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# Free Field Representation For Massive Integrable Models

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**Abstract.** A new approach to massive integrable models is considered. It allows one to find symmetry algebras which define the spaces of local operators and to get general integral representations for form-factors in the $SU(2)$ Thirring and Sine–Gordon models.

## 1. Introduction

Two dimensional integrable field theory today is among the most advanced topics in relativistic field theory. The reason essentially lies in specific two-dimensional symmetries which lead to exact solutions of the quantum field dynamics.

In a massive theory these symmetries show up as a drastically simplified scattering theory called Factorized Scattering Theory (FST). This structure was first observed in non-relativistic scattering of spin waves [1] and quantum particles with point-like interaction [2, 3], and also in classical scattering of solitons in nonlinear field models [4, 5]. Factorized scattering preserves the number of particles and the set of their on-mass-shell momenta. This conservation is ensured by an infinite series of commuting integrals of motion [6, 7]. The computation of the exact factorized S-matrix may be performed by combining the standard requirements of unitarity and crossing symmetry together with the symmetry properties of the model [8–10]. Large variety of the factorized scattering theories was constructed explicitly (see e.g. [11–16]).

The two particle S-matrix uniquely specifies a structure of a space of local operators for integrable models. In other words, its knowledge can be used to compute off-shell quantities, like correlation functions of elementary or composite fields of the integrable models under investigation. This can be achieved by considering the form-factors of local fields, which are matrix elements of operators between asymptotic states [17, 18]. A very important step in this direction was taken in a series of papers [19–21], where it was shown that general properties of unitarity, analyticity

★ On leave of absence from L.D. Landau Institute for Theoretical Physics, Kosygina 2, Moscow, Russia and locality lead, in the case of factorized scattering, to a system of functional equations on form-factors which is powerful enough to permit the reconstruction of the matrix elements (see also [22, 23, 16]).

In this article I discuss a method to describe the space of local operators for massive integrable models. It can be considered as a generalization of the Feigin-Fuchs representation in two dimensional Conformal Field Theory (CFT). The bosonization in CFT allows one to express local operators in terms of simple boson fields. Using well known examples of FST, I shall demonstrate that the Smirnov equations can be rewritten as a set of requirements for some boson field. The properties of its two point function naturally generalize the properties of the free boson Green function in CFT. Methods of the bosonization permit one to find symmetry algebras defining the structure of the space of local operators and to get general integral representations for form-factors in the $SU(2)$ Thirring and Sine–Gordon models.

The paper is organized as follows.

Section 2 contains well known facts from FST. For specialists it can be useful only as a list of necessary notations. Here I introduce the central object of the investigation: the formal Zamolodchikov–Faddeev algebra [10, 24]. The Hilbert space of a massive integrable model is a space of representation of this algebra. Such representations will be denoted as $\pi_A$. General ideas of this work are illustrated by the $SU(2)$ invariant Thirring model ($SU(2)$ TM) [25–27] and the Sine–Gordon model (SGM) [28, 29]. So I recall essential properties of these theories. At the end of the section, the definition of form-factors is introduced and the Smirnov equations are formulated.

In the next section, I argue the main idea of the work. From the mathematical point of view, the Smirnov equations are a kind of Riemann–Hilbert problem. Its solution is based on the following observation; Form-factors can be expressed as some traces over the proper representation of the Zamolodchikov–Faddeev algebra. I shall denote these representations as $\pi_Z$. They essentially differ from representations $\pi_A$. One can regard $\pi_Z$ as a space of angular quantization of an integrable model [30]. I should note that a similar idea was used for solving the quantum Knizhnik–Zamolodchikov equation [31]. This is not surprising since the Smirnov and quantum Knizhnik–Zamolodchikov equations have close structures. Moreover, they are equivalent for some cases [32]. Closely similar techniques have been used in a remarkable series of works on lattice models [33–36].

In Sects. 4–8 the formal constructions from Sect. 3 are illustrated by $SU(2)$ TM. I discuss this model in detail as the simplest example where the general bosonization technique can be applied. In this case the construction is equivalent to the Frenkel-Jing bosonization of the affine quantum algebra $U_q(\widehat{sl(2)})$ [37]. In Sect. 6 the symmetry algebra of the space of local operators is found. Then using the well known method from String Theory [38], I get the integral representation for form-factors generating functions. One can expect that they form a general solution of the Smirnov equations for $SU(2)$ TM.

In Sect. 9 the method is applied to SGM. I show that the Zamolodchikov–Faddeev algebra for SGM admits the free boson representation. The classical limit of this representation was first obtained in the work [39]. The bosonization for SGM is the natural generalization of the Feigin–Fuchs representation in 2D CFT [40, 41]. I believe also that it has the same base as the bosonizations of the quantum affine algebra $U_q(\widehat{sl(2)})$ for general level [42–44].

As in the $SU(2)$ TM the bosonization permits one to get the integral representation for form-factors of SGM.

Some details of the construction are described in the Appendices.

The main technical results of this work are represented by the theorem and Proposition 4 in Sect. 7, together with the explicit bosonization rules.

## 2. Preliminaries

*2.1. Zamolodchikov–Faddeev Algebra.* We start with a brief description of the basics of FST.

The massive character of a spectrum means that the interaction in a theory is short-distance and, asymptotically for $t \to \pm\infty$, the particles behave as free ones. So, two natural different bases can be chosen for the Hilbert space of a massive field theory: initial (in) and final (out) bases of states. Any in- and out-state

$$|A_{a_1}(p_1)\ \dots\ A_{a_n}(p_n)\rangle_{\text{in}}, \quad |A_{a_1}(p_1)\ \dots\ A_{a_n}(p_n)\rangle_{\text{out}}\ ;$$

$$p_1^x < p_2^x < \cdots < p_n^x \tag{2.1}$$

is characterized by a set of particles $\{A_a\}$ and their two- momenta $p_a^\mu(\mu = x, t)$:

$$(p_a^t)^2 - (p_a^x)^2 = m_a^2\ . \tag{2.2}$$

Here index $a$ denotes some quantum numbers specifying different types of particles $A_a$ with the masses $m_a$. The in- and out-bases should be connected by a proper unitary matrix, which is called S-matrix

$$|A_{a_1}(p_1')\dots A_{a_n}(p_n')\rangle_{\text{out}} = S_{a_1\dots a_n}^{b_1\dots b_n}(p_1',\dots p_n'|\,p_1,\dots p_n)|A_{b_1}(p_1)\dots A_{b_n}(p_n)\rangle_{\text{in}}. \tag{2.3}$$

Since the dynamics of integrable models is governed by an infinite number of nontrivial conservation laws the scattering processes are purely elastic and a general n-particle element of the S-matrix is factorizable into the two-particle S-matrices

$$|A_{a_1}(p_1)A_{a_2}(p_2)\rangle_{\text{out}} = S_{a_1a_2}^{b_1b_2}(p_1, p_2)|A_{b_1}(p_1)A_{b_2}(p_2)\rangle_{\text{in}}\ . \tag{2.4}$$

It is convenient to parameterize the energy-momentum spectrum (2.2) in terms of the rapidity variable $\beta$

$$p_a^t = m_a \cosh\beta, \quad p_a^x = m_a \sinh\beta\ . \tag{2.5}$$

By Lorentz invariance, the two particle scattering amplitude will be a function of the rapidity difference $\beta = \beta_1 - \beta_2$ only.

What can be said about the matrix function $S_{a_1a_2}^{b_1b_2}(\beta)$? In order to avoid some technical complication we will consider FST which contains only particles of the same mass in the spectrum[1]. One can suppose that they are arranged in a multiplet of some finite dimensional (quantum) group $G$. In this case general principles of Quantum Field Theory and factorization condition lead to the following requirements for the two-particle S-matrix [8–10]:

1. The matrix function $S_{a_1a_2}^{b_1b_2}(\beta)$ must be analytic in the physical strip $0 \leqq \Im m\beta \leqq \pi$.

[1] We assume that there are no bound states.

2. Unitarity condition

$$S^{b_1 b_2}_{a_1 a_2}(\beta) S^{c_1 c_2}_{b_1 b_2}(-\beta) = \delta^{c_1}_{a_1} \delta^{c_2}_{a_2} \,. \tag{2.6}$$

3. Crossing symmetry

$$S^{b_1 b_2}_{a_1 a_2}(i\pi - \beta) = \mathbf{C}_{a_1 c} S^{c b_2}_{d a_2}(\beta) \mathbf{C}^{d b_1} \,. \tag{2.7}$$

Here $\mathbf{C}_{ab}$ is the charge conjugation matrix and

$$\mathbf{C}_{ab}\mathbf{C}^{bc} = \delta^c_a \,.$$

4. Yang–Baxter equation:

$$S^{c_1 c_2}_{a_1 a_2}(\beta_1 - \beta_2) S^{b_1 c_3}_{c_1 a_3}(\beta_1 - \beta_3) S^{b_2 b_3}_{c_2 c_3}(\beta_2 - \beta_3) \\ = S^{c_1 b_3}_{a_1 c_3}(\beta_1 - \beta_3) S^{c_2 c_3}_{a_2 a_3}(\beta_2 - \beta_3) S^{b_1 b_2}_{c_1 c_2}(\beta_1 - \beta_2) \,. \tag{2.8}$$

To describe a space of states in a massive integrable model, it is convenient to introduce the formal Zamolodchikov–Faddeev algebra [10, 24]. It is generated by the operators $V_a(\beta)$ which satisfy the commutation relation

$$V_{a_1}(\beta_1) V_{a_2}(\beta_2) = S^{b_1 b_2}_{a_1 a_2}(\beta_1 - \beta_2) V_{b_2}(\beta_2) V_{b_1}(\beta_1) \,. \tag{2.9}$$

Asymptotic states (2.1) form the space of representation of the Zamolodchikov–Faddeev algebra. We will denote it as $\pi_A$ and

$$A_a(\beta) = \pi_A[V_a(\beta)] \,.$$

One can interpret $A_a(\beta)$ as particle creation operators, so

$$|A_{a_1}(p_1) \dots A_{a_n}(p_n)\rangle_{\text{out}} = A_{a_1}(\beta_1) \dots A_{a_n}(\beta_n) |\text{vac}\rangle \,,$$

$$|A_{a_1}(p_1) \dots A_{a_n}(p_n)\rangle_{\text{in}} = A_{a_n}(\beta_n) \dots A_{a_1}(\beta_1) |\text{vac}\rangle \,, \tag{2.10}$$

where $|\text{vac}\rangle$ is the vacuum state (the state without any particle) and $\beta_1 < \beta_2 < \dots < \beta_n$.

The conjugate operators annihilate the vacuum state

$$[A_a(\beta)]^+ |\text{vac}\rangle = 0 \,. \tag{2.11}$$

They satisfy the commutation relations:

$$[A_{a_1}(\beta_1)]^+ A_{a_2}(\beta_2) = A_{b_2}(\beta_2) S^{a_1 b_2}_{b_1 a_2}(\beta_2 - \beta_1) [A_{b_1}(\beta_1)]^+ + 2\pi \delta^{a_1}_{a_2} \delta(\beta_1 - \beta_2) \,. \tag{2.12}$$

Equation (2.12) specifies the structure of the Hilbert space on $\pi_A$.

Another important class of operators acting in the space $\pi_A$ is an infinite set $\{I_s\}$ of local commutative integrals of motion (IM) [6, 7]. The index $s$ denotes the spin of the conserved charge $I_s$. The asymptotic states (2.10) diagonalise local IM

$$I_s |\text{vac}\rangle = 0 \,,$$

$$I_s |A_{a_1}(p_1) \dots A_{a_n}(p_n)\rangle_{\text{in, out}} = \gamma^{(s)} \sum_{k=1}^{n} \exp(\beta_k s) |A_{a_1}(p_1) \dots A_{a_n}(p_n)\rangle_{\text{in, out}} \,, \tag{2.13}$$

where $\gamma^{(s)}$ are some numbers. It is convenient to consider the generating function $I(\alpha)$ such that

$$I(\alpha) = \sum_{s>0} I_s \exp(-\alpha s), \exp(\alpha) \to \infty ,$$
$$I(\alpha) = \sum_{s>0} I_{-s} \exp(\alpha s), \exp(\alpha) \to 0 . \tag{2.14}$$

The formula (2.13) means that the generating function (2.14) obeys the following commutation relation with operators $A_a(\beta)$:

$$[I(\alpha), A_a(\beta)] = \partial_\alpha \ln s(\alpha - \beta) A_a(\beta) . \tag{2.15}$$

The scalar function $s(\alpha)$ is an important characteristic of an integrable model.

Before ending this subsection, I wish to point out that the formal Zamolodchikov–Faddeev algebra (2.9) may have other interesting types of representations when relation (2.11) is not satisfied and the operators $\pi[V_a(\beta)]$ do not admit such a simple physical meaning as $A_a(\beta)$ [34].

*2.2. The Factorized Scattering Theories for* $SU(2)$ TM *and* SGM. Now, we shall consider examples of FST which contain only two particles $A_a (a = \pm 1)$ in their spectrum.

Suppose that the formal Zamolodchikov–Faddeev algebra (2.9) is defined by the two particle S-matrix with the following non-trivial elements [10, 45]

$$S_{++}^{++}(\beta) = S_{--}^{--}(\beta) = S(\beta) ,$$
$$S_{+-}^{+-}(\beta) = S_{-+}^{-+}(\beta) = S(\beta)\frac{\beta}{i\pi - \beta} ,$$
$$S_{+-}^{-+}(\beta) = S_{-+}^{+-}(\beta) = S(\beta)\frac{i\pi}{i\pi - \beta} , \tag{2.16}$$

here

$$S(\beta) = \frac{\Gamma\left(\frac{1}{2} - \frac{i\beta}{2\pi}\right)\Gamma\left(\frac{i\beta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{i\beta}{2\pi}\right)\Gamma\left(-\frac{i\beta}{2\pi}\right)} . \tag{2.17}$$

The S-matrix (2.16) satisfies all the axioms of FST, if the charge conjugation matrix $\mathbf{C}_{ab}$ is given by

$$\mathbf{C}_{ab} = \delta_{a+b,0} . \tag{2.18}$$

One can define an action of the Lie algebra $sl(2)$ on the Hilbert space $\pi_A$ corresponding to (2.16) as follows [46, 47];

1. We assume that the vacuum state $|\text{vac}\rangle$ is a $sl(2)$ singlet:

$$Q^{\pm}|\text{vac}\rangle = Q^0|\text{vac}\rangle = 0 , \tag{2.19}$$

where

$$Q^{\pm} = \pi_A[X^{\pm}], \quad Q^0 = \pi_A[H]$$

and $\{X^{\pm}, H\}$ is the Cartan–Weyl basis of $sl(2)$:

$$[H, X^{\pm}] = \pm\sqrt{2}X^{\pm} ,$$
$$[X^+, X^-] = \sqrt{2}H . \tag{2.20}$$

2. One particle states $|A_a(\beta)\rangle$ are identified with the fundamental representation $\mathscr{V}$ of the Lie algebra $sl(2)$:

$$
\begin{aligned}
Q^0|A_a(\beta)\rangle &= \frac{a}{\sqrt{2}}|A_a(\beta)\rangle \\
Q^{\pm}|A_{\pm}(\beta)\rangle &= 0\,, \\
Q^{\pm}|A_{\mp}(\beta)\rangle &= |A_{\pm}(\beta)\rangle\,.
\end{aligned} \tag{2.21}
$$

3. The $n$-particle in-states and out-states (2.10) are regarded respectively as the spaces $\mathscr{V}_n \otimes \cdots \otimes \mathscr{V}_1$ and $\mathscr{V}_1 \otimes \cdots \otimes \mathscr{V}_n$. Actions of the generators $Q^{\pm}, Q_0$ are specified by the coproduct:

$$\Delta(H) = 1 \otimes H + H \otimes 1\,, \tag{2.22}$$

$$\Delta(X^{\pm}) = X^{\pm} \otimes 1 - 1 \otimes X^{\pm}\,. \tag{2.23}$$

It is easy to check that this prescription introduces a structure of $sl(2)$ representation on the space $\pi_A$.

Precisely speaking, the unusual choice of the sign in the coproduct (2.23) implies that we are considering the representation of the quantum algebra $U_{-1}(sl(2))$. Let us discuss the reason of our choice. First of all, note that the in- and out-spaces $\mathscr{V}_2 \otimes \mathscr{V}_1, \mathscr{V}_1 \otimes \mathscr{V}_2$ are isomorphic, and that the two particles S-matrix defines the operator

$$\check{S} : \mathscr{V}_2 \otimes \mathscr{V}_1 \to \mathscr{V}_1 \otimes \mathscr{V}_2\,. \tag{2.24}$$

Using the explicit form of the S-matrix (2.16), one can represent this operator in the form:

$$\check{S} = S(\beta)\left(P_1 + \frac{i\pi + \beta}{i\pi - \beta}P_0\right)\,, \tag{2.25}$$

where $P_1$ and $P_0$ are projectors on three and one dimensional $U_{-1}(sl(2))$-irreducible components in the tensor product $\mathscr{V}_2 \otimes \mathscr{V}_1$. As it follows from (2.25), $\check{S}$ commutes with charges $Q^{\pm}, Q^0$ (2.19) and they are integrals of motion.

