Generalized Ray–Singer Conjecture.
I. A Manifold with a Smooth Boundary

S. M. Vishik

Max-Planck Institut für Mathematik, Gottfried-Claren-Strasse 26, D-53225 Bonn, Germany.
Permanent address: Department of Mathematics, Temple University, Philadelphia, PA19122, U.S.A. E-mail address: senia@euclid.math.temple.edu

Dedicated to my parents

Abstract: This paper is devoted to a proof of a generalized Ray–Singer conjecture for a manifold with boundary (the Dirichlet and the Neumann boundary conditions are independently given on each connected component of the boundary and the transmission boundary condition is given on the interior boundary). The Ray–Singer conjecture [RS] claims that for a closed manifold the combinatorial and the analytic torsion norms on the determinant of the cohomology are equal. For a manifold with boundary the ratio between the analytic torsion and the combinatorial torsion is computed. Some new general properties of the Ray–Singer analytic torsion are found. The proof does not use any computation of eigenvalues and their asymptotic expansions or explicit expressions for the analytic torsions of any special classes of manifolds.

Contents

Torsion invariants for manifolds which are not simply connected were introduced by K. Reidemeister in [Re1, Re2], where he obtained with the help of such invariants a full *PL*-classification of three-dimensional lens spaces. These invariants were generalized by W. Franz to multi-dimensional *PL*-manifolds in [Fr]. As the result of this generalization he obtained a *PL*-classificaiton of lens spaces of any dimension. (These torsions were the first invariants of manifolds which are not homotopy invariants.) J.H.C. Whitehead in [Wh] and G. de Rham in [dR3] intro-duced torsion invariants for smooth manifolds. G. de Rham in [dR3] proved that a spherical Clifford–Klein manifold (i.e., the quotient of a sphere under a fixed-point free action of a finite group of rotations) is determined up to an isometry by its fundamental group and by its Reidemeister torsions. The Whitehead torsion for a homotopy equivalence between finite cell complexes was introduced in [Wh] as a generalization of the Reidemeister torsion invariants defined in [Re1], [Fr], and [dR3]. (Its values are in the Whitehead group $\text{Wh}(\pi_1)$ of the funadmental group π_1.) The Whitehead torsion is connected with Whitehead's theory of simple ho-motopy types ([Wh, dRMK], [Mi], Sect. 7). Some modifications of Reidemeister torsions were considered by J. Milnor in [Mi], Sects. 8, 12, and by V. Turaev in [T], Sect. 3. The scalar Reidemeister torsion is a global invariant of a cell decom-position of a manifold and of an acyclic representation of its fundamental group. It is an invariant of a *PL*-structure of a manifold. The Reidemeister torsion for an arbitrary finite-dimensional unimodular representation of the fundamental group can be defined as a canonical norm on the determinant line of the cohomology of a manifold (with the coefficients in the local system defined by this representaion). It is some kind of multiplicative analog of the Euler characteristic in the case of odd-dimensional manifolds. (The Euler characteristic of a closed manifold is trivial in the odd-dimensional case.) Formulas for the Reidemeister torsions of a direct prod-uct of manifolds ([KwS]) are analogous to the multiplicative property of the Euler characteristic.

The Ray–Singer analytic torsion was introduced in [RS] for a closed Rieman-nian manifold (M, g_M) with an acyclic orthogonal representation of the fundamen-tal group $\pi_1(M)$. It is equal to a product of the corresponding powers of the

determinants of the Laplacians on differential forms $DR^\bullet(M)$. These determinants are regularized with the help of the zeta-functions of the Laplacians. (The scalar Reidemeister torsion also can be written by the analogous formula, where Riemannian Laplacians are replaced by the combinatorial ones.) The Ray–Singer analytic torsion is defined with the help of a Riemannian metric g_M but it is independent of g_M in the acyclic case. (This assertion was proved in [RS], Theorem 2.1) So it is an invariant of a smooth structure on M. It has properties analogous to the properties of the Reidemeister torsion ([RS], Sects. 2, 7). The Ray–Singer conjecture ([RS]) claims that for an acyclic representation ρ of the fundamental group of a closed manifold M the Reidemeister torsion of (M, ρ) (which is defined for any smooth triangulation of M) is equal to the Ray–Singer analytic torsion of (M, ρ). This conjecture was independently proved by W. Müller in [Mü1] and by J. Cheeger in [Ch] for closed manifolds. The Ray–Singer analytic torsion can also be defined for any finite-dimensional unitary representation ρ of $\pi_1(M)$. In this case the Ray–Singer torsion is the norm on the determinant line $\det H^\bullet(M, \rho)$. For instance, it is defined for a trivial one-dimensional representaion. So the analytic torsion norm provides us with a canonical norm on the determinant line of the de Rham complex of a manifold. (The Ray–Singer formula for an arbitrary finite-dimensional unitary representation ρ of $\pi_1(M)$ in the case, when M is a smooth closed manifold, claims that the Ray–Singer norm on $\det H^\bullet(M, \rho)$ is equal to the Reidemeister norm on $\det H^\bullet(M, \rho)$.)

Let (M, g_M) be a manifold with a smooth boundary ∂M and with the Dirichlet and the Neumann boundary conditions independently given on the connected components of ∂M. Let $Z \subset \partial M$ be a union of the components of ∂M where the Dirichlet boundary conditions are given. Let F_ρ be a local system with a fiber $\mathbb{C}^m$ defined by a unitary representation $\rho : \pi_1(M) \to U(m)$. Then the Ray–Singer torsion norm $T_0(M, Z; F_\rho)$ is defined on $\det H^\bullet(M, Z; F_\rho)$. It is independent of g_M (if g_M is a direct product metric near ∂M) and it depends on a flat Hermitian metric on the fibres F_ρ (for a general (M, Z)). A flat Hermitian structure on F_ρ defines a norm on the line $\det(F_x, M, Z) := \bigotimes_k (\det F_{x_k})^{\chi(M_k, Z \cap \partial M_k)}$, where the product is over the full set of representatives F_{x_k} of fibers of F_ρ over the connected components M_k of M (with one such a fiber F_{x_k} for each $M_k, x_k \in M_k$, $\det F_x := \wedge^{\max} F_x$). The tensor product of this norm and of $T_0(M, Z; F_\rho)$ is a modified Ray–Singer norm on $\det H^\bullet(M, Z; F_\rho) \otimes \det(F_x, M, Z)$ and it does not depend on g_M and on a flat Hermitian metric on F_ρ ([V1]). The Ray–Singer torsion norm for the Rham complex of (M, Z) with the coefficients in the direct sum of any finite-dimensional local system F_ρ and of the dual one $F_\rho^\vee$ is defined in [V2]. In this case, the Reidemeister torsion $\tau_0 (M, Z; F_\rho \oplus F_\rho^\vee)$ (i.e., the one for (M, Z) with the coefficients in $F_\rho \oplus F_\rho^\vee$) is well-defined, because the fibres of the line bundle $\det \left(F_\rho \oplus F_\rho^\vee \right)$ have the canonical norm in accordance with the local system structure. In this case, the Ray–Singer torsion differs from the Reidemeister torsion by an explicit factor (which is computed in [V2]) but this torsion does not depend on g_M (if g_M is a direct product metric near ∂M). This definition of the Ray–Singer torsion norm does not use a Hermitian structure in the fibers of F_ρ. In [Mü2] another Ray–Singer torsion was introduced for the de Rham complex of a closed (M, g_M) with the coefficients in a local system F_ρ, defined by a unimodular finite-dimensional representation ρ of $\pi_1(M)$. This torsion is defined with the help of an arbitrary Hermitian metric h_ρ in the fibres of F_ρ and it depends on this metric in general. (For a non-unitary representation ρ there are no Hermitian metrics on F_ρ, which are flat with respect

to the canonical flat structure.) It was proved in [Mü2] that in the case of an odd-dimensional M the Ray–Singer torsion, defined with the help of a Hermitian metric h_ρ, is independent of (h_ρ, g_M) and is equal to the Reidemeister torsion. (The Reidemeister torsion is canonically defined for any unimodular finite-dimensional representaion of π_1. In the case of an odd-dimensional closed M it is independent of a flat Hermitian metric on $\det F_\rho$, since the Euler characteristic in this case is equal to zero for each connected component of M.) The Ray–Singer torsion, defined with the help of h_ρ, depends on (h_ρ, g_M) for a general even-dimensional M. The definition of the Ray–Singer torsion for any finite-dimensional representation ρ of $\pi_1(M)$ for a closed (M, g_M) equipped with a Hermitian metric h_ρ (on the fibres of the corresponding vector bundle) is given in [BZ1, BZ2]. In [BZ2] the Ray–Singer metric on the determinant line, corresponding to a finite flat exact sequence $(F^\bullet, d_F)$ of finite-dimensional flat vector bundles over M, is computed (in terms of g_M and of Hermitian metrics on F^j).

The Gaussian integral of $\exp(-(Sx, x))$, where S is a positive self-adjoint operator in a finite-dimensional Hilbert space H, $\dim H = n$, is equal to $(2\pi)^{n/2} (\det S)^{-1/2}$. The Ray–Singer torsion appears naturally in the computations of asymptotic expansions for analogous infinite-dimensional integrals of $\exp(-ikI(A))$, where $I(A)$ possesses an infinite-dimensional symmetry group G ([Sc, Wi1, Wi2]). For instance, the Chern–Simons action

$$I(A) := (4\pi)^{-1} \int_M \operatorname{Tr}(A \wedge dA + 2/3 A \wedge A \wedge A)$$

on a trivialized principal G-bundle P_G over a closed orientable three-dimensional manifold (where $G = SU_N$ and Tr is the trace in the N-dimensional geometrical representation of G, and where A is a connection form) is invariant under the gauge transformations $A \to gAg^{-1} - dg \cdot g^{-1} =: A_g$ for a smooth $g: M \to G$ (where A_g is the same connection but with respect to another trivialization of P_G, i.e., with respect to another smooth section $G \to P_G$). Stationary points of $I(A)$ are the flat connections a_α (i.e., such that the curvature $F(A_\alpha)$ is equal to zero). The asymptotic of an integral of $\exp(-ikI(a))$ as $k \to +\infty, k \in \mathbb{Z}_+$, is computed by the stationary phase method. The principal term of the contribution of a point A_α into this integral (in the case when the flat connection A_α is an isolated one) has as its absolute value the square root of the Ray–Singer torsion of M with coefficients in the local system, defined by a flat connection A_α, with the Lie algebra $\mathfrak{g}$ of G as its fibres (see [Wi1]; [Wi2], 2.2; [BW], 2).

The Reidemeister torsion was essentially used in [Wi2], 4, for the computation of the volume of a moduli space $\mathcal{M}$ of the fundamental group representations for a closed two-dimensional surface. In this case the Reidemeister torsion is a section of $|\det|T^*\mathcal{M}$, i.e., it is a density on $\mathcal{M}$.

This paper is devoted to a proof of a generalized Ray–Singer conjecture for manifolds with a smooth boundary (and also for transmission boundary conditions given on the interior boundaries). We suppose that the local system is trivial. The proof of the Ray–Singer conjecture for non-unitary local systems and for manifolds with corners will be the subject of a subsequent paper.

Let (M, g_M) be a Riemannian manifold with a smooth boundary ∂M and let the Dirichlet and the Neumann boundary conditons be independently given on the connected components of ∂M. Let g_M be a direct product metric near ∂M. Then the Ray–Singer torsion of (M, g_M) is defined as a norm on the determinant line $\det H^\bullet(M, Z)$. (Here Z is the union of the connected components ∂M where the

Dirichlet boundary conditions are given.) This norm is independent of g_M (for direct product metrics g_M near ∂M). The Reidemeister torsion of (M, Z) is an invariant of the PL-structure of (M, Z) and it is a norm on the same determinant line. The torsion norms are defined in Sect. 1. The Ray–Singer norm differs from the Reidemeister norm on $\det H^\bullet(M, Z)$ for a general $\partial M \neq \emptyset$. Their ratio is computed in Theorem 1.4 below.

Let (M, g_M) be obtained by gluing two Riemannian manifolds (M_j, g_{M_j}) along the common component N of their boundaries, $M := M_1 \cup_N M_2$ (where N is a closed smooth manifold of codimension one in M). Let g_M be a direct product metric near N. Then, as it is proved in Theorem 1.1, the Ray–Singer torsion norm $T_0(M, Z)$ on $\det H^\bullet(M, Z)$ is equal to the tensor product of the Ray–Singer norms $T_0(M_1, Z_1 \cup N) \otimes T_0(M_2, Z_2 \cup N) \otimes T_0(N)$ ($Z_k := Z \cap \overline{\partial M_k}$, where the line $\det H^\bullet(M, Z)$ is identified with the tensor product of the lines $\det H^\bullet(M_1, Z_1 \cup N) \otimes \det H^\bullet(M_2, Z_2 \cup N) \otimes \det H^\bullet(N)$ by the short exact sequence of the de Rham complexes

$$0 \to DR^\bullet(M_1, Z_1 \cup N) \oplus DR^\bullet(M_2, Z_2 \cup N) \to DR^\bullet(M, Z) \to DR^\bullet(N) \to 0 , \quad (0.1)$$

where $DR^\bullet(M, Z)$ is the relative de Rham complex of smooth forms with the zero geometrical restrictions to Z, the left arrow is the natural inclusion, and the right arrow is $\sqrt{2}$ times a geometrical restriction. For the Reidemeister norm this assertion is also true and the identification of the determinant lines is given by the analogous exact sequence of cochain complexes. However in this case the right arrow is the geometrical restriction of cochains (without additional factor $\sqrt{2}$). Let (M, Z) be obtained by gluing two manifolds (M_1, Z_1) and (M_2, Z_2) along the common component N of their boundaries, $M := M_1 \cup_N M_2$. Then the ratio of the square of the Ray–Singer norm and the square of the Reidemeister norm for (M, Z) is equal to the product of the same ratios for $(M_1, Z_1 \cup N), (M_2, Z_2 \cup N)$, and for N with an additional factor $2^{-\chi(N)}$. So the assertion of Theorem 1.1 claims that it is possible to calculate the Ray–Singer norm by cutting of a manifold into pieces which are manifolds with smooth boundaries. The main theorems of this paper are consequences of Theorem 1.1. This theorem provides us with the gluing formula for the Ray–Singer torsion norms. Such a gluing formula is a new one.

In the case of a manifold with a smooth boundary, the Ray–Singer torsion $T_0(M, Z)$ is a function not only of (M, Z) but also of the phase θ of a cut of the spectral plane $\mathbb{C}$ (because the zeta-functions $\zeta_j(s)$ for the Laplacians Δ_j on $DR^j(M, Z)$ are defined for $\mathrm{Re}\, s > (\dim M)/2$ as the sums $\sum \lambda^{-s}$ over the nonzero eigenvalues, and λ^{-s} is defined as $\lambda_{(\theta)}^{-s} := \exp(-s \log_{(\theta)} \lambda)$, where $\theta - 2\pi < \mathrm{Im}\, \log_{(\theta)} \lambda < \theta, \theta \notin 2\pi\mathbb{Z}$). In fact, $T_0(M, Z; \theta)$ (as well as $\zeta_j(s)$) depends only on $[\theta/2\pi]$. The zeta-function regularization of the $\det'(\Delta_j)$ (i.e., of the product of all the nonzero eigenvalues of Δ_j, including their multiplicities) is defined as $\exp(-\partial_s \zeta_j(s)|_{s=0})$. The analytic continuation of $\zeta_j(s)$ is regular at zero. The zeta-function $\zeta_j(s; m)$ depends on $m := [\theta/2\pi], \theta \notin 2\pi\mathbb{Z}$, as follows:

$$\zeta_j(s; m+1) = \exp(-2\pi i s)\zeta_j(s; m) ,$$

$$\det'(\Delta_j; m+1) = \exp(2\pi i \zeta_j(0))\det'(\Delta_j; m) .$$

The number $\zeta_j(0)$ is independent of m, and the number $\zeta_j(0) + \dim \mathrm{Ker}\, \Delta_j$ can be interpreted as the regularized dimension of the space $DR^j(M)$. This regularized dimension depends not only on the space $DR^j(M)$ but it also depends on a positive

definite self-adjoint elliptic differential operator of a positive order, which acts in $DR^j(M)$. This dimension is a real number but it is not an integer in the case of the Laplacians on $DR^\bullet(M)$ for a general closed even-dimensional (M, g_M). Hence, $\det(\Delta_j; m)$ depends on m for such (M, g_M). The number $\zeta_j(0)$ is an integer for a generalized Laplacian on a closed odd-dimensional (M, g_M), according to [BGV], Theorem 2.30, or to [Gr], Theorem 1.6.1. It is equal to zero when M is closed, $\dim M$ is odd, $\dim \operatorname{Ker} \Delta_j = 0$.

Even in such a simple case as for an interval $(I, \partial I)$ with the Dirichlet boundary conditions the dependence of $T_0(M, Z; m)$ on m is nontrivial. The ratio of the torsion $T_0(M, Z; [\theta/2\pi])$ and the Reidemeister torsion norm is computed in Theorem 2.2.

The paper is organized as follows. In Sect. 1 we deduce a generalization of the Ray–Singer conjecture from the gluing formula for Ray–Singer torsion norms. This formula is proved in Theorem 1.1. The proof uses v-transmission interior boundary conditions on N, where $v = (\alpha, \beta) \in \mathbb{R}^2 \backslash (0, 0)$. These interior boundary problems give us a smooth in v family of spectral problems on M. Such a problem for $v = (1, 1)$ coincides (in a spectral sense) with the spectral problem for a glued M. For $v = (0, 1)$ or for $v = (1, 0)$ it is a direct sum of spectral problems on M_1 and on M_2, i.e., the two pieces of M are completely disconnected. So this family provides us with a smooth process of cutting (in a spectral sense) of M in two pieces M_1 and M_2. Let $M = M_1 \cup_N M_2$ be obtained by gluing M_1 and M_2 along the common component N of their boundaries. Then the Ray–Singer norm $T_0(M_v, Z)$ on the determinant line $\det H^\bullet(M_v, Z)$ for the de Rham complex $DR^\bullet(M_v, Z)$ with v-transmission conditions on N is defined. The short exact sequence for $DR^\bullet(M_v, Z)$, similar to (0.1), has the same the first and third terms as (0.1). The homomorphisms $r_v : DR^\bullet(M_v, Z) \to DR^\bullet(N)$ are of the form $r_v = (\alpha i_1^* + \beta i_2^*)/|v|$, where $i_j^* \omega_j$ are the geometrical restrictions to N for the components ω_j of $\omega = (\omega_1, \omega_2) \in DR^\bullet(M_v, Z)$. Note that $r_{(1,1)} = \sqrt{2} i^*$. (This is the reason of the appearance of $\sqrt{2} i^*$ in the exact sequence (0.1) connected with the gluing formula.) In Lemma 1.2 we prove that the gluing property for analytic torsion norms (Theorem 1.1) is equivalent to the independence of v of the norms on $\det H^\bullet(M_1, Z_1 \cup N) \otimes \det H^\bullet(M_2, Z_2 \cup N) \otimes \det H^\bullet(N)$ induced by $T_0(M_v, Z)$. (Here the identification of the determinant lines is defined by the short exact sequence for $DR^\bullet(M_v, Z)$.) The latter assertion is proved in Sect. 2. First we prove that the norm induced by the Ray–Singer torsion $T_0(M_v, Z)$ is locally independent of v in the case when $\alpha\beta \neq 0$ (where $v = (\alpha, \beta)$). We do this in Sects. 2.3, 2.5, and 2.6 with the help of explicit variation formulas for the scalar Ray–Singer torsion $T(M_v, Z)$ (if v depends smoothly on a parameter). We define a family (in v) of homomorphisms to identify finite-dimensional subcomplexes $W_a^\bullet(v)$ of $DR^\bullet(M_v, Z)$. (The complexes $W_a^\bullet(v)$ are spanned by the eigenforms of the Laplacians with eigenvalues less than a fixed number $a > 0$. We suppose that a is not an eigenvalue of $\Delta_j(M_v, Z)$ for $0 \leq j \leq n$.) Then we compute the actions of these homomorphisms on the determinant lines. These identifications are not canonical; we choose some particular (quite natural) identifications for v sufficiently close to v_0 such that $\alpha_0\beta_0 \neq 0$.

Then it is enough to prove the continuity in $v \in \mathbb{R}^2 \backslash (0, 0)$ of the norm on $\det H^\bullet(M_1, Z_1 \cup N) \otimes \det H^\bullet(M_2, Z_2 \cup N) \otimes \det H^\bullet(N)$, which is induced by the Ray–Singer norm $T_0(M_v, Z)$. We prove in Sect. 2.7 that the truncated scalar analytic torsion $T(M_v, Z; a)$, corresponding to the eigenvalues λ of $\Delta_j(M_v, Z)$ which are greater than a, is locally continuous in v. Then we prove that the norm, induced by the analytic torsion norm $T_0(W_a^\bullet(v))$ of a finite-dimensional complex $W_a^\bullet(v)$, is

locally continuous in v. The latter assertion is proved in Sects. 2.2, 2.4, and 2.7 with the use of the cone of the homomorphism $R_v^\bullet(a)$: $W_a^\bullet(v) \to C^\bullet(X_v, Z \cap X)$ (where $R_v^\bullet(a)$ is the integration of differential forms from $W_v^\bullet$ over the simplexes of a given smooth triangulation X of M, and $C^\bullet(X_v, Z \cap X)$ is the corresponding cochain complex). This homomorphism is a quasi-isomorphism for any $v \in \mathbb{R}^2 \backslash (0,0)$ (Proposition 2.3). We can conclude that the analytic torsion norm on the $\det H^\bullet$ (Cone $R_v(a)$) $= \mathbb{C}$ (for a fixed v) corresponds to an acyclic finite-dimensional complex and is defined by the derivatives at zero of the zeta-functions for self-adjoint finite-dimensional invertible operators. So these norms are locally continuous in v. (This is proved in Sect. 2.7.) Then the local continuity of the norm induced by $T_0(W_a^\bullet(v))$ on $\det H^\bullet(M_1, Z_1 \cap N) \otimes \det H^\bullet(M_2, Z_2 \cup N) \otimes \det H^\bullet(N)$ follows from the continuity of the norm (on the same determinant line) induced by $T_0(C^\bullet(X_v, Z \cap X))$ and from the identity:

$$T_0(W_a^\bullet) = T_0(C^\bullet(X_v, Z \cap X)) / \|1\|^2_{T_0(\text{Cone}^\bullet R_v(a))} .$$

This identity is proved in Lemma 2.4.

The use of the cone of $R_v^\bullet(a)$ allows us to avoid difficulties connected with the fact that some positive eigenvalues of the Laplacians $\Delta^\bullet(M_v, Z)$ tend to 0 as $v = (\alpha, \beta)$ tends to $v_0 = (1,0)$ (or to $v_0 = (0,1)$). The dimensions of $H^\bullet(M_v, Z)$ essentially change when $v, \alpha\beta \neq 0$, is replaced by v_0. (Only the Euler characteristic $\chi(H^\bullet(M_v, Z))$ does not change when v is replaced by v_0.) It is impossible to find for a general N the precise asymptotic expressions for the eigenvalues λ, which tend to zero as $v \to v_0$, and especially to find the asymptotics of the corresponding eigenforms ω_λ of $\Delta^\bullet(M_v, Z)$. So the continuity of the norm induced by $T_0(M_v, Z)$ (viewed as a function of v) at the point v_0 cannot be proved for a general M (obtained by gluing two pieces M_1 and M_2 along N) with the help of separate computations of the asymptotic expressions for the scalar torsion $T(M_v, Z)$ and for the measure on $\det H^\bullet(M_v, Z)$ defined by harmonic forms. The proof of the classical Ray–Singer conjecture in [Ch] and the proof in [Mü2] (in the case of unimodular representations of $\pi_1(M)$) are based on asymptotic computations of such quantities for a manifold with boundary $M_u := M \backslash S_u$, where S_u is a tubular neighborhood of an embedded sphere $S^k \hookrightarrow M^n$ as the radius u of the tubular neighborhood (in the normal to S^k direction) tends to zero. (It is also supposed in [Ch] that S_u is a direct product on $S^k \times D^{n-k}$ and that $g_M|_{S_u}$ is a direct product metric on $S^k \times D^{n-k}$.)

To give a rigorous proof of the assertions above used in the proof of the gluing formula, it is necessary to prove a lot of analytic propositions. We do it in Sects. 2.2, 2.6, 2.7, and in Sect. 3. The theory of ζ- and θ-functions in the case of v-transmission interior boundary conditions is elaborated in Sect. 3. The precise estimates of the corresponding ζ-functions in vertical strips are obtained in Sect. 3.4. These estimates allow us using the inverse Mellin transform to derive the information about the densities on M, N, and ∂M for the asymptotic expansions as $t \to +0$ of θ-functions from the properties of the densities for appropriate ζ-functions.

1. Analytic Torsion and Ray–Singer Conjecture

1.1. Analytic and Combinatorial Torsion Norms. The analytic torsion norm appears in the following finite-dimensional algebraic situation. Let $(A^\bullet, d)$ be a finite

complex of finite-dimensional Hilbert spaces. The *determinant of* $(A^\bullet, d)$ is the tensor product

$$\bigotimes_j (\Lambda^{\max} A^j)^{(-1)^{j+1}} =: \det(A^\bullet) \,,$$

where $\Lambda^{\max} A^j =: \det A^j$ is the top exterior power of the linear space A^j and where L^{-1} is the dual space $L^\vee$ for a one-dimensional vector space L over $\mathbb{C}$. The natural Hilbert norm $|| \cdot ||^2_{\det A}$ is defined by the Hilbert norms on A^j.

The determinant of the cohomology $\det H^\bullet(A)$ of $(A^\bullet, d)$ is also defined and there is a natural norm on it (since $H^j(A)$ is the subquotient of A^j). The differential d provides us with the identification

$$f(d) : \det(A^\bullet) \simeq \det H^\bullet(A) \,.$$

However in the general case this identification is not an isometry of the norms $|| \cdot ||^2_{\det A}$ and $|| \cdot ||^2_{\det H^\bullet(A)}$. For $f(d)$ to be an isometry it is necessary to multiply $|| \cdot ||^2_{\det H^\bullet(A)}$ by the *scalar analytic torsion* of a complex $(A^\bullet, d)$, which is defined as

$$T(A^\bullet, d) = \exp\left(\Sigma(-1)^j j \partial_s \zeta_j(s)|_{s=0}\right) \,. \tag{1.1}$$

Here $\zeta_j(s) = \sum' \lambda^{-s}$ is the sum[1] over all the nonzero eigenvalues $\lambda \neq 0$ (including their multiplicities) of the nonnegative (i.e., if $\lambda \neq 0$ then $\lambda > 0$) self-adjoint operator $(d^*d + dd^*)|A^j$. The derivative $\partial_s \zeta_j(s)|_{s=0}$ is equal to $-\log \det'((d^*d + dd^*)|_{A^j})$ (i.e., it is equal to the sum of $(-\log \lambda) \in \mathbb{R}$ over all the nonzero eigenvalues λ).

It is enough to prove the assertion (1.1) in the case of a two-terms complex $d : F_0 \rightleftarrows F_1$, where $\dim F_j = 1$, $e_j \in F_j$, $de_0 = \mu e_1$, $\mu \neq 0$, and where $||e_0||^2 = 1 = ||e_1||^2$. In this case the element $e_1 \otimes e_0^{-1} \in \det(F^\bullet)$ is of the unit norm and the square of the norm of the corresponding element $\mu^{-1} \in \mathbb{C}$ from $\mathbb{C} = \det 0 = \det H^\bullet(F)$ is equal to $|\mu^{-1}|^2$. If the norm $|\mu^{-1}|^2$ is multiplied by the scalar analytic torsion for $F^\bullet$, namely by $\exp(\log \det(d^*d)) = \exp(\log \det(dd^*)) = |\mu|^2$ then the isomorphism between $\det(F^\bullet)$ and $\mathbb{C} = \det 0$ (defined by d) becomes an isometry.

This finite-dimensional definition makes sense also for the infinite-dimensional de Rham complex of a closed smooth manifold. In this case the analytic torsion is the norm on the determinant of the cohomology of this manifold. Let $(DR^\bullet(M), d)$ be the de Rham complex of smooth differential forms (with the values in $\mathbb{C}$) on a closed manifold M. The scalar analytic torsion for a closed Riemannian manifold (M, g_M) is defined by the same formula (1.1), where $d^* = \delta$ (relative to g_M) and $(d^*d + dd^*)|DR^j(M)$ is the Laplace–Beltrami operator Δ_j. In this case the series, which defines $\zeta_j(s)$, converges for $\operatorname{Re} s > (\dim M)/2$. The analytic function $\zeta_j(s)$ can be analytically (meromorphically) continued to the whole complex plane. It is known that $\zeta_j(s)$ has simple poles and that it is regular at zero ([Se2]).

The cohomology $H^\bullet(DR(M))$ are canonically identified (by the integration of the forms over the simplexes) with the cohomology $H^\bullet(M)$ of M. This follows from the de Rham theorem. The Hodge theorem claims that each element of $H^j(DR(M))$ has one and only one representative in the space of harmonic forms $\operatorname{Ker} \Delta_j$. The natural norm on $\operatorname{Ker} \Delta_j$ (defined by the Riemannian metric g_M) provides us with the norm $|| \cdot ||^2_{\det H^\bullet(M)}$ on $\det H^\bullet(M)$. For an odd-dimensional M this norm depends on g_M.

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[1] The function λ^{-s} is defined as $\exp(-s \log \lambda)$ where $\log \lambda \in \mathbb{R}$ for $\lambda \in \mathbb{R}_+$.

Definition. *The analytic torsion norm $T_0(M)$ on $\det H^\bullet(M)$ is the norm*

$$T_0(M) := || \cdot ||^2_{\det H^\bullet(M)} \cdot \exp \left(\Sigma(-1)^j j \partial_s \zeta_j(s)|_{s=0} \right) . \tag{1.2}$$

The main property of this norm is its independence of a Riemannian metric g_M. So it is an invariant of a smooth structure on M. Let us suppose that $g_M = g_M(\gamma)$ depends smoothly on a parameter $\gamma \in \mathbb{R}^1$. Then the variation formulas in [RS], Theorems 2.1, 7.3 (or in [Ch], Theorem 3.10, (3.22)), claim that

$$\partial_\gamma \sum_j (-1)^j j \partial_s \zeta_{j,\gamma}(s)|_{s=0} = \sum_j (-1)^j \left(-\mathrm{Tr}(\exp(-t\Delta_{j,\gamma})\alpha)^0 + \mathrm{Tr}(\mathscr{H}_{j,\gamma}\alpha) \right) . \tag{1.3}$$

Here $\mathscr{H}_{i,\gamma}$ is the kernel of the orthogonal projection operator from $DR^j(M)$ onto $\mathrm{Ker}\,\Delta_j(M, g_M(\gamma))$, $\alpha := *_\gamma^{-1} \partial_\gamma(*_\gamma)$ ($*_\gamma$ corresponds to $g_M(\gamma)$) and $\mathrm{Tr}(\exp(-t\Delta_{j,\gamma})\alpha)^0$ is the constant coefficient in the asymptotic expansion as $t \to +0$ ($n := \dim M$):

$$\mathrm{Tr}(\exp(-t\Delta_{j,\gamma})\alpha) = \sum_{k=0}^{l} m_{j,k} t^{-n/2+k} + o\left(t^l\right) . \tag{1.4}$$

The existence of the asymptotic expansion (1.4) follows from [Gr], Theorem 1.6.1, or from [BGV], Theorem 2.30. For a family of norms $|| \cdot ||^2(\gamma)$ on $\det H^\bullet(M)$ defined by the harmonic forms $\mathrm{Ker}(\Delta_j(M, g_M(\gamma)))$ the following equality holds for any fixed $\mu \in \det H^\bullet(M)$, $\mu \neq 0$ ([RS], Sect. 7):

$$\partial_\gamma \log ||\mu||^2_{\det H^\bullet(M)}(\gamma) = -\sum(-1)^j \mathrm{Tr}(\mathscr{H}_{j,\gamma}\alpha) .$$

Hence, (1.3) implies the equality

$$\partial_\gamma \log T_0(M, g_M) = \sum(-1)^{j+1} m_{j,n/2} . \tag{1.5}$$

Since k in (1.4) are integers, we see that the right side of (1.5) is zero for odd n. For even n, $n = 2l$, the right side of (1.5) is also equal to zero, since $m_{j,l} = -m_{2l-j,l}$. This fact follows from the equalities

$$\partial_\gamma \left(*_\gamma^{-1} *_\gamma \right) = 0, \quad \alpha = -*\alpha *^{-1} ,$$

$$\mathrm{Tr}(\exp(-t\Delta_j)\alpha) = \mathrm{Tr}\left(\left(*\exp(-t\Delta_j)*^{-1} \right)(-\alpha) \right) = -\mathrm{Tr}(\exp(-t\Delta_{n-j})\alpha) ,$$

(since they imply the equalities $m_{j,k} = -m_{n-j,k}$, where n is even and $k \in \mathbb{Z}_+ \cup 0$).

The analytic torsion norm can be interpreted (in an intuitional sense) as the norm, corresponding to an element $v \in \det DR^\bullet(M)$ (v is defined up to a multiplicative constant $c \in \mathbb{C}$, $|c| = 1$, and its "torsion norm" is equal to one). The space $\det DR^\bullet(M)$ and L_2-norm on it are not defined but the space $\det H^\bullet(M)$ and the analytic torsion norm $T_0(M)$ on it are rigorously defined. For a finite-dimensional complex the analytic torsion norm on the deteminant of its cohomology corresponds to the norm on the determinant of the complex defined by the Hilbert structures on the terms of this complex. The analytic torsion norm is (in some sense) a multiplicative Euler characteristic useful for odd-dimensional manifolds.

The same definition of $T_0(M)$ makes sense also in the case when M is a compact Riemannian manifold with a smooth boundary $\partial M = \cup N_i$ and with the Dirichlet or the Neumann boundary conditions given independently on each connected component N_i of ∂M. Let the metric g_M be a direct product metric near ∂M. Then $T_0(M)$ is independent of g_M as in the case of a closed manifold (this is proved below).

Let X be a smooth triangulation of M and let $(C^\bullet(X), d_c)$ be a cochain complex of X (with complex coefficients). Then each $C^j(X)$ has the Hilbert structure defined by the orthonormal basis of basic cochains $\{\delta_e\}$, where $\delta_e(e_1)$ is 1 for $e_1 = e$ and 0 for $e_1 \neq e$. Hence the scalar torsion $T(C^\bullet(X), d_c)$ is also defined.

The *combinatorial torsion* $\tau_0(X)$ is defined as the following norm on the determinant of the cohomology $H^\bullet(C(X), d_c) = H^\bullet(M)$:

$$\tau_0(X) := \| \cdot \|^2_{\det H^\bullet(C(X))} \cdot T(C^\bullet(X), d_c) \tag{1.6}$$

(where $H^j(C(X))$ is the subquotient of $C^j(X)$ and so it has the natural Hilbert structure induced from $C^j(X)$). The norm (1.6) is invariant under any regular subdivisions of X. So this norm is an invariant of the combinatorial structure of M (which is completely defined by a smooth structure on M). This norm corresponds to the Hilbert norm on $\det C^\bullet(X)$ defined by the basic cochains.

Let M be a manifold with a smooth boundary $\partial M = \cup N_l$, where N_l are the connected components of ∂M. Let Z be the union of N_i where the Dirichlet boundary conditions are given. Set $V := X \cap Z$. Then (1.6) (where $H^\bullet(C(X)), d_c)$ and $T(C^\bullet(X), d_c)$ are replaced by $H^\bullet(C(X, V))$ and by $T(C^\bullet(X, V), d_c))$ provides us with the definition of the norm $\tau_0(X, V)$. This norm is an invariant of the combinatorial structure on (M, Z) ([Mi], Sects. 7, 8, 9).

1.2. Gluing Formulas. The Ray–Singer conjecture claims that for a closed smooth manifold M the norms $\tau_0(M)$ and $T_0(M)$ on the same one-dimensional space $\det H^\bullet(M)$ are equal[2]

$$\tau_0(M) = T_0(M) . \tag{1.7}$$

How to prove such a formula in a natural way? It is necessary to find a general property of the analytic torsion which implies the equality (1.7). Such a property can be formulated as follows. Let $(M, \partial M)$ be a Riemannian manifold with a smooth boundary and with the Dirichlet or the Newmann boundary conditions given independently on the connected components of ∂M. Let a closed codimension one submanifold N of M, $N \cap \partial M = \emptyset$, divide M in two pieces M_1 and M_2 (glued along N), $M = M_1 \cup_N M_2$, and let a metric g_M be a direct product metric near N and near ∂M. Let $T_0(M_k, N)$ be the analytic torsion norm for M_k (with the Dirichlet boundary conditions on N), and let the boundary conditions on the connected components of ∂M belonging to ∂M_k be the same as for $T_0(M)$. The following assertion is central in this paper.

Theorem 1.1 (Gluing property). *The analytic torsion norm $T_0(M, Z)$ is the tensor product of the analytic torsion norms for $(M_1, Z_1 \cup N), (M_2, Z_2 \cup N)$, and for N,*

$$\varphi_{\mathrm{an}} T_0(M, Z) = T_0(M_1, Z_1 \cup N) \otimes T_0(M_2, Z_2 \cup N) \otimes T_0(N) , \tag{1.8}$$

where $Z_k := Z \cap \partial \overline{M}_k$.

The identification φ_{an} (in (1.8)) of $\det H^\bullet(M, Z)$ with the tensor product of the three one-dimensional spaces:

$$\varphi_{\mathrm{an}} : \det H^\bullet(M, Z) \to \det H^\bullet(M_1, Z_1 \cup N) \otimes \det H^\bullet(M_2, Z_2 \cup N) \otimes \det H^\bullet(N)$$
$$=: \mathbf{Det}(M, N, Z) \tag{1.9}$$

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[2] The cohomology $H^\bullet(DR(M))$ and $H^\bullet(C(X))$ are identified (according to the de Rham theorem) by the homomorphism of the integration of forms from $DR^\bullet(M)$ over the simplexes of a smooth triangulation X of M.

is defined by the long cohomology exact sequence corresponding to the following short exact sequence of the de Rham complexes:

$$0 \to DR^\bullet(M_1, Z_1 \cup N) \oplus DR^\bullet(M_2, Z_2 \cup N) \to DR^\bullet(M_{1,1}, Z) \xrightarrow{r} DR^\bullet(N) \to 0 .$$
(1.10)

The relative de Rham complex $(DR^\bullet(M_k, Z_k \cup N), d)$ (where d is the exterior derivative of differential forms) consists of the smooth forms ω on $\overline{M}_k$, having the zero geometrical restriction to $N : i_k^* \omega = 0$ (where $i_k : N \subset \partial M_k \hookrightarrow M_k$) and also having the zero restrictions to the components of $\partial M \cap M_k$ where the Dirichlet boundary conditions are given (i.e., to Z_k). The complex $(DR(M_{1,1}), d)$ consists of the pairs (ω_1, ω_2) of smooth differential forms $\omega_k \in DR^\bullet(M_k, Z_k)$ (i.e., ω_k have the zero geometrical restrictions to the corresponding components of $\partial M \cap \overline{M}_k$), which have the same geometrical restrictions to N:

$$i_1^* \omega_1 = i_2^* \omega_2 .$$

The differential $d(\omega_1, \omega_2)$ in $DR^\bullet(M_{1,1})$ is defined as $(d\omega_1, d\omega_2)$. The left arrow in (1.10) is the natural inclusion of $\bigoplus_k DR^\bullet(M_k, Z_k \cup N)$ into $DR^\bullet(M_{1,1}, Z)$. The right arrow r in (1.10) is not a usual geometrical restriction but is the one multiplied by $\sqrt{2}$:

$$r(\omega_1, \omega_2) = \sqrt{2} i_k^* \omega_k \in DR^\bullet(N) .$$
(1.11)

To define φ_{an}, it is necessary to introduce a natural identification of $H^\bullet(DR(M, Z))$ with $H^\bullet(DR(M_{1,1}, Z))$. (The short exact sequence (1.10) provides us with the identification

$$\varphi_{an} : \det H^\bullet(DR(M_{1,1}, Z)) \widetilde{\to} \mathbf{Det}(M, N, Z) ,$$

but not with the identification of $\det H^\bullet(DR(M, Z))$ with $\mathbf{Det}(M, N, Z)$.) We show in Proposition 1.1 (for any given metric g_M) that not only all the eigenvalues with their multiplicities but also all the eigenforms of the natural Laplacian $\Delta_{1,1}$ on $DR^\bullet(M_{1,1}, Z)$ are the same as for the Laplacian on $DR^\bullet(M, Z)$. Thus, the operator $\Delta_{1,1}(g_M)$ in a very strict spectral sense is the same as $\Delta(g_M)$.

The homotopy operator between the identity operator on $DR^\bullet(M_{1,1}, Z)$ and the projection operator from $DR^\bullet(M_{1,1}, Z)$ onto $\mathrm{Ker}^\bullet \Delta_{1,1} = \mathrm{Ker}^\bullet \Delta$ is obtained with the help of the Green function $G_{1,1}$ for the operator $\Delta_{1,1}$ (Lemma 1.1). This homotopy operator provides us with the canonical identification of $H^\bullet(DR(M_{1,1}, Z))$ with $\mathrm{Ker}\,\Delta_{1,1}^\bullet$. So it defines the identification of $H^\bullet(DR(M_{1,1}, Z))$ with $\mathrm{Ker}\,\Delta^\bullet = H^\bullet(DR(M, Z))$ (since $\mathrm{Ker}\,\Delta^\bullet$ is canonically identified with $\mathrm{Ker}\,\Delta_{1,1}^\bullet$).

To prove Theorem 1.1 we introduce a family of interior boundary conditions on N and show that the induced norm $\varphi_{an}^v T_0(M_v, Z)$ on $\mathbf{Det}(M, N, Z)$ is independent of v (where $v = (\alpha, \beta) \in \mathbb{R}^2 \backslash (0, 0)$ are the parameters of interior boundary conditions on N). Namely

$$\varphi_v^{an} T_0(M_v, Z) = c_0 T_0(M_1, Z_1 \cup N) \otimes T_0(M_2, Z_2 \cup N) \otimes T_0(N)$$
(1.12)

with some positive c_0 which may depend on $(M, g_M, \partial M)$ and on the boundary conditions on ∂M but *does not depend* on the parameters $(\alpha, \beta) = v$. Suppose that the formula (1.12) holds for any gluing two pieces M_1 and M_2 along a closed N, $M = M_1 \cup_N M_2$, where the factor c_0 is independent of v. Then it is easy to conclude that $c_0 = 1$ (Lemma 1.2). In (1.12) $T_0(M_v, Z)$ is the analytic torsion norm for the de Rham complex $(DR^\bullet(M_v, Z), d)$. This complex consists of the pairs of smooth

forms (ω_1, ω_2) such that $\omega_k \in DR^\bullet\left(\overline{M}_k, Z_k\right)$ has the zero geometrical restrictions to $Z_k := Z \cap \partial M_k$ [3] and that the following transmission condition holds for the geometrical restrictions $i_k^* \omega_k$ of ω_k to N

$$\alpha i_1^* \omega_1 = \beta i_2^* \omega_2 \ . \tag{1.13}$$

The analytic torsion norm $T_0(M_\nu, Z)$ is defined for an arbitrary $\nu = (\alpha, \beta) \in \mathbb{R}^2 \backslash (0,0)$. There is a canonical identification of $H^j(DR^\bullet(M_\nu, Z))$ with the space of the corresponding harmonic forms $\mathrm{Ker}(\varDelta_\nu \mid DR^j(M_\nu, Z))$ (Lemma 1.1). This identification (similarly to the case of $DR^\bullet(M_{1,1}, Z)$) is obtained by the homotopy operator, which is defined using the Green function for the Laplacian $\varDelta_\nu$. (This Laplacian is an elliptic self-adjoint operator by Theorem 3.1.) The boundary conditions for $\varDelta_\nu$ on N and ∂M are elliptic (and differential). The Green function G_ν for $\varDelta_\nu$ exists (and depends smoothly on $\nu \neq (0,0)$) according to Theorem 3.1 and to Proposition 3.1. This identification provides us with the natural norms on $H^j(DR^\bullet(M_\nu, Z)) =: H^j(M_\nu, Z)$ and on $\det H^\bullet(M_\nu, Z)$. The scalar analytic torsion $T(M_\nu, Z)$ is defined by $\zeta_{\nu,j}(s) := \sum' \lambda_i^{-s}$ for $\mathrm{Re}\ s > (\dim M)/2$ (where the sum is over all the nonzero eigenvalues λ_i of the Laplacian $\varDelta_{\nu,j} := \varDelta_\nu \mid DR^j(M_\nu, Z)$ with their multiplicities). These functions $\zeta_{\nu,j}$ can be continued to meromorphic functions on the whole complex plane with simple poles and regular at zero. (This statement is proved in Theorem 3.1 and in Proposition 3.1 below.)

The *analytic torsion norm* on $\det H^\bullet(M_\nu, Z)$ is the norm

$$T_0(M_\nu, Z) = \| \cdot \|_{\det H^\bullet(M_\nu, Z)}^2 \ \exp\left(\sum(-1)^j j \partial_s \zeta_{\nu,j}(s)\mid_{s=0}\right) \ .$$

The identifiaction $\varphi_\nu^{\mathrm{an}}$ in (1.12) is defined by the short exact sequence of the de Rham complexes (where $Z_k := Z \cap \partial \overline{M}_k$):

$$0 \to DR^\bullet(M_1, Z_1 \cup N) \oplus DR^\bullet(M_2, Z_2 \cup N) \to DR^\bullet(M_{\alpha,\beta}, Z) \xrightarrow{r_{\alpha,\beta}} DR^\bullet(N) \to 0 \ . \tag{1.14}$$

The left arrow in (1.14) is the natural inclusion and the right arrow $r_{\alpha,\beta}$ is

$$r_{\alpha,\beta}(\omega_1, \omega_2) := (\alpha^2 + \beta^2)^{-1/2}(\beta i_1^* \omega_1 + \alpha i_2^* \omega_2) \ . \tag{1.15}$$

For $(\alpha, \beta) = (1,1)$ we have $r_{1,1} = \sqrt{2} i_k^* \omega_k$. This corresponds to (1.11). Hence, φ_{an} is equal to $\varphi_{\alpha,\beta}^{\mathrm{an}}$ for $(\alpha, \beta) = (1,1)$.

The complex $DR^\bullet(M_\nu, Z)$ for the values $(0,1)$ and $(1,0)$ of ν is the direct sum of the de Rham complexes of all the smooth forms (with the zero geometrical restriction to Z_k) on one of the manifolds M_k and of all the smooth forms with the zero geometrical restriction to $Z_j \cup N$ on another piece M_j of the manifold M. Thus, the two pieces of M are completely disconnected with respect to $DR^\bullet(M_\nu, Z)$ for these special values of ν. The family of spectral problems on $DR^\bullet(M_\nu, Z)$ for $\nu \in \mathbb{R}^2 \backslash (0,0)$ provides us with a smooth deformation between a spectral problem on M (without any interior boundary conditions) and the direct sum of spectral problems on (M_1, Z_1) and on $(M_2, Z_2 \cup N)$. So this family of interior boundary problems is (in a spectral sense) a kind of a smooth cutting of M in two disconnected pieces.

Let (M_1, N) be a compact smooth Riemannian manifold (M_1, g_{M_1}) with a smooth boundary ∂M_1 and let N be a union of some connected components of ∂M_1. Let a metric g_{M_1} be a direct product metric near the boundary. Then (as it follows from

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[3] Z is the union of the components of ∂M where the Dirichlet boundary conditions are given.

the equality (1.8)) the analytic torsion norm $T_0(M_1, N)$ on det $H^\bullet(DR(M_1, N))$ does not depend on g_{M_1}. To prove this, it is enough to take as (M, g_M) a closed manifold $M = M_1 \cup_N M_1$ with a mirror symmetric (with respect to N) Riemannian metric g_M which coincides with g_{M_1} on each piece M_1 of M (g_{M_1} is a direct product metric near N and so g_M is smooth on M). Since the torsions $T_0(M)$ and $T_0(N)$ are independent of g_{M_1} and of $g_N = g_{M_1}|_{TN}$, we see that $T_0(M_1, N)$ does not depend on g_{M_1}.

It follows from the equality (1.12) with $c_0 = 1$ that $T_0(M_v, Z)$ does not depend on g_M. Indeed, $T_0(M_J, Z_J \cup N)$ and $T_0(N)$ are independent of g_M, and the identification φ_v^{an} is also independent of g_M. (Here M is a manifold with a smooth boundary ∂M, $N \cap \partial M = \emptyset$, the Dirichlet boundary conditions are given on a union Z of some components of ∂M, the Neumann boundary conditions are given on $\partial M \backslash Z$, and g_M is a direct product metric near ∂M and near N, $Z_k := Z \cap \partial \overline{M}_k$.)

Since $DR^\bullet(M_{0,1}, Z)$ is the direct sum $DR^\bullet(M_1, Z_1) \oplus DR^\bullet(M_2, Z_2 \cup N)$ of the de Rham complexes ($Z_k := Z \cap \partial \overline{M}_k$), we see that the analytic torsion norm $T_0(M_{0,1})$ is canonically equal to the tensor product of norms:

$$T_0(M_{0,1}, Z) = T_0(M_1, Z_1) \otimes T_0(M_2, Z_2 \cup N). \qquad (1.16)$$

The determinant line in (1.16) is the tensor product

$$\det H^\bullet(M_{0,1}, Z) = \det H^\bullet(M_1, Z_1) \otimes \det H^\bullet(M_2, Z_2 \cup N)$$

(where $H^\bullet(M_1, Z_1)$ and $H^\bullet(M_2, Z_2 \cup N)$ are the relative cohomology).

