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Canonical Property of the Miura Maps Between the MKP and KP Hierarchies, Continuous and Discrete

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Abstract. Apart from the case of the KP hierarchy, all known Miura maps between integrable Hamiltonian systems had been proven to be canonical. The remaining KP case is settled below. As a corollary, it is shown that the KP hierarchy is a factor – hierarchy of the mKP one, with the kernel consisting of a single scalar field. A discrete mKP hierarchy and the associated Miura map are constructed, and the latter is shown to be canonical as well. As in the continuous case, this implies that one can extend the discrete KP hierarchy by a single new field into an extended discrete KP hierarchy in such a way that the extended discrete Miura map mKP $\rightarrow$ eKP is a canonical isomorphism.

1. Introduction

As a rule, modern theories of integrable dynamical systems, whether finite-dimensional, infinite-dimensional, continuous, discrete, etc. have two invariably present central ingredients: 1) The dynamical systems under consideration always turn out to be Hamiltonian systems, and 2) The morphisms in these theories, traditionally called the Miura maps (when they are not invertible), turn out to be Hamiltonian maps (also called canonical maps). The presence of these two ingredients, nowadays taken for granted as a sort of a metaphysical principle, has been by now established in a great variety of very general circumstances, including: scalar [1] and matrix [2] Lax equations and Kac–Moody–Lie-algebras-related equations [3–5] for continuous systems; and scalar Lax equations for discrete systems [6].

These two ingredients have highly unequal weights: while the Hamiltonian structure of the evolution equations can be established in a more-or-less routine fashion by employing the powerful Residue calculus (and its various generalizations) in modules of differential forms over rings of pseudo-differential operators [7], the problems of existence, the canonical property, and the interpretation of the Miura maps in contemporary theories are still a subject of mystery and speculation [8].

In the most important situation of the universal scalar differential Lax equations [9] which have become known as the KP hierarchy, the theory has a troubling defect in that it is still not known whether the associated Miura map is canonical (as conjectured in [10]) or not. One of the main results of this paper is a proof that the Miura map for the KP hierarchy is indeed canonical . This allows one to extend the KP hierarchy by a single field in such a way that the extended (= eKP) hierarchy is isomorphic to the mKP one, and this isomorphism is also canonical. Thus, KP is the factor-system of mKP, with the Kernel consisting of precisely one scalar field. The second group of results concerns discrete equations: I construct the modified KP (mKP) hierarchy and associated Miura map into the KP hierarchy, and then prove that this Miura map is also Hamiltonian. Again, this allows one to extend the KP hierarchy by a single field in such a way that the extended eKP hierarchy is canonically isomorphic to the mKP one.

The paper is organized as follows. In the next section we recall the Hamiltonian formalism for the mKP and KP hierarchies and write down an infinite system of differential identities whose validity, to be verified, is equivalent to the property of the Miura map being canonical . The checking of these identities is divided between Sects. 3 and 4. Section 5 is devoted to the construction of the eKP hierarchy and a proof of the isomorphism eKP $\approx$ mKP. In Sect. 6 the discrete mKP hierarchy is constructed and its Hamiltonian form is derived. Section 7 is devoted to a construction of the Miura map and to showing that it is a canonical map. In the last Sect. 8, the discrete eKP hierarchy is constructed and is shown to be canonically isomorphic to the mKP one.

2. Hamiltonian Formalism for the KP and MKP Hierarchies

In this section we summarize, following [10] but in notation adjusted to our purposes, the constructions of the mKP and KP hierarchies, of their Hamiltonian forms, and of the Miura map which maps the mKP hierarchy into the KP one.

Let

$$L = \xi + \sum_{i=0}^{\infty} A_i \xi^{-i-1} , \tag{2.1}$$

be the Lax operator of the KP hierarchy whose n^{th} flow has the form

$$L_{,t} = [(L^n)_{\geqq 0}, L] = [L, (L^n)_{<0}] . \tag{2.2}$$

Here ξ is the algebraic version of $\partial = \partial/\partial x$, and all the pseudodifferential notations are the usual ones (see [7, 11]). Let

$$\mathscr{L} = \xi + \sum_{i=0}^{\infty} a_i \xi^{-1} \tag{2.3}$$

be the Lax operator of the mKP hierarchy whose n^{th} flow has the form

$$\mathscr{L}_{,t} = \left[\left(((\mathscr{L}^n)^\dagger)_{\geqq 1} \right)^\dagger , \mathscr{L} \right] = \left[\mathscr{L}, \left(((\mathscr{L}^n)^\dagger)_{\leqq 0} \right)^\dagger \right] , \tag{2.4}$$

where "$\dagger$" stands for the "adjoint".

For a pseudodifferential operator

$$\theta = \sum \theta_s \xi^s ,$$

set

$$\operatorname{Res}(\theta) = \theta_{-1} .$$

The residue formula [7]

$$d[\mathrm{Res}(\theta^n)] \sim n\mathrm{Res}(\theta^{n-1}d\theta) \sim n\mathrm{Res}(d\theta \cdot \theta^{n-1}), \tag{2.5}$$

where d is the differential and "$\sim$" stands for the equality modulo $\mathrm{Im}\,\partial$, can be applied to the operators L (2.1) and $\mathscr{L}$ (2.3) in the following way. Set

$$H_n = n^{-1}\mathrm{Res}(L^n), \quad \mathscr{H}_n = n^{-1}\mathrm{Res}(\mathscr{L}^n), \tag{2.6a}$$

$$L^n = \sum_s \xi^s p_s(n), \quad \mathscr{L}^n = \sum_s \xi^s \pi_s(n). \tag{2.6b}$$

Then

$$d(H_{n+1}) \sim \mathrm{Res}(dL \cdot L^n) = \mathrm{Res}\left(\sum dA_i \xi^{-i-1}\xi^s p_s(n)\right) = \sum dA_i p_i(n),$$

whence

$$p_i(n) = \frac{\delta H_{n+1}}{\delta A_i}. \tag{2.7}$$

Similarly,

$$\begin{aligned} d(\mathscr{H}_{n+1}) \sim \mathrm{Res}(d\mathscr{L} \cdot \mathscr{L}^n) &= \mathrm{Res}\left(\sum da_i \xi^{-i}\xi^s \pi_s(n)\right) \\ &= \sum da_i \pi_{i-1}(n), \end{aligned}$$

so that

$$\pi_{i-1}(n) = \frac{\delta \mathscr{H}_{n+1}}{\delta a_i}. \tag{2.8}$$

Substituting (2.7) into (2.2) we obtain

$$A_{i,t} = \sum_j B_{ij}^{\mathrm{KP}}(\delta H_{n+1}/\delta A_j),$$

where

$$B_{ij}^{\mathrm{KP}} = \sum_{s \geqq 0}\left[\binom{j}{s}\partial^s A_{i+j-s} - \binom{i}{s} A_{i+j-s}(-\partial)^s\right] \tag{2.9}$$

is the (first) Hamiltonian structure of the KP hierarchy. (Formula (2.9) was derived by Watanabe [14].) Similarly, substituting (2.8) into (2.4) we get

$$a_{i,t} = \sum_j B_{ij}^{\mathrm{mKP}}(\delta \mathscr{H}_{n+1}/\delta a_j),$$

where

$$B^{\mathrm{mKP}} = \begin{array}{c|ccc} & a_0 & a_1 & w_j \\ a_0 & 0 & \partial & \\ a_1 & \partial & 0 & 0 \\ w_i & & 0 & B_{ij}^1 \end{array}, \quad w_k = a_{k+2},$$

$$B_{ij}^1 = \sum_{s \geqq 0}\left[\binom{j+1}{s}\partial^s w_{i+j+1-s} - \binom{i+1}{s} w_{i+j+1-s}(-\partial)^s\right] \tag{2.10}$$

is the (first) Hamiltonian structure of the mKP hierarchy. The Miura map between the two hierarchies, in the language of functions, is:

$$\Phi^* : \mathscr{L} = \xi + \sum a_i \xi^{-i} \mapsto L = e^{\int a_0} \mathscr{L} e^{-\int a_0} = \mathscr{L}|_{\xi \mapsto \xi - a_0}$$

$$= \xi + \sum_{i \geqq 0} a_{i+1} (\xi - a_0)^{-(i+1)} \quad \text{[by (2.12) below]}$$