Thus the Quantum Field Theory corresponding to FST (2.16) contains conserved currents $J_{\mu}^{m}, \{m = 0, \pm; \mu = x, t\}$ and

$$
\begin{aligned}
Q^0 &= \int_{-\infty}^{+\infty} dx J_t^0(x,t)\,, \\
Q^{\pm} &= \int_{-\infty}^{+\infty} dx J_t^{\pm}(x,t)\,.
\end{aligned} \tag{2.26}
$$

One can expect that they are local operators in the theory. It means that commutators

$$[J_{\mu}^{m}(x), J_{\mu}^{m}(y)]$$

vanish on the spacelike Minkowski interval $(x^{\mu} - y^{\mu})^2 < 0$. Moreover, from the formulas for the coproduct (2.22), (2.23) we conjecture that the current $J_{\mu}^{0}$ is a local operator and currents $J_{\mu}^{\pm}$ are semilocal ones with respect the "elementary" field $\Psi_a(x)$. The "elementary" field $\Psi_a(x)$ means any field with nonzero matrix elements between vacuum and one particle states.

Let me recall here the definition of a mutual locality index. Consider the operator product

$$A(x^{\mu})B(y^{\mu})\,.$$

As a function of the variable $x$ it can have nontrivial monodromic properties. Suppose

$$\mathscr{A}_C[A(x^\mu)B(y^\mu)] = \exp(2\pi i\omega(A,B))A(x^\mu)B(y^\mu)\,. \tag{2.27}$$

Here the symbol $\mathscr{A}_C$ means the analytical continuation along a counterclockwise contour $C$ around the point $y^\mu$. Then the number $\omega(A,B)$ is the mutual locality index for the fields $A$ and $B$. In the present case

$$\omega(J^0_\mu, \Psi) = 0, \quad \omega(J^\pm_\mu, \Psi) = \frac{1}{2}\,. \tag{2.28}$$

It is well known that FST (2.16) corresponds to $SU(2)$ TM, which can be defined by the Lagrangian [25–27]

$$L = \int_{-\infty}^{+\infty} dx(i\bar{\psi}\gamma^\mu\partial_\mu\psi - gJ^m_\mu J^\mu_m)\,. \tag{2.29}$$

The fields $\psi = \{\psi^a_i\}$ are Dirac spinors with the spinor index $\{i = 1,2\}$ and isotopic index $\{a = 1,2\}$ and the currents $J^m_\mu(x)$ are equal to

$$J^0_\mu = \frac{1}{\sqrt{2}}\bar{\psi}\gamma_\mu\sigma^3\psi\,,$$

$$J^\pm_\mu = \frac{1}{2}\bar{\psi}\gamma_\mu(\sigma^1 \pm i\sigma^2)\psi\,. \tag{2.30}$$

Here the Pauli matrices $\sigma^k, \{k = 1,2,3\}$ act on isotopic indices of spinors.

On the classical level the theory (2.29) is conformally invariant. However, its quantum spectrum contains a free massless boson and two massive kinks $A_a$ forming the fundamental multiplet of the isotopic group $SU(2)$. The two-particles S-matrix for the kinks is determined by (2.16).

Another non-trivial example of FST is connected with SGM [28, 29], whose Lagrangian is given by

$$L = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}(\partial_\mu\varphi)^2 + \frac{m_0^2}{b^2}\cos(b\varphi)\right]\,. \tag{2.31}$$

If the parameter

$$\xi = \frac{b^2}{8\pi - b^2} \tag{2.32}$$

is more than one, the quantum spectrum of SGM contains only solitons $A_a$ possessing an internal degree of freedom $a = \pm 1$ (soliton-antisoliton). Their two-particle $S$-matrix was found in the pioneering work [8]. It reads explicitly

$$S^{++}_{++}(\beta) = S^{--}_{--}(\beta) = S(\beta)\,,$$

$$S^{+-}_{+-}(\beta) = S^{-+}_{-+}(\beta) = S(\beta)\frac{\sinh\frac{\beta}{\xi},}{\sinh\frac{i\pi-\beta}{\xi}}\,,$$

$$S^{-+}_{+-}(\beta) = S^{+-}_{-+}(\beta) = S(\beta)\frac{\sinh\frac{i\pi}{\xi}}{\sinh\frac{i\pi-\beta}{\xi}}\,. \tag{2.33}$$

Here

$$S(\beta) = -\frac{\Gamma\left(\frac{1}{\xi}\right)\Gamma\left(1+\frac{i\beta}{\pi\xi}\right)}{\Gamma\left(\frac{1}{\xi}+\frac{i\beta}{\pi\xi}\right)} \prod_{p=1}^{\infty} \frac{R_p(\beta)R_p(i\pi-\beta)}{R_p(0)R_p(i\pi)}, \tag{2.34}$$

$$R_p(\beta) = \frac{\Gamma\left(\frac{2p}{\xi}+\frac{i\beta}{\pi\xi}\right)\Gamma\left(1+\frac{2p}{\xi}+\frac{i\beta}{\pi\xi}\right)}{\Gamma\left(\frac{2p+1}{\xi}+\frac{i\beta}{\pi\xi}\right)\Gamma\left(1+\frac{2p-1}{\xi}+\frac{i\beta}{\pi\xi}\right)}.$$

The charge conjugation matrix $\mathbf{C}_{ab}$ for this FST has the form (2.18).

The Hilbert space of asymptotic states for SGM possesses a symmetry described by the quantum algebra $U_q(sl(2))$ [48, 49, 47] with

$$q = \exp\left(i\pi\frac{1+\xi}{\xi}\right). \tag{2.35}$$

To fix the notation let me recall that the algebra $U_q(sl(2))$ is generated by operators $E^{\pm}, H$ with the commutation relations [50, 51]:

$$[H, E^{\pm}] = \pm\sqrt{2}E^{\pm},$$
$$[E^+, E^-] = \frac{q^{\sqrt{2}H} - q^{-\sqrt{2}H}}{q - q^{-1}} \tag{2.36}$$

and coproduct

$$\Delta(H) = 1 \otimes H + H \otimes 1,$$
$$\Delta(E^{\pm}) = q^{-\frac{\sqrt{2}}{2}H} \otimes E^{\pm} + E^{\pm} \otimes q^{\frac{\sqrt{2}}{2}H}. \tag{2.37}$$

The standard realization of $U_q(sl(2))$-invariance in SGM implies that we have to identify the vectors

$$\hat{A}_a(\beta)|\text{vac}\rangle,$$

where

$$\hat{A}_a(\beta) = \exp\left(a\frac{\beta}{2\xi}\right)A_a(\beta), \tag{2.38}$$

with the basis of the fundamental representation of $U_q(sl(2))$. The commutation relations for the operators (2.38) are defined by the matrix

$$\hat{S}_{ab}^{cd}(\beta) = \exp\left(\frac{(a-c)\beta}{2\xi}\right)S_{ab}^{cd}(\beta). \tag{2.39}$$

One can prove that the operator $\check{S}$ (2.24) corresponding to the matrix (2.39) commutes with the action of the charges

$$Q^{\pm} = \pi_A\left[E^{\pm}q^{\pm\frac{\sqrt{2}}{2}H}\right],$$
$$Q^0 = \pi_A[H]. \tag{2.40}$$

It is necessary to point out that there is an essential difference between $SU(2)$ TM and SGM. The currents $J_\mu^{\pm}$ corresponding to the conserved charges $Q^{\pm}$ are not local operators in SGM. At the same time the current $J^0_\mu$ generating the $U(1)$-charge $Q^0$ is a local one. It is also mutually local with respect to the "elementary" field.

The structures of local commutative IM $I_s$ are identical for $SU(2)$ TM and SGM. The function $s(\alpha)$ (2.15) was found in the work [52]

$$s(\alpha) = \coth \frac{\alpha}{2} \, . \tag{2.41}$$

As it follows from this formula, the spins of the local commutative IM are $s = 1$ (mod 2). Note that the first ones

$$I_1 - I_{-1}, \quad I_1 + I_{-1}$$

are the momentum and energy charges.

*2.3. Form-Factors.* In many cases the FST data are sufficient to reconstruct matrix elements

$$F^{b_1 \dots b_m}_{a_1 \dots a_n}(\beta'_1, \dots \beta'_m | \beta_1, \dots \beta_n) = {}_{\text{out}}\langle A^{b_m}(\beta'_m) \dots A^{b_1}(\beta'_1) | O(0) | A_{a_1}(\beta_1) \dots A_{a_n}(\beta_n) \rangle_{\text{in}} \, , \tag{2.42}$$

of an hermitian local operator $O(x)$ between asymptotic states [17, 22]. It is convenient to introduce the following functions, called form-factors

$$F_{a_1 \dots a_n}(\beta_1, \dots \beta_n) = \langle \text{vac} | O(0) | A_{a_1}(\beta_1), \dots A_{a_n}(\beta_n) \rangle_{\text{in}} \, , \tag{2.43}$$

which are matrix elements of an operator $O(x)$ at the origin between an n-particle in-state and the vacuum state. Crossing symmetry implies that a general matrix element (2.42) is obtained by an analytical continuation of (2.43), and equals [17]

$$F^{b_1 \dots b_m}_{a_1 \dots a_n}(\beta'_1, \dots \beta'_m | \beta_1, \dots \beta_n) = \mathbf{C}^{b_1 c_1} \dots \mathbf{C}^{b_m c_m} F_{a_1 \dots a_n c_1 \dots c_m} \times (\beta_1, \dots \beta_n, \beta'_1 + i\pi, \dots \beta'_m + i\pi) \, . \tag{2.44}$$

Thus, to describe the local operator one should present a set of tensor valued functions (2.43). Their reconstruction is based on the following system of axioms [22, 23, 16]:

## Axioms

1. *Function $F_{a_1 \dots a_n}(\beta_1, \dots \beta_n)$ is analytic in variables $\beta_{ij} = \beta_i - \beta_j$ inside the strip $0 < \Im m\beta < 2\pi$ except for simple poles. It becomes the physical matrix elements* (2.43) *when all $\beta_i$ are real and ordered as follows*:

$$\beta_1 < \beta_2 < \dots < \beta_n \, .$$

2. *Relativistic invariance demands that form-factors satisfy the equation*

$$F_{a_1 \dots a_n}(\beta_1 + \theta, \beta_2 + \theta, \dots \beta_n + \theta) = \exp(\theta S(O)) F_{a_1 \dots a_n}(\beta_1, \dots \beta_n) \, , \tag{2.45}$$

*where $s(O)$ is the spin of the local operator $O(x)$.*

3. *Form-factors should satisfy the symmetry property* (*Watson's theorem*)

$$
\begin{aligned}
&F_{a_1\dots a_{j+1}a_j\dots a_n}(\beta_1,\dots\beta_{j+1},\beta_j,\dots\beta_n) \\
&= S^{c_jc_{j+1}}_{a_ja_{j+1}}(\beta_j-\beta_{j+1})F_{a_1\dots c_jc_{j+1},\dots a_n}(\beta_1,\dots\beta_j,\beta_{j+1},\dots\beta_n)\,.
\end{aligned} \tag{2.46}
$$

4. *Form-factors satisfy the equation*

$$
\begin{aligned}
&F_{a_1\dots a_n}(\beta_1,\dots\beta_{n-1},\beta_n+2\pi i) \\
&= \exp(2\pi i\omega(O,\Psi))F_{a_na_1\dots a_{n-1}}(\beta_n,\beta_1,\dots\beta_{n-1})\,,
\end{aligned} \tag{2.47}
$$

*where the shift by* $2\pi i$ *is understood as an analytical continuation and the number* $\omega(O,\Psi)$ *means the mutual locality index* (2.27) *for the operator* $O(x)$ *and the "elementary" field* $\Psi$.

5. *Form-factors* $F_{a_1\dots a_n}(\beta_1,\dots\beta_n)$ *being considered as a function of* $\beta_n$ *have simple poles at the points* $\beta_n=\beta_j+i\pi$ *with the following residues:*

$$
\begin{aligned}
iF_{a_1\dots a_n}(\beta_1,\dots\beta_n) = \mathbf{C}_{a_na'_j}&\frac{F_{a'_1\dots\hat{a}'_j\dots a'_{n-1}}(\beta_1,\dots\hat{\beta}_j,\dots\beta_{n-1})}{\beta_n-\beta_j-\pi i} \\
&\times[\delta^{a'_1}_{a_1}\dots\delta^{a'_{j-1}}_{a_{j-1}}S^{a'_{n-1}a'_j}_{a_{n-1}c_1}(\beta_{n-1}-\beta_j)S^{a'_{n-2}c_1}_{a_{n-2}c_2}(\beta_{n-2}-\beta_j) \\
&\times\dots S^{a'_{j+1}c_{n-j-2}}_{a_{j+1}a_j}(\beta_{j+1}-\beta_j) \\
&-\exp(2\pi i\omega(O,\Psi))S^{a'_ja'_1}_{c_1a_1}(\beta_j-\beta_1) \\
&\times\dots S^{c_{j-3}a'_{j-2}}_{c_{j-2}a_{j-2}}(\beta_j-\beta_{j-2})S^{c_{j-2}a'_{j-1}}_{a_ja_{j-1}}(\beta_j-\beta_{j-1}) \\
&\times\delta^{a'_{j+1}}_{a_{j+1}}\dots\delta^{a'_{n-1}}_{a_{n-1}}]+\cdots\,.
\end{aligned} \tag{2.48}
$$

*In the absence of bound states these poles are the only singularities of* $F_{a_1\dots a_n}(\beta_1,\dots\beta_n)$ *in the strip* $0<\Im m\beta_n<2\pi$ *for real* $\beta_1,\dots\beta_{n-1}$.

It was shown [21, 22] that the operators $O(x)$ defined by the matrix elements (2.43) satisfy locality relations provided the form-factors satisfy 1–5.

Once the form factors of the theory are known, correlation functions of local operators can be written as an infinite series over multi-particle intermediate states. For instance, the two point function of an operator $O(x)$ for the spacelike Minkowski interval $(x-y)^2=-r^2$ is given by

$$
\begin{aligned}
&\langle O(x)O(y)\rangle = \\
&\sum_{n=0}^{\infty}\int\frac{d\beta_1\dots d\beta_n}{n!(2\pi)^n}F_{a_1\dots a_n}(\beta_1,\dots\beta_n)F^{a_n\dots a_1}(\beta_n,\dots\beta_1)\exp(-rm\sum_{k=1}^{n}\cosh\beta_k)\,.
\end{aligned} \tag{2.49}
$$

All the integrals are non-singular and convergent. The series is expected to be convergent as well. Similar expressions can be derived for multi-point correlators.

## 3. Form-Factors Reconstruction

The system of form-factors axioms (1–5) is a complicated Riemann–Hilbert problem. Its solution can be based on the following idea.

Let us assume that we have the representation $\pi_Z$ of the formal Zamolodchikov–Faddeev algebra, which satisfies the requirements;

1. In the space $\pi_Z$ an action of the operators $Z_a(\beta) = \pi_Z[V_a(\beta)]$ is defined

$$Z_a(\beta_1)Z_b(\beta_2) = S_{ab}^{cd}(\beta_1 - \beta_2)Z_d(\beta_2)Z_c(\beta_1) . \tag{3.1}$$

2. The singular part of the operator product

$$Z_a(\beta_2)Z_b(\beta_1) ,$$

being considered as a function of the complex variable $\beta_2$ for real $\beta_1$ in the upper half plane $\Im m\beta_2 \geqq 0$, contains only simple pole with the residue

$$iZ_a(\beta_2)Z_b(\beta_1) = \frac{\mathbf{C}_{ab}}{\beta_2 - \beta_1 - \pi i} + \cdots . \tag{3.2}$$

It means that there is only one singularity (3.2) depending on the real parameter $\beta_1$ for a general matrix element

$$\langle u|Z_a(\beta_2)Z_b(\beta_1)|v\rangle, |u\rangle, |v\rangle \in \pi_Z .$$

Of course, this matrix element may also have other singularities for $\Im m\beta_2 \geqq 0$, but their positions are defined by the vectors $|u\rangle$ and $|v\rangle$ only.

3. There is a unique $G$-invariant[2] vector $|0\rangle$(vacuum state) in the space $\pi_Z$ such that the two point function

$$G_{ab}(\beta_1, \beta_2) = \langle 0|Z_a(\beta_2)Z_b(\beta_1)|0\rangle , \tag{3.3}$$

satisfies the following constraints:

a. It depends only on the difference $\beta = \beta_1 - \beta_2$.

b. As a function of the complex variable $\beta$, it is analytic in the lower half plane $\Im m\beta \leqq 0$ except for one simple pole (3.2).

c. It is a bounded function for $\beta \to \infty, \Im m\beta \leqq 0$[3]

$$G_{ab}(\beta) = O(1), \beta \to \infty(\Im m\beta \leqq 0) . \tag{3.4}$$

The representation $\pi_Z$ will be completely specified if all vacuum matrix elements

$$G_{a_1\dots a_n}(\beta_1, \dots \beta_n) = \langle 0|Z_{a_n}(\beta_n)\dots Z_{a_1}(\beta_1)|0\rangle \tag{3.5}$$

are defined. The problem of their reconstruction is close to the Riemann–Hilbert problem (1–5).

To clarify this connection let us suppose that the following additional structures are present in the space $\pi_Z$:

1. The operator $K$, which obeys the commutation relation

$$Z_a(\beta + \theta) = \exp(-\theta K)Z_a(\beta)\exp(\theta K) . \tag{3.6}$$

---

[2] Recall that G is a finite dimensional symmetry group of the model

[3] On this requirement see the comment at the end of the next section

2. The map

$$O \to \Lambda(O) \in \mathrm{End}[\pi_Z] \tag{3.7}$$

from the space of local operators of the model under investigation to the endomorphism algebra of $\pi_Z$, which satisfies the conditions:

$$\Lambda(O)Z_a(\beta) = \exp(2\pi i\omega(O,\Psi))Z_a(\beta)\Lambda(O)\,,$$
$$\exp(\theta K)\Lambda(O)\exp(-\theta K) = \exp(\theta S(O))\Lambda(O)\,, \tag{3.8}$$

where the number $\omega(O,\Psi)$ coincides with the mutual locality index (2.27) of the "elementary" field $\Psi(x)$ and the local operator $O(x)$ with the spin $s(O)$.