The formula (1.8) claims for $v = (0, 1)$ that

$$\varphi_{0,1}^{\text{an}} T_0(M_{0,1}, Z) = T_0(M_1, Z_1 \cup N) \otimes T_0(M_2, Z_2 \cup N) \otimes T_0(N). \qquad (1.17)$$

It follows from the definition of the exact sequence (1.14) that $\varphi_{0,1}^{\text{an}}$ is the identity on the component $\det H^\bullet(M_2, Z_2 \cup N)$ of $\det H^\bullet(M_{0,1}, Z)$. The following theorem is an immediate consequence of (1.16) and (1.17). Let N be a union of some connected components of ∂M_1, let M_1 be a compact Riemannian manifold with a smooth boundary ∂M_1 and let Z_1 be a union of some connected components of ∂M_1 not belonging to N. Suppose that the metric g_{M_1} is a direct product metric near ∂M_1.

Theorem 1.2 (Gluing of Boundary Components). *The equality holds*

$$\varphi_{\text{an}} T_0(M_1, Z_1) = T_0(M_1, Z_1 \cup N) \otimes T_0(N). \qquad (1.18)$$

The identification of the determinant lines in (1.18),

$$\varphi_{\text{an}} : \det H^\bullet(M_1, Z_1) \xrightarrow{\sim} \det H^\bullet(M_1, Z_1 \cup N) \otimes \det H^\bullet(N), \qquad (1.19)$$

is defined by the short exact sequence of the de Rham complexes:

$$0 \to DR^\bullet(M_1, Z_1 \cup N) \to DR^\bullet(M_1, Z_1) \to DR^\bullet(N) \to 0, \qquad (1.20)$$

where the left arrow is the natural inclusion, and the right arrow is the geometrical restriction.

Example 1.1. Formula (1.18) contains the Lerch formula ([WW], 13.21, 12.32) for the derivative at zero of the zeta-functions of Riemann $\zeta(s)$ (defined for $\text{Re } s > 1$

as $\sum_{n \geq 1} n^{-s}$) :

$$\partial_s \zeta(s)|_{s=0} = -2^{-1} \log 2\pi .$$

Indeed, let M be an interval $(0, b] \subset \mathbb{R}$ with the Dirichlet boundary conditions at 0 and the Neumann conditions at b. Let N be the point b. Then the formula (1.18) claims in this case that

$$T_0((0, b]) = T_0((0, b)) \otimes T_0(b) . \tag{1.21}$$

The cohomology $H^\bullet((0, b]) = H^\bullet([0, b], 0)$ are trivial. The scalar analytic torsion $T((0, b])$ is equal to $\exp(-\partial_s \zeta_1(s; M)|_{s=0})$, where $\zeta_1(s; M)$ is the zeta-functions for the Laplacian on $DR^1((0, b])$. This zeta-function for $\operatorname{Re} s > 1/2$ is defined by the series

$$\zeta_1(s; M) = \sum_{n \geq 0} \left(((\pi/2b)(2n + 1))^2 \right)^{-s} .$$

So $\zeta_1(s; M) = (\pi/2b)^{-2s}(1 - 2^{-2s})\zeta(2s)$ for $\operatorname{Re} s > 1/2$, where $\zeta(s)$ is the zeta-function of Riemann. Hence, the latter equality between the analytic continuations of $\zeta_1(s; M)$ and of $\zeta(2s)$ holds for all $s \in \mathbb{C}$, and $\partial_s \zeta_1(s)|_{s=0} = 2\zeta(0) \log 2$.

The determinant line $\det H^\bullet(M)$ on the left in (1.21) is canonically isomorphic to $\mathbb{C}$, and the $T_0(M)$-norm of the element $1 \in \mathbb{C}$ is equal to

$$\|1\|_{T_0(M)}^2 = \exp(-\zeta_1'(0; M)) = \exp(-2\zeta(0) \log 2) = 2 .$$

(Note, that the function $2\zeta(2s)$ is the zeta-function for the Laplacian $\Delta = (-\partial^2/\partial x^2)$ on functions on the circle of the length 2π. As the circle is odd-dimensional, then the value of $2\zeta(2s)$ at zero is equal to $-\dim \operatorname{Ker} \Delta = -1$. Hence, $2\zeta(0) = -1$.)

The scalar analytic torsion $T((0, b))$ is equal to $\exp(-\partial_s \zeta_1(s; M, N))$, where $\zeta_1(s; M, N)$ for $\operatorname{Re} s > 1/2$ is defined by the series

$$\zeta_1(s; M, N) = \sum_{n \geq 1} \left(((\pi/b)n)^2 \right)^{-s} = (\pi/b)^{-2s}\zeta(2s) .$$

Hence, this equality holds for all $s \in \mathbb{C}$, and the scalar analytic torsion is equal to

$$T((0, b)) = \exp(-2\zeta'(0) + 2\zeta(0) \log(\pi/b)) = \exp(-\log(\pi/b) - 2\zeta'(0)) .$$

The identification of the determinant lines on the right and on the left in (1.21) is defined by the cohomology exact sequence

$$0 \to H^0(b) \to H^1((0, b)) \to 0 . \tag{1.22}$$

The element $1 \in H^0(b)$ (of the norm 1) is mapped by (1.22) to the element (dx/b) of the norm $\|dx/b\|^2 = b^{-1}$. The element $h = 1^{-1} \otimes (dx/b)$, corresponding to the element $1 \in \mathbb{C} = \det H^\bullet((0, b])$, has the norm b^{-1}. So the equality (1.21) claims that

$$\log 2 = -\log b - \log(\pi/b) - 2\zeta'(0) .$$

Thus the equality $\zeta'(0) = -2^{-1}\log(2\pi)$ is a particular case[4] of Theorem 1.2.

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[4] In this paper the proofs of the equality (1.18), of Theorem 1.1, and of the equality (1.12) with $c_0 = 1$ do not use the Lerch formula. So we have obtained (by the way) a new proof of the Lerch formula.

The natural L_2-norm on $\bigoplus_j DR^\bullet(\overline{M}_j)$ is defined by

$$(v_1, v_1) := \sum \int_M (v_1 \wedge *\overline{v}_1) \,, \qquad (1.23)$$

where $(v_1 \wedge *\overline{v}_1)$ is a real density on M corresponding to $v_1 \wedge *\overline{v}_1$.

Lemma 1.1 *The Green functions G_v for the Laplacians $\Delta_v^\bullet$ provide us with the homotopy operator in the complex $DR^\bullet(M_v, Z)$*

$$K_v := \delta G_v \qquad (1.24)$$

between the orthogonal projection operator $p_{\mathscr{H}} : DR^\bullet(M_v, Z) \to \mathrm{Ker}(\Delta_v^\bullet)$ and the identity operator on $DR^\bullet(M_v, Z)$. The following equality holds in $DR^\bullet(M_v, Z)$:

$$dK_v + K_v d = \mathrm{id} - p_{\mathscr{H}} \,,$$

Proof. The Green function for $\Delta_v^\bullet$ maps the L_2-completion $(DR^\bullet(M))_2$ of $DR^\bullet(M)$ [5] into the $\mathrm{Dom}(\Delta_v^\bullet)$ (Theorem 3.1). The $\mathrm{Dom}(\Delta_v^\bullet)$ is defined as the domain of definition $D(\Delta_v^\bullet)$ for $\Delta_v^\bullet$ in $DR^\bullet(M_v, Z)$ completed with respect to the graph topology norm $\|\omega\|_{\mathrm{graph}}^2 := \|\omega\|_2^2 + \|\Delta_v^\bullet \omega\|_2^2$ for $\omega \in D(\Delta_v^\bullet)$ (where $\|\omega\|_2^2 := (\omega, \omega)$ is the L_2-norm (1.23)). The Green function G_v maps $DR^\bullet(M_v, Z)$ into $D(\Delta_v^\bullet)$ (since, by Theorem 3.1, $\Delta_v^\bullet$ is a nonnegative elliptic differential operator with elliptic boundary conditions). The definition of the Green function claims that

$$\Delta_v^\bullet G_v = \mathrm{id} - p_{\mathscr{H}} \,, \qquad (1.25)$$

on $(DR^\bullet(M))_2$ (where $\Delta_v^\bullet \omega$ for $\omega \in \mathrm{Dom}(\Delta_v^\bullet)$ is defined as $\lim_i \Delta_v^\bullet \omega_i$ for $\omega_i \in D(\Delta_v^\bullet)$, $\|\omega - \omega_i\|_{\mathrm{graph}}^2 \to 0$). In particular, this equality holds on $DR^\bullet(M_v, Z) \subset (DR^\bullet(M))_2$.

The $D(\Delta_v^\bullet) \subset (DR^\bullet(M))_2$ is defined as follows. The adjoint to d_v operator δ_v in $\bigoplus_j DR^\bullet(\overline{M}_j)$ is defined on elements $v_2 = (\omega_1, \omega_2)$, where ω_k are smooth differential forms on $\overline{M}_k$ and the linear functional

$$l_{v_2}(v_1) = \langle dv_1, v_2 \rangle = \int_M (dv_1 \wedge *\overline{v}_2)$$

in continuous in $DR^\bullet(M_v, Z)$ with respect to the L_2-norm (1.23) of $v_1 \in DR^\bullet(M_v, Z)$. For such an element $v_2 = (\omega_1, \omega_2)$ the form $*v_2 = (*\omega_1, *\omega_2)$ has the zero geometrical restriction to $\partial M \backslash Z$, and the following transmission condition has to hold on N for v_2:

$$\beta i_{N,1}^*(*\omega_1) = \alpha i_{N,2}^*(*\omega_2) \,, \qquad (1.26)$$

where $i_{N,k} : N \subset \partial M_k \hookrightarrow M_k$. (These boundary conditions for v_2 are consequences of Stokes' formula.)

The domain $D(\Delta_v^\bullet) \subset DR^\bullet(M_v, Z)$ is defined as the set of $\omega \in DR^\bullet(M_v, Z)$ such that

$$\omega \in D(\delta_v), \quad d\omega \in D(\delta_v), \quad \delta\omega \in DR^\bullet(M_v, Z) \,. \qquad (1.27)$$

Note that $dG_v \omega = G_v d\omega$ for $\omega \in DR^\bullet(M_v, Z)$ (this equality follows from Stokes' formula). Hence the identity (1.25) can be represented on $DR^\bullet(M_v, Z)$ as

$$Kd + dK = \mathrm{id} - p_{\mathscr{H}} \,.$$

Thus the lemma is proved. $\square$

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[5] $(DR^\bullet(M))_2$ coincides with the L_2-completion of $DR^\bullet(M_v, Z)$ and with the L_2-completion of $\bigoplus_j DR^\bullet(\overline{M}_j)$.

Corollary 1.1 *The homotopy operator* (1.24) *defines a canonical identification between the cohomology* $H^\bullet(DR(M_v, Z))$ *and the space of harmonic forms* $\mathrm{Ker}^\bullet(\Delta_v)$.

Let for simplicity g_M be a direct product metric near N. Let the Dirichlet boundary conditions be given on a union Z of some connected components of ∂M and the Neumann conditions be given on $\partial M \backslash Z$. Then the following holds.

Proposition 1.1 *The eigenforms of* $\Delta(M, Z; g_M)$ *are the same as the eigenforms of* $\Delta_{1,1}$.

Proof. Let v be equal to $(1,1)$. The conditions (1.27) for $(\omega_1, \omega_2) \in DR^\bullet(M_{1,1}, Z)$ are equivalent on N to the following ones:

$$i^*_{N,1}\omega_1 = i^*_{N,2}\omega_2, \quad i^*_{N,1}(*\omega_1) = i^*_{N,2}(*\omega_2)\,, \tag{1.28}$$

$$i^*_{N,1}(*d\omega_1) = i^*_{N,2}(*d\omega_2), \quad i^*_{N,1}(*d*\omega_1) = i^*_{N,2}(*d*\omega_2)\,, \tag{1.29}$$

where $*$ is the star-operator for Riemannian metric g_M.

The equalitites (1.28) claim that the restrictions to N of the forms ω_1 and ω_2 are the same (i. e., they are the same sections of $\wedge^\bullet T^* M|_N$). The equalitites (1.28) and (1.29) are equivalent to the assertion that the following pairs of forms have the same restrictions to N (as the smooth sections of $\wedge^\bullet T^* M|_N$):

$$\{d\omega_1, d\omega_2\}, \quad \{\delta\omega_1, \delta\omega_2\}, \quad \{\omega_1, \omega_2\}\,. \tag{1.30}$$

Any eigenform for $\Delta^\bullet(M, Z; g_M)$ belongs to $D(\Delta^\bullet_{1,1})$. So it is an eigenform for $\Delta^\bullet_{1,1}$. Let $\omega = (\omega_1, \omega_2) \in D(\Delta^\bullet_{1,1})$ be an eigenform for $\Delta^\bullet_{1,1}$:

$$\Delta^\bullet_{1,1}\omega = (\Delta^\bullet\omega_1, \Delta^\bullet\omega_2) = \lambda(\omega_1, \omega_2)\,. \tag{1.31}$$

Then[6] ω_k are C^∞-forms on $\overline{M}_k$ and (as it follows from (1.30), (1.31)) the restrictions of the following pairs of forms are the same as the sections of $\wedge^\bullet T^* M|_N$ (for $k = 0, 1, 2...$):

$$\{\Delta^k\omega_1, \Delta^k\omega_2\}, \quad \{d\Delta^k\omega_1, d\Delta^k\omega_2\}, \quad \{\Delta^k\omega_1, \delta\Delta^k\omega_2\}\,. \tag{1.32}$$

So $\omega = (\omega_1, \omega_2)$ is a C^∞-form on $M = M_1 \cup_N M_2$. In fact, it follows from (1.28) and from the identity of the restrictions to N of $\Delta\omega_1$ and $\Delta\omega_2$ that $(\Delta_I \otimes \mathrm{id})\omega_k$ have (for $k = 1, 2$) the same restrictions from M_k to N. (The Laplacian Δ is equal to $\mathrm{id}_I \otimes \Delta_N + \Delta_I \otimes \mathrm{id}_N$ with respect to the direct product structure $I \times N$ in the neighborhood of $N = 0 \times N \hookrightarrow I \times N \hookrightarrow M, 0 \in I\backslash\partial I$.) Hence, according to (1.28) and (1.29), the 2-jets of ω_1 and of ω_2 are the same on N. The identity between the $(2k + 1)$-jets of ω_k on N follows (by induction) from (1.32). Thus, ω is an eigenform for $\Delta_M : \Delta_M\omega = \lambda\omega$. The proposition is proved. $\square$

1.3. Properties of Analytic and Combinatorial Torsion Norms. One of the main properties of the analytic torsion norm is as follows. Let M be a manifold $M_1 \times M_2$ with a direct product metric. One of these Riemannian manifolds, for instance M_1, can have a nonempty boundary ∂M_1. In this case let g_{M_1} be a direct product metric near ∂M_1, and let the Dirichlet boundary conditions be given on the components

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[6] All the eigenforms of $\Delta^\bullet_v$ (for $v \in \mathbb{R}^2\backslash(0,0)$) are C^∞-smooth on $\overline{M}_k$, as it follows from Theorem 3.1

$Z = Z_1 \times M_2$ of $(\partial M_1) \times M_2 = \partial(M_1 \times M_2)$. Let the Neumann boundary conditions be given on $\partial(M_1 \times M_2) \backslash (Z_1 \times M_2) = (\partial M_1 \backslash Z_1) \times M_2$.

The Künneth formula for the cohomology claims that

$$H^J(M,Z) = \bigoplus_{i+k=j} H^i(M_1,Z_1) \otimes H^k(M_2) . \tag{1.33}$$

So the determinant of the cohomology of $DR^\bullet(M,Z)$ is the tensor product

$$\det H^\bullet(M,Z) = (\det H^\bullet(M_1,Z_1))^{\chi(M_2)} \otimes \det H^\bullet(M_2)^{\chi(M_1,Z_1)} . \tag{1.34}$$

Proposition 1.2. *The identification* (1.34) *induces the isomorphism of the analytic torsion norm* $T_0(M,Z)$ *with the tensor product*

$$T_0(M,Z) = T_0(M_1,Z_1)^{\otimes \chi(M_2)} \otimes T_0(M_2)^{\otimes \chi(M_1,Z_1)} , \tag{1.35}$$

where $\chi(M_1,Z_1), \chi(M_2)$ *are the Euler characteristics.*

Remark. 1.1 It is shown above that the analytic torsion norms $T_0(M,Z), T_0(M_1,Z_1)$, and $T_0(M_2)$ are independent of Riemannian metrics g_M, g_{M_k} which are supposed to be direct product metrics near $\partial M, \partial M_1$. So, if the equality (1.35) holds for a direct product metric on $(M_1, \partial M_1) \times M_2$ (where g_{M_1} is a direct product metric near ∂M_1), then this equality holds for any metric g_M (which is supposed to be a direct product metric near ∂M).

Proof. The scalar analytic torsion $T(M)$ for a direct product metric on $M = M_1 \times M_2$ is equal to

$$T(M,Z) = T(M_1,Z_1)^{\chi(M_2)} T(M_2)^{\chi(M_1,Z_1)} . \tag{1.36}$$

This statement is proved in [RS], Theorem 2.5, in the case of an acyclic local system over M_1. In the general case, (1.36) follows from the proof of Theorem 2.5 in [RS] and from the following equality (where $\lambda \neq 0, m(i,\lambda,M_2)$ is the dimension of the λ-eigenspace for $\Delta_{M_2,i}$, $m(j,0,M_1) := \dim \operatorname{Ker} \Delta_{M_1,Z_{1,j}}$) :

$$\sum(-1)^{i+j}(i+j)m(i,\lambda,M_2)m(j,0,M_1) = (\sum(-1)^i i \, m(i,\lambda,M_2))\chi(M_1,Z_1) ,$$

which holds, since the alternating sum over i of $m(i,\lambda,M_2)$ is equal to zero (for any nonzero λ).

For such a metric on M the following canonical identifications are the isometries bewteen the natural Hilbert structure on the space of harmonic forms $\operatorname{Ker}\Delta_j(M,Z)$ and the tensor products (and the direct sums) of the Hilbert structures on harmonic forms for $\Delta_\bullet(M_1,Z_1)$ and $\Delta_\bullet(M_2)$:

$$\operatorname{Ker}\Delta_j(M,Z) = \bigoplus_{i+k=J} \operatorname{Ker}\Delta_i(M_1,Z_1) \otimes \operatorname{Ker}\Delta_k(M_2) . \tag{1.37}$$

These Hilbert structures induce the norms on

$$\det H^\bullet(M,Z) = \det \operatorname{Ker}\Delta_\bullet(M,Z; g_M), \quad \det H^\bullet(M_1,Z_1) = \det \operatorname{Ker}\Delta_\bullet(M_1,Z_1; g_{M_1})$$

and on $\det H^{\bullet}(M_2) = \det \operatorname{Ker} \Delta_{\bullet}(M_2, g_{M_2})$ such that the identification (1.34) of the determinant lines is an isometry:

$$\| \cdot \|^2_{\det \operatorname{Ker} \Delta_M} = \left(\| \cdot \|^2_{\det \operatorname{Ker} \Delta_{M_1, Z_1}} \right)^{\chi(M_2)} \left(\| \cdot \|^2_{\det \operatorname{Ker} \Delta_{M_2}} \right)^{\chi(M_1, Z_1)} . \qquad (1.38)$$

The equality (1.35) follows from (1.36), (1.34), and (1.38). □

The following lemma makes it possible to use the variations in v in the proof of Theorem 1.1. Let (M, g_M) be a compact Riemannian manifold with a smooth boundary ∂M and let $N \hookrightarrow M \backslash \partial M$ be a smooth closed codimension one submanifold of M with a trivial normal bundle $(TM|_N)/TN$ such that $M = M_1 \cup_N M_2$ is obtained by gluing two its pieces M_1 and M_2 along N. Let g_M be a direct product metric near ∂M and near N. Let Z be a union of some connected components of ∂M where the Dirichlet boundary conditions are given and let the Neumann boundary conditions be given on $\partial M \backslash N$.

Lemma 1.2. *Let us suppose that the norm $\varphi_v^{\mathrm{an}} T_0(M_v, Z)$ is independent of $v \in \mathbb{R}^2 \backslash (0,0)$ for any such (M, g_M, N, Z)[7] (where the identification φ_v^{an} is defined by the exact sequence (1.14) of the de Rham complexes and by Lemma 1.1). Then the factor c_0 in the gluing formula (1.12) for $\varphi_v^{\mathrm{an}} T_0(M_v, Z)$ is equal to one.*

Remark. 1.2. Theorem 1.1 is a direct consequence of Lemma 1.2 and of the assertion that $\varphi_v^{\mathrm{an}} T_0(M_v, Z)$ is independent of $v \in \mathbb{R}^2 \backslash (0,0)$. Indeed, $T_0(M_{1,1}, Z)$ coincides with $T_0(M, Z)$ (according to Proposition 1.1) and the identifications φ_v^{an} and φ_{an} are the same. Hence the formula (1.12), where c_0 is equal to one and $v = (1, 1)$, is the gluing formula of Theorem 1.1.

Remark. 1.3. The assertion that the norm $\varphi_v^{\mathrm{an}} T_0(M_v, Z)$ does not depend on v is equivalent to the independence of v of the factor c_0 in (1.12).

Proof. The factor c_0 in (1.12) lies in $\mathbb{R}_+$. If c_0 is independent of v for (M, g_M, N, Z) then

$$\varphi_{1,0}^{\mathrm{an}} T_0(M_{1,0}, Z) = \varphi_{0,1}^{\mathrm{an}} T_0(M_{0,1}, Z) . \qquad (1.39)$$

It follows from (1.39) and from (1.16), (1.19), and (1.14) that there are the equalitites with the same positive constant c_0 as in (1.12) for (M, g_M, N, Z) (where $Z_k := Z \cap \partial M_k$):

$$\varphi_{\mathrm{an}} T_0(M_1, Z_1) = c_0 T_0(M_1, Z_1 \cup N) \otimes T_0(N) , \qquad (1.40)$$

$$\varphi_{\mathrm{an}} T_0(M_2, Z_2) = c_0 T_0(M_2, Z_2 \cup N) \otimes T_0(N) . \qquad (1.41)$$

We can conclude from (1.40) and (1.41) that the factor $c_0 = c_0(N, g_N)$ is defined by N, g_N and that it *does not depend* on M_1, M_2, M, g_M (it is independent also of v).

Let M_1 in (1.40) be a manifold $M_1 = N \times I$ with a direct product metric. Then $\partial M_1 = N \cup N$ and (1.40) claims in this case that

$$\varphi_{\mathrm{an}} T_0(N \times I) = c_0(N, g_N)^2 T_0(N \times I, N \times \partial I) \otimes T_0(N)^{\otimes 2} , \qquad (1.42)$$

---

[7] The equivalent formulation is as follows. Let M be obtained by gluing along N, i.e., $M = M_1, \cup_N M_2$, and let it be equipped with a Riemannian metric g_M, which is a direct product metric near ∂M and near N. Then it is supposed that the norm $\varphi_v^{\mathrm{an}} T_0(M_v, Z)$ is independent of $v \in \mathbb{R}^2 \backslash (0, 0)$.

where the identification φ_{an} is defined by the exact sequence (1.20). It follows from (1.42) and from the multiplicative property (1.35) that

$$T_0(N)^{\chi(I)} \otimes T_0(I)^{\chi(N)} = c_0^2 T_0(N)^{\chi(I,\partial I)} \otimes T_0(I,\partial I)^{\chi(N)} \otimes T_0(N)^2 \,, \qquad (1.43)$$

where $c_0 := c_0(N, g_N)$ depends on N and g_N only. Then the following equality is a consequence of (1.43) and of the identification (1.19) (defined by the exact sequence (1.20)):

$$T_0(I)^{\chi(N)} = c_0(N, g_N)^2 T_0(I,\partial I)^{\chi(N)} \otimes T_0(\partial I)^{\chi(N)} \,. \qquad (1.44)$$

Note that $T_0(\partial I)$ is the standard norm on $\det H^\bullet(\partial I)$ which is canonically identified with $\mathbb{C}$ (up to a possible factor (-1) in the identification). Namely $\|1\|_{T_0}^2 = 1$ for $1 \in \mathbb{C}$. An immediate consequence of the equality (1.41) for $M_1 = I$, $N = \partial I$ and of (1.44) is the following:

$$c_0(N, g_N)^2 = c_0(\partial I)^{\chi(N)} \,. \qquad (1.45)$$

Hence, it is enough to prove that $c_0(\partial I) = 1$, and it will be done now.

Let I be an interval $[0, a]$. The scalar analytic torsions for I and for $(I, \partial I)$ are equal, $T(I) = T(I, \partial I)$, since

$$\zeta_1(s; I) = \zeta_0(s; I, \partial I) \,, \qquad (1.46)$$

$$\zeta_1(s; I, \partial I) = \zeta_0(s; I, \partial I) \qquad (1.47)$$

(where $\zeta_j(s; M, Z)$ is the ζ-function of the Laplacian on $(DR^j(M, Z), g_M)$). The equality (1.47) follows from the identification of the eigenforms, defined by the exterior derivative d, and the equality (1.46) follows from the identificaiton of the eigenforms defined by the Riemannian $*$ on I.

The cohomology exact sequence for the pair $(I, \partial I)$ is

$$0 \to H^0(I) \to H^0(\partial I) \to H^1(I, \partial I) \to 0 \,. \qquad (1.48)$$

The complex (1.48) is acyclic and so the determinant D of its cohomology is canonically isomorphic to $\mathbb{C}$. The components of (1.48) are equipped with the natural Hilbert structures (because they are the spaces of harmonic forms on the interval $I \subset \mathbb{R}$ with the standard metric). Hence, there is the induced norm $\| \cdot \|_D$ on $D = \mathbb{C}$. We have to prove that $\|1\|_D^2 = 1$ for $1 \in \mathbb{C} = D$. This equality is equivalent to the assertion that $c_0(\partial I)$ is equal to one.

The norm of the element $a^{-1/2} \cdot 1 \in DR^0(I)$ is equal to 1. (It is a harmonic form and it represents an element from $H^0(I)$). Its image in $H^0(\partial I)$ is as follows:

$$-a^{-1/2} \cdot [0] + a^{-1/2} \cdot [a] \in H^0(\partial I) = H^0(0 \cup a) \,.$$

The norm of the element $a^{-1/2} \cdot dx \in DR^1(I, \partial I)$ is equal to 1 in $H^1(I, \partial I)$ and an element $-a^{1/2} \cdot [0]$ is mapped by the differential of the exact sequence (1.48) to the harmonic form $a^{-1/2} dx \in H^1(I, \partial I)$. (The arrows in (1.48) are of the topological nature. So the latter statement is obtained using

$$a^{1/2} = \int_{[0,a]} a^{-1/2} dx = \left(a^{-1/2} dx, (I, \partial I) \right) \,,$$

where $(I, \partial I)$ is the fundamental class of $H_1(I, \partial I)$.)

The corresponding volume element $\left(-a^{-1/2}[0] + a^{-1/2}[a]\right) \wedge \left(-a^{1/2}[0]\right) = [0] \wedge$ $[a]$ in $\det H^0(\partial I)$ is an element with the norm one. Hence, $c_0(\partial I) = 1$. The equality $c_0(N, g_N) = 1$ (for a union N of some connected components of ∂M_1) follows from the equality $c_0(\partial I) = 1$ and from (1.45). The lemma is proved. $\square$

Let $M = M_1 \cup_N M_2$ be obtained by gluing M_1 and M_2 along N (as in Theorem 1.1), and let X be a smooth triangulation of M such that M_k and N are invariant under X. Namely $X = X_1 \cup_W X_2$, where X_k is a smooth triangulation of a manifold M_k with a smooth boundary $\partial M_k = N \cup \left(\partial M \cap \overline{M}_k\right)$. (Here $N \subset M$ is a smooth closed manifold of codimension one in M such that N divides M in two pieces M_1 and M_2 as in Theorem 1.1, $N \cap \partial M = \emptyset$, and $W := X_k \cap N$.)

Let Z be a union of the connected components of ∂M where the Dirichlet boundary conditions are given. Set $V := X \cap Z$, $Z_k := \partial M_k \cap Z$, $V_k := \cap Z_k$. The exact sequence of cochain complexes

$$0 \to \bigoplus_{k=1,2} C^\bullet(X_k, W \cup V_k) \to C^\bullet(X, V) \xrightarrow{i_N^*} C^\bullet(W) \to 0 \qquad (1.49)$$

(where the left arrow is the natural inclusion and the right arrow is the geometrical restriction of cochains) provides us with the identification

$$\varphi_c : \det H^\bullet(X, V) \xrightarrow{\sim} \det H^\bullet(X_1, W \cup V_1) \otimes \det H^\bullet(X_2, W \cup V_2) \otimes \det H^\bullet(W).$$

By the definition of the combinatorial torsion norm on the determinant line (determined by the preferred basises of the basic cochains) the following statement holds.

Proposition 1.3. *Under the conditions above, the combinatorial torsion norms are equal*:
$$\varphi_c \tau_0(X, V) = \tau_0(X_1, W \cup V_1) \otimes \tau_0(X_2, W \cup V_2) \otimes \tau_0(W). \qquad (1.50)$$

This combinatorial equality is analogous to the gluing formula of Theorem 1.1. But it is necessary to note as follows.

Remark. 1.4. The formulas (1.50) and (1.8) correspond to the *different identifi-cations* φ_c and $\varphi_{\mathrm{an}} = \varphi_{1,1}^{\mathrm{an}}$ between one pair of the canonically identified[8] one-dimensional spaces
$$\det H^\bullet(X, V) = \det H^\bullet(M, Z),$$

and the triple tensor products of three other pairs of the canonically identified spaces

$$\det H^\bullet(X_k, W \cup V_k) = \det H^\bullet(M_k, N \cup Z_k), \quad \det H^\bullet(W) = \det H^\bullet(N).$$

(Note that φ_c is defined by the exact sequence (1.49), where the right arrow i_N^* is the restriction of the cochains. However, in the exact sequence (1.10), which defines φ_{an}, the right arrow is equal to $\sqrt{2}i_N^*$ for the common geometrical restriction i_N^* to N of pairs $\omega = (\omega_1, \omega_2)$ of smooth differential forms ω_k on $\overline{M}_k$ such that $i_{N,1}^* \omega_1 = i_{N,2}^* \omega_2$.)

---

[8] The cohomology are identified according to the de Rham theorem by the integration over the simplexes of X of the corresponding differential forms. The spaces of harmonic forms $\operatorname{Ker} \varDelta_\bullet(M, Z)$ and $\operatorname{Ker} \varDelta_\bullet(M_{1,1}, Z)$ are canonically identified by Proposition 1.1.

Let X be a smooth triangulation of a compact manifold with boundary $(M, \partial M)$. Let Z and Y be disjoint unions of some connected components of ∂M such that $Z \cap Y = \emptyset$. Let $V = X \cap Z$, $F = X \cap Y$. Then the exact sequence

$$0 \to C^{\bullet}(X, V \cup F) \to C^{\bullet}(X, V) \to C^{\bullet}(F) \to 0$$

(where the left arrow is the natural inclusion of cochains and the right arrow is the restriction of cochains) defines the identification

$$\varphi_c^F : \det H^{\bullet}(X, V) \overset{\sim}{\to} \det H^{\bullet}(X, V \cup F) \otimes \det H^{\bullet}(F) \, .$$

The following assertion is an immediate consequence of the definition of the combinatorial torsion norm.

Proposition 1.4. *The combinatorial torsion norm of (X, V) is equal to the tensor product of the following combinatorial torsion norms*:

$$\varphi_c \tau_0(X, V) = \tau_0(X, V \cup F) \otimes \tau_0(F) \, .$$

This combinatorial equality is similar to the gluing formula of Theorem 1.2.

Let $e(M, Z)$ be the logarithm of the ratio between the analytic and the combinatorial torsion norms:

$$e(M, Z) := \log_2(T_0(M, Z) / \tau_0(X, V))$$

(where $T_0(M, Z)/\tau_0(X, V) := ||l||^2_{T_0(M,Z)} / ||l||^2_{\tau_0(X,V)}$ for an arbitrary nonzero element l of the determinant line $\det H^{\bullet}(M, Z) = \det H^{\bullet}(X, V)$).

Remark. 1.5. It is proved above that $e(M, \partial M)$ does not depend on a metric g_M, if g_M is a direct product metric near ∂M.

Lemma 1.3. 1. *Let (S, g_S) be a closed Riemannian manifold. Then the following identity holds, if $g_{M \times S}$ is a direct product metric near $\partial(M \times S) = \partial M \times S$:*

$$e(M \times S, Z \times S) = \chi(M, Z)e(S) + e(M, S)\chi(S) \tag{1.51}$$

($\chi(M, Z)$ is the relative Euler characteristic of M modulo $Z \subset \partial M$).
2. *Let Y be a union of some connected components of $\partial M \backslash Z$. Then*

$$e(M, Z) = e(M, Y \cup Z) + e(Y) \, . \tag{1.52}$$

Proof. The equality (1.52) follows from Theorem 1.2 and from Proposition 1.4. (In this case, $\varphi_c = \varphi_{\text{an}}$.) The equality (1.51) follows from Proposition 1.2 and from the multiplicative property of the combinatorial torsion norms. Namely let K be a smooth triangulation of S and let $V = X \cap Z$. Then the identification of the determinants of the cohomology defined by (1.33) and (1.34) is an isometry of the combinatorial torsion norms:

$$\tau_0(X \times K, V \times K) = \tau_0(X, V)^{\chi(K)} \otimes \tau_0(K)^{\chi(X,V)} \, .$$

The same identification of the cohomology is the isometry (1.35) of the analytic torsion norms, if the metric $g_{M \times S}$ is a direct product metric near $\partial(M \times S)$. Hence, the identity (1.51) holds for such metrics $g_{M \times S}$. $\square$

Remark. 1.6. It follows from (1.52) and from Remark 1.5 that $e(M,Z)$ does not depend on g_M for any union Z of the connected components of ∂M (in particular for $Z = \emptyset$).

1.4. Generalized Ray–Singer Conjecture.

1.4.1. Properties of the ratio of the analytic and the combinatorial torsion norms.

Lemma 1.2 claims that Theorem 1.1 follows from (1.12) with c_0 independent of v. So it is enough to prove that the norm $\varphi_{\mathrm{an}}^{\mathrm{an}} T_0(M_v, Z)$ is independent of $v \in \mathbb{R}^2 \backslash (0,0)$ (under the same conditions on M, g_M, N, and Z as in (1.12) and in Lemma 1.2). The latter assertion is proved in Sect.2. In the remaining part of Sect.1 we prove a generalization of the Ray–Singer conjecture for manifolds with boundary (and with the transmission condition (1.13) on the interior boundary) using the gluing formula of Theorem 1.1. This formula has the following consequence.

Let $M = M_1 \cup_N M_2$ be obtained by gluing M_1 and M_2 along N.

Lemma 1.4. *Under the conditions of Lemma 1.2, on (g_M, N, Z) the following holds*:

$$e(M,Z) = e(M_1, Z_1 \cup N) + e(M_2, Z_2 \cup N) + e(N) - \chi(N)\,.$$

Proof. This identity is an immediate consequence of Theorem 1.1 and of the following commutative diagram:

$$
\begin{array}{ccccc}
\det H^\bullet(M,Z) & \xrightarrow[\widetilde{}]{\varphi_{\mathrm{an}} = \varphi_{1,1}^{\mathrm{an}}} & & & \mathbf{Det}(M,Z,N) \\[2mm]
\| \, R & & & & \| \; \downarrow R \\[2mm]
\det H^\bullet(X,V) & \xrightarrow[\widetilde{}]{\varphi_c} \mathbf{Det}(X,V,W) & \xrightarrow[\widetilde{}]{A_W^H} & \mathbf{Det}(X,V,W) \\[2mm]
\uparrow \wr d_c & \uparrow \wr d_c & & \uparrow \wr d_c \\[2mm]
\det C^\bullet(X,V) & \xrightarrow[\widetilde{}]{(1.49)} \mathbf{Det}C^\bullet(X,V,W) & \xrightarrow[\widetilde{}]{A_W} & \mathbf{Det}C^\bullet(X,V,W)
\end{array}
\qquad (1.53)
$$

Here

$$\mathbf{Det}(M,Z,N) := \left(\bigotimes_{k=1,2} \det H^\bullet(M_k, N \cup Z_k) \right) \bigotimes \det H^\bullet(N)\,,$$

$$\mathbf{Det}(X,V,W) := \left(\bigotimes_{k=1,2} \det H^\bullet(X_k, W \cup V_k) \right) \bigotimes \det H^\bullet(W)\,,$$

$$\qquad (1.54)$$

$$\mathbf{Det}C^\bullet(X,V,W) := \left(\bigotimes_{k=1,2} \det C^\bullet(X_k, W \cup V_k) \right) \bigotimes \det C^\bullet(W)\,,$$

$$A_W := \mathrm{id}_X \bigoplus \sqrt{2}\,\mathrm{id}_W \in \mathrm{Aut}\left(\bigotimes_{k=1,2} C^\bullet(X_k, W \cup V_k) \right) \bigoplus \mathrm{Aut}C^\bullet(W)\,,$$

A_W^H is the induced by A_W operator on the determinant of the cohomology, R is the identification induced by the integration of differential forms over the simplexes of X (by the de Rham theorem). φ_c and φ_{an} are the identifications induced by (1.49) and by (1.10) in a view of Proposition 1.1.

The commutativity of (1.53) follows from the commutativity of the diagram

$$
\begin{array}{ccccccc}
0 \to \displaystyle\bigoplus_{k=1,2} DR^\bullet(M_k, N \cup Z_k) & \to & DR^\bullet(M_{1,1}, Z) & \xrightarrow{\sqrt{2}i_N^*} & & DR^\bullet(N) & \to 0 \\
\downarrow R & & \downarrow R & & & \downarrow R & \\
0 \to \displaystyle\bigoplus_{k=1,2} C^\bullet(X_k, W \cup V_k) & \to & C^\bullet(X, V) & \xrightarrow{i_N^*} C^\bullet(W) & \xrightarrow{\sqrt{2}\,\mathrm{id}} & C^\bullet(W) & \to 0
\end{array}
$$

The induced action of $\sqrt{2}\mathrm{id}$ on $l_W \in \det C^\bullet(W)$ is $l_W \to 2^{-\chi(W)/2}l_W$ (where $\chi(W) = \chi(N)$ is the Euler characteristic). So the induced action of A_W and of A_W^H on $l \in \mathbf{Det}(M, Z, N) = (\bigotimes_{k=1,2} \det C^\bullet(X_k, W \cup V_k)) \otimes \det C^\bullet(W)$ is

$$ l \to 2^{-\chi(N)/2}l \, . \tag{1.55} $$

(The identification of the determinant lines is defined by R and by d_c in the right column of (1.53).)

For an arbitrary nonzero $m \in \mathbf{Det}(M, Z, N)$ the following equality is deduced from (1.55) and from the commutativity of (1.53):

$$ (\varphi_{an} T_0(M, Z))(m) = 2^{-\chi(N)}(\varphi_c T_0(M, Z))(m) \, . \tag{1.56} $$

Theorem 1.1 and Proposition 1.3 claim that

$$ \varphi_{an} T_0(M, Z) = T_0(M_1, N \cup Z_1) \otimes T_0(M_2, N \cup Z_2) \otimes T_0(N) \, , $$
$$ \varphi_c \tau_0(X, V) = \tau_0(X_1, W \cup V_1) \otimes \tau_0(X_2, W \cup V_2) \otimes \tau_0(W) \, . \tag{1.57} $$

The isometries (1.56) and (1.57) imply the equality

$$ e(M, Z) = \log_2(T_0(M, Z)/\tau_0(X, V)) $$
$$ = -\chi(N) + \left(\sum_{k=1,2} \log_2(T_0(M_k, N \cup Z_k)/\tau_0(X_k, W \cup V_k)) \right) $$
$$ + \log_2(T_0(N)/\tau_0(W)) \, . $$

Thus the lemma is proved. $\square$

Let $v = (\alpha, \beta) \in \mathbb{R}^2 \setminus (0,0)$ and let $(C^\bullet(X_v, V), d_c)$ be the complex of pairs of cochains (c_1, c_2), $c_k \in C^\bullet(X_k, V_k)$, with the v-transmission boundary condition (similar to (1.13)) on $W \subset \partial X_k$ between their geometrical restrictions

$$ \alpha i_{W,1}^* c_1 = \beta i_{W,2}^* c_2 \, . \tag{1.58} $$

The integration over the simplexes provides us with a quasi-isomorphism of the complexes:
$$ R_v : (DR^\bullet(M_v, Z), d) \to (C^\bullet(X_v, V), d_c) $$

(i.e., R_v induces an isomorphism between the corresponding cohomology).

The morphism of complexes $r_{v,c} : (C^\bullet(X_v, V), d_c) \to C^\bullet(W, d_c)$ is defined by analogy with the definition of r_v. Its value on each element $(c_1, c_2) \in C^\bullet(X_v, V)$ is

$$r_{v,c}(c_1, c_2) = (\alpha^2 + \beta^2)^{-1/2}(\beta i^*_{W,1} c_1 + \alpha i^*_{W,2} c_2) \,.$$

The vertical arrows in the following diagram of complexes are quasi-isomorphisms[9]

$$
\begin{array}{ccccccccc}
0 \to & \bigoplus_{k=1,2} DR^\bullet(M_k, N \cup Z_k) & \to & DR^\bullet(M_v, Z) & \xrightarrow{r_v} & DR^\bullet(N) & \to 0 \\
& \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R_v} & & \downarrow{\scriptstyle R} & & (1.59) \\
0 \to & \bigoplus_{k=1,2} C^\bullet(X_k, W \cup V_k) & \xrightarrow{J} & C^\bullet(X_v, V) & \xrightarrow{r_{v,c}} & C^\bullet(W) & \to 0
\end{array}
$$

This diagram is commutative. The left horizontal arrows in it are the natural inclusions. Let φ_v^c be the identification

$$\varphi_v^c : \det H^\bullet(C(X_v, V)) \xupmath{\overset{R_v}{=}} \det H^\bullet(M_v, Z) \to \mathbf{Det}(M, Z, N) \,, \qquad (1.60)$$

defined by the bottom row of this diagram.

Remark. 1.7. The equality

$$\varphi_v^c = \varphi_v^{an} \qquad (1.61)$$

follows from the commutativity of (1.59). But $\varphi_{1,1}^c \neq \varphi_c$ (in contrast with the identity $\varphi_{1,1}^{an} = \varphi_{an}$). According to (1.56) it holds that

$$\varphi_{1,1}^c = \varphi_{an} = 2^{-\chi(N)}\varphi_c \,.$$

The space $C^j(X_v, V)$ is a subspace of $\bigoplus_{k=1,2} C^j(X_k, V_k)$. The Hilbert structure on $C^j(X_k, V_k)$ is defined by the orthonormal bases of cochains $\{\delta_e\}$ (parametrized by j-dimensional simplexes e of $X_k \backslash V_k$). So the Hilbert structures on $C^\bullet(X_v, V)$ and on $\det C^\bullet(X_v, V)$ are defined. The *scalar combinatorial torsion* is defined as in (1.1):

$$T(C^\bullet(X_v, V), d_c) := \exp\left(\sum(-1)^j j \partial_s \zeta_{j,v}^c(s)|_{s=0}\right) \,,$$

where $\zeta_{j,v}^c(s) := \mathrm{Tr}'\left((\Delta_{j,v}^c)^{-s}\right)$ is the sum $\sum' \lambda^{-s}$ over all the nonzero eigenvalues λ of the finite-dimensional operator $\Delta_{j,v}^c = (d_c^* d_c + d_c d_c^* | C^j(X_v, V))$ (with their multiplicities), d_c^* is adjoint to d_c in $C^j(X_v, V)$ with respect to the Hilbert structure in $C^\bullet(X_v, V)$.

The combinatorial torsion is the following norm on $\det H^\bullet(C(X_v, V))$[10]:

$$\tau_0(X_v, Z) := \| \cdot \|^2_{\det H^\bullet(C(X_v,V))} \cdot T(C^\bullet(X_v, V), d_c) \,, \qquad (1.62)$$

where the norm on $\det^\bullet(C(X_v, V))$ is defined by the Hilbert structures on the subquotients $H^j(C(X_v, V))$ of the Hilbert spaces $C^j(X_v, V)$.

Remark. 1.8. For each $v = (\alpha, \beta) \in \mathbb{R}^2 \backslash (0,0)$ the combinatorial torsion $\tau_0(X_v, V)$ is an invariant of the combinatorial structure defined by a smooth triangulation of

---

[9] R_v is a quasi-isomorphism according to Proposition 2.3.

[10] It is isomorphic to $\det H^\bullet(M_v, Z)$ under the quasi-isomorphism R_v.

the triplet $[(M, \partial M); Z; N]$, where M is a manifold with a smooth boundary ∂M, Z is a union of some connected components of ∂M, and N is a smooth codimension one closed submanifold of M with a trival normal bundle $(TM|_N)/TN$.

Proposition 1.5. *The combinatorial torsion norm* $\tau_0(X_v, Z)$ *is isometric under the identification* (1.60) *to the tensor product of the combinatorial torsion norms:*

$$\varphi_v^c \tau_0(X_v, V) = \tau_0(X_1, W \cup V_1) \otimes \tau_0(X_2, W \cup V_2) \otimes \tau_0(W).$$

Proof. Under the identification (1.60), the Hilbert space $C^\bullet(X_v, V)$ is isometric to the tensor product of the Hilbert spaces $\det C^\bullet(X_k, W \cup V_k)$ (for $k = 1, 2$) and $\det C^\bullet(W)$. (The Hilbert structures on $C^\bullet(X_v, V), C^\bullet(X_k, W \cup V_k)$, and on $C^\bullet(W)$ are defined above.) Indeed, let $\rho_{v,c} : C^j(W) \to C^j(X_v, V)$ be linear maps defined for $w \in C^j(W)$ by

$$\rho_{v,c}(w) = (\alpha^2 + \beta^2)^{-1/2}(\beta w, \alpha w) \in C^j(X_v, V). \tag{1.63}$$

Then $r_{v,c}\rho_{v,c} = \mathrm{id}$ on $C^\bullet(W)$, $\rho_{v,c}$ is an isometry between $C^j(W)$ and $\mathrm{Im}\rho_{v,c}$, and $\mathrm{Im}\rho_{v,c}$ is the orthogonal complement in $C^\bullet(X_v, V)$ to the image of the natural inclusion $j : \bigoplus_{k=1,2} C^\bullet(X_k, W \cup V_k) \hookrightarrow C^\bullet(X_v, V)$ (where j is an isometry onto $\mathrm{Im} j$). So the identification φ_v^c is the isometry of the combinatorial torsion norms. $\square$

The number $e(M_v, Z) \in \mathbb{R}$ is defined as the logarithm of the ratio between the analytic and the combinatorial torsion norms:

$$e(M_v, Z) := \log_2(T_0(M_v, Z)/\tau_0(X_v, V)).$$

Corollary 1.2. *Under the conditions of Lemma* 1.2, *the equality holds:*

$$e(M_v, Z) = e(M_1, Z_1 \cup N) + e(M_2, Z_2 \cup N) + e(N), \tag{1.64}$$

where Z is a union of some connected components of ∂M and $Z_k = Z \cap \partial M_k$.

Corollary 1.3. $e(M_v, Z)$ *is independent of* $v \in \mathbb{R}^2 \backslash (0,0)$.

Corollary 1.4. *For an arbitrary* $v \in \mathbb{R}^2 \backslash (0,0)$ *the equality holds:*

$$e(M_v, Z) - e(M, Z) = \chi(N). \tag{1.65}$$

This equality follows from Lemma 1.4 and from (1.64).

Remark. 1.9. Even for $v = (1,1)$ the number $e(M_v, Z)$ differs from $e(M, Z)$ in the case $\chi(N) \neq 0$.

1.4.2. Ratio of the analytic torsion norm and the combinatorial torsion norm for spheres and disks. Spherical Morse surgeries. The values of $e(M)$ and $e(M, \partial M)$, where M is a sphere S^n or a disk D^n (with a direct product metric near $\partial D^n = S^{n-1}$) are deduced now from Lemma 1.4.

Lemma 1.5. 1.*For all the spheres,* $e(S^n)$ *is zero.*
2. *For even-dimensional disks,* $e(D^{2n})$ *and* $e(D^{2n}, \partial D^{2n})$ *are zero.*
3. *For all odd-dimensional disks,* $e(D^{2n+1})$ *and* $e(D^{2n+1}, \partial D^{2n+1})$ *are equal to one.*

Proof. A closed interval D^1 is obtained by gluing two intervals $D^1 = D^1 \cup_{pt} D^1$ in their common boundary point. Lemma 1.4 claims in this case that

$$e(D^1) = 2e(D^1, pt) + e(pt) - \chi(pt). \tag{1.66}$$

Since $e(pt) = 0$, we see that (1.52) implies the equalities

$$e(D^1) = e(D^1, pt) = e(D^1, \partial D^1). \tag{1.67}$$

Hence (1.66) implies $e(D^1) = \chi(pt) = 1$.