$$= \xi + \sum a_{i+1} \binom{-i-1}{\alpha} Q_\alpha(-a_0) \xi^{-i-1-\alpha} = \xi + \sum_{r \geqq 0} A_r \xi^{-r-1} \ , \quad (2.11')$$

or, in the dual algebraic language,

$$\Phi(L) = \Phi(\xi + \sum A_r \xi^{-r-1}) = \xi + \sum a_i \xi^{-i} = \mathscr{L} \ ,$$

whence

$$\Phi(A_r) = \sum_{i+\alpha=r} a_{i+1} \binom{-i-1}{\alpha} Q_\alpha(-a_0) \quad \text{[by (2.14) below]}$$

$$= \sum_{i+\alpha=r} a_{i+1} \binom{i+\alpha}{\alpha} (-1)^\alpha Q_\alpha(-a_0) = \sum_{i+\alpha=r} \binom{r}{\alpha} (-1)^\alpha a_{i+1} Q_\alpha(-a_0)$$

$$= \sum_{\alpha=0}^{r} \binom{r}{\alpha} (-1)^\alpha Q_\alpha(-a_0) a_{r+1-\alpha} = (-1)^r Q_r(-a_0) a_1 \quad (2.11a)$$

$$+ \sum_{\alpha=0}^{r-1} \binom{r}{\alpha} (-1)^\alpha Q_\alpha(-a_0) a_{r+1-\alpha} \ , \quad (2.11b)$$

where we used the formulae [10]

$$(\xi + u)^m = \sum_{\alpha \geqq 0} \binom{m}{\alpha} Q_\alpha(u) \xi^{m-\alpha}, \quad m \in \mathbf{Z} \ , \quad (2.12)$$

$$Q_\alpha(u) = (\partial + u)^\alpha (1) \ , \quad \alpha \in \mathbf{Z}_+ \ , \quad (2.13)$$

and the binomial relation

$$\binom{-i-1}{\alpha} = (-1)^\alpha \binom{i+\alpha}{\alpha} \ , \quad i, \alpha \in \mathbf{Z}_+ \ . \quad (2.14)$$

The Q_α's are the classical Faà di Bruno polynomials; many formulae involving these polynomials can be found in [2].

Transforming the expression (2.11b) by changing α into $r-1-\alpha$ we get

$$\sum_{\alpha=0}^{r-1} \binom{r}{r-1-\alpha} (-1)^{r-1-\alpha} Q_{r-1-\alpha}(-a_0) a_{\alpha+2}$$

$$= \sum_{\alpha=0}^{r-1} \binom{r}{\alpha+1} (-1)^{r-1-\alpha} Q_\alpha(-a_0) a_{\alpha+2}$$

$$= \sum_{\alpha=0}^{r-1} \binom{r}{\alpha+1} (-1)^{r-1-\alpha} Q_{r-1-\alpha}(-a_0) w_\alpha \ ,$$

so that formula (2.11) becomes

$$\Phi(A_i) = (-1)^i Q_i(-a_0) a_1 + \sum_{\alpha=0}^{i-1} \binom{i}{\alpha+1} (-1)^{i+1+\alpha} Q_{i-1-\alpha}(-a_0) w_\alpha \ . \quad (2.15)$$

This is the Miura map. The claim is that this map is canonical between the Hamiltonian matrices B^{KP} (2.9) and B^{mKP} (2.10). Thus, the canonical property is equivalent to the equality [1]

$$JB^{\mathrm{mKP}}J^{\dagger} = \Phi(B^{\mathrm{KP}})\,, \tag{2.16}$$

where J is the Fréchet Jacobian of the homomorphism Φ (dual to the Miura map Φ^*):

$$J_{ij} = \frac{D\Phi(A_i)}{Da_j}\,. \tag{2.17}$$

Here $\frac{D}{Da}$ is the usual Fréchet derivative:

$$\frac{D(\,\cdot\,)}{Da} = \sum_{s \geqq 0} \frac{\partial(\,\cdot\,)}{\partial a^{(s)}} \partial^s\,, \tag{2.18}$$

and $a^{(s)} = \partial^s(a)$.

The next two sections are devoted to the proof of the identity (2.16).

Remark. 2.19. Fix a positive integer $N \geqq 2$. Then the constraints

$$(L^N)_- = 0 \tag{2.20a}$$

and

$$_{\geqq 1}(\mathscr{L}^N) = \left(((\mathscr{L}^N)^{\dagger})_{\geqq 1}\right)^{\dagger} = 0 \tag{2.20b}$$

are preserved by the flows (2.2) and (2.4), respectively. The Miura map Φ^* (2.11) preserves these constraints and thus provides a Miura map into the scalar Lax hierarchy based on the Lax operator

$$\mathscr{L}_N = \xi^N + \sum_{i=0}^{N-2} u_i \xi^i\,. \tag{2.21}$$

For $N = 2$ one gets the usual KdV and mKdV hierarchies, as proved in [10]. However, for $N > 2$, the hierarchy (2.20b) is clearly *not* isomorphic to the modified Lax hierarchy constructed in [1]. It is not even clear whether it's isomorphic to the degenerate modified Lax hierarchy constructed in [15].

3. The a_1 – Part

In the Hamiltonian matrix B^{mKP} (2.10), the variable a_1 is separated from the w's. In this section we prove that part of the equality (2.16) which involves a_1. The next section addresses the contribution of the w's.

Let us introduce the new variables

$$\bar{A}_i = (-1)^i A_i, \quad \bar{a}_i = (-1)^{i+1} a_i\,, \tag{3.1}$$

so that

$$\overline{w}_i = (-1)^{i+1} w_i\,. \tag{3.2}$$

(In other words, we work with $L^{\dagger}$ and $\mathscr{L}^{\dagger}$ instead of L and $\mathscr{L}$.) In the new variables, the *negative* of the Hamiltonian matrices B^{KP} (2.9) and B^{mKP} (2.10) take the form

$$-B^{\mathrm{KP}}_{ij} = \sum_{s\geqq 0}\left[\binom{i}{s} A_{i+j-s}\partial^s - \binom{j}{s}(-\partial)^s A_{i+j-s}\right], \tag{3.3}$$

$$-B^{\mathrm{mKP}} = \begin{array}{c|c} & \begin{array}{ccc} a_0 & a_1 & w_j \end{array} \\ \begin{array}{c} a_0 \\ a_1 \\ w_i \end{array} & \begin{pmatrix} 0 & \partial & \\ \partial & 0 & \\ & & \sum\limits_{s\geq 0}\left[\binom{i+1}{s} w_{i+j+1-s}\partial^s - \binom{j+1}{s}(-\partial)^s w_{i+j+1-s}\right] \end{pmatrix} \end{array}, \tag{3.4}$$

where the bars have been dropped off the variables $\overline{A}_i$'s, $\overline{a}_i$'s, $\overline{w}_i$'s; and the Miura homomorphism (2.15) takes the form

$$\Phi(A_i) = Q_i(a_0)a_1 + \sum_{\alpha=0}^{i-1}\binom{i}{\alpha+1} Q_{i-1-\alpha}(a_0)w_\alpha . \tag{3.5}$$

The next step in proving the identity (2.16) is to prepare formulae for the Fréchet Jacobian J. From formula (3.5) we have, with Q_α standing for $Q_\alpha(a_0)$:

$$J_{k0} = a_1 D(Q_k) + \sum_\alpha \binom{k}{\alpha+1} w_\alpha D(Q_{k-1-\alpha}), \tag{3.6a}$$

where

$$D(Q_\alpha) = \frac{DQ_\alpha(a_0)}{Da_0};$$

$$J_{k1} = Q_k ; \tag{3.6b}$$

$$J_{kw_i} = \frac{D\Phi(A_k)}{Dw_i} = \binom{k}{i+1} Q_{k-1-i} . \tag{3.6c}$$

Hence, for the matrix $J(-B^{\mathrm{mKP}})$ we obtain

$$\begin{array}{ccc} a_0 & a_1 & w_j \\ \end{array}$$
$$\left(Q_k\partial \,\middle|\, a_1 D(Q_k)\partial + \sum_\alpha \binom{k}{\alpha+1} w_\alpha D(Q_{k-1-\alpha})\partial \,\middle|\, \sum_i \binom{k}{i+1} Q_{k-1-i}(-B^{\mathrm{mKP}}_{w_i w_j}) \right)$$

