Commun. Math. Phys. 167, 237 – 254 (1995)

Communications in
Mathematical
Physics

The Fractal Volume of the Two-Dimensional Invasion Percolation Cluster

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Received: 5 March 1993/in revised form: 10 December 1993

Abstract. We consider both invasion percolation and standard Bernoulli bond percolation on the Z^2 lattice. Denote by $\mathscr{V}$ and $\mathscr{C}$ the invasion cluster and the occupied cluster of the origin, respectively. Let $\mathscr{V}_n = \mathscr{V} \cap [-n,n]^2$, and

$$\pi_n = P_{p_c}(\mathscr{C} \cap \text{ the boundary of } [-n,n]^2 \neq \emptyset) .$$

Let $\varepsilon > 0$ be given. Here we show that, with a probability tending to 1,

$$n^{2-\varepsilon}\pi_n \leqq |\mathscr{V}_n| \leqq n^{2+\varepsilon}\pi_n .$$

Assuming the existence of an exponent $1/\rho$ for π_n, it can be seen that with probability tending to one

$$n^{2-1/\rho-\varepsilon} \leqq |\mathscr{V}_n| \leqq n^{2-1/\rho+\varepsilon} .$$

Moreover, by den Nijs' and Nienhuis et al's computations,

$$n^{1.8958389583\ldots-\varepsilon} = n^{1+\frac{43}{48}-\varepsilon} \leqq |\mathscr{V}_n| \leqq n^{1+\frac{43}{48}+\varepsilon} = n^{1.8958389583\ldots+\varepsilon}$$

with a probability tending to one. The result matches Wilkinson and Willemsen's numerical computation $\mathscr{V}_n \sim n^{1.89}$. The method allows us also to show the same argument for any planar graph. Therefore, any two planar invasion clusters have the same fractal dimension $2 - \frac{1}{\rho}$ if one believes "universality."

Furthermore, the escape time of the invasion cluster is considered in this paper. More precisely, denote by h_n the first time that the invasion cluster escapes from $[-n,n]^2$. We here can show that with a probability tending to one

$$n^{2-\varepsilon}\pi_n \leqq h_n \leqq n^{2+\varepsilon}\pi_n .$$

Finally, invasion percolation with trapping is considered in this paper. Denote by

$$\mathscr{R}_n = \{\text{the number of bonds trapped by the invasion cluster before time } n\} .$$

★ Supported in part by NSF Grant DMS 9400467

Here we show that with a probability tending to one

$$n^{2/(2-\alpha_n)-\varepsilon} \leqq |\mathcal{R}_n| \leqq n^{2/(2-\alpha_n)+\varepsilon},$$

where $\alpha_n = -\frac{\log \pi_n}{\log n}$. By assuming the existence of ρ and $\rho = \frac{48}{5}$ again, we can show that

$$n^{1.054945054945...+\varepsilon} = n^{2\rho/(2\rho-1)+\varepsilon} \leqq |\mathcal{R}_n| \leqq n^{2\rho/(2\rho-1)+\varepsilon} = n^{1.054945054945...+\varepsilon}$$

with a probability tending to one.

1. Introduction

Invasion percolation was introduced in de Gennes and Guyon [1], modified by Lenormand and Bories [2] and Chandler, Koplik, Lerman and Willemsen [3], Chayes and Chayes [4], Kesten [5] and Grimmett [6], and studied further by Wilkinson and Willemsen [7], Willemsen [8], Chayes, Chayes and Newman [9] and Chayes, Chayes and Newman [10]. The simple setup is as follows. Consider the bonds in the Z^d lattice. Let $\{X(e) : e \text{ is a bond in } Z^d\}$ be independent random variables, each having the uniform distribution on [0,1]. Let $\mathcal{P}$ be the corresponding product probability measure. More precisely, $\mathcal{P} = \prod_{e\in Z^d} \mu_e$, where μ_e is uniform measure on [0,1]. Expectation with respect to $\mathcal{P}$ is denoted by $\mathcal{E}$. We then construct a set sequence $\{S_i : i \geqq 0\}$ of random connected subgraphs of the lattice by means of the $\{X(e) : e \in Z^d\}$. The graph S_0 contains the origin and no bonds. Having defined S_i, we consider the bond boundary of S_i being the set of bonds not in S_i but incident to at least one vertex of S_i. We write ΔS_i for the bond boundary of S_i. We simply select the bond in ΔS_i with the smallest value, and add the bond to S_i. We then obtain a larger connected set S_{i+1} which is called the invasion cluster at time $i+1$. The invasion cluster is denoted by $\mathcal{V} = \bigcup_{i=0}^{\infty} S_i$. The original motivation was to describe the displacement of one fluid by another. Indeed, consider the methods (see [7]) which attempt to recover oil by pumping water into ground. In this model one assigns to each bond e a value $X(e) \geqq 0$. We think of e as a capillary, and $X(e)$ as the minimal pressure which the water must have to force the oil out of this capillary. If water is pumped in only at $\mathbf{0}$, then nothing happens until the pressure reaches

$$\min\{X(e) : e \text{ is incident to } \mathbf{0}\}.$$

Assume that there exists a unique bond e_1 incident to $\mathbf{0}$ for which the minimum above is taken on, and set $S_1 = e_1$. If the pressure is increased, oil is first forced out of S_1, and nothing else happens until the pressure reaches

$$\min\{X(e) : e \text{ is incident to } S_1\}.$$

After that the oil is forced out of a bond e_2 for which the minimum above is achieved. Inductively, we will obtain the invasion cluster which contains $\mathbf{0}$. The resulting model is called invasion percolation. Perhaps the most important question is to understand the geometry of the invasion cluster. A first step toward understanding the geometry of $\mathcal{V}$ is to estimate the density of $\mathcal{V}$. If we write

$$\mathcal{V}_n = \mathcal{V} \cap [-n,n]^d,$$

some numerical work by Wilkinson and Willemsen indicates that

$$|\mathscr{V}_n| \sim n^{1.89} \text{ for } d = 2\,, \tag{1}$$

$$|\mathscr{V}_n| \sim n^{2.52} \text{ for } d = 3\,, \tag{2}$$

where $|A|$ denotes the number of bonds of A and $a \sim b$ means that $\frac{\log a}{\log b}$ tends to 1 in the appropriate limit (as $n \to \infty$ in (1) and (2)) for some numbers a and b. After that, Chayes, Chayes and Newman (1985) proved rigorously that the invasion region has zero volume fraction with probability one, i.e.,

$$\lim_{n\to\infty} \frac{1}{n^d}|\mathscr{V}_n| = 0 \text{ a.s}\,.$$

for $d \geqq 2$, provided $\theta(p_c) = 0$ (see the definition below). However, based on the numerical work of [7], it is believed that (see [7, 9, and 5]) that the density of $\mathscr{V}$ in $B(n)$ behaves like n^c for some constant c as $n \to \infty$. The number c is sometimes referred to as the *fractal dimension* or the *Hausdorff dimension* of $\mathscr{V}$. In this paper we shall mainly discuss the fractal dimension of $\mathscr{V}$. In fact, we find that the fractal dimension of $\mathscr{V}$ is related to the critical exponents in percolation. Before stating our precise results, we need to introduce some basic knowledge of percolation and the power law hypothesis since our proofs entirely depend on them.