A circle S^1 is obtained by gluing two intervals, namely $S^1 = D^1 \cup_{\partial D^1} D^1$. So, according to Lemma 1.4 and to (1.67), we have

$$e(S^1) = 2e(D^1) + e(\partial D^1) - \chi(\partial D^1) = 0. \tag{1.68}$$

Suppose (by the induction hypothesis) that $e(S^m) = 0$ for $m \leqq n - 1$. The sphere S^n (for $n \geqq 2$) is the union $(D^{n-1} \times S^1) \bigcup_{S^{n-2} \times S^1} (D^2 \times S^{n-2}) = S^n$. Indeed, $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_j^2 = 1\}$, the disk D^2 in $D^2 \times S^{n-2}$ in the decomposition above corresponds to $\{(x_1, x_2) : x_1^2 + x_2^2 \leqq \varepsilon\}$ and $S^{n-2} = \{(x_j) \in S^n, x_1 = x_2 = 0\}$. Lemma 1.4 claims in this case that (since $\chi(S^{n-2} \times S^1) = 0$)

$$e(S^n) = e(D^{n-1} \times S^1, S^{n-2} \times S^1) + e(D^2 \times S^{n-2}, S^1 \times S^{n-2}) + e(S^{n-2} \times S^1).$$

The equalities below are deduced from the induction hypothesis, from Lemma 1.3 ((1.52), (1.51)), and from (1.68):

$$e(D^{n-1} \times S^1, S^{n-2} \times S^1) = e(D^{n-1} \times S^1) - e(S^{n-2} \times S^1),$$

$$e(D^2 \times S^{n-2}, S^1 \times S^{n-2}) = e(D^2 \times S^{n-2}) - e(S^1 \times S^{n-2}),$$

$$e(S^1 \times S^{n-2}) = 0, \quad e(D^{n-1} \times S^1) = 0,$$

$$e(D^2 \times S^{n-2}) = \chi(S^{n-2})e(D^2). \tag{1.69}$$

Hence the combinatorial torsion norm is equal to the analytic torsion norm for all odd-dimensional spheres S^{2m+1}.

$$e(S^{2m+1}) = 0, \quad T_0(S^{2m+1}) = \tau_0(S^{2m+1}). \tag{1.70}$$

It follows from Lemma 1.3 and from (1.68) that $e(D^2) = e(D^2, \partial D^2)$. It is deduced from Lemma 1.4 and from (1.68) that $e(S^2) = 2e(D^2)$. According to (1.69) the equality $e(S^{2m}) = 0$ for all even-dimensional spheres is a consequence of the equality $e(S^2) = 0$.

Let (M, g_M) be any closed Riemannian manifold of even dimension $2n$. Then the scalar analytic torsion $T(M, g_M)$ is equal to 1. (This equality was proved in [RS], Theorem 2.1, with the help of the equality

$$\sum (-1)^j j m(\lambda, j) = 0,$$

where λ is an arbitrary nonzero eigenvalue of Δ_j on $DR^j(M)$ and $m(\lambda, j)$ is its multiplicity. The latter assertion follows from the symmetry relation $m(\lambda, j) = m(\lambda, 2n - j)$, which is obtained applying the operator $*$ for a Riemannian metric g_M to the λ-eigenforms for Δ_j.) So (in particular) the torsion norm $T_0(S^2)$ is equal to $\| \cdot \|_{\det H^\bullet(S^2)}^2$, where the norm on $H^\bullet(S^2)$ is the norm defined by g_M on the

harmonic forms $\operatorname{Ker}\Delta^\bullet$. (The induced norm $\|\cdot\|^2_{\det H^\bullet(S^2)}$ does not depend on the metric g_{S^2}, as it follows from the invariance of $T_0(M, g_M)$ with respect to g_M, proved above.)

Let v be a volume of S^2 relative to a Riemannian metric g_{S^2}. Then the element $h \in \det H^\bullet(S^2)$ defined below is of the norm 1:

$$h = \left(v^{-1/2}\cdot 1_{S^2}\right)^{-1} \otimes \left(v^{-1/2}(*1_{S^2})\right)^{-1}, \quad \|h\|^2_{\det H^\bullet(S^2)} = 1$$

(here 1_{S^2} is the constant $1 \in DR^0(S^2)$ and $*1_{S^2}$ is the g_{S^2}-volume form).

The sphere S^2 has a cell decomposition[11] $X_{S^2} : X := D^2 \cup_{\partial D^2} pt$. Hence the element $h_c \in \det C^\bullet(X_{S^2})$ defined below is of the norm 1:

$$h_c = (\delta_{pt})^{-1} \otimes (\delta_{D^2})^{-1}, \quad \|h_c\|^2_{\det C^\bullet(X_{S^2})} = 1 \ .$$

(For this cell-decomposition $d_c = 0$, and so $\det C^\bullet(X_{S^2})$ is the same as $\det H^\bullet(S^2)$ without the d_c-identification. The cochains δ_{pt}, δ_{D^2} are the basic elements in $H^0(S^2)$, $H^2(S^2)$.)

The integration homomorphism $R : DR^\bullet(S^2) \to C^\bullet(X_{S^2})$ maps 1_{S^2} to δ_{pt} and $*1_{S^2}$ to $v\cdot\delta_{D^2}$. So $R(h) = h_c$ and we have

$$e(S^2) = 0, \quad e(S^{2m}) = 0 \ . \tag{1.71}$$

The equalities below follow from Lemmas 1.3, 1.4, and from (1.70), (1.71):

$$0 = e(S^n) = 2e(D^n, \partial D^n) + e(S^{n-1}) - \chi(S^{n-1}) \ ,$$
$$e(D^n, \partial D^n) = e(D^n) - e(S^{n-1}) = e(D^n) \ ,$$
$$e(D^n, \partial D^n) = 2^{-1}\chi(S^{n-1}) = e(D^n) \ .$$

Lemma 1.5 is proved. □

The equality $e(D^{m+1} \times S^n) = e(D^{m+1} \times S^n, \partial(D^{m+1} \times S^n))$ holds by Lemmas 1.3 and 1.5.

Corollary 1.5. *For arbitrary $n, m \geqq 0$ the equality holds*:

$$e(D^{m+1} \times S^n) = e(S^m \times D^{n+1}) \ . \tag{1.72}$$

(According to Lemma 1.5, each side of (1.72) is equal to 2 in the case of a pair of even numbers (m, n) and it is equal to zero for other pairs (m, n).)

Let M be a compact maniifold with a smooth boundary ∂M and let Z be a union of some connected components of ∂M. Let $\widetilde{M}$ be obtained by some spherical Morse surgery (with a trivial normal bundle) of M (i.e., there exists a manifold $(M_1, \partial M_1) \subset M\backslash\partial M$, $M_1 \simeq D^{m+1} \times S^n, m + n + 1 = \dim M$, with $\partial M_1 = S^m \times S^n$, $M = M_1 \cup_{\partial M} M_2$, such that $\widetilde{M} = \widetilde{M}_1 \cup_{\partial \widetilde{M}_1} M_2$ is obtained by gluing $\widetilde{M}_1 = S^m \times D^{n+1}$ and M_2 by a diffeomorphism $f : \partial\widetilde{M}_1 \widetilde{\approx} \partial M_2$).

---

[11] This CW-complex (cell stratification) has a subdivision which is a C^1-triangulation of S^2. So as the combinatorial torsion is defined also for CW-complexes and as it is invariant under subdivisions, $\tau(S^2)$ can be computed from this cell stratification ([Mi], Sects. 7,8,12.3).

Let the metrics g_M and $g_{\widetilde{M}}$ be direct product metrics near ∂M and $\partial \widetilde{M}$. (It is proved above that the numbers $e(M, Z)$ and $e(\widetilde{M}, Z)$ do not depend on the metrics g_M, $g_{\widetilde{M}}$ supposed to be direct product metrics near ∂M and near $\partial \widetilde{M}$.)

Lemma 1.6. *The number $e(M, Z)$ is invariant under spherical Morse surgeries with a trivial normal bundle, i.e., the equality holds*

$$e(M, Z) = e(\widetilde{M}, Z) . \tag{1.73}$$

Proof. The metrics g_M and $g_{\widetilde{M}}$ can be replaced by Riemannian metrics on M and $\widetilde{M}$ which are direct product metrics on $\partial M_1 \times I$ and $\partial \widetilde{M}_1 \times I$ near $\partial M_1 \subset M$ and near $\partial \widetilde{M}_1 \subset \widetilde{M}_1$ (and which are direct product metrics near ∂M and $\partial \widetilde{M}$). Lemma 1.4 claims in this case that

$$e(M, Z) = e(M_1, \partial M_1) + e(M_2, \partial M_1 \cup Z) + e(\partial M_1) - \chi(\partial M_1) ,$$

$$e(\widetilde{M}, Z) = e(\widetilde{M}_1, \partial \widetilde{M}_1) + e(M_2, \partial \widetilde{M}_1 \cup Z) + e(\partial \widetilde{M}_1) - \chi(\partial \widetilde{M}_1) . \tag{1.74}$$

The smooth closed manifolds $\partial \widetilde{M}_1$ and ∂M_1 are diffeomorphic. Hence

$$e(\partial M_1) = e(\partial \widetilde{M}_1), \quad \chi(\partial M_1) = \chi(\partial \widetilde{M}_1), \quad e(M_2, \partial M_1 \cup Z) = e(M_2, \partial \widetilde{M}_1 \cup Z) .$$

Corollary 1.5 and Lemmas 1.3 and 1.5 claim that $e(M_1, \partial \widetilde{M}_1) = e(\widetilde{M}_1, \partial \widetilde{M}_1)$. So the equality (1.73) follows from (1.74). □

1.4.3. Proof of the generalized Ray–Singer conjecture.

Theorem 1.3 (Classical Ray–Singer Conjecture). *For any closed Riemannian manifold (M, g_M) its analytic torsion norm is equal to the combinatorial torsion norm*

$$T_0(M) = \tau_0(M) .$$

Proof. There is a smooth Morse function f on a direct product $M \times I$ (i.e., a function with the nondegenerate isolated critical points with different critical values) such that the following holds. Its minimum value is equal to zero, $f^{-1}(0) = M \times \partial I$, and the zero is not a critical value of f. Its maximum value $\max_{M \times I} f$ equals 1 and the maximum value level is the only one point. Namely $f^{-1}(1)$ is an interior point of $M \times (I, \partial I)$.

As $f^{-1}(1 - \varepsilon)$ (where $\varepsilon > 0$ is very small) is a sphere S^n (n=dim M), there exists a sequence of spherical Morse surgeries (given by transformations of levels $f^{-1}(x)$, $x \in (0, 1 - \varepsilon)$ for x divided by critical values) such that their composition is a transformation of a manifold[12] $M \cup M = M \times \partial I = f^{-1}(0)$ into $S^n = f^{-1}(1 - \varepsilon)$.

As a consequence of Lemma 1.6 in this case we get

$$2e(M) = e(M \cup M) = e(S^n) .$$

Lemma 1.5 claims that $0 = e(S^n) = e(M)$. Thus, the Ray–Singer conjecture is proved. □

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[12] The manifold M is not supposed to be orientable.

Let (M, g_M) be a compact Riemannian manifold with a smooth boundary ∂M. Let Z be a union of some connected components of ∂M and let g_M be a direct product metric near ∂M. The following two theorems are generalizations of the Ray–Singer conjecture.

Theorem 1.4. *Under the conditions above, the following equality holds for a manifold with a smooth boundary*:

$$T_0(M, Z) = 2^{\chi(\partial M)/2} \tau_0(M, Z) . \tag{1.75}$$

Proof. Lemma 1.3 claims that $e(M, Z) = e(M, \partial M) + e(\partial M \backslash Z)$. According to Theorem 1.3, $e(\partial M \backslash Z)$ is equal to zero. Hence $e(M, Z) = e(M, \partial M)$. In the case of $\partial M \neq \emptyset$ there is a mirror-symmetric closed Riemannian manifold $P = M \cup_{\partial M} M$ obtained by gluing two copies of (M, g_M) along ∂M. According to Lemma 1.4, we have

$$e(P) = 2e(M, \partial M) + e(\partial M) - \chi(\partial M) .$$

Theorem 1.3 claims that $e(\partial M) = 0 = e(P)$. Thus, we get

$$e(M, Z) = e(M, \partial M) = 2^{-1} \chi(\partial M) ,$$

which is equivalent to (1.75). $\square$

Let (M, Z, g_M) be as in Theorem 1.4. Let N be a codimension one in M two-sided in M closed submanifold $N \subset M \backslash \partial M$. Let M be obtained by gluing M_1 and M_2 along N. Let g_M be a direct product metric near N and let the v-transmission boundary conditions (1.13) be given on N (where $v = (\alpha, \beta) \subset \mathbb{R}^2 \backslash (0, 0)$).

Theorem 1.5. *The analytic torsion norm is expressed by the combinatorial torsion norm* (*in the case of the v-transmission interior boundary condition on N*) *as follows:*

$$T_0(M_v, Z) = 2^{\chi(\partial M)/2 + \chi(N)} \tau_0(M_v, Z) .$$

Proof. The equality (1.65) claims that $e(M_v, Z) = e(M, Z) + \chi(N)$. So the assertion of the theorem follows from Theorem 1.4 and from the equality (1.65). $\square$

Remark. 1.10. This proof of the generalization of the Ray–Singer conjecture does not use any explicit expressions for the scalar analytic torsions of any special classes of manifolds. The proof in [Mü1] of the classical Ray–Singer conjecture essentially used the explicit expressions for the scalar analytic torsions for spheres and lens spaces. (The latter expressions were obtained by D.B.Ray in [Ra]. He computed there the scalar analytic torsion for lens spaces and spheres with homogeneous metrics by explicit calculations of the ζ-functions for the corresponding Laplacians using Gegenbauer's polynomials.) The proof in [Mü1] used precise estimates of [DP] for the eigenvalues of the corresponding combinatorial Laplacians. In our proof we don't use the Lerch formula for the derivative at zero of the zeta-function of Riemann [WW], 13.21, 12.32. (Its new proof is obtained here.) Our proof of the generalized Ray–Singer formula is based on a gluing property for the analytic torsion norms. This property is proved here for a general gluing two Riemannian manifolds by a diffeomorphism of some connected components of their boundaries. It is proved without any computations of asymptotics of eigenvalues and eigenforms for the corresponding Laplacians.

2. Gluing Formula for Analytic Torsion Norms. Proof of Theorem 1.1

2.1. Strategy of the proof. In Sect.1 the generalized Ray–Singer conjecture for a manifold with a smooth boundary is deduced from Theorem 1.1. Namely it is deduced from the gluing formula

$$\varphi_{\mathrm{an}} T_0(M, Z) = T_0(M_1, Z_1 \cup N) \otimes T_0(M_2, Z_2 \cup N) \otimes T_0(N), \qquad (2.1)$$

which holds under the conditions of Theorem 1.1 (where $Z_k := Z \cap \partial M_k$). The identification φ_{an} in (2.1) is defined in (1.9) with the help of the exact sequence (1.10) of the de Rham complexes. It is proved in Lemma 1.2 that the equality (2.1) follows from the assertion that under the conditions of Theorem 1.1, the induced analytic torsion norm[13] $\varphi_v^{\mathrm{an}} T_0(M_v)$ does not depend on a parameter v of the interior boundary conditions. The latter statement means that the equality

$$\varphi_v^{\mathrm{an}} T_0(M_v, Z) = c_0 T_0(M_1, Z_1 \cup N) \otimes T_0(M_2, Z_2 \cup N) \otimes T_0(N), \qquad (2.2)$$

holds with a positive constant c_0 which is independent of $v \in \mathbb{R}^2 \backslash (0, 0)$. (However, it *is not supposed* in Lemma 1.2 that c_0 *is independent of* M, N, g_M, and Z.)

The strategy of the proof of the equality (2.2) is as follows. First we prove that c_0 is constant on each of four connected components

$$U_j \subset U := \{ (\alpha, \beta) \in \mathbb{R}^2 : \alpha\beta \neq 0 \}. \qquad (2.3)$$

Then it is enough to prove that $c_0(v)$ is continuous as a function of v for $v \in \mathbb{R}^2 \backslash (0, 0)$. These two assertions provide us with a proof of the equality (2.2).

Let $v_0 \in U$ and let $a > 0$ be a number not belonging to the spectrum $S(v_0) := \bigcup_i \operatorname{Spec}\Delta_i(M_{v_0}, g_M) \subset \mathbb{R}_+$ of the Laplacians on $DR^\bullet(M_v, Z)$. This spectrum is discrete according to Theorem 3.1. In particular, each eigenvalue is of a finite multiplicity. Let $W_a^i(v)$ be a subspace of $DR^i(M_v, Z)$, spanned by all the eigenforms ω_λ for $\Delta_{v,i} := \Delta_i(M_v, g_M)$ with their eigenvalues $\lambda \leq a$. Then $dW_a^i(v) \subset W_a^{i+1}(v)$. So $(W_a^\bullet(v), d)$ is a finite-dimensional subcomplex of $(DR^\bullet(M_v, Z), d)$ equipped with the natural Hilbert structures on $W_a^\bullet(v) \hookrightarrow DR^\bullet(M_v, Z)$ (defined by g_M).

Let $\| \cdot \|_{\det W_a^\bullet(v)}^2$ be the induced norm on $\det W_a^\bullet(v)$. For v very close to v_0 it holds also that $a \notin S(v)$ (Proposition 3.1). By the definition of $W_a^\bullet(v)$, its cohomology $H^j(W_a^\bullet(v))$ are canonically identified with the space of harmonic forms $\operatorname{Ker}\Delta_j(M_v, g_M)$. The differential d in $W_a^\bullet(v)$ induces the identification

$$d_W : \det W_a^\bullet(v) \widetilde{\rightarrow} \det \operatorname{Ker} \Delta_\bullet(M_v, g_M). \qquad (2.4)$$

According to Lemma 1.1 there is a canonical identification between the harmonic forms and the cohomology of the de Rham complex (the latter one is independent of g_M):

$$\operatorname{Ker} \Delta_i(M_v, g_M) = H^i(DR(M_v, Z)). \qquad (2.5)$$

So there is the induced canonical identification of the determinant lines:

$$\det \operatorname{Ker} \Delta_\bullet(M_v, g_M) = \det H^\bullet(DR(M_v, Z), d). \qquad (2.6)$$

Let $\| \cdot \|_{\det H^\bullet(M_v)}^2$ be a norm on $\det H^\bullet(M_v, Z) := \det H^\bullet(DR(M_v, Z), d)$ induced by the identifications (2.5) and (2.6) from the Hilbert structure on the harmonic forms $\operatorname{Ker} \Delta_\bullet(M_v, g_M)$. (This structure is defined by the Riemannian metric g_M.)

---

[13] The identification φ_v^{an} is defined by the short exact sequence (1.14).

The identification (2.4) is not an isometry of the norms $\| \cdot \|^2_{\det H^\bullet(M_v)}$ and $\| \cdot \|^2_{\det W_a^\bullet(v)}$ in general. The norm $\| \cdot \|^2_{\det H^\bullet(M_v)}$ has to be multiplied by an additional factor for the identification (2.4) to become an isometry. This factor is the *scalar analytic torsion* of a complex $(W_a^\bullet(v), d)$, defined by the general formula (1.1). We can conclude that the *analytic torsion norm* $T_0(M_v, Z)$ on the determinant of $H^\bullet(DR(M_v, Z))$ is isometric (under the identifications (2.4) and (2.6)) to the norm

$$T_0(M_v, Z; a) := \| \cdot \|^2_{\det W_a^\bullet(v)} \exp\left(\sum(-1)^j j \partial_s \zeta_{v,j}(s; a)|_{s=0}\right) . \qquad (2.7)$$

The zeta-function $\zeta_{v,j}(s; a)$ is defined for Re $s > (\dim M)/2$ by the series $\sum_{\lambda > 0} \lambda^{-s}$, where the sum is over all the eigenvalues λ of $\Delta_j((M_v, g_M))$ (including their multiplicities) such that $\lambda > a$. This ζ-function can be continued meromorphically to the whole complex plane $\mathbb{C}$ and it is regular at zero. The latter assertion follows from Theorem 3.1 and from the equality (which is obvious for Re $s > (\dim M)/2$):

$$\zeta_{v,j}(s; a) = \zeta_{v,j}(s) - \sum_{0 < \lambda \leq a} \lambda^{-s} . \qquad (2.8)$$

(The series for $\zeta_{v,j}(s)$, Re $s > (\dim M)/2$, is the sum over all the nonzero eigenvalues of $\Delta_j(M_v, g_M)$ with their multiplicities, where $\lambda^{-s} := \exp(-s \log \lambda)$ and $\log \lambda \in \mathbb{R}$ for $\lambda > 0$).

The identifications d_W (2.4) and φ^v_{an} (the latter one is defined with the help of (1.14)) provide is (under the conditions of Lemma 1.2) with the identification:

$$\varphi^{\text{an}}_v(a) : \det W_a^\bullet(v) \tilde{\rightarrow} \mathbf{Det}(M, N, Z) , \qquad (2.9)$$

($\mathbf{Det}(M, N, Z)^{14}$ is defined in (1.19)). The assertion that $c_0(v)$ is independent of v on each connected component U_j of U (2.3) is equivalent to the following one. The analytic torsion norm $T_0(M_v, Z; a)$ is transformed (under the identification (2.9)) into the norm on $\mathbf{Det}(M, N, Z)$:

$$\varphi^{\text{an}}_v(a) \circ T_0(M_v, Z; a) = c_0(v) T_0(M_1, Z_1 \cup N) \otimes T_0(M_2, Z_2 \cup N) \otimes T_0(N) , \qquad (2.10)$$

where c_0 is *constant on each connected component U_j*.

The action of $\varphi^v_{\text{an}}(a)$ is as follows (by its definition):

$$\varphi^v_{\text{an}}(a) T_0(M_v, Z; a) = T(M_v, Z; a) \varphi^v_{\text{an}}(a) \circ \| \cdot \|^2_{\det W_a^\bullet(v)} ,$$

where the scalar analytic torsion $T(M_v, Z; a)$ is defined as the scalar factor in (2.7):

$$T(M_v, Z; a) := \exp\left(\sum(-1)^j j \partial_s \zeta_{v,j}(s; a)|_{s=0}\right) . \qquad (2.11)$$

Let $v(\gamma), \gamma \in (\varepsilon, \varepsilon) \subset \mathbb{R}$, be a smooth curve on U (2.3) and let $v(0) = v_0$. Let $\prod_j(v_0; a)$ be an orthogonal projection operator from $(DR^j(M))_2$ onto $W_a^j(v_0)$ (with respect to the natural Hilbert structure (1.23) in $(DR^j(M))_2$). Let p_1 be a linear operator in $(DR^\bullet(M))_2$, mapping $(\omega_1, \omega_2) \in (DR^\bullet(M))_2$ to $(\omega_1, 0)$. (Respectively p_2 maps (ω_1, ω_2) to $(0, \omega_2)$.)

Let v and v_0 be arbitrary points from U. Then the following isomorphism of the de Rham complexes is defined (where $k_v := \alpha/\beta$ for $v = (\alpha, \beta) \in U$):

$$v_v = v_{vv_0} : DR^\bullet(M_{v_0}, Z) \tilde{\rightarrow} DR^\bullet(M_v, Z), \quad v_v(\omega_1, \omega_2) := (\omega_1, (k_v/k_{v_0}) \cdot \omega_2) . \qquad (2.12)$$

---

[14] To recall, Z is the union of the connected components of ∂M, where the Dirichlet boundary conditions are given. The Neumann boundary conditions are given on $\partial M \backslash Z$.

Thus the induced isormorphism is defined:

$$v_{v*} : H^{\bullet}(DR(M_{v_0}, Z)) \to H^{\bullet}(DR(M_v, Z)) .$$

Let a be a positive number from $\mathbb{R}_+ \backslash S(v_0)$. Then for v very close to v_0 the number a is also from $\mathbb{R}_+ \backslash S(v)$ (Proposition 3.1). The complexes $W_a^{\bullet}(v)$ and $W_a^{\bullet}(v_0)$ are isomorphic as abstract finite-dimensional complexes (and (2.12) provides us with a natural but not canonical isomorphism of these complexes). We have to compute the action of φ_{an}^v on the norms $|| \cdot ||_{\det W_a^{\bullet}(v)}^2$ for v very close to v_0. However $|| \cdot ||_{\det W_a^{\bullet}(v)}^2$ are the norms on different complexes. So it is necessary to define some isomorphism between $W_a^{\bullet}(v_0)$ and $W_a^{\bullet}(v)$ and then to compute its action on $|| \cdot ||_{\det W_a^{\bullet}(v_0)}^2$ and on the space $\mathbf{Det}(M, N, Z)$. The choice (2.13) of such an identification is done below.

For v very close to v_0 the subspaces $W_a^{\bullet}(v)$ and $W_a^{\bullet}(v_0)$ are very close in the L_2-completion $(DR^{\bullet}(M))_2$ of $DR^{\bullet}(M_v, Z) =: DR^{\bullet}(v)$, according to Proposition 3.1. So the following isomorphism of these finite-dimensional complexes is well-defined:

$$g_v = \Pi^{\bullet}(v; a) \cdot v_{vv_0} \cdot j_{v_0} :$$

$$W_a^{\bullet}(v_0) \hookrightarrow (DR^{\bullet}(v_0), d) \xrightarrow[\sim]{v_{vv_0}} (DR^{\bullet}(v), d) \xrightarrow[\Pi^{\bullet}(v;a)]{} (W_a^{\bullet}(v), d) , \qquad (2.13)$$

where j_{v_0} is the natural inclusion of $W_a^{\bullet}(v_0)$ and $\Pi^{\bullet}(v; a)$ is the orthogonal projection operator onto $W_a^{\bullet}(v)$. Its action on the norm $|| \cdot ||_{\det W_a^{\bullet}(v_0)}^2$ is computed by the following lemma.

Lemma 2.1. *Let l be an arbitrary nonzero element of $\det W_a^{\bullet}(v_0)$. Then the equality holds for any smooth variation $v(\gamma)$ of $v_0 = v(0)$:*

$$\partial_{\gamma} \log ||g_v l||_{\det W_a^{\bullet}(v)}^2 |_{\gamma=0} = -2 \partial_{\gamma} \log(k_v)|_{\gamma=0} \left(\sum (-1)^l \mathrm{Tr} \left(p_2 \Pi^j(v_0; a) \right) \right) . \qquad (2.14)$$

In (2.14) the rank (i.e., the dimension of the image) of the operator $p_2 \Pi^j(v_0; a)$ is less or equal to $\dim W_a^{\bullet}(v_0)$. This operator acts in $(DR^{\bullet}(M))_2$.

Then the following lemma provides us with the variation formula for $T(M_v, Z; a)$.

Lemma 2.2. *For $\gamma = 0$ the equality holds:*

$$\partial_{\gamma} \log T(M_v, Z; a) = 2 \partial_{\gamma} \log(k_v)|_{\gamma=0} \sum (-1)^j b_{1,j}(M_{v_0}, Z; a) ,$$

where $k_v := \alpha/\beta$ for $v = (\alpha, \beta) \in U$. Here $b_{1,j}(M_{v_0}, Z; a)$ is a constant coefficient (i.e., t^0-coefficient q_0) in the asymptotic expansion as $t \to +0$ of the trace of the operator below (acting in $(DR^j(M))_2$):

$$\mathrm{Tr}(p_1 \{ \exp(-t\Delta_{v_0,j})(1 - \Pi^j(v_0; a)) \}) \sim q_{-n} t^{-n/2} + q_{-n+1} t^{-(n-1)/2} + \cdots + q_0 t^0 + \cdots$$
$$(2.15)$$

Remark. 2.1. The operators $\exp(-t\Delta_{v_0,j})$ and $\Pi^j(v_0; a)$ acting in the L_2-completion $(DR^j(M))_2$ of $DR^j(M_{v_0}, Z)$ (which coincides with the L_2-completion of $(DR^j(M))$ have their images in the domain of definition of the Laplacian $D(\Delta_{v_0,j}) \subset DR^j(M_{v_0}, Z)$. The existence of the asymptotic expansion (2.15) follows from Theorem 3.2. The coefficients q_m with $m \leq -1$ in (2.15) are independent of a. The coefficients $\tilde{q}_m$ of the asymptotic expansion for $\mathrm{Tr}(p_1 \exp(-t\Delta_{v_0,j}))$ are equal to the

sums of the integrals over M_1 and over $\partial M_1 \supset N$ of the locally defined densities on M_1 and on ∂M_1 (by Theorem 3.2). However, in the general case we cannot represent $\mathrm{Tr}(p_1 \Pi^j(v_0; a))$ as an integral of a locally defined density (because there are no universal local formulas for the eigenforms ω_λ of $\Delta_{v_0,j}$). Hence there is no universal local formula for a coefficient q_0 in (2.15) but there are such formulas for $q_m = \tilde{q}_m$ with $m < 0$.

Corollary 2.1. *For an arbitrary nonzero* $l \in \det W_a^\bullet(v_0)$ *the equality holds*

$$\partial_\gamma \log ||g_v l||^2_{T_0(M_v, Z)} = 2\partial_\gamma (\log k_v)|_{\gamma=0} (\sum (-1)^j (b_{1,j}(M_{v_0}, Z) - \dim W_a^j)) , \quad (2.16)$$

where $b_{1,j}(M_{v_0}, Z)$ *is a constant coefficient (i.e., the* t^0*-coefficient) of the asymptotic expansion of* $\mathrm{Tr}(p_1 \exp(-t\Delta_{v_0,j}))$ *relative to* $t \to +0$ *and* $p_1 \exp(-t\Delta_{v_0,j}))$ *is the operator acting in* $(DR^j(M))_2$.

Remark. 2.2. Note that in the right side of (2.16) there are the Euler characteristic $\chi(M_v, Z) := \sum (-1)^j \dim W_a^j$ and the alternating sum of the integrals $b_{1,j}(M_{v_0}, Z)$ (over M_1 and over ∂M_1) of the locally defined densities (Remark 2.1). (Here Z is the union of the connected components of ∂M where the Dirichlet boundary conditions are given). The number $\chi(M_v, Z)$ is also equal to the sum of the integrals over M, N, and over ∂M of the locally defined densities.

Let $v(\gamma)$ be a smooth variation of a point $v_0 \in U$ (2.3). Let $l(\gamma) \in \det W_a^\bullet(v_0)$ be a variation of an arbitrary nonzero element $l \in \det W_a^\bullet(v_0)$ such that $\varphi_{v(\gamma)}^{an}(a) \circ l(\gamma)$ is a fixed (nonzero) element of $\mathbf{Det}(M,N,Z)$. Then the equality (2.10) (where the factor $c_0(v)$ is constant on each connected component of U (2.3)) is equivalent to the assertion that for any such a variation $l(\gamma)$ its analytic torsion norm is independent of γ:

$$\partial_\gamma ||l(\gamma)||^2_{T_0(M_{v(\cdot)}, Z, a)}|_{\gamma=0} = 0 . \quad (2.17)$$

Corollary 2.1 provides us with the formula (2.16) for a variation of the analytic torsion norm $||g_{v(\gamma)} l||^2_{T_0(M_{v(\cdot)}, Z; a)}$ (where $l \in \det W_a^\bullet(v_0)$). The assertion (2.17) is equivalent to the following identity:

$$\partial_\gamma \log ||g_{v(\gamma)} l||^2_{T_0(M_{v(\cdot)}, Z; a)}|_{\gamma=0} = \partial_\gamma \log ||g_{v*} f||^2|_{\gamma=0} , \quad (2.18)$$

where f is an arbitrary nonzero element of $\mathbf{Det}(M,N,Z)$ (for instance, $f = \varphi_{v_0}^{an}(a) \circ l$) and $g_{v*} = \varphi_v^{an}(a) \circ g_v \circ (\varphi_{v_0}^{an}(a))^{-1}$ is defined by the following commutative diagram, where $v \in U$ is very close to v_0:

$$
\begin{array}{ccc}
\mathbf{Det}(M,N,Z) & \xrightarrow{\ \ g_{v*}\ \ } & \mathbf{Det}(M,N,Z) \\[4pt]
{\scriptstyle \varphi_{v_0}^{an}(a)} \big\uparrow {\scriptstyle \wr} & & {\scriptstyle \wr} \big\uparrow {\scriptstyle \varphi_v^{an}(a)} \\[4pt]
\det W_a^\bullet(v_0) & \xrightarrow[\ g_v = g_{vv_0}\ ]{} & \det W_a^\bullet(v)
\end{array}
$$

The norm on the right in the equality (2.18) is an arbitrary Hilbert norm in one-dimensional space $\mathbf{Det}(M,N,Z)$. The value of the expression on the right in (2.18) is independent of such a norm.

The action of the isomorphism $g_v = g_{vv_0} : W_a^\bullet(v_0) \to W_a^\bullet(v)$ on $\mathbf{Det}(M,N,Z)$ is described by the following lemma.

Lemma 2.3. *For an arbitrary element* $f \in \mathbf{Det}(M, N, Z)$ *the equality holds*:

$$\partial_\gamma \log\|g_{v*}f\|^2 \,|_{\gamma=0} = -2\partial_\gamma(\log k_v)|_{\gamma=0} \sum(-1)^j b_{2,j}(M_{v_0}, Z) , \qquad (2.19)$$

where $b_{2,j}(M_{v_0}.Z)$ *is the constant coefficient (i.e., the* t^0*-coefficient) in the asymptotic expansion (relative to* $t \to +0$*) for the trace of the operator* $p_2 \exp(-t\Delta_{v_0,j})$ *acting in* $DR^j(M))_2$.

Here p_2 is the operator $p_2 : (\omega_1, \omega_2) \to (0, \omega_2)$ for $\omega_k \in (DR(\overline{M}_k))_2$.

Remark. 2.3. Note that $\mathrm{Tr} \exp(-t\Delta_{v_0,j}) = \sum_j \mathrm{Tr}(p_j \exp(-t\Delta_{v_0,j}))$. So we have

$$-\sum(-1)^j b_{2,j}(M_{v_0}, Z) = \sum(-1)^j b_{1,j}(M_{v_0}, Z) - \chi(M_{v_0}, Z) .$$

Hence the equality (2.18) follows from (2.16) and (2.19).

Thus Lemmas 2.1-2.3 provide us with a proof of the assertion that the factor $c_0(v)$ is independent of v on each connected component U_j of U (2.3).

2.2. Continuity of Analytic Torsion Norms.

To prove that $c_0(v)$ is independent of $v \in \mathbb{R}^2 \setminus (0,0)$, it is enough[15] to show that the norm $\varphi_v^{\mathrm{an}} \circ T_0(M_v)$ on $\mathbf{Det}(M, N, Z)$ is continuous in $v \in \mathbb{R}^2 \setminus (0,0)$. The following norms on $\mathbf{Det}(M, N, Z)$ are the same for an arbitrary $a \geqq 0$:

$$\varphi_v^{\mathrm{an}}(a) \circ T_0(M_v, Z; a) = \varphi_v^{\mathrm{an}} \circ T_0(M_v, Z) . \qquad (2.20)$$

Let us prove the continuity of $\varphi_v^{\mathrm{an}} T_0(M_v, Z)$ as a function of v at a point $v_0 \in \mathbb{R}^2 \setminus (0,0)$. (The series of lemmas above provides us with the proof of this assertion in the case when $v_0 \in U$ (2.3). But now this will be proved at an arbitrary $v_0 \in \mathbb{R}^2 \setminus (0,0)$, for instance, at $v_0 \in \mathbb{R}^2 \setminus (U \cup (0,0))$.) By (2.20), it is enough to obtain the continuity in v at $v = v_0$ of the norm $\varphi_v^{\mathrm{an}}(a) \circ T_0(M_v; a)$ on $\mathbf{Det}(M, N, Z)$ for a fixed $a > 0$ such that $a \notin S(v_0) := \cup_j \mathrm{Spec}(\Delta_{v_0,j})$. Since $a \notin S(v_0)$, we see that $a \notin S(v)$ for v very close to v_0. (The latter assertion follows from Proposition 3.1. It claims that the resolvents $G_\lambda^\bullet(v) := (\Delta_v^\bullet - \lambda)^{-1}$ for $\lambda \notin \mathrm{Spec}(\Delta_{v,\bullet})$ form a smooth in (λ, v) family of bounded operators in $(DR^\bullet(M))_2$, and that $\mathrm{Spec}(\Delta_{v,\bullet})$ is discrete. As $G_a^\bullet(v_0)$ is bounded in $(DR^\bullet(M))_2$, the operator $G_a^\bullet(v)$ is also bounded for v close to v_0, and so $a \notin \mathrm{Spec}(\Delta_{v,\bullet})$ for such v.) The assertion below claims that the truncated scalar analytic torsion (2.11) is a locally continuous function.[16]

Proposition 2.1. *The scalar analytic torsion* $T(M_v, Z; a)$ *is continuous in* v *at* v_0.

Thus, the continuity of $\varphi_v^{\mathrm{an}} T_0(M_v, Z)$ (as a function of v) at v_0 is equivalent to the condition that the norm on $\mathbf{Det}(M, N, Z)$

$$\varphi_v^{\mathrm{an}}(a) \circ \| \cdot \|_{\det W_a^\bullet(v)}^2 \qquad (2.21)$$

is continuous in v at v_0. The continuity of the norm (2.21) is deduced from the following finite-dimensional algebraic lemma. Let

$$f : (A^\bullet, d_A) \to (V^\bullet, d_V) \qquad (2.22)$$

---

[15] The factor $c_0(v)$ is constant on each connected componenet U_j of U (2.3), and U is dense in $\mathbb{R}^2 \setminus (0,0)$.

[16] This truncated scalar analytic torsion is a continuous function on the set of $v \in \mathbb{R}^2 \setminus (0,0)$ such that $a \notin \cup_i \mathrm{Spec}(\Delta_{v,i})$.

be a quasi-isomorphism of finite complexes of finite-dimensional Hilbert spaces. Let $f_* : \det H^\bullet(A) \widetilde{\to} \det H^\bullet(W)$ be the induced identification of the determinant lines. Let $T_0(A^\bullet)$ and $T_0(V^\bullet)$ be the analytic torsion norms (1.2) on the determinant lines identified by $f_* : \det H^\bullet(A) = \det H^\bullet(V)$. Let $(\mathrm{Cone}^\bullet f, d)$, $\mathrm{Cone}^j f = A^{j-1} \oplus V^j$, be a simple complex, associated with the bicomplex (2.22):

$$d_{\mathrm{Cone}} : \mathrm{Cone}^j \to \mathrm{Cone}^{j+1}, \ d_{\mathrm{Cone}}(x, y) = (-d_A x, fx + d_V y) \qquad (2.23)$$

(for $(x, y) \in A^{j+1} \oplus V^j$). Then $\mathrm{Cone}^\bullet f$ is an acyclic finite complex of finite-dimensional Hilbert spaces ($\mathrm{Cone}^j f$ is the direct sum of Hilbert spaces A^{j+1} and V^j), $H^\bullet(\mathrm{Cone} f) = 0$. Hence $\det H^\bullet(\mathrm{Cone} f)$ is canonically identified with $\mathbb{C}$ and the analytic torsion norm for $\mathrm{Cone}^\bullet f$ is a norm on $\mathbb{C}$. The ratio $T_0(V)/T_0(A) \in \mathbb{R}_+$ is defined as the ratio between the two norms on the one-dimensional spaces $\det H^\bullet(V)$ and $\det H^\bullet(A)$ identified by f_*.

Lemma 2.4. *Under the conditions above, the equality holds*:

$$\|1\|^2_{T_0(\mathrm{Cone}^\bullet f)} = T_0(V^\bullet)/T_0(A^\bullet) \,, \qquad (2.24)$$

where the left side is the analytic torsion norm of $1 \in \mathbb{C} = \det H^\bullet(\mathrm{Cone}\ f)$.

Let $a > 0$ be a number from $\mathbb{R}_+ \setminus S(v_0)$. Then there exists an open neighborhood $U_{v_0}(a)$ of $v_0 \in U_{v_0}(a) \subset \mathbb{R}^2 \setminus (0, 0)$ such that $a \notin S(v)$ for $v \in U_{v_0}(a)$ (Proposition 3.1). The family of complexes $(W_a^\bullet(v), d)$ of Hilbert spaces is continuous on $U_{v_0}(a)$ in the following sense.

The operator $\Pi_a^j(v) := \Pi^j(v; a)$ is a finite rank projection operator in $(DR^j(M))_2$ with its image $W_a^j(v)$:

$$\Pi_a^j(v) : (DR^j(M))_2 \to W_a^j(v) \subset DR^j(M_v, Z) \subset (DR^j(M))_2 \,.$$

Proposition 2.2. *The family of operators* $\Pi_a^\bullet(v)$ *is continuous in* v *for* $v \in U_{v_0}(a)$ *with respect to the operator norm in* $(DR^\bullet(M))_2$. *The same is true for the families*

$$d\Pi_a^\bullet(v) : (DR^\bullet(M))_2 \to DR^{\bullet+1}(M_v) \subset (DR^{\bullet+1}(M))_2 \,.$$

$$\delta\Pi_a^\bullet(v) : (DR^\bullet(M))_2 \to DR^{\bullet-1}(M_v) \subset (DR^{\bullet-1}(M))_2 \,.$$

These are the families of finite rank operators.

Proof. It follows from Proposition 3.1 that if $a \notin S(V_0)$ then there exists an $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \cap S(v) = \emptyset$ for v sufficiently close to v_0. Hence $\{\lambda : a - \varepsilon < |\lambda| < a + \varepsilon\} \cap S(v) = \emptyset$ for such v (since $S(v) \subset \mathbb{R}_+ \cup 0$ by Theorem 3.1). Thus, according to Proposition 3.1, the operators

$$\Pi_a^\bullet(v) = \frac{i}{2\pi} \int_{\Gamma_a} G_\lambda^\bullet(v) d\lambda$$

form a smooth in v (for such v) family of finite rank operators in $(DR^\bullet(M))_2$ (where the circle $\Gamma = \{\lambda : |\lambda| = a\}$ is oriented opposite to the clockwise). The operators $d\Pi_a^\bullet(v) : (DR^\bullet(M))_2 \to (DR^{\bullet+1}(M))_2$ form (for such v) a smooth in v family of finite rank operators (according to Proposition 3.1.) $\square$

Corollary 2.2. *For v sufficiently close to v_0 the family of operators $\Pi_a^\bullet(v)$ identifies the graded linear spaces $W_a^\bullet(v_0) := \operatorname{Im} \Pi_a^\bullet(v_0)$ and $W_a^\bullet(v)$. Such an identification nearly commutes with d in the following sense*:

$$\|d\Pi_a^\bullet(v)w - \Pi_a^{\bullet+1}(v)dw\|_2 \leqq c(v,v_0)\|w\|_2 \qquad (2.25)$$

(for any $w \in W_a^\bullet(v_0)$), where $c(v,v_0) \to +0$ as $v \to v_0$. This identification also nearly commutes with δ :

$$\|\delta\Pi_a^\bullet(v)w - \Pi_a^{\bullet-1}(v)\delta w\|_2 \leqq c(v,v_0)\|w\|_2$$

for $w \in W_a^\bullet(v_0)$ ($\|\cdot\|_2$ is the L_2-norm in $(DR^\bullet(M))_2$).

The estimate (2.25) follows from the continuity (in v) of the families $d\Pi_a^\bullet(v)$ and $\Pi_a^{\bullet+1}(v)$ since the following operator norms tend to zero as $v \to v_0$:

$$\|d\Pi_a^\bullet(v) - d\Pi_a^\bullet(v_0)\|_2 \to +0, \quad \|\Pi_a^{\bullet+1}(v) - \Pi_a^{\bullet+1}(v_0)\|_2 \to +0 \ .$$

Indeed, for an arbitrary $w \in W_a^\bullet(v_0)$ we have $d\Pi_a^\bullet w = dw$. Hence the estimates

$$\|\Pi_a^{\bullet+1}(v)dw - dw\|_2 \leqq \|\Pi_a^{\bullet+1}(v) - \Pi_a^{\bullet+1}(v_0)\|_2 \cdot \|dw\|_2$$
$$\leqq C \cdot \|\Pi_a^{\bullet+1}(v) - \Pi_a^{\bullet+1}(v_0)\|_2 \cdot \|w\|_2$$

are true because the differential $d : W_a^\bullet(v_0) \to W_a^{\bullet+1}(v_0)$ of a finite complex of finite-dimensional spaces is bounded (with respect to the Hilbert norm induced from $(DR^\bullet(M))_2$).

For each $v \in \mathbb{R}^2 \setminus (0,0)$ the combinatorial cochain complex $(C^\bullet(X_v, V), d)$ (with $V := X \cap Z$) is defined by the v-transmission condition (1.58). A homomorphism of the integration of forms from $W_a^\bullet(v)$ over the simplexes of X

$$R_v(a) : (W_a^\bullet(v), d) \to (C^\bullet(X_v, V), d) \qquad (2.26)$$

is also defined for all $v \in \mathbb{R}^2 \setminus (0,0)$. For every such v the following variant of the de Rham theorem holds.

Proposition 2.3. *$R_v(a)$ is a quasi-isomorphism.*

Proof. 1. Let $R_v : (DR^\bullet(M_v, Z), d) \to (C^\bullet(X_v, V), d)$ be the integration homomorphism of pairs of forms $(\omega_1, \omega_2) \in DR^\bullet(M_v, Z)$ over the simplexes of $X_j \setminus V_j$. Then R_v is a *quasi-isomorphism*.[17]

Indeed, in the commutative diagram (1.59) the left and the right vertical arrows are quasi-isomorphisms according to the de Rham theorem for a closed manifold N and for manifolds M_1 and M_2 with smooth boundaries. (The proof of the latter

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[17] This assertion claims that the analogy of the classical de Rham theorem is true in the case of the v-transmission interior boundary conditions. The classical de Rham theorem for smooth closed manifolds was proved in [dR1] (see also [dR4], Ch. IV, [W], Ch. IV, Sect. 29). The explicit isomorphism between the Čech cohomology for a good cover of a smooth closed M and the de Rham cohomology of M is defined with the help of the de Rham-Čech complex ([BT], Ch. II Sect. 9).

one is given in [RS], Proposition 4.2.) The cohomology exact sequences provide us with the commutative diagram

$$
\begin{array}{ccccccc}
\overset{\partial_D}{\to} & H^*\left(\bigoplus_{k=1,2} DR^\bullet(M_k, N \cup Z_k)\right) & \to & H^*(DR^\bullet(M_v, Z)) & \underset{(r_v)_*}{\to} & H^*(DR^\bullet(N)) & \to \\
 & \quad\downarrow R_* & & \downarrow (R_v)_* & & \downarrow R_* & \\
\overset{\partial_c}{\to} & H^*\left(\bigoplus_{k=1,2} C^\bullet(X_k, W \cup V_k)\right) & \underset{j^*}{\to} & H^*(C^\bullet(X_v, V)) & \overset{(r_{v,c})}{\to} & H^*(C^\bullet(W)) & \to
\end{array}
$$

$$(2.27)$$

with the exact rows, where the vertical arrows R_* on the left and on the right are isomorphisms (according to the de Rham theorem) and where $\partial_D = \partial_c$ under the identifications R_*. Hence $(R_v)_*$ is also an isomorphism.

The exactness of the top row in (2.27) can be interpreted and proved as follows. The sheaf $F_v^\bullet := DR_v^\bullet$ ($v = (\alpha, \beta) \in \mathbb{R}^2 \setminus (0,0)$) of germs (ω_1, ω_2) of pairs of C^∞-forms ω_j on M_j such that[18] $\alpha i_1^* \omega_1 = \beta i_2^* \omega_2$ (here i_j^* are the geometrical restrictions from M_j to $N \hookrightarrow \partial M_j$) is a c-soft sheaf. (The latter notion means that the restriction $\Gamma(M, F_v^j) \to \Gamma(K, i_K^{-1} F_v^j)$ is surjective for any compact $i_K : K \hookrightarrow M$, [KS], Definition 2.5.5.) The sheaf F_v is c-soft since appropriate smooth partitions of unity exist on M. The sequence of complexes of global sections

$$0 \to \Gamma_c(M \setminus N, F_v^\bullet) \to \Gamma(M, F_v^\bullet) \to \Gamma(M, i_{N,*} i_N^{-1} F_v^\bullet) \to 0$$

(here $i_N : N \hookrightarrow M$) has the terms which possess the following properties:

1) $\Gamma(M, F_v^\bullet) = DR^\bullet(M_v, Z)$,

2) $\Gamma_c(M \setminus N, F_v^\bullet)$ is a subcomplex of $\bigoplus_{k=1,2} DR^\bullet(M_k, N \cup Z_k)$ and its natural inclusion is a quasi-isomorphism. Indeed, if $\omega \in DR^\bullet(M_k, N \cup Z_k)$ is a closed form then $\omega = dv$ in a neighborhood of N in M_k (where v is a smooth form with the zero geometrical restriction to N). So $\omega - d(\varphi v) = 0$ in some neighborhood of N in M_k (φ is an appropriate cutting function). We have $\Gamma_c(M \setminus N, F_v^\bullet) = \Gamma(M, j_! j^{-1} F_v^\bullet)$, where $j : M \setminus N \hookrightarrow M$ and $j_!$ is the direct image with proper supports, $j^{-1} F_v^\bullet \simeq DR^\bullet|_{M \setminus N}$. The sheaf $j_! j^{-1} F_v^\bullet$ is c-soft according to [KS], Proposition 2.5.7.

3) $\Gamma(M, i_{N,*} i_N^{-1} F_v^\bullet)$ has a natural homomorphism $q_v := r_v \circ (i_1^*, i_2^*)$ onto $DR^\bullet(N)$ (where r_v is defined in (1.15)) and q_v is a quasi-isomorphism. In fact, if the form $u = dt \wedge \omega_N(t)$ on $I \times N$ is closed then it is exact because then $d_N \omega_N(t) = 0$ and so $u = d \int_0^t \omega_N(\tau) d\tau$. (Here t is the coordinate on I and $t = 0$ is the equation of $N = 0 \times N \hookrightarrow I \times N$, $0 \in I \setminus \partial I$.) Hence q_v is a quasi-isomorphism. (This assertion follows also from the Poincaré lemma.) The sheaf $i_{N,*} i_N^{-1} F_v^\bullet$ is c-soft by [KS], Proposition 2.5.7.