Similarly, for the matrix $J^\dagger$ we get

$$J^\dagger = \begin{array}{c|c} & \Phi(A_s) \\ \begin{array}{c} a_0 \\ a_1 \\ w_j \end{array} & \begin{pmatrix} D(Q_s)^\dagger a_1 + \sum\limits_\beta \binom{s}{\beta+1} D(Q_{s-1-\beta})^\dagger w_\beta \\ Q_s \\ \binom{s}{j+1} Q_{s-1-j} \end{pmatrix} \end{array}, \tag{3.8}$$

so that, finally

$$
\begin{aligned}
&[J(-B^{\mathrm{mKP}})J^{\dagger}]_{ks} \\
&= Q_k\partial\left[D(Q_s)^{\dagger}a_1 + \sum_{\beta}\binom{s}{\beta+1}D(Q_{s-1-\beta})^{\dagger}w_{\beta}\right] && (3.9a)\\
&+\left[a_1 D(Q_k) + \sum_{\alpha}\binom{k}{\alpha+1}w_{\alpha}D(Q_{k-1-\alpha})\right]\partial Q_s && (3.9b)\\
&+\sum_{i,j}\binom{k}{i+1}\binom{s}{j+1}Q_{k-1-i}\sum_{\mu}\left[\binom{i+1}{\mu}w_{i+j+1-\mu}\partial^{\mu}\right.\\
&\left.-\binom{j+1}{\mu}(-\partial)^{\mu}w_{i+j+1-\mu}\right]Q_{s-1-j}\,. && (3.9c)
\end{aligned}
$$

For the right-hand-side of formula (2.16) we obtain

$$
\begin{aligned}
\Phi(-B^{\mathrm{KP}}_{ks}) &= \sum_{\nu\geqq 0}\left[\binom{k}{\nu}\Phi(A_{k+s-\nu})\partial^{\nu} - \binom{s}{\nu}(-\partial)^{\nu}\Phi(A_{k+s-\nu})\right]\\
&= \sum_{\nu}\left\{\binom{k}{\nu}\left[Q_{k+s-\nu}a_1 + \sum_{\gamma}\binom{k+s-\nu}{\gamma+1}Q_{k+s-\nu-1-\gamma}w_{\gamma}\right]\partial^{\nu}\right. && (3.10a)\\
&\left.-\binom{s}{\nu}(-\partial)^{\nu}\left[Q_{k+s-\nu}a_1 + \sum_{\gamma}\binom{k+s-\nu}{\gamma+1}Q_{k+s-\nu-1-\gamma}w_{\gamma}\right]\right\}\,. && (3.10b)
\end{aligned}
$$

For each fixed pair of indices (k, s), we have to verify the equality $\{(3.9) = (3.10)\}$. Each of these expressions is a differential operator linearly dependent upon a_1 and the w's. Thus, the desired equality breaks into separate subequalities for a_1 and each of the w's. The a_1-terms, entering at the beginning of the expression (3.9a, b) and (3.10a, b), combine into the following identity to be verified:

$$
\begin{aligned}
&Q_k\partial D(Q_s)^{\dagger}a_1 + a_1 D(Q_k)\partial Q_s && (3.11l)\\
&= \sum_{\nu}\left[\binom{k}{\nu}Q_{k+s-\nu}a_1\partial^{\nu} - \binom{s}{\nu}(-\partial)^{\nu}Q_{k+s-\nu}a_1\right]\,. && (3.11r)
\end{aligned}
$$

Using formulae ((4.20) and (4.18) in [10])

$$D(Q_s)\partial = (\partial + u)^s - Q_s\,, \tag{3.12}$$

$$\sum_{\alpha\geqq 0}\binom{m}{\alpha}Q_{m+r-\alpha}\partial^{\alpha} = (\partial + u)^m Q_r\,, \tag{3.13}$$

we can transform the expression (3.11l) as follows:

$$\begin{aligned}
&- [a_1 D(Q_s)\partial Q_k]^\dagger + a_1 D(Q_k)\partial Q_s \quad \text{[by (3.12)]}\\
&= a_1[(\partial + u)^k - Q_k]Q_s - \{a_1[(\partial + u)^s - Q_s]Q_k\}^\dagger\\
&= a_1(\partial + u)^k Q_s - [a_1(\partial + u)^s Q_k]^\dagger \quad \text{[by (3.13)]}\\
&= a_1 \sum_\alpha \binom{k}{\alpha} Q_{k+s-\alpha}\partial^\alpha - \sum_\alpha \binom{s}{\alpha} (-\partial)^\alpha Q_{k+s-\alpha} a_1 ,
\end{aligned}$$

which is (3.11r).

Thus, the a_1-terms are taken care of. Now Fix $\gamma \in \mathbf{Z}_+$. For $w = w_\gamma$, the w-terms in the expressions (3.9), (3.10) contribute the following identity to be verified in order for the canonical property of the Miura map to be proven:

$$\begin{aligned}
&Q_k \partial \binom{s}{\gamma+1} D(Q_{s-1-\gamma})^\dagger w + w \binom{k}{\gamma+1} D(Q_{k-1-\gamma})\partial Q_s\\
&+ \sum_{i+j=\gamma+\mu} \binom{k}{i+1}\binom{s}{j+1} Q_{k-1-i} \Bigg[\binom{i+1}{\mu+1} w\partial^{\mu+1}\\
&- \binom{j+1}{\mu+1} (-\partial)^{\mu+1} w \Bigg] Q_{s-1-j}\\
&= \sum_\nu \binom{k+s-\nu}{\gamma+1} \left[\binom{k}{\nu} Q_{k+s-1-\gamma-\nu} w \partial^\nu - \binom{s}{\nu} (-\partial)^\nu Q_{k+s-1-\gamma-\nu} w \right] .
\end{aligned} \tag{3.14}$$

The next section is devoted to a proof of this identity.

4. The w – Part

We consider firstly the case when $s = 0$. The identity (3.14) reduces in this case to the identity

$$w \binom{k}{\gamma+1} D(Q_{k-1-\gamma})\partial = \sum_{\nu>0} \binom{k-\nu}{\gamma+1}\binom{k}{\nu} w Q_{k-1-\gamma-\nu}\partial^\nu . \tag{4.1}$$

For the left-hand-side of (4.1) we have:

$$w \binom{k}{\gamma+1} D(Q_{k-1-\gamma})\partial \quad \text{[by (3.12)]}$$

$$\begin{aligned}
&= w \binom{k}{\gamma+1} [(\partial + u)^{k-1-\gamma} - Q_{k-1-\gamma}] \quad \text{[by (2.12)]}\\
&= w \binom{k}{\gamma+1} \sum_{\nu>0} \binom{k-1-\gamma}{\nu} Q_{k-1-\gamma-\nu}\partial^\nu ,
\end{aligned} \tag{4.2}$$

which is the same as the right-hand-side of (4.1) provided we use the identity

$$\binom{k}{\gamma+1}\binom{k-\gamma-1}{\nu} = \binom{k-\nu}{\gamma+1}\binom{k}{\nu} . \tag{4.3}$$

Thus, (3.14) is true for $s = 0$. Similarly, it is true for $k = 0$, which amounts to the adjoint of {formula (4.1) with s replacing k}. From now on, we consider the case

$$k > 0, \quad s > 0 \tag{4.4}$$

of the identity (3.14) to be verified. The first two terms on the left-hand-side of (3.14) we transform as follows:

$$w\binom{k}{\gamma+1} D(Q_{k-1-\gamma})\partial Q_s - \left[w\binom{s}{\gamma+1} D(Q_{s-1-\gamma})\partial Q_k\right]^{\dagger} \text{[by(3.12)]}$$

$$= w\binom{k}{\gamma+1} \left[(\partial+u)^{k-\gamma-1} - Q_{k-\gamma-1}\right] Q_s$$

$$- \left\{w\binom{s}{\gamma+1} \left[(\partial+u)^{s-\gamma-1} - Q_{s-\gamma-1}\right] Q_k\right\}^{\dagger} \text{[by(3.13)]}$$

$$= w\binom{k}{\gamma+1}\left[\sum_{\nu}\binom{k-\gamma-1}{\nu} Q_{k-\gamma-1-\nu+s}\partial^{\nu} - Q_{k-\gamma-1}Q_s\right]$$

$$- \binom{s}{\gamma+1}\left[\sum_{\nu}\binom{s-\gamma-1}{\nu}(-\partial)^{\nu} Q_{s-\gamma-1-\nu+k} - Q_{s-\gamma-1}Q_k\right] w \quad \text{[by (4.3)]}$$

$$= \sum_{\nu}\left[\binom{k-\nu}{\gamma+1}\binom{k}{\nu} w Q_{k+s-1-\gamma-\nu}\partial^{\nu} - \binom{s-\nu}{\gamma+1}\binom{s}{\nu}(-\partial)^{\nu} Q_{k+s-1-\gamma-\nu} w\right] \tag{4.5a}$$

$$+ w\left[\binom{s}{\gamma+1} Q_k Q_{s-\gamma-1} - \binom{k}{\gamma+1} Q_s Q_{k-\gamma-1}\right] . \tag{4.5b}$$

