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On Additional Symmetries of the KP Hierarchy and Sato's Backlund Transformation

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Abstract. A short proof is given to the fact that the additional symmetries of the KP hierarchy defined by their action on pseudodifferential operators according to Fuchssteiner–Chen–Lee–Lin–Orlov–Shulman coincide with those defined by their action on τ-functions as Sato's Bäcklund transformations. A new simple formula for the generator of additional symmetries is also presented.

0. The so-called "additional symmetries" of the KP hierarchy, i.e., symmetries which are not contained in the hierarchy itself and play such a crucial role for the string equation independently appeared twice in remote areas of the theory of integrable systems. On the one hand, they were introduced in works by Fuchssteiner, Chen, Lee and Lin, Orlov and Shulman, and others who explicitly wrote the action of the additional symmetries on the pseudodifferential operators and their wave functions. On the other hand, they were found as Bäcklund transformations of τ-functions by Sato and other mathematicians of the Kyoto school. During a long period of time there was no general evidence that these two kinds of symmetries coincide. This problem received a great practical significance. As it was said, the symmetries of the first kind are related to the string equation. They provide the Virasoro and higher W-constraints. It is very important to know how they act on τ-functions since the latter ones have a direct physical meaning as partition functions in matrix models.

For lower additional symmetries the problem was solved in the positive sense by direct though laborious calculations. The general proof that two types of symmetries are, in fact, the same was given by Adler, Shiota and van Moerbeke [1].

The main goal of this note is to give a new, and possibly short, proof to this important result (Theorem 2 below). The proof is based on a new expression for a generator of additional symmetries (Theorem 1). The formula is very simple and good looking, it generalizes an earlier obtained expression (see [3]) for resolvents, generators of symmetries belonging to the hierarchy. It was our next goal to present this formula.

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1. Here we summarize well-known definitions and properties of the KP hierarchy.

The KP hierarchy is generated by a pseudodifferential operator (ΨDO)

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \cdots, \quad \partial = d/dx .$$

This operator can be represented in a dressing form

$$L = \phi\partial\phi^{-1} ,$$

where ϕ is a ΨDO$\phi = \sum_0^\infty w_i\partial^{-i}$ with $w_0 = 1$. Assuming that w_i depend on some "time variables" t_m, where $m = 1, 2, \ldots$ the hierarchy is the totality of equations

$$\partial_m\phi = -L_-^m\phi, \quad \partial_m = \partial/\partial t_m .$$

Let $\xi(t,z) = \sum_1^\infty t_i z^i$. Put

$$w(t,z) = \phi \exp\xi(t,z) = \sum_0^\infty w_i z^i \exp\xi(t,z) = \hat{w}(t,z)\exp\xi(t,z) .$$

This is the (formal) *Baker, or wave, function.*

Let ϕ^* be the formal adjoint to ϕ (by definition, $(f\partial)^* = -\partial f$). The function $w^*(t,z) = (\phi^*)^{-1}\exp(-\xi(t,z)) = \hat{w}^*(t,z)\exp(-\xi(t,z))$ is called the *adjoint Baker function.*

Further we need a very simple and useful lemma (see [3]). One can consider two types of residues, that of formal series in ∂ : $\mathrm{res}_\partial \sum a_i\partial^i = a_{-1}$, and that of formal series in z : $\mathrm{res}_z \sum a_i z^i = a_{-1}$.

Lemma 1. *Let P and Q be two ΨDO, the*

$$\mathrm{res}_z[(Pe^{xz}) \cdot (Qe^{-xz})] = \mathrm{res}_\partial PQ^* ,$$

where Q^* is the formal adjoint to Q.

The proof is in a straightforward vertification. We also need another simple lemma:

Lemma 2. *If $f(z) = \sum_{-\infty}^\infty a_i z^{-i}$ then*

$$\mathrm{res}_z[\zeta^{-1}(1 - z/\zeta)^{-1} + z^{-1}(1 - \zeta/z)^{-1}]f(z) = f(\zeta) .$$

(Here $(1 - z/\zeta)^{-1}$ is understood as a series in ζ^{-1} while $(1 - \zeta/z)^{-1}$ is a series in z^{-1}.)