For a compact manifold M the category of c-soft sheaves on M is injective with respect to the functor of global sections $\Gamma(M; \cdot)$ ([KS], Proposition 2.5.10). The complex $F_v^\bullet$ is a c-coft resolvent of a constructible sheaf ([KS], Chapter VIII) $\mathbb{C}_v$ on M, which is isomorphic to $\mathbb{C}_{M \setminus N}$ on $M \setminus N$ and to $\mathbb{C}_N$ on N (where $\mathbb{C}_X$ is a constant sheaf on X), and the gluing map for $\mathbb{C}_v$ is $|v|^{-1/2}(\alpha, \beta) : \mathbb{C}_N \to i_N^{-1} j_* \mathbb{C}_{M \setminus N} = \mathbb{C}_N \oplus \mathbb{C}_N$ (i.e., $c \to |v|^{-1}(\beta c, \alpha c)$). The complexes $j_! j^{-1} F_v^\bullet$ and $i_{N,*} i_N^{-1} F_v^\bullet$ are c-soft resolvents of constructible sheaves $j_! j^{-1} \mathbb{C}_v = j_! \mathbb{C}_{M \setminus N}$ and of $i_{N,*} i_N^{-1} \mathbb{C}_v$. (The latter one is isomorphic to $i_{N,*} \mathbb{C}_N$ under r_v.) So the exactness of the cohomology sequence in the top row of (2.27) follows from [KS], (2.6.33), Remark 2.6.10.

---

[18] It is supposed that ω_j has the zero geometrical restriction to Z_k (at points $x \in Z_k \subset \partial M_k$).

2. The projection operator $p_{\mathcal{H}} : DR^\bullet(M_v, Z) \hookrightarrow (DR^\bullet(M))_2 \to \mathrm{Ker}(\Delta_v^\bullet)$ provides us with the isomorphism $p_{\mathcal{H}*} : H^\bullet(DR(M_v, Z)) \to \mathrm{Ker}(\Delta_v^\bullet)$ (by Lemma 1.1). So the inclusion $i_a : (W_a^\bullet(v), d) \hookrightarrow (DR^\bullet(M_v, Z), d)$ is a quasi-isomorphism and $(i_a)_* : \mathrm{Ker}(\Delta_v^\bullet) \widetilde{\to} H^\bullet(DR(M_v, Z))$ is equal to $(p_{\mathcal{H}*})^{-1}$ (since $p_{\mathcal{H}} i_a = \mathrm{id}$ on $\mathrm{Ker}(\Delta_v^\bullet)$). From an obvious equality $R_v(a) = R_v i_a$ it follows that $R_v(a)$ is a quasi-isomorphism. $\square$

Thus the assertion of Lemma 2.4 can be applied to the bicomplex (2.26). The result is as follows.

Corollary 2.3. *The equality holds*:

$$T_0(C^\bullet(X_v, V)) / \|1\|^2_{T_0(\mathrm{Cone}^\bullet R_v(a))} = T_0(W_a^\bullet) . \tag{2.28}$$

The identifications $\varphi_v^{\mathrm{an}}(a)$ (for an arbitrary $a > 0$) and φ_v^{an} are defined such that the following norms on $\mathbf{Det}(M, N, Z)$ are equal:

$$\varphi_v^{\mathrm{an}}(a) \circ \| \cdot \|^2_{\det W_a^\bullet(v)} = \varphi_v^{\mathrm{an}} \circ T_0(W_a) . \tag{2.29}$$

Hence, as it follows from (2.28), (2.29), we have

$$\varphi_v^{\mathrm{an}}(a) \circ \| \cdot \|^2_{\det W_a^\bullet(v)} = (\varphi_v^{\mathrm{an}} \circ T_0(C^\bullet(X_v, V))) \circ (\|1\|^2)^{-1}_{T_0(\mathrm{Cone}^\bullet(R_v(a)))} . \tag{2.30}$$

Proposition 2.4. *The factor* $(\|1\|^2_{T_0(\mathrm{Cone}^\bullet(R_v(a)))})^{-1}$ *in* (2.30) *is a continuous function of* $v \in U_{v_0}(a)$.

Proof. The complex $\mathrm{Cone}^\bullet(R_v(a))$ is acyclic according to Proposition 2.3. Its scalar analytic torsion

$$\|1\|^2_{T_0 \mathrm{Cone}^\bullet(R_v(a)))} := \exp\left(\sum_{j \geqq -1} (-1)^j j \zeta_j'(0) \right) \tag{2.31}$$

is defined as in (1.1) by the ζ-functions of the "Laplacians" $L_v := d_v^* d_v + d_v d_v^*$ of the complex $(\mathrm{Cone}^\bullet(R_v(a)), d_v)$.[19] Since the complex $\mathrm{Cone}^\bullet(R_v(a))$ is acyclic, we see that these Laplacians are positive definite. (So they have the zero kernels.) Their determinants $\det(\Delta_v^\bullet)$ are continuous positive functions of v on $U_{v_0}(a)$ (and so the expression on the right in (2.31) is a continuous function of $v \in U_{v_0}(a)$). The latter statement is derived as follows.

Proposition 2.5. *Let* $m \in \mathbb{Z}_+$ *and* $m \geqq m_0 := 1 + \min\{k \in \mathbb{Z}_+ : 4k \geqq \dim M\}$. *Then there exists a positive constant* $C = C(M, N, Z, g_M)$ *independent of* $v \in \mathbb{R}^2 \setminus (0,0)$ *(and of m also) such that the following estimate holds uniformly with respect to* $x \in \overline{M}_1 \cup \overline{M}_2$:

$$|\omega(x)|^2 < C \sum_{i=0}^m \|\Delta_v^i \omega\|_2^2 \tag{2.32}$$

---

[19] The spaces $W_{(v)}^\bullet$ are equipped with the Hilbert structure from $((DR^\bullet(M))_2, g_M)$. The spaces $C^\bullet(X_v, V) \subset C^\bullet(X_1) \oplus C^\bullet(X_2)$ are equipped with the Hilbert structure defined by the basic cochains in $\oplus C^\bullet(X_k)$ and $\mathrm{Cone}^\bullet(R_v) = W_a^{\bullet-1}(v) \oplus C^\bullet(X_v, V)$ is the orthogonal direct sum of Hilbert spaces.

for all ω such that[20]

$$\omega \in DR^\bullet(M_v, Z), \quad \omega \in D(\Delta_v^\bullet), \quad \Delta_v \omega \in D(\Delta_v^\bullet), \dots, \Delta_v^m \omega \in D(\Delta_v^\bullet) . \tag{2.33}$$

(Here $|\omega(x)|^2$ is the norm at $\wedge^\bullet T_x M$ defined by g_M and $\| \cdot \|_2^2$ is the L_2-norm in $(DR^\bullet(M)_2)$.)

Corollary 2.4. *If $\omega \in W_a^\bullet(v)$ then $w \in D(\Delta_v^m)$ for an arbitrary $m \in \mathbb{Z}_+$. So the following estimate holds uniformly with respect to $x \in \overline{M}_1 \cup \overline{M}_2$ and to $v \in \mathbb{R}^2 \setminus (0,0)$*

$$|w(x)|^2 < C_1 \|w\|_2^2 , \tag{2.34}$$

where $C_1 = C_1(M, N, Z, g_M) > 0$.

The graded Hilbert space $C^\bullet(X_v, V)$ is isomorphic to the direct sum

$$C^\bullet(X_v, V) = C^\bullet(X, N_X \cup V) \oplus C^\bullet(N_X) , \tag{2.35}$$

where $V := X \cap Z$, $N_X := X \cap N$, $C^\bullet(X, N_X \cap V)$ is a graded linear subspace of $C^\bullet(X_v, V)$ (with respect to the natural inclusion), and the inclusion $j_v : C^\bullet(N_X) \hookrightarrow C^\bullet(X_v, V) \subset \bigoplus_i C^\bullet(X_i)$ is defined as $j_v := (\alpha^2 + \beta^2)^{-1}(\beta \, \mathrm{id}, \alpha \, \mathrm{id})$. The space on the right in (2.35) is independent of v. Hence (2.35) provides us with the isometric identification of the graded Hilbert spaces

$$p_v : C^\bullet(X_{v_0}, V) \overrightarrow{\sim} C^\bullet(X_v, V) . \tag{2.36}$$

Corollary 2.5. *Let $v \in U_{v_0}(a)$ be sufficiently close to v_0. Let $W_a^\bullet(v)$ be identified with $W_a^\bullet(v_0)$ by $\Pi_a^\bullet(v) : W_a^\bullet(v_0) \to W_a^\bullet(v)$. Let $C^\bullet(X_{v_0}, V)$ be identified with $C^\bullet(X_v, V)$ by p_v (2.36). Then the estimate (2.34) implies that for such v the family of homomorphisms of the integration over the simplexes of X*

$$R_v^\bullet(a) : (W_a^\bullet(v), d) \to (C^\bullet(X_v, V), d_c)$$

is a continuous in v family of quasi-isomorphisms between finite complexes of finite-dimensional Hilbert spaces.

Let $f_v : (F^\bullet(v), d_F(v)) \to (K^\bullet(v), d_K(v))$ be a family of homomorphisms between finite complexes of finite-dimensional Hilbert spaces. Let the trivialization of these two families of complexes be defined by the identifications of the graded linear spaces

$$\Pi_v : F^\bullet(v_0) \to F^\bullet(v), \quad p_v : K^\bullet(v_0) \to K^\bullet(v) .$$

Let these identifications be chosen such that f_v becomes a continuous family of the homomorphisms

$$f_v : (F^\bullet, d_F(v)) \to (K^\bullet, d_K(v))$$

between the continuous families of complexes with the fixed underlying graded linear spaces $F^\bullet := F^\bullet(v_0)$ and $K^\bullet := K^\bullet(v_0)$. Let the Hilbert structures of F^j and K^j are continuous functions of v for all j. In this case, f_v is called *a continuous family*. Then the following assertion is true

Proposition 2.6. *Let f_v be a continuous family. Then the determinants $\det(L_v^\bullet)$ of the Laplacians $L_v^\bullet = d_v^* d_v + d_v d_v^*$ on $(\mathrm{Cone}^\bullet f_v, d_v)$ are continuous functions of v.*

---

[20] The domain of definition of $D(\Delta_v^\bullet)$ for $\Delta_v^\bullet$ is defined by (1.27) and (1.26).

Proof. The operator d_v^* adjoint to the differential d_v of $\mathrm{Cone}^\bullet f_v$ (with respect to the Hilbert structure $\mathrm{Cone}^\bullet f_v = F^{\bullet-1} \oplus K^\bullet)^{21}$ is defined on the whole finite-dimensional space $\mathrm{Cone}^\bullet f_v$. Since d_v tends to d_{v_0} (for instance, in the operator norm[22]) as $v \to v_0$, we see that d_v^* also tends to $d_{v_0}^*$. Thus $L_v^\bullet \to L_{v_0}^\bullet$ as $v \to v_0$ and $\det L_v^\bullet \to \det L_{v_0}^\bullet$ (since the space $\mathrm{Cone}^\bullet f_v$ is finite-dimensional). $\square$

Corollary 2.6. *The functions* $\det(L_v^\bullet)$ *of* v *for* $f_v = R_v^\bullet(a)$ *are continuous and positive.*

The positivity of $\det(L_v^\bullet)$ is equivalent to the acyclicity of $(\mathrm{Cone}^\bullet R_v(a), d_v)$ (where $d_v := d_{\mathrm{Cone}(R_v(a))}$).
    Proposition 2.4 is proved. $\square$

Remark. 2.4. Propositions 2.2, 2.5, and Corollary 2.5 claim that under the identifications (2.36) and $\Pi_a^\bullet(v)$, the Hilbert structures on $\det \mathrm{Cone}^\bullet(R_v(a))$ and the differentials d_v in $\mathrm{Cone}^\bullet(R_v(a))$ are continuous in v at v_0. Hence the analytic torsion norms $\| \cdot \|_{T_0(\mathrm{Cone}^\bullet(R_v(a)))}^2$ on $\mathbb{C} = \det H^\bullet(\mathrm{Cone}(R_v(a))) = \det 0$ are also continuous in v at v_0.
    According to (1.61) we have $\varphi_v^{\mathrm{an}} = \varphi_v^c$, where φ_v^c is defined by the bottom row of the commutative diagram (1.59). So the continuity of the norm $\varphi_v^{\mathrm{an}}(a) \circ \| \cdot \|_{\det W_a^\bullet(v)}^2$ on $\mathbf{Det}(M, N, Z)$ can de deduced from (2.30) and from the following lemma.

Lemma 2.5. *The norm* $\varphi_v^c T_0(C(X_v, V))$ *on* $\mathbf{Det}(M, N, Z)$ *does not depend on* $v \in U_{v_0}(a)$.

    The continuity in v of the norm $\varphi_v^{\mathrm{an}} T_0(M_v, Z)$ on $\mathbf{Det}(M, N, Z)$ follows from (2.20), (2.21), and from the continuity of the norm $\varphi_v^{\mathrm{an}}(a) \circ \| \cdot \|_{\det W_a^\bullet(v)}^2$. (The latter assertion is proved above.) The equality (1.12) holds with $c_0(v)$ which is constant and positive on each connected component U_j of U (2.3). Because the norm $\varphi_v^{\mathrm{an}} T_0(M_v)$ on $\mathbf{Det}(M, N, Z)$ is continuous in $v \in \mathbb{R}^2 \setminus (0,0)$, the equality (1.12) holds for all such v with c_0 *independent of* v. Theorem 1.1 follows from (1.12) and from the assertion of Lemma 1.2. $\square$

Remark. 2.5. It is not important for the proofs of Theorem 1.1 and of (1.12) that the family of finite-dimensional complexes $(C^\bullet(X_v, V), d_c)$ in (2.30) is of a combinatorial nature. It is enough for the proof to have a family of finite-dimensional complexes $(F_v^\bullet, d_F)$ which are defined locally in v (i.e., for v in a neighborhood of an arbitrary $v_0 \in \mathbb{R}^2 \setminus (0,0)$) together with the data as follows. Continuous families of quasi-isomorphisms $f_v(a) : (W_a^\bullet(v), d) \to (F_v^\bullet, d_F)$ and of Hilbert structures h_v on $F_v^\bullet$ are defined. A family $(F_v^\bullet, h_v)$ may depend on a and on v_0 but it has to possess the property as follows. The norm $\varphi_v^{\mathrm{an}} \circ (f_v(a)_*)^{-1} \circ T_0(F_v^\bullet, h_v)$ on $\mathbf{Det}(M, N, Z)$ is continuous in v at v_0. (Here $f_v(a)_* : \det H^\bullet(M_v, Z) \rightleftarrows \det H^\bullet(F_v, d_F)$ is the induced identification.)

---

[21] $\mathrm{Cone}^\bullet f$ is the direct sum of Hilbert spaces $F^{\bullet+1} \oplus K^\bullet$ (with the Hilbert structures on $F^{\bullet+1}$ and $K^\bullet$ depending continuously on v).
[22] As $\mathrm{Cone}^\bullet f_v$ is a finite-dimensional space, the weak convergence of the operators acting in it is equivalent to the convergence with respect to the operator norm.

Proof of Lemma 2.5. Let ψ_v be the identification of the determinant lines defined by the bottom row of the commutative diagram (1.59):

$$\psi_v : \det C^\bullet(X_v, V) \xrightarrow{\sim} \left(\bigotimes_{k=1,2} \det C^\bullet(X_k, W \cup V_k) \right) \otimes \det C^\bullet(W) =: \mathbf{Det}\, C^\bullet(X, V, W)$$

(where V is the induced smooth triangulation $Z \cap X$ of $Z \subset \partial M$, $V_k := V \cap \partial M_k$ and $W := X \cap N = N_X$). The following diagram is commutative:

$$
\begin{array}{ccc}
\det C^\bullet(X_v, V) & \xrightarrow[\sim]{\psi_v} & \mathbf{Det}\, C^\bullet(X, V, W) \\[4pt]
{\scriptstyle d_c} \downarrow {\scriptstyle \wr} & & {\scriptstyle d_c} \downarrow {\scriptstyle \wr} \\[4pt]
\det H^\bullet(X_v, V) & \xrightarrow[\sim]{\varphi_v^c} & \mathbf{Det}(X, V, W) \\[4pt]
\| {\scriptstyle R} & & \| {\scriptstyle R} \\[4pt]
\det H^\bullet(M_v, Z) & \xrightarrow[\sim]{\varphi_v^{\mathrm{an}}} & \mathbf{Det}(M, Z, N)
\end{array}
\qquad (2.37)
$$

(The determinant lines on the right in (2.37) are defined by (1.54). The identification d_c on the right in (2.37) is a triple tensor product of the identifications induced by d_c on $C^\bullet(X_k, W \cup V_k)$ and on $C^\bullet(W)$. The identification R is defined by the integration over the simplexes of X.) The commutativity of the diagram (2.37) is equivalent to the definition (1.60) of φ_v^c. Since the identification d_c on the right in (2.37) is independent of v, we see that the statement of Lemma 2.5 is a consequence of the following proposition.

Proposition 2.7. *The identification ψ_v in (2.37) is an isometry between the combinatorial norm $\| \cdot \|_{\det C^\bullet(X_v, V)}^2$ and the triple tensor product of the combinatorial norms on $\det C^\bullet(X_k, W \cup V_k)$ $(k = 1, 2)$ and on $\det C^\bullet(W)$.*

(The Hilbert structures on $\bigoplus_{k=1,2} C^\bullet(X_k, V_k)$ and on $C^\bullet(W)$ are defined by the orthonormal basis of the basic cochains.)

Proof. Let $\rho_{v,c} : C^\bullet(W) \to C^\bullet(X_v, V)$ be defined by (1.63). Then $r_{v,c}\rho_{v,c} = \mathrm{Id}$ and $\rho_{v,c}$ is an isometry onto $\mathrm{Im}(\rho_{v,c})$ (relative to the Hilbert structures, defined above). The subspace $\mathrm{Im}(\rho_{v,c})$ is the orthogonal complement to $\mathrm{Im}\, j(\bigoplus_{k=1,2} C^\bullet(X_k, W \cup V_k))$ in $C^j(X_v, V)$ and j is an isometry onto $\mathrm{Im}\, j$. (Here, $r_{v,c}$ and j are the same as in the bottom row of (1.59).)[23]
Thus Lemma 2.5 is proved. $\square$

2.2.1. Uniform Sobolev inequalities for v-transmission interior boundary conditions. Proof of Proposition 2.5. Let $I \times N \subset M$ (where $I = [-1, 1]$) be a neighborhood of $N = 0 \times N \subset M$ and let g_M be a direct product metric on $I \times N$. Proposition 2.5 is a consequence of the assertions as follows.

Proposition 2.8. *The inequality* (2.32) *holds uniformly with respect to $v \in \mathbb{R}^2 \setminus (0,0)$ for all $\omega \in DR^\bullet(M_v)$ of the class* (2.33) *and such that* $\mathrm{supp}\,\omega \subset [-4/5, 4/5] \times N \subset M$.

---

[23] This proposition is essentially equivalent to Proposition 1.5, Sect. 1.

Proposition 2.9. *The inequality* (2.32) *holds for all* $\omega \in DR^\bullet(M, Z)$ *such that* $\mathrm{supp}\,\omega \subset M \setminus ([-1/3, 1/3] \times N)$ *and such that*[24]

$$\omega \in D(\Delta_{M,Z}), \Delta\omega \in D(\Delta_{M,Z}), \ldots, \Delta^m \omega \in D(\Delta_{M,Z}) \,. \tag{2.38}$$

The last assertion is well known ([Ch], Sect. 5).

Let f be a smooth function on $M, 0 \leqq f \leqq 1, f \equiv 1$ on $[-1/2, 1/2] \times N$ and $f \equiv 0$ on $M \setminus ([-3/4, 3/4] \times N)$ The $2m$-Sobolev norm on the right in (2.32), defined as

$$\|\omega\|_{(2m)}^2 := \sum_{k=0}^{m} \|\Delta_v^k \omega\|_2^2 \,,$$

is equivalent uniformly in $v \in \mathbb{R}^2 \setminus (0, 0)$ (i.e., with constants $c_3, c_4 > 0$ independent of v and of ω) to the norm[25] $\|\omega\|_{(2m),f}^2 := \sum_{k=0}^{m}(\|\Delta_v^k(f\omega)\|_2^2 + \|\Delta^k((1-f)\omega)\|_2^2)$:

$$c_3 \|\omega\|_{(2m)}^2 < \|\omega\|_{(2m),f}^2 \leqq c_4 \|\omega\|_{(2m)}^2 \,. \tag{2.39}$$

It is enough to verify the upper estimate (with c_4 independent of v) for $f\omega$. It is true for $m = 1$, since the estimate holds:

$$\|\Delta_v(f\omega)\|_2^2 \leqq C_1 \cdot (\|\Delta_v\omega\|_2^2 + \|\omega\|_2^2 + \|d\omega\|_2^2 + \|\delta\omega\|_2^2)$$

$$\leqq 3/2 C_1(\|\Delta_v\omega\|_2^2 + \|\omega\|_2^2) \,,$$

where C_1 depends on f but it is *independent of* v. Hence the following estimate holds for $\omega \in D(\Delta_v^k)$ (with C_2 *independent of* v *and of* ω):

$$\|\Delta_v^k(f\omega)\|_2^2 \leqq C_2 \cdot (\|\Delta_v^k\omega\|_2^2 + \|\Delta_v^{k-1} - \omega\|_2^2 + \cdots + \|\omega\|_2^2) \,.$$

The upper estimate is done. Thus Proposition 2.5 follows from (2.39) and from Propositions 2.8 and 2.9.

Proof of Proposition 2.8. The form ω on $I \times N$ is the sum $\omega_0 + \omega_1$, where ω_i is an i-form in the direction of I (where $I = [-1, 1]$). It is enough to prove the inequality (2.32) separately for ω_0 and for ω_1. Let us prove it for ω_0. For $v = (\alpha, \beta) \in \mathbb{R}^2 \setminus (0, 0)$ the Green function $G(v)$ for the Laplacian $\Delta_{v,I}$ on functions on I with the v-transmission boundary condition at $0 \in I$ and the Dirichlet boundary conditions on $\partial I = \{-1, 1\}$ is given by the kernel

$$(G_I(v))_{x_1, x_2} = g_{x_1, x_2} + \frac{\beta^2 - \alpha^2}{\alpha^2 + \beta^2} g_{-x_1, x_2} \quad \text{for } x_1, x_2 \in Q_1 = [-1, 0] \,,$$

$$(G_I(v))_{x_1, x_2} = g_{x_1, x_2} + \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} g_{-x_1, x_2} \quad \text{for } x_1, x_2 \in Q_2 = [0, 1] \,,$$

$$(G_I(v))_{x_1, x_2} = \frac{2\alpha\beta}{\alpha^2 + \beta^2} g_{x_1, x_2} \quad \text{for } x_1, x_2 \text{ from different } Q_k \,. \tag{2.40}$$

---

[24] For ω with $\mathrm{supp}\,\omega \subset M \setminus N$ the conditions (2.33) and (2.38) are equivalent. The domain $D(\Delta_{M,Z})$ of $\Delta_{M,Z}$ consists of smooth forms on M with the Dirichlet boundary conditions on Z and the Neumann ones on $\partial M \setminus Z$.

[25] The lower estimate with c_3 in (2.39) is obvious. Note that $\mathrm{supp}((1 - f)\omega) \subset M \setminus ([-1/3, 1/3] \times N)$. Then the upper estimate with a constant c_4' for $\|(1 - f)\omega\|_{(2m)}$ by $\|\omega\|_{(2m)}$ is well known ([Hö], Appendix B and Proposition 20.1.11).

Here, g_{x_1,x_2} is the Green function for the Laplacian on functions on I with the Dirichlet boundary conditions on ∂I:

$$g_{x_1,x_2} = \begin{cases} c \cdot (x_2 + 1)(1 - x_1), & -1 \leqq x_2 \leqq x_1 \leqq 1, \\ c \cdot (x_1 + 1)(1 - x_2), & -1 \leqq x_1 \leqq x_2 \leqq 1, \end{cases} \qquad (2.41)$$

where $c \neq 0$ is a constant.

It follows from (2.40) and (2.41) that $G_I(v)$ has a continuous kernel on $\overline{Q}_{r_1} \times \overline{Q}_{r_2}$ and that it is estimated uniformly with respect to $v \in \mathbb{R}^2 \setminus (0,0)$ and to x_1, x_2:

$$\sup_{x_1, x_2, v} |(G_I(v))_{x_1, x_2}| < c_2. \qquad (2.42)$$

Since supp $\omega_0 \subset (I \setminus \partial I) \times N$ and since the Laplacian $\Delta_{v,I}$ has the zero kernel on functions with the Dirichlet boundary conditions on ∂I, we have

$$\omega_0 = (\mathrm{Id}_I \otimes \Pi_0^\bullet(N))\omega_0 + G_I(v) \otimes G_N^{m_2}((\Delta_{v,I} \otimes \Delta_N^{m_2})\omega_0), \qquad (2.43)$$

where G_N is the Green function for $\Delta_N^\bullet$ and where $\Pi_0^\bullet(N)$ is the orthogonal projection operator in $(DR^\bullet(N))_2$ onto Ker $\Delta_N^\bullet$. The operator $G_N^{m_2}$ on a closed Riemannian manifold (N, g_N) has a square-integrable kernel (relative to the second argument) for $m_2 > (n-1)/4$ (where $n - 1 = \dim N$) and it has a continuous on $N \times N$ kernel for $m_2 > (n-1)/2$.

The following estimate holds uniformly with respect to $v \in \mathbb{R}^2 \setminus (0,0)$ for any $m_2 \in \mathbb{Z}_+$, $m_2 > (n-1)/4$. From (2.43), (2.42), and from the Cauchy inequality we have

$$|\omega_0(x)|^2 \leqq c_5(\|\omega_0\|_2^2 + \|(\Delta_{v,I} \otimes \mathrm{id})\omega_0\|_2^2 + \|(\Delta_{v,I} \otimes \Delta_N^{m_2})\omega_0\|_2^2). \qquad (2.44)$$

Indeed, the following two Banach norms on the finite-dimensional space Ker $\Delta_N^\bullet$

$$\|h\|_B^2 := \max_{x \in N}|h(x)|^2 \quad \text{and} \quad \|h\|_{2,N}^2$$

are equivalent. So we get (where $x = (x_1, x_N) \in I \times N$ and $I = [-1,1]$):

$$|((\mathrm{id}_I \otimes \Pi_0^\bullet(N))\omega_0)(x)|^2 \leqq \|(\Pi_0^\bullet(N)\omega_0(x_1, *))\|_B^2 \leqq c_6\|\omega_0(x_1, *)\|_{2,N}^2, \qquad (2.45)$$

$$\|\omega_0(x_1, *)\|_{2,N}^2 \leqq 2\sup_{x_2}|(G_I(v))_{x_1,x_2}|^2 \cdot \|(\Delta_{v,I} \otimes \mathrm{id})\omega_0\|_{2,M}^2$$

$$\leqq 2c_2^2\|(\Delta_{v,I} \otimes \mathrm{id})\omega_0\|_2^2. \qquad (2.46)$$

The following estimate is obtained by the similar method:

$$|((G_I(v) \otimes G_N^{m_2})(\Delta_{v,I} \otimes \Delta_N^{m_2})\omega_0)(x_1, x_N)|^2$$

$$\leqq 2c_2^2\sup_{y_1}|(G_N^{m_2})_{y_1,*}\|_{2,N}^2 \cdot \|(\Delta_{v,I} \otimes \Delta_N^{m_2})\omega_0\|_{2,M}^2. \qquad (2.47)$$

Hence the estimate (2.44) holds for ω_0 (even without the first term on the right in (2.44)), as it follows from (2.43), (2.45), (2.46) and (2.47).

Since $\Delta_v = \mathrm{id}_I \otimes \Delta_N + \Delta_v(I) \otimes \mathrm{id}_N$ and since Δ_N and $\Delta_{v,I}$ are nonnegative self-adjoint operators, we have for $m_2 \in \mathbb{Z}_+$:

$$\|(\Delta_{v,I} \otimes \Delta_N^{m_2})\omega_0\|_2 \leqq \|\Delta_v^{m_2+1}\omega_0\|_2. \qquad (2.48)$$

The inequality (2.32) for ω_0 follows from (2.44) and (2.48). For ω_1 the analogous to (2.43) equality holds:

$$\omega_1 = (\Pi_0^1(I_v) \otimes (\mathrm{id}_N - \Pi_0^{\bullet-1}(N)))\omega_1 + ((\mathrm{id}_I - \Pi_0^1(I_v)) \otimes \Pi_0^{\bullet-1}(N))\omega_1$$

$$+ (\Pi_0^1(I_v) \otimes \Pi_0^{\bullet-1}(N))\omega_1 + (G_I(v) \otimes G_N^{m_2})(\Delta_{v,I} \otimes \Delta_N^{m_2}\omega_1) , \qquad (2.49)$$

where $\Pi_0^1(I_v)$ is the projection operator of $(DR^1(I))_2$ onto the one-dimensional space $c \cdot dx_1$ and $G_I(v)_1$ is the Green function for the Laplacian $\Delta_{v,I}$ on $DR^1(I_v)$ (with the Dirichlet boundary conditions on $\partial I = \{-1, 1\}$ and with the v-transmission boundary conditions at 0). The kernel $G_I(v)_1$ is continuous on $\overline{Q}_{r_1} \times \overline{Q}_{r_2}$ because it can be written in a form similar to (2.40). It is written through the Green function g_1 of Δ_I on $DR^1(I)$ with the Dirichlet boundary conditions on ∂I (the kernel $(g_1)_{x_1,x_2}$ of g_1 is continuous on $I \times I$). Hence the second term on the right in (2.49) is estimated similarly to (2.45) and to (2.46). The kernel $\Pi_0^1(I_v)_{x_1,x_2}$ is expressed in a form analogous to (2.40) through the kernel $2^{-1}dx_1 \otimes dx_2$ on $I \times I$ (corresponding to $\Pi_0^1(I_{1,1})$). So the kernel of $\Pi_0^1(I_v)$ is continuous on $\overline{Q}_{r_1} \times \overline{Q}_{r_2}$, and it satisfies the estimate (2.42) (with the upper bound c). The first and the third terms in (2.49) are estimated as follows:

$$|(\Pi^1(I_v) \otimes (\mathrm{id}_N - \Pi_0^{\bullet-1}(N)))\omega_1(x_1, x_N)|^2$$

$$\leqq 2c^2 \sup_{y_1} \|(G_N^{m_2})_{y_1,*}\|_{2,N}^2 \|(\mathrm{id} \otimes \Delta_N^{m_2})\omega_1\|_{2,M}^2 ,$$

$$|(\Pi^1(I_v) \otimes \Pi_0^{\bullet-1}(N))\omega_1(x)|^2 \leqq 2c^2 c_6 \|\omega_1\|_{2,M}^2 .$$

Hence the estimate (2.44) holds uniformly with respect to $v \in \mathbb{R}^2 \setminus (0,0)$ for any $m \in \mathbb{Z}_+$, $m \geqq m_0 := 1 + \min\{k \in \mathbb{Z}_+, 4k \geqq n\}$. Thus Proposition 2.8 is proved. $\square$

2.3. Actions of the Homomorphisms of Identifications on the Determinant. Proof of Lemma 2.3.

The most simple method to compute the action of $g_{v,*}$ on $\mathrm{Det}(M, N, Z)$[26] is to obtain the expression for the action of v_{v*}^c on the determinant line $\mathrm{Det}\,C^\bullet(X, V, W)$ (1.54), induced by the identifications of the corresponding cochain complexes $v_v^c = v_{vv_0}^c : C^\bullet(X_{v_0}, V) \to C^\bullet(X_v, V)$ (where $v_v^c(c_1, c_2) := (c_1, (k_v/k_{v_0})c_2)$ for $v, v_0 \in U$ (2.3)), and then to use Proposition 2.10 below. The action of v_{v*}^c is defined by identifications ψ_v and ψ_{v_0}, where $\psi_v : \det C^\bullet(X_v, V) \overset{\sim}{\rightleftarrows} \mathrm{Det}\,C^\bullet(X, V, W)$ are defined by the exact sequence in the bottom row of the diagram (1.59). The following diagram of the identifications is commutative:

$$
\begin{array}{ccc}
\det C^\bullet(X_{v_0}, V) & \overset{v_v^c}{\underset{\sim}{\longrightarrow}} & \det C^\bullet(X_v, V) \\[2mm]
\psi_{v_0} \downarrow \wr & & \psi_v \downarrow \wr \\[2mm]
\mathbf{Det}\,C^\bullet(X, V, W) & \overset{v_{v*}^c}{\underset{\sim}{\longrightarrow}} & \mathbf{Det}\,C^\bullet(X, V, W) .
\end{array}
$$

Proposition 2.10. *Under the conditions of Lemma 2.3, the equality holds:*

$$g_{v*} = v_{v*}^c . \qquad (2.50)$$

---

[26] This action is multiplying by a nonzero factor.

The proof of the equality (2.50) is done just after the end of the proof of Lemma 2.3. The expression for the action of v_{v*}^c on the determinant line can be obtained as follows. Let $v \in U$ and let j be the natural inclusion $j :$ $\bigoplus_{k=1,2} C^\bullet(X_k, W \oplus V_k) \to C^\bullet(X_v, V)$. Then v_v^c acts on $C^\bullet(X_1, W \cup V_1)$ as the identity operator and it acts on $C^\bullet(X_2, W \cup V_2)$ as the operator (k_v/k_{v_0})id. Proposition 2.7 claims that the identification ψ_v is an isometry between the combinatorial norm on $\det C^\bullet(X_v, V)$ and the triple tensor product of the combinatorial norms on the components of $\mathbf{Det}\, C^\bullet(X, W, V)$. It is enough to compute the action of v_{v*}^c on the component $\det C^\bullet(W)$ of the tensor product $\mathbf{Det}\, C^\bullet(X, W, V)$. The inclusion $\rho_{v,c} : C^\bullet(W) \hookrightarrow C^\bullet(X_v, V)$ (defined by (1.63)) is an isometry onto orthogonal complement to $\operatorname{Im} j$ and $r_{v,c}\rho_{v,c} = \operatorname{id}$ on $C^\bullet(W)$. So the action of v_v^c on this orthogonal complement $(\operatorname{Im} j)^\perp$ (identified with $C^\bullet(W)$ by $r_{v_0,c}$ and by $r_{v,c}$) can be expressed as the composition

$$
m \in C^\bullet(W) \xrightarrow{\rho_{v_0,c}} \left((\beta_0, \alpha_0) \Big/ \sqrt{\alpha_0^2 + \beta_0^2} \right) m \xrightarrow{v_v^c} \left((\beta_0, \alpha_0(k_v/k_{v_0})) \Big/ \sqrt{\alpha_0^2 + \beta_0^2} \right) m
$$

$$
= \left((1, k_v) \Big/ \sqrt{1 + k_{v_0}^2} \right) m \xrightarrow{r_{v,c}} \left(\sqrt{1 + k_v^2} \Big/ \sqrt{1 + k_{v_0}^2} \right) m \in C^\bullet(W) . \tag{2.51}
$$

(Here, the signs are written for positive β and β_0.[27]) The expression for v_{v*}^c follows from (2.51) and from the assertion that v_v^c acts on $C^\bullet(X_2, W \cup V_2)$ as (k_v/k_{v_0})id. Namely.

$$
v_{v*}^c l = (k_v/k_{v_0})^{-\chi(M_2, N \cup Z_2)} ((1 + k_v^2)/(1 + k_{v_0}^2))^{-\chi(N)/2} l \tag{2.52}
$$

for $l \in \mathbf{Det}\, C^\bullet(X, V, W)$. It follows from (2.52) that the equality holds (for $l \neq 0$):

$$
\partial_\gamma \log \|v_{v*}^c l\|^2 = -2\chi(M_2, N \cup Z_2)\partial_\gamma \log (k_v) - 2\chi(N)(1 + k_v^{-2})^{-1}\partial_\gamma \log (k_v) . \tag{2.53}
$$

Proposition 2.10 claims that the same identity holds also for the action of g_{v_0} on $\mathbf{Det}(M, N, Z)$.

The right side in the formula (2.19) (i.e., in the assertion of Lemma 2.3) is defined in analytic terms while the right side in (2.53) is defined in topological terms. Each $b_{2,j}(M_{v_0}, Z)$ on the right in (2.19) is the sum of integrals over M_2 and over N of the locally defined densities, according to Theorem 3.2. So it is enough to compute (in topological terms) the expression on the right in (2.19) in the case of a mirror-symmetric $\widetilde{M} = M_2 \cup_N M_2$ with a mirror-symmetric metric $g_{\widetilde{M}}$ (which is a direct product metric neat $\partial \widetilde{M}$ and near N) and with mirror-symmetric boundary conditions on the connected componenets of $\partial \widetilde{M}$. In this case, the expression in (2.19) is the same as for a general M (if the piece M_2 of M, $g_{M|TM_2}$, and the boundary conditions on $\partial M \cap M_2$ are the same as in the mirror-symmetric case on each piece M_2 of $\widetilde{M}$). It is supposed from now on in the proof of Lemma 2.3 that M and all the data on M are mirror-symmetric relative to N. In this case the kernel $E_{t,x,y}^\bullet(v)$ $(v \in \mathbb{R}^2 \setminus (0,0))$ of the operator $\exp(-t\Delta_v^\bullet)$ with the Dirichlet boundary conditions on $Z = Z_2 \cup Z_2 \subset \partial M$ and with the Neumann conditions on $\partial M \setminus N$ is

---

[27] The signs are not important for the transformations of the norm on the determinant line under the actions of v_{v*}^c.

expressed through the fundamental solution $E^\bullet_{t,x,y}$ for $\partial_t + \Delta^\bullet$ on $DR^\bullet(M, Z)$ (with the same boundary conditions on ∂M)[28] as follows[29]:

$$E^\bullet_{t,x,y}(v) = E^\bullet_{t,x,y} + ((\alpha^2 - \beta^2)/(\alpha^2 + \beta^2))(\sigma^*_1 E^\bullet)_{t,x,y} \quad \text{for } x \in M_2, y \in M_2 \,, \quad (2.54)$$

$$E^\bullet_{t,x,y}(v) = (2\alpha\beta/\alpha^2 + \beta^2)E^\bullet_{t,x,y} \quad \text{for } x \in M_1, y \in M_2 \,. \quad (2.55)$$

Note that the kernel $(E_t + \sigma^*_1 E_t)_{x,y} =: E^{j,\text{Neu}}_{t,x,y}$ for $x, y \in M_2$ is the fundamental solution for $\partial_t + \Delta^\bullet_{M_2}$, where $\Delta^\bullet_{M_2}$ is the Laplacian on $DR^\bullet(M_2, Z_2)$ with the Neumann boundary conditions on N and the kernel $(E_t - \sigma^*_1 E_t)_{x,y} =: E^{\text{Dir}}_{t,x,y}$ is the kernel of $\exp(-t\Delta^\bullet_{M_2,N})$, where $\Delta^\bullet_{M_2,N}$ is the Laplacian on $DR^\bullet(M_2, N \cup Z_2)$ (i.e., with the Dirichlet boundary conditions on N). It follows from (2.54) that the alternating sum of zero-order terms (in the asymptotic expansions of the traces of the heat equation operator relative to $t \to +0$) on the right in (2.19) can be represented in the following form (where $m_{v_0} := 2^{-1}(1 - k^{-2}_{v_0})/(1 + k^{-2}_{v_0})$):

$$\sum(-1)^j b_{2,j}(M_{v_0}, Z) = \sum(-1)^j \int_{M_2} \text{tr}\,(E^j_{t,x,y}(v_0))^0$$

$$= 2^{-1}\sum(-1)^j \int_M \text{tr}\,(E^j_{t,x,y})^0$$

$$+ m_{v_0}\sum(-1)^j \int_{M_2} (\text{tr}(E^{j,\text{Neu}}_{t,x,y})^0 - \text{tr}(E^{j,\text{Dir}}_{t,x,y})^0)$$

$$= 2^{-1}\chi(M, Z) + m_{v_0}(\chi(M_2, Z_2) - \chi(M_2, Z_2 \cup N))$$

$$= \chi(M_2, Z_2 \cup N) + (1 + k^{-2}_{v_0})^{-1}\chi(N) \,. \quad (2.56)$$

Hence the expression on the right in (2.53) is equal to the right side of (2.19), and the assertion of Lemma 2.3 follows from Proposition 2.10. The zero superscripts in (2.56) denote the densities on $M_j, N, \partial M$, corresponding to the constant terms (i.e., the t^0-coefficients) in the asymptotic expansions as $t \to +0$ for $\text{Tr}(p_j \exp(-t\Delta^\bullet))$, where $\Delta^\bullet$ is the Laplacian with appropriate boundary conditions. In (2.56) $\int_{M_j} \text{tr}(\,.\,)^0$ denotes the sum of the integrals over M_j, N, and over $\partial M_j \setminus N$ of the corresponding densities. We use the following equalities to produce (2.56):

$$\sum(-1)^j \int_M \text{tr}(E^j_{t,x,x})^0 = \chi(M, Z) \,,$$

$$\sum(-1)^j \int_{M_2} \text{tr}(E^{j,\text{Neu}}_{t,x,x})^0 = \chi(M_2, Z_2) \,,$$

$$\sum(-1)^j \int_{M_2} \text{tr}(E^{j,\text{Dir}}_{t,x,x})^0 = \chi(M_2, N \cup Z_2) \,. \quad (2.57)$$

These equalities are consequences of the analogous equalities without the zero superscripts and of the existence of asymptotic expansions in powers of t for the

---

[28] It is proved in Proposition 1.1 that $\Delta^\bullet$ on $DR^\bullet(M, Z)$ for M obtained by gluing two pieces, $M = M_1 \cup_N M_2$, has the same eigenvalues (including their multiplicities) and eigenforms as $\Delta^\bullet_{1,1}$ in $DR^\bullet(M_{1,1}, Z)$. The analogous assertion is true for the operators $\exp(-t\Delta^\bullet_{1,1})$ and $\exp(-t\Delta^\bullet)$ in $(DR^\bullet(M))_2$ and for their kernels.

[29] These formulas are analogous to (2.40).

corresponding traces as $t \to +0$ ([Se2], Theorem 3, or Theorem 3.2 below).   □

Proof of Proposition 2.10. The identifications

$$\mathbf{Det}\, C^\bullet(X, V, W) \xrightarrow[\sim]{d_c} \mathbf{Det}(X, V, W) \xleftarrow[\sim]{R} \mathbf{Det}(M, N, Z) \tag{2.58}$$

do not depend on v (the determinant lines in (2.58) are defined in (1.54)). So the actions of v^c_{v*} on $\mathbf{Det}(X, V, W)$ and on $\mathbf{Det}\, C^\bullet(X, V, W)$ are the same (i.e., they multiply by the same number). To prove (2.50) it is enough to show that the corresponding operators on $\mathbf{Det}(M, N, Z)$ are the same (i.e., that $g_{v*} = v^c_{v*}$ on $\mathbf{Det}(M, N, Z)$).

The proof of Proposition 2.10 uses the following assertion.

Proposition 2.11. *Let* $\varphi : (F^\bullet_0, d_{F_0}) \to (F^\bullet_1, d_{F_1})$ *be an isomorphism of finite complexes of finite-dimensional linear spaces. Then the diagram is commutative:*

$$
\begin{array}{ccc}
\det F^\bullet_0 & \xrightarrow[\sim]{\varphi} & \det F^\bullet_1 \\[2pt]
d \Big\downarrow \wr & & d \Big\downarrow \wr \\[2pt]
\det H^\bullet(F_0) & \xrightarrow[\sim]{\varphi_*} & \det H^\bullet(F_1)
\end{array}
\tag{2.59}
$$

Proof. This identifications $\det F^\bullet_j \overset{d}{\underset{\sim}{\to}} \det H^\bullet(F_j)$ are defined with the help of differentials $d = d_{F_j}$. Hence the commutativity of (2.59) holds.   □

The commutativity of the following diagram of the identifications (for v sufficiently close to v_0 such that v_v is an isomorphism) follows from (2.59):

$$
\begin{array}{ccccc}
\det W^\bullet_a(v_0) & \xrightarrow[\sim]{v_v} & \det v_v(W^\bullet_a(v_0)) & \xrightarrow[\sim]{\Pi_a} & \det W^\bullet_a(v_0) \\[2pt]
d \Big\downarrow \wr & & d \Big\downarrow \wr & & \Big\downarrow \wr\, d \\[2pt]
\det H^\bullet(M_{v_0}, Z) & \xrightarrow[v_{v*}]{} & \det H^\bullet(M_v, Z) & = & \det H^\bullet(M_v, Z)
\end{array}
\tag{2.60}
$$

where the identification $j_* : H^\bullet(v_v(W^\bullet_a(v_0))) \to H^\bullet(DR(M_v, Z)) = H^\bullet(M_v, Z)$ is defined by the natural inclusion $j : v_v(W^\bullet_a(v_0)) \hookrightarrow DR^\bullet(M_v, Z)$ of a quasi-isomorphic subcomplex. The commutativity of the left square in (2.60) follows from (2.59). The commutativity of the right square in (2.60) also follows from (2.59) because the operator induced by the projection operator Π_a on $H^\bullet(DR(M_v, Z))$ is the identity operator.

The commutativity of the following diagram is a consequence of the commutativity of the diagram (2.60):

$$
\begin{array}{ccccc}
\det W^\bullet_a(v_0) & = & \det W^\bullet_a(v_0) & \xrightarrow[\sim]{v_v} & \det v_v(W^\bullet_a(v_0)) \\[2pt]
& & d \Big\downarrow \wr & & d \Big\downarrow \wr \\[2pt]
\varphi_{v_0}(a) \Big\downarrow \wr & & \det H^\bullet(M_{v_0}, Z) & \xrightarrow[v_{v*}]{} & \det H^\bullet(M_v, Z) \\[2pt]
& & \varphi^{\mathrm{an}}_{v_0} \Big\downarrow \wr & & \varphi^{\mathrm{an}}_v \Big\downarrow \wr \\[2pt]
\mathbf{Det}(M, N, Z) & = & \mathbf{Det}(M, N, Z) & \xrightarrow[\sim]{g_{v*}} & \mathbf{Det}(M, N, Z)
\end{array}
\tag{2.61}
$$

The action of $v_{v*} : H^{\bullet}(M_{v_0}, Z) \to H^{\bullet}(M_v, Z)$ coincides with the combinatorial action $v_{v*}^c : H^{\bullet}(X_{v_0}, V) \to H^{\bullet}(X_v, V)$ under the identification of the cohomology $R : H^{\bullet}(M_v, Z) \to H^{\bullet}(X_v, V)$ induced by the integration R of closed differential forms over the simplexes of X. Hence the commutativity of (2.60) implies also the commutativity of the diagram:

$$
\begin{array}{ccccc}
& \det W_a^{\bullet}(v_0) & \xrightarrow[\widetilde{\;\;}]{v_v} & \det v_v(W_a^{\bullet}(v_0)) & \\[1ex]
& d\upharpoonright\wr & & d\upharpoonright\wr & \\[1ex]
\det H^{\bullet}(M_{v_0}, Z) & = \det H^{\bullet}(M_{v_0}, Z) & \xrightarrow[v_{v*}]{} & \det H^{\bullet}(M_v, Z) & \\[1ex]
& d_c\upharpoonright\wr & & d_c\upharpoonright\wr & \\[1ex]
\varphi_{v_0}^c = \varphi_{v_0}^{\mathrm{an}} \Big\downarrow \wr & \det C^{\bullet}(X_{v_0}, V) & \xrightarrow[v_v^c]{} & \det C^{\bullet}(X_v, V) & \quad (2.62)\\[1ex]
& \psi_{v_0}\upharpoonright\wr & & \psi_v\upharpoonright\wr & \\[1ex]
& \mathbf{Det}\, C^{\bullet}(X, V, W) & \xrightarrow[v_{v*}^c]{} & \mathbf{Det}\, C^{\bullet}(X, V, W) & \\[1ex]
& (2.58)\upharpoonright\wr & & (2.58)\upharpoonright\wr & \\[1ex]
\mathbf{Det}\,(M, N, Z) & = \mathbf{Det}\,(M, N, Z) & \xrightarrow[v_{v*}^c]{} & \mathbf{Det}\,(M, N, Z) &
\end{array}
$$

The equality (2.50) follows immediately from the commutativity of the right bottom square in (2.61) and from the commutativity of (2.62). Proposition 2.10 is proved. $\square$

2.4. Analytic Torsion Norm on the Cone of a Morphism of Complexes. Proof of Lemma 2.4.

Lemma 2.4 is a particular case of the following assertion. Let f be a morphism (2.22) of finite complexes of finite-dimensional Hilbert spaces.[30] Then $\mathrm{Cone}^{\bullet} f$ is defined by (2.23). The exact sequence of complexes:[31]

$$
0 \to V^{\bullet} \to \mathrm{Cone}^{\bullet} f \xrightarrow{p} A^{\bullet}[1] \to 0 \,,
\tag{2.63}
$$

(where the left arrow maps $y \in V^{\bullet}$ into $(0, y) \in \mathrm{Cone}^{\bullet} f$ and $p(x, y) = x$ for $(x, y) \in A^{j+1} \oplus V^j$) defines the identification of the determinants of its cohomology:

$$
\varphi_{\mathrm{Cone}^{\bullet} f}^H : \det H^{\bullet}(\mathrm{Cone}\ f) \xrightarrow{\sim} \det H^{\bullet}(V) \otimes (\det H^{\bullet}(A))^{-1} \,.
\tag{2.64}
$$

Let the Hilbert spaces $\mathrm{Cone}^j f$ be the direct sums $A^j \oplus V^{j+1}$ of the Hilbert spaces.