We now transform the remaining third summand in the left-hand-side of (3.14):

$$\sum_{i+j=\gamma+\nu-1}\binom{k}{i+1}\binom{s}{j+1} Q_{k-1-i}\left[\binom{i+1}{\nu} w\partial^{\nu} - \binom{j+1}{\nu}(-\partial)^{\nu} w\right] Q_{s-1-j}$$

$$= \sum_{\substack{\alpha+\beta=\gamma+\nu+1 \\ \alpha,\beta>0}}\binom{k}{\alpha}\binom{s}{\beta} Q_{k-\alpha}\left[\binom{\alpha}{\nu} w\partial^{\nu} - \binom{\beta}{\nu}(-\partial)^{\nu} w\right] Q_{s-\beta}$$

$$= \sum_{\alpha+\beta=\gamma+\nu+1}\binom{k}{\alpha}\binom{s}{\beta} Q_{k-\alpha}\left[\binom{\alpha}{\nu} w\partial^{\nu} - \binom{\beta}{\nu}(-\partial)^{\nu} w\right] Q_{s-\beta} \tag{4.6a}$$

$$- \sum\binom{s}{\gamma+\nu+1} Q_k\left[w\delta_0^{\nu} - \binom{\gamma+\nu+1}{\nu}(-\partial)^{\nu} w\right] Q_{s-\gamma-\nu-1} \tag{4.6b}$$

$$- \sum\binom{k}{\gamma+\nu+1} Q_{k-\gamma-\nu-1}\left[\binom{\gamma+\nu+1}{\nu} w\partial^{\nu} - w\delta_0^{\nu}\right] Q_s . \tag{4.6c}$$

The first term in (4.6b) and the second term in (4.6c) together cancel out the expression (4.5b). By using the identity

$$\binom{k}{\gamma+\nu+1}\binom{\gamma+\nu+1}{\nu}=\binom{k}{\gamma+1}\binom{k-\gamma-1}{\nu}, \tag{4.7}$$

the remaining first term in (4.6c) can be transformed as

$$-\binom{k}{\gamma+1}w\sum_\nu\binom{k-\gamma-1}{\nu}Q_{k-\gamma-1-\nu}\partial^\nu Q_s \quad \text{[by (2.12), (3.13)]}$$

$$=-\binom{k}{\gamma+1}w\sum_\nu\binom{k-\gamma-1}{\nu}Q_{k+s-\gamma-1-\nu}\partial^\nu \quad \text{[by (4.3)]}$$

$$=-\sum_\nu\binom{k-\nu}{\gamma+1}\binom{k}{\nu}wQ_{k+s-\gamma-1-\nu}\partial^\nu, \tag{4.8a}$$

while the remaining second term in (4.6b) is minus adjoint of {(4.8a) with the indices k and s interchanged}:

$$\sum_\nu\binom{s+\nu}{\gamma+1}\binom{s}{\nu}(-\partial)^\nu Q_{k+s-\gamma-1-\nu}w. \tag{4.8b}$$

But (4.8a, b) cancel out (4.5a). Thus, only the expression (4.6a) is left from the left-hand-side of the sought after identity (3.14), which now simplifies to

$$\sum_{\alpha+\beta=\gamma+\nu+1}\binom{k}{\alpha}\binom{s}{\beta}Q_{k-\alpha}\left[\binom{\alpha}{\nu}w\partial^\nu-\binom{\beta}{\nu}(-\partial)^\nu w\right]Q_{s-\beta}$$

$$=\sum_\nu\binom{k+s-\nu}{\gamma+1}\left[\binom{k}{\nu}Q_{k+s-1-\gamma-\nu}w\partial^\nu-\binom{s}{\nu}(-\partial)^\nu Q_{k+s-1-\gamma-\nu}w\right]. \tag{4.9}$$

Recall that k,s,γ are fixed. The identity (4.9) follows from the following slightly more general identity:

$$\sum_{\alpha+\beta=\nu+\gamma}\binom{k}{\alpha}\binom{s}{\beta}Q_{k-\alpha}\left[\binom{\alpha}{\nu}w\partial^\nu-\binom{\beta}{\nu}(-\partial)^\nu w\right]Q_{s-\beta}$$

$$=\sum_\nu\binom{k+s-\nu}{\gamma}\left[\binom{k}{\nu}Q_{k+s-\gamma-\nu}w\partial^\nu-\binom{s}{\nu}(-\partial)^\nu Q_{k+s-\gamma-\nu}w\right], \tag{4.10}$$

which, in turn, results when the identity

$$\sum_{\alpha+\beta=\nu+\gamma}\binom{k}{\alpha}\binom{s}{\beta}Q_{k-\alpha}\binom{\alpha}{\nu}\partial^\nu Q_{s-\beta} \tag{4.11l}$$

$$=\sum_\nu\binom{k+s-\nu}{\gamma}\binom{k}{\nu}Q_{k+s-\gamma-\nu}\partial^\nu \tag{4.11r}$$

is subtracted from {its adjoint, with the indices k and s interchanged }. We transform (4.11l) as follows:

$$\sum_{\alpha+\beta=\nu+\gamma} \binom{k}{\alpha}\binom{s}{\beta}\binom{\alpha}{\nu} Q_{k-\alpha}\partial^\nu Q_{s-\beta}$$

$$= \sum_{\alpha+\beta=k+s-\nu-\gamma} \binom{k}{\alpha}\binom{s}{\beta}\binom{k-\alpha}{\nu} Q_\alpha \partial^\nu Q_\beta$$

$$= \sum_{\beta,\nu} \binom{s}{\beta}\binom{k}{\gamma+\nu+\beta-s}\binom{\gamma+\nu+\beta-s}{\nu} Q_{k+s-\nu-\gamma-\beta}\partial^\nu Q_\beta \quad \text{[by (4.7)]}$$

$$= \sum_{\beta} \binom{s}{\beta}\binom{k}{\gamma+\beta-s}\sum_{\nu}\binom{k+s-\beta-\gamma}{\nu} Q_{k+s-\beta-\gamma-\nu}\partial^\nu Q_\beta \quad \text{[by (3.13)]}$$

$$= \sum_{\beta} \binom{s}{\beta}\binom{k}{\gamma+\beta-s}\sum_{\nu}\binom{k+s-\beta-\gamma}{\nu} Q_{k+s-\gamma-\nu}\partial^\nu$$

$$= \sum_{\nu} Q_{k+s-\gamma-\nu}\partial^\nu \sum_{\beta} \binom{s}{\beta}\binom{k}{\gamma+\beta-s}\binom{k+s-\gamma-\beta}{\nu},$$

and comparing this expression to the expression (4.11r) we see that we are left with checking the combinatorial identity

$$\binom{k+s-\nu}{\gamma}\binom{k}{\nu} = \sum_{\beta}\binom{s}{\beta}\binom{k}{\gamma+\beta-s}\binom{k+s-\gamma-\beta}{\nu}. \tag{4.12}$$

Since

$$\binom{k}{\alpha}\binom{k-\alpha}{\nu} = \binom{k-\nu}{\alpha}\binom{k}{\nu}, \tag{4.13}$$

the right-hand-side of (4.12) becomes

$$\sum_{\beta}\binom{s}{\beta}\binom{k-\nu}{\gamma+\beta-s}\binom{k}{\nu},$$

so that the identity (4.12) finally reduces to the identity

$$\binom{k+s-\nu}{\gamma} = \sum_{\beta}\binom{s}{\beta}\binom{k-\nu}{\gamma+\beta-s}, \tag{4.14}$$

which in turn, results upon picking out the t^γ-coefficients in the equality

$$(1+t)^{k+s-\nu} = (1+t)^s(1+t)^{k-\nu}. \tag{4.15}$$

The canonical property of the Miura map is proven.

