig. 3). We come to the same variables modulo an S^7 transformation, i.e. we stay within a given equivalence class. The transformations (2.48) between diagrams can be modified to include also an arbitrary S^7 transformation, since it does not alter the "link invariants" of (2.41). It can be shown that no such transformtions may be imposed to remove the residual S^7 phase obtained from the loop in Fig. 3. This means that (2.48) only gives well-defined transition functions between partial equivalence classes, and not between the homogeneous coordinates. (ξ_i, η_i, ζ_i) with $|\xi_i|^2 + |\eta_i|^2 + |\zeta_i|^2 = 1$ for $1 \leqq i \leqq 3$ do not define coordinate patches on the twentythree-sphere. This is consistent with the point of view that the discontinuity of (2.43) seen as a map $S^{23} \to \mathbb{O}P^2$ is of topological nature and can not be smeared out. An attempt to generalize the procedure to $\mathbb{O}P^3$ fails. There are 16 inequivalent diagrams transforming four fields (12 linear and 4 star-shaped). Any closed curve in the diagram of diagrams here leads to an inconsistency. It can be shown that this property, forcing also the diagram of diagrams to be a tree diagram, excludes some overlaps that would have been needed in order to define $\mathbb{O}P^3$ in this way.

We hope to find applications to the homogeneous coordinates (2.43), (2.47). It should be possible to realize $E_6 \approx SL(3; \mathbb{O})$ [Fr, Sud] on them in a "spinor-like" manner, much like $SO(10) \approx SL(2; \mathbb{O})$ acts on its 16-dimensional spinor representations that play the role of homogeneous coordinates for $\mathbb{O}P^1$ (see the following section). That would open for a twistor transform [Jo1, PMcC, BeCed, Berk1, Ced1, Ced2] for elements in $J_3(\mathbb{O})$ (the exceptional Jordan algebra of 3×3 hermitean octonionic matrices [Jo2, JNW, GPR, Alb]) with zero Freudenthal product [Fr] – a known realization of $\mathbb{O}P^2$ [Jo1, Fr, Po]. Then one would have a direct analogy to the twistor transform of the masslessness condition in $SL(2; \mathbb{O})$ [Berk1, Ced1] that leads to $\mathbb{O}P^1$ as the projective light-cone (see reference [Ced2]).

2.8. Example: Twistors in Ten Dimensions. In ten-dimensional Minkowski space, the mass shell-constraint for a bosonic particle is $P_\mu P^\mu = 0$. According to the isomorphism $SO(1,9) \approx SL(2;\mathbb{O})$ [Sud], P may be viewed as an element in the Jordan algebra $J_2(\mathbb{O})$ of 2×2 hermitian octonionic matrices, and the constraint becomes that of scale-invariant idempotency [Ced2].

$$P^2 = P \operatorname{tr} P \,. \tag{2.49}$$

This is a well known realization of $\mathbb{O}P^1$ [Fr, Scha]. The Lorentz group is the structure group of $J_2(\mathbb{O})$ [Scha, Sud]. The two rows (or columns) in P fulfilling (2.49) contain the two charts (2.38) (up to a real scale). This makes it possible to perform a *twistor transform* [PMcC, BeCed, Berk1, Ced1, Ced2], which amounts to a change of the parametrization of $\mathbb{O}P^1$ from the Jordan algebra one to homogeneous coordinates. In $SL(2;\mathbb{O})$ language, the correspondence reads ($\lambda = [\xi, \eta]^t$)

$$P = \lambda\lambda^\dagger = \begin{bmatrix} \xi\xi^* & \xi\eta^* \\ \eta\xi^* & \eta\eta^* \end{bmatrix}, \tag{2.50}$$

where we immediately recognize the homogeneous coordinates and the two charts (2.38) in the rescaled columns. The similarity transformations on the homogeneous coordinates are the S^7 transformations (2.40) (and *not* the traditional transformations where the components of λ are subject to right multiplication with the same parameter). The scheme may be described $SO(1,9)$-covariantly, demanding a two-component current in a spinor representation [Berk1, Ced1], which provides a covariant solution replacing (2.40) of the singularity in the current. We expect something similar to be possible for the case of $J_3(\mathbb{O})$ described in the previous section. The treatment of supersymmetric particles [Fe, Sh, BeCed, Berk1, Ced1] introduces fermions into $\mathfrak{J}$ along the lines described earlier, but that falls outside the scope of this paper.