Using the cyclic properties of a matrix trace and the relations (3.1), (3.2), (3.8) it is easy to verify that the function

$$F_{a_1\dots a_n}(\beta_1,\dots\beta_n) = \mathrm{Tr}_{\pi_Z}[\exp(2\pi iK)\Lambda(O)Z_{a_n}(\beta_n)\dots Z_{a_1}(\beta_1)] \tag{3.9}$$

formally satisfy the axioms (2.45)–(2.48). For example, let us get the formula (2.48). To do this, it is convenient to consider the more general trace than (3.9):

$$\mathrm{Tr}_{\pi_Z}[\exp(i(2\pi+\delta)K)\Lambda(O)Z_{a_n}(\beta_n)\dots Z_{a_1}(\beta_1)]\,.$$

As a function of $\beta_n$, this trace has poles at the points $\beta_n = \beta_j + i\pi$ and $\beta_n = \beta_j + i(\pi+\delta), j = 1,\dots n-1$. Their residues are defined by the operator decomposition (3.2) and the commutation relation (3.1). After taking the limit $\delta \to 0$, we obtain (2.48).

Certainly our observation is not the rigorous method to solve the Riemann–Hilbert problem (1–5), since we have no proof that the trace (3.9) exists and satisfies axiom 1. Nevertheless, it can be substantiated in some cases. In this paper it will be demonstrated for $SU(2)$ TM and SGM.

At the conclusion of this section let us briefly argue a physical interpretation of the space $\pi_Z$ [30]. First of all, note that the space $\pi_A$ is associated with an infinite line $t=$ const. The formula (3.9) implies that $\pi_A$ is obtained by "gluing" two copies of $\pi_Z$. So we can associate the space $\pi_Z$ with half infinite line. In other words one should consider $\pi_Z$ as a space of angular quantization of a massive integrable model [53, 54].

## 4. Two Point Vacuum Averages

To describe the representation $\pi_Z$ of the Zamolodchickov–Faddeev algebra for $SU(2)$ TM and SGM we have to reconstruct all vacuum averages

$$G_{a_1\dots a_n}(\beta_1,\dots\beta_n) = \langle 0|Z_{a_n}(\beta_n)\dots Z_{a_1}(\beta_1)|0\rangle\,.$$

Due to the unbroken $U(1)$-symmetry present in the models under investigation, they are non-trivial only for even n. In this section the two point functions

$$G_{ab}(\beta_1-\beta_2) = \langle 0|Z_a(\beta_2)Z_b(\beta_1)|0\rangle \tag{4.1}$$

will be found.

First, let us consider the case of $SU(2)$ TM. The commutation relation (3.1) means that (4.1) obeys the functional equation:

$$G_{ab}(-\beta) = S_{ab}^{cd}(\beta) G_{dc}(\beta) . \tag{4.2}$$

It is convenient to introduce the function

$$g(\beta) = k^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} + \frac{i\beta}{2\pi}\right)}{\Gamma\left(\frac{i\beta}{2\pi}\right)} , \tag{4.3}$$

where $k$ is an arbitrary number. It satisfies the following:

a. It is an analytical function the lower half plan $\Im m\beta \leqq 0$;

b. If $\beta \to \infty (\Im m\beta \leqq 0)$, then

$$g(\beta) = \kappa^{\frac{1}{2}} \left(i \frac{\beta}{2\pi}\right)^{\frac{1}{2}} \left(1 + O\left(\frac{1}{\beta}\right)\right) . \tag{4.4}$$

c. The function $g(\beta)$ obeys the functional equation:

$$S(\beta) = \frac{g(-\beta)}{g(\beta)} . \tag{4.5}$$

As it follows from (2.25), the general solution of Eq. (4.2) is given by:

$$G_{ab}(\beta) = \frac{g(\beta)}{g(-i\pi)} \left( \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{a}{2} & \frac{b}{2} & 0 \end{bmatrix}_{-1} \frac{A(\beta)}{i\pi + \beta} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{a}{2} & \frac{b}{2} & \frac{a+b}{2} \end{bmatrix}_{-1} B(\beta) \right) . \tag{4.6}$$

Here $A(\beta)$ and $B(\beta)$ are arbitrary even functions, and the symbol

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q$$

denotes Clebsch–Gordan coefficient for the quantum universal enveloping algebra $U_q(sl(2))$ [55][4], in particular

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{a}{2} & \frac{b}{2} & 0 \end{bmatrix}_{-1} = \frac{\delta_{a+b,0}}{\sqrt{2}} ,$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{a}{2} & \frac{b}{2} & \pm 1 \end{bmatrix}_{-1} = \delta_{a+b,\pm 1} ,$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{a}{2} & \frac{b}{2} & 0 \end{bmatrix}_{-1} = (-1)^{\frac{1-a}{2}} \frac{\delta_{a+b,0}}{\sqrt{2}} . \tag{4.7}$$

Due to the $U_{-1}(sl(2))$-invariance of the vacuum state $|0\rangle$, the function $B(\beta)$ (4.6) must be zero,

$$B(\beta) \equiv 0 . \tag{4.8}$$

At the same time, the analyticity condition implies that the even function $A(\beta)$ is analytic in the whole complex plane and

---

[4] In this work the normalization of Clebsh-Gordan coefficients is chosen as in Ref. [55]

$$A(-i\pi) = i\sqrt{2}\,.$$

Moreover, the boundary condition (3.4) provides the unique reconstruction of this function:

$$A(\beta) \equiv i\sqrt{2}\,. \tag{4.9}$$

We should note at this point that the boundary condition (3.4) for the two point function $G_{ab}(\beta)$ is closely connected with the unbroken symmetry condition. Indeed if the function $B(\beta)$ (4.6) is non-zero, we could not impose (3.4).

In this way, the two point function for SU(2) TM is given by

$$G_{ab}(\beta) = \frac{g(\beta)}{g(-i\pi)} \frac{i\delta_{a+b,0}}{i\pi + \beta}\,. \tag{4.10}$$

One can find the two point function for SGM in a similar fashion. I have to make the following comment only: To apply the boundary condition (3.4) one has to use the basis of the Zamolodchikov–Faddeev operators which conforms to the $G$-invariance of the theory. So, in the case of SGM we have to consider the following simple redefinition of the Zamolodchikov–Faddeev operators (compare with (2.38)):

$$\hat{Z}_a(\beta) = \exp\left(\frac{a\beta}{2\xi}\right) Z_a(\beta)\,. \tag{4.11}$$

Then the function

$$\hat{G}_{ab}(\beta) = \exp\left(-\frac{a\beta}{2\xi}\right) G_{ab}(\beta) \tag{4.12}$$

will satisfy (3.4). Here is its explicit form

$$\hat{G}_{ab}(\beta) = (-1)^{\frac{a+1}{2}} q^{\frac{a}{2}} \delta_{a+b,0} \frac{g(\beta)}{g(-i\pi)} \frac{\Gamma(1+\frac{2}{\xi})\Gamma(-\frac{1}{\xi}+\frac{i\beta}{\pi\xi})}{i\pi\xi\Gamma(1+\frac{1}{\xi}+\frac{i\beta}{\pi\xi})}\,, \tag{4.13}$$

where

$$g(\beta) = \left[\frac{\kappa}{\Gamma\left(\frac{1}{\xi}\right)}\right]^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{\xi}+\frac{i\beta}{\pi\xi}\right)}{\Gamma\left(\frac{i\beta}{\pi\xi}\right)} \prod_{p=1}^{+\infty} \frac{[R_p(i\pi)R_p(0)]^{\frac{1}{2}}}{R_p(\beta)}\,. \tag{4.14}$$

The functions $R_p(\beta)$ are defined by Eq. (2.34) and $\kappa$ is an arbitrary constant.

## 5. Bosonization Technique for Massive Integrable Models

In the following I shall have to introduce a lot of new functions and constants. Their explicit expressions are complicated enough and essentially depend on a model. In order to avoid bulky formulas I will try to use unique symbolic notations for these objects and point out their universal properties. One can find the explicit forms of the introduced functions and constants in Appendices 2, 3.

*5.1. Free Boson Field.* Consider the formal operator-valued function $\phi(\beta)$ which obeys the commutation relation for real $\beta$:

$$[\phi(\beta_1), \phi(\beta_2)] = \ln S(\beta_2 - \beta_1)\,. \tag{5.1}$$

In order to construct the representation of (5.1) it is necessary to specify the two point correlation function

$$\langle 0|\phi(\beta_1)\phi(\beta_2)|0\rangle = -\ln g(\beta_2-\beta_1)\rangle. \tag{5.2}$$

The compatibility of (5.2) and (5.1) means that the functions $g(\beta), S(\beta)$ are connected by the relation:

$$S(\beta) = \frac{g(-\beta)}{g(\beta)}\,. \tag{5.3}$$

We will demand that $g(\beta)$ satisfies also the proper analyticity and boundary conditions, which are the natural generalization of properties of the free boson Green function in CFT.

a. It is an analytical function without zeroes and poles in the lower half plane $\Im m\beta \leqq 0$ except a simple zero at $\beta = 0$.

b. If $\beta \to \infty(\Im m\beta \leqq 0)$, then

$$\partial_\beta \ln g(\beta) = O\left(\frac{1}{\beta}\right)\,.$$

Equation (5.3) supplemented with the conditions (a–b) is the simplest Riemann–Hilbert problem. It has the unique solution (4.3).

Now, let us introduce the field

$$\bar{\phi}(\beta) = \phi\left(\beta + i\frac{\pi}{2}\right) + \phi\left(\beta - i\frac{\pi}{2}\right)\,. \tag{5.4}$$

It should be considered as a more fundamental object than the field $\phi(\beta)$, since it has more simple and universal properties. For example, using the unitarity and crossing symmetry equations

$$\begin{aligned} S(\beta)S(-\beta) &= 1\,,\\ S(i\pi-\beta) &= \frac{\beta}{i\pi-\beta}S(\beta)\,, \end{aligned} \tag{5.5}$$

one can get the commutation relations for this field

$$[\bar{\phi}(\beta_1), \bar{\phi}(\beta_2)] = \ln\frac{\beta_2-\beta_1-i\pi}{\beta_2-\beta_1+i\pi}\,, \tag{5.6}$$

$$[\bar{\phi}(\beta_1), \phi(\beta_2)] = \ln\frac{\frac{i\pi}{2}-\beta_2+\beta_1}{\frac{i\pi}{2}+\beta_2-\beta_1}\,. \tag{5.7}$$

The two point functions

$$\langle 0|\bar{\phi}(\beta_1)\phi(\beta_2)|0\rangle = \ln w(\beta_2-\beta_1)\,, \tag{5.8}$$

$$\langle 0|\bar{\phi}(\beta_1)\bar{\phi}(\beta_2)|0\rangle = -\ln\bar{g}(\beta_2-\beta_1) \tag{5.9}$$

also have simple forms:

$$\begin{aligned} w(\beta) &= k^{-1}\frac{2\pi}{i\left(\beta+i\frac{\pi}{2}\right)}\,,\\ \bar{g}(\beta) &= -k^2\frac{\beta(\beta+i\pi)}{4\pi^2}\,. \end{aligned} \tag{5.10}$$

Notice that the functions $g(\beta), \bar{g}(\beta), w(\beta)$ are connected as follows:

$$w(\beta) = \left[g\left(\beta + i\frac{\pi}{2}\right) g\left(\beta - i\frac{\pi}{2}\right)\right]^{-1} ,$$
$$\bar{g}(\beta) = \left[w\left(\beta + i\frac{\pi}{2}\right) w\left(\beta - i\frac{\pi}{2}\right)\right]^{-1} . \tag{5.11}$$

*5.2. Elementary Vertex Operators.* Consider the elementary vertex operators

$$V(\beta) = \exp(i\phi(\beta)) = (g(0))^{\frac{1}{2}} : \exp(i\phi(\beta)) : ,$$
$$\bar{V}(\beta) = \exp(-i\bar{\phi}(\beta)) = (\bar{g}(0))^{\frac{1}{2}} : \exp(-i\bar{\phi}(\beta)) : , \tag{5.12}$$

here the dots implies the normal ordering of exponents.[5] The function $g(\beta)$ and $\bar{g}(\beta)$ have simple zeroes when $\beta = 0$, hence the vertex operators (5.12) should be regularized. We shall do it in a similar fashion as in 2D CFT. It means that regularized values of the functions $g(\beta)$ and $\bar{g}(\beta)$ for $\beta = 0$ are the limits

$$g_{\mathrm{reg}}(0) = \lim_{\beta\to 0} \frac{g(\beta)}{\beta} \equiv \rho^2 ,$$
$$\bar{g}(0)_{\mathrm{reg}} = \lim_{\beta\to 0} \frac{\bar{g}(\beta)}{\beta} \equiv \bar{\rho}^2 . \tag{5.13}$$

The correlations functions (5.2), (5.8), (5.9) define expressions of the operator products:

$$V(\beta_1)V(\beta_2) = \rho^2 g(\beta_2 - \beta_1) : V(\beta_1)V(\beta_2) : ,$$
$$\bar{V}(\beta_1)V(\beta_2) = \rho\bar{\rho} w(\beta_2 - \beta_1) : \bar{V}(\beta_1)V(\beta_2) : ,$$
$$\bar{V}(\beta_1)\bar{V}(\beta_2) = \bar{\rho}^2 \bar{g}(\beta_2 - \beta_1) : \bar{V}(\beta_1)\bar{V}(\beta_2) : . \tag{5.14}$$

*5.3. "Screening Charge."* In the space of representation $\pi_Z$ of the algebra (5.1) (it will be accurately described later) one can introduce the operator $\mathscr{X}$

$$\langle u|\mathscr{X}|v\rangle \equiv \eta^{-1} \left\langle u\middle|\int_C \frac{d\gamma}{2\pi} \bar{V}(\gamma)\middle|v\right\rangle , \tag{5.15}$$

where $\eta$ is an irrelevant constant. The contour of integration $C$ is specified as follows:

First of all, we assume that matrix elements

$$\langle u|\bar{V}(\gamma)|v\rangle$$

are meromorphic functions decreasing infinitely faster than $\gamma^{-1}$ for all vectors $|u\rangle, |v\rangle \in \pi_Z$. Then the contour $C$ goes from $\Re\mathrm{e}\,\gamma = -\infty$ to $\Re\mathrm{e}\,\gamma = +\infty$. It lies above all singularities whose positions depend on the vector $|v\rangle$, but below singularities depending on $|u\rangle$.

To illustrate this definition let us calculate the following matrix elements of the operator $\mathscr{X}$:

---

[5] Since the two point functions are $c$-numbers Wick theorem can be applied

$$J(\beta_1-\beta_2)=\langle 0|V(\beta_2)\mathscr{X}V(\beta_1)|0\rangle\,,$$
$$J_1(\beta_1-\beta_2)=\langle 0|V(\beta_2)V(\beta_1)\mathscr{X}|0\rangle\,,$$
$$J_2(\beta_1-\beta_2)=\langle 0|\mathscr{X}V(\beta_2)V(\beta_1)|0\rangle\,. \tag{5.16}$$

From the formula (5.14) we immediately obtain the integral representation for these matrix elements

$$J(\beta_1-\beta_2)=\eta^{-1}\rho^2\bar{\rho}g(\beta_1-\beta_2)\int_C\frac{d\gamma}{2\pi}w(\gamma-\beta_2)w(\beta_1-\gamma)\,,$$
$$J_1(\beta_1-\beta_2)=\eta^{-1}\rho^2\bar{\rho}g(\beta_1-\beta_2)\int_{C_1}\frac{d\gamma}{2\pi}w(\gamma-\beta_2)w(\gamma-\beta_1)\,,$$
$$J_2(\beta_1-\beta_2)=\eta^{-1}\rho^2\bar{\rho}g(\beta_1-\beta_2)\int_{C_2}\frac{d\gamma}{2\pi}w(\beta_2-\gamma)w(\beta_1-\gamma)\,. \tag{5.17}$$

In the case at hand the contours of integration are specified by the prescription (see formula (5.10)):

The contour $C$ lies above the pole $\gamma=\beta_1+i\frac{\pi}{2}$, but below $\gamma=\beta_2-i\frac{\pi}{2}$;
The contour $C_1$ lies below the poles $\gamma=\beta_{1,2}-i\frac{\pi}{2}$;
The contour $C_2$ lies above the poles $\gamma=\beta_{1,2}+i\frac{\pi}{2}$.
The functions (5.17) are given by:

$$J(\beta)=\frac{2\pi\bar{\rho}}{\eta k}\frac{g(\beta)}{g(-i\pi)}\frac{1}{\beta+i\pi}\,,$$
$$J_1(\beta)=J_2(\beta)=0\,. \tag{5.18}$$

It is convenient to chose the parameter $\eta$ such that $\frac{2\pi\bar{\rho}}{\eta k}=1$, so

$$\eta=\sqrt{\frac{\pi}{i}}\,. \tag{5.19}$$

Note that there is the simple relation between the functions (4.10) and (5.18),

$$G_{ab}(\beta)=iJ(\beta)\delta_{ab}\,. \tag{5.20}$$

*5.4. Free field representation for the Zamolodchikov–Faddeev algebra of $SU(2)$* TM. Define the operators $Z_a(\beta)$ for $SU(2)$ TM by the formulas:

$$Z_+(\beta)=V(\beta)\,,$$
$$Z_-(\beta)=i(\mathscr{X}V(\beta)+V(\beta)\mathscr{X})\,. \tag{5.21}$$

The following propositions describe commutation relations and analytical properties of $Z_a(\beta)$.