5. The Extended KP Hierarchy

In this section we use the canonical property of the mKP $\to$ KP Miura map to construct the eKP hierarchy, and to show that it is a Hamiltonian system isomorphic to the mKP hierarchy.

The n^{th} flow of the eKP hierarchy consists of the n^{th} flow (2.2) of the KP hierarchy and an evolution equation for one extra variable u:

$$u_t = \partial e^{\int u}[(L^n)^\dagger]_+ e^{-\int u}(1) . \tag{5.1}$$

We first transform (5.1) into a more manageable form. By formulae (2.6), (2.7),

$$[(L^n)^\dagger]_{\geqq 0} = \sum_{s\geqq 0} p_s(n)(-1)^s \xi^s = \sum_{s\geqq 0} (-1)^s \frac{\delta H_{n+1}}{\delta A_s} \xi^s ,$$

and since

$$e^{\int u}\partial^s e^{-\int u}(1) = Q_s(-u) , \tag{5.2}$$

(5.1) becomes

$$u_t = \partial \left(\sum_s (-1)^s Q_s(-u) \frac{\delta H_{n+1}}{\delta A_s} \right) . \tag{5.3}$$

We see that the eKP hierarchy can be written in the form

$$\begin{cases} u_t = \sum_s B_{uA_s}(\delta H/\delta A_s) , \\ A_{k,t} = \sum_s B^{\mathrm{KP}}_{ks}(\delta H/\delta A_s) , \end{cases} \tag{5.4}$$

with $H = H_{n+1}$ for the flow #n.

Theorem 5.5. (i)*The matrix* B^{eKP} *entering the system* (5.4):

$$B^{\mathrm{eKP}} = \begin{array}{c|cc} & u & A_s \\ u & 0 & \partial(-1)^s Q_s(-u) \\ A_k & (-1)^k Q_k(-u)\partial & B^{\mathrm{KP}}_{ks} \end{array} , \tag{5.6}$$

is Hamiltonian; (ii) *The Miura map*

$$\Phi(u) = a_0 , \tag{5.7 a}$$

$$\Phi(A_i) = (-1)^i a_1 Q_i(-a_0) + \sum_{\alpha=0}^{i-1} \binom{i}{\alpha+1} (-1)^{i+1+\alpha} w_\alpha Q_{i-1-\alpha}(-a_0) , \tag{5.7 b}$$

is canonical between the Hamiltonian structures B^{mKP} (2.10) *and* B^{eKP} (5.6); (iii) *The Miura map* (5.7) *is invertible*; (iv) *Under the invertible Miura map* (5.7), *the* n^{th} *flow of the eKP hierarchy transforms into the* n^{th} *flow of the mKP hierarchy.*

Proof. (iii) We have, from (5.7a) and (5.7b) with $i = 0$:

$$\Phi(u) = a_0, \quad \Phi(A_0) = a_1 . \tag{5.8}$$

For $i > 1$, the remaining system (5.7b), considered as a system of equations on the $w'_\alpha s$, is lower-triangular with the 1's on the diagonal. Hence, the Miura Φ map is invertible. In fact, we can write down an explicit formula for the inverse map Φ^{-1}. Indeed, recalling that the formula (5.7b) was obtained in Sect. 2 as formula (2.11), we can reverse the map Φ as follows:

$$a_0 = u ,$$

and then

$$L = \xi + \sum A_i \xi^{-i-1} \mapsto \mathscr{L} = e^{-\int a_0} L e^{\int a_0} = L|_{\xi \mapsto \xi + a_0}$$

$$= \xi + u + \sum_{i \geqq} A_i (\xi + u)^{-i-1} \quad \text{[by(2.12)]}$$

$$= \xi + u + \sum_{i,\alpha} A_i \binom{-i-1}{\alpha} Q_\alpha(u) \xi^{-i-1-\alpha} = \xi + \sum_{j \geqq 0} a_j \xi^{-j} ,$$

so that

$$\Phi^{-1}(a_0) = u ,$$

$$\Phi^{-1}(a_{j+1}) = \sum_{i+\alpha=j} \binom{-i-1}{\alpha} A_i Q_\alpha(u) ; \tag{5.9}$$

(ii) Since

$$\frac{D\Phi(u)}{Da_i} = \delta_i^0 , \tag{5.10}$$

in checking the canonical property of the Miura map in terms of the identity to be verified:

$$\Phi(B^{\mathrm{eKP}}) = JB^{\mathrm{mKP}} J^\dagger , \tag{5.11}$$

we have to check only the equality of the entries in the u-row of the matrix in the left-hand-side of (5.11): the u-column will follow from that by taking the adjoint, and the remaining submatrix B^{KP} of the matrix B^{eKP} has been checked before (the check being equivalent to the canonical property of the mKP $\to$ KP Miura map). Now, from (5.10) and (2.10) we obtain:

$$(JB^{\mathrm{mKP}})_{u(\,\cdot\,)} = \begin{array}{cc} & \begin{array}{ccc} a_0 & a_1 & w_i \end{array} \\ u & \left(\begin{array}{c|c|c} 0 & \partial & 0 \end{array} \right) \end{array} , \tag{5.12}$$

while, by (5.7b),

$$(J^\dagger) = \begin{array}{cc} & \begin{array}{cc} u & A_s \end{array} \\ \begin{array}{c} a_0 \\ a_1 \\ w_i \end{array} & \begin{pmatrix} 1 & * \\ 0 & (-1)^s Q_s(-a_0) \\ 0 & * \end{pmatrix} \end{array} , \tag{5.13}$$

where the entries marked by "$*$" are inconsequential. Multiplying (5.12) and (5.13) we get

$$(JB^{\mathrm{mKP}} J^\dagger)_{u(\,\cdot\,)} = \begin{array}{cc} & \begin{array}{cc} u & A_s \end{array} \\ u & \left(\begin{array}{cc} 0 & \partial(-1)^s Q_s(-a_0) \end{array} \right) \end{array} ,$$

which is precisely the u-row of the matrix B^{eKP} (5.6) when a_0 is identified with $\Phi(u)$;

(i) Since the matrix B^{eKP} (5.6) is connected with the Hamiltonian matrix B^{mKP} (2.10) by an invertible canonical map, the matrix B^{eKP} is also Hamiltonian; (iv) The n^{th} eKP flow is generated by the Hamiltonian matrix B^{eKP} (5.6) and the Hamiltonian (2.6a)

$$H_{n+1} = \mathrm{Res}(L^{n+1})/(n+1) . \tag{5.14a}$$

The n^{th} mKP flow is generated by the Hamiltonian matrix B^{mKP} (2.10) and the Hamiltonian (2.6a)

$$\mathscr{H}_{n+1} = \text{Res}(\mathscr{L}^{n+1})/(n+1)\,. \tag{5.14b}$$

By (ii), the Hamiltonian matrices are connected by the Miura homomorphism Φ (5.7). But the Hamiltonians H_{n+1} and $\mathscr{H}_{n+1}$ are also Φ-related:

$$(n+1)\Phi(H_{n+1}) = \Phi[\text{Res}L^{n+1})] = \text{Res}[\Phi(L^{n+1})]$$

$$= \text{Res}[(e^{-\int a_0} L e^{\int a_0})^{n+1}] = \text{Res}(\mathscr{L}^{n+1}) = (n+1)\mathscr{H}_{n+1}\,.$$

Thus, the KP hierarchy is a subhierarchy of the mKP hierarchy, with the complement of the projection mKP $\rightarrow$ KP being the single scalar field a_0. This is purely infinite-component effect: in the familiar single-component mKdV $\rightarrow$ KdV situation, the KdV equation is also a projection from the mKdV one [12], but the Kernel is not another field.