Consider standard (Bernoulli) bond percolation on Z^d, in which all bonds are independently occupied with probability p and vacant with probability $1 - p$, the corresponding probability measure and expectation on the configuration of occupied and vacant bonds are denoted by P_P and E_P respectively. The cluster of the vertex x, $\mathscr{C}(x)$, consists of all vertices which are connected to x by an occupied path on Z^d. An occupied path here is a nearest neighbor path on Z^d, whose bonds are occupied. The percolation probability is

$$\theta(p) = P_P(|\mathscr{C}(0)| = \infty)\,,$$

and the critical probability is

$$p_c = p_c(Z^d) = \sup\,\{p : \theta(p) = 0\}\,.$$

It is well known that $0 < p_c < 1$. For any two sets of vertices A and B we write $A \leftrightarrow B$ for the event that there exists an occupied path from some vertex in A to some vertex in B. We set

$$B(n) = [-n, n]^d\,,$$

and its boundary or surface is

$$\partial B(n) = \{x \in Z^d : \|x\| = n\}\,,$$

where

$$\|x\| := \max_{1 \leqq i \leqq d} |x_i| \text{ for } x = (x_1, \ldots x_d)\,.$$

Throughout this paper, C or C_i stands for a strictly positive finite constant which may depend on k, t and m but not n, whose value is of no significance to us. In fact the value of C or C_i may change from appearance to appearance. Besides the percolation probability, other important quantities are defined by:

$$\begin{aligned}\xi(p) &= \text{ correlation length}\\ &= \lim_{n\to\infty}\frac{1}{n}\log P_P(0\leftrightarrow\partial B(n), |\mathscr{C}(0)|<\infty) \text{ for } p\neq p_c\,; \end{aligned}\tag{3}$$

$$\pi(p,n) = P_P(0\leftrightarrow\partial B(n))\,. \tag{4}$$

Note that the existence of the limit in (3) follows simply from a subadditive argument. In particular, we denote by $\pi_n = \pi(p_c, n)$. With these definitions and notations, it is widely believed (see [11, 5 and 6]) that various quantities in percolation behave like powers of $|p-p_c|$ as p approaches the critical probability p_c. More precisely, the principal conjectures concerning above quantities are as follows:

$$\theta(p)\approx(p-p_c)^{\beta}\,, \tag{5}$$

$$\xi(p)\approx(p-p_c)^{-\nu}\,, \tag{6}$$

$$\pi_n\approx n^{\frac{-1}{\rho}} \tag{7}$$

for some constants β,ν and ρ which are called critical exponents, where $f(x)\approx g(x)$ means that $C_1g(x)\leqq f(x)\leqq C_2g(x)$ for all x. These conjectures are usually called the power law hypothesis in percolation. It has been proved rigorously (see [11]) for the Z^2 lattice that

$$(p-p_c)^{\delta_1}\leqq\theta(p)\leqq(p-p_c)^{\delta_2}\,, \tag{8}$$

$$(p-p_c)^{-\delta_3}\leqq\xi(p)\leqq(p-p_c)^{-\delta_4}\,, \tag{9}$$

$$n^{\frac{-1}{3}}\leqq\pi_n\leqq n^{-\delta_5}\,, \tag{10}$$

for some strictly positive constants $\delta_1,\delta_2,\delta_3,\delta_4$ and δ_5. In particular, $\delta_1<1$, and $\delta_3>1$ have been proved (see [12]). It also follows from (10) that

$$\rho\geqq 3\,, \tag{11}$$

provided (7) holds. However, as far as we know none of (5)–(7) has been proved for percolation. The computations in [13] and [14] indicate that

$$\beta=\frac{5}{36},\quad \nu=\frac{4}{3}\quad\text{and}\quad \rho=\frac{48}{5}\,. \tag{12}$$

Now we return to discuss the invasion percolation. With the knowledge of percolation, we find the fractal dimension of $\mathscr{V}$ is related to π_n. Furthermore, with the power law hypothesis, we can show that $\mathscr{V}$ has a fractal dimension $2-1/\rho$. More precisely, our results are as follows.

Theorem 1. *For $d=2$, any $\varepsilon>0$ and integer $m>0$, there exist constants C and $a>0$ such that*

$$\mathscr{P}(|\mathscr{V}_n|\leqq n^{2+\varepsilon}\pi_n)\geqq 1-Cn^{-m} \tag{13}$$

and

$$\mathscr{P}(|\mathscr{V}_n|\geqq n^{2-\varepsilon}\pi_n)\geqq 1-Cn^{-a}\,. \tag{14}$$

In particular, for any integer $t \geqq 1$, *there exists a constant* C *such that*

$$\mathscr{E}|\mathscr{V}_n|^t \geqq C(n^2\pi_n)^t \tag{15}$$

and

$$\mathscr{E}|\mathscr{V}_n|^t \leqq C(n^{2+\varepsilon}\pi_n)^t \,. \tag{16}$$

Unlike the general percolation model which is static, invasion percolation is a dynamical model. Therefore, time plays an important role in the model. It is interesting to consider "escape" time or "hitting" time problems. More precisely, we write

$$h_n = \min\{m : S_m \cap \partial B(n) \neq \emptyset\}$$

for the escape time from the box $B(n)$. In other words, the invasion cluster $\mathscr{V}$ has to use at least time h_n to occupy some bond outside of the box $B(n)$. With this definition, we have the following result.

Theorem 2. *For* $d = 2$ *and any* $\varepsilon > 0$ *there exist* $a > 0$ *and* C *such that*

$$\mathscr{P}(n^{2-\varepsilon}\pi_n \leqq h_n \leqq n^{(2+\varepsilon)}\pi_n) \geqq 1 - Cn^{-a} \,. \tag{17}$$

It is also important to take the phenomenon of trapping into account. A region $\mathscr{R}_n$ becomes trapped by S_n if $\mathscr{R}_n$ is separated from ∞ by S_n. More precisely,

$$R_n = \{e \in Z^2 \backslash S_n : \text{ any path from } e \text{ to } \infty \text{ has to use at least one vertex of } S_n\} \,.$$

Let $\mathscr{R} = \cup_n \mathscr{R}_n$. We can still consider $\mathscr{R}$ as the oil region trapped by water. Once $\mathscr{R}$ is trapped by water no oil from $\mathscr{R}_n$ can be displaced. In fact, $\mathscr{R}$ is one region of the phenomenon of "residual oil," a great economic problem in the oil industry. Thus it is natural to ask (see [5, 7 and 4]) what the volume fraction of the trapped region is. More precisely, what is

$$\lim_{n\to\infty} \frac{1}{d(2n)^d} |\mathscr{R} \cap [-n,n]^d| \tag{18}$$

and

$$\text{what is the behavior of } \mathscr{R}_n \text{ as } n \to \infty? \tag{19}$$

By a simple percolation argument (see proof of Theorem 3), every bond has to be either in S_n or trapped by S_n for large n if $d = 2$. On the other hand, $|\mathscr{V}_n| \ll n^2$ by the result in [9] or our Theorem 1. Hence we have the following theorem.