Lemma 2.6. *The analytic torsion norm on the determinant of the cohomology of* $\mathrm{Cone}^{\bullet} f$ *is isometric under the identification* (2.64) *to the tensor product of the analytic torsions norms*[32]:

$$
T_0(\mathrm{Cone}^{\bullet} f) = T_0(V^{\bullet}) \otimes T_0(A^{\bullet})^{-1} \,.
\tag{2.65}
$$

Remark. 2.6. Let $h \in \det H^{\bullet}(\mathrm{Cone}\ f)$ be identified by (2.64) with $h_1 \otimes h_2^{-1}$ for $h_1 \in \det H^{\bullet}(V)$ and $h_2 \in \det H^{\bullet}(A)$. Namely $\varphi_{\mathrm{Cone}^{\bullet} f} h = h_1 \otimes h_2^{-1}$, where h_2^{-1} is an

---

[30] The morphism f *is not supposed* to be a quasi-isomorphism.
[31] $A^{\bullet}[1]$ is a complex with components $A[1]^j = A^{j+1}$ and with $d_{A[1]} = -d_A$. There are the canonical identifications: $\det A^{\bullet}[1] = (\det A^{\bullet})^{-1}$ and $\det H^{\bullet}(A[1]) = (\det H^{\bullet}(A))^{-1}$.
[32] The analytic torsion norm on $\det H^{\bullet}(A[1]) = \det H^{\bullet}(A)^{-1}$ is the dual norm $T_0(A^{\bullet})^{-1}$.

element of the dual one-dimensional space $\det H^\bullet(A)^{-1}$ such that $h_2^{-1}(h_2) = 1$. In this case, the equality (2.65) claims that

$$\|h\|^2_{T_0(\mathrm{Cone}^\bullet f)} = \|h_1\|^2_{T_0(V^\bullet)} \big/ \|h_2\|^2_{T_0(A^\bullet)} . \tag{2.66}$$

Remark. 2.7. The identity (2.65) (Lemma 2.6) and the equality (2.66) also hold under weaker assumptions. Let the Hilbert structures on $A^\bullet, V^\bullet$, and on $\mathrm{Cone}^\bullet f$ be such that the identification

$$\varphi_{\mathrm{Cone}^\bullet f} : \det \mathrm{Cone}^\bullet f \rightleftarrows \det V^\bullet \otimes (\det A^\bullet)^{-1} \tag{2.67}$$

(induced by the exact sequence (2.63)) is an isometry. Then the equality (2.66) holds.

Corollary 2.7. *Let* $f : A^\bullet \to V^\bullet$ *be a quasi-isomorphism. Then* $H^\bullet(\mathrm{Cone} f) = 0$. *Hence* $\det H^\bullet(\mathrm{Cone} f)$ *is canonically* $\mathbb{C}$ *and* $1 \in \mathbb{C} = \det H^\bullet(\mathrm{Cone} f)$ *is identified by* (2.64) *with* $h_1 \otimes h_2^{-1}$. *Here,* $f_* h_2 = h_1$ *under the identification induced by* f:

$$f_* : \det H^\bullet(A) \rightleftarrows \det H^\bullet(V) . \tag{2.68}$$

In this case, the equality (2.66) *claims that*

$$\|1\|^2_{T_0(\mathrm{Cone}^\bullet f)} = \|h_1\|^2_{T_0(V^\bullet)} \big/ \|h_2\|^2_{T_0(A^\bullet)} = T_0(V^\bullet) \big/ T_0(A^\bullet) , \tag{2.69}$$

where $T_0(V^\bullet)/T_0(A^\bullet)$ *is the ratio of the two norms on the same determinant line (since* $\det H^\bullet(V)$ *and* $\det H^\bullet(A)$ *are identified by* (2.68)).

The equality (2.69) is the assertion of Lemma 2.4.

Proof of Lemma 2.6. The identification (2.67) is an isometry of norms on the determinant lines. Let u be a nonzero element of $\det \mathrm{Cone}^\bullet f$ and let

$$\varphi_{\mathrm{Cone}^\bullet f} u = u_1 \otimes u_2^{-1} , \tag{2.70}$$

where $u_1 \in \det V^\bullet$ and $u_2 \in \det A^\bullet$. Let h, h_1, h_2 be the images of u, u_1, u_2 under the identifications (definied by the differentials of the corresponding complexes):

$$\det \mathrm{Cone}^\bullet f \xrightarrow{d_{\mathrm{Cone}^\bullet f}} \det H^\bullet(\mathrm{Cone}\ f) ,$$

$$\det A^\bullet \xrightarrow{d_A} \det H^\bullet(A), \quad \det V^\bullet \xrightarrow{d_V} \det H^\bullet(V) .$$

Then by the definition of $\varphi^H_{\mathrm{Cone}^\bullet f}$ we have $\varphi^H_{\mathrm{Cone}^\bullet f} h = h_1 \otimes h_2^{-1}$.

The analytic torsion norm on the determinant of the cohomology of a finite-dimensional complex is the norm, corresponding to the L_2-norm on the determinant of this complex defined by the Hilbert structures on its components. Hence the equalities hold:

$$\|h\|^2_{T_0(\mathrm{Cone}^\bullet f)} = \|u\|^2_{\det(\mathrm{Cone}^\bullet f)} ,$$

$$\|h_1\|^2_{T_0(V^\bullet)} = \|u_1\|^2_{\det(V^\bullet)}, \quad \|h_2\|^2_{T_0(A^\bullet)} = \|u_2\|^2_{\det(A^*)} . \tag{2.71}$$

Since the identification (2.67) is an isometry, we see that the equality

$$\|h\|^2_{T_0(\mathrm{Cone}^\bullet f)} = \|h_1\|^2_{T_0(V^\bullet)} \big/ \|h_2\|^2_{T_0(A^\bullet)} \tag{2.72}$$

follows from (2.71) and from (2.70). $\square$

The equality (2.24) in Lemma 2.4 is a particular case of (2.72) (by Corollary 2.7) corresponding to the case of a quasi-isomorphism f. Lemma 2.4 is proved. □

2.5. Variation Formula for Norms of Morphisms of Identifications. Proof of Lemma 2.1.

The assertion (2.14) of Lemma 2.1 can be deduced from the definition of g_v and from the following proposition.

Proposition 2.12. *There is a neighborhood $U_{v_0}, v_0 \in U_{v_0} \subset \mathbb{R}^2 \setminus (0,0)$, such that a family of finite rank projection operators $\Pi_a^j(v) := \Pi_j(v; a)$ in $(DR^j(M))_2$ is smooth on $U_{v_0} \ni v$.*

Its proof follows just after the proof of Lemma 2.1. Let $l \in \det W_a^\bullet(v_0), l \neq 0$, and let $l = \otimes_j l_j^{(-1)^{j+1}}$, where $l_j \in \det W_a^j(v_0), l_j \neq 0$. Then we have

$$\log \|g_v l\|^2_{\det W_a^\bullet(v)} = \sum (-1)^{j+1} \log \|g_v l_j\|^2_{\det W_a^j(v)} . \tag{2.73}$$

Proposition 2.13. *For every j the following equality holds (under the conditions of Lemma 2.1):*

$$\partial_\gamma \log \|g_v l\|^2_{\det W_a^j(v)}\Big|_{\gamma=0} = 2\partial_\gamma \log(k_v)\Big|_{\gamma=0} \mathrm{Tr}\left(p_2 \Pi_a^j(v_0) \right) . \tag{2.74}$$

Thus the assertion (2.14) of Lemma 2.1 is a consequence of (2.74) and of (2.73).

Proof of Proposition 2.13. It is enough to prove the equality (2.74) in the case when $\|l_j\|^2_{\det W_a^\bullet(v_0)} = 1$, i.e., when $l = e_1 \wedge \ldots \wedge e_m$, where $\{e_i\}$ is an orthonormal basis in $W_a^\bullet(v_0)$ ($v_0 = v(0)$). In this case, we have

$$\partial_\gamma \log \|g_v l\|^2_{\det W_a^j(v)}\Big|_{\gamma=0} = \mathrm{tr}(\partial_\gamma(a_{lj}(v)))\Big|_{\gamma=0} , \tag{2.75}$$

where $a(v) = (a_{lj}(v))$ is a matrix of scalar products in $(DR^j(\overline{M}_1) \oplus DR^j(\overline{M}_2), g_M)$ of the images of the basis elements

$$a_{lj}(v) := \langle g_v e_i, g_v e_j \rangle .$$

The formula (2.75) is deduced as follows. Since $\|g_v l\|^2_{\det W_a^j(v)} = \det a(v)$, we have

$$\partial_\gamma \log \det a(v)\Big|_{\gamma=0} = \mathrm{tr}(\partial_\gamma a \cdot a^{-1}(v))\Big|_{\gamma=0} = \mathrm{tr}(\partial_\gamma a)\Big|_{\gamma=0} .$$

The family of the operators $\Pi_a^j(v)$ is smooth in v for $v \in U_{v_0}$ (Proposition 2.12). Hence the operator $\partial_\gamma \Pi_a^j(v)$ exists. Since $\Pi_a^j(v)$ are projection operators, we have

$$\Pi_a^2 = \Pi_a, \quad \partial_\gamma \Pi_a \cdot \Pi_a = (\mathrm{id} - \Pi_a)\partial_\gamma \Pi_a .$$

So e_i are orthogonal to $\partial_\gamma \Pi_a|_{\gamma=0} e_j$ and we get

$$\begin{aligned}
\partial_\gamma a_{lj}(v)\big|_{\gamma=0} &= \partial_\gamma \langle \Pi_a g_v e_i, \Pi_a g_v e_j \rangle\big|_{\gamma=0} \\
&= \langle \partial_\gamma (\Pi_a g_v) e_i, e_j \rangle\big|_{\gamma=0} + \langle e_i, \partial_\gamma (\Pi_a g_v) e_j \rangle\big|_{\gamma=0} \\
&= \partial_\gamma (\log k_v)\big|_{\gamma=0}(\langle p_2 e_i, e_j \rangle + \langle e_i, p_2 e_j \rangle) . \tag{2.76}
\end{aligned}$$

Since $\{e_i\}$ is an orthonormal basis in $W_a^\bullet(v_0)$, we have

$$\mathrm{tr}(\langle p_2 e_i, e_j \rangle) = \mathrm{tr}(p_2 \Pi_a^j(v_0)) . \tag{2.77}$$

Hence (2.74) follows from (2.75), (2.76), and (2.77). Thus Proposition 2.13 and Lemma 2.1 are proved. $\square$

Proof of Proposition 2.12. Let $U_{v_0} \subset \mathbb{R}^2 \setminus (0,0)$ be the set of $v \in \mathbb{R}^2 \setminus (0,0)$ such that the Laplacians $\Delta_{v,j}$ on $DR^j(M_v, Z)$ (for all j) have no eigenvalues λ from $(a - \varepsilon, 1 + 2\varepsilon) \subset \mathbb{R}_+$ (where $\varepsilon > 0$ is small enough). Then for $v \in U_{v_0}$ the projection operator $\Pi_a^j(v)$ from $(DR^j(M))_2$ onto a linear space $W_a^j(v)$ (spanned by the eigenforms of $\Delta_{v,j}$ with the eigenvalues λ from $[0, a]$) is equal to the contour integral

$$\Pi_a^j(v) = i/2\pi \int_{\Gamma_{a+\varepsilon}} G_\lambda^j(v) d\lambda ,$$

where $G_\lambda^j(v) = (\Delta_{v,j} - \lambda)^{-1}$ is the resolvent for the Laplacian $\Delta_{v,j}$ and $\Gamma_{a+\varepsilon}$ is a circle $\Gamma_{a+\varepsilon} = \{\lambda : |\lambda| = a + \varepsilon\}$ oriented opposite to the clockwise. For $\lambda \in \Gamma_{a+\varepsilon}$ and $v \in U_{v_0}$ the operators $G_\lambda^j(v)$ form a smooth in (λ, v) family of bounded operators in $(DR^j(M))_2$. (It is an immediate consequence of Proposition 3.1 below. Indeed, $\mathrm{Spec}(\Delta_{v,j})$ is discrete and it is a subset of $\mathbb{R}_+ \cup 0$, according to Theorem 3.1. Thus if $(a - \varepsilon, a + 2\varepsilon) \subset \mathrm{Spec}(\Delta_{v,j}) = \emptyset$ then $\Gamma_{a+\varepsilon} \cap \mathrm{Spec}(\Delta_{v,j}) = \emptyset$ and $G_\lambda^j(v)$ form a smooth in $v \in U_{v_0}$ and in $\lambda \in \Gamma_{a+\varepsilon}$ family of bounded in $(DR^j(M))_2$ operators by Proposition 3.1.) Proposition 2.13 is proved. $\square$

2.6. Variation Formula for the Scalar Analytic Torsion. Proof of Lemma 2.2. First the lemma is proved in the case when $a > 0$ is less than $4^{-1}\lambda_1(v_0)$ (where $\lambda_1(v)$ is the minimal positive eigenvalue of the Laplacian Δ_v on $(\bigoplus_j DR^j(M_v, Z), g_M)$). Let $U_1(a)$ be a neighborhood of v_0, $v_0 \in U_1(a) \subset U$ (2.3), such that for $v \in U_1(a)$ we have $a < 2^{-1}\lambda_1(v)$. (Such a neighborhood exists according to Theorem 3.1 and to Proposition 3.1.)

Let $v(\gamma)$ be a smooth local map $(\mathbb{R}^1, 0) \to (U(a), v_0)$.

Proposition 2.14. *For $t > 0$ the following variation formula holds*[33]

$$\partial_\gamma \sum (-1)^j j \mathrm{Tr} \exp(-t\Delta_{v,j})|_{\gamma=0}$$
$$= 2\partial_\gamma \log(k_v)|_{\gamma=0} \sum (-1)^j (-t\partial/\partial t) \mathrm{Tr}(p_1 \exp(-t\Delta_{v_0, j})) , \tag{2.78}$$

where $k_v := \alpha/\beta$ for $v = (\alpha, \beta) \in U$ (i.e., for $\alpha\beta \neq 0$).

Proposition 2.15. *Let $\mathrm{Re}\, s > (\dim M)/2$ and let $0 < a < \lambda_1(v_0)$. Then the following equalities hold:*

---

[33] The operator $\exp(-t\Delta_{v,j})$ acts from $(DR^j(M))_2$ into the domain $D(\Delta_{v,j})$ of $\Delta_{v,j}$ defined by (1.27). The operators $\exp(-t\Delta_{v,j})$ and $p_1 \exp(-t\Delta_{v,j})$ are of trace class. Their traces are equal to the integrals over the diagonal of the traces of their kernels restricted to the diagonals (by Proposition 3.8).

$$\partial_\gamma \left[\Gamma(s)^{-1} \int_0^\infty t^{s-1} \sum (-1)^j j (\mathrm{Tr}\exp(-t\varDelta_{v,j}) - \dim \mathrm{Ker}\, \varDelta_{v,j}) dt \right] \Bigg|_{\gamma=0}$$

$$= \Gamma(s)^{-1} \int_0^\infty t^s \sum (-1)^j (-\partial_t) \mathrm{Tr}(p_1(\exp(-t\varDelta_{v_0,j}) - \Pi_a^j(v_0))) dt$$

$$= 2\partial_\gamma \log(k_v)|_{\gamma=0}(s/\Gamma(s)) \sum (-1)^j$$

$$\times \int_0^\infty t^{s-1} \mathrm{Tr}(p_1(\exp(-t\varDelta_{v_0,j}) - \Pi_a^j(v_0))) dt\,. \tag{2.79}$$

Proof of Proposition 2.15. To produce the second row of (2.79) from (2.78), it is enough to prove ([Bo], II.26, Proposition 7) that for $\mathrm{Re}\,s > (\dim M)/2$ the integral

$$\int_0^\infty t^s(-\partial_t)\mathrm{Tr}(p_1 \exp(-t\varDelta_{v,j}))\, dt = \int_0^\infty t^s \mathrm{Tr}(p_1 \varDelta_{v,j} \exp(-t\varDelta_{v,j})) dt \tag{2.80}$$

converges uniformly in v for v from a small neighborhood U_{v_0} of v_0 and to prove the convergence of the integral

$$\int_0^\infty t^{s-1}(\mathrm{Tr}(\exp(-t\varDelta_{v_0,j})) - \dim \mathrm{Ker}\, \varDelta_{v_0,j}) dt \tag{2.81}$$

together with the uniform convergence in v for an arbitrary $v_1 \in U_{v_0}$ (as $v \to v_1$) of the function

$$t^{s+1} \sum (-1)^j \mathrm{Tr}(p_1 \varDelta_{v,j} \exp(-t\varDelta_{v,t})) \to t^{s+1} \sum (-1)^j \mathrm{Tr}(p_1 \varDelta_{v_1,j} \exp(-t\varDelta_{v_1,t})) \tag{2.82}$$

for t from any closed finite interval $t \in I \subset (0, +\infty)$. (To apply the theorem from [Bo], quoted above, it is useful first to do the transformation $\mathbb{R}_+ \ni t \to h = \log t \in \mathbb{R}, dt \to t\, dh$.)

The following estimates are satisfied

$$|\mathrm{Tr}(p_1 \varDelta_{v,j} \exp(-t\varDelta_{v,j}))| \leqq \mathrm{Tr}(\varDelta_{v,j} \exp(-t\varDelta_{v,j}))$$

$$\leqq \|\varDelta_{v,j} \exp(-t\varDelta_{v,j}/2)\|_2 (\mathrm{Tr}(\exp(-t\varDelta_{v,j}/2))$$

$$- \dim\, \mathrm{Ker}\, \varDelta_{v,j}))\,, \tag{2.83}$$

$$\|\varDelta_{v,j} \exp(-t\varDelta_{v,j}/2)\|_2 \leqq \max_{\lambda \geqq 0}(\lambda \exp(-t\lambda/2)) = 2/(te)\,, \tag{2.84}$$

(where $\| \cdot \|_2$ are the operator norms in $(DR^j(M))_2$). The first estimate in (2.83) follows from the Mercer Theorem. (Applying this theorem here is similar to its application in the proofs of Propositions 3.8 and 3.9 below.)

Let t_0 be any positive number. Then for $t \geqq 2t_0$ and for v sufficiently close to v_0 we have the following uniform with respect to v estimate

$$\mathrm{Tr}\exp(-t\varDelta_{v,j}/2) - \dim W_a^j(v) < C \exp(-c_1 t)\,, \tag{2.85}$$

where C and c_1 are positive constants. Indeed, according to Theorem 3.2, for any $t_0 > 0$ there is a constant $L > 0$ (depending on t_0) such that the inequality

$$\mathrm{Tr}\, \exp(-t_0 \varDelta_{v,j}/2) \leqq L$$

holds uniformly with respect to $v \in \mathbb{R}^2 \setminus (0,0)$. So for all $v \in U$ sufficiently close to v_0 and such that $\lambda_1(v) > 2a$, the estimate (2.85) is true for $t \geqq 2t_0$ with $C = L$ and $c_1 = a/4$:

$$\mathrm{Tr} \exp(-t\Delta_{v,j}/2) - \dim W_a^j(v) \leqq L \exp(-ta/4) \,. \tag{2.86}$$

The uniform convergence (with respect to v) of the integral (2.80) for $\mathrm{Re}\,s > (\dim M)/2$ follows from the asymptotic expansion (2.15) in powers of t as $t \to +0$ and from the estimates (2.83), (2.84), and (2.85). The convergence of the integral (2.81) for $\mathrm{Re}\,s > (\dim M)/2$ follows from (2.85) and from the existence of the asymptotic expansion of the trace at $t \to +0$ (by Theorem 3.2):

$$\mathrm{Tr} \exp(-t\Delta_{v_0,j}) = f_{-\dim M} t^{-(\dim M)/2} + f_{1-\dim M} t^{(1-\dim M)/2} + \cdots +$$
$$+ f_k t^{k/2} + O(t^{(k+1)/2}) \,, \tag{2.87}$$

where $k \in \mathbb{Z}_+$ and $f_t := f_t(v; j)$ are smooth in $v \in \mathbb{R}^2 \setminus (0,0)$. This asymptotic expansion (2.87) is differentiable with respect to v, according to Proposition 3.2.

The uniform convergence of (2.82) for $t \in I \subset (0, \infty)$ (if s is fixed and $\mathrm{Re}\,s > (\dim M)/2$) follows from Proposition 3.8 and from the uniform convergence (with respect to v) of the functions of t

$$t^{s+1}\partial_t \int_{M_1} \mathrm{Tr} E_{t,x,x}^j(v) \to t^{s+1}\partial_t \int_{M_1} \mathrm{tr} E_{t,x,x}^j(v_1)$$

for $t \in I$. (The latter assertion follows from Proposition 3.2 and from Theorem 3.2.)

The last equality in (2.79) is true for $\mathrm{Re}\,s > (\dim M)/2$ according to the asymptotic expansion (2.87), to the estimate (2.86), to the absolute convergence of the integral (2.80), and to the following estimates (where $0 < a < \lambda_1(v_0)$):

$$0 < \mathrm{Tr}(p_1(\exp(-t\Delta_{v_0,j}) - \Pi_a^j(v_0))) \leqq \mathrm{Tr}(\exp(-t\Delta_{v_0,j}) - \Pi_a^j(v_0)) \,. \tag{2.88}$$

(These estimates are deduced from the Mercer theorem the same way as in the proofs of Propositions 3.8 and 3.9.) Thus Proposition 2.15 is proved. $\square$

Proposition 2.16. *For* $0 < a < \lambda_1(v_0)$ *the assertion of Lemma* 2.1 *is true, i.e., it holds:*

$$\partial_\gamma \log T(M_v, Z)|_{\gamma=0} = 2\partial_\gamma \log(k_v)|_{\gamma=0} \sum (-1)^j \mathrm{Tr}(p_1(\exp(-t\Delta_{v_0,j}) - \Pi_a^j(v_0)))^0 \,. \tag{2.89}$$

(The zero superscript denotes the constant coefficient in the asymptotic expansion as $t \to +0$ for the operator trace.)

Proof. The equality (2.79) claims that for $\mathrm{Re}\,s > (\dim M)/2$ and for $0 < a < \lambda_1(v_0)/4$ we have

$$\partial_\gamma \left(\sum (-1)^j j \zeta_{v,j}(s) \right)\Big|_{\gamma=0}$$
$$= 2\partial_\gamma \log(k_v)|_{\gamma=0} (s/\Gamma(s)) \sum (-1)^j$$
$$\times \int_0^\infty t^{s-1} \mathrm{Tr}(p_1(\exp(-t\Delta_{v_0,j}) - \Pi_a^j(v_0))) dt \,. \tag{2.90}$$

The final expressions in (2.90) and (2.79) are analytic functions of s for $\operatorname{Re} s >$ $(\dim M)/2$ according to (2.88), (2.86), and to (2,15). The meromorphic continuation to the whole complex plane $\mathbb{C} \ni s$ of this function of s can be produced with the help of the asymptotic expansion (2.15) (or of the expansion (2.91) below for $\operatorname{Tr}(p_1(\exp(-t\Delta_{v_0,j}) - \Pi_a^j(v_0)))$) using the estimates (2.88) and (2.86). Let the asymptotic expansion as $t \to +0$ for the trace of the operator below be as follows ($n := \dim M$):

$$\operatorname{Tr}(p_1(\exp(-t\Delta_{v_0,j}) - \Pi_a^j(v_0))) = q_{-n}t^{-n/2} + \cdots + q_0 t^0 + \hat{q}_1 t^{1/2}$$
$$+ \cdots + \hat{q}_k t^{k/2} + r_{k,j}(t), \qquad (2.91)$$

where $r_{k,j}(t)$ is $O(t^{(k+1)/2})$ as $t \to +0$ and $r_{k,j}(t)$ is a $C^{[k/2]}$-smooth functions of $t \in [0,1]$.[34] Then the analytic continuation to $\operatorname{Re} s > -(k+1)/2$ of the integral on the right in (2.90) is given by the expression

$$\int_0^\infty t^{s-1} \operatorname{Tr}(p_1(\exp(-t\Delta_{v_0,j}) - \Pi_a^j(v_0))) dt$$

$$= q_{-n}/(-n/2 + s) + q_{-n+1}/(-n/2 + 1/2 - s) + \cdots + q_0/s$$

$$+ \hat{q}_1/(1/2 + s) + \cdots + \hat{q}_k/(k/2 + s) + \int_0^1 t^{s-1} r_{k,j}(t) dt$$

$$+ \int_1^\infty t^{s-1} \operatorname{Tr}(p_1(\exp(-t\Delta_{v_0,j}) - \Pi_a^j(v_0))) dt. \qquad (2.92)$$

The latter integral in (2.92) is an analytic function on the whole complex plane $\mathbb{C} \in s$ (according to (2.88) and (2.86)). The integral of $r_{k,j}$ in (2.92) is an analytic function of s for $\operatorname{Re} s > -(k+1)/2$. The asymptotic expansions (2.87) for $\operatorname{Tr} \exp(-t\Delta_{v,j})$ can be differentiated with respect to v according to Proposition 3.2. They provide us with the analytic continuation of $\sum(-1)^j j \zeta_{v,j}(s)$ to $\operatorname{Re} s > -(k+1)/2$ as follows:

$$\sum(-1)^j j \zeta_{v,j}(s) = \Gamma(s)^{-1}(F_{-n}/(-n/2 + s) + \cdots + F_k/(k/2 + s)$$

$$+ \int_0^1 t^{s-1} m_{k,v}(t) dt + \int_1^\infty t^{s-1} \sum(-1)^j j \operatorname{Tr}(\exp(-t\Delta_{v,j}))$$

$$- \dim \operatorname{Ker} \Delta_{v,j}) dt), \qquad (2.93)$$

where $F_k := \sum_j (-1)^j j(f_k(v,j) - \delta_{0,k} \dim \operatorname{Ker} \Delta_{v,j})$ and the functions $m_{k,v}(t)$ are $C^{[k/2]}$-smooth in $t \in [0,1]$ and in v (for $v \in \mathbb{R}^2 \setminus (0,0)$) and are such that $m_{k,v} = O(t^{(k+1)/2})$ uniformly with respect to v as $t \to +0$.

The latter integral in (2.93) is an analytic function of $s \in \mathbb{C}$. We obtain its derivative with respect to γ taking into account (2.78), (2,86), and (2,88):

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[34] This asymptotic expansion exists and is differentiable with respect to t by Proposition 3.2.

$$\partial_\gamma \int_1^\infty t^{s-1} \sum_j (-1)^j j \mathrm{Tr}(\exp(-t\Delta_{v,j}) - \dim \ \mathrm{Ker}\,\Delta_{v,j}) dt|_{\gamma=0}$$

$$=2\partial_\gamma \log(k_v)|_{\gamma=0}\{s\int_1^\infty t^{s-1} \sum_j (-1)^j \mathrm{Tr}(p_1(\exp(-t\Delta_{v_0,j}) - \Pi_0^j(v_0)))dt$$

$$+ \sum_j (-1)^j \mathrm{Tr}(p_1(\exp(-\Delta_{v_0,j}) - \Pi_a^j(v_0)))\} \,,$$

where $0 < a < \lambda_1(v_0)$.[35] Using (2.87), (2.93), and (2.92), we get the equalities (where $q_k(j,v_0) := q_k$ for $k \leqq 0$, $q_k(j,v_0) := \hat{q}_k$ for $k > 0$ and $q_k, \hat{q}_k$ are from (2.91)):

$$\partial_\gamma F_l|_{\gamma=0} = 2\partial_\gamma \log(k_v)|_{\gamma=0}(-l/2)\sum (-1)^j q_l(j,v_0) \,,$$

$$\partial_\gamma m_{k,v}(t)|_{\gamma=0} = 2\partial_\gamma \log(k_v)|_{\gamma=0}(-t)\partial_t \sum_j (-1)^j r_{k,j}(t) \,. \qquad (2.94)$$

Hence the equality (2.90) holds on the whole complex plane $\mathbb{C} \ni s$. In particular, we obtain

$$\partial_\gamma \log \ T(M_v, Z)|_{\gamma=0} = 2\partial_\gamma \log(k_v)|_{\gamma=0} \sum (-1)^j q_0(j, v_0) \,.$$

Thus Proposition 2.16 is proved. □

Remark. 2.8. A consequence of (2.94) is as follows. For any smooth local map $v(\gamma) : (\mathbb{R}^1, 0) \to (U, 0)$ (where U is defined by (2.3)) the identity $\partial_\gamma F_0 \equiv 0$ holds according to (2.94). Hence the function on U

$$F_0(v) = \sum (-1)^j j(f_0(v, j) - \dim \ \mathrm{Ker}\,\Delta_{v,j})$$

is independent of v. The dimension of $\mathrm{Ker}\,\Delta_{v,j}$ is independent of v for $v \in U$ as it follows from the cohomology exact sequence in the top row of (2.27). Indeed, for $v \in U$ the dimension of $\mathrm{Im}\,\partial_D$ (where $\partial_D : H^i(N, \mathbb{C}) \to \bigoplus_{k=1,2} H^{i+1}(M_k, Z_k \cup N; \mathbb{C})$ is a differential in this exact sequence) is independent of v because for different $v = v_0$ and v_1 from U the maps $\partial_{D,k}(v) : H^i(N, \mathbb{C}) \to H^{i+1}(M_k, N \cup Z_k; \mathbb{C})$ for fixed $k = 1, 2$ (and for a fixed i) differ by the nonzero scalar constant factor (depending on v_0 and v_1). Hence $\sum(-1)^j j f_0(v, j)$ is a constant function on U.

Proof of Proposition 2.14. Let $E_{t,x,y}^j(v)$ (where $t > 0$) be the heat kernel of the operator $\exp(-t\Delta_{v,j})$. By the Duhamel principle, a variation in v of $E_{t,x,y}^\bullet(v)$ can be written as follows. Let $(E_{t_1,x_1,*}^\bullet(v_1), E_{t_2,*,y}^\bullet(v))_M =: \int_M E_{t_1,x_1,z}^\bullet(v_1) \wedge *_z E_{t_2,z,x_2}^\bullet(v)$ be a scalar product (1.23) (with respect to the variable z). We have

$$E_{t,x,y}^\bullet(v) - E_{t,x,y}^\bullet(v_0) = \lim_{\varepsilon \to +0} \int_\varepsilon^{t-\varepsilon} d\tau \partial/d\tau (E_{\tau,x,*}^\bullet(v), E_{t-\tau,*,y}^\bullet(v_0))$$

$$= \lim_{\varepsilon \to +0} \int_\varepsilon^{t-\varepsilon} d\tau [(-\Delta_* E_{\tau,x,*}^\bullet(v), E_{t-\tau,*,y}^\bullet(v_0))$$

$$+ (E_{\tau,x,*}^\bullet(v), \Delta_* E_{t-\tau,*,y}^\bullet(v_0))] \,. \qquad (2.95)$$

---

[35] For such a the operator $\Pi_a^j(v_0)$ is the projection operator from $(DR^j(M))_2$ onto the space of harmonic forms $\mathrm{Ker}(\Delta_{v_0,j})$.

Stokes' formula claims that for any two smooth forms $\omega_1, \omega_2 \in DR^j(\overline{M}_1)$ on a manifold M_1 with boundary we have

$$(d\omega_1, \omega_2)_{M_1} - (\omega_1, \delta\omega_2)_{M_1} = (r\omega_1, A\omega_2)_{\delta M_1} := \int_{\partial M_1} r\omega_1 \wedge *_{\partial M_1} A\omega_2 \,, \qquad (2.96)$$

where the density $r\omega_1 \wedge *_{\partial M} A\omega_2$ on $\partial M_1 =: N$ and the operators r and A are defined as follows. Let any local orientations be chosen on $TM|_N$ and on TN. Then the following forms on N are locally defined:

$$r\omega_1 := [N : M_1]\, i^*_{N,M_1} \omega_1, \quad A\omega_2 := *_N^{-1} i^*_{N,M_1} *_{M_1} \omega_2 \,, \qquad (2.97)$$

where $i^*_{N,M_1} : DR^\bullet(\overline{M}_1) \to DR^\bullet(N)$ is the geometrical restriction to N (and $[N : M_1] = 1$ if N is locally oriented as ∂M_1 and $[N : M_1] = -1$ in the opposite case). The locally defined form $r\omega_1 \wedge *_{\partial M_1} A\omega_2$ is a globally defined density on $N = \partial M$[36]. So the equality (2.95) can be written in the form

$$\begin{aligned}
E^\bullet_{t,x,y}(v) - E^\bullet_{t,x,y}(v_0) = \lim_{\varepsilon \to +0} \int_\varepsilon^{t-\varepsilon} d\tau \sum_{k=1,2} &([[-(r\delta)_* E^\bullet(v), A_* E^\bullet_{t-\tau,*,y}(v_0))_{\partial M_k} \\
&+ (A_* E^\bullet_{\tau,x,*}(v), (r\delta)_* E^\bullet_{t-\tau,*,y}(v_0))_{\partial M_k}] \\
&+ [((Ad)_* E^\bullet_{\tau,x,*}(v), r_* E^\bullet_{t-\tau,*,y}(v_0))_{\partial M_k} \\
&- (r_* E^\bullet_{\tau,x,*}(v), (Ad)_* E^\bullet_{t-\tau,*,y}(v_0))_{\partial M_k}] \,, \qquad (2.98)
\end{aligned}$$

where $r = r_k$, $A = A_k$ are the corresponding operators for pairs $(M_k, \partial M_k)$.

Let any local orientations be chosen on $TM|_N$ and on TN. Then the conditions (1.27) for the domain $D(\Delta_v^\bullet)$ claim that for the kernels $E^\bullet(v)$ and $E^\bullet(v_1)$ (where $v = (\alpha, \beta)$ and $v_1 = (\alpha_1, \beta_1)$ are from $\mathbb{R}^2 \setminus (0,0)$) the following equalities hold on N:

$$\begin{aligned}
\alpha r_z(M_1, N) \circ E^\bullet_{t,x,z}(v) &= -\beta r_z(M_2, N) \circ E^\bullet_{t,x,z}(v) \,, \\
\beta A_z(M_1, N) \circ E^\bullet_{t,x,z}(v) &= \alpha A_z(M_2, N) \circ E^\bullet_{t,x,z}(v) \,, \\
\alpha r_z(M_1, N) \circ \delta_z E^\bullet_{t,x,z}(v) &= -\beta r_z(M_2, N) \circ \delta_z E^\bullet_{t,x,z}(v) \,, \\
\beta A_z(M_1, N) \circ d_z E^\bullet_{t,x,z}(v) &= \alpha A_z(M_2, N) \circ d_z E^\bullet_{t,x,z}(v) \,. \qquad (2.99)
\end{aligned}$$

The analogous equalities hold for $E^\bullet_t(v_1)$ (where (α, β) are replaced by (α_1, β_1)). Hence the equality (2.98) for $v \in U(a) \subset U$ can be written in the form

$$\begin{aligned}
E^\bullet_{t,x,y}(v) - E^\bullet_{t,x,y}(v_0) = \lim_{\varepsilon \to +0} \int_\varepsilon^{t-\varepsilon} d\tau [&-(1 - (k_v/k_{v_0}))((r\delta)_* E^\bullet_{\tau,x,*}(v), A_* E^\bullet_{t-\tau,*,y}(v_0))_{\partial M_1} \\
&+ (1 - (k_{v_0}/k_v))(A_* E^\bullet_{\tau,x,*}(v), (r\delta)_* E^\bullet_{t-\tau,*,y}(v_0))_{\partial M_1} \\
&+ (1 - (k_{v_0}/k_v))(A_* d_* E^\bullet_{\tau,x,*}(v), r_* E^\bullet_{t-\tau,*,y}(v_0))_{\partial M_1} \\
&- (1 - (k_v/k_{v_0}))(r_* E^\bullet_{\tau,x,*}(v), A_* d_* E^\bullet_{t-\tau,*,y}(v_0))_{\partial M_1}] \,. \quad (2.100)
\end{aligned}$$

---

[36] It does not depend on a local orientation on TM and if a local orientation on TN changes to the opposite then this local form changes its sign.

The kernel $E^\bullet_{t,x,y}(v)$ is smooth in $v \in \mathbb{R}^2 \setminus (0,0)$ as a smooth double form on $\overline{M}_{r_1} \times \overline{M}_{r_2}$ (where the limit values on $N \subset \partial\overline{M}_r$ are taken from the side of M_r) according to Proposition 3.2. So we can conclude from (2.100) that

$$
\partial_\gamma E^\bullet_{t,x,y}(v)|_{\gamma=0} = \partial_\gamma \log(k_v)|_{\gamma=0} \lim_{\varepsilon\to+0} \int_\varepsilon^{t-\varepsilon} d\tau [((r\delta)_* E^\bullet_{\tau,x,*}(v_0), A_* E^\bullet_{t-\tau,*,y}(v_0))_{\partial M_1}
$$
$$
+ (A_* E^\bullet_{\tau,x,*}(v_0), (r\delta)_* E^\bullet_{t-\tau,*,y}(v_0))_{\partial M_1}
$$
$$
+ (A_* d_* E^\bullet_{\tau,x,*}(v_0), r_* E^\bullet_{t-\tau,*,y}(v_0))_{\partial M_1}
$$
$$
+ (r_* E^\bullet_{\tau,x,*}(v_0), A_* d_* E^\bullet_{t-\tau,*,y}(v_0))_{\partial M_1}] , \qquad (2.101)
$$

where the limits of the kernels are taken from the side of M_1.

By Proposition 3.8, we have the equality

$$
\operatorname{Tr} \exp(-t\Delta_{v_0,j}) = \sum_{r=1,2} \int_{\overline{M}_r} i^*_{M_r} E^j_{t,x_1,x_2}(v_0) , \qquad (2.102)
$$

where the exterior product of double forms on the diagonals $i_{M_r} : \overline{M}_r \hookrightarrow \overline{M}_r \times \overline{M}_r$ are implied.

We deduce from the semigroup property for the kernels of the operators $\exp(-t\Delta_{v_0,j})$ that

$$
(Q_{1,z_1} E^j_{t_1,x,z_1}(v_0), Q_{2,z_2} E^j_{t_2,z_2,x}(v_0))_M :=
$$
$$
= \int_M Q_{1,z_1} E^j_{t_1,x,z_1}(v_0) \wedge_x *_x Q_{2,z_2} E^j_{t_2,z_2,x}(v_0)
$$
$$
= Q_{1,z_1} Q_{2,z_2} E^j_{t_1+t_2,z_2,z_1}(v_0) , \qquad (2.103)
$$

where Q_{i,z_i} are differential operators acting on differential forms, $t_i > 0$ and the integration is with respect to the variable x. We deduce the following variation formula from (2.101) using (2.102) and (2.103):

$$
\partial_\gamma \operatorname{Tr} \exp(-t\Delta_{v,j})|_{\gamma=0} = 2\partial_\gamma \log(k_v)|_{\gamma=0} t[(A_{z_1}(M_1,N) d_{z_1} r_{z_2}(M_1,N) E^j_{t,z_1,z_2}(v_0))_N
$$
$$
+ (A_{z_1}(M_1,N) r_{z_2}(M_1,N) \delta_{z_2} E^j_{t,z_1,z_2}(v_0))_N] , \qquad (2.104)
$$

where the summands are the integrals of the densities on N (as it is defined in (2.96) and (2.97)). For instance,[37]

$$
(A_{z_1} d_{z_1} r_{z_2} E^j_{t,z_1,z_2}(v_0))_N := \int (A_{z_1} d_{z_1} \wedge *_{z_2,N} r_{z_2} E^j_{t,z_1,z_2}(v_0)) . \qquad (2.105)
$$

The local forms $F_1 = (i^*_{M_1} *_{z_2} d_{z_1} d_{z_2} E^\bullet_{t,z_1,z_2}(v_0))$ and $F_2 = (i^*_{M_1} *_{z_2} \delta_{z_1} d_{z_1} E^{\bullet+1}_{t,z_1,z_2}(v_0))$ are smooth on the diagonal $i_{M_1} : \overline{M}_1 \hookrightarrow \overline{M}_1 \times \overline{M}_1$. (The limit values on $N \subset \partial M_1$ of these double forms are taken from the side of M_1 and the exterior product of these double forms is implied). The local form $i^*_N *_{N,z_2} A_{z_1} d_{z_1} r_{z_2} E^\bullet_{t,z_1,z_2}(v_0)$ is a smooth density on the diagonal $N \subset \partial M_1$, $i_N : N \hookrightarrow N \times N$. Hence we can apply Stokes' formula (2.96) to the expression on the right in (2.104). The result transformed with the help of the equality $d_x E^\bullet_{t,x,y}(v) = \delta_y E^{\bullet+1}_{t,x,y}(v)$ is as follows:

---

[37] We assume (but do not write) the restriction to the diagonal $N \hookrightarrow N \times N$ in (2.105).

$$\partial_\gamma \mathrm{Tr}\exp(-t\Delta_{\nu,J}|_{\gamma=0} = 2\partial_\gamma\log(k_\nu)\Big|_{\gamma=0} t[-(\delta_{z_1}d_{z_1}E^j_{t,z_1,z_2}(\nu_0))_{M_1} + (d_{z_1}\delta_{z_1}E^{j+1}_{t,z_1,z_2}(\nu_0))_{M_1}$$

$$+ (d_{z_2}\delta_{z_2}E^j_{t,z_1,z_2}(\nu_0))_{M_1} - (\delta_{z_2}d_{z_2}E^{j-1}_{t,z_1,z_2}(\nu_0))_{M_1}]\,, \qquad (2.106)$$

where the summands in (2.106) are the integrals of the densities of the type:

$$(\delta_{z_1}d_{z_1}E^\bullet_{t,z_1,z_2}(\nu_0))_{M_1} := \int_{M_1} i^*_{M_1} *_{z_2} \delta_{z_1}d_{z_1}E^\bullet_{t,z_1,z_2}(\nu_0) \qquad (2.107)$$

(and the exterior product of double forms is implied in (2.107) and (2.106)).

The formula (2.78) is an immediate consequence of (2.106) if we take into account the equalities

$$\partial_t E^\bullet_{t,x,y} = -\Delta_x E^\bullet_{t,x,y} = -\Delta_y E^\bullet_{t,x,y}, \quad E^\bullet_{t,x,y} = E^\bullet_{t,y,x}$$

and use the formula (3.67) in Proposition 3.8. Thus Proposition 2.14 is proved. Hence Lemma 2.2 is proved in the case of $a < 4^{-1}\lambda_1(\nu_0)$. $\square$

Let now $a > 0$ be an arbitrary number such that $a \notin \cup_J \mathrm{Spec}(\Delta_{\nu_0,J})$. Then for any nonzero element $l \in \det W^\bullet_a(\nu_0)$ we have by Lemma 2.1 that

$$\partial_\gamma\log\|g_\nu l\|^2_{\det W^\bullet_a(\nu_0)}|_{\gamma=0} = -2\partial_\gamma\log(k_\nu)|_{\gamma=0}\sum(-1)^j\mathrm{Tr}(p_2\Pi^j_a(\nu_0))\,,$$

where $g_\nu = \Pi^\bullet_a \nu$ is defined by (2.13). We know that under the identifications (2.4) and (2.6), the analytic torsion norm $T_0(M_\nu, Z; a)$ on $\det W^\bullet_a(\nu)$ is transformed into the analytic torsion norm $T_0(M_\nu, Z)$ on $\det H^\bullet(M_\nu, Z)$, $\|g_\nu l\|^2_{T_0(M_\nu,Z;a)} = \|(g_\nu l)_H\|^2_{T_0(M_\nu,Z)}$. (Here, $(g_\nu l)_H \in \det H^\bullet(M_\nu, Z)$ corresponds to $g_\nu l$ under these identifications.) Let $(g_\nu l)_H$ be fixed. Then the analytic torsion norm of $g_\nu l \in \det W^\bullet_a(\nu)$

$$\|g_\nu l\|^2_{T_0(M_\nu,Z;a)} := \|g_\nu l\|^2_{\det W^\bullet_a(\nu)} \cdot T(M_\nu, Z; a) \qquad (2.108)$$

is independent of $a > 0$. Let $\nu_0 = (\alpha_0, \beta_0) \in U$ (i.e., $\alpha_0\beta_0 \neq 0$). Then using (2.108), (2.89) we obtain ($\mu := \partial_\gamma\log(k_\nu)|_{\gamma=0}$):

$$\partial_\gamma\log T(M_\nu, Z; a)|_{\gamma=0} = \lim_{\varepsilon\to+0}\partial_\gamma\log T(M_\nu, Z; \varepsilon)|_{\gamma=0}$$

$$+ 2\mu(-\lim_{\varepsilon\to+0}\sum(-1)^j\mathrm{Tr}(p_2\Pi^j_\varepsilon(\nu_0))$$

$$+ \sum(-1)^j\mathrm{Tr}(p_2\Pi^j_a(\nu_0)))$$

$$= 2\mu[\sum(-1)^j(\mathrm{Tr}\,p_1\exp(-t\Delta_{\nu_0,J}))^0 - \sum(-1)^j\mathrm{Tr}\,\Pi^j_\varepsilon(\nu_0)$$

$$+ \sum(-1)^j\mathrm{Tr}(p_2\Pi^j_a(\nu_0))]\,. \qquad (2.109)$$

Note that for arbitrary $c > 0$, $a > 0$ we have

$$\sum(-1)^j\mathrm{Tr}\,\Pi^j_c(\nu_0) = \chi(M_{\nu_0}; Z) = \sum(-1)^j\mathrm{Tr}((p_1 + p_2)\Pi^j_a(\nu_0))\,.$$

So the final expression in (2.109) is equal to

$$2\partial_\gamma\log(k_\nu)|_{\gamma=0}\sum(-1)^j\mathrm{Tr}(p_1(\exp(-t\Delta_{\nu_0,J}) - \Pi^j_a(\nu_0)))\,.$$

Thus Lemma 2.2 is proved for an arbitrary $a > 0$ such that $a \notin \cup_J\mathrm{Spec}(\Delta_{\nu_0,J})$. $\square$

2.7. Continuity of the Truncated Scalar Analytic Torsion. Proof of Proposition 2.1.

Taking into account the definition (2.11) of the truncated scalar analytic torsion $T(M_v, Z; a)$, we see that Proposition 2.1 is a consequence of the assertion as follows. For $\text{Re}\, s > (\dim M)/2$ the truncated ζ-function (2.8) for $\Delta_{v,j}$ is defined by the integral[38]

$$\zeta_{v,j}(s;a) = \Gamma(s)^{-1} \int_0^\infty t^{s-1} \text{Tr}(\exp(-t\Delta_{v,j})(1 - \Pi_a^j(v)))dt \;. \tag{2.110}$$

Proposition 2.17. *For $a > 0$ the truncated determinant for the v-transmission interior boundary conditions*

$$\det(\Delta_v^\bullet; a) := \exp(-\partial/\partial s \zeta_{v,\bullet}(s;a)|_{s=0})$$

is a continuous function of v for $v \in \mathbb{R}^2 \setminus (0,0)$ such that $a \notin \text{Spec}(\Delta_v^\bullet)$.

Proof. Let $E_{t,x,y}^\bullet(v)$, $t > 0$, be the kernel of $\exp(-t\Delta_v^\bullet)$. According to Proposition 3.8 we have

$$\text{Tr}\exp(-t\Delta_v^\bullet) = \sum_{k=1,2} \int_{\overline{M}_k} \text{tr}(*_{x_2} i_k^* E_{t,x_1,x_2}^\bullet) \;, \tag{2.111}$$

where $i : \overline{M}_k \hookrightarrow \overline{M}_k \times \overline{M}_k$ are the diagonal immersions. (The exterior product of the restriction to the diagonal of double forms is implied in (2.111).) Set $I = [-1,1]$. Let $I \times N \subset M$ be the inclusion of the neighborhood of $N = 0 \times N$ into M and let g_M be a direct product metric on $I \times N$. Let $\Delta_{v,0}^\bullet$ be the Laplacian on $DR^\bullet(I \times N)$ with the v-transmission boundary conditions on $N = 0 \times N$ and with the Dirichlet boundary conditions on $\partial I \times N$. Let $E_{t,x,y}^\bullet(v;0)$, $t > 0$, be the kernel of $\exp(-t\Delta_{v;0}^\bullet)$. The equality analogous to (2.111) holds also for $\text{Tr}\exp(-t\Delta_{v;0}^\bullet)$ (where $\overline{M}_k$ is replaced by $Q_k \times N$, $Q_1 := [-1,0]$, $Q_2 := [0,1]$).

Let $v_0 \neq (0,0)$ and let $a \notin \text{Spec}(\Delta_{v_0}^\bullet)$ be a fixed positive number. Then from Theorem 3.2, Proposition 3.1, and from the estimate (2.86) it follows that for an arbitrary $\varepsilon > 0$ there are a neighborhood $U_0(\varepsilon)$ of v_0 and $T > 0$ such that for $v \in U_0(\varepsilon)$ and for $t \geqq T$ the estimate holds

$$|\text{Tr}(\exp(-t\Delta_v^\bullet)(1 - \Pi_a^\bullet(v)))| \leqq \varepsilon \exp(-at/2) \;. \tag{2.112}$$

To prove the continuity of $\det(\Delta_v^\bullet)$ in v at $v = v_0$, it is enough to obtain the following estimate.[39] For a given $b, 1 > b > 0$, and for an arbitrary $\varepsilon > 0$ there exists a neighborhood $U_\varepsilon \ni v_0$ such that the estimate holds for $v \in U_\varepsilon, -(1-b) < s < 1$:

$$\int_0^T |\text{Tr}(\exp(-t\Delta_v^\bullet)(1 - \Pi_a^\bullet(v))) - \text{Tr}(\exp(-t\Delta_{v_0}^\bullet)(1 - \Pi_a^\bullet(v_0)))| t^{s-1} dt < \varepsilon \;. \tag{2.113}$$

---

[38] The analytic continuation of this integral from $\text{Re}\, s > (\dim M)/2$ to the whole complex plane coincides with the meromorphic continuation of $\zeta_{v,j}(s;a)$.