6. Discrete MKP Hierarchy

Let $\Delta : K \rightarrow K$ be an automorphism of a commutative ring K. (The latter may be thought of as Fun($\mathbf{Z}$), with $(\Delta f)(n) = f(n+1)$ for $f \in K$.) Let $C_A = K[A_i^{(s)}], i \in \mathbf{Z}_+, s \in \mathbf{Z}$, be the free polynomical ring with the action of the automorphism Δ extended to C_A via the rule

$$\Delta(A_i^{(s)}) = A_i^{(s+1)}\,.$$

Consider the ring $\tilde{C}_A = C_A((\zeta^{-1}))$, with the commutation relations

$$\zeta^s c = \Delta^s(c)\zeta^s, \quad s \in \mathbf{Z}\,,$$

and let $L \in \tilde{C}_A$ be the following Lax operator:

$$L = \zeta + \sum_{i=0}^{\infty} A_i \zeta^{-i}\,. \tag{6.1}$$

The n^{th} flow in the discrete KP hierarchy has the form

$$L_t = [(L^n)_{\geqq 0}, L] = [L, (L^n)_{<0}]\,. \tag{6.2}$$

So far everything is very similar to the continuous case. Define

$$\text{Res}\left(\sum_s \theta_s \zeta^s\right) = \theta_0\,, \tag{6.3}$$

(not θ_{-1} !) and let us write $c_1 \sim c_2$ when $(c_1 - c_2) \in \text{Im}(\Delta - 1)$. The Residue formula (2.5) is still valid [6]. Denote

$$L^n = \sum_s \zeta^s p_s(n)\,, \tag{6.4}$$

and set

$$H_n = \text{Res}(L^n)/n, \; n \in \mathbf{N}\,. \tag{6.5}$$

Then the Residue formula yields

$$d(H_{n+1}) \sim \mathrm{Res}(dL \cdot L^n) = \mathrm{Res}\left(\sum dA_i\zeta^{-i}\zeta^s p_s(n)\right) = \sum dA_i p_i(n)\,,$$

whence

$$p_i(n) = \frac{\delta H_{n+1}}{\delta A_i}\,. \tag{6.6}$$

Substituting (6.6) into (6.2) we obtain the (first) Hamiltonian form of the discrete KP hierarchy [13]:

$$A_{i,t} = \sum_j B_{ij}^{\mathrm{KP}}(\delta H/\delta A_j), \quad H = H_{n+1}\,,$$

where

$$B_{ij}^{\mathrm{KP}} = \Delta^j A_{i+j} - A_{i+j}\Delta^{-i}, \quad i,j \in \mathbf{Z}_+\,. \tag{6.7}$$

We now describe the discrete mKP hierarchy, which is a discrete version of the $k = 1$ – nonstandard (continuous) integrable systems from [10]. Let $C_{Q,a} = K[Q^{(s)}, a_i^{(s)}]$, and pick the following Lax operator $\mathscr{L} \in \tilde{C}_{Q,a} = C_{Q,a}((\zeta^{-1}))$:

$$\mathscr{L} = Q\zeta + \sum_{i=0}^{\infty} a_i\zeta^{-i}\,. \tag{6.8}$$

The discrete mKP hierarchy has the n^{th} flow

$$\mathscr{L}_t = [(\mathscr{L}^n)_{\geqq 1}, \mathscr{L}] = [\mathscr{L}, (\mathscr{L}^n)_{\leqq 0}]\,. \tag{6.9}$$

From the second equality in (6.9) we see that the flows are well defined. Since the flow is in the Lax form, it has an infinite number of integrals

$$\mathscr{H}_n = \mathrm{Res}(\mathscr{L}^n)/n, \quad n \in \mathbf{N}\,. \tag{6.10}$$

The commutativity of the discrete mKP flows doesn't follow from the general theory [13, 16], but follows instead from the Hamiltonian formalism derived below. (Alternatively, one can modify the arguments in [6] to provide a purely algebraic proof of commutativity, bypassing the Hamiltonian formalism.)

Set

$$\mathscr{L}^n = \sum_s \zeta^s \pi_s(n)\,. \tag{6.11}$$

The Residue formula yields

$$\begin{aligned} d(\mathscr{H}_{n+1}) \sim \mathrm{Res}(d\mathscr{L} \cdot \mathscr{L}^n) &= \mathrm{Res}\left[\left(dQ\zeta + \sum da_i\zeta^{-i}\right)\sum \zeta^s\pi_s(n)\right] \\ &= dQ\pi_{-1}(n) + \sum da_i\pi_i(n)\,, \end{aligned}$$

so that

$$\pi_{-1}(n) = \frac{\delta\mathscr{H}_{n+1}}{\delta Q}, \quad \pi_0(n) = \frac{\delta\mathscr{H}_{n+1}}{\delta a_0}\,, \tag{6.12}$$

$$\pi_{i+1}(n) = \frac{\delta\mathscr{H}_{n+1}}{\delta a_{i+1}}, \quad i \in \mathbf{Z}_+\,. \tag{6.13}$$

Substituting formula (6.12) into the second equality in (6.9) we obtain

$$\begin{aligned} (\mathscr{L}_t)_{\geqq 0} &= Q_t\zeta + a_{0,t} = [\mathscr{L}, (\mathscr{L}^n)_{\leqq 0}]_{\geqq 0} \\ &= [Q\zeta + a_0 + \ldots, \pi_0(n) + \zeta^{-1}\pi_{-1}(n) + \ \ldots]_{\geqq 0} = [Q\zeta, \pi_0(n) + \zeta^{-1}\pi_{-1}(n)] \\ &= [Q(\Delta - 1)(\pi_0(n))]\zeta + (1 - \Delta^{-1})[Q\pi_{-1}(n)]\,, \end{aligned}$$

so that

$$\begin{pmatrix} Q_t \\ a_{0,t} \end{pmatrix} = \begin{pmatrix} 0 & Q(\Delta - 1) \\ (1 - \Delta^{-1})Q & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta Q \\ \delta H/\delta_{a_0} \end{pmatrix} , \tag{6.14}$$

with $H = \mathscr{H}_{n+1}$. The matrix entering the right-hand-side of (6.14) is Hamiltonian since in the variables $(q = \ln Q, a_0)$ it becomes

$$\begin{pmatrix} 0 & \Delta - 1 \\ 1 - \Delta^{-1} & 0 \end{pmatrix} ,$$

which is skewsymmetric constant-coefficient. Similarly, the first equality in (6.9) yields

$$(\mathscr{L}_t)_{<0} = \sum_{i \geqq 0} a_{i+1,t} \zeta^{-i-1} = [(\mathscr{L}^n)_{\geqq 1}, \mathscr{L}]_{<0}$$

$$= \left[\sum_{s \geqq 0} \zeta^{s+1} \pi_{s+1}(n), Q\zeta + a_0 + \sum_{k \geqq 0} a_{k+1} \zeta^{-k-1} \right]_{\leqq -1}$$

$$= \left[\sum_{s \geqq 0} \zeta^{s+1} \pi_{s+1}(n), \sum_{k \geqq 0} a_{k+2} \zeta^{-k-2} \right]_{\leqq -1}$$

$$= \sum_{s,k} \left([\pi_{s+1}(n) a_{k+2}]^{(s+1)} \zeta^{s-k-1} - a_{k+2} [\pi_{s+1}(n)]^{(s-k-1)} \zeta^{s-k-1} \right)_{\leqq -1} ,$$

and using formulae (6.13) we obtain

$$a_{i+1,t} = \sum_{s \geqq 0} (\Delta^{s+1} a_{i+s+2} - a_{i+s+2} \Delta^{-i-1})(\delta H/\delta a_{s+1}) . \tag{6.15}$$

We see that the variables Q, a_0 split from the rest. Denoting

$$w_i = a_{i+1} , \tag{6.16}$$

the form (6.15) becomes

$$w_{i,t} = \sum_{j \geqq 0} (\Delta^{j+1} w_{i+j+1} - w_{i+j+1} \Delta^{-i-1})(\delta H/\delta w_j) . \tag{6.17}$$