We apologize for not referring to many important papers concerning twistors – we have limited ourselves to contributions strictly relevant to the division algebra twistor program.

2.9. Relation to Malcev Algebras. A Lie algebra $\mathfrak{L}$ with antisymmetric product $[x, y]$ fulfills the Jacobi identities

$$J(x, y, z) \equiv [[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \tag{2.51}$$

for all elements $x, y, z \in \mathfrak{L}$. J is by definition alternating, i.e. completely antisymmetric in the arguments. Note the analogy with the associator (1.2) of an alternative algebra. In the same way as the concept of associative algebras can be weakened to alternativity, leading to alternative algebras, including the octonionic algebra $\mathbb{O}$, the Jacobi identities may be relaxed in favour of a weaker version,

$$J(x, y, [x, z]) = [J(x, y, z), x] \,. \tag{2.52}$$

These are the Malcev identities, and an algebra fulfilling them is called a *Malcev algebra* [Ma, Sa, My, Kuz]. In view of the analogy of J to the associator, the correspondence of the Malcev identities is the Moufang identity (1.5). The analogy goes further: the only central simple non-Lie Malcev algebra is the commutator algebra of imaginary octonions [Kuz].

Malcev algebras have been considered for physical applications in connection with Malcev–Kac-Moody [Os1, Os2] and related $N = 8$ superconformal algebras [ESTPS, DTH]. In this context, they have disadvantage that they cannot be realized in terms of Poisson brackets or commutators, since then (2.51) automatically is fulfilled.

Consider the S^7 algebra in the form (2.5), and evaluate the right-hand side at the north pole $X = 1$ (or at any other specific X). The so obtained algebra is then a Malcev algebra. We see that what distinguishes our S^7 algebra from this Malcev algebra is the transformation of the parameter field occurring in the X-product, see (2.11), or more generally, the transformation of the path products. It is exactly this associator term that cancels the Malcev J. Note that J for the octonionic commutator algebra is [Zorn]

$$J(x, y, z) = 6[x, y, z] . \tag{2.53}$$

In tensor formalism, the equation responsible for the cancellation is

$$\delta_i T_{jkl}(X) = 2T_{mi[j}(X) T_{kl]m}(X) \equiv 2R_{ijkl}(X) \tag{2.54}$$

(which is equivalent to the zero-curvature condition (1.15)), R being the completely antisymmetric X-associator that at the north pole reduces to ρ of (1.13), so that

$$\begin{aligned} J(\delta_i, \delta_j, \delta_k) &= 2[T_{ijl}(X)\delta_l, \delta_k] + \text{cycl.} \\ &= 4T_{ijl}(X) T_{lkm}(X)\delta_m - 2(\delta_k T_{ijl})(X)\delta_l + \text{cycl} = 0 . \end{aligned} \tag{2.55}$$

Omitting the last term gives the Malcev algebra.

3. Seven-Sphere Kac–Moody Algebra

3.1. Current Algebra and Schwinger Terms. A Lie algebra $\mathfrak{L}$ may be lifted to a Kac–Moody algebra [Moo, GoOl] $\hat{\mathfrak{L}}$ consisting of the mappings $S^1 \to \mathfrak{L}$ by applying the Lie product pointwise on the circle. The interesting feature of this structure is of course that it allows for non-trivial central extensions, or Schwinger terms [Schw]. The classical version of our "S^7 Kac–Moody algebra," $\widehat{S^7}$, is therefore trivial – in a conformal field theory language we simply have

$$\underbrace{\mathfrak{J}_\alpha(z)\mathfrak{J}_\beta(\zeta)} = \frac{1}{z - \zeta}\mathfrak{J}_{[\alpha,\beta]_X} , \tag{3.1}$$

ignoring potential normal ordering terms.