Theorem 3. *If* $d = 2$, *then we have*

$$\lim_{n\to} \frac{1}{2(2n)^2} |\mathscr{R} \cap [-n,n]^2| = 1 \ a.s. \,. \tag{20}$$

However, nothing is known for $d > 2$. In fact, it may not be very difficult to show that the corresponding limit in (20) is strictly less than one but the challenge is to find the exact limit for $d > 2$. Concerning the behavior of $\mathscr{R}_n$, we shall show its exact growth rate as follows.

Theorem 4. *For* $d = 2$ *and each* $\varepsilon > 0$ *there exist* $a > 0$ *and* C *such that*

$$\mathscr{P}(n^{(2-\varepsilon)/(2-\alpha_n)} \leqq |\mathscr{R}_n| \leqq n^{(2+\varepsilon)/(2-\alpha_n)}) \geqq 1 - Cn^{-a} \,, \tag{21}$$

where $\alpha_n = -\frac{\log \pi_n}{\log n}$.

Remarks. (i) Our proofs are restricted to the Z^2 lattice. However, extending the results to any planar graph poses no serious difficulties. For each graph, the critical exponents (if they exist) depend of course on d, but it is believed that they do not depend on the particular lattice structure. This conjecture is called "universality." Thus it follows from our Theorem 1 and the universality that the fractal dimension of the invasion cluster in each planar graph does not depend on the particular lattice structure too. In other words, the invasion cluster in any planar graph has the same fractal dimension $2 - 1/\rho$. We believe that this also holds for $d > 2$. In fact Theorem 1 provides an important relation between the critical exponent ρ and the fractal dimension of the invasion cluster. We believe that the fractal dimension of the invasion cluster is much easier to handle at least from the point of simulations. For example, computing h_n in any regular graph is easier than computing π_n since h_n is not related to p_c which is unknown for almost all graphs.

(ii) Assuming the existence of ρ in (7), it follows from Theorem 1 and Themorem 3 with probability tending to one that

$$|\mathscr{V}_n| \sim n^{2-1/\rho} \text{ and } |\mathscr{R}_n| \sim n^{2\rho/(2\rho-1)} .$$

Therefore, the fractal dimensions of $\mathscr{V}$ and $\mathscr{R}$ are $2 - 1/\rho$ and $\frac{2\rho}{2\rho-1}$ respectively. Note that (see (12)) $\rho = \frac{48}{5}$ so that fractal dimensions of $\mathscr{V}$ and $\mathscr{R}$ are 1.8958389583... and 1.054945054945..., respectively. Comparing with (1), it matches Wilkinson and Willemsen's numerical result perfectly. However, without assuming (7) it follows from Theorem 1, Theorem 4, (10) and (11) that for any $\varepsilon > 0$ with probability tending to one

$$|\mathscr{V}_n| \geqq n^{5/3-\varepsilon} \text{ and } |\mathscr{V}_n| \leqq n^{2-\delta_5-\varepsilon}$$

and

$$|\mathscr{R}_n| \geqq n^{2/(2-\delta_5)-\varepsilon} \text{ and } |\mathscr{R}_n| \leqq n^{6/5-\varepsilon} .$$

(iii) From (13) and the Borel-Cantelli Lemma, $|\mathscr{V}_n| \leqq n^{2+\varepsilon}\pi_n$ almost surely. In fact, we may even improve the upper bound above to $Cn^2\pi_n \log^2 n$ almost surely. However, it seems that the lower bound in the probability estimate in (14) is hard to improve to $\frac{1}{n^m}$ for a large m though we believe such an estimate to be true.

(iv) J. Chayes and L. Chayes proposed $\mathscr{V}$ as the incipient infinite cluster (see [4]). Later, H. Kesten (see [15]) proposed another definition of the incipient infinite cluster. He defined the probability measure υ_p by

$$\upsilon_p(A) = P_P(A \mid |\mathscr{C}(0)| = \infty) \tag{22}$$

for a cylinder event A. After that he proved that the limit of $\upsilon_p(A)$ exists as $p \downarrow p_c$: Denote the limit by $\upsilon(A)$. Under the measure υ, there exists an infinite occupied cluster W containing the origin with probability one. H. Kesten also showed that the expected number of $|W \cap [-n,n]^2|$ satisfies that

$$E_\upsilon|W \cap [-n,n]^2| \approx n^2\pi_n . \tag{23}$$

Although it is not clear what the relation is between the different definitions of the incipient infinite cluster, we at least, can show that both definitions of the incipient infinite cluster have the following relation:

$$E_\upsilon|W \cap [-n,n]^2| \sim \mathscr{E}\mathscr{V}_n .$$

Our argument for Theorem 1 is divided into two parts: the lower bound and the upper bound estimates. Both of them rely on some percolation results. We collect these preliminary percolation results in Sect. 2. Then we complete Theorem 1 in Sect. 3. Sect. 4 will discuss the proofs of Theorem 2–4.

2. Preliminaries

We begin with the introduction of duality. Define Z^* as the dual graph of Z^2 with vertices $\{v + (\frac{1}{2}, \frac{1}{2})\}$ and bonds joining all pairs of vertices which are a unit distance apart. For any bond set $A \subset Z^2$, we write $A^* \subset Z^*$ for the corresponding bonds of the dual graph A. For each bond $e^* \in Z^*$, we declare it is occupied or vacant if e is occupied or vacant. In other words, if e^* crosses an occupied (vacant) bond in Z^2 then e^* is occupied (vacant). With this definition, we can obtain (see [16] for detail) that if there exists a vacant dual circuit surrounding some set A^* on Z^*, then no occupied path can connect a vertex of A to ∞.

We define a left-right (respectively top-bottom) occupied crossing of a rectangle B to be an occupied path in B which joins some vertex on the left (respectively upper) side of B to some vertex on the right (respectively lower) side of B, but which uses no bonds joining two vertices in the boundary of B. Similarly, we can define a left-right vacant dual crossing of a rectangle. Let

$$\sigma(n, p) = P_P(\exists \text{ a left-right occupied crossing on } [-n,n]^2)\,.$$

Then we define that

$$L(p) = L(p,\delta) = \begin{cases} \min\,\{n : \sigma(n,p)\} \geqq 1 - \delta \text{ if } p > p_c, \\ \min\,\{n : \sigma(p,n)\} \leqq \delta \text{ if } p < p_c \end{cases} \tag{24}$$

for some strictly positive constant δ whose precise value is not important. $L(p)$ is also called the correlation length. Indeed, it follows from [11] that

$$\xi(p) \approx L(p) \text{ and } L(p_c + \eta, \delta) \approx L(p_c - \eta, \delta)\,. \tag{25}$$

With this definition, [11] proved the following lemma:

Lemma 1. *There exist C_1 and C_2 such that*

$$C_1 \leqq \frac{\pi(p, n)}{\pi(p_c, n)} \leqq C_2 \textit{ for all } n \leqq L(p,\delta)\,. \tag{26}$$

For $p < p_c$, let

$$\upsilon(p, n) = -\frac{1}{n} \log P_p(0 \to \partial B(n))\,.$$

It follows from Theorem 5.10 in [6] that

$$\xi^{-1}(p) - \frac{C_1 \log n}{n} \leqq \upsilon(p, n) \leqq \xi^{-1}(p) + \frac{C_2 \log n}{n} \tag{27}$$

for some constants C_1 and C_2. Clearly, $\upsilon(p, n)$ is continuous in p for a fixed n since it is a polynomial in p. On the other hand, for fixed n, $\upsilon(p, n) \to \infty$ as

$p \downarrow 0$. Thus it follows from the intermediate value theorem, (9) and (27) that for any $0 < \tau < 1$ there exists $p_n < p_c$ such that

$$\upsilon(p_n, n) = \left(\frac{1}{n}\right)^{1-\tau} \tag{28}$$

Clearly, it follows from (26) and (28) that

$$L(p_n) \geqq Cn^{1-\tau} \tag{29}$$

for some constant C.