To show that (6.17) is a Hamiltonioan form, set, for any $r \in \mathbf{Z}_+$,

$$B^r_{ij} = \Delta^{j+r} w_{i+j+r} - w_{i+j+r} \Delta^{-i-r} . \tag{6.18}$$

For $r = 1$, the matrix B^1 enters formula (6.17); for $r = 0$, the matrix B^0 is the Hamiltonian matrix B^{KP} (6.7). Since the matrix B^r is linear in its arguments w's, to show that it's Hamiltonian we have to exhibit the corresponding Lie algebra [6]. So,

$$\sum_{ij} X_i B^r_{ij}(Y_j) = \sum X_i (\Delta^{j+r} w_{i+j+r} - w_{i+j+r} \Delta^{-i-r})(Y_j)$$

$$\sim \sum_{ij} w_{i+j+r} [X_i^{(-j-r)} Y_j - X_i \Delta^{-i-r}(Y_j)] .$$

Thus,

$$[\mathbf{X}, \mathbf{Y}]_k = \sum_{i+j=k-r} [Y_j \Delta^{-(j+r)}(X_i) - X_i \Delta^{-i-r}(Y_j)] ,$$

which is the commutator on the Lie algebra formed from the associative algebra of discrete operators of the form $\{X = \sum_{i \geqq 0} X_i \zeta^{-i-r}\}$. Thus, the discrete mKP hierarchy is Hamiltonian, with the corresponding Hamiltonian matrix (6.14), (6.15):

$$B^{\mathrm{mKP}} = \begin{array}{c|c} & \begin{array}{ccc} Q & a_0 & a_{j>0} \end{array} \\ \begin{array}{c} Q \\ a_0 \\ a_{i>0} \end{array} & \begin{pmatrix} 0 & Q(\Delta - 1) & 0 \\ (1 - \Delta^{-1})Q & 0 & 0 \\ 0 & 0 & \Delta^j a_{i+j} - a_{i+j}\Delta^{-i} \end{pmatrix} \end{array} . \tag{6.19}$$

7. Discrete Miura Map

In this section we first construct a map from the discrete mKP hierarchy into the discrete KP one. Then this map will be shown to be canonical .

Deriving formula (6.14) we saw that

$$Q_t = Q(\Delta - 1)(\pi_0(n)) , \tag{7.1}$$

which implies that

$$(\ln Q)_t = (\Delta - 1)(\pi_0(n)) . \tag{7.2}$$

Consider a ring $C_{w,a}$ and the homomorphism $h : C_{Q,a} \to C_{w,a}$, which is identical on the a's and acts on Q as

$$h(Q) = e^{(\Delta-1)(w)} = e^{w^{(1)} - w} = e^{w^{(1)}}/e^w , \tag{7.3}$$

so that

$$h(\ln Q) = (\Delta - 1)(w) . \tag{7.4}$$

We can lift up the n^{th} mKP flow from the ring $C_{Q,a}$ into the ring $C_{w,a}$ by the rule

$$w_t = h(\pi_0(n)) . \tag{7.5}$$

Formulae (7.2) and (7.4) then show that the w-version of the n^{th} mKP flow is h-related to the n^{th} mKP flow itself. Now consider the map

$$\begin{aligned} \Phi^* : \mathscr{L} = Q\zeta + \textstyle\sum a_i \zeta^{-i} \mapsto L &= \zeta + \textstyle\sum A_i \zeta^{-i} = e^w \mathscr{L} e^{-w} \\ &= e^w Q \zeta e^{-w} + \textstyle\sum a_i e^w \zeta^{-i} e^{-w} . \end{aligned} \tag{7.6}$$

By formula (7.3),

$$e^w Q \zeta e^{-w} = \zeta , \tag{7.7}$$

and

$$e^w \zeta^{-i} e^{-w} = R_i \zeta^{-i} , \tag{7.8}$$

where $R_0 = 1$ and, for $i > 0$,

$$\begin{aligned} R_i = R_i(Q) &= e^{w - w^{(-i)}} = e^{w - w^{(-1)}} e^{w^{(-1)} - w^{(-2)}} \dots e^{w^{-(i-1)} - w^{(-i)}} \\ &= Q^{(-1)} \dots Q^{(-i)} , \end{aligned}$$

so that, for all i,

$$R_i = \sqcap_{s=0}^{i} Q^{(-s)}/Q\,. \tag{7.9}$$

Thus, the Miura map (7.6) has the form

$$\Phi(A_i) = R_i(Q)a_i, \quad i \in \mathbf{Z}_+\,. \tag{7.10}$$

The argument involving w can be now erased, if desired, – the final product is the Miura homomorphism $Q : C_A \rightarrow C_{Q,a}$ (7.10), free from w-attributes.

We are now going to show that this map is canonical between the Hamiltonian structures B^{KP} (6.7) and B^{mKP} (6.19). For this, we have to verify the criterion of canonicity

$$JB^{\mathrm{mKP}}J^{\dagger} = \Phi(B^{\mathrm{KP}})\,, \tag{7.11}$$

where the Fréchet Jacobian J is computed from the Miura map Φ (7.10):

$$\begin{array}{cccc} & Q & a_0 & a_{i>0} \\ J = \Phi(A_n)\,(& a_nD(R_n)| & R_n\delta_n^0| & R_n\delta_n^i\;)\,, \end{array} \tag{7.12}$$

$$J^{\dagger} = \begin{array}{c} \\ Q \\ a_0 \\ a_{j>0} \end{array} \begin{array}{c} \Phi(A_m) \\ \begin{pmatrix} D(R_m)^{\dagger}a_m \\ R_m\delta_m^0 \\ R_m\delta_m^j \end{pmatrix} \end{array}. \tag{7.13}$$

Multiplying through formulae (7.12), (6.19), and (7.13), for the left-hand-side of the equality (7.11) we obtain:

$$\begin{array}{cccc} & Q & a_0 & a_{j>0} \\ JB^{\mathrm{mKP}} = \Phi(A_n)\,(& \delta_n^0(1-\Delta^{-1})Q| & a_n\,D(R_n)Q(\Delta-1)| & R_n\,(\Delta^j a_{n+j} - a_{n+j}\Delta^{-n}), n>0\,)\,, \end{array}$$

$$(JB^{\mathrm{mKP}}J^{\dagger})_{nm} = \delta_n^0(1-\Delta^{-1})QD(R_m)^{\dagger}a_m \tag{7.14a}$$

$$+\,\delta_m^0 a_nD(R_n)Q(\Delta-1)R_m \tag{7.14b}$$

$$+\,R_n(\Delta^m a_{n+m} - a_{n+m}\Delta^{-n})R_m \;\; (n,m>0) \tag{7.14c}$$

This is to be compared with the right-hand-side of the criterion (7.11):

$$\Phi(B_{nm}^{\mathrm{KP}}) = \Delta^m R_{n+m}a_{n+m} - R_{n+m}a_{n+m}\Delta^{-n} \tag{7.15}$$

Let us compare the (n0)-entry in (7.14), which is (7.14b), with the (n0)-entry in (7.15): we have to show that

$$a_nD(R_n)Q(\Delta-1) = R_na_n(1-\Delta^{-n})\,, \tag{7.16}$$

which is the same as

$$D(R_n)Q(\Delta-1) = R_n(1-\Delta^{-n})\,, \tag{7.17}$$

which is equivalent to

$$D(R_n) = R_n(1 - \Delta^{-n})(\Delta - 1)^{-1}Q^{-1} . \tag{7.18}$$

Lemma 7.19.

$$D(R_n) = R_n \frac{1 - \Delta^{-n}}{\Delta - 1} \frac{1}{Q} . \tag{7.20}$$

Proof. For $n = 0$, $R_0 = 1$ by (7.9), so both sides of (7.20) vanish. Let now $n > 0$. Then, by (7.9),

$$\begin{aligned} D(R_n) &= D(Q^{(-1)} \dots Q^{(-n)}) = R_n \sum_{s=1}^{n} \tfrac{1}{Q^{(-s)}} D(Q^{(-s)}) \\ &= R_n \sum_{s=1}^{n} \frac{1}{Q^{(-s)}} \Delta^{-s} = R_n \sum_{s=1}^{n} \Delta^{-s} \frac{1}{Q} = R_n \Delta^{-1} \frac{1 - \Delta^{-n}}{1 - \Delta^{-1}} \frac{1}{Q} = R_n \frac{1 - \Delta^{-n}}{\Delta - 1} \frac{1}{Q} . \quad \blacksquare \end{aligned}$$

Thus, the A_0-columns in both sides of the criterion (7.11) coincide. Since both sides of (7.11) are skew-symmetric matrices, the A_0-rows coincide too. Thus, it remains to show that the expression (7.14c) (for $n, m > 0$) is the same as (7.15), which amounts to the identity

$$R_n \Delta^m a_{n+m} R_m = \Delta^m R_{n+m} a_{n+m} ,$$

which is equivalent to

$$R_n \Delta^m R_n = \Delta^m R_{n+m} , \tag{7.21}$$

which is equivalent to

$$R_n^{(-m)} R_m = R_{n+m} , \tag{7.22}$$

which can be seen as follows: by (7.9),

$$\begin{aligned} R_n^{(-m)} R_m &= R_m \Delta^{-m} (Q^{(-1)} \dots Q^{(-n)}) \\ &= Q^{(-1)} \dots Q^{(-m)} [Q^{(-m-1)}] \dots Q^{(-m-n)}] = Q^{(-1)} \dots Q^{(-m-n)} = R_{n+m} . \end{aligned}$$

Thus, the Miura map is canonical .