Now we have a set of structure functions (the torsion tensor) that varies over S^7, so it can be expected that the Schwinger terms can exhibit a similar behaviour (this is why we in the generic case avoid the notion of "central extensions"). This is easily demonstrated.

Take the simplest realization, where only the parameter field ξ and its momentum π transform, and the current is

$$\mathfrak{J} = \{\pi^* \xi\} . \tag{3.2}$$

We use a conformal field theory language, i.e. let the fields be holomorphic in a complex variable z, and postulate the fundamental correlator

$$\underbrace{\xi_o(z)\pi_{o'}(\zeta)}_{} = \frac{\hbar[o^* o']}{z-\zeta}\,. \tag{3.3}$$

Then the correlator of two currents is easily evaluated as

$$\underbrace{\mathfrak{J}_\alpha(z)\mathfrak{J}_\beta(\zeta)}_{} = -\frac{8\hbar^2[\alpha\beta]}{(x-\zeta)^2} + \frac{\hbar}{z-\zeta}\mathfrak{J}_{[\alpha,\beta]_X}\,. \tag{3.4}$$

In this simplest example, the Schwinger term is obviously a central extension. This is not so in general. If we stay with the free fields ξ ($X = \xi/|\xi|$) and π and the correlator (3.3), but let the current get a "quantum correction" according to

$$\mathfrak{J} = \{\pi^*\xi\} + \sigma\hbar X^*\partial X, \quad \sigma \in \mathbb{R}\,, \tag{3.5}$$

the algebra acquires a non-central extension:

$$\underbrace{\mathfrak{J}_\alpha(z)\mathfrak{J}_\beta(\zeta)}_{} = -\frac{(8+2\sigma)\hbar^2[\alpha\beta]}{(z-\zeta)^2} + \frac{\sigma\hbar^2}{z-\zeta}(X^*\partial X)_{[\alpha,\beta]_X} + \frac{\hbar}{z-\zeta}\mathfrak{J}_{[\alpha,\beta]_X}\,. \tag{3.6}$$

The currents of Eq. (3.5) for different values of σ carry the same field content and the algebras (3.6) can therefore be taken as equivalent up to a (quantum) redefinition of the current. The central extension of (3.4) can be taken as a representative of this class.

Let us now turn to currents constructed from several octonions, as in (2.31), and for simplicity we first treat the case (2.15) of two transforming fields. It is easily seen that any double contraction (giving rise to Schwinger terms) must take place between identical terms in the two currents, since these are linear in momenta. The Schwinger terms are therefore "additive" – introducing new terms in $\mathfrak{J}$ corresponding to new transforming fields only gives rise to extra Schwinger terms arising from double contractions of these terms with themselves. The second term of (2.15) gives a field dependent double contraction, so that the quantum current algebra now takes the form (from now on, $\hbar$ is suppressed)

$$\underbrace{\mathfrak{J}_\alpha(z)\mathfrak{J}_\beta(\zeta)}_{} = -\frac{16[\alpha\beta]}{(z-\zeta)^2} + \frac{4}{z-\zeta}(X^*\partial X)_{[\alpha,\beta]_X} + \mathfrak{J}_{[\alpha,\beta]_X}\,. \tag{3.7}$$