**Proposition 1.** *The operators* (5.21) *satisfy the Zamolodchikov–Faddeev commutation relations* (3.1).

**Proposition 2.** *The singular part of the operator product*

$$Z_a(\beta_2)Z_b(\beta_1)$$

*considered as a function of the complex variable* $\beta_2$ *for real* $\beta_1$ *in the upper half plane* $\Im m\beta_2 \geqq 0$ *contains only one simple pole with residue equals* $\frac{1}{i}\mathbf{C}_{ab}$.

We have proved these propositions for vacuum matrix elements in the last subsection. The main steps of the general proof can be found in Appendix 1.

The formulas (5.21) define the integral representation for n-point correlation functions (3.5) for $SU(2)$ TM. They have simple group theoretic meaning. Indeed, the operators $Z_a(\beta)$ can be regarded as the basis of the fundamental representation of $U_{-1}(sl(2))$ (2.20). If we identify $\mathcal{X}$ with the $\pi_Z[X^-]$, it is easy to recognize the coproduct (2.23) in the definition (5.21). In the next section the generators $\pi_Z[X^+]$ and $\pi_Z[H]$ will be constructed.

## 6. Fock Realization of the Representation $\pi_Z$

In this section it will be shown that the space $\pi_Z$ can be represented in the form:

$$\pi_Z = \lim_{\varepsilon\to 0} \pi_Z^{\varepsilon} . \tag{6.1}$$

Here the space $\pi_Z^{\varepsilon}$ admits a decomposition into a direct sum of Fock modules and $\varepsilon$ is a parameter of the ultraviolet regularization. In the case of $SU(2)$ TM the below construction is equivalent to the Frenkel–Jing bosonization of the quantum affine algebra $U_q(\widehat{sl(2)})$ [37, 35].

*6.1. Oscillator Decomposition.* Trying to get an oscillator decomposition for the field $\phi(\beta)$ (5.1), we run into the well known problem. The function $\ln S(\beta)$ does not tend to zero when $\beta \to \pm\infty$. So, the commutation relation (5.1) is not compatible with the decreasing boundary condition for the field $\phi(\beta)$ and it cannot be decomposed into a Fourier integral.

A possible way out is to consider the field $\phi_\varepsilon(\beta)$ which is defined on the finite interval

$$-\frac{\pi}{\varepsilon} \leqq \beta \leqq \frac{\pi}{\varepsilon} \tag{6.2}$$

and satisfies the commutation relations:

$$[\phi_\varepsilon(\beta_1), \phi_\varepsilon(\beta_2)] = \ln S_\varepsilon(\beta_2 - \beta_1) . \tag{6.3}$$

The function $S_\varepsilon(\beta)$ must tend to $S(\beta)$ when $\varepsilon \to 0$ for finite $\beta$ and

$$S_\varepsilon\left(-\frac{\pi}{\varepsilon}\right) = S(-\infty) \equiv \exp(i\pi s) . \tag{6.4}$$

Then the field $\phi_\varepsilon(\beta)$ admits the following decomposition

$$\phi_\varepsilon(\beta) = s^{\frac{1}{2}}(Q - \varepsilon\beta P) + \phi_\varepsilon^{\mathrm{osc}}(\beta) , \tag{6.5}$$

where the field $\phi_\varepsilon^{\mathrm{osc}}(\beta)$ is periodic in the interval (6.2). Zero modes $P, Q$ obey the canonical commutation relation

$$[P, Q] = \frac{1}{i} \tag{6.6}$$

and commute with the oscillator part $\phi_\varepsilon^{\mathrm{osc}}(\beta)$. One should consider the operator $\phi(\beta)$ as a properly regularized limit of $\phi_\varepsilon(\beta)$.

Now let us apply this simple idea to our problem. In the present case the regularization (6.2) has a clear physical meaning. Indeed we regard the parameter $\beta$ as rapidity of physical excitations in massive models, so (6.2) is the ultraviolet cut-off.

Introduce the following expansion:

$$\phi_\varepsilon(\beta) = \sqrt{s}(Q - \varepsilon\beta P) + \sum_{m \neq 0} \frac{a_m}{i \sinh(\pi m \varepsilon)} \exp(im\varepsilon\beta) \,. \tag{6.7}$$

I wish to emphasize that this representation for the field $\phi_\varepsilon(\beta)$ will be used in the SGM model also. In the case of $SU(2)$ TM we have

$$s = \frac{1}{2}\,, \tag{6.8}$$

and the oscillator modes $a_m$ satisfy the commutation relation

$$[a_m, a_n] = \frac{\sinh \frac{\pi m \varepsilon}{2} \sinh \pi m \varepsilon}{m} \exp \frac{\pi |m| \varepsilon}{2} \delta_{m+n,0} \,. \tag{6.9}$$

The operator (6.7) commutes as (6.3) and the function $S_\varepsilon$ obeys the necessary requirements. It can be represented in the form:

$$S_\varepsilon(\beta) = \exp(-i\varepsilon s \beta) \frac{g_\varepsilon(-\beta)}{g_\varepsilon(\beta)} \,. \tag{6.10}$$

The function $g_\varepsilon(\beta)$ defines the commutator of the positive and negative frequency parts $\phi_\varepsilon^{\pm}(\beta)$ of the field $\phi_\varepsilon(\beta)$

$$[\phi_\varepsilon^+(\beta_1), \phi_\varepsilon^-(\beta_2)] = -\ln g_\varepsilon(\beta_2 - \beta_1)\,,$$

$$g_\varepsilon(\beta) = \exp\left[-\sum_{m=1}^{\infty} \frac{\exp \frac{\pi m \varepsilon}{2}}{2m \cosh \frac{\pi m \varepsilon}{2}} \exp(-im\varepsilon\beta)\right] \,. \tag{6.11}$$

The sum (6.11) converges only for $\Im m\beta < 0$. It can be analytically continued to the whole complex plane by the following:

$$g_\varepsilon(\beta) = [1 - \exp(-2\pi\varepsilon)]^{\frac{1}{2}} \frac{\Gamma_\varepsilon\left(\frac{1}{2} + \frac{i\beta}{2\pi}\right)}{\Gamma_\varepsilon\left(\frac{i\beta}{2\pi}\right)}\,, \tag{6.12}$$

here

$$\Gamma_\varepsilon(x) = [1 - \exp(-2\pi\varepsilon)]^{1-x} \prod_{k=1}^{+\infty} \frac{1 - \exp(-2\pi\varepsilon k)}{1 - \exp(-2\pi\varepsilon(x + k - 1))} \,. \tag{6.13}$$

In the limit $\varepsilon \to 0$ the "quantum" $\Gamma$-function (6.13) becomes the usual one and the functions $S_\varepsilon(\beta), g_\varepsilon(\beta)$ tend to $S(\beta), g(\beta)$. Note that the constant $k$ in the formula (4.3) is connected with the ultraviolet regularization of the theory. So all the final formulas should not depend on it.

Let us consider regularized versions of the operators introduced in the last section,

$$\begin{aligned}
\bar{\phi}_\varepsilon(\beta) &= \phi_\varepsilon\left(\beta + i\frac{\pi}{2}\right) + \phi_\varepsilon\left(\beta - i\frac{\pi}{2}\right), \\
V_\varepsilon(\beta) &= \exp(i\phi_\varepsilon(\beta)), \\
\bar{V}_\varepsilon(\beta) &= \exp(-i\bar{\phi}_\varepsilon(\beta)), \\
\mathcal{X}_\varepsilon &= \eta_\varepsilon^{-1} \int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}} \frac{d\gamma}{2\pi} \bar{V}_\varepsilon(\gamma).
\end{aligned} \tag{6.14}$$

The following proposition explains our choice of the regularization.

**Proposition 3.** *The operators*

$$\begin{aligned}
Z_{\varepsilon+}(\beta) &= \exp\left(\frac{i\varepsilon\beta}{4}\right) V_\varepsilon(\beta), \\
Z_{\varepsilon-}(\beta) &= i\exp\left(-\frac{i\varepsilon\beta}{4}\right)\left[\exp\frac{\pi\varepsilon}{4}\mathcal{X}_\varepsilon V_\varepsilon(\beta) + \exp\left(-\frac{\pi\varepsilon}{4}\right) V_\varepsilon(\beta)\mathcal{X}_\varepsilon\right],
\end{aligned} \tag{6.15}$$

*obey the Zamolodchikov–Faddeev commutation relations* (3.1) *with the S-matrix*:

$$\begin{aligned}
S_{++}^{++}(\beta) &= S_{--}^{--}(\beta) = S_\varepsilon(\beta), \\
S_{+-}^{+-}(\beta) &= S_{-+}^{-+}(\beta) = S_\varepsilon(\beta)\frac{\sinh\frac{i\varepsilon\beta}{2}}{\sinh\frac{i\varepsilon(i\pi-\beta)}{2}}, \\
S_{+-}^{-+}(\beta) &= S_{-+}^{+-}(\beta) = -S_\varepsilon(\beta)\frac{\sinh\frac{\pi\varepsilon}{2}}{\sinh\frac{i\varepsilon(i\pi-\beta)}{2}}.
\end{aligned} \tag{6.16}$$

*The operator product*

$$Z_{\varepsilon a}(\beta_2)Z_{\varepsilon b}(\beta_1)$$

*has a simple pole at the point* $\beta_2 = \beta_1 + i\pi$. *If the constant* $\eta_\varepsilon$ *reads*

$$\eta_\varepsilon = \left[\frac{2}{i\varepsilon}\sinh\frac{\pi\varepsilon}{2}\right]^{\frac{1}{2}}, \tag{6.17}$$

*then the residue is given by*

$$iZ_{\varepsilon a}(\beta_2)Z_{\varepsilon b}(\beta_1) = \frac{\delta_{a+b,0}}{\beta_2 - \beta_1 - i\pi} + \cdots. \tag{6.18}$$

*6.2. Second "Screening Charge."* As discussed already, we regard the operator $\mathcal{X}$ as the generator $\pi_Z[X^-]$ of the quantum algebra $U_{-1}(sl(2))$. Let us now define the actions of other generators. In order to do this, we have to introduce a set of new fields and vertex operators. First of all, consider the fields

$$\begin{aligned}
\phi'_\varepsilon(\alpha) &= -\sqrt{s'}(Q - \varepsilon\alpha P) - \sum_{m\neq 0}\frac{a'_m}{i\sinh(\pi m\varepsilon)}\exp(im\varepsilon\alpha), \\
\bar{\phi}'_\varepsilon(\alpha) &= \phi'_\varepsilon\left(\alpha + i\frac{\pi}{2}\right) + \phi'_\varepsilon\left(\alpha - i\frac{\pi}{2}\right).
\end{aligned} \tag{6.19}$$

In the case of $SU(2)$ TM the parameter $s'$ is given by

$$s' = s = \frac{1}{2} \tag{6.20}$$

and the normal modes $a'_m$ satisfy the commutation relation:

$$[a'_m, a'_n] = \frac{\sinh \frac{\pi m \varepsilon}{2} \sinh \pi m \varepsilon}{m} \exp\left(-\frac{\pi |m| \varepsilon}{2}\right) \delta_{m+n,0} . \tag{6.21}$$

They are simply connected with the oscillators $a_m$ (6.9),

$$a'_m \exp \frac{\pi |m| \varepsilon}{4} = a_m \exp\left(-\frac{\pi |m| \varepsilon}{4}\right) . \tag{6.22}$$

Define the operators $V'_\varepsilon(\alpha), \bar{V}'_\varepsilon(\alpha)$ and $\mathcal{X}'_\varepsilon$ by analogy with (6.14),

$$\begin{aligned} V'_\varepsilon(\alpha) &= \exp(i\phi'_\varepsilon(\alpha)) , \\ \bar{V}'_\varepsilon(\alpha) &= \exp(-i\bar{\phi}'_\varepsilon(\alpha)) , \\ \mathcal{X}_\varepsilon &= \eta'^{-1}_\varepsilon \int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}} \frac{d\delta}{2\pi} \bar{V}'_\varepsilon(\delta) . \end{aligned} \tag{6.23}$$

By working out the operator product expansion, one may check that $\mathcal{X}_\varepsilon$, $\mathcal{X}'_\varepsilon$ and $P$ satisfy the commutation relations:

$$\begin{aligned} [P, \mathcal{X}_\varepsilon] &= -\sqrt{2}\mathcal{X}_\varepsilon , \\ [P, \mathcal{X}'_\varepsilon] &= \sqrt{2}\mathcal{X}'_\varepsilon , \\ [\mathcal{X}'_\varepsilon, \mathcal{X}'_\varepsilon] &= \frac{\sinh \frac{\sqrt{2}}{2}\pi\varepsilon P}{\sinh \frac{\pi\varepsilon}{2}} . \end{aligned} \tag{6.24}$$

As follows from the explicit form of $Z_{\varepsilon a}(\beta)$ (6.15), the coproduct for the quantum algebra (6.24) reads

$$\begin{aligned} \Delta(P) &= 1 \otimes P + P \otimes 1 , \\ \Delta(\mathcal{X}_\varepsilon) &= \mathcal{X}_\varepsilon \otimes 1 - \exp\left(-\frac{\sqrt{2}}{2}P\pi\varepsilon\right) \otimes \mathcal{X}_\varepsilon , \\ \Delta(\mathcal{X}'_\varepsilon) &= \mathcal{X}'_\varepsilon \otimes \exp\left(\frac{\sqrt{2}}{2}P\pi\varepsilon\right) - 1 \otimes \mathcal{X}'_\varepsilon . \end{aligned} \tag{6.25}$$

If we conjecture the identification

$$\mathcal{X}' = \pi_Z[X^+], \quad \mathcal{X} = \pi_Z[X^-], \quad P = \pi_Z[H], \tag{6.26}$$

then the commutation relations (6.24) and the coproduct (6.25) become (2.20), (2.22), (2.23) in the limit $\varepsilon \to 0$.

*6.3. Fock Decomposition for $SU(2)$ TM.* At the conclusion of this section let us discuss a structure of the regularized space $\pi_Z^\varepsilon$ for $SU(2)$ TM. Define the highest vector $|p\rangle$ by the system of equations

$$a_m|p\rangle = 0, \quad m > 0\,, \\ P|p\rangle = p|p\rangle\,, \tag{6.27}$$

where $p$ is some number. Fock space $F_p$ is generated by all possible vectors:

$$a_{-m_1}\ldots a_{-m_n}|p\rangle, m_1,\ldots m_n > 0\,, \tag{6.28}$$

and it is an irreducible representation of the algebra (5.1). Any representation $\pi$ of the universal enveloping algebra generated by operators $a_m, P, Q$ can be decomposed into a direct sum of Fock spaces

$$\pi = \bigoplus_p F_p\,. \tag{6.29}$$

Here the parameter $p$ specifies components in the decomposition.

To describe the decomposition of the $\pi_Z^\varepsilon$, we have to find admissible values of the parameter $p$ in the direct sum (6.29). First of all, let us determine the value $p_0$ for the vacuum state $|0\rangle$. From the definition of $\pi_Z$, its vacuum vector must be $U_{-1}(sl(2))$-invariant. It is natural to assume that the vacuum state for finite $\varepsilon$ satisfies the system of equations:

$$\mathscr{X}'_\varepsilon|0\rangle = \mathscr{X}_\varepsilon|0\rangle = P|0\rangle = 0\,. \tag{6.30}$$

The highest vector $|p_0\rangle$ with the eigenvalue $p_0 = 0$ is the unique solution of (6.30). The operators $Z_{\varepsilon a}(\beta)$ (6.15) change an eigenvalue $p$ to $p \pm \frac{1}{\sqrt{2}}$. Hence the space $\pi_Z^\varepsilon$ is represented by the direct sum:

$$\pi_Z^\varepsilon = \bigoplus_{l\in Z} F_{\frac{l}{\sqrt{2}}}\,. \tag{6.31}$$

In the space (6.31) the action of the following operator is well defined:

$$K_\varepsilon = i\varepsilon H_C - i\frac{\sqrt{2}}{4}\varepsilon P. \tag{6.32}$$

Here

$$H_C = \frac{P^2}{2} + \sum_{m=1}^{+\infty} \frac{m^2}{\sinh\frac{\pi m\varepsilon}{2}\sinh \pi\varepsilon m} a'_{-m}a_m\,.$$

It is easy to check that the operator $K_\varepsilon$ generates an infinitesimal shift of the variable $\beta$ for $Z_{\varepsilon a}(\beta)$. Hence we can consider the operator $K$ (3.6) acting in the space $\pi_Z$ as the limit of $K_\varepsilon$ when $\varepsilon \to 0$.

## 7. Additional Structures in the Space $\pi_Z$

In this section I shall introduce all operators which are necessary for reconstruction of form-factors in $SU(2)$ TM. There are analogical structures in SGM. So, statements will be formulated in universal forms.

*7.1. Principle Theorem.* Let us define the following operators in the space $\pi_Z$.

$$Z'_+(\alpha) = V'(\alpha)\,,$$
$$Z'_-(\alpha) = i(\mathscr{X}'V'(\alpha) + V'(\alpha)\mathscr{X}')\,. \tag{7.1}$$

Principal Theorem summarizes properties of the algebra generated by the operators $Z_a(\beta), Z'_a(\alpha)$.