[39] The integrals (2.110) for the values v_0 and v of the transmission parameter have the analytic contunuations from $\text{Re}\, s > (\dim M)/2$ to the whole complex plane. It follows from the estimates (2.113) and (2.112) that the difference of these integrals multiplied by $\Gamma(s)$ is an absolutely convergent integral for $\text{Re}\, s > -1$. Hence this difference is an analytic function of s for such s and it is equal to the difference of the analytic continuations of the integrals (2.110) for v and v_0.

The spectrum $\mathrm{Spec}(\varDelta_v^\bullet)$ is discrete and it depends continuously on v (by Proposition 3.1). Since $a \notin \mathrm{Spec}(\varDelta_v^\bullet)$, we see that $\mathrm{Tr}\,\Pi_a^\bullet(v) = \mathrm{Tr}\,\Pi_a^\bullet(v_0)\,(=\mathrm{rk}\,\Pi_a^\bullet)$ in a neighborhood of v_0 and that the following estimate is satisfied uniformly with respect to $s, -1/2 < s < 1$, and to v for v sufficiently close to v_0:

$$\int_0^T |\mathrm{Tr}\,(\exp(-t\varDelta_v^\bullet)\Pi_a^\bullet(v)) - \mathrm{Tr}\,(\exp(-t\varDelta_{v_0}^\bullet)\Pi_a^\bullet(v_0))|t^{s-1}dt < \varepsilon/2 . \qquad (2.114)$$

The inequality

$$\int_0^T |\mathrm{Tr}\,(\exp(-t\varDelta_v^\bullet)) - \mathrm{Tr}\,\exp(-t\varDelta_{v_0}^\bullet)|t^{s-1}dt < \varepsilon/2 \qquad (2.115)$$

for v sufficiently close to v_0 and for $-1/2 < s < 1$ is obtained as follows. According to Proposiion 3.1, $\mathrm{Tr}\exp(-t\varDelta_v^\bullet)$ is equal to the integral over $\cup \bar M_j$ of the density defined by the restriction to the diagonal of the corresponding kernel. So it is enough to estimate in (2.115) the integrals of the difference between the densities defined by $\exp(-\varDelta_v^\bullet)$ and by $\exp(-t\varDelta_{v_0}^\bullet)$ separately over a fixed neighborhood U of $N = 0 \times N \hookrightarrow M$ and over $M \setminus U$. The estimate of the integral over $U \subset N$ is obtained with the help of the kernel $E_{t,x,y}^\bullet(v;0)$ of $\exp(-t\varDelta_{v;0}^\bullet)$. Set $e_{t,x,y}^\bullet(v) := E_{t,x,y}^\bullet(v) - E_{t,x,y}^\bullet(v;0)$ for $x,y, \in I \times N$.

Proposition 2.18. *For an arbitrary $m \in \mathbb{Z}_+$ there is a neighborhood of v_0 such that for all $x,y \in M_{[-1,/2,1/2]} := [-1/2,0] \times N \cup [0,1/2] \times N \hookrightarrow M_1 \cup M_2$ and for $t \in (0,1]$ the estimate is satisfied uniformly with respect to $v \in \mathbb{R}^2 \setminus (0,0)$*

$$|E_{t,x,y}^\bullet(v)| \leqq c_m t^m , \qquad (2.116)$$

(where c_m is independent of t and of v).

Proposition 2.19. *The following estimate holds uniformly with respect to s for $-(1-b) < s < 1$ and to v for v sufficiently close to v_0*

$$\int_0^T t^{s-1}dt \bigg|_{M_{[-1/2,1/2]}} \int \mathrm{tr}(*i_k^* e_{t,x_1,x_2}^\bullet(v)) - \mathrm{tr}(*i_k^* e_{t,x_1,x_2}^\bullet(v_0))\bigg| < \varepsilon/4 . \qquad (2.117)$$

Remark. 2.9. For $x,y \in [-1,1] \times N$ the equalities hold (analogous to (2.54), (2.55), (2.40)):

$$E_{t,x,y}^\bullet(v;0) = \begin{cases} E_{t,x,y}^\bullet + (\beta^2 - \alpha^2/\beta^2 + \alpha^2)(\sigma_1^* E^\bullet)_{t,x,y} & \text{for } x,y \in Q_1 \times N, \\ E_{t,x,y}^\bullet + (\alpha^2 - \beta^2/\beta^2 + \alpha^2)(\sigma_1^* E^\bullet)_{t,x,y} & \text{for } x,y \in Q_2 \times N, \quad (2.118) \\ (2\alpha\beta/\alpha^2 + \beta^2)E_{t,x,y}^\bullet & \text{for } x,y \text{ from different } Q_k \times N . \end{cases}$$

Here, $E_{t,x,y}^\bullet$ is the kernel of $\exp(-t\varDelta^\bullet)$ on $I \times N$ with the Dirichlet boundary conditions on $\partial I \times N$ and σ_1 is the mirror symmetry with respect to $N = 0 \times N$ acting on the variable x of the kernel. So we get

$$\int_{M_{[-1/2,1/2]}} \mathrm{tr}(*_{x_2}i_k^* e_{t,x_1,x_2}^\bullet(v)) = \int_{M_{[-1/2, 1/2]}} \mathrm{tr}(*_{x_2}i_k^*(E_{t,x_1,x_2}^\bullet(v) - E_{t,x_1,x_2}^\bullet)) . \qquad (2.119)$$

From this equality and from the estimate (2.117) it follows that the integral over $M_{-1/2,1/2]}$ of the difference between the kernels for v and for v_0 gives the term in (2.115) which is less than $\varepsilon/4$.

Proposition 2.20. *The following estimate holds uniformly with respect to s, $-(1-b) < s < 1$, and to v for v sufficiently close to v_0:*

$$\int_0^T \left| \int_{M\setminus M_{[-1/2,1/2]}} \mathrm{tr}(*_{x_2} i^* E^\bullet_{t,x_1,x_2}(v)) - \mathrm{tr}(*_{x_2} i^* E^\bullet_{t,x_1,x_2}(v_0)) \right| dt < \varepsilon/4 . \qquad (2.120)$$

The estimate (2.115) is a consequence of (2.117), (2.119), and (2.120). The estimate (2.113) follows from (2.114), (2.115). The estimate (2.115) together with (2.112) gives us the continuity of $\Delta^\bullet(v;a)$ in v at v_0. Thus we get the proofs of Propositions 2.17 and 2.1. $\square$

Proof of Proposition 2.18. The following equality is obtained similarly to (2.98) by using of (2.99):

$$e_{t,x,y}(v) = - \lim_{\varepsilon \to +0} \int_\varepsilon^{t-\varepsilon} d\tau \partial/\partial\tau \int_{\partial I \times N} [((r\delta)_* E^\bullet_{\tau,x,*}(v), A_* E^\bullet_{t-\tau,*,y}(v;0))_{\partial I \times N}$$

$$+ (r_* E^\bullet_{\tau,x,*}(v), (Ad)_* E^\bullet_{t-\tau,*,y}(v;0))_{\partial I \times N}] , \qquad (2.121)$$

where the operators r and A correspond to the pair $(I \times N, \partial I \times N)$. So the estimate (2.116) follows from the analogous estimates for the kernels

$$r_z E^\bullet_{t,x,z}(v), \quad (r\delta)_z E^\bullet_{t,x,z}(v), \quad A_z E^\bullet_{t,z,y}(v;0), \quad (Ad)_z E^\bullet_{t,z,y}(v;0) , \qquad (2.122)$$

where $x, y \in M_{[-1/2,1/2]}$ and $z \in \partial I \times N = \{-1,1\} \times N \hookrightarrow M$. Such estimates are derived with the help of Proposition 2.5 for $\Delta^\bullet_v$ and $\Delta^\bullet_{v;0}$ as follows.

Let $m \in \mathbb{Z}_+$ be taken large enough. Then there is an approximate fundamental solution $P^{\bullet(m)}(v)$ for $(\partial_t + \Delta^\bullet_{v,x})$ which is the sum of an interior term $P^{\bullet(m)}_{\mathrm{int}}$ and of terms, defined near the boundaries ∂M and N. The kernel $P^{\bullet(m)}(v)$ is a good approximation for $E^\bullet_{t,x,y}(v)$ for small $t > 0$. Its interior term is defined as follows. For any closed Riemannian (M, g_M) there is a locally defined *parametrix* $p^{\bullet(m)}_{t,x,y}$ (i.e., an approximate fundamental solution for $(\partial_t + \Delta^\bullet_M)$) such that the difference $(p^\bullet(m) - E^\bullet)_{t,x,y}$ (where $E^\bullet_{t,x,y}$ is the fundamental solution for $(\partial_t + \Delta^\bullet_M)$) is a C^∞-double form for $t > 0$ and such that the following estimates hold uniformly with respect to $(x, y) \in M \times M$ and to $t \in (0, T]$:

$$|p^{\bullet(m)}_{t,x,y} - E^\bullet_{t,x,y}| \leqq c_m t^{-n/2+m+1} ,$$

$$|(\Delta^\bullet_x)^k (p^{\bullet(m)}_{t,x,y} - E^\bullet_{t,x,y})| \leqq c_{m,k} t^{-n/2+m+1-k}$$

$$|(\partial_t + \Delta^\bullet_M) p^{\bullet(m)}_{t,x,y}| < c_{(m)} t^{-n/2+m} ,$$

$$|(\Delta^\bullet_x)^k (\partial_t + \Delta^\bullet_M) p^{\bullet(m)}_{t,x,y}| < c_{(m,k)} t^{-n/2+m-k} \qquad (2.123)$$

([RS], Proposition 5.3, [BGV], Theorems 2.20, 2.23, 2.26, 2.30). Such a parametrix can be represented in the following form ($n := \dim M$):

$$p^{\bullet(m)}_{t,x,y} = (4\pi t)^{-n/2} \exp\left(-d(x,y)^2/4t\right) f(d(x,y)) \sum_{i=0}^m t^i \Phi_i(x,y) , \qquad (2.124)$$

where $d(x, y)$ is the geodesic distance between x and y, $f \in C_0^\infty(\mathbb{R}_+), f(\tau) \equiv 1$ for $0 \leqq \tau \leqq \varepsilon$ and $f \equiv 0$ for $\tau > 2\varepsilon$. The injectivity radius $i(M, g_M)$ is supposed to be greater than 2ε, i.e., the exponential map $\exp_x B_{2\varepsilon}$ is a diffeomorphism on its image for any $x \in M$ (where $B_{2\varepsilon} := \{\xi \in T_x M, |\xi| \leqq 2\varepsilon\}$). The coefficients $\Phi_i(x, y)$ in (2.124) are smooth double forms on $M \times M$ whose germs on the diagonal $M \hookrightarrow M \times M$ are unique. The principal term $\Phi_0(x, y)$ is the kernel of the parallel transport in $\wedge^\bullet TM$ along the geodesic line $\exp_y \xi = x$ from y into x (and it is defined for $d(x, y) < i(M, g_M)$). Each $\Phi_i(x, y)$ is determined through $\Phi_{i-1}(x, y)$ in differential geometry terms and it is well-defined for $d(x, y) < i(M, g_M)$ ([RS], Sect. 5, [BGV], Theorem 2.26, Lemma 2.49).

Let $0 \times N \hookrightarrow I \times N \hookrightarrow M$ be a neighborhood of $N = 0 \times N \hookrightarrow M$, where the metric g_M is a direct product. The fundamental solution for $(\partial_t + \Delta_v^\bullet)$ on $I \times N$ (with the Dirichlet boundary conditions on $\partial I \times N$) is

$$\mathscr{P}^\bullet(v) = \sum_{i=0,1} E_{I,t}^i(v) \otimes E_{N,t}^{\bullet - i}, \tag{2.125}$$

where $E_{I,t}^\bullet(v)$ is defined by the formulas completely analogous to (2.40) and (2.118). (Here, $E_{N,t}^\bullet$ is the fundamental solution for $(\partial_t + \Delta_N^\bullet)$. The operator corresponding to the kernel $E_{I,t}^i(v) \otimes E_{N,t}^{\bullet - i}$ acts on $DR^i(I, \partial I) \otimes DR^{\bullet - i}(N)$. The kernel $E_{I,t}^i(v)$ corresponds to the Laplacian on I with the Dirichlet boundary conditions on ∂I and with the v-transmission boundary conditions at $0 \in I$. The term in (2.125) with $i = 1$ is equal to zero for $\bullet = 0$.)

The parametrix $P_{t,x,y}^{\bullet(m)}(v)$ for $(\partial_t + \Delta_v^\bullet)$ on M is defined by

$$P_{t,x,y}^{\bullet(m)}(v) = \psi \mathscr{P}_{t,x,y}^\bullet(v) \varphi + \psi_1 p_{t,x,y}^{\bullet(m)}(1 - \varphi). \tag{2.126}$$

Here, $\varphi = \varphi(y_1)$, $\psi = \psi(x_1)$ (in the coordinates $(x_1, x') = x$ and $(y_1, y') = y$ of points in $I \times N$), $\varphi, \psi \in C_0^\infty(I, \partial I)$; $\varphi, \psi \geqq 0$, $\varphi(y_1) \equiv 1$ for $|y_1| \leqq 2^{-1} + \varepsilon$, $\varphi(y_1) \equiv 0$ for $|y_1| \geqq 5/8$, $\psi \equiv 1$ in a neighborhood of supp φ, and $\psi \equiv 0$ for $|x_1| \geqq 3/4$, $\psi_1 \in C_0^\infty(M \backslash N)$, $\psi_1 \equiv 1$ in a neighborhood of supp $(1 - \varphi) \subset M \backslash M_{[-1/2, 1/2]}$. Hence the parametrix $P_{t,x,y}^{\bullet(m)}$ is equal to zero for $y \in M_{[-1/2, 1/2]}$, $x \in M \backslash ([-3/4, 3/4] \times N)$. The term $\psi_1 p^{\bullet(m)}(1 - \varphi)$ in (2.126) is defined from now on as $P_{\text{int}}^{\bullet(m)}$. (In the case of $\partial M \neq \emptyset$ the terms, completely analogous to $\mathscr{P}^\bullet(v)$ for $v \in \{(0, 1), (1, 0)\}$ have to be added to $P^{\bullet(m)}$. Their supports are in $([0, 1] \times \partial M)^2 \hookrightarrow M \times M$ and g_M is a direct product metric on $[0, 1] \times \partial M$.)

Proposition 2.21. 1. *The boundary condition for* $P_{t,x,y}^{\bullet(m)}(v)$ *on* N *and on* ∂M *and the boundary condition for* $(\Delta_x^\bullet)^k P_{t,x,y}^{\bullet(m)}(v)$ $(k \in \mathbb{Z}_+)$ *are the same as for* $E_{t,x,y}^\bullet(v)$ *and for* $(\Delta_{v,x}^\bullet)^k E_{t,x,y}^\bullet(v)$. *Namely* $P_{t,x,y}^{\bullet(m)}(v)$ *is a smooth in* $t > 0$ *and in* $(x, y) \in \overline{M}_{j_1} \times \overline{M}_{j_2}$ *kernel,* $\Delta_{v,x}^k P_{t,x,y}^{\bullet(m)}(v) \subset D(\Delta_{v,x}^\bullet)$ *for fixed* $y, t > 0$, *and for any* $k \in \mathbb{Z}_+ \cup 0$. *Here* $D(\Delta_{v,x}^\bullet) \subset DR^\bullet(M_v, Z)$ *is the domain of definition of* $\Delta_{v,x}^\bullet$ *on pairs* (ω_1, ω_2) *of smooth forms on* $\overline{M}_j$. *It is defined by* (1.27)).

2. *The following estimates (analogous to* (2.123)) *hold for* $t \in (0, T]$ *uniformly with respect to* $v \in \mathbb{R}^2 \backslash (0, 0)$ *and to* $(x, y) \in \overline{M}_{j_1} \times \overline{M}_{j_2}$ *(with* $C_m, C_{m,k}$ *independent of* t):

$$|(\partial_t + \Delta_{v,x}^\bullet) P_{t,x,y}^{\bullet(m)}(v)| < C_m t^{-n/2+m}, \tag{2.127}$$

$$|(\Delta_{v,x}^\bullet)^k (\partial_t + \Delta_{v,x}^\bullet) P_{t,x,y}^{\bullet(m)}(v)| < C_{m,k} t^{-n/2+m-k}. \tag{2.128}$$

The kernel $r^{\bullet(m)}_{t,x,y}(v) := (\partial_t + \Delta^\bullet_{v,x})P^{\bullet(m)}_{t,x,y}(v)$ *is smooth in* $(x,y) \in M \times \overline{M}_j$ *and its* C^{2l}-norm on $M \times \overline{M}_j$ *is estimated through* $c_{m,l} t^{-n/2+m-l}$. *For any linear differential operator* F *of order* $d = d(F)$, *acting on double forms on* $M \times \overline{M}_j$, *and for any* $T > 0$ *the kernel* $F \circ r^{\bullet(m)}_{t,x,y}(v)$ *satisfies the estimate as follows when* $t \in (0,T]$. *It holds uniformly with respect to* $(x,y) \in M \times \overline{M}_j$ *and to* $v \in \mathbb{R}^2 \setminus (0,0)$,

$$|F \circ r^{\bullet(m)}_{t,x,y}(v)| < c(F)t^{-(n+d)/2+m} . \tag{2.129}$$

3. *For* $0 \leq k \leq [-n/2] + m - 1$ *the following condition is satisfied uniformly with respect to* $v \in \mathbb{R}^2 \setminus (0,0)$ *and to* $(x,y) \in \overline{M}_{j_1} \times \overline{M}_{j_2}$:

$$\lim_{t \to +0}(\Delta^\bullet_{v,x})^k(E^\bullet - P^{\bullet(m)})_{t,x,y}(v) = 0 . \tag{2.130}$$

Corollary 2.8. 1. *For* $k \in \mathbb{Z}_+$, $0 \leq k \leq [-n/2] + m - 1$, *the following estimates of* L_2-norms (*with respect to the variable* x) *of the kernel* $(P^{\bullet(m)}(v) - E^\bullet(v))_{t,x,y}$ *hold for* $t \in (0,T]$ *uniformly in* $y \in \cup_{j=1,2}\overline{M}_j$ *and in* $v \in \mathbb{R}^2 \setminus (0,0)$, *where* C_m *and* $C_{m,k}$ *are the same as in* (2.127) *and* (2.128):

$$\|(E^\bullet(v) - P^{\bullet(m)}(v))_{t,*,v}\|_2 \leq C_m(-n/2 + m + 1)^{-1}t^{-n/2+m+1} , \tag{2.131}$$

$$\|\Delta^k_{v,*}(E^\bullet(v) - P^{\bullet(m)}(v))_{t,*,v}\|_2 \leq C_{m,k}(-n/2 + m + 1 - k)^{-1}t^{-n/2+m+1-k} . \tag{2.132}$$

2. *The following estimate for* $E^\bullet_{t,x,y}(v)$ *holds for* $t \in (0,T]$ *and for an arbitrary* $q \in \mathbb{Z}_+$ *uniformly with respect to* $v \in \mathbb{R}^2 \setminus (0,0)$, *to* $x \in M \setminus ([-3/4, 3/4] \times N)$, *and to* $y \in M_{[-1/2,1/2]}$:

$$|E^\bullet_{t,x,y}(v)| \leq C_q t^{-n/2+q} . \tag{2.133}$$

It holds according to (2.131), (2.132), *and* (2.32). (*Indeed, for such* x, y *and for an arbitrary* $m \in \mathbb{Z}_+$ *we have* $E^\bullet_{t,x,y}(v) = (E^\bullet(v) - P^{\bullet(m)}(v))_{t,x,y}$. *Hence* m *can be chosen large enough to get* (2.133).)

The estimate (2.131) is a consequence of (2.127) and of the following equality:

$$(E^\bullet - P^{\bullet(m)})_{t,x,y}(v) = \lim_{\varepsilon \to +0} \int_\varepsilon^{t-\varepsilon} d\tau \partial_\tau (E^\bullet_{t-\tau,x,*}(v), (E^\bullet - P^{\bullet(m)})_{\tau,*,y}(v))_M$$

$$= \lim_{\varepsilon \to +o} \int_\varepsilon^{t-\varepsilon} d\tau (E^\bullet_{t-\tau,x,*}(v), (\partial_\tau + \Delta^\bullet_{v,*})(E^\bullet - P^{\bullet(m)})_{\tau,*,y}(v))_M . \tag{2.134}$$

This equality follows from the assertions that $(E^\bullet - P^{\bullet(m)})_{t,x,y}(v) \to 0$ as $t \to +0$ and that $(E^\bullet - P^{\bullet(m)})_{\tau,x,y}(v) \in D(\Delta^\bullet_{v,x})$ with respect to the variable x.

The estimate (2.131) is a consequence of (2.134), (2.127), and (2.135), since the operator $\exp(-t\Delta^\bullet_v)$ for $t > 0$ is bounded in $(DR^\bullet(M))_2$ and its operator norm is less or equal to one:

$$\|\exp(-t\Delta^\bullet_v)\|_2 \leq 1 . \tag{2.135}$$

(This inequality follows from Theorems 3.1 and 3.2. They claim that $\Delta^\bullet_v$ is a non-negative self-adjoint unbounded operator in $(DR^\bullet(M))_2$ and that $\exp(-t\Delta^\bullet_v)$ is a trace class operator.)

The inequality (2.132) for $1 \leqq k \leqq [-n/2] + m - 1$ is a consequence of (2.127), (2.135), and of the following equality (which is a generalization of (2.134)):

$$\Delta_{v,x}^k (E^\bullet - P^{\bullet(m)})_{t,x,y}(v)$$

$$= \lim_{\varepsilon \to +0} \int_\varepsilon^{t-\varepsilon} (E^\bullet_{t-\tau,x,*}(v), (\Delta_{v,*}^\bullet)^k (\partial_\tau + \Delta_{v,*})(E^\bullet - P^{\bullet(m)})_{\tau,*,y}(v))_M .$$

This equality holds since $\Delta_{v,x}^k (E^\bullet - P^{\bullet(m)})_{t,x,y}(v) \to 0$ as $t \to +0$ (for $0 \leqq k \leqq [-n/2] + m - 1$) and since $(\Delta_{v,x}^\bullet)^k (E^\bullet - P^{\bullet(m)})_{t,x,y}(v) \in D(\Delta_{v,x}^\bullet)$ for fixed y and $t > 0$.

The proof of Proposition 2.21 is preceded by the proof of Proposition 2.18.

The estimates analogous to (2.133) (with $t \in (0,T]$ and $q \in \mathbb{Z}_+$) hold also for the kernel $E^\bullet_{t,x,y}(v; 0) = \mathscr{P}^\bullet_{t,x,y}$ of $\exp(-t\Delta_{v;0}^\bullet)$, where $x \in (I \setminus [-3/4, 3/4]) \times N$, $y \in M_{[-1/2,1/2]}$. Indeed, such estimates are true (n is replaced by 1) for $(E^1_{l,t}(v))_{x_1,y_1}$ with $x_1 \in I \setminus [-3/4, 3/4]$ and $y_1 \in [-1/2, 1/2]$, and the kernel $(E^{\bullet-1}_N)_{x',y'}$ is $O(t^{-(n-1)/2})$ for $t \in (0,T]$.

The desired estimates for $(r\delta)E^\bullet(v)$ and $Ad\, E^\bullet(v; 0)$ are obtained from the estimates (2.131), (2.132), and from the generalization of the inequality (2.32) as follows. Let K be an arbitrary first order differential operator acting on $DR^\bullet(\overline{M}_j)$. Let $\omega \in DR^\bullet(M_v, Z)$ obeys the conditions (2.33) with $m_1 = 1 + \min\{l : 4l \geqq \dim M + 1\}$. Then the inequality is satisfied uniformly with respect to ω and to $x \in \overline{M}_1 \cup \overline{M}_2$:

$$|K\omega(x)|^2 < C_K \sum_{l=0}^{m_1} \|\Delta_v^l \omega\|_2^2 , \tag{2.136}$$

where $C_K > 0$ is independent of $v \in \mathbb{R}^2 \setminus (0,0)$. The proof of (2.136) is exactly the same as the proof of (2.32) given above except the kernel $(G_I(v) \otimes G_N^{m_2})_{x,y}$ has to be replaced by $K_x(G_I(v) \otimes G_N^{m_2})_{x,y}$. Thus Proposition 2.18 is proved. $\square$

Proof of Proposition 2.21. 1. For x from a neighborhood of $N = 0 \times N \hookrightarrow M$, where $\psi_1 \equiv 0$, the parametrix $P^{\bullet(m)}_{t,x,*}(v)$ is equal to $\mathscr{P}^{\bullet(m)}_{t,x,*}(v)\varphi(*)$. So $P^{\bullet(m)}_{t,x,*}(v) \in D((\Delta_{v,x}^\bullet)^k)$ with respect to the variable x, since $\mathscr{P}^{\bullet(m)}_{t,x,*}(v) \in D((\Delta_{v,x}^\bullet)^k)$ for $k \in \mathbb{Z}_+$.

2. The estimates (2.127) and (2.128) hold for the term $\psi \mathscr{P}^\bullet(v)\varphi$ of $P^{\bullet(m)}(v)$ with an arbitrary $m \in \mathbb{Z}_+$, since $(\partial_t + \Delta_{v,x}^\bullet)\mathscr{P}^\bullet(v) = 0$ (for $x \in (I \setminus \partial I) \times N$) and since $\min_{x_1 \in A}\min_{y_1 \in \text{supp}\varphi}(|x_1 - y_1|, |x_1 + y_1|) > \delta > 0$ (where $A := \text{supp}(\partial_{x_1}\psi)$ and the number $\delta > 0$ is fixed). For $x_1 \in A$ and $y_1 \in \text{supp }\varphi$ the estimate (2.133) (n is replaced by 1) holds with an arbitrary $q \in \mathbb{Z}_+$ for the fundamental solution $(E^\bullet_{l,t})_{x,y}$ of $(\partial_t + \Delta_I^\bullet)$ with the Dirichlet boundary conditions on ∂I. The same estimate for such x_1, y_1 holds for the kernels $(\sigma_1^* E^\bullet_{l,t})_{x_1,y_1}$ and $(E^\bullet_{l,t}(v))_{x_1,y_1}$. So this estimate holds also for the kernel $(\mathscr{P}^\bullet(v))_{(x_1,x'),(y_1,y')}$ (defined by (2.126)), since the kernel $(E^\bullet_{N,t})_{x',y'}$ is $O(t^{-(n-1)/2})$ uniformly with respect to (x', y') ([RS], Proposition 5.3, [BGV], Theorem 2.23). By the analogous reasons, for such x_1, y_1 the estimate (2.133) with an arbitrary $q \in \mathbb{Z}_+$ holds for the kernel $(F \circ \mathscr{P}^\bullet(v))_{(x_1,x'),(y_1,y')}$, where F is a linear differential operator of finite order $d(F)$ on $M \times M$, acting on double forms on $\overline{M}_{j_1} \times \overline{M}_{j_2}$, and n in (2.133) is replaced by $n + d(F)$.

So the estimate (2.133) with an arbitrary $q \in \mathbb{Z}_+$ is satisfied by $(\partial_t + \Delta_{v,x}^\bullet)(\psi \mathscr{P}^\bullet \varphi)$ and by $(\Delta_{v,x}^\bullet)^k (\partial_t + \Delta_{v,x}^\bullet)(\psi \mathscr{P}^\bullet \varphi)$ with $k \in \mathbb{Z}_+$.

The estimates (2.127) and (2.128) hold for $\psi_1 p^{\bullet(m)}(1 - \varphi) =: P^{\bullet(m)}_{\text{int}}$, since they hold for $p^{\bullet(m)}_{t,x,y}$ and since the distance on M between the closure $\overline{B}$ of B (where

$B := \{x : d_x \psi_1 \neq 0\}$) and the support supp $(1 - \varphi)$ is greater than a positive number δ. Hence the uniform with respect to $(x, y) \in \bar{B} \times$ supp $(1 - \varphi)$ estimate (2.133) (with an arbitrary $q \in \mathbb{Z}_+$) is satisfied by $p_{t,x,y}^{\bullet(m)}$. This estimate holds also for the kernel $F p_{t,x,y}^{\bullet(m)}$, where F is a linear differential operator of finite order $d(F)$, acting on the smooth kernels, defined on $M \times M$. (For instance, the function $f(d(x, y))$ in the definition (2.124) of $p_{t,x,y}^{\bullet(m)}$ can be chosen such that $f(\tau) \equiv 0$ for $\tau \geqq \delta$. These estimates follow also from [RS], Proposition 5.3, estimates (5.5), and from [BGV], Theorem 2.23(2).)[40]

3. The difference $(E^\bullet - P^{\bullet(m)})_{t,x,y}(v)$ can be written as the Volterra series (analogous to [BGV], 2.4):

$$(E^\bullet - P^{\bullet(m)})_{t,x,y}(v)$$
$$= \sum_{l \geqq 1} (-t)^l \int_{\Delta_l} \int_{(y_1,\dots,y_l)\in(\overline{M}_1 \cup \overline{M}_2)^l} P_{\sigma_0 t,x_1,y_1}^{\bullet(m)}(v) r_{\sigma_1 t,y_1,y_2}^{(m)}(v) \dots r_{\sigma_l t,y_l,y}^{(m)}(v) , \quad (2.137)$$

where $\Delta_l = \{(\sigma_0, \dots, \sigma_l) : 0 \leqq \sigma_i \leqq 1, \sum \sigma_i = 1\}$ and $r_{t,x,y}^{(m)}(v) := (\Delta_{v,x} + \partial_t) P_{t,x,y}^{\bullet(m)}(v)$ (a scalar product tr$(\omega_1 \wedge *\omega_2)$ with its values in densities on M is assumed in (2.137)). The assertion (2.130) follows from the convergence of the series (2.137) in the topology of uniform convergence of smooth kernels on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ together with their partial derivatives of orders $\leqq 2k$ on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ (i.e., in the C^{2k}-topology on $\overline{M}_{j_1} \times \overline{M}_{j_2}$).

Indeed, the definition of $P_{t,x,y}^{\bullet(m)}$ (2.126) implies that the kernel $r_{t,x,y}^{(m)}(v)$ is equal to zero for x from a fixed (independent of $t, v,$ and m) neighborhood of $N = 0 \times N \hookrightarrow M$ in M. So $r_{t,x,y}^{(m)}(v)$ is a smooth kernel on $M \times \overline{M}_j$, and the inequalities (2.129) claim that the C^{2k}-norm of $r_{t,x,y}^{(m)}(v)$ is $O(t^{-n/2+m-k})$ for $t \in (0, T]$ uniformly with respect to $v \in \mathbb{R}^2 \setminus (0,0)$. It is $O(t)$ for $0 \leqq k \leqq [-n/2] + m - 1$, and the series (2.137) is convergent in the C^{2k}-topology for such k, since the volume of Δ_l is $(l!)^{-1}$ and since the following assertion is true. For any $T > 0$ the parametrix $P_{t,x,y}^{\bullet(m)}(v)$ defines a family of bounded operators from the space of smooth forms $DR^\bullet(M)$ (equipped with a C^{2k}-norm) into the space $\bigoplus_{k=1,2} DR^\bullet(\overline{M}_j)$ (equipped with a C^{2k}-norm on $DR^\bullet(\overline{M}_j)$). These operators are bounded uniformly with respect to $t \in (0, T]$ and to $v \in \mathbb{R}^2 \setminus (0,0)$. This assertion for the operators, corresponding to $P_{\text{int},t}^{\bullet(m)}$, is proved in [BGV], Lemma 2.49. It is also true for the operators corresponding to $\psi \mathscr{P}_t^{\bullet(m)}(v)\varphi$. Indeed, it holds for the operators $\exp(-t\Delta_N^\bullet)$ in $DR^\bullet(N)$ (equipped with a C^{2k}-norm and for the operators $\psi \exp(-t\Delta_I^\bullet)\varphi$ in $DR^\bullet(I)$ equipped with a C^{2k}-norm. (Here $\Delta_I^\bullet$ is defined on forms with the Dirichlet boundary conditions on ∂I.) It holds also for the operators with the kernels $\psi(x_1)(\sigma_1^* E_{I,t}^\bullet)_{x_1,y_1} \varphi(y_1)$, acting from smooth forms on I into smooth forms on $[0, \pm 1]$ (where the Dirichlet boundary conditions are assumed on ∂I and σ_1 is the mirror symmetry with respect to $0 \in I$).

The C^{2k}-norm of the kernel $(P_{N,t}^{\bullet(m)} - E_{N,t}^\bullet)$ on $N \times N$ is $O(t^{m-(n-1)/2-k})$ for $t \in (0, T]$, where $P_{N,t}^{\bullet(m)}$ and $E_{N,t}^\bullet$ are the parametrix of the type (2.124) and the fundamental solution for $(\partial_t + \Delta_N^\bullet)$ ([BGV], Theorem 2.30). The operators in $DR^\bullet(N)$ corresponding to $P_{N,t}^{\bullet(m)}$ are uniformly bounded for $t \in (0, T]$ with respect to a C^{2k}-norm ([BGV], Lemma 2.49). So the operators $\exp(-t\Delta_N^\bullet)$ in $DR^\bullet(N)$ are uniformly

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[40] For the sake of brievity the proof of Proposition 2.21 is written in the case of $\partial M = \emptyset$.

bounded for $t \in (0.T]$ with respect to a C^{2k}-norm. The convergence of the series on the right in (2.137) with the respect to C^{2k}-norms ($k \leqq [-n/2]+m-1$) for the kernels on $\overline{M}_{J_1} \times \overline{M}_{J_2}$ implies also a proof of the equality (2.137) (in the case of $m \geqq -[-n/2]+2$). Indeed, we have[41]

$$(\partial_t + \Delta^{\bullet}_{v,x})(P^{\bullet(m)}_l)_{t,x,y} = (r^{\bullet(m)}_l + r^{\bullet(m)}_{l+1})_{t,x,y} , \tag{2.138}$$

where $(-1)^m P^{\bullet(m)}_l$ is the term with the number l in the right side of (2.137), $(-1)^m r^{\bullet(m)}_l$ is the same term in which $P^{\bullet(m)}_{\sigma_0 t}$ is replaced by $r^{\bullet(m)}_{\sigma_0 t}, P^{\bullet(m)}_0 := P^{\bullet(m)}$. For any fixed y and $t > 0$ and for any $k \in \mathbb{Z}_+ \cup 0$ we have $(\Delta^k_{v,x})(P^{\bullet(m)}_l)_{t,x,y} \subset D(\Delta^{\bullet}_{v,x})$. The series $P^{\bullet}_t = \sum_{l \geqq 0}(-1)^m P^{\bullet(m)}_l$ for $m \geqq [-n/2]+2$ is the fundamental solution for $(\partial_t + \Delta^{\bullet}_v)$ since $(\partial_t + \Delta^{\bullet}_{v,x})P^{\bullet}_t = 0$ for $t > 0$ and since the operator corresponding to $P^{\bullet(m)}_t$ tends in a weak sense to the identity operator in $(DR^{\bullet}(M))_2$ as $t \to +0$ (i.e., $P^{\bullet(m)}_t \omega \to \omega$ as $t \to +0$ for $\omega \in (DR^{\bullet}(M))_2$). The latter assertion holds for $P^{\bullet(m)}_{t,int}(1 - \varphi)\omega$ and for $\mathscr{P}^{\bullet(m)}\varphi\omega$. Proposition 2.21 is proved. □

Proof of Proposition 2.19. Proposition 2.18 implies the following conclusion. For any $\varepsilon > 0$ there exist a $\delta > 0$ and a neighborhood $U := U(v_0, \varepsilon) \subset \mathbb{R}^2 \setminus (0,0)$ of v_0 such that the estimate holds uniformly with respect to $v \in U$ and to s, $-1 < s < 1$:

$$\left| \int\limits_0^{\delta} t^{s-1}dt \int\limits_{M_{[-1/2,1/2]}} \sum_{k=1,2} \mathrm{tr}(*i^*_k e^{\bullet}_t(v)) \right| < \varepsilon/20 .$$

So it is enough to prove the existence of a neighborhood U_1 of v_0 such that for any $v \in U_1$ the following estimate holds uniformly with respect to s for $-(1 - b) < s < 1$ ($b, 0 < b < 1$, is fixed):

$$\int\limits_{\delta}^{T} t^{s-1}dt \left| \int\limits_{M_{[-1/2,1/2]}} \sum_{k=1,2} (\mathrm{tr}(*i^*_k e^{\bullet}_t(v)) - \mathrm{tr}(*i^*_k e^{\bullet}_t(v_0))) \right| < \varepsilon/10 . \tag{2.139}$$

For $e(v)$ and $e(v_0)$ the equalities (2.121) hold. So the estimate (2.139) takes place for v sufficiently close to v_0 since the convergence

$$E^{\bullet}_{t,x,y}(v) \to E^{\bullet}_{t,x,y}(v_0) \tag{2.140}$$

is uniform with respect to $t \in [\delta_1, T]$ (where $\delta_1 > 0$ is fixed), to $x \in M \setminus ((-3/4, 3/4) \times N)$, and to $y \in M_{[-1/2,1/2]}$. The convergence of the kernels

$$d_x E^{\bullet}_{t,x,y}(v) \to d_x E^{\bullet}_{t,x,y}(v_0), \quad \delta_x E^{\bullet}_{t,x,y}(v) \to \delta_x E^{\bullet}_{t,x,y}(v_0) \tag{2.141}$$

is a uniform one for such (t,x,y). All the double forms in (2.140) and (2.141) are uniformly bounded on the set of such (t,x,y) and their norms at (t,x,y) satisfy the upper estimate for $t \in (0, \delta_1]$ (obtained in Proposition 2.18 above) through $c_m t^m$ with an arbitrary $m \in \mathbb{Z}_+$ and with c_m independent of v. The uniform convergence of the kernels in (2.140) and (2.141) on the compact set of (t,x,y) defined above follows

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[41] The proof of (2.138) is analogous to the one given in [BGV], Lemma 2.22. It follows from the formula for $\partial_t \int_0^t f(x,t)dx$.

from the continuity in t, x, y and v for $(x, y) \in \overline{M}_{j_1} \times \overline{M}_{j_2}$ of the corresponding double forms. (See Proposition 3.2, where it is proved that these double forms are C^∞-smooth in t, x, y and v for $\operatorname{Re} t > 0$ and $v \neq (0, 0)$.) $\square$

Proof of Proposition 2.20. If $(\alpha_0, \beta_0) = v_0 \in U$ (i.e., if $\alpha_0 \cdot \beta_0 \neq 0$) then we can suppose that $v \in U$ in (2.120). In this case, the identity (2.100) holds for the difference

$$(E^\bullet(v) - E^\bullet(v_0))_{t, x_1, x_2} .\tag{2.142}$$

Let $v_0 \in \mathbb{R}^2 \setminus (U \cup (0, 0))$. For example, let $v_0 := (\alpha_0, 0), \alpha_0 \neq 0$. Then the identities (2.98) and (2.99) claim in the cases of v_0 and of $v := (\alpha, \beta)$ that (2.142) is equal to

$$\left(\frac{\beta}{\alpha} \right) \lim_{\varepsilon \to +0} \int_\varepsilon^{t-\varepsilon} d\tau [-(r_{*, \partial M_2} \delta E^\bullet_{\tau, x_1, *}(v), A_{*, \partial M_1} E^\bullet_{t-\tau, *, x_2}(v_0))_N$$

$$+ (A_{*, \partial M_1} E^\bullet_{\tau, x_1, *}(v), r_{*, \partial M_2} \delta_* E^\bullet_{t-\tau, *, x_2}(v_0))_N$$

$$+ (A_{*, \partial M_1} d_* E^\bullet_{\tau, x_1, *}(v), r_{*, \partial M_2} E^\bullet_{t-\tau, *, x_2}(v_0))_N$$

$$- (r_{*, \partial M_2} E^\bullet_{\tau, x_1, *}(v), A_{*, \partial M_1} d_* E^\bullet_{t-\tau, *, x_2}(v_0))_N] .\tag{2.143}$$

The factor $k_v^{-1} = \beta/\alpha$ in (2.143) tends to zero as v tends to v_0. The factors $(1 - k_v/k_{v_0})$ and $(1 - k_{v_0}/k_v)$ in (2.100)) also tend to zero as $v \to v_0$ in the case $v_0, v \in U$. The estimate (2.120) follows from (2.143) and (2.100). Indeed, there are the uniform with respect to v upper estimates (analogous to (2.116)) for the kernels (2.122), where $x, y \in M \setminus M_{[-1/2, 1/2]}$ and $z \in N = 0 \times N \subset \partial M_j$. These estimates follows from (2.32) and their proofs are completely analogous to the proof of (2.116). The main step in these proofs is using the parametrix $P^{(m)}$[42] and the estimates (2.127), (2.128), and (2.132) for $t \in (0, T]$. $\square$

2.8. Dependence on the Phase of a Cut of the Spectral Plane. Analytic Torsions as Functions of the Phase of a Cut. Gluing Formula for Analytic Torsions. The scalar analytic torsion (2.11) depends not only on (M, g_M, Z, v) but also on the phase θ of a cut on the spectral plane $\mathbb{C} \ni \lambda$. A zeta-function $\zeta_{v, \bullet}(s; \theta)$ is defined for $\operatorname{Re} s > n/2$ ($n := \dim M$) as the sum of absolutely convergent series $\sum m(\lambda_j) \lambda_{j, \theta}^{-s}$, where the sum is over nonzero $\lambda_j \in \operatorname{Spec}(\Delta_v^\bullet)$ and $m(\lambda_j)$ are the multiplicities of λ_j. The function $\lambda_{j, \theta}^{-s} := \exp(-s \log_{(\theta)} \lambda_j)$, $\theta > \operatorname{Im}(\log_{(\theta)} \lambda_j) > \theta - 2\pi$, is defined for $\theta \notin \arg \lambda_j + 2\pi\mathbb{Z}$. (For positive self-adjoint operators this condition means that $\theta \notin 2\pi\mathbb{Z}$.) All the results for the analytic torsion norm are obtained above in the case of $0 < \theta < 2\pi$ (for instance, for $\theta = \pi$).

The zeta-function $\zeta_{v, \bullet}(s; \theta)$ does not depend on $\theta \notin 2\pi\mathbb{Z}$, if $[\theta/2\pi]$ does not change. However we have

$$\zeta_{v, \bullet}(s; \theta + 2\pi) = \exp(-2\pi i s) \zeta_{v, \bullet}(s; \theta) \quad \text{for } \operatorname{Re} s > n/2 .\tag{2.144}$$

Since $\zeta_{v, \bullet}(s; \pi)$ can be meromorphically continued to the complex plane $\mathbb{C} \ni s$ (Theorem 3.1 below), we see that $\zeta_{v, \bullet}(s; \theta)$ for $\theta \notin 2\pi\mathbb{Z}$ also can be meromorphically

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[42] The parametrix $P^{(m), \bullet}(v)$ for $E^\bullet(v)$ can be chosen such that $P^{(m), \bullet}_{t, \lambda, z}(v) = 0$ for $x \notin [-1/3, 1/3] \times N \hookrightarrow M$ and $z \in [-1/6, 1/6] \times N \hookrightarrow M$.

continued to $\mathbb{C}$ with at most simple poles at $s_j := (n - j)/2$. The continuation of $\zeta_{v,\bullet}(s;\theta)$ is regular at s_j for $(-s_j) \in \mathbb{Z}_+ \cup 0$. So the equality (2.144) holds for all $\theta \notin 2\pi\mathbb{Z}$, $s \in \mathbb{C}$. Hence for such θ we have

$$\zeta_{v,\bullet}(s;\theta) = \zeta_{v,\bullet}(s)\exp(-2\pi ism) \,, \qquad (2.145)$$

where $\zeta_{v,\bullet}(s) := \zeta_{v,\bullet}(s;\pi)$ and $m := [\theta/2\pi], \theta \notin 2\pi\mathbb{Z}$. From now on this ζ-function will be denoted as $\zeta_{v,\bullet}(s;m)$ with $m = [\theta/2\pi]$. The value $\zeta_{v,\bullet}(0,m)$ is independent of m (according to (2.145)).

The dependence of the scalar analytic torsion (2.11) on m is given by

$$T(M_v, Z; m) = T(M_v, Z)\exp(-2\pi imF(M_v, Z)) \,, \qquad (2.146)$$

where[43] $F(M_v, Z) := \sum(-1)^j j\zeta_{v,\bullet}(0)(\mathrm{mod}\ \mathbb{Z})$ and $T(M_v, Z) := T(M_v, Z; 0)$, i.e., $T(M_v, Z)$ corresponds to $\theta = \pi$ and it is the scalar analytic torsion defined by (2.11). Here Z is the union of the connected components of ∂M where the Dirichlet boundary conditions are given. The Laplacian $\Delta_v^\bullet$ is defined on $\omega \in DR^\bullet(M_v, Z)$ with the v-transmission boundary conditions (1.27) on N, with the Dirichlet and the Neumann boundary conditions on Z and on $\partial M \setminus Z$. The equality (2.146) is obtained by using

$$\partial_s\zeta_{v,\bullet}(s;m)|_{s=0} = -2\pi im\zeta_{v,\bullet}(0) + \partial_s\zeta_{v,\bullet}(s)|_{s=0} \,.$$

The number $F(M, Z)$ is defined also in the case of a manifold M without an interior boundary N. In this case, $\zeta_{v,\bullet}(0)$ in the definition of $F(M, Z)$ is replaced by $\zeta_\bullet(0)$ for the Laplacians on $DR^\bullet(M, Z)$. The dependence of $T(M, Z; m)$ on m is given by (2.146) with $F(M_v, Z)$ replaced by $F(M, Z)$. In particular, $F(M)$ is defined for a closed M and also in the case $\partial M \neq \emptyset$, $Z = \emptyset$. Let M be obtained by gluing two pieces M_1 and M_2 along the common component N of their boundaries, $M = M_1 \cup_N M_2$, where $N \subset M$ is closed and of codimension one. Then $F(M, Z) = F(M_{1,1}, Z)$, according to Proposition 1.1.

The class of $F(M_v, Z)$ in $\mathbb{C}/\mathbb{Z}$ is the same as the class $(\mathrm{mod}\,\mathbb{Z})$ of the number $F_1(M_v, Z) \in \mathbb{C}$, where

$$F_1(M_v, Z) := \sum(-1)^i j(\zeta_{v,j}(0) + \dim\mathrm{Ker}(\Delta_{v,j})) \,.$$

The Laplacian $\Delta_{v,j}(M_v, Z)$ with its domain $\mathrm{Dom}\,(\Delta_{v,j}) \subset (DR^j(M))_2$ is self-adjoint according to Theorem 3.1. For $\mathrm{Re}\,s > 2^{-1}\dim M$ the zeta-function $\zeta_{v,j}(s)$ is defined by the absolutely convergent series $\sum m(\lambda_k)\exp(-s\log\lambda_k)$ ($[\theta/2\pi] = 0, \theta \neq 0$), where the sum is over $\lambda_k \in \mathrm{Spec}(\Delta_{v,j})$, $\lambda_k \neq 0$, and with the branch of logarithm $-\pi < \mathrm{Im}\log\lambda < \pi$. Because $\log\lambda_k \in \mathbb{R}$ for $\lambda_k > 0$, the function $\zeta_{v,j}(s)$ is real for real s. Hence $F_1(M_v, Z) \in \mathbb{R}$. It is supposed from now on that a metric g_M on $M = M_1 \cup_N M_2$ is a direct product metric near N and near ∂M.

Proposition 2.22 1. *For a closed manifold M the number $F_1(M)$ is an integer.*

2. *Let $M = M_1 \cup_N M_2$ be obtained by gluing two its pieces M_1 and M_2 along the common component N of their boundaries. Let the v-transmission boundary conditions (1.27) be given on N, the Dirichlet boundary conditions be given on a union Z of some connected components of ∂M and the Neumann boundary*

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[43] By the definition, $F(M_v, Z) \in \mathbb{C}/\mathbb{Z}$ but it can be also defined as $\sum(-1)^j j\zeta_{v,j}(0) \in \mathbb{C}$.

conditions be given on $\partial M \setminus Z$. Let L be a closed Riemannian manifold. Then the following holds. [44]

$$F_1(M_v \times L, Z \times L) = \chi(L)F_1(M_v, Z) + F_1(L)\chi(M_v, Z) . \tag{2.147}$$

3. Let $K \subset \partial M \setminus Z$ be a union of some connected components of ∂M. Then the following holds under the conditions on M above:

$$F_1(M_v, Z) = F_1(M_v, Z \cup K) + F_1(K) + 2^{-1}\chi(K) . \tag{2.148}$$

4. Under the conditions on M above, the number $F_1(M_v, Z)$ obeys a gluing property analogous to the gluing property (2.1) for the analytic torsion norms. Namely the following holds:

$$F_1(M_v, Z) = F_1(M_1, Z_1 \cup N) + F_1(M_2, Z_2 \cup N) + F_1(N) + 2^{-1}\chi(N) , \tag{2.149}$$

where $Z_k := Z \cap \partial \overline{M}_k$.