Here we have a field dependent Schwinger term. Notice that the anomaly is in the one-parameter class of (3.6) with $\sigma = 4$, so that adding a quantum correction $-4X^*\partial X$ to $\mathfrak{J}$ gives back (3.4). In fact, the second transforming octonion has not changed the numerical coefficient of the central extension. This result generalizes to any number of fields – adding a quantum correction to (2.31) to obtain

$$\mathfrak{J}_\alpha = -\sum_k [\pi_k^*(\mathfrak{r}_k \underset{\mathfrak{P}[\mathfrak{r}_k]}{\circ} \alpha)] + 4\sum_{k\neq 0}[X_{k-1}^*\partial X_{k-1}\alpha_k]\,, \tag{3.8}$$

with α_k defined as in (2.33), gives back the correlator (3.4). We obtain the surprising result, contrasting to the situation in (Lie) Kac–Moody algebras, that the coefficient of the central extension is *unique* up to redefinitions of the current. This statement of course holds given that the field content is that of Sect. 2.6. A more general uniqueness proof will have to wait for the general representation theory to be completed.

Any conjugate pair of fermions (at an endpoint of the tree diagram) contributes to the Schwinger term with the same term as a boson in the same position would have done, but with opposite sign, just as for Lie Kac–Moody algebras. If the fermion is self-conjugate, the coefficient is divided by two. The forms of the quantum corrections needed to render the algebra extension central are completely analogous.

3.2. BRST Operator. "*Anomaly Cancellation.*" In this section we would like to address the question of anomaly cancellation: under what circumstances is the Schwinger term "quantum mechanically consistent," i.e. when is the BRST operator quantum mechanically nilpotent, and what actual exact form of the Schwinger term is needed? This question will be of relevance if the algebra is considered as an algebra of gauge constraints, e.g. in some twistor string model. It will be shown that quantum mechanical consistency is compatible with one member of the class of anomalies obtained above.

The first thing to do is to examine how to construct a (classical) BRST operator for the S^7 algebra with field-dependent structure functions. This turns out to be extremely simple. The BRST operator takes the same form as for a Lie algebra, namely

$$\mathcal{Q} = c^i \mathfrak{J}_i - T_{ij}^k(X) c^i c^j b_k \,, \tag{3.9}$$

where b_i and c^i are fermionic ghosts with $b_i(z) c^j(\zeta) \sim \delta_i^j (z - \zeta)^{-1}$. Higher order ghost terms are not present since the Jacobi identities hold, due to (2.54). This makes BRST analysis quite manageable.

Then, turning to $\widehat{S^7}$ and the quantum algebra, a somewhat lengthy calculation shows that the current must obey (3.6) with $\sigma = 8$ in order for Ω (with (3.9) as the BRST charge density) to be quantum mechanically nilpotent:

$$\mathcal{Q}^2 = 0 \Leftrightarrow \underbrace{\mathfrak{J}_\alpha(z) \mathfrak{J}_\beta(\zeta)} = -\frac{24\,[\alpha\beta]}{(z-\zeta)^2} + \frac{1}{z-\zeta}(\mathfrak{J} + 8X^* \partial X)_{[\alpha,\beta]_X} \,. \tag{3.10}$$

We have thus demonstrated the non-trivial fact that $\mathcal{Q}$ *may* be nilpotent, and that $\widehat{S^7}$ may be used as a gauge algebra. Normally, one would have expected $\mathcal{Q}^2 = 0$ to put a constraint on the number of transforming octonionic fields, but that is not the case at hand. Instead one is permitted, for any field content, to adjust the numerical coefficient of $X^* \partial X$ in $\mathfrak{J}$ in order to fulfill that relation.[4]

One may remark that if one restricts one self to the plain generators of (2.31) without quantum corrections, the BRST charge is nilpotent for the (linear) diagram of three bosonic fields, with the parameter field in the middle. We suspect that it is more than a coincidence that this is the number of fields transforming under $\widehat{S^7}$ in the string twistor model of [Berk2], where one has the pair of octonions making

[4] One can however imagine that other covariance properties, for example Lorentz symmetry, puts a restriction on the quantum corrections, so that (3.10) becomes predictive for the field conten.

the twistor variable λ (see Sect. 2.8) and the ghosts for the eight supercharges of the associated superconformal algebra [Berk2, BCP, CedPr]. We have not carried out the the detailed analysis, however.