Throughout this paper, we always denote by $p_n < p_c$ and $q_n = 1 - p_n$ the numbers such that

$$\upsilon(p_n, n) = \left(\frac{1}{n}\right)^{1-\tau}$$

for a given $0 < \tau < 1$. Let

$$E(m, n) = \{\exists \text{ a left-right occupied crossing of } [0, m] \times [0, n]\} .$$

Lemma 2. *For each* $0 < \tau < 1$ *and integer* $k > 0$, *then there exist some constants* C_1 *and* C_2 *such that*

$$P_{q_n}(E(kn, n)) \geqq 1 - C_1 k \exp\{-n^{C_2}\} . \tag{30}$$

Proof. Suppose that $E(kn, n)$ does not occur. Then there exists a vacant dual path on Z^* from the top to the bottom in $[0, kn] \times [0, n]^*$. However, it follows from (28) and the definition of duality that

$$\begin{aligned} &P_{q_n}(\exists \text{ a vacant path from the top to the bottom of } [0, kn] \times [0, n]^*) \\ &= P_{p_n}(\exists \text{ an occupied path from the top to the bottom of } [0, kn] \times [0, n]) \\ &\leqq kn \exp\left\{-\left(\frac{n}{n^{1-\tau}}\right)\right\} . \end{aligned} \tag{31}$$

Since $\tau < 1$, Lemma 2 is proved by (31). □

With Lemma 2, it is easy to obtain by the FKG inequality that

Corollary 3. *There exist some constants* C_1 *and* C_2 *such that*

$$P_{q_n}(\exists \text{ an occupied circuit in } B(2n)\backslash B(n)) \geqq 1 - C_1 \exp\{-n^{C_2}\} . \tag{32}$$

If we are only interested in $p = p_c$, the following principal lemma was proved by Russo (see [17]), and Seymour and Welsh (see [18]). We state it as the RSW lemma.

RSW Lemma. *For any integer* $k > 0$, *there exists a constant* C_k *such that*

$$P_{p_c}(E(kn, n)) \geqq C_k \tag{33}$$

for all n.

In particular, it follows from the FKG inequality and the duality that for any $k > 1$ and $\varepsilon > 0$ there exist C_k (depends on k) and $a > 0$ (depends on ε) such that

$$P_{p_c}(\exists \text{ an occupied (a vacant) circuit in } B(kn)\backslash B(n)) \geqq C_k \tag{34}$$

and

$$P_{p_c}(\exists \text{ an occupied (a vacant) circuit in } B(n^{1+\varepsilon})\backslash B(n)) \geqq 1 - \frac{1}{n^a}\,. \tag{35}$$

Lemma 4. *Let $H = \{(x, y) : y \geqq 0\}$ be the upper space. Then there exists a constant C such that*

$$P_{q_n}(\exists \text{ an occupied path from } [-n, n] \times \{0\} \text{ to } \infty \text{ in } H) \geqq 1 - \exp\{-n^C\}\,. \tag{36}$$

Proof. Suppose that there does not exist such an infinite occupied path. Then there exists a vacant path on Z^* from some point on $((-\infty, -n) \times \{0\})^*$ to some point on $(\{(n, \infty) \times \{0\}\})^*$. Suppose that these two points are $(m_1 + \frac{1}{2}, \frac{1}{2})$ and $(m_2 + \frac{1}{2}, \frac{1}{2})$. By the definition of correlation length and (28),

$$\begin{aligned}
&P_{q_n}(\nexists \text{ an occupied path from } [-n, n] \times \{0\} \text{ to } \infty \text{ in } H)\\
&\quad = P_{q_n}(\exists \text{ a vacant path on } H^* \text{ from } (m_1 + \tfrac{1}{2}, \tfrac{1}{2})\\
&\qquad \text{to } (m_2 + \tfrac{1}{2}, \tfrac{1}{2}) \text{ with } m_1 < -n \text{ and } m_2 > n)\\
&\quad \leqq \sum_{m_1 < -n} \sum_{m_2 > n} P_{p_n}(\exists \text{ an occupied path on } H \text{ from } (m_1, 0) \text{ to } (m_2, 0))\\
&\quad \leqq \sum_{m_1 < -n} \sum_{m_2 > n} \exp\left\{\frac{m_2 - m_1}{n^{1-\tau}}\right\}\\
&\quad \leqq \exp\{-n^C\}\,.
\end{aligned} \tag{37}$$

Therefore, Lemma 4 is proved. □

Lemma 5. *There exists a constant C such that for any $p \geqq p_c$ and integers m and n with $0 < m < n$, $P_P(\partial B(n)$ and $\partial B(m)$ are connected by an occupied path inside $B(n)\backslash B(m)) \geqq C(\frac{m}{n})$.*

Proof. For any $m < n$, define

$$S(m, n) = \{[-m, m] \times [-n, n]\}\,.$$

By convention we assume that $\frac{n}{m}$ is an integer, otherwise we always can use $\lfloor \frac{n}{m} \rfloor$ instead of $\frac{n}{m}$. Clearly, $S(m, n)$ is formed by some squares $D_1 \cup D_2, \ldots, \cup D_{\frac{n}{m}}$, where $D_1 = [-m, m] \times [-n, -n + m], \ldots, D_i = [-m, m] \times [-n + im, -n + (i + 1)m]$. We also denote by v_i the center vertex of D_i. If there exists an occupied left-right crossing in $[-n, n]^2$, then the lowest left-right occupied crossing has to cross $S(m, n)$. Therefore, the lowest left-right occupied crossing intersects as least one of $D_1, \ldots, D_{\frac{n}{m}}$. By (33) with $k = 1$,

$$\begin{aligned}
C &\leqq P_p(\exists \text{ a left-right occupied crossing in } [-n, n]^2)\\
&\leqq \sum_{i=1}^{\frac{n}{m}} P_p(\text{ the lowest left-right occupied crossing intersects } \partial D_i)
\end{aligned}$$

$$\leqq \sum_{i=1}^{\frac{n}{m}} P_p(\exists \text{ an occupied path from } \partial D_i \text{ to } v_i + \partial B(n))$$

$$\leqq \frac{n}{m} P_p(\partial B(n) \text{ and } \partial B(m) \text{ are connected by an occupied path inside}$$

$$B(n)\backslash B(m)) \text{ (by the translation invariance).}$$

Therefore, Lemma 5 is proved. □

By Lemma 5 we can also obtain the following lemma.