## Theorem

1. *The operators* $Z_a(\beta), Z'_a(\alpha)$ *satisfy the commutation relations*:

$$Z_a(\beta_1)Z_b(\beta_2) = S_{ab}^{cd}(\beta_1 - \beta_2)Z_d(\beta_2)Z_c(\beta_1)\,,$$
$$Z'_a(\alpha_1)Z'_b(\alpha_2) = R_{ab}^{cd}(\alpha_1 - \alpha_2)Z'_d(\alpha_2)Z'_c(\alpha_1)\,,$$
$$Z_a(\beta)Z'_b(\alpha) = ab\tan\left(\frac{\pi}{4} + i\frac{\beta-\alpha}{2}\right)Z'_b(\alpha)Z_a(\beta)\,. \tag{7.2}$$

*In the case of* $SU(2)$ TM *the matrix* $S_{ab}^{cd}(\beta)$ *is given by* (2.16) *and*

$$R_{ab}^{cd}(\alpha) = -S_{ab}^{cd}(-\alpha)\,. \tag{7.3}$$

2. *The singular part of the operator product*

$$Z_a(\beta_2)Z_b(\beta_1)$$

*considered as a function of the complex variable* $\beta_2$ *for real* $\beta_1$ *in the upper half plane* $\Im m\beta_2 \geqq 0$, *contains only one simple pole with the residue*

$$iZ_a(\beta_2)Z_b(\beta_1) = \frac{\mathbf{C}_{ab}}{\beta_2 - \beta_1 - \pi i} + \cdots. \tag{7.4}$$

3. *The operator product*

$$Z'_a(\alpha_2)Z'_b(\alpha_1)$$

*considered as a function of the complex variable* $\alpha_2$ *for real* $\alpha_1$ *is regular for* $\Im m\alpha_2 \geqq -\pi$ *and*

$$\mathbf{C}_{ab}Z'_a(\alpha + i\pi)Z'_b(\alpha) = i\,,$$
$$Z'_a(\alpha - i\pi)Z'_b(\alpha) = i\mathbf{C}_{ab}\,. \tag{7.5}$$

4. *The following combination*

$$\frac{\Gamma\left(\frac{3}{4} + i\frac{\beta-\alpha}{2\pi}\right)}{\Gamma\left(\frac{1}{4} + i\frac{\beta-\alpha}{2\pi}\right)}Z'_a(\alpha)Z_b(\beta)$$

*considered as a function of the variable* $\beta$ *is regular in the whole complex plane.*

5. *The commutation relations* (7.2) *and operator products* (7.4), (7.5) *are consistent with the following conjugation conditions*:

$$[Z_a(\beta)]^+ = \mathbf{C}^{ab}Z_b(\beta + i\pi), \Im m\beta = 0\,;$$
$$i[Z'_a(\alpha)]^+ = \mathbf{C}^{ab}Z'_b(\alpha + i\pi), \Im m\alpha = 0\,. \tag{7.6}$$

The proof of the theorem is close to the proofs of Propositions 1, 2 from Sect. 5.4. It is based on explicit expressions for operator products of the vertices

$$V(\beta), \bar{V}(\gamma), V'(\alpha), \bar{V}'(\delta)\,.$$

To get them, it is necessary to apply the oscillator representation from Sect. 6 and then consider the limit $\varepsilon \to 0$. The result of the calculations is presented in Appendix 2.

*7.2. Symmetry Algebra of the Space of Local Operators.* As it follows from discussion in Sect. 3, the problem of a description of the space of local operators is reduced to finding operators $\Lambda(O) \in \mathrm{End}[\pi_Z]$ satisfying (3.8). The operators $Z'_a(\alpha)$ allow us to solve this problem as follows.

In the case of $SU(2)$ TM let us introduce the set of operators:

$$T(\alpha) = \frac{1}{i}\mathbf{C}_{ab} Z'_a\left(\alpha + i\frac{\pi}{2}\right)\partial_\alpha Z'_b\left(\alpha - i\frac{\pi}{2}\right)\,, \tag{7.7}$$

$$\Lambda_m(\alpha) = \frac{i}{\eta'}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{a}{2} & \frac{b}{2} & m \end{bmatrix}_{-1} Z'_a\left(\alpha + i\frac{\pi}{2}\right) Z'_b\left(\alpha - i\frac{\pi}{2}\right)\,. \tag{7.8}$$

Note that due to formula (7.5) the operators (7.7), (7.8) can be represented in the equivalent forms:

$$T(\alpha) = \frac{i}{2}\mathbf{C}_{ab} Z'_a\left(\alpha - i\frac{\pi}{2}\right)\partial_\alpha Z'_b\left(\alpha + i\frac{\pi}{2}\right) + \mathrm{const}\,, \tag{7.9}$$

$$\Lambda(\alpha) = \frac{\eta'}{i}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{a}{2} & \frac{b}{2} & m \end{bmatrix}_{-1} Z'_a\left(\alpha - i\frac{\pi}{2}\right)\partial_\alpha Z'_b\left(\alpha + i\frac{\pi}{2}\right)\,, \tag{7.10}$$

where the irrelevant constant reads

$$\mathrm{const} = \frac{i}{2\pi}(1 - 4\ln 2).$$

We shall need also the bosonic forms of $\Lambda_m(\alpha)$,

$$\Lambda_1(\alpha) = \tilde{V}'(\alpha)\,,$$

$$\Lambda_0(\alpha) = \frac{i}{\sqrt{2}}[\mathscr{X}'\tilde{V}'(\alpha) - \tilde{V}'(\alpha)\mathscr{X}']\,,$$

$$\Lambda_{-1}(\alpha) = \frac{1}{2}[\mathscr{X}'^2\tilde{V}'(\alpha) - 2\mathscr{X}'\tilde{V}'(\alpha)\mathscr{X}' + \tilde{V}'(\alpha)\mathscr{X}'^2]\,. \tag{7.11}$$

Here the new vertex operator

$$\tilde{V}'(\alpha) = \exp(i\bar{\phi}'(\alpha))\,. \tag{7.12}$$

is introduced.

The essential properties of the operators $T(\alpha), \Lambda_m(\alpha)$ are direct consequences of Principle Theorem:

**Proposition 4.**

1. *The operators* $T(\alpha), \Lambda_m(\alpha)$ *generate the quadratic algebra:*

$$\begin{aligned}\Lambda_{m_1}(\alpha_1)\Lambda_{m_2}(\alpha_2) = &\mathscr{R}^{m_3 m_4}_{m_1 m_2}(\alpha_1 - \alpha_2)\Lambda_{m_4}(\alpha_2)\Lambda_{m_3}(\alpha_1) \\ &+ \mathscr{A}_{m m_1 m_2}(\alpha_1 - \alpha_2)(\Lambda_m(\alpha_1) + \Lambda_m(\alpha_2)) \\ &+ \mathscr{A}_{m_1 m_2}(\alpha_1 - \alpha_2)\,, \end{aligned} \tag{7.13}$$

$$\begin{aligned}[\Lambda_m(\alpha_1), T(\alpha_2)] = &- \mathscr{A}_{m m_2 m_1}(\alpha_1 - \alpha_2)\Lambda_{m_2}(\alpha_2)\Lambda_{m_1}(\alpha_1) \\ &+ \mathscr{A}(\alpha_1 - \alpha_2)\Lambda_m(\alpha_1) \\ &+ \mathscr{B}(\alpha_1 - \alpha_2)\Lambda_m(\alpha_2)\,, \end{aligned} \tag{7.14}$$

$$-\eta'^2[T(\alpha_1), T(\alpha_2)] = \mathscr{A}_{m_2 m_1}(\alpha_1 - \alpha_2)\Lambda_{m_2}(\alpha_2)\Lambda_{m_1}(\alpha_1) + \mathscr{C}(\alpha_1 - \alpha_2)\,. \tag{7.15}$$

*One can find explicit expressions for the structure functions* $\mathscr{R}$, $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ *in Appendix 3.*

2. *The operators* $T(\alpha), \Lambda_m(\alpha)$ *obey the following commutation relations with* $Z_a(\beta)$*:*

$$\Lambda_m(\alpha)Z_a(\beta) = (-1)^m Z_a(\beta)\Lambda_m(\alpha)\,, \tag{7.16}$$

$$[T(\alpha), Z_a(\beta)] = \partial_\alpha \ln s(\alpha - \beta) Z_a(\beta)\,, \tag{7.17}$$

*here*

$$s(\alpha) = \coth\frac{\alpha}{2}\,.$$

3. *The combination*

$$(\alpha - \beta)\Lambda_m(\alpha)Z_a(\beta) \tag{7.18}$$

*considered as a function of the variable* $\beta$ *is a regular one in the whole complex plane.*

It is useful to compare (7.16) with the formula (3.8). The operators $\Lambda_m(\alpha)$ satisfy the proper commutation relations with $Z_a(\beta)$. In the next section it will be explained that they define generating functions for form-factors. Here I wish to note only that the sign factor in the formula (7.16) is connected with the mutual locality indices (2.28).

To clarify the meaning of the operator $T(\alpha)$ let us return to the general consideration from Sect. 3. Note that if $O(x)$ is a local operator then its commutators with IM (2.13)

$$O(x, s) = [O(x), I_s] \tag{7.19}$$

are also local operators. The generating function for the form-factors of (7.19) reads

$$F_{a_1 \dots a_n}(\alpha|\beta_1, \dots \beta_n) = \sum_{k=1}^{n} \partial_\alpha \ln s(\alpha - \beta_k) F_{a_1 \dots a_n}(\beta_1, \dots \beta_n)\,, \tag{7.20}$$

where the function $s(\alpha)$ is defined by Eq. (2.15) and $F_{a_1 \dots a_n}(\beta_1, \dots \beta_n)$ is the form-factor of the operator $O(x)$. Suppose that the function $F_{a_1 \dots a_n}(\beta_1, \dots \beta_n)$ can be represented by the formula (3.9) with a some $\Lambda(O) \in \mathrm{End}[\pi_Z]$. Then the generating function (7.20) is also given by the trace

$$F_{a_1 \dots a_n}(\alpha|\beta_1, \dots \beta_n) = \mathrm{Tr}_{\pi_Z}[\exp(2\pi i K)\Lambda(O, \alpha)Z_{a_n}(\beta_n) \dots Z_{a_1}(\beta_1)]\,, \tag{7.21}$$

where

$$\Lambda(O,\alpha) = \Lambda(O)T(\alpha) - T(\alpha + 2i\pi)\Lambda(O)\,, \tag{7.22}$$

and the operator $T(\alpha)$ satisfies Eq. (7.17).

Thus the algebra (7.13)–(7.15) determines a structue of the space of local operators in the theory.

## 8. Trace Calculations

In this section we will get the integral representation for form-factors in $SU(2)$ TM. Consider the following functions:

$$\begin{aligned}&\mathscr{F}^{m_1\dots m_k}_{a_1\dots a_n}(\alpha_1,\dots\alpha_k|\beta_1,\dots\beta_n)\\ &\quad= \mathrm{Tr}_{\pi_Z}[\exp(2\pi iK)\Lambda_{m_k}(\alpha_k)\dots\Lambda_{m_1}(\alpha_1)Z_{a_n}(\beta_n)\dots Z_{a_1}(\beta_1)]\,. \end{aligned}\tag{8.1}$$

Due to the boson representation for the operators $Z_a(\beta)$ and $\Lambda_m(\alpha)$, they are given by combinations of multiple contour integrals. The integrands are functions like

$$\begin{aligned}&R(\alpha_1,\dots\alpha_k|\delta_1,\dots\delta_p|\beta_1,\dots\beta_n|\gamma_1,\dots\gamma_r)\\ &= \mathrm{Tr}_{\pi_Z}[\exp(2\pi iK)\tilde{V}(\alpha_k)\dots\tilde{V}(\alpha_1)\bar{V}(\delta_p)\dots\bar{V}(\delta_1)V(\beta_n)\dots V(\beta_1)\bar{V}(\gamma_r)\dots\bar{V}(\gamma_1)]\,. \end{aligned}\tag{8.2}$$

Here $\{\delta_i\},\{\gamma_l\}$ are variables of integrations. To treat the traces (8.2) it is useful to consider the space $\pi_Z$ as the limit (6.1),

$$R(\alpha_1,\dots) = \lim_{\varepsilon\to 0}\mathrm{Tr}_{\pi_Z^\varepsilon}[\exp(2\pi iK_\varepsilon)\tilde{V}_\varepsilon(\alpha_k),\dots]\,. \tag{8.3}$$

According to the formula (6.31) the trace over $\pi_Z^\varepsilon$ is a product of the traces over Fock module $F(\mathrm{Tr}_F)$ and the space of zero modes ($\mathrm{Tr}_0$). First, let us consider the second one,

$$\mathrm{Tr}_0[\exp(2\pi iK_\varepsilon)\tilde{V}_\varepsilon(\alpha_k)\dots] = \delta_{n-2r+2p-2k,0}\,f\left(\sum_{j=1}^{n}\beta_j - 2\sum_{j=1}^{r}\gamma_j + 2\sum_{j=1}^{p}\delta_j - 2\sum_{j=1}^{k}\alpha_j\right)\,. \tag{8.4}$$

Here

$$f(\beta) = \sum_{l=-\infty}^{+\infty}\exp\left(-\frac{\pi\varepsilon l^2}{2} - \frac{i\varepsilon l}{2}(\beta + \pi i)\right)\,.$$

The limit $\varepsilon\to 0$ can be calculated directly if we apply the Poisson formula:

$$\sum_{n=-\infty}^{+\infty} u(n\delta) = \delta^{-1}\sum_{n=-\infty}^{+\infty} v(2\pi n/\delta)\,, \tag{8.5}$$

where

$$v(k) = \int_{-\infty}^{+\infty} dx\exp(ikx)u(x)\,.$$

After a little algebra one can find that the function $f(\beta)$ (8.4) equals to (infinite) constant. Hence we can set up

$$\mathrm{Tr}_0[\exp(2\pi iK)\tilde{V}(\alpha_k)\dots] = \delta_{n-2r+2p-2k,0}\,. \tag{8.6}$$

The calculation of the trace over Fock module is simplified by the technique of Clavelli and Shapiro [38, 35]. Their prescription is as follows: Introduce a copy of bosons $b_n$ satisfying $[a_m, b_n] = 0$ and the same commutation relations as the $a_n$. Let

$$\tilde{a}_m = \frac{a_m}{1-\exp(-2\pi m\varepsilon)} + b_{-m}(m > 0), \tilde{a}_m = a_m + \frac{b_{-m}}{\exp(-2\pi m\varepsilon)-1}(m < 0)\,. \tag{8.7}$$

For a linear operator $\mathcal{O}(\{a_n\})$ on the Fock space $F[a]$, let $\tilde{\mathcal{O}} = \mathcal{O}(\{\tilde{a}_n\})$ be the operator on $F[a] \otimes F[b]$ obtained by substituting $\tilde{a}_m$ for $a_m$. We have then

$$\mathrm{Tr}_F[\exp(2\pi i K_\varepsilon)\mathcal{O}] = \frac{\langle 0|\tilde{\mathcal{O}}|0\rangle}{\prod_{m=1}^{\infty}(1-\exp(-2\pi m\varepsilon))}\,, \tag{8.8}$$

where $\langle 0|\tilde{\mathcal{O}}|0\rangle$ denotes the usual expectation value with respect to the Fock vacuum $|0\rangle \in F[a]\otimes F[b], \langle 0|0\rangle = 1$.

Here is the final result of calculations of the functions (8.2):

$$\begin{aligned}
&R(\alpha_1,\dots\alpha_k|\delta_1,\dots\delta_p|\beta_1,\dots\beta_n|\gamma_1,\dots\gamma_r)\\
&= \mathscr{C}_1^{-\frac{n}{2}}\mathscr{C}_2^{\frac{n+2r}{4}}\mathscr{C}'^{\frac{p+k}{2}}_2\, 2^{-r-\frac{n}{2}} i^{\frac{n}{2}+p+k}\eta^r\eta'^{-p-k}\, T_0(\{\alpha_i\},\{\delta_i\},\{\beta_i\},\{\gamma_i\})\\
&\times \prod_{1\leqq i<j\leqq n} G(\beta_i-\beta_j)\prod_{1\leqq i<j\leqq r}\bar{G}(\gamma_i-\gamma_j)\prod_{\substack{1\leqq i\leqq r\\1\leqq j\leqq n}} W(\gamma_i-\beta_j)\\
&\times \prod_{1\leqq i<j\leqq k}\bar{G}'(\alpha_i-\alpha_j)\prod_{1\leqq i<j\leqq p}\bar{G}'(\delta_i-\delta_j)\prod_{\substack{1\leqq i\leqq p\\1\leqq j\leqq k}}\bar{G}'^{-1}(\delta_i-\alpha_j)\\
&\times \prod_{\substack{1\leqq i\leqq n\\1\leqq j\leqq k}} U^{-1}(\beta_i-\alpha_j)\prod_{\substack{1\leqq i\leqq n\\1\leqq j\leqq p}} U(\beta_i-\delta_j)\\
&\times \prod_{\substack{1\leqq i\leqq r\\1\leqq j\leqq k}}\bar{H}^{-1}(\gamma_i-\alpha_j)\prod_{\substack{1\leqq i\leqq r\\1\leqq j\leqq p}}\bar{H}(\gamma_i-\delta_j)\,.
\end{aligned} \tag{8.9}$$

For the case of $SU(2)$ TM the functions and constants in the formula (8.9) have the following explicit expressions:

$$\begin{aligned}
&T_0(\{\alpha_i\},\{\delta_i\},\{\beta_i\},\{\gamma_i\}) = \delta_{n-2r+2p-2k,0}\,,\\
&G(\beta) = i\mathscr{C}_1\sinh\frac{\beta}{2}\exp\left[\int_0^{+\infty}\frac{dt}{t}\frac{\sinh^2 t(1-i\frac{\beta}{\pi})\exp(-t)}{\sinh 2t\cosh t}\right]\,,\\
&W(\beta) = (\pi^3\mathscr{C}_2)^{-\frac{1}{2}}\Gamma\left(-\frac{1}{4}+i\frac{\beta}{2\pi}\right)\Gamma\left(\frac{3}{4}-i\frac{\beta}{2\pi}\right)\,,\\
&\bar{G}(\beta) = -\frac{\mathscr{C}_2}{4}(\beta+i\pi)\sinh\beta\,,\\
&\bar{G}'(\alpha) = -\mathscr{C}'_2\frac{\sinh\alpha}{\alpha+i\pi}\,,\\
&U(\alpha) = i\sinh\frac{\alpha}{2}\,,
\end{aligned}$$

$$\bar{H}(\alpha) = -\frac{2}{\cosh\alpha}\,,$$

$$\mathscr{C}_1 = \exp\left[-\int\limits_0^{+\infty}\frac{dt}{t}\frac{\sinh^2\frac{t}{2}\exp(-t)}{\sinh 2t\cosh t}\right]\,,$$

$$\mathscr{C}_2 = \mathscr{C}'^{-1}_2 = \frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^3}\,. \qquad (8.10)$$

Now, to complete the construction of integral representations for the functions (8.1), we have to describe the contours of integrations. The rules are: If all $\{\beta\}_i, \{\alpha\}_i$ are real then contours of integrations over the variables $\{\delta\}_j, \{\gamma\}_j$ lie in the strip $-i\pi < \{\delta\}_j, \{\gamma\}_j < i\pi$ exactly in the same way as in the integral representation for the vacuum averages,

$$\langle 0|\Lambda_{m_k}(\alpha_k)\ldots\Lambda_{m_1}(\alpha_1)Z_{a_n}(\beta_n)\ldots Z_{a_1}(\beta_1)|0\rangle\,.$$

Let me recall that contours for vacuum averages are taken according to the definition of the action of the operators $\mathscr{X}, \mathscr{X}'$ (see Sect. 5.3).