In an ordinary algebra with structure *constants*, one can let $\mathcal{Q}$ act on the b field to obtain a modified, BRST-exact, current $\tilde{\mathfrak{J}}$ containing the ghost fields. If $\mathcal{Q}^2 = 0$, then the algebra of $\tilde{\mathfrak{J}}$ is non-anomalous. This is not so here. Since the Jacobi identities hold, it is easy to show that a Poisson bracket of two BRST-exact operators is BRST-exact, but for the S^7 algebra one obtains

$$\{\{b_i, \mathcal{Q}\}, \{b_j, \mathcal{Q}\}\} = \{2T_{ij}^k(X)b_k, \mathcal{Q}\} \mp 2T_{ij}^k(X)\{b_k, \mathcal{Q}\} . \tag{3.11}$$

This specific subset of the BRST-exact operators does not close to an algebra. It seems that one has to conclude that the S^7 or $\widehat{S^7}$ ghosts do not come in an S^7 representation. This is also confirmed by an attempt to construct a representation (other than scalar) for imaginary octonions, which turns out to be impossible. This was already hinted at when dealing with tensor composition of spinors in Sect. 2.3.

3.3. Sugawara Construction. The similarity of S^7 to a group manifold and the results of Osipov [Os2] for Malcev–Kac-Moody algebras let us expect that a Sugawara construction [Sug] is possible also for $\widehat{S^7}$. This is not difficult to verify for the simplest current $\mathfrak{J} = \{\pi^* \xi\}$ and the associated energy-momentum tensor

$$\begin{aligned} L &= -\frac{1}{8} :\mathfrak{J}_j \mathfrak{J}_j: \\ &= \frac{7}{8}[\xi^* \partial\pi - \partial\xi^* \pi] + \frac{1}{8}[\xi^* \pi e_j][\xi^* \pi e_j] , \end{aligned} \tag{3.12}$$

where we have normal-ordered the currents in the standard way:

$$:\mathfrak{J}_j(\zeta)\mathfrak{J}_k(\zeta): \; - \lim_{z\to\zeta} \left(\mathfrak{J}_j(z)\mathfrak{J}_k(\zeta) - \frac{56}{(z-\zeta)^2} \right) . \tag{3.13}$$

In order to generalize this result to currents with arbitrary field content, one has to be careful about normal ordering. Up to this point we have implicitly assumed free-field normal ordering on the right-hand side of the current-current commutation relations. Even though the torsion tensor $T_{ijk}(X)$ commutes with $\mathfrak{J}_k$, the product of those two operators obeys

$$\begin{aligned} T_{ijk}(X)\mathfrak{J}_k &= \,:T_{ijk}(X)\mathfrak{J}_k: - 8T_{ijk}(X)[X^* \partial X e_k] \\ &= \,:T_{ijk}(X)\mathfrak{J}_k: + T_{ikl}(X)\partial_z T_{jkl}(X) , \end{aligned} \tag{3.14}$$

where the left-hand side free-field normal ordered and the first term on the right-hand side is current normal ordered. It is useful to employ the technology described by Bais, Bouwknegt, Surridge and Schoutens [BPSS] to show the following result: for currents that obey

$$\underbrace{\mathfrak{J}_i(z)\mathfrak{J}_j}(\zeta) = \frac{k\delta^{ij}}{(z-\zeta)^2} + \frac{2}{z-\zeta} : T_{ijk}(X)(\mathfrak{J}_k - 8[X^* \partial X e_k])(\zeta): , \tag{3.15}$$

the Sugawara energy-momentum tensor is given by

$$L = \frac{1}{2k - 24} : \mathfrak{J}_i \mathfrak{J}_j : \tag{3.16}$$

and has a central charge of

$$c = \frac{7k}{k - 12}, \tag{3.17}$$

in accordance with the known results for Kac–Moody algebras. Equations (3.16) and (3.17) are identical to Osipov's formulas [Os2]. We note that in contrast to the Kac–Moody case, the requirement that $\mathfrak{J}_i(z) + \alpha[X^* \partial X e_i](z)$ should transform like a dimension 1 current under the action of the Sugawara energy-momentum tensor fixes only the constant α. There is a three-parameter family of candidate Sugawara tensors which satisfy this condition. The operator product of $L(z)$ with itself determines the parameters. There are two solutions: (3.16) for any k and another solution which would require a complex value of k. The Sugawara construction is therefore unique.