Lemma 6. *There exists a constant C such that for any $p \geqq p_c$ and integers m and n with $m < n$,*

$$\pi(p, m) \leqq C\left(\frac{n}{m}\right)\pi(p, n), \tag{38}$$

$$\pi(p, m) \leqq C\left(\frac{n}{m}\right)^2 \pi(p, n). \tag{39}$$

Proof. Denote by

$$A = \{0 \leftrightarrow \partial B(m)\},$$

$$B = \left\{\exists \text{ an occupied circuit in } B(m)\backslash B\left(\frac{m}{2}\right)\right\},$$

$$C = \left\{\text{two boundaries of } B(n) \text{ and } B\left(\frac{m}{2}\right) \text{ are}\right.$$

$$\left.\text{connected by an occupied path}\right\}.$$

Clearly, $\{0 \leftrightarrow \partial B(n))\}$ occurs if $A \cap B \cap C$ occurs. Thus (38) is implied by the FKG inequality, the RSW lemma and Lemma 5. Equation (39) is obvious since $\frac{n}{m} \geqq 1$. □

For each $x \in B(n)$, Let $I_x(n)$ be the indicator of the event that there exists an occupied path from x to $\partial B(n)$. Now we begin to estimate the first and the moments of $\sum_{x\in B(n)} I_x(n)$ for any $p \geqq p_c$.

Lemma 7. *For any $p \geqq p_c$ there exist constants C_1 and C_2 such that*

$$C_1 n^2 \pi(p, n) \leqq E_p \sum_{x\in B(\frac{n}{2})} I_x(n) \leqq E_p \sum_{x\in B(n)} I_x(n) \leqq C_2 n^2 \pi(p, n). \tag{40}$$

For any integer $t \geqq 1$ there exists a constant C_t such that

$$E_p\left(\sum_{x\in B(n)} I_x(n)\right)^t \leqq C_t\left(E_p \sum_{x\in B(n)} I_x(n)\right)^t. \tag{41}$$

Proof. We shall not prove Lemma 7 here. Indeed, it is easy to adapt the proof of Theorem 8 in [15] to show (40). In addition, (41) has been proved by B. Nguyen (see the lemma in [20]). H. Kesten also gave a similar estimate of (41) (see Theorem 8 in [15]). □

We then can obtain the following corollary by Lemma 7 and the Cauchy–Schwarz inequality.

Corollary 8. *There exist C_1 and $\mu > 0$ such that*

$$P_P\left(\sum_{x\in B(n)} I_x(n) \geqq C_1 n^2 \pi_n\right) \geqq \mu > 0\,. \tag{42}$$

Finally, we end this section with another lemma.

Lemma 9. *Given $\varepsilon > 0$, there exist constants C_1, C_2 and C_3 such that*

$$\begin{aligned} &P_{p_c}(\exists\, occupied\ circuit\ \mathscr{C}_n\ in\ B(n)\backslash B(n^{1-\varepsilon}), but\ the\ number\ of\ vertices \\ &\quad x \in B(n)\ such\ that\ x \leftrightarrow \mathscr{C}_n\ in\ B(n)\ is\ less\ than\ n^{2-3\varepsilon}\pi(n)) \\ &\quad \leqq C_1\ exp\ (-C_2 n^{C_3\varepsilon})\,. \end{aligned} \tag{43}$$

Proof. We shall not prove Lemma 9 here too since the proof is the same as the proof of Lemma 3.24 in [21].

3. Proof of Theorem 1

Before the proof of Theorem 1 we present a heuristic explanation of Theorem 1. The explanation is divided into two parts as follows.

1. The heuristic of the lower bounds in Theorem 1. A bond e is called p-open if $X(e) \leqq p$, and p-closed if $X(e) > p$. When $p = p_c$, by the RSW lemma, there should be many p_c-open circuits surrounding the origin in $B(n)$ for a large n. On the other hand, there exists a p_c-closed dual circuit surrounding these p_c-open circuits since $\theta(p_c) = 0$. We denote by Γ_n the bonds of these p_c-open circuits and the bonds connected to these p_c-open circuits by a p_c-open path. Note that $\mathscr{V}$ has to first cross these p_c-open circuits and then it has to use at least a bond of the p_c-closed dual circuit eventually. Note also that $X(e_1) > X(e_2)$ if e_1, is a p_c-closed and e_2 is a p_c-open so that $\mathscr{V}$ has to occupy all the bonds of Γ_n before occupying any bond of the p_c-closed dual circuit. Therefore, we can use $|\Gamma_n|$ as a lower bound of $|\mathscr{V}_n|$. With this observation, the rest work is to estimate $|\Gamma_n|$ by lemmas in Sect. 2.

2. The heuristic of the upper bounds in Theorem 1. The upper bounds in Theorem 1 is more complicated. Note that for each $p > p_c$ there exists a unique infinite cluster $\mathscr{C}(p)$ of p-open bonds. Once the invasion cluster reaches $\mathscr{C}(p)$, all future invasion bonds will stay in $\mathscr{C}(p)$. If we only consider $d = 2$, by a standard percolation argument there exists a finite circuit in $\mathscr{C}(p)$ surrounding the origin. In other words, $\mathscr{V}$ has to reach $\mathscr{C}(p)$ in finite time. Then $|\mathscr{C}(p) \cap B(n)|+$ constant is an upper bound of $|\mathscr{V}_n|$. However, we may lose too much if we only use $|\mathscr{C}(p) \cap B(n)|+$ constant as an upper bound since $|\mathscr{C}(p) \cap B(n)| \approx n^2$ for any $p > p_c$. [22] considered an inhomogeneous site percolation model as follows. They use $p_c + f(x)$ instead of p for each vertex $x \in Z^2$ and show that there exists an infinite cluster $\mathscr{C}(p_c + f(x))$ if $f(x)$ decays to zero slower than $\|x\|^{-1/v}$ as $\|x\| \to \infty$. Therefore, we can certainly control $\mathscr{V}$ by $\mathscr{C}(p_c + f(x))$. However, the result of [22] depends on the assumption of the power law hypothesis. The rest diffculty is to estimate

$|\mathscr{C}(p_c + f(x)) \cap B(n)|$ by lemmas in Sect. 2 without using the power law hypothesis.