Now it is useful to consider examples of (8.1). I calculated explicitly the simplest ones,

$$\mathscr{F}_{ab}(\beta_1,\beta_2) = 0\,,\quad \mathscr{F}^m(\alpha) = 0\,, \qquad (8.11)$$

$$\mathscr{F}^m_{ab}(\alpha|\beta_1,\beta_2) = \frac{\eta'}{2\pi i}\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ \frac{b}{2} & \frac{a}{2} & m\end{bmatrix}_{-1}\mathscr{C}_1^{-1}\frac{G(\beta_1-\beta_2)}{\sinh\frac{\beta_1-\alpha}{2}\sinh\frac{\beta_2-\alpha}{2}}\,. \qquad (8.12)$$

Let us analyze these formulas. One should expect that the functions $\mathscr{F}_{ab}(\beta_1,\beta_2)$ are two particle form-factors of the unit operator. Hence it must be zero for real $\beta_1-\beta_2$. Then, we can see that $\mathscr{F}^m_{ab}(\alpha|\beta_1,\beta_2)$, considered as functions of $\exp(\alpha)$ admit decompositions into the series in the neighborhood of the points $\exp(\alpha)=0$ and $\exp(\alpha)=\infty$:

$$\mathscr{F}^m_{ab}(\alpha|\beta_1,\beta_2) = \sum_{s=1}^{+\infty}\exp(s\alpha)F^m_{ab}(s|\beta_1,\beta_2)\,,$$

$$\mathscr{F}^m_{ab}(\alpha|\beta_1,\beta_2) = \sum_{s=-1}^{-\infty}\exp(s\alpha)F^m_{ab}(s|\beta_1,\beta_2)\,. \qquad (8.13)$$

The coefficients $F^m_{ab}(s|\beta_1,\beta_2)$ are form-factors of operators which are $SU(2)$ vectors and have Lorentz spins $s$. Moreover, $F^{\pm}_{ab}(s|\beta_1,\beta_2)$ and $F^0_{ab}(s|\beta_1,\beta_2)$ correspond to operators which are respectively semilocal and local with respect to the "elementary" field of $SU(2)$ TM. This follows from the commutation relations (7.16). So, we conjecture that the functions $F^m_{ab}(s|\beta_1,\beta_2), (s=\pm1, m=0,\pm)$ are the form-factors of the currents $-\frac{2}{\eta'}\,[sJ^m_t(x)+J^m_x(x)]$ [17, 21]. Note that the constant of normalization is fixed by Eq. (2.26).

From this simple example we have learned that $\mathscr{F}^m_{ab}(\alpha|\beta_1,\beta_2)$ are the generating functions for form-factors of local operators in the theory. As a matter of fact it is the general property of the functions (8.1). Indeed, using Proposition 4 from Sect. 7.2, we can show that any function (8.1) has the form:

$$\mathscr{F}^{m_1\ldots m_k}_{a_1\ldots a_n}(\alpha_1,\ldots\alpha_k|\beta_1,\ldots\beta_n)$$
$$= A^{m_1\ldots m_k}(\alpha_1,\ldots\alpha_k)B_{a_1\ldots a_n}(\beta_1,\ldots\beta_n)C(\alpha_1,\ldots\alpha_k|\beta_1,\ldots\beta_n)\,, \qquad (8.14)$$

where

$$C(\alpha_1,\dots,\alpha_j+2\pi i,\dots\alpha_k|\beta_1,\dots\beta_n)=C(\alpha_1,\dots,\alpha_j,\dots\alpha_k|\beta_1,\dots\beta_n),\quad j=1,\dots k\,.$$

Scalar periodic functions $C(\alpha_1,\dots\alpha_k|\beta_1,\dots\beta_n)$ admit, in the neighborhood of the points $\exp(\alpha_j)=0,\infty$, decompositions into the series. Hence the functions (8.14) can be represented as follows:

$$\begin{aligned}&\mathscr{F}^{m_1\dots m_k}_{a_1\dots a_n}(\alpha_1,\dots\alpha_k|\beta_1,\dots\beta_n)\\&=\sum_{\{s_j\}}F'^{m_1\dots m_k}(\alpha_1,\dots\alpha_k|\{s_j\})F_{a_1\dots a_n}(\{s_j\}|\beta_1,\dots\beta_n)\exp(s_1\alpha_1)\dots\exp(s_k\alpha_k)\,.\end{aligned}\tag{8.15}$$

The coefficients $F_{a_1\dots a_n}(\{s_j\}|\beta_1,\dots\beta_n)$ satisfy the axioms (1–5) from Sect. 3, so they are form-factors of local operators in the theory. One can expect that this huge set of functions is a general solution of the Riemann–Hilbert problem (1–5) for $SU(2)$ TM.

## 9. Free Field Representation for the Sine–Gordon Model

In this section the representation $\pi_Z$ for SGM will be investigated. The main steps of the construction have been already discussed for the case of $SU(2)$ TM, so we shall focus only on essential differences.

*9.1. Bosonization of the Zamolodchikov–Faddeev Algebra.* Let us begin with a definition of the space $\pi_Z^{\xi}$. Consider the set of oscillators $a_m$, which satisfy the commutation relation:

$$[a_m,a_n]=\sinh\frac{\pi m\varepsilon}{2}\sinh\pi m\varepsilon\frac{\sinh\frac{\pi m\varepsilon}{2}(\xi+1)}{m\sinh\frac{\pi m\varepsilon}{2}\xi}\delta_{n+m,0}\,.\tag{9.1}$$

It is also convenient to introduce the set of "dual" oscillators $a'_m$ connected with $a_m$ as follows:

$$a'_m\sinh\frac{\pi m\varepsilon}{2}(\xi+1)=a_m\sinh\frac{\pi m\varepsilon}{2}\xi\,.\tag{9.2}$$

They obey the commutation relations:

$$[a'_m,a'_n]=\sinh\frac{\pi m\varepsilon}{2}\sinh\pi m\varepsilon\frac{\sinh\frac{\pi m\varepsilon}{2}\xi}{m\sinh\frac{\pi m\varepsilon}{2}(\xi+1)}\delta_{n+m,0}\,.\tag{9.3}$$

Note the duality between the operators $a_m$ and $a'_m$; $Z_2$-transformation

$$\xi\to-1-\xi\tag{9.4}$$

transforms the commutation relation (9.3) into (9.1) and vice versa. As we shall see below, this duality has the same nature as the "$\alpha_+\leftrightarrow\alpha_-$" one in the minimal model of 2D CFT [56, 41].

Using the oscillators $a_m, a'_m$, we define the fields $\phi(\beta)$ and $\phi'(\alpha)$ in the same way as have been done for $SU(2)$ TM (6.7), (6.19). One should point out only that the parameters $s$ and $s'$ must be chosen as

$$s = \frac{\xi + 1}{2\xi}, \quad s' = \frac{\xi}{2(\xi + 1)} . \tag{9.5}$$

It may be useful to recall that $s$ is connected with an asymptotic behavior of the two particle S-matrix (6.4).

Introduce the operators

$$V_\varepsilon(\beta), \bar{V}_\varepsilon(\gamma), V'_\varepsilon(\alpha), \bar{V}'_\varepsilon(\delta), \mathscr{X}_\varepsilon, \mathscr{X}'_\varepsilon$$

by the formulas (6.14), (6.23). Using $\mathscr{X}_\varepsilon$ and $\mathscr{X}'_\varepsilon$, the vacuum state can be specified as follows:

$$\mathscr{X}_\varepsilon|0\rangle = \mathscr{X}'_\varepsilon|0\rangle = 0 . \tag{9.6}$$

Their unique solution is the highest vector $|p_0\rangle$ with

$$p_0 = \frac{\alpha_+ + \alpha_-}{\sqrt{2}} , \tag{9.7}$$

where we have conveniently used the notations:

$$\alpha_+ = -\alpha_-^{-1} = \sqrt{\frac{\xi + 1}{\xi}} . \tag{9.8}$$

The vertices $V_\varepsilon(\beta), \bar{V}_\varepsilon(\gamma), V'_\varepsilon(\alpha), \bar{V}'_\varepsilon(\delta)$ shift an eigenvalue $p$ of the operator $P$ to, respectively,

$$p + \frac{\alpha_+}{\sqrt{2}}, \quad p - \sqrt{2}\alpha_+, \quad p + \frac{\alpha_-}{\sqrt{2}}, \quad p - \sqrt{2}\alpha_- .$$

Hence one can expect that the space $\pi_Z^\varepsilon$ admits the following decomposition:

$$\pi_Z^\varepsilon = \bigoplus_{\{l,l'\}\in Z} F_{\frac{\gamma_+ l + \gamma_- l'}{\sqrt{2}}} . \tag{9.9}$$

The operator $K_\varepsilon$ is defined by

$$K_\varepsilon = i\varepsilon H_C + \frac{\alpha_+ + \alpha_-}{\sqrt{2}} P ; \tag{9.10}$$

here

$$H_C = \frac{P^2 - p_0^2}{2} + \sum_{m=1}^{+\infty} \frac{m^2}{\sinh \frac{\pi m \varepsilon}{2} \sinh \pi m \varepsilon} a'_{-m} a_m .$$

It satisfies the proper commutation relation with $V_\varepsilon(\beta)$

$$\exp(-\theta K_\varepsilon) V_\varepsilon(\beta) \exp(\theta K_\varepsilon) = V_\varepsilon(\beta + \theta) \exp\left(-\frac{\theta}{2\xi}\right) . \tag{9.11}$$

The space $\pi_Z$ must be considered as the limit (6.1). It allows to determine forms of necessary operator products. They are listed in Appendix 2.

Now, we can introduce the following set of operators acting in the space $\pi_Z$:

$$
\begin{aligned}
Z_+(\beta) &= \exp\left(-\frac{\beta}{2\xi}\right) V(\beta), \\
Z_-(\beta) &= \exp\left(\frac{\beta}{2\xi}\right) [q^{\frac{1}{2}}\mathcal{X} V(\beta) - q^{-\frac{1}{2}} V(\beta)\mathcal{X}], \\
Z'_+(\alpha) &= \exp\left(-\frac{\alpha}{2(\xi+1)}\right) V'(\alpha), \\
Z'_-(\alpha) &= \exp\left(\frac{\alpha}{2(\xi+1)}\right) [q'^{\frac{1}{2}}\mathcal{X}' V(\alpha) - q'^{-\frac{1}{2}} V'(\alpha)\mathcal{X}'],
\end{aligned}
\tag{9.12}
$$

Here

$$
q = \exp i\pi\alpha_+^2, \quad q' = \exp i\pi\alpha_-^2 . \tag{9.13}
$$

Note that the integrals associated with the action of the operator $\mathcal{X}$ in (9.12) can be calculated explicitly at the free fermion point ($\xi = 1$). For this case the total contour of the integration closes and the operators $Z_a(\beta)$ read as:

$$
\begin{aligned}
Z_+(\beta) &= \exp\left(-\frac{\beta}{2}\right) \exp(i\phi(\beta)), \\
Z_-(\beta) &= \exp\frac{\beta}{2}[\exp(-i\phi(\beta+i\pi)) - \exp(-i\phi(\beta-i\pi))] .
\end{aligned}
\tag{9.14}
$$

Principle Theorem from Sect. 7.1 describes the essential properties of the operators (9.12). In the case of SGM the S-matrix defining the commutation relations of the two operators $Z_a(\beta)$ is given by (2.33). At the same time the matrix $R_{ab}^{cd}(\alpha)$ in the formula (7.2) has the following nontrivial elements:

$$
\begin{aligned}
R_{++}^{++}(\alpha) &= R_{--}^{--}(\alpha) = R(\alpha), \\
R_{+-}^{+-}(\alpha) &= R_{-+}^{-+}(\alpha) = -R(\alpha)\frac{\sinh\frac{\alpha}{\xi+1}}{\sinh\frac{i\pi+\alpha}{\xi+1}}, \\
R_{+-}^{-+}(\alpha) &= R_{-+}^{+-}(\alpha) = R(\alpha)\frac{\sinh\frac{i\pi}{\xi+1}}{\sinh\frac{i\pi+\alpha}{\xi+1}} .
\end{aligned}
\tag{9.15}
$$

Here the function $R(\alpha)$ is represented by:

$$
\begin{aligned}
R(\alpha) &= \frac{\Gamma\left(\frac{1}{\xi+1}\right)\Gamma\left(1-\frac{i\alpha}{\pi(\xi+1)}\right)}{\Gamma\left(\frac{1}{\xi+1}-\frac{i\alpha}{\pi(\xi+1)}\right)} \prod_{p=1}^{\infty} \frac{R'_p(-\alpha)R'_p(i\pi+\alpha)}{R'_p(0)R'_p(i\pi)}, \\
R'_p(\alpha) &= \frac{\Gamma\left(\frac{2p}{\xi+1}+\frac{i\alpha}{\pi(\xi+1)}\right)\Gamma\left(1+\frac{2p}{\xi+1}+\frac{i\alpha}{\pi(\xi+1)}\right)}{\Gamma\left(\frac{2p+1}{\xi+1}+\frac{i\alpha}{\pi(\xi+1)}\right)\Gamma\left(1+\frac{2p-1}{\xi+1}+\frac{i\alpha}{\pi(\xi+1)}\right)} .
\end{aligned}
\tag{9.16}
$$

The proof of Principle Theorem for SGM is based on the ideas given in Appendix 1. I wish to comment only the following commutation relation:

$$
Z_-(\beta)Z'_-(\alpha) = \tan\left(\frac{\pi}{4} + i\frac{\beta-\alpha}{2}\right) Z'_-(\alpha)Z_-(\beta) . \tag{9.17}
$$

To prove it we have to show that, for all vectors $|u\rangle, |v\rangle \in \pi_Z$,

$$|u\rangle, |v\rangle = \mathscr{Z}_{a_n} \dots \mathscr{Z}_{a_1}|0\rangle, \mathscr{Z}_{a_i} = \{Z_{a_i}(\beta_i) \text{ or } Z'_{a_i}(\alpha_i)\} ,$$

a general matrix element

$$\langle u|[\mathscr{X}', \mathscr{X}]|v\rangle$$

vanishes. The structure of the operator products provides exactly this important property. Here we have the essential difference from the $SU(2)$ TM, where commutation relations

$$[\mathscr{X}', \mathscr{X}] = \sqrt{2}P$$

hold.

*9.2. Connection with the Feigin–Fuchs bosonization.* Now let us argue a group theoretic meaning of the formulas (9.12). As has been mentioned in Sect. 4 the operators

$$\hat{Z}_a(\beta) \equiv \exp\left(a\frac{\beta}{2\xi}\right) Z_a(\beta)$$

can be regarded as the basis of the fundamental representation of $U_q(sl(2))$. Then the definition (9.12) is equivalent to the coproduct (2.37), if the operator $\mathscr{X}$ is considered as

$$\mathscr{X} = \pi_z\left[E^- q^{-\frac{\sqrt{2}}{2}H}\right] . \tag{9.18}$$

The definition of the operators $Z_a(\alpha)$ admits the same interpretation. Hence we conclude that there are two quantum algebras ($U_q(sl(2))$ and $U_{q'}(sl(2))$) in SGM. Their quantum parameters $q$ and $q'$ are given by Eq. (9.13). Analogous phenomena takes place in 2D Conformal Field Theory [41, 57].