Corollary 2.9. 1. For a closed M the scalar analytic torsion $T(M, [\theta/2\pi])$ is independent of $\theta \notin 2\pi\mathbb{Z}$.
2. Under the conditions of $(2.147), (2.148)$, and (2.149), the following holds in $\mathbb{R}/\mathbb{Z}$:

$$F(M_v \times L, Z \times L) = \chi(L)F(M_v, Z) ,$$

$$F(M_v, Z) = F(M_v, Z \cup K) + 2^{-1}\chi(K) ,$$

$$F(M_v, Z) = F(M_1, Z_1 \cup N) + F(M_2, Z_2 \cup N) + 2^{-1}\chi(N) .$$

Example 2.1. The number $F_1(S^1)$ is equal to $-f_{0,1} = -f_{0;0}$. [45] The latter one is equal to zero because the asymptotic expansion for $\mathrm{Tr}\exp(-t\Delta_0(S^1))$ as $t \to +0$ (where Δ_0 is the Laplacian on functions) is $f_{-1;0}t^{-1/2} + f_{1;0}t^{1/2} + f_{3;0}t^{3/2} + \dots$

Example 2.2. The number $F_1(I, \partial I)$ is equal to $-f_{0,1}(I, \partial I) = -f_{0;0}(I)$. Since S^1 has a mirror symmetry relative to its diameter, we have, taking into account (2.54) and (2.118),

$$f_{0,0}(S^1) = f_{0;0}(I) + f_{0;0}(I, \partial I) = 0 . \tag{2.150}$$

Since $f_{0;0}(I, \partial I) - f_{0;1}(I, \partial I) = \chi(I, \partial I) = 1$ and (2.150) holds, we see that

$$F_1(I, \partial I) = -f_{0;1}(I, \partial I) = -f_{0;0}(I) = f_{0;0}(I, \partial I) = -2^{-1} . \tag{2.151}$$

Remark 2.10. The equality (2.151) means that the analytic torsion $T(I, \partial I; [\theta/2\pi])$ is multiplied by the factor $\exp(-2\pi i \cdot (-2^{-1})) = -1$, if θ is replaced by $\theta + 2\pi$ $(\theta \notin 2\pi\mathbb{Z})$.
It is necessary to note the following. The scalar analytic torsion $T(I, \partial I) := T(I, \partial I; \pi)$ is the factor in the analytic torsion norm. But the latter one is the *square*

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[44] The Euler characteristic $\chi(M_v, Z) := \sum(-1)^i \dim H^i(M_v, Z)$ is equal to the Euler characteristic of the complex $(C^\bullet(X_v, X \cap Z), dc)$ (as it follows from Proposition 2.3). Hence it is equal to $\chi(M, Z)$ and is independent of $v \in \mathbb{R}^2 \setminus (0, 0)$.

[45] The coefficients $f_{k,j} := f_{k,j}(M, Z)$ are the coefficients in the asymptotic expansion $\sum f_{k,j}t^{k;2}$ $(k \geq -n)$ for $\mathrm{Tr}\exp(-t\Delta_j(M, Z))$ as $t \to +0$ for the Laplacian on $DR^j(M, Z)$ $(n := \dim M)$.

of the norm on the determinant line $\det H^1(I, \partial I)$. So the factor, corresponding to the *norm* itself, is multiplied in the case of $(M, Z) = (I, \partial I)$ by the factor

$$\exp(\pi i \zeta_1(I, \partial I)|_{s=0}) = \exp(-\pi i/2) = -i ,$$

if θ is replaced by $\theta + 2\pi$, $\theta \notin 2\pi\mathbb{Z}$. Indeed, we have

$$-2^{-1} = F_1(I, \partial I) = -(\dim H^1(I, \partial I) + \zeta_1(I, \partial I)\|_{s=0}) ,$$

and so $\zeta_1(I, \partial I)\|_{s=0} = -2^{-1} = -F(I, \partial I)$. For $M = S^1$ it holds that $-F(S^1) = \zeta_1(S^1)|_{s=0} = -1$, and so $\exp(-\pi i F(S^1)) = -1$.

It follows from Proposition 2.23 below that $F(M_v, Z)$ and $F(M, Z)$ have a form $(1/2) + \mathbb{Z}$, if the numbers $n := \dim M$ and $\chi(M, Z)$ are odd. So in this case the factors $\exp(-\pi i F(M_v, Z))$ and $\exp(-\pi i F(M, Z))$ are equal to $\{\pm i\}$.

Proof of Proposition 2.22. 1. Theorem 3.2.1 claims that the number $\zeta_{v,j}(0) + \dim \mathrm{Ker}(\Delta_{v,j})$ is equal to the constant term $f_{0;j}(M_v, Z)$ in the asymptotic expansion (2.87) for $\mathrm{Tr}\exp(-t\Delta_{v,j})$ as $t \to +0$. So according to Theorem 3.2.1, the number $F_1(M_v, Z)$ is equal to the sum of the integrals over M_1, M_2, N, and ∂M of the locally defined densities. Then we have

$$F_1(M_v, Z) = \sum (-1)^j j f_{0;j}(M_v, Z) . \tag{2.152}$$

If (M, g_M) is a closed Riemannian manifold then $f_{0;j}(M) = f_{0,n-j}(M)$. Hence taking into account (2.152) and (2.57), we get (for even $n := \dim M$)

$$F_1(M) = \sum (-1)^j j f_{0;j}(M) = (n/2) \sum (-1)^j f_{0;j}(M) = (n/2)\chi(M) . \tag{2.153}$$

Let n be odd. Then $f_{0;j}(M)$ is equal to zero since the asymptotic expansion for $\mathrm{tr}\exp(-t\Delta_j(M, g_M))$ (as $t \to +0$) is $t^{-n/2} \sum t^l f_{2l-n;j}(M, g_M)$, where the sum is over $l \in \mathbb{Z}_+ \cup 0$ ([Gr], Theorem 1.6.1; [BGV], Theorem 2.30). Hence $F_1(M) = 0 = (n/2)\chi(M)$ for an odd n also.

This number $(n/2)\chi(M)$ is an integer for any closed M. (The assertion that $F_1(M)$ is an integer follows also from the equality which holds for any closed even-dimensional Riemannian (M, g_M):

$$\sum j(-1)^j \zeta_j(M, s) = \sum j(-1)^j \zeta_{n-j}(M, s) = (n/2) \sum (-1)^j \zeta_j(M, s) = 0 ,$$

because $\zeta_j(M, s) = \zeta_{n-j}(M, s)$.)

2. Let $\lambda \in \mathrm{Spec}(\Delta_{v,j}(M, Z))$, $\mu \in \mathrm{Spec}(\Delta_i(L))$ and let $m_\lambda(j; M_v, Z), m_v(i; L)$ be their multiplicities. If $\lambda \neq 0$ and $\mu \neq 0$ then we have

$$\sum (-1)^{j+i}(j+i) m_\lambda(j; M_v, Z) m_\mu(i; L) = 0 , \tag{2.154}$$

since the subcomplexes $(V_\lambda^\bullet(M_v, Z), d) \hookrightarrow (DR^\bullet(M_v, Z), d)$ and $(V_\mu^\bullet(L), d) \hookrightarrow (DR^\bullet(L), d)$, corresponding to the λ-eigenforms for $\Delta_{v,\bullet}(M, Z)$ and to the μ-eigenforms for $\Delta_\bullet(L)$, are acyclic. If $\lambda \neq 0$ but $\mu = 0$ then the right side of (2.154) is equal to

$$\left(\sum (-1)^j j m_\lambda(j; M_v, Z)\right)\left(\sum (-1)^i \dim \mathrm{Ker}\Delta_i(L)\right) = \chi(L) \sum (-1)^j j m_\lambda(j; M_v, Z) .$$

So under the conditions of 2, by using (1.37), we have

$$\sum (-1)^j j\zeta_{v,j}(M_v \times L, Z \times L)|_{s=0}$$

$$= \chi(L)\sum (-1)^j j\zeta_{v,j}(M_v, Z)|_{s=0} + \chi(M_v, Z)\sum (-1)^j j\zeta_j(L)|_{s=0}, \quad (2.155)$$

$$\sum (-1)^j j\dim \mathrm{Ker}(\varDelta_{v,j}(Mv \times L, Z \times L))$$

$$= \chi(L)\sum (-1)^j j\dim \mathrm{Ker}(\varDelta_{v,j}(M_v, Z)) + \chi(M_v, Z)\sum (-1)^j j\dim \mathrm{Ker}(\varDelta_j(L)), \quad (2.156)$$

The equality (2.147) follows now from (2.155) and (2.156).

3. The numbers $F_1(M_v, Z)$ and $F_1(M_v, Z \cup K)$ are the sums of the integrals over M_1, M_2, N, and ∂M of the locally defined densities (as it follows from (2.152) and from Theorem 3.2). The densities, corresponding to the pairs (M_v, Z) and $(M_v, Z \cup K)$, differ only on K. So the difference $F_1(M_v, Z) - F_1(M_v, Z \cup K)$ depends only on K and on g_M near K. Thus taking into account that g_M is a direct product metric near K, we get

$$2(F_1(M_v, Z) - F_1(M_v, Z \cup K)) = F_1(K \times I) - F_1(K \times (I, \partial I)) \quad (2.157)$$

for any fixed metric on K in all the terms of this equality.

According to (2.147) we have

$$F_1(K \times I) = F_1(K)\chi(I) + F_1(I)\chi(K),$$

$$F_1(K \times (I, \partial I)) = F_1(K)\chi(I, \partial I) + F_1(I, \partial I)\chi(K),$$

$$F_1(K \times I) - F_1(K \times (I, \partial I)) = 2F_1(K) + \chi(K)(F_1(I) - F_1(I, \partial I)). \quad (2.158)$$

Since $\zeta_1(s; I) = \zeta_1(s; I, \partial I)$ (on the same I) we have

$$F_1(I) - F_1(I, \partial I) = -\zeta_1(I)|_{s=0} + \zeta_1(I, \partial I)|_{s=0} + \dim H^1(I, \partial I) = 1.$$

By (2.157) and (2.158) we get

$$F_1(M_v, Z) = F_1(M_v, Z \cup K) + F_1(K) + 2^{-1}\chi(K).$$

4. The number $F_1(M_v, Z)$ is the sum of the integrals over M_1, M_2, N, and over ∂M of the locally defined densities. (It is a consequence of Theorem 3.2). So the densities on M_j, N, and on $\partial M \cap \overline{M}_j$ are the same as for the number $F_1(M_{j,v}^{(2)}, Z_j^{(2)})$, where $M_j^{(2)} := M_j \cap_N M_j$, $Z_j^{(2)} := Z_j \cap Z_j$, and $g_{M_j^{(2)}}$ are mirror symmetric with respect to N (the v-transmission boundary conditions are given on $N \hookrightarrow M_j^{(2)}$) and $g_{M_j^{(2)}}|_{M_j} = g_M|_{M_j}$. Thus we have

$$2F_1(M_v, Z) = \sum_{j=1,2} F_1(M_{j,v}^{(2)}, Z_j^{(2)}).$$

Since pairs $(M_j^{(2)}, Z_j^{(2)})$ are mirror symmetric with respect to N, it follows from (2.54) and (2.55) that

$$F_1(M_{j,v}^{(2)}, Z_j^{(2)}) = F_1(M_j^{(2)}, Z_j^{(2)}) = F_1(M_j, Z_j) + F_1(M_j, Z_j \cup N). \quad (2.159)$$

$$F_1(M_v, Z) = 2^{-1} \sum_{j=1,2} (F_1(M_j, Z_j) + F_1(M_j, Z_j \cup N)). \quad (2.160)$$

The equality (2.149) follows from (2.148) with $M = M_j$, $Z = Z_j$, $K = N$, and $v = (1, 1)$ and from (2.160), because $F_1(M_j, Z_j) = F_1(M_j, Z_j \cup N) + F_1(N) + 2^{-1}\chi(N)$. Thus Proposition 2.22 is proved. □

The analytic torsion $T_0(M_v, Z; m)$ (where $Z \subset \partial M$ is a union of some connected components of ∂M, $m := [\theta/2\pi]$, $\theta \notin 2\pi\mathbb{Z}$) is defined as the *product of the norm* $\| \cdot \|^2_{\det H^\bullet(M_v, Z)}$ (given by the natural norm on harmonic forms for $\Delta_v^\bullet(M_v, Z)$) and of the *scalar analytic torsion* $T(M_v, Z; m)$:

$$T_0(M_v, Z; m) := \| \cdot \|^2_{\det H^\bullet(M_v, Z)} T(M_v, Z; m) . \tag{2.161}$$

The analytic torsion $T_0(M, Z; m)$ is the norm $\| \cdot \|^2_{\det H^\bullet(M, Z)} T(M, Z; m)$, where the norm on the determinant line is given by the harmonic forms for $\Delta_\bullet(M, Z)$.

(If N is the interior boundary and if g_M is a direct product metric near N then $T_0(M, Z; m) = T_0(M_{1,1}, Z; m)$ according to Proposition 1.1.)

Theorem 2.1. 1. *Let M be obtained by gluing two pieces along N, $M = M_1 \cup_N M_2$, where N is a closed of codimension one submanifold in M with a trivial normal bundle $TM|_N/TN$ and g_M is a direct product metric near N and near ∂M. Then for $v \in \mathbb{R}^2 \setminus (0, 0)$ the following gluing formula holds:*

$$\varphi_v^{\mathrm{an}} T_0(M_v, Z; m) :=$$
$$= (-1)^{m\chi(N)} T_0(M_1, Z_1 \cup N; m) \otimes T_0(M_2, Z_2 \cup N; m) \otimes T_0(N; m) , \tag{2.162}$$

where the identification φ_v^{an} of the determinants lines is defined by the short exact sequence (1.14) of the de Rham complexes and by Lemma 1.1, $Z_k := Z \cap \partial \overline{M}_k$. The factor $T_0(N; m) := T_0(N)$ is independent of m (according to Proposition 2.22.1).
    2. *Let $K \subset \partial M \setminus Z$ be a union of some connected components of ∂M. Then the formula holds for gluing K and $(M_v, Z \cup K)$:*

$$\varphi_{\mathrm{an}} T_0(M_v, Z; m) = (-1)^{m\chi(K)} T_0(M_v, Z \cup K; m) \otimes T_0(K; m) . \tag{2.163}$$

Here the identification φ_{an} is defined by the short exact sequence (analogous to (1.20)):

$$0 \to DR^\bullet(M_v, Z \cup K) \to DR^\bullet(M_v, Z) \to DR^\bullet(K) \to 0 \tag{2.164}$$

(the left arrow in (2.164) is the natural inclusion and the right arrow is the geometrical restriction) and by Lemma 1.1. The factor $T_0(K; m) := T_0(K)$ is independent of m. The analogous formula holds for gluing K and $(M, Z \cup K)$:

$$\varphi_{\mathrm{an}} T_0(M, Z; m) = (-1)^{m\chi(K)} T_0(M, Z \cup K; m) \otimes T_0(K) , \tag{2.165}$$

where φ_{an} is defined by the short exact sequence (2.164) with M_v replaced by M.

Proof. 1. For $T_0(M_v, Z) := T_0(M_v, Z; 0)$ the following gluing formula holds (according to (1.12) and to Lemma 1.2):

$$\varphi_v^{\mathrm{an}} T_0(M_v, Z) = T_0(M_1, Z_1 \cup N) \otimes T_0(M_2, Z_2 \cup N) \otimes T_0(N) . \tag{2.166}$$

By the definition of $F(M_v, Z)$ we have

$$T_0(M_v, Z; m + 1) = \exp(-2\pi i F(M_v, Z)) T_0(M_v, Z; m) .$$

(Analogous equalities are true for $T_0(M_j, Z_j \cup N; m)$. The differences $F_1(M_v, Z) - F(M_v, Z)$, $F_1(M_j, Z_j \cup N) - F(M_j, Z_j \cup N)$, and $F_1(N) - F(N)$ are integers and $F_1(N)$ is an integer (according to Proposition 2.22.1). Hence (2.162) is consequence of (2.149) and (2.166).

2. The gluing formula holds for $T_0(M, Z) := T_0(M, Z; 0)$ according to (1.18) (Theorem 1.2): $\varphi_{\mathrm{an}} T_0(M, Z) = T_0(M, Z \cup K) \otimes T_0(K)$. So the gluing formula (2.165) follows from (2.148) since $T_0(M, Z; m + 1) = \exp(-2\pi i F(M, Z)) T_0(M, Z; m)$ and since the difference $F_1(M, Z) - F(M, Z)$ is an integer.

Let $N \subset M$ be a disjoint union $N_1 \cup N_2$ of two closed codimension one submanifolds of M with trivial normal bundles and let the v_j-transmission interior boundary conditions be given on N_j. Let $M = M_1 \cup_{N_1} M_2$ and let $N_2 \subset M_1$. Under these conditions, the equality (1.12) and the assertion of Lemma 1.2 are also true. Their proofs are similar to that given above. The resulting formula is

$$\varphi_{v_1}^{\mathrm{an}} T_0(M_{v_1, v_2}, Z) = T_0(M_{1, v_2}, Z_1 \cup N_1) \otimes T_0(M_2, Z_2 \cup N_1) \otimes T_0(N_1) . \qquad (2.167)$$

(Here $Z \subset \partial M$ is a union of some connected components of ∂M, $Z_k = Z \cap \partial \overline{M}_k$ and g_M is a direct product metric near N_j and ∂M.) As a consequence of (2.167) (obtained by the same method as Theorem 1.2 is obtained from (1.16) and (1.17)) we get the following equality

$$\varphi_{\mathrm{an}} T_0(M_{1, v_2}, Z_1) = T_0(M_{1, v_2}, Z_1 \cup N_1) T_0(N_1) . \qquad (2.168)$$

The equality (2.163) follows from (2.168) (where M_{1, v_2}, Z_1, N_1 are replaced by M_v, Z, K) and from (2.148) since $T_0(M_v, Z; m + 1) = \exp(-2\pi i F(M_v, Z)) T_0(M_v, Z; m)$. Theorem 2.1 is proved. $\square$

Proposition 2.23. *Let $M = M_1 \cup_N M_2$ be obtained by gluing along N and let the v-transmission boundary conditions ($v \in \mathbb{R}^2 \setminus (0,0)$) be given on N. Set $n := \dim M$. Then the number $F_1(M_v, Z)$ for the scalar analytic torsion $T(M_v, Z; m)$ is expressed by*

$$F_1(M_v, Z) = 2^{-1} n \chi(M_v, Z) = 2^{-1} n \chi(M, Z) . \qquad (2.169)$$

The number $F(M_v, Z) := \sum (-1)^j j \zeta_{v, j}(0)$ is as follows:

$$F(M_v, Z) = \sum (-1)^j (-j + 2^{-1} n) \dim H^j(M_v, Z) .$$

Proof. 1. Proposition 2.3 claims that $\chi(M_v, Z)$ (i.e., the Euler characteristic $\sum (-1)^j \times \dim H^j(DR(M_v, Z))$) is equal to the Euler characteristic for the finite-dimensional complex $(C^\bullet(X_v, Z \cap X), d_c)$. Note that $\dim C^j(X, Z \cap X)$ is equal to $\dim C^j(X_v, Z \cap X)$. Hence $\chi(M_v, Z) = \sum (-1)^j \dim C^j(X, Z \cap X)$. This sum is equal to $\chi(M, Z)$ by the de Rham theorem ([RS], Proposition 4.2). Thus $\chi(M_v, Z) = \chi(M, Z)$.

2. According to (2.149), the gluing formula holds:

$$F_1(M_v, Z) = F_1(M_1, Z_1 \cup N) + F_1(M_2, Z_2 \cup N) + F_1(N) + 2^{-1} \chi(N) .$$

For the Euler characteristics the analogous formula holds:

$$\chi(M_v, Z) = \chi(M, Z) = \chi(M_1, Z_1 \cup N) + \chi(M_2, Z_2 \cup N) + \chi(N) . \qquad (2.170)$$

The number $F_1(N)$ for a closed manifold N is equal to $\chi(N)(\dim N)/2$ by (2.153). So $F_1(N) + 2^{-1} \chi(N) = n \chi(N)/2$. Let (M, g_M) be a closed Riemannian

manifold, mirror symmetric with respect to $N = \partial M_1$, $M = M_1 \cup_N M_1$. Let g_M be a direct product metric near N. Then the equality $F_1(M_1, N) = 2^{-1} n \chi(M_1, N)$ follows from (2.153), which claims that $F_1(M) = 2^{-1} n \chi(M)$, and from (2.170). So (2.169) holds for pairs (M, Z), where $Z = \partial M$. For any union Z of some connected components of ∂M the equality (2.169) follows from (2.148) for $K = \partial M \setminus Z$, as $F_1(K) + 2^{-1} \chi(K) = n \chi(K)/2$, according to (2.153). The equality (2.169) for $F_1(M_v, Z)$ follows from its particular cases for $F_1(M_j, Z_j \cup N)$ by using (2.149) and (2.170). $\square$

Corollary 2.10. 1. *The analytic torsion* $T_0(M_v, Z; m)$ *(defined by* (2.161)) *is the following function of* $m = [\theta/2\pi]$ *(*$\theta \notin 2\pi\mathbb{Z}$*):*

$$T_0(M_v, Z; m) = (-1)^{mn\chi(M,Z)} T_0(M_v, Z) . \qquad (2.171)$$

Here $T_0(M_v, Z) := T_0(M_v, Z; 0)$, $n := \dim M$.

2. *The analytic torsion* $T_0(M, Z; m)$ *is equal to* $T(M_{v_0}, Z; m)$ *for* $v_0 = (1, 1)$ *according to Proposition* 1.1. *The formula* (2.171) *holds also for* $T_0(M, Z; m)$, *where* $T_0(M_v, Z)$ *is replaced by* $T_0(M, Z) := T_0(M, Z; 0)$.

Let (M, g_M) be obtained by gluing two Riemannian manifolds M_1 and M_2 along a common component N of their boundaries, $M = M_1 \cup_N M_2$. Let g_M be a direct product metric near N and near ∂M and let $Z \subset \partial M$ be a union of some connected components of ∂M. The following main theorem is an immediate consequence of Theorems 1.4, 1.5 and of Corollary 2.10.

Theorem 2.2 (Generalized Ray–Singer Conjecture). 1. *The analytic torsion* $T_0(M_v, Z; m)$ *is expressed through the combinatorial torsion norm* (1.62) *as follows*:

$$T_0(M_v, Z; m) = 2^{\chi(\partial M) + \chi(N)} (-1)^{mn\chi(M,Z)/2} \tau_0(M_v, Z)$$

(where $m = [\theta/2\pi]$, $\theta \notin 2\pi\mathbb{Z}$ *is the phase of a cut of the spectral plane* $\mathbb{C} \ni \lambda$ *and* $n = \dim M$*).*

2. *The analytic torsion* $T_0(M, Z; m)$ *is expressed through the combinatorial torsion norm*:

$$T_0(M, Z; m) = 2^{\chi(\partial M)} (-1)^{mn\chi(M,Z)/2} \tau_0(M, Z) .$$

Remark 2.11. The combinatorial torsion norms $\tau_0(M, Z)$ and $\tau_0(M_{v_0}, Z)$ (where $v_0 = (1, 1)$) on the determinant line $\det H^\bullet(M_{v_0}, Z) = \det H^\bullet(M, Z)$ are different, if $\chi(N) \neq 0$ (by Remarks 1.7 and 1.9). The canonical identifications $H^\bullet(M_{v_0}, Z) = \mathrm{Ker}(\Delta^\bullet_{v_0}) = \mathrm{Ker}(\Delta^\bullet) = H^\bullet(M, Z)$ are given by Proposition 1.1 and by the de Rham theorem.

3. Zeta- and Theta-Functions for the Laplacians with v-Transmission Interior Boundary Conditions

3.1. Properties of Zeta- and Theta-Functions for v-Transmission Boundary Conditions. Let M be a compact manifold with boundary obtained by gluing manifolds M_1 and M_2 along a common component N of their boundaries, $M = M_1 \cup_N M_2$ ($N \subset M$ is a closed codimension one submanifold of M with a trivial normal bundle $TM|_N/TN$). Let g_M be a direct product metric near $N = 0 \times N \hookrightarrow I \times N \hookrightarrow M$. Let the Dirichlet boundary conditions be given on a union Z of some

connected components of ∂M, the Neumann ones be given on $\partial M \setminus Z$ and the v-transmission interior boundary conditions (1.27) be given on N.

The operator $\Delta_v^\bullet$ is originally defined on the set $D(\Delta_v^\bullet)$ of all the pairs of smooth forms $\omega = (\omega_1, \omega_2) \in DR^\bullet(\overline{M}_1) \oplus DR^\bullet(\overline{M}_2)$ such that the Dirichlet boundary conditions hold for ω on Z, the Neumann boundary conditions hold on $\partial M \setminus Z$ and the interior boundary conditions (1.27) hold for ω on N. Let $\mathrm{Dom}\,(\Delta_v^\bullet)$ be the closure of the $D(\Delta_v^\bullet)$ in $(DR^\bullet(M))_2$ in the topology given by the graph norm [46] $\|\omega\|_2^2 + \|\Delta_v^\bullet \omega\|_2^2 =: \|\omega\|_{\mathrm{graph}}^2$. The closure of the operator $\Delta_v^\bullet$ (with respect to the graph norm) is an operator with the domain of definition $\mathrm{Dom}\,(\Delta_v^\bullet)$. If $\omega_j \to \omega$ in the graph norm topology, $\omega_j \in D(\Delta_v^\bullet)$, then $\Delta_v^\bullet(\omega)$ is defined as $\lim_j \Delta_v \omega_j$ in the L_2-topology in $(DR^\bullet(M))_2$.

Theorem 3.1. 1. *The operator $\Delta_v^\bullet$ with the domain $\mathrm{Dom}\,(\Delta_v^\bullet)$ is self-adjoint in $(DR^\bullet(M))_2$. Its spectrum $\mathrm{Spec}\,(\Delta_v^\bullet) \subset \mathbb{R}_+ \cup 0$ is discrete.* [47]
2. Its zeta-function is defined for $\mathrm{Re}\,s > (\dim M)/2$ by the absolutely convergent series (including the multiplicities) $\zeta_{v,\bullet}(s) := \sum_{\lambda_j \in \mathrm{Spec}(\Delta_v^\bullet)\setminus 0} \lambda_j^{-s}$. This series converges uniformly for $\mathrm{Re}\,s \geq (\dim M)/2 + \varepsilon$ (for an arbitrary $\varepsilon > 0$). The zeta-function $\zeta_{v,\bullet}$ can be continued to a meromorphic function on the whole complex plane with at most simple poles at the points $s_j := (j - \dim M)/2$, $j = 0, 1, 2, \ldots$. It is regular at $s = 0, 1, 2, \ldots$.
3. The residues $\mathrm{res}_{s=s_j} \zeta_{v,\bullet}(s)$ and the values $\zeta_{v,\bullet}(m) + \delta_{m,0} \dim \mathrm{Ker}(\Delta_v^\bullet)$ are equal to the sums of the integrals over $M, \partial M$, and N of the densities locally defined on these manifolds.

Proposition 3.1. 1. *Let $\lambda \notin \mathrm{Spec}\,(\Delta_v^\bullet)$. Then the resolvent $G_v^\bullet(v) := (\Delta_v^\bullet - \lambda)^{-1}$, $G_\lambda^\bullet(v) : (DR^\bullet(M))_2 \rightleftarrows \mathrm{Dom}\,(\Delta_v^\bullet) \hookrightarrow (DR^\bullet(M))_2$, is the isomorphism (in algebraic and topological senses) onto the closure $\mathrm{Dom}\,(\Delta_v^\bullet)$ of $D(\Delta_v^\bullet)$ with respect to the graph norm.* [48] *The operators $G_\lambda^\bullet(v)$ for pairs (λ, v) such that $\lambda \notin \mathrm{Spec}\,(\Delta_v^\bullet)$ form a smooth in (λ, v) family of bounded operators in $(DR^\bullet(M))_2$.*
2. The families $d \circ G_\lambda^\bullet(v)$ and $\delta \circ G_\lambda^\bullet(v)$ for $\lambda \notin \mathrm{Spec}\,(\Delta_v^\bullet)$ form a smooth in (λ, v) family of bounded operators $(DR^\bullet(M))_2 \to (DR^{\bullet\pm 1}(M))_2$.

Theorem 3.2. 1. *The operator $\exp(-t\Delta_v^\bullet)$ in $(DR^\bullet(M))_2$ for an arbitrary $t > 0$ is of trace class. For its trace the asymptotic expansion (2.87) (relative to $t \to +0$) holds. The coefficients $f_{-\dim M+j}$ of this expansion are the sums of the integrals over $M, \partial M$, and N of the locally defined densities. If $j \neq \dim M + 2m$, $m \in \mathbb{Z}_+ \cup 0$, the densities on $M, \partial M$, and on N for $f_{-\dim M+j}$ are the same as for*

$$\Gamma((\dim M - j)/2)\mathrm{res}_{s=s_j}\zeta_{v,\bullet}(s)\,.$$

If $j = \dim M + 2m$, $m \in \mathbb{Z}_+ \cup 0$, these densities are the same as for

$$(m!)^{-1}(-1)^m(\zeta_{v,\bullet}(m) + \delta_{m,0}\dim \mathrm{Ker}(\Delta_v^\bullet))\,.$$

2. *Let $p_1 : (DR^\bullet(M))_2 \to (DR^\bullet(M_1))_2 \hookrightarrow (DR^\bullet(M))_2$ be the composition of the restriction to M_1 and of the extension by zero of L_2-forms. Then the operator*

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[46] The L_2-completion $(DR^\bullet(M))_2$ of $DR^\bullet(M)$ coincides with the L_2-completion of $DR^\bullet(\overline{M}_1) \oplus DR^\bullet(\overline{M}_2)$.
[47] A spectrum is discrete if it consists entirely of isolated eigenvalues with finite multiplicities.
[48] The topology on $(DR^\bullet(M))_2$ is given by $\|\omega\|_2^2$, and on $\mathrm{Dom}\,(\Delta_v^\bullet)$ it is given by $\|\omega\|_{\mathrm{graph}}^2$.

$p_1 \exp(-t\Delta_v^\bullet)$ in $(DR^\bullet(M))_2$ for $t > 0$ is of trace class. For its trace the asymptotic expansion relative to $t \to +0$ holds

$$\text{Tr}(p_1 \exp(-t\Delta_v^\bullet)) = q_{-n}t^{-n/2} + \cdots + q_0 t^0 + q_1 t^{1/2} + \cdots + q_m t^{m/2} + r_m(t), \quad (3.1)$$

where $r_m(t)$ is $O(t^{(m+1)/2})$ uniformly with respect to v and it is smooth in t for $t > 0$ ($n := \dim M$). The coefficients q_i are equal to the sums of the integrals over M_1 and over $\partial M_1 = N \cup (\partial M \cap \overline{M}_1)$ of the locally defined densities. The coefficients q_j in (3.1) depend only on $(j, M_1, g_M \setminus_{TM_1}, Z \cap \partial\overline{M}_1, N, v)$ and do not depend on M_2 and $Z \cap \partial\overline{M}_2, g_M \setminus_{TM_2}$.

3. For any $t > 0$ the traces of $\exp(-t\Delta_v^\bullet)$ are bounded uniformly with respect to $v \in \mathbb{R}^2 \setminus (0,0)$:

$$|\text{Tr} \exp(-t\Delta_v^\bullet)| < C(t). \quad (3.2)$$

The traces $\text{Tr}(p_j \exp(-t\Delta_v^\bullet))$ are also bounded uniformly with respect to v for any $t > 0$.

Proposition 3.2. 1. The kernel $E_{t,x_1,x_2}^\bullet(v)$ for $\exp(-t\Delta_v^\bullet)$ (where $t > 0$) is smooth in $x_j \in \overline{M}_{r_j}, t$, and in $v \in \mathbb{R}^2 \setminus (0,0)$.

2. The asympttotic expansions (3.1), (2.87) are differentiable with respect to $v \in \mathbb{R}^2 \setminus (0,0)$.

3.2. Zeta-Functions for the Laplacians with v-Transmission Interior Boundary Conditions. Proofs of Theorem 3.1 and of Proposition 3.1.

Let A be an elliptic differential operator on a manifold with boundary $(M, \partial M)$. Let the differential elliptic boundary conditions be given for A on ∂M such that A with these boundary conditions satifies Agmon's condition (formulated below) for λ from a section $\theta_1 < \arg \lambda < \theta_2$ in the spectral plane $\mathbb{C} \ni \lambda$. Properties of zeta-functions for A with these boundary conditions can be investigated with the help of the parametrix for $(A - \lambda)^{-1}$. The analogous statement is true also for elliptic interior boundary conditions. [49] The parametrix P_λ^m for $(\Delta_v^\bullet - \lambda)^{-1}$ is defined locally in coordinate charts. Namely

$$P_\lambda^m = \sum \psi_j P_{\lambda, U_j}^m \varphi_j, \quad (3.3)$$

where φ_j is a partition of unity subordinate to a finite cover $\{U_j\}$ of M by coordinate charts, $\psi_j \varphi_j = \varphi_j, \psi_j \in C_0^\infty(U_j)$. If $U_j \cap (\partial M \cup N) = \emptyset$ then the operator P_{λ, U_j}^m is a pseudodifferential operator (PDO) with parameter λ ([Sh], Chapter II, Sect. 9) and its symbol is equal to $\theta(\xi, \lambda)s_{(m)}((\Delta^\bullet - \lambda)^{-1})(x, \xi, \lambda)$. This symbol is defined as follows. Let $s(\Delta^\bullet - \lambda) = ((b_2 - \lambda)\text{id} + b_1)(x, \xi)$ be the symbol of $\Delta^\bullet - \lambda$ (where $\Delta^\bullet$ is the Laplacian on $DR^\bullet(U_j)$) and let

$$s((\Delta^\bullet - \lambda)^{-1}) := \sum_{j \in \mathbb{Z}_+ \cup 0} a_{-2-j}(x, \xi, \lambda)$$

be the symbol of $(\Delta^\bullet - \lambda)^{-1}$ as of a PDO with parameter (a_{-k} is positive homogeneous of degree $-k$ in $(\xi, \lambda^{1/2})$). Set $s_{(m)}((\Delta^\bullet - \lambda)^{-1}) := \sum_{j=0}^m a_{-2-j}(x, \xi, \lambda)$. The condition $s(\Delta^\bullet - \lambda) \circ s((\Delta^\bullet - \lambda)^{-1}) = 1$, where $\circ$ is the compositon of symbols

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[49] Theorem 3.1 is analogous to the results of [Se1, Se2] with modifications connected with the v-transmission interior boundary conditions. In [Sb], Ch. II, the theory [Se1. Se2] of the zeta-functions is written in detail in the case of a closed manifold.

with parameter ([Sh], Sect. 11.1), is equivalent to the system of equalities

$$a_{-2}(x, \xi, \lambda) = (b_2 - \lambda)^{-1},$$

$$a_{-3} = -(b_2 - \lambda)^{-1}[b_1 a_{-2} + \sum_i D_{\xi_i} b_2 \partial_{x_i} a_{-2}],$$

$$a_{-2-j} = -(b_2 - \lambda)^{-1} \sum_{|\gamma|+i+l=j} \frac{1}{\gamma!} D_\xi^\gamma b_{2-i} \partial_x^\gamma a_{-2-l}. \tag{3.4}$$

The sum in the last equation of (3.4) is over (γ, i, l) such that $\gamma = (\gamma_1, \ldots, \gamma_n) \in (\mathbb{Z}_+ \cup 0)^n$, $|\gamma| := \gamma_1 + \cdots + \gamma_n$, $0 \leqq |\gamma| \leqq j$ for b_j, $|\gamma| + i \geqq 1$ $(D := i^{-1}\partial)$. The function $\theta(\xi, \lambda)$ (in the symbol of P_λ^m) is smooth, $\theta(\xi, \lambda) \equiv 1$ for $|\xi|^2 + |\lambda| \geqq 1$, and θ is equal to zero for $|\xi|^2 + |\lambda| \leqq \varepsilon$.

Let $U_j \cap N \neq \emptyset$. Then the term $P_{\lambda,U}^m$ of the parametrix is the sum of the interior term (which is a PDO with parameter and its symbol is defined with the help of (3.4)) and of the correction terms. (Here $U := U_j$.) The latter terms correspond to the v-transmission interior boundary conditions on N and to the Dirichlet and the Neumann boundary conditions, given on the connected components of ∂M. First of all we'll verify that these v-transmission boundary conditions are *Agnon's conditions* on any ray $\arg \lambda = \varphi$ in the spectral plane not coinciding with $\mathbb{R}_+$.

Let $(t, y) \in I \times U_N$ be the coordinates on $U := U_j$ near $N = 0 \times N \hookrightarrow I \times N \hookrightarrow M$, $I = [-2, 2]$, and let $t > 0$ on M_1. From now on it is supposed that $\varphi_j(t, y) = \varphi_{j,I}(t)\varphi_{j,N}(y)$ and that $\varphi_{j,I}(t) \equiv 1$ for $|t| \leqq 1$. It is supposed also that $\psi_j(t, y) = \psi_{j,I}(t)\psi_{j,N}(y)$ and that $\varphi_{j,I}, \psi_{j,I}$ are even functions: $\varphi_{j,I}(-t) = \varphi_{j,I}(t)$, $\psi_{j,I}(-t) = \psi_{j,I}(t)$. The forms dy^c and $dt dy^f$ (where $c = (c_1 \ldots, c_{n-1})$, $f = (f_1, \ldots, f_{n-1}), c_i$, $f_i \in \{0, 1\}$) provide us with a trivialization of $\wedge^\bullet TM|_{I \times U_N}$. Namely $\omega_j = \sum_{|c|=\bullet} \omega_{j,c} dy^c + \sum_{|f|+1=\bullet} \omega_{j,(1,f)} dt dy^f$. Let $\omega = (\omega_1, \omega_2) \in D(\Delta_v^\bullet) \subset DR^\bullet(M_v)$. Then on $U_N = 0 \times U_N \hookrightarrow I \times U_N$ the conditions $\omega \in D(\Delta_v^\bullet)$ can be written as follows. Set $|v| = (\alpha^2 + \beta^2)^{1/2}$. Let $\mathscr{L}$ be the transformation $(t \geqq 0)$

$$v_{1,c}(t, y) := |v|^{-1}(\alpha \omega_{1,c}(t, y) - \beta \omega_{2,c}(-t, y)).$$

$$v_{2,c}(t, y) := |v|^{-1}(\beta \omega_{1,c}(t, y) + \alpha \omega_{2,c}(-t, y)),$$

$$w_{2,(1,f)}(t, y) := |v|^{-1}(\alpha \omega_{1,(1,f)}(t, y) + \beta \omega_{2,(1,f)}(-t, y)),$$

$$w_{1,(1,f)}(t, y) := |v|^{-1}(-\beta \omega_{1,(1,f)}(t, y) + \alpha \omega_{2,(1,f)}(-t, y)). \tag{3.5}$$

Then the conditions $\omega \in D(\Delta_v^\bullet)$ are equivalent on U_N to

$$v_{1,c}(0, y) = 0, \quad w_{1,(1,f)}(0, y) = 0,$$

$$\partial_t v_{2,c}|_{t=0} = 0, \quad \partial_t w_{2,(1,f)}|_{t=0} = 0. \tag{3.6}$$

The inverse to (3.5) transformation $\mathscr{L}^{-1}$ is

$$\begin{pmatrix} \omega_{1,c}(t, y) \\ \omega_{2,c}(-t, y) \end{pmatrix} = L^{-1} \begin{pmatrix} v_{1,c} \\ v_{2,c} \end{pmatrix}(t, y), \begin{pmatrix} \omega_{2,(1,f)}(-t, y) \\ \omega_{1,(1,f)}(t, y) \end{pmatrix} = L^{-1} \begin{pmatrix} w_{1,(1,f)} \\ w_{2,(1,f)} \end{pmatrix}(t, y), \tag{3.7}$$

$$L := |v|^{-1} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

Agmon's conditions on a ray $l := \{\arg \lambda = \varphi\}$ in the case of v-transmission boundary conditions claim that for $(\xi', \lambda) \neq (0, 0)$ and $\lambda \in l$ the equation on $\mathbb{R}_+ \ni t$

$$(-\partial_t^2 + b_2(y, \xi') - \lambda)v(t) = 0, \quad v(t) \to 0 \text{ for } t \to +\infty, \tag{3.8}$$

has a unique solution for each of the initial conditions $v_{t=0} = v_0$ or $\partial_t v|_{t=0} = v_1$.
(Here ξ' are dual to y and $b_2(\varDelta_N) = b_2(y, \xi')$id is the scalar principal symbol of
$\varDelta_N^\bullet$ on U_N.) Agmon's conditions for the v-transmission boundary value problem
are satisfied on each ray $\arg \lambda = \varphi$ not coinciding with $\mathbb{R}_+$ because eq. (3.8) with
each of the initial conditions given above has a unique solution for any $\lambda \notin \mathbb{R}_+$,
$(\xi', \lambda) \notin (0,0)$.

It is convenient to compute the contribution to $P_{\lambda,U}^m$ from v-transmission bound-
ary conditions in the coordinates $v_{j,c}(t, y)$, $w_{J,(1,f)}(t, y)$ defined by (3.5) with
$t \geqq 0$. (Then the v-transmission boundary conditions are transformed into the
conditions (3.6).) These contributions are defined with the help of the symbol
$d = \sum_{j \in \mathbb{Z}_+ \cup 0} d_{-2-j}(t, y, \tau, \xi', \lambda)$, which is the solution of the equation [50]

$$(-\partial_t^2 + (b_2(y, \xi') - \lambda)\mathrm{id} + b_1(y, \xi')) \circ d(t, y, \tau, \xi', \lambda) = 0 \qquad (3.9)$$

(with the composition $\circ$ of symbols of (y, ξ') in it). Equation (3.9) holds for $t \neq 0$.
The boundary conditions for (3.9) are: $d_{-k} \to 0$ as $|t| \to \infty$ and

$$(\mathscr{L}d_{-k})_{1,c}|_{t=0} = (\mathscr{L}a_{-k})_{1,c}|_{t=0}, \quad (\mathscr{L}d_{-k})_{1,(1,f)}|_{t=0} = (\mathscr{L}a_{-k})_{1,(1,f)}|_{t=0},$$

$$\partial_t(\mathscr{L}d_{-k})_{2,c}|_{t=0} = i\tau(\mathscr{L}a_{-k})_{2,c}|_{t=0}, \partial_t(\mathscr{L}d_{-k})_{2,(1,f)}|_{t=0} = i\tau(\mathscr{L}a_{-k})_{2,(1,f)}|_{t=0} .$$

$$(3.10)$$

Here the transformation $\mathscr{L}$ acts on the columns of the matrix-valued functions
d, a (depending on t and on τ). [51]

Equation (3.9) is the recurrent system

$$-\partial_t^2 d_{-k} + (b_2 - \lambda)d_{-k} + \sum \frac{1}{\gamma!}D_{\xi'}^\gamma b_i \partial_y^\gamma d_{-m} = 0 , \qquad (3.11)$$

where the sum is over $m < k$ and γ such that $m + |\gamma| + 2 - i = k$, $0 \leqq |\gamma| \leqq i$ for
b_i.

For $t = 0$ the symbol d_{-k} over $M_j \cap U_N$ is positive homogeneous of degree $(-k)$
in $(\tau, \xi', \lambda^{1/2})$. The boundary contribution to $P_{\lambda,U}^m$ is an operator $\mathscr{D}_m$ corresponding
to [52] $\theta_1(\xi, \lambda)\sum_{j=0}^m d_{-2-j}(t, y, \tau, \xi', \lambda)$. This operator acts on $f \in DR_c^\bullet(\mathbb{R}_t \times \mathbb{R}_y^{n-1})$
such that $\operatorname{supp} f \cap (0 \times \mathbb{R}_y^{n-1}) = \emptyset$ as follows:

$$(\mathscr{D}_m f)(y,t) = (2\pi)^{-n} \int\int \exp(i(y, \xi'))\sum_0^m \theta_1 d_{-2-j}(t, y, \tau, \xi', \lambda)(\mathscr{F}f)(\tau, \xi')d\xi'd\tau$$

$$(3.12)$$

(where $(\mathscr{F}f)(\tau, \xi') = \int\int \exp(-i(t\tau + (x, \xi')))f(t,x)dxdt$ is the Fourier transform
of f). The term of the parametrix, corresponding to U (if $U \cap N \neq \emptyset$) is defined
by

$$P_{\lambda,U}^m = P_{\lambda,\mathrm{int}}^m - \mathscr{D}_m , \qquad (3.13)$$

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[50] Here $b_2(y, \xi')$id $+ b_1(y, \xi)$ is the symbol $s(\varDelta_N^\bullet)$ on U_N of the Laplacian on $DR^\bullet(N)$ for the
components $\omega_{\tan,N}(t, y)$ and on $DR^{\bullet-1}(N)$ for $\omega_{\mathrm{norm},N}(t, y)$. The variable τ is dual to t.

[51] Note that the function $a_{-2-j}(t, y, \tau, \xi', \lambda)$ is continuous in N and nonsingular for $\lambda \notin \mathbb{R}_+$
and $(\tau, \xi', \lambda) \notin (0,0,0)$. (It is also independent of t for $|t|$ small enough.) So the right sides of
(3.10) can be simplified for a_{-k}. In (3.9)-(3.11) it is used that g_M is a direct product metric on
$I \times N \hookrightarrow M$.

[52] $\theta_1(\xi', \lambda) \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{C})$, $\theta_1 \equiv 0$ for $|\xi'|^2 + |\lambda| < \varepsilon$ and $\theta_1 \equiv 1$ for $|\xi'|^2 + |\lambda| \geqq 1$; $n :=$
$\dim M$.

where $P_{\lambda,\text{int}}$ is the PDO with the symbol $s_{(m)}(t, y, \tau, \xi', \lambda)$ defined by (3.4) (x is replaced by (t, y) and $\xi = (\tau, \xi')$).

For $U_j \cap \partial M \neq \emptyset$ the boundary term in P_{λ,U_j}^m for the Dirichlet or the Neumann boundary conditions on the connected components of ∂M is defined similarly.

The following assertions are true:

1. For $m \geqq n$ the operator $(\Delta^\bullet - \lambda)P_\lambda^m - \text{id}$ (where $(\Delta^\bullet - \lambda)$ acts on the restrictions of forms to $\overline{M}_1$ and to $\overline{M}_2$) has a continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ kernel which is $O((1 + |\lambda|^{1/2})^{n-m})$ for $\lambda \in \Lambda_\varepsilon := \{\lambda \neq 0, \varepsilon < \arg \lambda < 2\pi - \varepsilon\}$, where $\pi > \varepsilon > 0$ is fixed ([Se1], Lemma 5, p. 901). This estimate is satisfied uniformly with respect to v since the families d_{-2-j} are smooth in $v \in \mathbb{R}^2 \setminus (0,0)$ and since the estimates for $\mathcal{L}d_{-2-j}$ by [Se1], (29), p. 900, are uniform with respect to $v \neq (0,0)$.

2. Let $m \geqq n$ and $v = (\alpha, \beta) \in \mathbb{R}^2 \setminus (0,0)$. Let $A_j := A(M_j, N)$ be the same as in (2.97) and (2.99) and R_j be the geometrical restrictions to $N \subset \partial M_j$ of forms on M_j. Then the operators

$$|v|^{-1}(\alpha R_1 - \beta R_2)P_\lambda^m, \quad |v|^{-1}(\beta A_1 - \alpha A_2)P_\lambda^m,$$

$$|v|^{-1}(\alpha R_1 \delta - \beta R_2 \delta)P_\lambda^m, \quad |v|^{-1}(\beta A_1 d - \alpha A_2 d)P_\lambda^m$$

have smooth kernels on $N \times \overline{M}_j$ which are $O((1 + |\lambda|^{1/2})^{n-m})$ for $\lambda \in \nabla_\varepsilon$, where $\pi > \varepsilon > 0$ and ε is fixed ([Se1], Lemma 6). These estimates are uniform with respect to $v \in \mathbb{R}^2 \setminus (0,0)$.

3. Set $B_{1,v} := |v|^{-1}(\alpha R_1 - \beta R_2) : \bigoplus_j DR^\bullet(\overline{M}_j) \to DR^\bullet(N)$. Let $p_1 : [0,1] \times N \to N$, $p_2 : [-1,0] \times N \to N$ be the natural projections. The operator

$$q_{1,v} : DR^\bullet(N) \to \oplus DR^\bullet(M_j), \; q_{1,v}(\omega_N) := |v|^{-1}\varphi(t)(\alpha p_1^* \omega_N, -\beta p_2^* \omega_N) \quad (3.14)$$

(where $\varphi(t) \in C_0^\infty(I)$, $\varphi(t) \equiv 1$ for $t \in [-1/2, 1/2]$) is the right inverse to $B_{1,v}$ since $B_{1,v}q_{1,v} = \text{id}$. The analogous right inverse operators $q_{k,v}$ are naturally defined for $B_{k,v}$

$$B_{2,v} := |v|^{-1}(\beta A_1 - \alpha A_2), \quad B_{3,v} := |v|^{-1}(\alpha R_1 \delta - \beta R_2 \delta),$$
$$B_{4,v} := |v|^{-1}(\beta A_1 d - \alpha A_2 d), \quad\quad (3.15)$$

$$B_{i,v}q_{j,v} = \delta_{ij} \cdot \text{id} \quad\quad (3.16)$$

For instance, $q_{3,v} \circ \omega_N := \varepsilon|v|^{-1}\varphi(t)t(\alpha dt \wedge p_1^* \omega_N, -\beta dt \wedge p_2^* \omega_N)$, $\varepsilon = \pm 1$. Set $B_v := (B_{j,v})$, $q_v := (q_{j,v})$, and $q_v B_v := (q_{j,v} B_{j,v})$. For $m \geqq n$ the operator is defined[53]

$$R_\lambda^m := P_\lambda^m - q_v B_v P_\lambda^m . \quad\quad (3.17)$$

It maps (according to [Se1], Lemma 12, p. 912) C^∞-forms $\omega \in DR^\bullet(M)$, $\text{supp}\,\omega \cap N = \emptyset$, to C^∞-forms on M_j. Moreover $R_\lambda^m : DR_c^\bullet(M \setminus N) \to D(\Delta_v^\bullet)$.