Now we begin to give a rigorous proof of Theorem 1. We first estimate the lower bound of Theorem 1. As we pointed before, a bond e is called p-open if $X(e) \leqq p$ and p-closed if $X(e) > p$. Since $\theta(p_c) = 0$, there must exist a dual circuit M_n on Z^* outside $B(n)$ with p_c-closed bonds. We write $\mathscr{M}_n$ for such an event. On the other hand, it follows from Lemma 3 that there exists a p_c-open circuit in $B(n)\backslash B(\frac{n}{2})$ with a positive probability. We write $\mathscr{D}_n$ for the event. On the event $\mathscr{D}_n$ we can select such a p_c-open circuit and denote by D_n. For convenience, we always select the innermost open circuit as D_n. Let us consider each vertex $x \in B(\frac{n}{2})$. If both D_n and M_n exist, then x is in $\mathscr{V}$ if there exists a p_c-open path connecting x to D_n. Therefore,

$$
\begin{aligned}
\mathscr{E}|\mathscr{V}_n| &= \sum_{x \in B(n)} \mathscr{P}(x \in \mathscr{V}) \\
&\geqq \sum_{x \in B(\frac{n}{2})} \mathscr{P}(x \in \mathscr{V}, \mathscr{D}_n, \mathscr{M}_n) \\
&\geqq \sum_{x \in B(\frac{n}{2})} P_{p_c}(x \leftrightarrow D_n, \mathscr{D}_n, \mathscr{M}_n) \\
&= \sum_{x \in B(\frac{n}{2})} P_{p_c}(x \leftrightarrow D_n, \mathscr{D}_n) P(\mathscr{M}_n) \\
&\quad (\text{ note that } \mathscr{M}_n \text{ and } \{x \leftrightarrow D_n, \mathscr{D}_n\} \text{ only depend on} \\
&\quad \{X(e)\} \text{ for } e \in Z^2\backslash B(n) \text{ and } e \in B(n) \text{ respectively }) \\
&\geqq C_2 \sum_{x \in B(\frac{n}{2})} P_{p_c}(x \to \partial B(n)) \\
&\quad (\text{by the FKG inequality and note that} \\
&\quad \{x \leftrightarrow \partial B(n)\} \subset \{x \leftrightarrow D_n\} \text{ for } x \in B(\frac{n}{2})) \\
&\geqq C_3 n^2 P_{p_c}(0 \leftrightarrow \partial B(n)) \text{ (by (40))} \\
&= C_3 n^2 \pi_n \,.
\end{aligned} \tag{44}
$$

Equation (15) follows from Jensen's inequality and (44), that is

$$
\mathscr{E}|\mathscr{V}_n|^t \geqq (\mathscr{E}|\mathscr{V}_n|)^t \geqq C(n^2\pi_n)^t
$$

for some constant C. Now we turn to show (14) in Theorem 1 about the probability estimate. Given $\varepsilon > 0$, we will estimate $\mathscr{P}(|\mathscr{V}_n| \leqq n^{2-\varepsilon}\pi_n)$. Let $\mathscr{D}$ be the event that there exists a p_c-open circuit in $B(n)\backslash B(n^{1-\varepsilon/4})$. On the event $\mathscr{D}$ select D as the innermost open circuit. By (35) there exists $a > 0$ such that

$$
\mathscr{P}(\mathscr{D}) \geqq 1 - \frac{1}{n^a} \,.
$$

Furthermore, on the event $\mathscr{D}$, denote by J_x the indicator of the event that there exists a p_c-open path from x to D in $B(n)$ for each $x \in B(n)$. With the definition of J_x it can be seen that x has to be connected to D by a p_c-open path if D exists and $J_x = 1$. Note that there always exists a p_c-closed dual circuit on Z^* surrounding $B(n)$ and $\mathscr{V}$ has to cross D somewhere, so that $D \subset \mathscr{V}_n$ if D exists. Similarly, it can also be seen that $x \in \mathscr{V}_n$ if D exists and $J_x = 1$. Then $\sum_{x \in B(n)} J_x \leqq n^{2-\varepsilon}\pi_n$ if there exists D and $\{|\mathscr{V}_n| \leqq n^{2-\varepsilon}\pi_n\}$. By Lemma 9

$$\begin{aligned}
&\mathscr{P}(|\mathscr{V}_n| \leqq n^{2-\varepsilon}\pi_n) \\
&\leqq \mathscr{P}(|\mathscr{V}_n| \leqq n^{2-\varepsilon}\pi_n, \mathscr{D}) + \frac{1}{n^a} \\
&\leqq \mathscr{P}\left(\sum_{x \in B(n)} J_x \leqq n^{2-\varepsilon}\pi_n, \mathscr{D}\right) + \frac{1}{n^a} \\
&\leqq P_{p_c}(\exists \text{ the innermost occupied circuit } \mathscr{C} \text{ in } B(n)\backslash B(n^{1-\varepsilon/4}), \\
&\qquad \text{but the number of vertices } x \in B(n) \text{ such that } x \leftrightarrow \mathscr{C} \text{ in } B(n) \\
&\qquad \text{is less than } n^{2-\varepsilon}\pi(n)) \\
&\leqq C_1 \exp\,(-C_2 n^{C_3\varepsilon}) + \frac{1}{n^a}\,.
\end{aligned} \tag{45}$$

Therefore, the lower bound of Theorem 1 is proved.

Next we will show the upper bound of Theorem 1. We begin with the estimate of the expected $\mathscr{V}_n$. For any n and $1 > \tau > 0$, set

$$k_0 = n,\ k_1 = k_0^{1-\tau}, k_2 = k_1^{1-\tau}, \ldots, k_r = k_{r-1}^{1-\tau},\ k_{r+1} = k_r^{1-\tau}\,,$$

where r is the smallest integer such that $(1-\tau)^r \leqq \frac{3}{4}$. For each r and j with $0 < j \leqq r$, denote by $p_{k_j} = 1 - q_{k_j} < p_c$ the number which satisfies

$$\upsilon(p_{k_j}, k_j) = \left(\frac{1}{k_j}\right)^{1-\tau}\,.$$

By (29), such a p_{k_j} exists. Clearly, there exists a constant C such that

$$|\mathscr{V}_{k_r}| \leqq C n^{2(1-\tau)^r} \leqq C n^{3/2}\,. \tag{46}$$

On the other hand, by (15) and (10),

$$\mathscr{E}|\mathscr{V}_n|^t \geqq C n^{(5t/3)} \geqq 2n^{3t/2} \tag{47}$$

for large n. Set $\Gamma = B(n)\backslash B(k_r)$. By Minkowski's inequality and (46)

$$\begin{aligned}
(\mathscr{E}|\mathscr{V}_n|^t)^{1/t} &= (\mathscr{E}(|\mathscr{V}_n \cap \Gamma| + |\mathscr{V}_{k_r}|)^t)^{1/t} \\
&\leqq (\mathscr{E}(|\mathscr{V}_n \cap \Gamma|)^t)^{1/t} + C n^{3/2}\,.
\end{aligned} \tag{48}$$

Then it follows from (47) and (48) that

$$\mathscr{E}|\mathscr{V}_n|^t \leqq C\mathscr{E}(|\mathscr{V}_n \cap \Gamma|)^t \tag{49}$$

for some C. We let $\mathscr{D}_{j+1}$ be the event that there exists a $q_{k_{j+1}}$-open circuit inside $B(k_j)\backslash B(k_{j+1})$ for $0 \leqq j \leqq r$. We also let $\mathscr{L}_j$ be the event that there exists a $q_{k_{j+1}}$-open path from the left edge to the right edge of $[k_{j+1}, k_{j-1}] \times [-k_{j+1}, k_{j+1}]$ for $1 \leqq j \leqq r$, and $\mathscr{L}_0$ be the event that there exists a q_{k_1}-open path from $\{k_1\} \times [-k_1, k_1]$