The Baker function can be expressed in terms of the τ-function as

$$\hat{w}(t,z) = \frac{\tau(t_1 - 1/z, t_2 - 1/(2z^2), t_3 - 1/(3z^3), \ldots}{\tau(t_1, t_2, t_3, \ldots)} = \frac{\tau(t - [z^{-1}])}{\tau(t)} .$$

A τ-function is determined up to a multiplication by $c\exp\sum_1^\infty c_i t_i$, where $c, c_1, c_2, \ldots$ are arbitrary constants. For the adjoint Baker function we have

$$\hat{w}^*(t,z) = \frac{\tau(t_1 + 1/z, t_2 + 1/(2z^2), t_3 + 1/(3z^3), \ldots)}{\tau(t_1, t_2, t_3, \ldots)} = \frac{\tau(t + [z^{-1}])}{\tau(t)} .$$

2. The above formulas can be rewritten in terms of the so-called vertex operators. One can write $f(t-[z^{-1}]) = \exp(-\sum_1^\infty \partial_i/(iz^i))f(t)$. Then

$$\hat{w}(t,z) = \frac{\exp(\sum_1^\infty t_i z^i)\exp(-\sum_1^\infty \partial_i/(iz^i))\tau(t)}{\tau(t)} .$$

Moreoever, for two non-commuting operators A and B an equation can be written

$$e^A \cdot e^B =: e^{A+B} : ,$$

where the symbol of normal ordering ": :" means that in all monomials the operator A must be placed to the left of all B's.

Let

$$p_i = \begin{cases} \partial_i, & i > 0 \\ |i| t_{|i|}, & i \leqq 0 \end{cases} .$$

These are Heisenberg generators. Then the above formula for $\hat{w}$ is

$$\hat{w}(t,z) = \frac{: \exp(-\sum_{-\infty}^\infty p_i/(iz^i)) : \tau(t)}{\tau(t)}$$

The symbol of normal ordering means here that p_i with negative i must be placed to the left of positive ones. The operator $X(z) =: \exp \sum_{-\infty}^\infty p_i/(iz^i)$: is called a vertex operator. Similarly,

$$\hat{w}^*(t,z) = \frac{: \exp(-\sum_{-\infty}^\infty p_i/(iz^i)) : \tau(t)}{\tau(t)}$$

Another vertex operator can be introduced:

$$X(\lambda,\mu) =: \exp \sum_{-\infty}^\infty \left(\frac{p_i}{i\lambda^i} - \frac{p_i}{i\mu^i}\right) : .$$

Proposition 1 (*Sato*). *The operator $X(\lambda,\mu)$ acts as an infinitesimal operator in the space of τ-functions, i.e., solving the differential equation $\partial\tau/\partial t^*_{\lambda,\mu} = X(\lambda,\mu)\tau$, where $t^*_{\lambda,\mu}$ is a variable, we obtain for each $t^*_{\lambda,\mu}$ a new τ function. In other words this yields symmetries of the KP hierarchy.*

The operator $X(\lambda,\mu)$ can be considered as a generator of infinitesimal symmetries if expanded in double series, in $\mu-\lambda$ and λ:

$$X(\lambda,\mu)\tau = \sum_{m=0}^\infty \frac{(\mu-\lambda)^m}{m!} \sum_{n=-\infty}^\infty \lambda^{-n-m} W_n^{(m)}(\tau) .$$

Differential operators $W_n^{(m)}$ can be taken as generators of a Lie algebra which is called $W_{1+\infty}$

The procedure of application of $X(\lambda,\mu)$ to τ is called Sato's (infinitesimal) Bäcklund transformation.

3. Now we define additional symmetries in a form given them by Orlov and Shulman [4] (see also [3]). Let

$$\Gamma = \sum_1^\infty t_i i \partial^{i-1} \quad \text{and} \quad M = \phi \Gamma \phi^{-1}\,.$$

Proposition 2 (*Orlov and Shulman*). *The differential equation*

$$\partial^*_{lm} \phi = -(M^m L^l)_- \phi\,,$$

where ∂^*_{lm} symbolizes a derivative with respect to some additional variable t^*_{lm} gives a symmetry of the KP hierarchy.

One can consider a generator of these symmetries

$$Y(\lambda, \mu) = \sum_{m=0}^\infty \frac{(\mu-\lambda)^m}{m!} \sum_{l=-\infty}^\infty \lambda^{-m-l-1} (M^m L^{m+l})_-\,.$$

Theorem 1. *The formula*

$$Y(\lambda, \mu) = w(t, \mu) \cdot \partial^{-1} \cdot w^*(t, \lambda)$$

holds.

Remark. If $\lambda = \mu$ then $Y(\lambda, \lambda) = \sum_{-\infty}^\infty \lambda^{-l-1} L^l_-$ is the so-called resolvent which is a generator of the symmetries belonging to the hierarchy; in this special case Theorem 1 was proven in [3].