As a matter of fact, the algebra generated by the operators $Z_a(\beta), Z'_a(\alpha)$ is the natural generalization of the vertex operator algebra in the minimal models. In order to clarify this statement, let us consider n-point vacuum functions

$$\hat{G}_{a_1 \dots a_n}(\beta_1, \dots \beta_n) = \langle 0|\hat{Z}_{a_n}(\beta_n) \dots \hat{Z}_{a_1}(\beta_1)|0\rangle . \tag{9.19}$$

The vacuum state $|0\rangle$ is a $U_q(sl(2))$-scalar, so matrix elements (9.19) have the following structure:

$$G_{a_1 \dots a_n}(\beta_1, \dots \beta_n) = \sum_{\{m_k\},\{j_s\}} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & j_1 \\ \frac{a_2}{2} & \frac{a_1}{2} & m_1 \end{bmatrix}_q \begin{bmatrix} \frac{1}{2} & j_1 & j_2 \\ \frac{a_3}{2} & m_1 & m_2 \end{bmatrix}_q \dots \begin{bmatrix} \frac{1}{2} & j_{n-1} & 0 \\ \frac{a_n}{2} & m_{n-1} & 0 \end{bmatrix}_q \Im_{j_1 \dots j_{n-1}}(\beta_1, \dots, \beta_n) . \tag{9.20}$$

Functions $\Im_{j_1 \dots j_{n-1}}(\beta_1, \dots \beta_n)$ are vacuum averages of $U_q(sl(2))$ scalars. To clear up their property it is useful to rewrite the variables $\beta_i$ in the form

$$\beta_i = \sigma_i L \tag{9.21}$$

and take a limit $L \rightarrow +\infty$. Let us consider the commutation relation for the operators $\hat{Z}_a(\beta)$ in this limit. Using the explicit expression for the matrix $\hat{S}(\beta)$ (2.33), (2.39), one can derive the formula:

$$\hat{S}_{ab}^{cd}(L\sigma) \rightarrow R_{ab}^{cd}(q)\Theta(-\sigma) + (R^{-1})_{ba}^{dc}(q)\Theta(\sigma), L \rightarrow +\infty , \tag{9.22}$$

where

$$\Theta(\sigma) = \begin{cases} 1, & \text{if } \sigma > 0\,; \\ 0, & \text{otherwise} \end{cases}$$

and $R^{cd}_{ab}(q)$ are matrix elements of the universal R-matrix [51] in the tensor product of two fundamental representations of $U_q(sl(2))$. Hence the commutation relation for the operators $\hat{Z}_a(\beta)$ becomes

$$\chi_a(\sigma_1)\chi_b(\sigma_2) = [R^{cd}_{ab}(q)\Theta(\sigma_2 - \sigma_1) + (R^{-1})^{dc}_{ba}(q)\Theta(\sigma_1 - \sigma_2)]\chi_d(\sigma_2)\chi_c(\sigma_1)\,. \quad (9.23)$$

Here the limit of the operator $\hat{Z}_a(L\sigma)(L \to \infty)$ is denoted as $\chi_a(\sigma)$. Now it is easy to see that functions

$$\Im^{\infty}_{j_1 \cdots j_{n-1}}(\sigma_1, \ldots \sigma_n) = \lim_{L \to +\infty} \Im_{j_1 \cdots j_{n-1}}(\sigma_1 L, \ldots \sigma_n L)$$

satisfy the same equations as conformal blocks; their braiding is described by the quantum 6-j symbols [57–60]. Braiding properties do not uniquely specify conformal blocks. For example, the chiral correlation functions of conformal descendants have the same monodromic properties as correlators of primary fields. The braiding will uniquely determine conformal blocks if we describe their singularities at coincident points $\sigma_i \to \sigma_j$. In the present case one can find the character of singularities from the explicit form of the two point function (4.13). In the limit $\beta \to \infty(\Im m\beta \leqq 0)$ the function $g(\beta)$ (4.14) has the following asymptotic behavior:

$$g(\beta) = k^{\frac{1}{2}} \left(i\frac{\beta}{\pi\xi}\right)^{\frac{\gamma_+^2}{2}} \left(1 + O\left(\frac{1}{\beta}\right)\right)\,. \quad (9.24)$$

Hence the conformal dimension of the field $\chi_a(\sigma)$ is given by:

$$\bar{\Delta}[\chi_a(\sigma)] = \bar{\Delta}_{(2,1)}\,, \quad (9.25)$$

where the standard notation [41] for the Kac spectrum is used

$$\bar{\Delta}_{l,l'} \equiv \frac{(\alpha_+ l + \alpha_- l')^2 - (\alpha_+ + \alpha_-)^2}{4}\,. \quad (9.26)$$

In this way we identify the functions $\Im^{\infty}_{j_1 \cdots j_{n-1}}(\sigma_1, \ldots \sigma_n)$ with the $n$-point conformal blocks of the fields $\Phi_{(2/1)}(\sigma)$ [56]. The corresponding central charge is expressed by the parameter $\xi$ as follows:

$$c = 1 - 6(\alpha_+ + \alpha_-)^2\,. \quad (9.27)$$

Similar arguments show that the operator

$$\hat{Z}_a(\alpha) = \exp\left(a\frac{\alpha}{2(\xi + 1)}\right) Z_a(\alpha) \quad (9.28)$$

can be regarded as the field $\Phi_{(1/2)}$ in the limit $\alpha \to \infty$.

In spite of the remarkable connection between operators (9.12) and the fields $\Phi_{(2/1)}, \Phi_{(1/2)}$, it is important to understand their physical differences. We consider the variable $\beta$ as a rapidity of physical excitations in massive models and it cannot be identified with the holomorphic coordinate in the 2D CFT. From this point of view the discussed connection seems to be a puzzle.

*9.3. Integral Representation for Form-Factors in* SGM. In order to get the integral representation for form-factors we have to introduce an analogue of the operators $\Lambda_m(\alpha), T(\alpha)$ for SGM. It can be done by using the operators $\hat{Z}_a(\alpha)$ (9.28). The following formulas are a generalization of (7.7),

$$T(\alpha) = \frac{(2\cos\pi\nu)^{\frac{1}{2}}}{i}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{a}{2} & \frac{b}{2} & 0 \end{bmatrix}_{q'} \hat{Z}'_a\left(\alpha + i\frac{\pi}{2}\right)\partial_\alpha \hat{Z}'_b\left(\alpha - i\frac{\pi}{2}\right), \tag{9.29}$$

$$\Lambda_m(\alpha) = \frac{i}{\eta'}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{a}{2} & \frac{b}{2} & m \end{bmatrix}_{q'} \hat{Z}'_a\left(\alpha + i\frac{\pi}{2}\right)\hat{Z}'_b\left(\alpha - i\frac{\pi}{2}\right), \tag{9.30}$$

here $\nu \equiv \frac{1}{\xi+1}$. Using this definition, one can obtain the bosonic representation for $\Lambda_m(\alpha)$, which is necessary for evaluations of form-factors.

$$\begin{aligned}
\Lambda_1(\alpha) &= \tilde{V}'(\alpha)\,,\\
\Lambda_0(\alpha) &= -\frac{i}{(2\cos\pi\nu)^{\frac{1}{2}}}[q'\mathscr{X}'\tilde{V}'(\alpha) - q'^{-1}\tilde{V}'(\alpha)\mathscr{X}']\,,\\
\Lambda_{-1}(\alpha) &= -\frac{1}{2\cos\pi\nu}[q'\mathscr{X}'^2\tilde{V}'(\alpha) - (q'+q'^{-1})\mathscr{X}'\tilde{V}'(\alpha)\mathscr{X}' + q'^{-1}\tilde{V}'(\alpha)\mathscr{X}'^2]\,.
\end{aligned} \tag{9.31}$$

As in the case of $SU(2)$ TM, the properties of the operators $\Lambda_m(\alpha), T(\alpha)$ are described by Proposition 4 from Sect. 7. So, $\mathscr{F}^{m_1\dots m_k}_{a_1\dots a_n}(\alpha_1,\dots\alpha_k|\beta_1,\dots\beta_n)$ (8.1) will be the generating functions for form-factors in SGM. The evaluations of traces can be done by the technique discussed in Sect. 9. I should make the following remark only: the trace over the zero modes essentially depends on arithmetical properties of the interaction constant $\xi$. The same phenomena take place when conformal blocks on a torus are calculated [61]. To avoid difficult problems connected with a reducibility of the representation $\pi_Z$, we shall consider a general case when the parameter $\xi$ is an irrational number greater than one. Then traces of the vertex operators (8.2) are represented by the formula (8.9), where the functions and constants have the following forms:

$$\begin{aligned}
&T_0(\{\alpha_i\},\{\delta_i\},\{\beta_i\},\{\gamma_i\})\\
&\quad = \delta_{k,p}\delta_{2n,r}\exp\left(\frac{1}{2\xi}\left[\sum_{j=1}^{n}\beta_j - 2\sum_{j=1}^{r}\gamma_j\right] + \frac{1}{\xi+1}\left[\sum_{j=1}^{p}\delta_j - \sum_{j=1}^{k}\alpha_j\right]\right),
\end{aligned}$$

$$G(\beta) = i\mathscr{C}_1\sinh\frac{\beta}{2}\exp\left[\int_0^{+\infty}\frac{dt}{t}\frac{\sinh^2 t\left(1-i\frac{\beta}{\pi}\right)}{\sinh 2t\cosh t}\frac{\sinh t(\xi-1)}{\sinh t\xi}\right],$$

$$W(\beta) = -\frac{2}{\cosh\beta}\exp\left[-2\int_0^{+\infty}\frac{dt}{t}\frac{\sinh^2 t\left(1-i\frac{\beta}{\pi}\right)}{\sinh 2t}\frac{\sinh t(\xi-1)}{\sinh t\xi}\right],$$

$$\bar{G}(\beta) = -\frac{\mathscr{C}_2}{4}\xi \sinh\frac{(\beta+i\pi)}{\xi}\sinh\beta\,,$$

$$\bar{G}'(\alpha) = -\mathscr{C}'_2\frac{\sinh\alpha}{(\xi+1)\sinh\frac{\alpha+i\pi}{\xi+1}}\,,$$

$$U(\alpha) = i\sinh\frac{\alpha}{2}\,,$$

$$\bar{H}(\alpha) = -\frac{2}{\cosh\alpha}\,,$$

$$\mathscr{C}_1 = \exp\left[-\int_0^{+\infty}\frac{dt}{t}\frac{\sinh^2\frac{t}{2}}{t\sinh 2t\cosh t}\frac{\sinh t(\xi-1)}{\sinh t\xi}\right] = G(-i\pi)\,,$$

$$\mathscr{C}_2 = \exp\left[4\int_0^{+\infty}\frac{dt}{t}\frac{\sinh^2\frac{t}{2}}{\sinh 2t}\frac{\sinh t(\xi-1)}{\sinh t\xi}\right] = 4\left[W\left(i\frac{\pi}{2}\right)\xi\sin\frac{\pi}{\xi}\right]^{-2}\,,$$

$$\mathscr{C}'_2 = \exp\left[-4\int_0^{+\infty}\frac{dt}{t}\frac{\sinh^2\frac{t}{2}}{\sinh 2t}\frac{\sinh t\xi}{\sinh t(\xi+1)}\right]\,. \tag{9.32}$$

In Appendix 4 it is shown that two particle form-factors of the unit operator are equal to zero. Evaluations of integrals which define nontrivial generating functions (8.1) is a complicated enough problem. I am going to discuss it in a separate publication.

## 10. Conclusion

At the end I wish to point out that functions

$$\mathscr{F}^{m_1\dots m_k}(\alpha_1,\dots\alpha_k) = \mathrm{Tr}\pi_z[\exp(2\pi iK)\Lambda_{m_k}(\alpha_k)\dots\Lambda_{m_1}(\alpha_1)] \tag{10.1}$$

are of special interest. One can expect that they and more general objects

$$\mathrm{Tr}\pi_z[\exp(2\pi iK)Z'_{a_n}(\alpha_n)\dots Z'_{a_1}(\alpha_1)] \tag{10.2}$$

represent some vacuum correlation functions in SGM. The simplest of them have the following explicit forms:

$$\mathrm{Tr}\pi_z[\exp(2\pi iK)Z'_a(\alpha_2)Z'_b(\alpha_1)] = \mathbf{C}_{ab}\frac{\exp\frac{av}{2}(\alpha_1-\alpha_2-i\pi)}{4v\cos\pi v}\frac{\sinh v(\alpha_1-\alpha_2+i\pi)}{\cosh\frac{\alpha_1-\alpha_2}{2}}\frac{G'(\alpha_1-\alpha_2)}{G'(-i\pi)}\,, \tag{10.3}$$

where $G'(\alpha)$ is given by

$$G'(\alpha) = \exp\left[\int_0^{+\infty}\frac{dt}{t}\frac{\sinh^2 t\left(1-i\frac{\alpha}{\pi}\right)}{\sinh 2t\cosh t}\frac{\sinh t\xi}{\sinh t(\xi+1)}\right]\,, \tag{10.4}$$

and

$$\mathrm{Tr}\pi_z[\exp(2\pi iK)\Lambda_m(\alpha_2)\Lambda_n(\alpha_1)] = \delta_{m+n,0}(-1)^{m+1}q'^m\frac{\sin 2\pi v}{16}\frac{\theta(\alpha_1-\alpha_2+i\pi)}{\sinh v(\alpha_1-\alpha_2+i\pi)}\,. \tag{10.5}$$

Fourier transformation of the function $\theta(\alpha)$ for $\xi > 2$ reads:

$$\theta(\alpha) = \int_{-\infty}^{+\infty} dt \frac{\tanh \frac{\pi t}{2\nu} \exp(i\alpha t)}{\cos \frac{\pi}{2}(\nu - it) \cos \frac{\pi}{2}(\nu + it) \cos \frac{\pi}{2}(3\nu - it) \cos \frac{\pi}{2}(3\nu + it)} . \tag{10.6}$$

Here $\nu \equiv \frac{1}{\xi+1}$.

It seems important to clear up a physical meaning of these functions.

**Note Added in Proof.** After submitting this paper for publication the author has found the interpretation of the functions (10.2) [53]. They are nothing but correlators of the Jost functions. At the same time the correlators of elements of the monodromy matrix for SGM can be expressed in terms of (10.1). In [53] a physical meaning of the space $\pi_Z$ is also discussed.

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## 11. Appendix 1

Here I give a draft of the proofs of Propositions 1,2 from Sect. 5.

Let us begin with Proposition 1. The commutation relation

$$Z_+(\beta_1)Z_+(\beta_2) = S^{++}_{++}(\beta_1 - \beta_2)Z_+(\beta_2)Z_+(\beta_1) \tag{11.1}$$

is evident from the definition (5.1). To prove the formula

$$Z_+(\beta_1)Z_-(\beta_2) = S^{+-}_{+-}(\beta_1 - \beta_2)Z_-(\beta_2)Z_+(\beta_1) + S^{+-}_{-+}(\beta_1 - \beta_2)Z_+(\beta_2)Z_-(\beta_1) , \tag{11.2}$$

we have to use the relation:

$$\mathscr{X}V(\beta_2)V(\beta_1) = \frac{i\pi + \beta_1 - \beta_2}{\beta_2 - \beta_1} V(\beta_2)\mathscr{X}V(\beta_1) + \frac{i\pi + \beta_2 - \beta_1}{\beta_1 - \beta_2} S(\beta_2 - \beta_1)V(\beta_1)\mathscr{X}V(\beta_2) - V(\beta_2)V(\beta_1)\mathscr{X} . \tag{11.3}$$

It may be derived in the following way. Consider the algebraic identity:

$$(\beta_1 - \beta_2)\left(i\frac{\pi}{2} - \beta_1\right)\left(i\frac{\pi}{2} - \beta_2\right) + (i\pi + \beta_1 - \beta_2)\left(i\frac{\pi}{2} - \beta_1\right)\left(i\frac{\pi}{2} + \beta_2\right) = (i\pi + \beta_2 - \beta_1)\left(i\frac{\pi}{2} + \beta_1\right)\left(i\frac{\pi}{2} - \beta_2\right) + (\beta_2 - \beta_1)\left(i\frac{\pi}{2} + \beta_1\right)\left(i\frac{\pi}{2} + \beta_2\right) . \tag{11.4}$$

It is equivalent to the relation between the functions $g(\beta)$ and $w(\beta)$:

$$w(\beta_2 - \gamma)w(\beta_1 - \gamma)g(\beta_1 - \beta_2) = \frac{i\pi + \beta_1 - \beta_2}{\beta_2 - \beta_1} w(\gamma - \beta_2)w(\beta_1 - \gamma)g(\beta_1 - \beta_2)$$

$$+\frac{i\pi+\beta_2-\beta_1}{\beta_1-\beta_2}S(\beta_2-\beta_1)w(\beta_2-\gamma)w(\gamma-\beta_1)g(\beta_2-\beta_1)$$
$$-w(\gamma-\beta_2)w(\gamma-\beta_1)g(\beta_2-\beta_1)\,. \tag{11.5}$$

Hence

$$\begin{aligned}\bar V(\gamma)V(\beta_2)V(\beta_1) = & \frac{i\pi+\beta_1-\beta_2}{\beta_2-\beta_1}V(\beta_2)\bar V(\gamma)V(\beta_1)\\ & +\frac{i\pi+\beta_2-\beta_1}{\beta_1-\beta_2}S(\beta_2-\beta_1)V(\beta_1)\bar V(\gamma)V(\beta_2)\\ & - V(\beta_2)V(\beta_1)\bar V(\gamma)\,.\end{aligned} \tag{11.6}$$

One can integrate both parts of this equation over the variable $\gamma$ and get (11.3) after proper deformations of the integration contours.

Using the formula (11.3) and the definition (5.21), it is easy to derive (11.2). Other commutation relations for the Zamolodchikov–Faddeev algebra can be obtained by similar arguments.