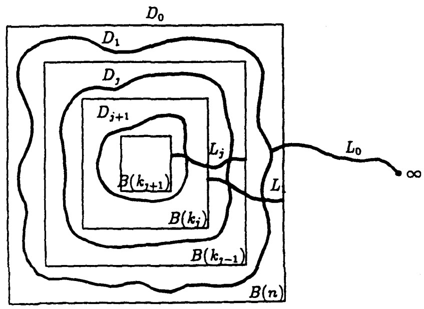

Fig. 1. The event F_n

to ∞ (see Fig 1). On the event $\mathscr{D}_j$ or $\mathscr{L}_j$ we can select such a circuit or path and denote by such a circuit or path $\mathscr{D}_J$ or $\mathscr{L}_j$ respectively. Denote by

$$F_n = \bigcap_{j=1}^{r+1}\{\mathscr{D}_j\}\bigcap_{j=0}^{r}\{\mathscr{L}_j\}\,.$$

Note that $k_r \geqq n^c$ for some $c > 0$. Then by Lemma 2, Cor. 3, Lemma 4 and the FKG inequality,

$$\mathscr{P}(F_n) \geqq 1 - \exp\{-n^C\}\,. \tag{50}$$

For each $x \in B(k_{j-1})\backslash B(k_j)$ set $T_x(j)$ to be the indicator of the event that there exists a $q_{\max}(j)$-open path from x to $\partial B(k_{j-1})$, where

$$q_{\max}(j) = \max\{q_{k_{j+1}}, q_{k_j}, \ldots, q_{k_0}\}\,.$$

Note that

$$L(q_{\max}(j)) \geqq C\,k_{j+1}^{1-\tau}\,. \tag{51}$$

Let

$$G_j = \begin{cases} \sum_{x\in B(k_{j-1})\backslash B(k_j)} T_x(j) & \text{if } F_n \text{ occurs}\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

Note that $\mathscr{V}$ must cross D_{j+1} and D_{j+1} is connected to ∞ by a $q_{\max}(j)$-open path if F_n occurs. Once the invasion cluster reaches D_{j+1}, all future invaded edges e_k will have $X(e_k) \leqq q_{\max(j)}$. Thus if $x \in \mathscr{V} \cap \{B(k_{j-1})\backslash B(k_j)\}$, there must exist a $q_{\max}(j)$-open path from D_{j+1} to x. In other words if $x \in \mathscr{V} \cap \{B(k_{j-1})\backslash B(k_j)\}$, then $T_x(j) = 1$. On the event F_n, we have

$$G_0 + G_1 + \ldots + G_r \geqq |\mathscr{V}_n \cap \Gamma|\,.$$

For integers l, t and j with $1 \leqq l \leqq t$ and $j \leqq r$,

$$\begin{aligned}\mathscr{E}(G_j)^l &\leqq \mathscr{E}\left(\sum_{x\in B(k_{j-1})\backslash B(k_j)} T_x(j)\right)^l\\ &\leqq CE_{q_{\max}(j)}\left(\sum_{x\in B(k_{j-1})} I_x(k_{j-1})\right)^l\end{aligned}$$

$$\begin{aligned}
&\leqq C_1\left(E_{q_{\max}(j)}\textstyle\sum_{x\in B(k_{j-1})}I_x(k_{j-1})\right)^l \quad \text{(Cor. 8 (a))}\\
&\leqq C_1(k_{j-1}^2\pi(q_{\max}(j),k_{j-1}))^l \text{ (by Lemma 7)}\\
&\leqq C_1(k_{j-1}^2\pi(q_{\max}(j),k_{j+1}^{1-\tau}))^l \text{ (note that } k_{j-1}\geqq(k_{j+1})^{1-r})\\
&\leqq C_2(k_{j-1}^2\pi_{k_{j+1}^{1-\tau}})^l \text{ (by (53) and Lemma 1)}\\
&\leqq C_3(k_{j-1}^{2+3\tau}\pi_{k_{j-1}})^l \text{ (by Lemma 6)}\\
&\leqq C_3 n^{(2+3\tau)l}\pi_n^l \text{ (by (39))}\,.
\end{aligned}\tag{52}$$

Note that G_i and G_j are independent if $i \neq j$, so that for any $0 \leqq j_1, j_2, \ldots, j_k \leqq r$ and $i_1 + \ldots + i_k = t$,

$$\mathscr{E} G_{j_1}^{i_1} \cdot \ldots \cdot G_{j_k}^{i_k} \leqq C n^{(2+3\tau)t}\pi_n^t \tag{53}$$

for some constant C. Therefore, by (49), (50) and (53),

$$\begin{aligned}
\mathscr{E}|\mathscr{V}_n|^t &\leqq C_1\mathscr{E}|\mathscr{V}_n\cap\Gamma|^t\\
&\leqq C_1\mathscr{E}(G_1+G_2+\ldots+G_r)^t+(8n)^{2t}\exp(-n^C)\\
&\leqq C_2 r^{t+1}n^{(2+3\tau)t}\pi_n^t+(8n)^{2t}\exp(-n^C)\\
&\leqq C_3 r^{t+1}n^{(2+3\tau)t}\pi_n^t\,.
\end{aligned}\tag{54}$$

Therefore, (16) is implied by (54).

To show (13) use Markov's inequality,

$$\mathscr{P}(|\mathscr{V}_n|\geqq n^{2+\varepsilon}\pi_n)\leqq\frac{\mathscr{E}|\mathscr{V}_n|^t}{n^{(2+\varepsilon)t}\pi_n^t}\,. \tag{55}$$

Therefore, (13) is proved by choosing t large and τ small. Theorem 1 is proved.

4. Proof of Theorem 2–4

Proof of Theorem 2. We first estimate the lower bound of h_n in Theorem 2. We denote by $\mathscr{D}$ the event that there exists a p_c-open circuit in $B(n^{1-\varepsilon/8})\backslash B(n^{1-\varepsilon/4})$. On the event $\mathscr{D}$, let D be the innermost p_c-open circuit in $B(n^{1-\varepsilon/8})\backslash B(n^{1-\varepsilon/4})$. We also denote by $\mathscr{M}$ the event that there exists a p_c-closed dual circuit in $B(n)\backslash B(n^{1-\varepsilon/8})$. On the event $\mathscr{M}$ we can select a p_c-closed dual circuit M in $(B(n)\backslash B(n^{1-\varepsilon/8}))^*$. Clearly, M surrounds D in $B(n)$ if both of them exist. By (35), the probability of the existence of D and M in $B(n)\backslash B(n^{1-\frac{\varepsilon}{4}})$ is larger than $1-\frac{1}{n^a}$ for some constant $a>0$. On the event $\mathscr{D}\cap\mathscr{M}$, the invasion cluster $\mathscr{V}$ has to occupy all bonds of D before it occupies any bond of M. Of course, $\mathscr{V}$ must also occupy all possible vertices which are connected to D by a p_c-open path. Clearly, $\sum_{x\in B(n^{1-\varepsilon/8})}J_x\leqq n^{2-\varepsilon}\pi_n$ if $h_n\leqq n^{2-\varepsilon}\pi_n$ and both D and M exist (see the definition of J_x in Sect. 3). Therefore,

$$\begin{aligned}
\mathscr{P}(h_n\leqq n^{2-\varepsilon}\pi_n) &\leqq \mathscr{P}(h_n\leqq n^{2-\varepsilon}\pi_n,\mathscr{D},\mathscr{M})+\frac{1}{n^a}\\
&\leqq \mathscr{P}\left(\sum_{x\in B(n^{1-\varepsilon/8})}J_x\leqq n^{2-\varepsilon}\pi_n,\mathscr{D}\right)+\frac{1}{n^a}\\
&\leqq C_1\exp(-C_2 n^{C_3})+\frac{1}{n^a}\text{ (by 45))}\,.
\end{aligned}\tag{56}$$

Therefore the lower bound of Theorem 2 is proved. Since $h_n \leqq |\mathscr{V}_n|$, the upper bound of Theorem 2 holds by Theorem 1.