Proof of the theorem. We have

$$(M^m L^{m+l})_- = (\phi \Gamma^m \partial^{m+l} \phi^{-1})_- = \sum_1^\infty \partial^{-i} \mathrm{res}_\partial \partial^{i-1} \phi \Gamma^m \partial^{m+l} \phi^{-1}\,.$$

According to Lemma 1 this can be written as

$$(M^m L^{m+l})_- = \sum_1^\infty \partial^{-i} \mathrm{res}_z \partial^{i-1} \phi \Gamma^m \partial^{m+l} e^{\xi(t,z)} (\phi^*)^{-1} e^{-\xi(t,z)}\,.$$

Taking into account that

$$\Gamma \exp \xi(t,z) = \sum_1^\infty t_i i \partial^{i-1} \exp \xi(t,z) = \sum_1^\infty t_i i z^{i-1} \exp \xi(t,z) = \partial_z \exp \xi(t,z)$$

and that ϕ commutes with ∂_z we have

$$(M^m L^{m+l})_- = \mathrm{res}_z \sum_1^\infty \partial^{-i} (z^{m+l} \partial_z^m w)^{(i-1)} \cdot w^* = \mathrm{res}_z z^{m+l} \partial_z^m w \cdot \partial^{-1} \cdot w^*\,.$$

Now, using Lemma 2, we have

$$\begin{aligned} Y(\lambda, \mu) &= \mathrm{res}_z \sum_{m=0}^\infty \sum_{l=-\infty}^\infty \frac{z^{m+l}}{\lambda^{m+l+1}} \cdot \frac{1}{m!} (\mu - \lambda)^m \partial_z^m w \cdot \partial^{-1} \cdot w^* \\ &= \mathrm{res}_z \left[\frac{1}{z(1-\lambda/z)} + \frac{1}{\lambda(1-z/\lambda)} \right] \exp((\mu-\lambda)\partial_z) w(t,z) \cdot \partial^{-1} \cdot w^*(t,z) \\ &= \exp((\mu-\lambda)\partial_\lambda) w(t,\lambda) \cdot \partial^{-1} \cdot w^*(t,\lambda) = w(t,\mu) \cdot \partial^{-1} \cdot w^*(t,\lambda)\,. \qquad \square \end{aligned}$$

4. Our second result is a new proof of the following theorem:

Theorem 2 (*Adler, Shiota, van Moerbeke*).*The action of the infinitesimal operator $X(\lambda,\mu)$ on the Baker function $w(t,z)$ generated by its action on τ given by Sato's formula is connected with $Y(\lambda,\mu)$ by the formula*

$$X(\lambda,\mu)=(\lambda-\mu)Y(\lambda,\mu)\,.$$

This means that the additional symmetries in Fuchssteiner–Chen, Lee and Lin–Orlov and Shulman sense are the same as Sato's Bäcklund transformations.

Proof. We need two identities involving the τ-function, the Fay identity and its differential form, see, e.g., [5].

Fay identity. *The τ-function satisfies the identity*

$$\sum_{(s_1,s_2,s_3)}(s_0-s_1)(s_2-s_3)\tau(t+[s_0]+[s_1])\tau(t+[s_2]+[s_3])=0\,,$$

where (s_1,s_2,s_3) symbolizes cyclic permutations.

Differential Fay identity.

$$\begin{aligned}&\partial\tau(t-[s_1])\cdot\tau(t-[s_2])-\tau(t-[s_1])\cdot\partial\tau(t-[s_2])\\&\quad+(s_1^{-1}-s_2^{-1})\{\tau(t-[s_1])\tau(t-[s_2])-\tau(t)\tau(t-[s_1]-[s_2])\}\,.\end{aligned}$$

We have

$$\begin{aligned}X(\lambda,\mu)\frac{\tau(t-[z^{-1}])}{\tau(t)}&=\frac{\tau(t)X\tau(t-[z^{-1}])-\tau(t-[z^{-1}])X\tau(t)}{\tau^2(t)}\\&=\{\tau(t)\exp\xi(t,-\lambda+\mu)\left(1-\frac{\lambda}{z}\right)^{-1}\left(1-\frac{\mu}{z}\right)\tau(t-[z^{-1}]+[\lambda^{-1}]-[\mu^{-1}])\\&\quad-\exp\xi(t,-\lambda+\mu)\tau(t-[z^{-1}])\tau(t+[\lambda^{-1}]-[\mu^{-1}])\}/\tau^2(t)\\&=\exp\xi(t,-\lambda+\mu)(z-\lambda)^{-1}\{\tau(t'+[z^{-1}]+[\mu^{-1}])\tau(t'+[\lambda^{-1}])(z-\mu)\\&\quad-\tau(t'+[\mu^{-1}])\tau(t'+[\lambda^{-1}]+[z^{-1}])(z-\lambda)\}/r^2(t)\,,\end{aligned}$$