4. Summary and Discussion

We have discussed several aspects of the seven-sphere algebra and some related topics. We find it somewhat surprising that this algebra has received so little attention in the mathematical literature (compared to Malcev algebras), in spite of the fact that the parallelizability property has been known for a long time, and the simplicity of the argument in Sect. 2.1.

From a physical point of view, the S^7 algebra provides a natural generalization of the Lie algebra concept. We have demonstrated how it can be handled when arising as a gauge algebra of constraints (BRST procedure) and how it can be used as a generalized Kac-Moody-Lie algebra. For this last case, some unexpected features of the Schwinger term occur, distinguishing it from ordinary Kac–Moody-Lie algebras. The feasibility of a BRST procedure involving field-dependent structure functions and anomalies is not *a priori* ascertained, but has been demonstrated. The class of physical models closest in our minds for this kind of symmetry is string twistor theories. Different versions have been formulated, but at least one of them posseses an S^7 Kac–Moody gauge symmetry [Berk2]. Superstring twistors involve a super-extension to an $N = 8$ superconformal algebra [Berk2, BCP, CedPr], and we hope that it will be possible to give a similar treatment of the super-extensions to the one presented for S^7 in the present paper. Especially the problem of anomaly cancellation may gain some insight from our results.

A part of the structure of S^7 we have treated only fragmentarily is representation theory. We would like to return to that question later. It is not immediately clear even how to define a representation. We have quite strong feelings, though, that the spinorial representations and the adjoint, as described in this paper, in some sense are the only ones allowed, and that the spinor representation is the only one to which a variable can be freely assigned.

Most of this paper has been written without specific aim at physical applications, mostly because we felt that out mathematical understanding of the algebras we were dealing with in ten-dimensional superstring models was dragging behind. This means that some sections may be of little interest when studying a specific physical problem. On the other hand, we find some of our byproducts, e.g. those concerning infinitesimal generation of the octonionic Hopf map, quite appealing by themselves.

Acknowledgements. We would like to thank Lars Brink, Nathan Berkovits, Volodia Fock, Viktor Ogievetsky and Peter West for valuable discussions.