Proof of Theorem 3. By the duality and $\theta(p_c) = 0$, there exists with probability one a p_c-open circuit D_n and a p_c-closed dual circuit F_n outside of $B(n)$ such that D_n is surrounded by F_n. Once such two circuits exist, $\mathscr{V}$ has to first occupy every bond of D_n before it occupies any bond of F_n. Therefore, each bond in $B(n)$ is either trapped by $\mathscr{V}$ or in $\mathscr{V}$. However, by (13) the total number of bonds in $\mathscr{V}_n$ exceeds $n^{2+\varepsilon}\pi_n$ with a probability less than $\frac{1}{n^m}$ for some m. Therefore, the total number of trapped bonds by $\mathscr{V}$ in $[-n,n]^2$ is less than $|B(n)| - n^{2+\varepsilon}\pi_n = 2(2n)^2 - n^{2+\varepsilon}\pi_n$ with a probability less than $\frac{1}{n^m}$. By taking ε small and $m > 1$, the Borel-Cantelli lemma and (10) will imply that

$$\lim_{n\to\infty} \frac{1}{2(2n)^2} |\mathscr{R} \cap [-n,n]^2| = 1 \text{ a.s.} \tag{57}$$

Theorem 3 is proved.

Proof of Theorem 4. Denote by

$$T_n = \max\{m : S_n \cap \partial B(m) \neq \emptyset\}.$$

Clearly,

$$\{T_n \geqq l\} \text{ implies } h_l \leqq n \text{ and } \{T_n < l\} \text{ implies } h_l \geqq n.$$

Note that, by definition of $\alpha_n, \pi_n = n^{\log \pi_n/\log n} = n^{-\alpha_n}$. Then

$$n^2\pi_n = n^{2-\alpha_n}.$$

Given $\varepsilon > 0$, let

$$A_n = \{n^{1/(2-\alpha_n+\varepsilon/2)} \leqq T_n\}.$$

Then by Theorem 2,

$$\mathscr{P}(A_n \text{ does not occur }) \leqq \mathscr{P}(h_{n^{1/(2-\alpha_n+\varepsilon/2)}} \geqq n) \leqq \frac{1}{n^a} \tag{58}$$

for some constant $a > 0$. Similarly, let $\mathscr{D}_n$ and $\mathscr{M}_n$ be the events that there exist a p_c-open circuit D_n and a p_c-closed circuit M_n such that both of them are in $B(n^{1/(2-\alpha_n+\varepsilon/2)})\backslash B(n^{1/(2-\alpha_n+\varepsilon)})$ and D_n is also surrounded by M_n. Clearly, by (35),

$$\mathscr{P}(\mathscr{D}_n \cap \mathscr{M}_n) \geqq 1 - n^{-b} \tag{59}$$

for some constant $b > 0$. If D_n and M_n as above exist, then each bond in $B(n^{1/(2-\alpha_n+\varepsilon)})$ is either trapped by $\mathscr{V}$ or in $\mathscr{V}$ since $\mathscr{V}$ has to occupy all bonds of D_n before occupying any bond of M_n. However, if A_n occurs, D_n has to intersect S_n somewhere since $S_n \cap \partial B(n^{1/(2-\alpha_n+\varepsilon/2)}) \neq \emptyset$. On the other hand, note that M_n is in $B(n^{1/(2-\alpha_n+\varepsilon/2)})$ and $\mathscr{V}$ only use at most n bonds to come to the boundary of $B(n^{1/(2-\alpha_n+\varepsilon/2)})$ on the event A_n, so that $D_n \subset S_n$. With this observation, if A_n occurs and D_n and M_n exist, then every bond in $B(n^{1/(2-\alpha_n+\varepsilon)})$ is either in S_n or trapped by S_n. Note that $|S_n| = n$. Therefore, at least $n^{2/(2-\alpha_n+\varepsilon)} - n$ bonds are trapped by S_n. If we take ε small and n large such that $n^{2/(2-\alpha_n+\varepsilon)} - n \geqq \frac{1}{2}n^{2/(2-\alpha_n+\varepsilon)}$, then the total number of bonds trapped by S_n cannot be less than $\frac{1}{2}n^{2/(2-\alpha_n+\varepsilon)}$ if A_n occurs and D_n and M_n exist. Therefore,

$$
\begin{aligned}
&\mathscr{P}\left(\frac{1}{2}n^{2/(2-\alpha_n+\varepsilon)} \geqq |\mathscr{R}_n|\right) \\
&\quad \leqq \mathscr{P}\left(\frac{1}{2}n^{2/(2-\alpha_n+\varepsilon)} \geqq |\mathscr{R}_n|, A_n, \mathscr{D}_n, \mathscr{M}_n\right) + \frac{1}{n^a} + \frac{1}{n^b} \\
&\quad = \frac{1}{n^a} + \frac{1}{n^b} .
\end{aligned} \tag{60}
$$

Similarly, by Theorem 2,

$$\mathscr{P}(T_n \geqq n^{1/(2-\alpha_n-\varepsilon)}) \leqq \mathscr{P}(h_{n^{1/(2-\alpha_n-\varepsilon)}} \leqq n) \leqq n^{-d} \tag{61}$$

for some constant $d > 0$. On the event $\{T_n \leqq n^{1/(2-\alpha_n-\varepsilon)}\}$, the bonds trapped by S_n are at most $n^{2/(2-\alpha_n-\varepsilon)} - n$ in number. Then

$$
\begin{aligned}
&\mathscr{P}(n^{2/(2-\alpha_n-\varepsilon)} \leqq |\mathscr{R}_n|) \\
&\quad \leqq \mathscr{P}(n^{2/(2-\alpha_n+\varepsilon)} \leqq |\mathscr{R}_n|, T_n \leqq n^{1/(2-\alpha_n-\varepsilon)}) + \frac{1}{n^d} \\
&\quad = \frac{1}{n^d} .
\end{aligned} \tag{62}
$$

Equation (21) is implied by (60) and (62). Theorem 4 is proved.

Acknowledgements. The author would like to thank the referee for carefully reading this manuscript and numerous helpful comments and typographical corrections.

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Communicated by M. Aizenman