where $t'=t-[z^{-1}]-[\mu^{-1}]$. The expression in the braces can be transformed according to the Fay identity ($s_0=0,s_1=\lambda^{-1},s_2=\mu^{-1},s_3=z^{-1}$). it is equal to $-(\mu-\lambda)\tau(t'+[z^{-1}])\tau(t'+[\lambda^{-1}+[\mu^{-1}])$. In order to obtain the action of X on $w(t,z)$ we must multiply this by $\exp\xi(t,z)$.

Now, the equality

$$\begin{aligned}&\exp\xi(t,z)\exp\xi(t,-\lambda+\mu)(z-\lambda)^{-1}(\lambda-\mu)\tau(t-[\mu^{-1}])\tau(t+[\lambda^{-1}]-[z^{-1}])/\tau^2(t)\\&\quad=(\lambda-\mu)w(t,\mu)\partial^{-1}w^*(t,\lambda)\hat{w}(t,z)\exp\xi(t,z)\end{aligned}$$

must be proven. Dividing by $(\lambda-\mu)w(t,\tau)$ and multiplying by ∂ we have

$$\partial \exp\xi(t,z-\lambda)(z-\lambda)^{-1}\frac{\tau(t+[\lambda^{-1}-[z^{-1}])}{\tau(t)}$$

$$=\frac{\tau(t+[\lambda^{-1}])\tau(t-[z^{-1}])}{\tau^2(t)}\exp\xi(t,z-\lambda)$$

or

$$\partial\tau(t+[\lambda^{-1}]-[z^{-1}])\cdot\tau(t)-\tau(t+[\lambda^{-1}]-[z^{-1}])\cdot\partial\tau(t))$$
$$=-(z-\lambda)\{\tau(t+[\lambda^{-1}]-[z^{-1}])\tau(t)-\tau(t+[\lambda^{-1}])\tau(t-[z^{-1}])\}$$

which is the differential Fay identity, i.e. it is true. All the transformations are convertible. □

5. We are going to present one more formula for Sato's generator $X(\lambda,\mu)$ which can be useful. This formula appeared as a result of discussions with M. Niedermaier and, virtually, belongs to him. We merely simplified the proof. Let

$$X(\lambda,\mu)\tau=\sum_{m=0}^{\infty}((\mu-\lambda)^m/m!)W^{(m)}(\lambda),\quad W^{(m)}(\lambda)=\sum_{n-=\infty}^{\infty}\lambda^{-n-m}W_n^{(m)}\,.$$

Let

$$\theta(\lambda)=\sum_{-\infty}^{\infty}\frac{p_i}{i\lambda^i},\ \text{then}\ X(\lambda,\mu)=:\exp(\theta(\lambda)-\theta(\mu)):\ .$$

The normal ordering means that we can operate with all operators as if they commute. Then

$$X(\lambda,\mu)=:e^{\theta(\lambda)}e^{-\theta(\mu)}:=:\left[\sum_m\frac{(\mu-\lambda)^m}{m!}\partial_\mu^m e^{-\theta(\mu)}|_{\mu=\lambda}\right]e^{\theta(\lambda)}:\ .$$

Now,

$$W^{(m)}(\lambda)=:\partial_\lambda^m e^{-\theta(\lambda)}\cdot e^{\theta(\lambda)}:=Q_m(\lambda)\,.$$

Polynomials $Q_m(\lambda)$ satisfy the recursion relations

$$Q_0(\lambda)=1,Q_{m+1}(\lambda)=(\partial_\lambda-\theta'(\lambda))Q_m(\lambda)\,.$$

They are $Q_m(\lambda)=P_m(-\theta'(\lambda))$, where P_m are the so-called Faà di Bruno polynomials (see, e.g., [3]). It is easy to prove by induction using the above recursion formula that

$$W^{(m)}(\lambda)=\sum_{m_1+2m_2+\cdots+km_k=m}:\frac{m!}{m_1!m_2!\ldots m_k!}(-\partial_\lambda\theta/1!)^{m_1}(-\partial_\lambda^2\theta/2!)^{m_2}\ldots(-\partial_\lambda^k\theta/k!)^{m_k}:$$

(see [3], Eq.(7.5.7)).

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