Now let us prove Proposition 2. Its statement for the operator product

$$Z_+(\beta_2)Z_+(\beta_1)=\rho^2 g(\beta_1-\beta_2):V(\beta_2)V(\beta_1): \tag{11.7}$$

follows from the explicit form of the function $g(\beta)$ (4.3). Consider the operator product:

$$\begin{aligned}Z_+(\beta_2)Z_-(\beta_1) = & \, i\rho^2\bar\rho\eta^{-1}g(\beta_1-\beta_2)\\ & \left[\int_C\frac{d\gamma}{2\pi}w(\gamma-\beta_2)w(\beta_1-\gamma):V(\beta_1)V(\beta_2)\bar V(\gamma):\right.\\ & \left.+\int_{C_1}\frac{d\gamma}{2\pi}w(\gamma-\beta_1)w(\gamma-\beta_2):V(\beta_1)V(\beta_2)\bar V(\gamma):\right].\end{aligned} \tag{11.8}$$

Here the integration contours are the same as in the formula (5.17). There is one possibility of getting a singularity in the operator product (11.8). It appears when two integrand's poles clutch the integration contour. Using the form of the function $w(\beta)$ (5.10), we can find that the second term in (11.8) is regular and the first one has a simple pole with the proper residue for $\beta_2=\beta_1+i\pi$.

The analytical properties of the operator products $Z_-(\beta_2)Z_+(\beta_1)$ and $Z_-(\beta_2)Z_-(\beta_1)$ can be investigated in a similar manner.

## 12. Appendix 2

In this Appendix I list the explicit expressions for the functions and constants, which describe the operator products:

$$\begin{aligned}V(\beta_2)V(\beta_1) &= \rho^2 g(\beta_1-\beta_2):V(\beta_2)V(\beta_1):\,,\\ \bar V(\gamma)V(\beta) &= \rho\bar\rho w(\beta-\gamma):\bar V(\gamma)V(\beta):\,,\\ \bar V(\gamma_2)\bar V(\gamma_1) &= \bar\rho^2\bar g(\gamma_1-\gamma_2):\bar V(\gamma_2)\bar V(\gamma_1):\,,\end{aligned}$$

$$
\begin{aligned}
V'(\alpha_2)V'(\alpha_1) &= \rho'^2 g'(\alpha_1-\alpha_2) : V'(\alpha_2)V'(\alpha_1) : , \\
\bar{V}'(\delta)V'(\alpha) &= \rho'\bar{\rho}' w'(\alpha-\delta) : \bar{V}'(\delta)V'(\alpha) : , \\
\bar{V}'(\delta_2)\bar{V}'(\delta_1) &= \bar{\rho}'^2 \bar{g}'(\delta_1-\delta_2) : \bar{V}'(\delta_2)\bar{V}'(\delta_1) : , \\
V'(\alpha)V(\beta) &= \rho\rho' h(\beta-\alpha) : V(\beta)V'(\alpha) : , \\
\bar{V}'(\delta)V(\beta) &= \rho\bar{\rho}' u(\beta-\delta) : V(\beta)\bar{V}'(\delta) : , \\
V'(\alpha)\bar{V}(\gamma) &= \bar{\rho}\rho' u(\gamma-\alpha) : \bar{V}(\gamma)V'(\alpha) : , \\
\bar{V}'(\delta)\bar{V}(\gamma) &= \bar{\rho}\bar{\rho}' \bar{h}(\gamma-\delta) : \bar{V}(\gamma)\bar{V}'(\delta) : \ . 
\end{aligned} \tag{12.1}
$$

The functions $g, w, \bar{g}, g', w', \bar{g}', h, u, \bar{h}$ have the following forms for $SU(2)$ TM:

$$
\begin{aligned}
g(\beta) &= k^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}+\frac{i\beta}{2\pi}\right)}{\Gamma\left(\frac{i\beta}{2\pi}\right)} , \\
w(\beta) &= k^{-1} \frac{2\pi}{i(\beta + i\frac{\pi}{2})} , \\
\bar{g}(\beta) &= -k^2 \frac{\beta(\beta+i\pi)}{4\pi^2} , \\
g'(\alpha) &= k^{\frac{1}{2}} \frac{\Gamma\left(1+\frac{i\alpha}{2\pi}\right)}{\Gamma\left(\frac{1}{2}+\frac{i\alpha}{2\pi}\right)} , \\
w'(\alpha) &= k^{-1} \frac{2\pi}{i(\alpha - i\frac{\pi}{2})} , \\
\bar{g}'(\alpha) &= -k^2 \frac{\alpha(\alpha-i\pi)}{4\pi^2} , \\
h(\beta) &= k^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{4}+\frac{i\beta}{2\pi}\right)}{\Gamma\left(\frac{3}{4}+\frac{i\beta}{2\pi}\right)} , \\
u(\beta) &= k\frac{i\beta}{2\pi} , \\
\bar{h}(\beta) &= -k^{-2} \frac{4\pi^2}{\beta^2 + \frac{\pi^2}{4}} \ .
\end{aligned} \tag{12.2}
$$

The constants $\rho, \bar{\rho}, \rho', \bar{\rho}'$ are given by

$$
\begin{aligned}
\rho^2 &= i\left[\frac{k}{4\pi}\right]^{\frac{1}{2}} , \quad \bar{\rho}^2 = -i\frac{k^2}{4\pi} , \\
\rho'^2 &= \left[\frac{k}{\pi}\right]^{\frac{1}{2}} , \quad \bar{\rho}'^2 = i\frac{k^2}{4\pi} \ .
\end{aligned} \tag{12.3}
$$

The constant $k$ is connected with the parameter of the ultraviolet cut-off $\varepsilon$ (6.2) as follows:

$$
k = 1 - \exp(-2\pi\varepsilon) \tag{12.4}
$$

In the case of $SU(2)$ TM, the constants $\eta, \eta'$ in the definitions of the operators $\mathscr{X}, \mathscr{X}'$ equal:

$$\eta = [-i\pi]^{\frac{1}{2}}, \quad \eta' = [i\pi]^{\frac{1}{2}} . \tag{12.5}$$

Now, let us consider the case of SGM. The functions $g, w, \bar{g}, g', w', \bar{g}', h, u, \bar{h}$ are given by:

$$g(\beta) = \left[\frac{\kappa}{\Gamma\left(\frac{1}{\xi}\right)}\right]^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{\xi} + \frac{i\beta}{\pi\xi}\right)}{\Gamma\left(\frac{i\beta}{\pi\xi}\right)} \prod_{p=1}^{+\infty} \frac{[R_p(i\pi)R_p(0)]^{\frac{1}{2}}}{R_p(\beta)} ,$$

$$w(\beta) = \kappa^{-1} \frac{\Gamma\left(-\frac{1}{2\xi} + \frac{i\beta}{\pi\xi}\right)}{\Gamma\left(1 + \frac{1}{2\xi} + \frac{i\beta}{\pi\xi}\right)} ,$$

$$\bar{g}(\beta) = \kappa^2 \frac{i\beta}{\pi\xi} \frac{\Gamma\left(1 + \frac{1}{\xi} + \frac{i\beta}{\pi\xi}\right)}{\Gamma\left(-\frac{1}{\xi} + \frac{i\beta}{\pi\xi}\right)} ,$$

$$g'(\alpha) = \left[\kappa'\Gamma\left(\frac{1}{\xi+1}\right)\right]^{\frac{1}{2}} \frac{\Gamma\left(1 + \frac{i\alpha}{\pi(\xi+1)}\right)}{\Gamma\left(\frac{1}{\xi+1} + \frac{i\alpha}{\pi(\xi+1)}\right)} \prod_{p=1}^{\infty} \frac{R'_p(\alpha)}{[R'_p(i\pi)R'_p(0)]^{\frac{1}{2}}} ,$$

$$w'(\alpha) = \kappa'^{-1} \frac{\Gamma\left(\frac{1}{2(\xi+1)} + \frac{i\alpha}{\pi(\xi+1)}\right)}{\Gamma\left(1 - \frac{1}{2(\xi+1)} + \frac{i\alpha}{\pi(\xi+1)}\right)} ,$$

$$\bar{g}'(\alpha) = \kappa'^2 \frac{i\alpha}{\pi(\xi+1)} \frac{\Gamma\left(1 - \frac{1}{\xi+1} + \frac{i\alpha}{\pi(\xi+1)}\right)}{\Gamma\left(\frac{1}{\xi+1} + \frac{\alpha}{i\pi(\xi+1)}\right)} ,$$

$$h(\beta) = k^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{4} + \frac{i\beta}{2\pi}\right)}{\Gamma\left(\frac{3}{4} + \frac{i\beta}{2\pi}\right)} ,$$

$$u(\beta) = k \frac{i\beta}{2\pi} ,$$

$$\bar{h}(\beta) = -k^{-2} \frac{4\pi^2}{\beta^2 + \frac{\pi^2}{4}} . \tag{12.6}$$

Here $R_p(\beta)$ and $R'_p(\alpha)$ are defined by Eqs. (2.34), (9.16). The constants $\rho, \bar{\rho}, \rho', \bar{\rho}'$ have the following values for SGM:

$$\rho^2 = \frac{i}{\pi\xi} \left[\kappa\Gamma\left(\frac{1}{\xi}\right)\right]^{\frac{1}{2}} \prod_{p=1}^{+\infty} \left[\frac{R_p(i\pi)}{R_p(0)}\right]^{\frac{1}{2}} ,$$

$$\bar{\rho}^2 = \kappa^2 \frac{i}{\pi\xi} \frac{\Gamma\left(1 + \frac{1}{\xi}\right)}{\Gamma\left(-\frac{1}{\xi}\right)} ,$$

$$\rho'^2 = \left[\frac{\kappa'}{\Gamma\left(\frac{1}{\xi+1}\right)}\right]^{\frac{1}{2}} \prod_{p=1}^{+\infty} \left[\frac{R'_p(0)}{R'_p(i\pi)}\right]^{\frac{1}{2}},$$

$$\bar{\rho}'^2 = \kappa'^2 \frac{i}{\pi(\xi+1)} \frac{\Gamma\left(1-\frac{1}{\xi+1}\right)}{\Gamma\left(\frac{1}{\xi+1}\right)}. \quad (12.7)$$

The constants $\kappa, \kappa', k$ are connected with the parameter of the ultraviolet cut-off $\varepsilon$ as follows:

$$\kappa = [1-\exp(-\pi\xi\varepsilon)]^{\frac{\xi+1}{\xi}}, \kappa' = [1-\exp(-\pi(\xi+1)\varepsilon)]^{\frac{\xi}{\xi+1}}, k = 1-\exp(-2\pi\varepsilon). \quad (12.8)$$

In the case of SGM the constants $\eta$, $\eta'$ equal

$$\eta = \left[-i\xi \sin\frac{\pi}{\xi}\right]^{\frac{1}{2}}, \eta' = \left[i(\xi+1)\sin\frac{\pi}{\xi+1}\right]^{\frac{1}{2}}. \quad (12.9)$$

## 13. Appendix 3

In this Appendix the structure functions for the algebra (7.13)–(7.15) are listed in the case of SGM. The structure functions for $SU(2)$ TM can be obtained from the presented ones by taking a limit $\xi \to \infty$.

The matrix $\mathscr{R}_{ab}^{cd}$ is the R-matrix for the 19-vertex model [62]. It can be represented as follows:

$$\mathscr{R}_{m_1 m_2}^{m_3 m_4}(\alpha) = \sum_{\substack{j=0,1,2 \\ -j \leqq m \leqq j}} d_j(\alpha) \begin{bmatrix} 1 & 1 & j \\ m_1 & m_2 & m \end{bmatrix}_{q'} \begin{bmatrix} 1 & 1 & j \\ m_4 & m_3 & m \end{bmatrix}_{q'}. \quad (13.1)$$

Here

$$d_0(\alpha) = \frac{[\alpha - 2i\pi]}{[\alpha + 2i\pi]},$$

$$d_1(\alpha) = -\frac{[\alpha + i\pi][\alpha - 2i\pi]}{[\alpha - i\pi][\alpha + 2i\pi]},$$

$$d_2(\alpha) = \frac{[\alpha + i\pi]}{[\alpha - i\pi]}, \quad (13.2)$$

and

$$[x] \equiv \sinh\frac{x}{\xi+1}. \quad (13.3)$$

The structure functions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ read

$$\mathscr{A}_{mm_1m_2}(\alpha) = \frac{[i\pi][4i\pi]^{\frac{1}{2}}}{(\xi+1)^{\frac{1}{2}}[\alpha - i\pi][\alpha + 2i\pi]} \begin{bmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m \end{bmatrix}_{q'}, \quad (13.4)$$

$$\mathscr{A}_{m_1m_2}(\alpha) = \frac{([i\pi]^3[3i\pi])^{\frac{1}{2}}[2\alpha]}{(\xi+1)[\alpha - i\pi][\alpha][\alpha + i\pi][\alpha + 2i\pi]} \begin{bmatrix} 1 & 1 & 0 \\ m_1 & m_2 & 0 \end{bmatrix}_{q'}, \quad (13.5)$$

$$\mathcal{A}(\alpha) = \frac{[i\pi]^2[2i\pi]}{(\xi+1)[\alpha - i\pi][\alpha][\alpha + i\pi][\alpha + 2i\pi]}\,, \tag{13.6}$$

$$\mathcal{B}(\alpha) = -\frac{[i\pi]^2[2\alpha + 2i\pi]}{(\xi+1)[\alpha - i\pi][\alpha][\alpha + i\pi][\alpha + 2i\pi]}\,, \tag{13.7}$$

$$\mathcal{C}(\alpha) = -\frac{[i\pi]^3[\alpha + 3i\pi][2\alpha]}{(\xi+1)[2i\pi][\alpha - i\pi][\alpha][\alpha + i\pi]^2][\alpha + 2i\pi]} + (\xi+1)[i\pi]\partial_\alpha^2 \ln R(\alpha)\,, \tag{13.8}$$

where $R(\alpha)$ is defined by Eq. (9.16).

## 14. Appendix 4

Here it is shown that two particle form-factors of the unit operator are equal to zero.

According to the integral representation described in Sects. 8, 9 the functions

$$\mathcal{F}_{ba}(\beta_1 - \beta_2) = \mathrm{Tr}_{\pi_Z}[\exp(2\pi i K) Z_a(\beta_2) Z_b(\beta_1)]$$

can be represented as follows:

$$\mathcal{F}_{ba}(\beta) = -\mathbf{C}_{ab} \exp \frac{a}{2\xi}(\beta + i\pi)\frac{G(\beta)}{G(-i\pi)}[I(\beta) + I(-\beta - 2\pi i)]\,, \tag{14.1}$$

where

$$I(\beta) = \left[\xi \sin\frac{\pi}{\xi} W\left(i\frac{\pi}{2}\right)\right]^{-2} \int_{-\infty}^{+\infty} \frac{d\gamma}{2\pi} W(\gamma - i\pi) W(\gamma - \beta - i\pi) \exp\left(\frac{\beta - 2\gamma}{2\xi}\right)\,, \tag{14.2}$$

and $G(\beta), W(\beta)$ are given by the relations (9.32). Using the Fourier transformation let us rewrite (14.2) in the form:

$$I(\beta) = \int_{-\infty}^{+\infty} dz \mathcal{W}(z) \mathcal{W}\left(\frac{i}{\xi} - z\right) \exp\left[i\left(z - \frac{i}{2\xi}\right)\beta\right]\,,$$

$$\mathcal{W}(z) = \left[\xi \sin\frac{\pi}{\xi} W\left(i\frac{\pi}{2}\right)\right]^{-1} \int_{-\infty}^{+\infty} \frac{d\gamma}{2\pi} W(\gamma - i\pi) \exp(iz\gamma)\,. \tag{14.3}$$

The function $W(\beta)$ satisfies the following functional equations:

$$W(\beta - i\pi) = W(-\beta - i\pi)\,,$$

$$\frac{W(\beta - i\frac{\pi}{2})W(\beta + i\frac{\pi}{2})}{W^2(i\frac{\pi}{2})} = -\frac{\xi \sin^2\frac{\pi}{\xi}}{\sinh\beta \sinh\frac{\beta + i\pi}{\xi}}\,. \tag{14.4}$$

They make it possible to calculate $\mathcal{W}(z)$ (14.3) explicitly:

$$\mathcal{W}(z) = -\frac{1}{2\cosh\pi\left(\frac{z}{2} - i\frac{\xi-1}{4\xi}\right)} \prod_{p=0}^{+\infty} \frac{Q_p\left(\frac{\xi-1}{4\xi} + i\frac{z}{2}\right)}{Q_p\left(\frac{\xi+1}{4\xi} - i\frac{z}{2}\right)}\,, \tag{14.5}$$

where

$$Q_p(x) = \frac{\Gamma\left(\frac{2p+1}{2\xi} + \frac{1}{2} + x\right)\Gamma\left(\frac{2p+1}{2\xi} + x\right)}{\Gamma\left(\frac{p}{\xi} + \frac{1}{2} + x\right)\Gamma\left(\frac{p+1}{\xi} + x\right)} . \tag{14.6}$$

One can check that the function $\mathscr{W}(z)$ satisfies the equation:

$$\mathscr{W}(z)\mathscr{W}\left(\frac{i}{\xi} - z\right) = \frac{1}{2\cosh \pi(z - \frac{i}{2\xi})} . \tag{14.7}$$

With this formula at hand the calculation of the integral $I(\beta)$ (14.3) becomes trivial:

$$I(\beta) = \frac{1}{2\cosh \frac{\beta}{2}} . \tag{14.8}$$

From (14.1), (14.8) it follows that

$$Tr_{\pi_z}[\exp(2\pi i K)Z_a(\beta_2)Z_b(\beta_1)] = 0, \Im m(\beta_1 - \beta_2) = 0 . \tag{14.9}$$

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