8. The Discrete eKP Hierarchy

In this section we derive a decomposition which is similar to the continuous case: mKP $\approx$ KP + $\{u\}$.

The extended discrete KP hierarchy has, as its n^{th} flow, Eq. (6.2) and the equation

$$u_t = u(\Delta - 1)\mathrm{Res}(L^n) . \tag{8.1}$$

By formula (6.6), this can be rewritten as

$$\dot{u} = u(\Delta - 1)(\delta H_{n+1}/\delta A_0) , \tag{8.2}$$

so that the whole discrete eKP hierarchy is governed by the (suspected to be Hamiltonian) matrix

$$
\begin{array}{cc}
 & \begin{array}{cc} u & A_m \end{array} \\
B^{\mathrm{eKP}} = \begin{array}{c} u \\ A_n \end{array} & \begin{pmatrix} 0 & u(\Delta - 1)\delta^0_m \\ \delta^0_n(1-\Delta^{-1})u & \Delta^m A_{n+m} - A_{n+m}\Delta^{-n} \end{pmatrix}
\end{array} \tag{8.3}
$$

Similar to the continuous Theorem 5.5, we have

Theorem 8.4. (i) *The matrix* B^{eKP} (8.3) *is Hamiltonian*; (ii) *The Miura map* $\Phi : C_{u,A} \to C_{w,a}$,

$$\Phi(u) = Q, \quad \Phi(A_i) = a_i R_i(Q)\,, \tag{8.5}$$

is canonical between the Hamiltonian structures B^{eKP} (8.3) *and* B^{mKP} (6.19); (iii) *The Miura map* (8.5) *is invertible*; (iv) *Under this invertible Miura map the* n^{th} *flow of the eKP hierarchy transforms into the* n^{th} *flow of the mKP hierarchy.*

Proof. (iii) Formulae (8.5) are easily inverted:

$$\Phi^{-1}(Q) = u, \quad \Phi^{-1}(a_i) = A_i/R_i(u)\,; \tag{8.6}$$

(ii) We have to verify the criterion of canonicity

$$JB^{\mathrm{mKP}}J^{\dagger} = \Phi(B^{\mathrm{eKP}})\,, \tag{8.7}$$

where J is the Fréchet Jacobian of the map Φ (8.5). Since the discrete Miura map mKP $\to$ KP has been proven to be canonical in the previous section, we need to examine only the u-row in the matrix $\Phi(B^{\mathrm{eKP}})$, which is, by formula (8.3):

$$
\begin{array}{ccc}
 & \Phi(u) & \Phi(A_m) \\
\Phi(u) & (\,0\,| & Q(\Delta - 1)\delta^0_m\,)\;.
\end{array} \tag{8.8}
$$

On the other hand, the u-row of the left-hand-side of the relation (8.7) comes out of multiplying the matrices

$$
\begin{array}{cccc}
 & Q & a_0 & a_{j>0} \\
(JB^{\mathrm{mKP}})_{u(\,\cdot\,)} = & (\,0\,| & Q(\Delta-1)\,| & 0\,)
\end{array}
$$

and

$$
\begin{array}{cc}
 & \begin{array}{ccc} \Phi(u) & \Phi(A_0) & \Phi(A_{m>0}) \end{array} \\
J^{\dagger} = \begin{array}{c} Q \\ a_0 \\ a_{j>0} \end{array} & \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & * \end{pmatrix}
\end{array},
$$

so that

$$
\begin{array}{cccc}
 & \Phi(u) & \Phi(A_0) & \Phi(A_{m>0}) \\
(JB^{\mathrm{mKP}}J^{\dagger})_{u(\,\cdot\,)} = & (\,0\;\; | & Q(\Delta-1) & |\;\; 0\,)\,,
\end{array} \tag{8.9}
$$

which is the same as (8.8); (i) Therefore, the matrix B^{eKP} (8.3) is Hamiltonian, representing the same Hamiltonian structure as the Hamiltonian matrix B^{mKP} (6.19) but in different coordinates; (iv) The Hamiltonian structures of the flows #n in the mKP and eKP hierarchies respectively are related by the Homomorphism Φ (8.5). But their Hamiltonians are also Φ-related:

$$\begin{aligned}(n+1)\Phi(H_{n+1}) &= \Phi[\mathrm{Res}(L^{n+1})] = \mathrm{Res}[\Phi(L)^{n+1}] \quad [\text{by}(7.6)] \\ &= \mathrm{Res}[(e^{w}\mathscr{L}e^{-w})^{n+1}] = \mathrm{Res}(e^{w}\mathscr{L}^{n+1}e^{-w}) \\ &= \mathrm{Res}(\mathscr{L}^{n+1}) = (n+1)\mathscr{H}_{n+1} .\end{aligned}$$

Remark. 8.10. The basic variables in this paper have been considered as scalar fields, all mutually commuting. Most of the results can be extended into the vastly more general noncommutative case when the basic variables do not commute (e.g., being matrices). This is explained in [16].

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References

1. Kupershmidt, B.A. Wilson, G.: Modifying Lax Equations and the Second Hamiltonian Structure. Invent. Math. **62**, 403–436 (1981)
2. Dickey, L.A.: Soliton Equations And Hamiltonian Systems. Singapore: World Scientific 1991
3. Drinfel'd, V.G., Sokolov, V.V: Equations of KdV type and simple Lie algebras. Dokl. Akad. Nauk SSSR **258**, 11–16 (1981) [Sov. Math. Dokl. **23**, 457–462 (1981)]
4. Drinfel'd, V.G., Sokolov, V.V.: Lie algebras and the Korteweg-de Vries type equations. Itogi Nauki Tech., Ser. Sovr. Prob. Math. **24**, 81–180 (1984) [J. Sov. Math. **30**, 1975–2036 (1985)]
5. Wilson, G.: The modified Lax and two-dimensional Toda lattice equations associated with simple Lie algebras. Ergod. Th. Dynam. Syst. **1**, 361–380 (1981)
6. Kupershmidt, B.A.: Discrete Lax Equations And Differential – Difference Calculus. Paris: Asterisque 1985
7. Manin, Yu. I.: Algebraic aspects of non-linear differential equations. Itogi Nauki Tech., Ser. Sovr. Prob. Math. **11**, 5–152 (1978) [J. Sov. Math. **11** 1–122 (1979)]
8. Wilson, G.: On the antiplectic pair connected with the Adler–Gel'fand–Dikii bracket. Nonlinearity **5**, 109–131 (1992)
9. Wilson, G.: On two constructions of conservation laws for Lax equations. Quart. J. Math. Oxford (2) **34**, 491–512 (1981)
10. Kupershmidt, B.A.: Mathematics of dispersive water waves. Commun. Math. Phys. **99**, 51–73 (1985)
11. Wilson, G.: Commuting flows and conservation laws for Lax equations, Math. Proc. Cambr. Phil. Soc. **86**, 131–143 (1979)
12. Wilson, G.: On the quasi-Hamiltonian formalism of the KdV equation. Phys. Lett. **132A**, 445–450 (1988)
13. Kupershmidt, B.A.: On algebraic models of dynamical systems. Lett. Math. Phys. **6**, 85–89 (1982)
14. Watanabe, Y.: Hamiltonian structure of Sato's hierarchy of KP equations and a coadjoint orbit of a certain formal Lie group. Lett. Math. Phys. **7**, 99–106 (1983)
15. Kupershmidt, B.A.: Deformations of Integrable Systems. Proc. Roy. Irish Acad. **83** A No. 1, 45–74 (1983)
16. Kupershmidt, B.A.: KP or mKP (to appear)

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