References

[Ad] Adams, J.F.: Ann. Math. **75**, 603 (1962)
[Alb] Albert, A.A.: Ann. Math. **35**, 65 (1934)
[BeCed] Bengtsson, I., Cederwall, M.: Nucl. Phys. **B302**, 81 (1988)
[Berk1] Berkovits, N.: Phys. Lett. **247B**, 45 (1990)
[Berk2] Berkovits, N.: Nucl. Phys. **B358**, 169 (1991)
[BCP] Brink, L., Cederwall, M., Preitschopf, C.R.: Phys. Lett. **311B**, 76 (1993)
[BPSS] Bais, F.A., Bouwknegt, P. Surridge, M., Schoutens, K.: Nucl. Phys. **B304**, 348 (1988)
[BoMi] Bott, R. Milnor, J.: Bull. Am. Math. Soc. **64**, 87 (1958)
[Bor] Borel, A.: Compt. Rend. Acad. Sci. **230**, 1378 (1950)
[CaSch] Cartan, E. Schouten, J.A.: Proc. Kon. Wet. Amsterdam **29**, 803; 933 (1926)
[Cay] Cayley, A.: Phil. Mag. **3**, 210 (1845)
[Ced1] Cederwall, M.: J. Math. Phys. **33**, 388 (1992)
[Ced2] Cederwall, M.: Phys. Lett. **210B**, 169 (1988)
[CedPr] Cederwall, M. Preitschopf, C.R.: Göteborg-ITP-92-40, hep-th/9209107, Nucl. Phys. B (in press)
[DTH] Defever, F., Troost, W., Hasiewisz, Z.: J. Math. Phys. **32**, 2285 (1991)
[ESTPS] Englert, F., Sevrin, A., Troost, W., Van Proyen, A., Spindel, Ph.: J. Math. Phys. **29**, 281 (1988)
[Fe] Ferber, A.: Nucl. Phys. **B132**, 55 (1978)
[Fr] Freudenthal, H.: Advan. Math. **1** 145 (1964)
[GoOl] Goddard, P. Olive, D.: Int. J. Mod. Phys. **A1**, 303 (1986)
[GuTze] Gürsey, F., Tze, C.-H.: Phys. Lett. **127B**, 191 (1983)
[GPR] Günaydin, M., Piron, C., Ruegg, H.: Commun. Math. Phys. **61**, 69 (1978)
[Hopf] Hopf, H.: Math. Ann. **104**, 637, (1937), Fund. Math. **XXV** 427 (1935)
[Hur] Hurwitz, A.: Nachr. Ges. Wiss. Göttingen (1898) 309
[Hus] Husemoller, D.: Fibre bundles. New York, McGraw-Hill, 1966
[Jac] Jacobson, N.: Lie algebras, Wiley Interscience, New York (1962)
[Jo1] Jordon, P.: Abh. Math. Sem. Univ. Hamburg **16**, 74 (1949)
[Jo2] Jordon, P.: Nachr. Ges. Wiss. Göttingen 569 (1932)
[JNW] Jordon, P., Neumann, J.v., Wigner, E.: Ann. Math. **35**, 29 (1934)
[Ker] Kervaire, M., Proc. Nat. Acad. Sci. USA **44**, 280 (1958)
[Kuz] Kuzmin, E.N., Алгеőра и Логика **16**, 424 (1977)
[LuMi] Lukierski, J. Minnaert, P.: Phys. Lett. **129B**, 392 (1983)
[Ma] Malcev, A.I., Mat. C6. ("Mat. Sb." in russian letters), **36(78)**, 569 (1955)
[Moo] Moody, R.V.: Bull. Am. Math. Soc. **73**, 217 (1967)
[Mou] Moufang, R.: Abh. Math. Sem. Univ. Hamburg **9**, 207 (1932)
[My] Myung, H.C.: Malcev-admissible algebras, Birkhäuser, Basel (1986)
[Os1] Osipov, E.P.: Lett. Math. Phys. **18**, 35 (1989)
[Os2] Osipov, E.P.: Phys. Lett. **214B**, 371 (1988)
[PMcC] Penrose, R., McCallum, M.A.H.: Phys. Rep. **6**, 241 (1972) (and references therein)

[Po] Porteous, I.R.: Topological geometry, Cambridge University Press, Cambridge (1981)
[Roo] Rooman, M.: Nucl. Phys. **B236**, 501 (1984)
[Sa] Sagle, A.A.: Trans. Am. Math. Soc. **101**, 426 (1961)
[Scha] Schafer, R.D.: An introduction to nonassociative algebras, Academic Press, New York (1966)
[Schw] Schwinger, J.: Phys. Rev. Lett. **3**, 296 (1959)
[Sh] Shirafuji, T.: Progr. Theor. Phys. **70**, 18 (1983)
[Sud] Sudbery, A.: J. Phys. **A17**, 939 (1984)
[Sug] Sugawara, H.: Phys. Rev. **170**, 659 (1968)
[Wolf] Wolf, J.A.: J. Diff. Geom. **6** (1972) 317; **7**, 19 (1972)
[Zorn] Zorn, M.: Abh. Math. Sem. Univ. Hamburg **8**, 123 (1930)

Communicated by R.H. Dijkgraaf