4. For $m \geqq n$ and $\lambda \notin \mathbb{R}_+ \cup 0$ the operator R_λ can be continued to a bounded operator in $(DR^\bullet(M))_2$, $R_\lambda : (DR^\bullet(M))_2 \to \text{Dom}\,(\Delta_v^\bullet)$.

Indeed, for any fixed differential operator F of order $d = d(F) \leqq 2$ the operator FR_λ is bounded in $(DR^\bullet(M))_2$ with its norm $O((1 + |\lambda|^{1/2})^{d-2})$ in a sector $\lambda \in \Lambda_\varepsilon$ ($\pi > \varepsilon > 0$ and ε is fixed) according to [Se1], Lemmas 7, 13, 14. This estimate is uniform with respect to $v \in \mathbb{R}^2 \setminus (0,0)$. The continuation of R_λ to $(DR^\bullet(M))_2$ is

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[53] For simplicity we'll suppose from now on here that $\partial M = \emptyset$. For the Dirichlet or the Neumann boundary conditions on the components of ∂M the appropriate terms have to be added to R_j^m.

as follows. If $\omega_j \in DR_c^\bullet(M \setminus N)$ and $\omega_j \to \omega$ in $(DR^\bullet(M))_2$ then $R_\lambda \omega_j$ converges in $(DR^\bullet(M))_2$ and $R_\lambda \omega$ is defined as its limit. We see that $R_\lambda \omega_j \in D(\Delta_v^\bullet)$ and $(\Delta_v^\bullet - \lambda)R_\lambda \omega_j$ converges in $(DR^\bullet(M))_2$. Hence $R_\lambda \omega \in \mathrm{Dom}\,(\Delta_v^\bullet)$.

5. The operator $G_\lambda^\bullet(v) := (\Delta_v^\bullet - \lambda)^{-1} : (DR^\bullet(M))_2 \to \mathrm{Dom}\,(\Delta_v^\bullet)$ exists [54] for $\lambda \in \Lambda_\varepsilon := \{\lambda \neq 0 : \varepsilon < \arg\lambda < 2\pi - \varepsilon\}$ and $|\lambda|$ sufficiently large. Its operator norm is $O(|\lambda|^{-1})$ for such λ uniformly with respect to $v \in \mathbb{R}^2 \setminus (0,0)$ ([Se1], Lemma 15).

6. The Laplacian $\Delta_v^\bullet$ is a closed unbounded operator in $(DR^\bullet(M))_2$ with its domain of definition $\mathrm{Dom}\,(\Delta_v^\bullet)$. Actually, if $\{u_i\} \subset \mathrm{Dom}\,(\Delta_v^\bullet)$ and if the limits $\lim_i u_i =: u$ and $\lim_i ((\Delta_v^\bullet - \lambda)u_i) =: v$ exist in $(DR^\bullet(M))_2$ then for sufficiently large $\lambda \in \Lambda_\varepsilon$ we have $u = \lim_i G_\lambda^\bullet(v)((\Delta_v^\bullet - \lambda)u_i) = G_\lambda(v)v \in \mathrm{Dom}\,(\Delta_v^\bullet)$. Hence $(\Delta_v^\bullet - \lambda)u = (\Delta_v^\bullet - \lambda)(G_\lambda^\bullet(v)v) = v$, i.e., the operators $\Delta_v^\bullet - \lambda\mathrm{id}$ and $\Delta_v^\bullet$ are closed in $(DR^\bullet(M))_2$. The operator $\Delta_v^\bullet$ is defined on $\mathrm{Dom}\,(\Delta_v^\bullet)$. It is a self-adjoint unbounded operator in $(DR^\bullet(M))_2$. Indeed, the domain of definition $\mathrm{Dom}\,((\Delta_v^\bullet - \lambda)^*)$ of the adjoint operator $(\Delta_v^\bullet - \lambda)^*$ in $(DR^\bullet(M))_2$ is the set of $v \in (DR^\bullet(M))_2$ such that the linear functional $((\Delta_v^\bullet - \lambda)\omega, v)$ is continuous on $\mathrm{Dom}\,(\Delta_v^\bullet) \ni \omega$ in the L_2-topology of $(DR^\bullet(M))_2$. If $v \in \mathrm{Dom}\,(\Delta_v^\bullet)$ then for any $\omega \in \mathrm{Dom}\,(\Delta_v^\bullet)$ we have $((\Delta_v^\bullet - \lambda_0)\omega, v) = (\omega, (\Delta_v^\bullet - \lambda_0)v)$ for $\lambda_0 \in \mathbb{R}_-$. Indeed, for each ω and v from $\mathrm{Dom}\,(\Delta_v^\bullet)$ there exist sequences $\{\omega_j\}$ and $\{v_i\}$ of elements $D(\Delta_v^\bullet)$ whose limits in the graph norm topology are ω and v. Hence we have

$$\lim_j (\Delta_v \omega_j, v)_2 = \lim_j \lim_i (\Delta_v \omega_j, v_i)_2 = \lim_j \lim_i (\omega_j, \Delta_v v_i)_2 = (\omega, \Delta_v v)_2 \ .$$

So $((\Delta_v^\bullet - \lambda_0)u, v)$ is a continuous linear functional on $\mathrm{Dom}\,(\Delta_v^\bullet) \ni u$ with respect to the L_2-topology of $(DR^\bullet(M))_2$ for any $v \in \mathrm{Dom}\,(\Delta_v^\bullet)$. Hence $\mathrm{Dom}\,(\Delta_v^\bullet) \subset \mathrm{Dom}\,((\Delta_v^\bullet - \lambda_0)^*)$ and $(\Delta_v - \lambda_0)^* v = (\Delta_v - \lambda_0)v$ for $v \in \mathrm{Dom}\,(\Delta_v^\bullet)$.

Let $\lambda_0 \in \mathbb{R}_-$ and $|\lambda_0|$ be sufficiently large. Then for any $w \in \mathrm{Dom}\,((\Delta_v^\bullet - \lambda_0)^*)$ there exists an element $v \in \mathrm{Dom}\,(\Delta_v^\bullet)$ such that $(\Delta_v^\bullet - \lambda_0)^* w = (\Delta_v^\bullet - \lambda_0)v = (\Delta_v^\bullet - \lambda_0)^* v$ (since $\mathrm{Im}(\Delta_v^\bullet - \lambda_0) = (DR^\bullet(M))_2$). So $w - v \in \mathrm{Ker}((\Delta_v^\bullet - \lambda_0)^*)$ and for any $u \in \mathrm{Dom}\,(\Delta_v^\bullet)$ we have $0 = (u, (\Delta_v^\bullet - \lambda_0)^*(w - v)) = ((\Delta_v^\bullet - \lambda_0)u, w - v)$. Then $w - v = 0$, as $\mathrm{Im}(\Delta_v^\bullet - \lambda_0) = (DR^\bullet(M))_2$. Hence $\mathrm{Dom}\,((\Delta_v^\bullet - \lambda_0)^*) = \mathrm{Dom}\,(\Delta_v^\bullet) = \mathrm{Dom}\,((\Delta_v^\bullet)^*)$, and $\Delta_v^\bullet$ is a self-adjoint unbounded operator in $(DR^\bullet(M))_2$.

The operator $\Delta_v^\bullet$ is nonnegative, $(\Delta_v^\bullet \omega, \omega)_2 \geqq 0$ for any $\omega \in \mathrm{Dom}\,(\Delta_v^\bullet)$, since there exists a sequence $\{\omega_j\}, \omega_j \in D(\Delta_v^\bullet)$, such that its limit in the graph norm topology is ω. So we have $\lim_j (\Delta_v^\bullet \omega_j, \omega_j)_2 = \lim_j (d_v \omega_j, d_v \omega_j)_2 + \lim_j (\delta_v \omega_j, \delta_v \omega_j)_2 \geqq 0$.

7. The spectrum $\mathrm{Spec}(\Delta_v^\bullet)$ of the operator $\Delta_v^\bullet$ is discrete because the operator

$$(\Delta_v^\bullet - \lambda)(\Delta_v^\bullet - \lambda_0)^{-1} = \mathrm{id} + (\Delta_v^\bullet - \lambda_0)^{-1} \cdot (\lambda_0 - \lambda)$$

differs from the identity operator in $(DR^\bullet(M))_2$ by a compact operator. Here, $\lambda_0 \in \Lambda_\varepsilon$ and $|\lambda_0|$ is large enough. The assertion 5 above claims that $(\Delta_v^\bullet - \lambda_0)^{-1}$ exists for such λ_0. The operator $G_{\lambda_0}(v) := (\Delta_v^\bullet - \lambda_0)^{-1}$ is compact since it is bounded in $(DR^\bullet(M))_2$ and since the operators $I - (\Delta_v^\bullet - \lambda_0)R_{\lambda_0}^m$ (for $m \geqq n$) and $R_{\lambda_0}^m$ are compact in $(DR^\bullet(M))_2$ ([Se1], Lemmas 4,5,9 (iv)). So the operator

$$(\Delta_v^\bullet - \lambda_0)^{-1} = R_{\lambda_0}^m + (\Delta_v^\bullet - \lambda_0)^{-1}(I - (\Delta_v^\bullet - \lambda_0)R_{\lambda_0}^m)$$

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[54] This means that $\Delta_v^\bullet - \lambda$ maps $\mathrm{Dom}^\bullet(\Delta_v)$ one-to-one to $(DR^\bullet(M))_2$. It is equivalent to the existence of $(\Delta_v^\bullet - \lambda)^{-1} : (DR^\bullet(M))_2 \to \mathrm{Dom}\,(\Delta_v^\bullet)$, $(\Delta_v^\bullet - \lambda) \circ (\Delta_v^\bullet - \lambda)^{-1} = \mathrm{id}$ on $(DR^\bullet(M))_2$.

is compact in $(DR^\bullet(M))_2$. Since $G_{\lambda_0}(v)$ is a compact operator for $\lambda_0 \in \Lambda_\varepsilon, |\lambda_0| \gg 1$, and since $\Delta_v^\bullet$ is a closed operator in $(DR^\bullet(M))_2$, it follows that $\Delta_v^\bullet$ is an operator in $(DR^\bullet(M))_2$ with compact resolvent. So (according to [Ka], Ch. 3, Theorem 6.29) its spectrum $\mathrm{Spec}(\Delta_v^\bullet)$ consists of isolated eigenvalues with finite multiplicities (i.e., $\mathrm{Spec}(\Delta_v^\bullet)$ is discrete) and the operator $G_\lambda^\bullet(v)$ is compact in $(DR^\bullet(M))_2$ for $\lambda \in \mathbb{C} \setminus \mathrm{Spec}(\Delta_v^\bullet)$. The operator $\Delta_v^\bullet$ is nonnegative. Hence $\mathrm{Spec}(\Delta_v^\bullet) \subset \mathbb{R}_+ \cup 0$.

If $\lambda_0 \notin \mathrm{Spec}(\Delta_v^\bullet)$ then $\mathrm{Ker}(\Delta_v^\bullet - \lambda_0) = 0$ and $\mathrm{Im}(\Delta_v^\bullet - \lambda_0) = (DR^\bullet(M))_2$. Hence $\mathrm{ind}(\Delta_v^\bullet - \lambda_0)$ is equal to 0 (as the index of the operator from $(\mathrm{Dom}(\Delta_v^\bullet), \| \cdot \|^2_{\mathrm{graph}})$ into $((DR^\bullet(M))_2, \| \cdot \|_2^2)$). The operator $(\lambda_0 - \lambda)\mathrm{id}$ from $\mathrm{Dom}(\Delta_v^\bullet)$ into $(DR^\bullet(M))_2$ is compact (since $G_{\lambda_0}^\bullet(v)$ is a compact operator in $(DR^\bullet(M))_2$ and since it is a topological isomorphism $G_{\lambda_0}(v) : (DR^\bullet(M))_2 \widetilde{\to} \mathrm{Dom}(\Delta_v^\bullet)$). So $\mathrm{ind}(\Delta_v^\bullet - \lambda) = 0$ for an arbitrary $\lambda \in \mathbb{C}$ (according to [Ka], Ch. 4, Theorem 5.26, Remarks 1.12, 1.4).

8. The operator $(\Delta_v^\bullet)^{-s}$ for $\mathrm{Re}\, s > 0$ is defined by the integral

$$\frac{i}{2\pi}\int_\Gamma \lambda^{-s}(\Delta_v^\bullet - \lambda)^{-1}d\lambda =: T_{-s}(v)\,, \tag{3.18}$$

where the contour Γ is

$$\{\lambda = re^{i\pi}, \infty > r > \varepsilon\} \cup \{\lambda = \varepsilon e^{i\varphi}, \pi > \varphi > -\pi\} \cup \{\lambda = re^{-i\pi}, \varepsilon < r < \infty\}\,.$$

Here the number $\varepsilon > 0$ is such that $\mathrm{Spec}(\Delta_v^\bullet) \cap (0, \varepsilon] = \emptyset$. The integral (3.18) is absolutely convergent (with respect to the operator norm $\| \cdot \|_2$ in $(DR^\bullet(M))_2$) because the estimate $\|(\Delta_v^\bullet - \lambda)^{-1}\|_2 \leqq C|\lambda|^{-1}$ is satified as $\lambda \to -\infty$ for $\lambda \in \mathbb{R}_-$. So $T_{-s}(v)$ is a bounded operator in $(DR^\bullet(M))_2$ for $\mathrm{Re}\, s < 0$.

For $-k \geqq \mathrm{Re}\, s > -(k+1)$, $k \in \mathbb{Z}_+$, the operator T_{-s} is defined as $\Delta_v^{k+1}T_{-(s+k+1)}$. Its domain is $\mathrm{Dom}(T_{-s}) = \{\omega \in (DR^\bullet(M))_2, T_{-(s+k+1)}\omega \in \mathrm{Dom}((\Delta_v^\bullet)^{k+1})\}$, where

$$\mathrm{Dom}((\Delta_v^\bullet)^{k+1}) := \{\omega \in \mathrm{Dom}(\Delta_v^\bullet), \Delta_v^\bullet\omega \in \mathrm{Dom}(\Delta_v^\bullet), \ldots, (\Delta_v^\bullet)^k\omega \in \mathrm{Dom}(\Delta_v^\bullet)\}\,.$$

The restriction T_{-s}^0 of T_{-s} to the orthogonal complement L_0 of $\mathrm{Ker}(\Delta_v^\bullet)$ in $(DR^\bullet(M))_2$ is defined on $\mathrm{Dom}(T_{-s}^0) := D(T_{-s}) \cup L_0$, $T_{-s}^0 := T_{-s}|_{L_0}$. Then T_{-s} is the direct sum[55] of T_s^0 and of the zero operator on $\mathrm{Ker}(\Delta_v^\bullet)$. Theorem 1 in [Se2] claims that the family T_{-s}^0 of operators in the Hilbert space L_0 for $\mathrm{Re}\, s_2 > 0$ satisfies the equation $T_{-s_1}^0 T_{-s_2}^0 = T_{-(s_1+s_2)}^0$ and that the same is true for $-s_1 \in \mathbb{Z}_+$ and for each s_2. This theorem claims also that

$$T_0^0 = \mathrm{id} \text{ on } L_0, \ T_{-l}^0 = \left((\Delta_v^\bullet)^{-1}|_{L_0}\right)^l, \text{ for } l \in \mathbb{Z}_+, \text{ and } T_1^0 = \Delta_v^\bullet|_{L_0}$$

(the domain of T_1^0 is $\mathrm{Dom}(\Delta_v^\bullet) \cap L_0$) and that T_{-s}^0 for $\mathrm{Re}\, s > 0$ is a holomorphic function [56] with its values in a Banach space $B(L_0)$ of bounded operators in L_0, where the Banach norm is the operator norm (as the norm on $B(L_0)$).

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[55] If $v \in L_0$ and $\mathrm{Re}\, s > 0$ then we have $T_{-s}v \in L_0$ since for $h \in \mathrm{Ker}(\Delta_v^\bullet)$ and $\lambda \in \Gamma$ it holds $0 = (v, h) = ((\Delta_v^\bullet - \lambda)G_\lambda^\bullet(v)v, h) = -\lambda(G_\lambda^\bullet(v)v, h)$ and since the integral (3.18) is absolutely convergent. For $h \in \mathrm{Ker}(\Delta_v^\bullet)$ and for $\mathrm{Re}\, s > 0$ we have $T_{-s}h = 0$ because for such s the integral $\int_\Gamma \lambda^{-s-1}d\lambda$ is absolutely convergent and is equal to zero. Since $T_{-s}h = (\Delta_v^\bullet)^{k+1}T_{-(s+k+1)}h = 0$ for $-k \geqq \mathrm{Re}\, s > -(k+1)$, we get $T_{-s}h \equiv 0$ for all s.

[56] A function with the values in a Banach space is holomorphic in a strong sense if it is weakly holomorphic ([Ka], Ch. 3, Sect. 1, Theorem 1.37, p. 139).

9. For $\text{Re}\, s > n/2$ the kernel of $T_{-s}(v)$ is continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ and analytic in s ([Se2], Theorem 2(i)). For $\text{Re}\, s > n/2$ the zeta-function $\zeta_{v,\bullet}(s)$ is equal to the sum of integrals over the diagonals $\overline{M}_j \hookrightarrow \overline{M}_j \times \overline{M}_j$ ($j = 1, 2$) of the densities defined by the restrictions to these diagonals of the kernel $T_{-s}(v)$, according to Proposition 3.9 below. So $\zeta_{v,\bullet}(s)$ is holomorphic for $\text{Re}\, s > n/2$.

The operator $G^\bullet_\lambda(v) - P^m_\lambda$ for $m \geqq n$ (where P^m_λ is the parametrix (3.3)) is a bounded in $(DR^\bullet(M))_2$ operator with a continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ kernel $(r^m_\lambda)_{x_1 x_2}$ which is

$$(r^m_\lambda)_{x_1, x_2} = O\!\left((1 + |\lambda|^{1/2})^{-(2+m)+n} \right) \tag{3.19}$$

as $|\lambda| \to +\infty$, $\lambda \in \Lambda_\varepsilon$ ([Se1], Theorem 1, or also the assertions 5, 1, 2, 7 above). So the operator

$$\frac{i}{2\pi} \int_\Gamma \lambda^{-s} (G^\bullet_\lambda(v) - P^m_\lambda) d\lambda \tag{3.20}$$

for $\text{Re}\, s > (n - m)/2$ is of trace class and its kernel is continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ and analytic in s.

The trace of the operator (3.20) is holomorphic in s for $\text{Re}\, s > (n - m)/2$. Let us denote by $K^{\text{int}}_{x,y}(s)$ the kernel of the operator $(i/2\pi) \int_\Gamma \lambda^{-s} P^m_{\lambda,\text{int}} d\lambda$ (where $P^m_{\lambda,\text{int}} := \sum \psi_j P^m_{\lambda, U_j, \text{int}} \varphi_j$ is a term of (3.3) and $P^m_{\lambda, U_j, \text{int}}$ is a PDO with symbol $\theta s_{(m)}$, defined by (3.4)). This kernel is continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ for $\text{Re}\, s > n/2$. Off the diagonals $\overline{M}_j \hookrightarrow \overline{M}_j \times \overline{M}_j$ it extends to a kernel which is an entire function of $s \in \mathbb{C}$ equal to zero for $(-s) \in \mathbb{Z}_+ \cup 0$. The density on $\cup_j \overline{M}_j$ defined by the restriction of this kernel to the diagonals also can be continued to a meromorphic in $s \in \mathbb{C}$ density. This density has at most simple poles at $s_j = (n - j)/2$ for $(-s_j) \notin \mathbb{Z}_+ \cup 0$, $0 \leqq j \leqq m$, and it is regular at s_j for $(-s_j) \in \mathbb{Z}_+ \cup 0$.

The residue at $s = s_j$ is completely defined by the component $a_{-2-j}(x, \xi, \lambda)$ of the symbol $s((\Delta^\bullet - \lambda)^{-1})$ ([Se2], Lemma 1, or [Sh], Theorem 12.1). These components are given by (3.4). The value of this density at $s = s_j$ for $(-s_j) \in \mathbb{Z}_+ \cup 0$ is completely defined by a_{-2-j} (by the formulas (11), (12) in [Se2] with changing of the sign in (11) to the opposite one). Here, $j = n + 2m$, $m \in \mathbb{Z}_+ \cup 0$.

The kernel $K^\partial_{x,y}(s)$ of the operator[57] $(i/2\pi) \int_\Gamma \lambda^{-s} \mathscr{D}_{m,\lambda} d\lambda$ for $\text{Re}\, s > n/2$ is continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ and analytic in s ([Se2], Lemma 4). Let (x, y) be off the diagonals or let either x or y be not from $\cup_j \partial \overline{M}_j \supset N$. Then $K^\partial_{x,y}(s)$ is an entire function of $s \in \mathbb{C}$ and it is equal to zero at s for $(-s) \in \mathbb{Z}_+ \cup 0$ ([Se2], Lemma 4). For $\text{Re}\, s > n/2$ the densities defined by the restriction $K^\partial_{x,x}(s)$ of $K^\partial_{x,y}(s)$ to the diagonals $\overline{M}_j$ are integrable over the fibers of the natural projections $p_1 : [0, 1] \times N \to N$ and $p_2 : [-1, 0] \times N \to N$. These integrals are densities on N. They can be continued [58] to meromorphic in $s \in \mathbb{C}$ densities (on N) with at most simple poles at $s_j = (n - j)/2$, $1 \leqq j \leqq m$, such that $(-s_j) \notin \mathbb{Z}_+ \cup 0$. Their residues at s_j for $1 \leqq j \leqq m + 1$ are completely defined by a term d_{-2-j+1} in d ([Se2], Theorem

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[57] Here $\mathscr{D}_{m,\lambda} = \sum \psi_j \mathscr{D}_{m,U}(\lambda, v) \varphi_j$. The operator $\mathscr{D}_{m,U}$ from (3.13) is defined for $U \cap N \neq \emptyset$ by (3.12), (3.9), and (3.10).

[58] Let $\mathscr{L}\mathscr{D}_{m,U}(\lambda, v)$ be an operator acting on $\mathscr{L}\omega$ as $\mathscr{L}(\mathscr{D}_{m,U}(\lambda, v)\omega)$ (for any $\omega \in DR^\bullet_c(\mathbb{R}^n)$) such that $\text{supp}\,\omega \cap \mathbb{R}^{n-1} = \emptyset$, where $\mathbb{R}^{n-1}$ are local coordinates on N). All the assertions about the kernels analogous to $K^\cap_{x,y}(s)$ in the case of $\mathscr{L}\mathscr{D}_{m,U}(\lambda, v)$, and about the corresponding densities in this case, are proved in [Se2]. Thus the transformation (3.7) provides us with all the assertions about the kernel $K^\cap_{x,y}(s)$ (and about the corresponding densities) connected with $\mathscr{D}_{U,\prime}(v)$.

2(iv), formula (II)). The values of these densities at s_j for $(-s_j) \in \mathbb{Z}_+ \cup 0$ (where $n \leq j \leq m+1$ are also completely defined by d_{-2-j+1} ([Se2], Lemmas 2, 3, 4, Theorem 2(iv), formula (II') with changing of the sign to the opposite one).

10. The kernel of the operator (3.20) is the difference of the kernels

$$(T_{-s})_{\lambda, v} - (K^{\text{int}}_{x, v}(s) - K^{\hat{c}}_{x, v}(s)) \, . \tag{3.21}$$

For $\mathrm{Re}\, s > (n - m)/2$ it is holomorphic in s and continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2}$. The term $(K^{\text{int}} - K^{\hat{c}})_{\lambda, v}(s)$ is equal to zero for $x \neq y$ and $(-s) \in \mathbb{Z}_+ \cup 0$. The term $(T_{-s})_{x, v}$ is equal to zero for $x \neq y$ and $(-s) \in \mathbb{Z}_+$ (according to the assertions of 8 above, since $\Delta^\bullet_v$ is a differential operator). We have $(T_0)_{x, v} = -\mathscr{H}^\bullet_{x, v}$, where $\mathscr{H}^\bullet$ is the kernel of the orthogonal projection operator in $(DR^\bullet(M))_2$ onto $\mathrm{Ker}(\Delta^\bullet_v)$ (the assertion 8). The properties of $\zeta_{v, \bullet}(s)$ [59] formulated in Theorem 3.1, follow from the assertions of 9 and 10. The theorem is proved. $\quad\square$

Remark 3.1. The kernel (3.21) of the operator (3.20) is holomorphic in s and continuous in $(x, y) \in \overline{M}_{j_1} \times \overline{M}_{j_2}$ for $\mathrm{Re}\, s > (n - m)/2$. It is equal to zero for $x \neq y$ at $s = -k$, $k \in \mathbb{Z}_+$, and to $-\mathscr{H}^\bullet_{x, y}(v)$ at $s = 0$. So the analytic continuations of the densities on $\overline{M}_j$ and on N defined by the kernels $(T_{-s})_{x, v}$ and $(K^{\text{int}}(s) - K^{\hat{c}}(s))_{\lambda, v}$ have the same residues at $s = s_j$, $0 \leq j \leq m$, $(-s_j) \notin \mathbb{Z}_+ \cup 0$, and the same values at $s = s_j$, $(-s_j) \in \mathbb{Z}_+$, $n + 2 \leq j \leq m$. They differ at $s = 0$ (i.e., for $j = n$) by the densities on $\overline{M}_j$, defined by $-\mathscr{H}^\bullet_{x, x}(v)$. Hence the densities on $\overline{M}_j$ and on N, corresponding to the residues and to the values at $s = s_j$, $0 \leq j \leq m - 1$, of $(K^{\text{int}}(s) - K^{\hat{c}}(s))_{x, x}$ are the same for all the parametrixes P^m_λ defined by (3.3) (with different covers $\{U_j\}$, partitions of unity $\{\varphi_j\}$ subordinate to $\{U_j\}$, and $\{\psi_j\}$).

And back, the values and the residues of the analytic continuation for the integral $\int_{\overline{M}_j} \mathrm{tr}(i^*_j (T_{-s})) + \delta_{s, 0} \int_{\overline{M}_j} \mathrm{tr}(i^*_j \mathscr{H}^\bullet(v))$ at s_j, $0 \leq j \leq m$, are defined by an arbitrary parametrix P^m_λ.

Proof of Proposition 3.1. Let $m \geq n := \dim M$, $m \in \mathbb{Z}_+$, and $\lambda \in \Lambda_\varepsilon$. Then the parametrix R^m_λ for $G^\bullet_\lambda(v)$ [60] (defined by (3.17)) is a bounded operator [61] in $(DR^\bullet(M))_2$ with its norm estimated by $O((1 + |\lambda|^{1/2})^{-2})$ for $\lambda \in \Lambda_\varepsilon$. It holds that $R_\lambda : (DR^\bullet(M))_2 \to \mathrm{Dom}(\Delta^\bullet_v)$. For a linear differential operator F of order $d = d(F) \leq 2$ the operator FR_λ is defined on smooth forms $\omega \in DR^\bullet_c(M \setminus N)$ and its closure in $(DR^\bullet(M))_2$ is a bounded operator in $(DR^\bullet(M))_2$ with its norm estimated by $O((1 + |\lambda|^{1/2})^{d-2})$ for $\lambda \in \Lambda_\varepsilon$. (All these estimates are uniform with respect to $v \in \mathbb{R}^2 \setminus (0, 0)$.) The only terms of R^m_λ depending on v are the terms $\mathscr{L} - q_v B_v P^m_\lambda(v)$, where [62] $\mathscr{L} = \mathscr{L}(v) := -\sum \psi_j \mathscr{L}_{m, U_j} \varphi_j$ has the kernel with support

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[59] The values of $\zeta_{v, \bullet}(s)$ at $(-s) \in \mathbb{Z}_+$ and the residues of $\zeta_{v, \bullet}$ at $s = s_j$ can be also expressed in terms of noncommutative residues ([Wo] or [Kas]). The density on M whose integral over M is equal to a volume term in $\mathrm{Res}_{s = s_j} \zeta_{v, \bullet}(s)$ can be written as $2^{-1} \mathrm{res}\left(x, \Delta^{-s_j}_{\theta = \pi}\right)$. Here res is a noncommutative residue for the symbol of PDO $\Delta^{-s_j}_{\theta = \pi}$. This symbol is defined with the help of the symbol $\sum a_{-2-j}(x, \xi, \lambda)$ of $(\Delta^\bullet - \lambda)^{-1}$ ([Sh], 11.2). The boundary term in $\mathrm{Res}_{s = s_j} \zeta_{v, \bullet}(s)$ for $(-s_j) \notin \mathbb{Z}_+ \cup 0$ is expressed similarly.

[60] The statement that $G^\bullet_\lambda(v) : (DR^\bullet(M))_2 \to \mathrm{Dom}(\Delta^\bullet_v)$ is an isomorphism for $\lambda \notin \mathrm{Spec}(\Delta^\bullet_v)$ is proved in Theorem 3.1.

[61] The terms $P^{(m)}_{\lambda, \text{int}}$, $\psi_j \mathscr{L}_m \varphi_j$, and $q_v B_v P^m_\lambda$ in R^m_λ are bounded operators with the same estimate of their norms in $(DR^\bullet(M))_2$ for $\lambda \in \Lambda_\varepsilon$ (the proof of Theorem 3.1). For the sake of brevity the proof of Proposition 3.1 is given in the case of $\partial M = \emptyset$.

[62] $\mathscr{L}_m = \mathscr{L}_{m, U_j}$ is defined by (3.11), (3.10), and (3.12).

in the neighborhood $I \times N$ of the interior boundary $N \hookrightarrow M$. [63] The parametrix $P_\lambda^m(v) := P_{\lambda,\text{int}}^m - \mathscr{D}(v)$ is defined by (3.13)). We need the following assertion now.

Proposition 3.3. *The operators $\mathscr{D}(v)$ and $q_v B_v P_\lambda^m(v)$ depend smoothly on $v \in \mathbb{R}^2 \setminus (0,0)$ as bounded operators in $(DR^\bullet(M))_2$. For a C^∞-map $v = \varphi(\gamma) : [-a,a] \to \mathbb{R}^2 \setminus (0,0)$ the operator $\partial_\gamma \mathscr{D}(v)$ is a bounded operator in $(DR^\bullet(M))_2$ whose norm is uniformly with respect to γ estimated by $O((1 + |\lambda|^{1/2})^{-2})$ for $\lambda \in \Lambda_\varepsilon$. Let F be a linear differential operator of order $d = d(F) \leqq 2$ from $DR^\bullet(\overline{M}_j)$ into $DR^{\bullet+k}(\overline{M}_j)$, $k \in \mathbb{Z}$. Then $F\mathscr{D}(v)$, $F\partial_\gamma \mathscr{D}(v)$ are bounded operators from $(DR^\bullet(M))_2$ into $(DR^{\bullet+k}(M))_2$ whose norms are estimated by $O((1 + |\lambda|^{1/2})^{d(F)-2})$ for $\lambda \in \Lambda_\varepsilon$. The operator $\partial_\gamma(q_v B_v P_\lambda^m(v))$ is uniformly with respect to γ estimated by $O((1 + |\lambda|^{1/2})^{-1})$ for $\lambda \in \Lambda_\varepsilon$.*

Proof. The kernel of the operator $\mathscr{L}\psi_j \mathscr{D}_{m,U_j}\varphi_j \mathscr{L}^{-1}$ (where $\mathscr{L} = \mathscr{L}(v)$ and $\mathscr{L}^{-1}$ are defined by (3.5) and (3.7)) has a support in $((U_j \cap N) \times [0,1])^2$. The operator $\mathscr{D}_{m,U_j}$ is defined in $(U_j \cap N) \times \mathbb{R}_+$ by (3.11) and (3.10). The right sides of the boundary conditions (3.10) depend on v only by their dependence on $L(v)$ (where L is the matrix defined by (3.7)). Since g_M is a direct product metric near N, a mirror symmetry (relative to N) acts as the identity operator on the symbol $\sum a_{-2-j}(x, \xi)$ of the Laplacian $\Delta^\bullet$ on M for $x = (t, x')$ from the neighborhood $I \times N$ of N. The symbol $\sum a_{-2-j}(t, x', \tau, \xi', \lambda)$ is independent of t for $t \in I$.

So the symbol $\mathscr{L}\sum a_{-2-j}$ (for $t \in I$) is expressed as LaL^{-1}, where L and L^{-1} act on the components of a matrix-valued functions a_{-2-j} in the coordinates $\omega_{j,c}$ and $\omega_{j,(1,f)}$ as follows (according to (3.5)):

$$(La)_{1,c;*} = |v|^{-1}(\alpha - \beta)(a)_{c,*}, \quad (La)_{1,(1,f);*} = |v|^{-1}(-\beta + \alpha)(a)_{(1,f);*},$$

$$(La)_{2,c;*} = |v|^{-1}(\beta + \alpha)(a)_{c;*}, \quad (La)_{2,(1,f);*} = |v|^{-1}(\alpha + \beta)(a)_{(1,f);*}. \quad (3.22)$$

The boundary conditions (3.10) (according to (3.22)) depend on v only by the matrix transformation whose coefficients are independent of (t, x') and smooth in v. This transformation acts separately on each homogeneous component a_{-2-j}. The right sides in (3.22) are nonsingular in $(x', \tau, \xi', \lambda)$ for $b_2(x', \tau, \xi') - \lambda \neq 0$ (where b_2 is the principal symbol of the Laplacian on M for $(t, x') \in I \times N$). Hence Lemma 2 in [Se1] holds also for the symbol $\partial_\gamma(\mathscr{L}(v)\sum d_{-2-j})$. Thus the desired estimates for the norm of $\partial_\gamma \mathscr{D}(v)$ in $(DR^\bullet(M))_2$ and for the norms of $F\mathscr{D}(v)$ and of $F\partial_\gamma \mathscr{D}(v)$ are consequences of [Se1], Lemma 7.

The operators $q_{j,v}$ ($1 \leqq j \leqq 4$) from (3.14) and (3.16) (for $\varphi(t)$ even on t) can be defined such that [64]

$$\mathscr{L}q_v B_v P_\lambda^m(v)f(t)\mathscr{L}^{-1} = \sum q_j B_j \mathscr{L}P_\lambda^m(v)f(t)\mathscr{L}^{-1},$$

where $f \in C_0^\infty(I)$, $f(t) \equiv 1$ for $t \in [0, 1/2]$ and $f(t) \equiv 0$ for $t > 3/4$. The operators B_j and q_j are independent of $v \in \mathbb{R}^2 \setminus (0,0)$ and correspond to $B_{j,v}$ and $q_{j,v}$

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[63] In the case $\partial M \neq \emptyset$ the terms connected with the Dirichlet and the Neumann boundary conditions are added to $\mathscr{D}(v)$. Then the corresponding kernel has its support in a neighborhood of ∂M in M.

[64] The operator $B_v P_\gamma^m(v)$ has a continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ kernel which is estimated uniformly with respect to $(x_1, x_2) \in (N \cap \overline{M}_{j_1}) \times \overline{M}_{j_2}$ and to $v \in \mathbb{R}^2 \setminus (0,0)$ by $O((1 + |\lambda|^{1/2})^{n-m})$ for $\lambda \in \Lambda_\varepsilon$ ([Se1], Lemma 6). Such an estimate holds also for the kernel of $q_v B_v P_\gamma^m(v)$.

from (3.15), (3.16). (Here B_1, B_2, B_3, B_4 are the operators (3.6) acting respectively on $v_{1,c}$, $w_{2,(1,f)}$, $w_{1,(1,f)}$, $w_{2,c}$.) These operators are such that $B_i q_j = \delta_{ij} \mathrm{id}$.

The operator $\sum q_j B_j \mathscr{L}(\partial_\gamma P^m_\lambda(v)) f \mathscr{L}^{-1}$ is equal to $\sum q_j B_j \mathscr{L}(-\partial_\gamma \mathscr{D}(v)) f) \mathscr{L}$ (since $P^m_{\lambda,\mathrm{int}}$ is independent of v). The operator $B_j \mathscr{L}(\partial_\gamma \mathscr{D}(v)) \mathscr{L}^{-1}$ is defined on smooth forms $\omega \in DR^\bullet_c((0,1) \times N)$ and its L_2-norm is estimated by $O((1 + |\lambda|^{1/2})^{-1})$ for $\lambda \in \Lambda_\varepsilon$ ([Se1], Lemma 7). The operators $\partial_\gamma (q_v B_v) P^m_{\lambda,\mathrm{int}}$ are defined on smooth forms $\omega \in DR^\bullet_c(M \backslash N)$ and their operator L_2-norms are estimated by $O((1 + |\lambda|^{1/2})^{-1})$ for $\lambda \in \Lambda_\varepsilon$ uniformly with respect to v (according to [Se1], Lemma 7). The proposition is proved. □

Let $\lambda \in \Lambda_\varepsilon$ and $|\lambda|$ be large enough. Then the Green function $G^\bullet_\lambda(v)$ can be represented by the series

$$G^\bullet_\lambda(v) = R^m_\lambda \sum_{i=0}^\infty (L^m_\lambda)^i \,, \tag{3.23}$$

where $(L^m_\lambda) := \mathrm{id} - (\Delta^\bullet_v - \lambda) R^m_\lambda$ is a bounded operator in $(DR^\bullet(M))_2$ for $\lambda \in \Lambda_\varepsilon$. The norm of L^m_λ in $(DR^\bullet(M))_2$ is $O((1 + |\lambda|^{1/2})^{n-m+2})$ (where $n := \dim M$) because the norm of $(\mathrm{id} - (\Delta^\bullet - \lambda) P^m_\lambda)$ is $O((1 + |\lambda|^{1/2})^{n-m})$ and the norm of $(\Delta^\bullet - \lambda) q_v B_v P^m_\lambda$ is $O((1 + |\lambda|^{1/2})^{n-m+2})$ (according to the proof of Theorem 3.1). Hence if $m > n + 2$ and if $\lambda \in \Lambda_\varepsilon$ with $|\lambda|$ large enough then the series (3.23) is convergent with respect to the operator norm in $(DR^\bullet(M))_2$. The operator L^m_λ depends smoothly on $v \in \mathbb{R}^2 \backslash (0,0)$. The norm of $\partial_\gamma L^m_\lambda$ is estimated by $O(1 + |\lambda|^{1/2})$ (according to Proposition 3.3). Let $v : [-a,a] \to \mathbb{R}^2 \backslash (0,0)$ be a smooth map. Then the series for $\partial_\gamma G^\bullet_\lambda(v)$ is convergent (in the operator norm) if $\lambda \in \Lambda_\varepsilon$ and $|\lambda|$ is large enough. Hence for such λ the resolvent $G^\bullet_\lambda(v) := (\Delta^\bullet_v - \lambda)^{-1}$ depends smoothly on γ. So the family $G^\bullet_\lambda(v)$ of bounded operators in $(DR^\bullet(M))_2$ is smooth in (v,λ) for such λ. Their operator norms are estimated by $O(|\lambda|^{-1})$ uniformly with respect to $v \in \mathbb{R}^2 \backslash (0,0)$. Let F be a linear differential operator of degree $d(F) \leqq 2$. Then the operators $FG^\bullet_\lambda(v)$ for such λ are bounded in $(DR^\bullet(M))_2$ with their operator norms estimated by $O(|\lambda|^{(d-2)/2})$ uniformly with respect to v. These operators depend smoothly on v for such λ and we have

$$\partial_\gamma F G^\bullet_\lambda(v) = F \partial_\gamma G^\bullet_\lambda(v) \,. \tag{3.24}$$

Hence for a given $v_0 \in \mathbb{R}^2 \backslash (0,0)$ there exists $\lambda_1 \in \Lambda_\varepsilon$ such that $G^\bullet_{\lambda_1}(v)$ depends smoothly on v for v sufficiently close to v_0. For $\lambda \in \mathbb{C} \backslash \mathrm{Spec}(\Delta^\bullet_{v_0})$ the resolvent $G^\bullet_\lambda(v_0)$ can be represented as follows:

$$G^\bullet_\lambda(v_0) = -(\lambda - \lambda_1)^{-1} - (\lambda - \lambda_1)^{-2} R((\lambda - \lambda_1)^{-1}, G^\bullet_{\lambda_1}(v_0)) \,, \tag{3.25}$$

where $R(\eta, G^\bullet_{\lambda_1}(v_0)) := (G^\bullet_{\lambda_1}(v_0) - \eta)^{-1}$ is the resolvent of a bounded operator $G^\bullet_{\lambda_1}(v_0)$ in $(DR^\bullet(M))_2$ ([Ka], Ch. IV, (3.6), Ch. III, (6.18)). The bounded operator $R(\eta, B)$ is an analytic function of a bounded operator B and of η for $\eta \notin \mathrm{Spec}\, B$ (i.e., near (η_0, B_0), $\eta_0 \notin \mathrm{Spec}\, B_0$, it is locally defined by a convergent double power series in $(\eta - \eta_0)$ and $(B - B_0)$). The operator $G^\bullet_{\lambda_1}(v)$ depends smoothly on v for v sufficiently close to v_0. Then it follows from (3.25) that $G^\bullet_\lambda(v)$ depends smoothly on v for $\lambda \in \mathbb{C} \backslash \mathrm{Spec}(\Delta^\bullet_v)$ and for v sufficiently close to v_0.

Let $F : \bigoplus_j DR^\bullet(\overline{M}_j) \to \bigoplus_j DR^{\bullet+k}(\overline{M}_j)$, $k \in \mathbb{Z}$, be a linear differential operator of degree $d(F) \leqq 2$. Then for $\lambda \in \Lambda_\varepsilon$ and $|\lambda|$ large enough the operators

$FG_\lambda^\bullet(v) : (DR^\bullet(M))_2 \to (DR^{\bullet+k}(M))_2$ are defined, bounded, and smooth in $v \in \mathbb{R}^2\backslash(0,0)$. (It is proved above.) For example, $dG_\lambda^\bullet(v)$ and $\delta G_\lambda^\bullet(v)$ are smooth in v. According to (3.24) we have $\partial_\gamma(dG_\lambda^\bullet(v)) = d\partial_\gamma G_\lambda^\bullet(v)$, $\partial_\gamma(\delta G_\lambda^\bullet(v)) = \delta\partial_\gamma G_\lambda^\bullet(v)$.

The operators $dG_\lambda^\bullet(v)$ are defined for $\lambda \notin \mathrm{Spec}(\Delta_v^\bullet)$. From (3.24) and (3.25) we get

$$\partial_\gamma(dG_\lambda^\bullet(v))|_{\gamma=0} = d(\partial_\gamma G_\lambda^\bullet(v)|_{\gamma=0})$$

$$= -(\lambda - \lambda_1)^{-2} d\left(\frac{\partial}{\partial B}R((\lambda-\lambda_1)^{-1}, B)\Big|_{B=G_{\lambda_1}^\bullet(v_0)}\partial_\gamma G_{\lambda_1}^\bullet(v)\Big|_{\gamma=0}\right)$$

for a C^∞-local map $(\mathbb{R}_\gamma^1, 0) \to (\mathbb{R}^2\backslash(0,0), v_0)$, where $\lambda_1 \in \Lambda_\varepsilon$ with $|\lambda_1|$ large enough and $\lambda \notin \mathrm{Spec}(\Delta_{v_0}^\bullet)$. Proposition 3.1 is proved. $\square$

3.3. Theta-Functions for the Laplacians with v-Transmission Boundary Conditions. Proofs of Theorem 3.2 and of Proposition 3.2.

Let ε be fixed, $0 < \varepsilon < \pi/2$. The operator $\exp(-t\Delta_v^\bullet)$ is defined for $\mathrm{Re}\,t > 0$, $\pi/2 - \varepsilon > \arg t > -(\pi/2 - \varepsilon)$, by the integral

$$\exp(-t\Delta_v^\bullet) = \frac{i}{2\pi}\int_{\Gamma_{L,\varepsilon}} \exp(-\lambda t)G_\lambda^\bullet(v)d\lambda, \tag{3.26}$$

where $\Gamma_{L,\varepsilon} = \Gamma_{L,\varepsilon}^1 \cup \Gamma_{L,\varepsilon}^2$, $\Gamma_{L,\varepsilon}^1 = \{\lambda = -L + x\exp(i\varepsilon), +\infty > x \geqq 0\}$, $\Gamma_{L,\varepsilon}^2 = \{\lambda = -L + x\exp(-i\varepsilon), 0 \leqq x < +\infty\}, L > 0$. The integral (3.26) is absolutely convergent because the operator norm in $(DR^\bullet(M))_2$ of the operator $G_\lambda^\bullet(v)$ (which is bounded in $(DR^\bullet(M))_2$) is estimated by $O(|\lambda|^{-1})$ for $\lambda \in \Gamma_{L,\varepsilon}$, according to Theorem 3.1. [65] This integral is independent of $L > 0$ and of ε, $\pi/2 > \varepsilon > 0$, for t such that $|\arg t| < \pi/2 - \varepsilon$, since the spectrum of $\Delta_v^\bullet$ is discrete and since $\mathrm{Spec}(\Delta_v^\bullet) \subset \mathbb{R}_+ \cup 0$. With the help of the inverse *Mellin transform* $f \to M^{-1}f$,

$$(M^{-1}f)(t) := (2\pi i)^{-1}\int_{\mathrm{Re}\,s=c} \Gamma(s)t^{-s}f(s)ds,$$

it is possible to obtain the results about the asymptotic expansion for $\mathrm{Tr}\exp(-t\Delta_v^\bullet)$ as $\mathrm{Re}\,t \to +0$ (when $\pi/2 - \varepsilon > |\arg t|$) from the results about $\zeta_{v,\bullet}(-m)$, $m \in \mathbb{Z}_+ \cup 0$, and about $\mathrm{res}_{s=s_j}\zeta_{v,\bullet}(s)$ obtained in Theorem 3.1. The integral (3.26) can be transformed as follows:

$$\exp(-t\Delta_v^\bullet) = \mathcal{H}^\bullet(v) + \frac{i}{2\pi}\int_{\Gamma_{-\delta,\varepsilon}} \exp(-t\lambda)G_\lambda^\bullet(v)d\lambda,$$

where $\mathcal{H}^\bullet(v)$ is the kernel of the orthogonal projection operator of $(DR^\bullet(M))_2$ onto $\mathrm{Ker}\,\Delta_v^\bullet$ and where $\delta > 0$ and ρ, $\rho \geqq \delta$, is such that $\mathrm{Spec}(\Delta_v^\bullet) \cap (0,\rho] = \emptyset$. The operator $\exp(-t\Delta_v^\bullet)$ for $|\arg t| < \pi/2 - \varepsilon$ can be represented as follows (where Γ is the same as in (3.18) and $c > 0$):

$$\exp(-t\Delta_v^\bullet) = \mathcal{H}^\bullet(v) + \frac{i}{2\pi}\int_{\Gamma_{-\delta,\varepsilon}} \exp(-t\lambda)G_\lambda^\bullet(v)d\lambda$$

$$= \mathcal{H}^\bullet(v) + \frac{i}{2\pi}\int_{\Gamma_{-\delta,\varepsilon}} (2\pi i)^{-1}G_\lambda^\bullet(v)\left(\int_{\mathrm{Re}\,s=c}(\lambda t)^{-s}\Gamma(s)ds\right)d\lambda$$

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[65] The constant factor in this estimate depends on ε.

$$= \mathcal{H}^\bullet(v) + (2\pi i)^{-1} \int\limits_{\mathrm{Re}\,s=c} t^{-s} \Gamma(s) \left(\frac{i}{2\pi} \int\limits_{\Gamma_{-\delta,\varepsilon}} \lambda^{-s} G_\lambda^\bullet(v) d\lambda \right) ds$$

$$= \mathcal{H}^\bullet(v) + (2\pi i)^{-1} \int\limits_{\mathrm{Re}\,s=c} t^{-s} \Gamma(s) \left(\frac{i}{2\pi} \int\limits_{\Gamma} \lambda^{-s} G_\lambda^\bullet(v) d\lambda \right) ds$$

$$= \mathcal{H}^\bullet(v) + (2\pi i)^{-1} \int\limits_{\mathrm{Re}\,s=c} \Gamma(s) t^{-s} T_{-s}(v) ds \ . \tag{3.27}$$

Here, the integration is over $\mathrm{Re}\,s = c$ from $c - i\infty$ to $c + i\infty$ (where $c > 0$). The operator $T_{-s}(v)$ for $\mathrm{Re}\,s > 0$ is defined by the integral (3.18). The transformations we apply in (3.27) are correct by the Fubini theorem since the estimate

$$\|G_\lambda^\bullet(v)\|_2 < C \cdot |\lambda|^{-1}, \quad \lambda \in \Gamma_{-\delta,\varepsilon} \ ,$$

is satisfied by the operator norm of $G_\lambda^\bullet(v)$ in $(DR^\bullet(M))_2$ and since for $\mathrm{Re}\,s > 0$ the gamma-function can be estimated as follows. We have

$$\Gamma(s) = \int\limits_0^\infty t^{s-1} \exp(-t) dt = \int\limits_0^\infty t^{s-1} \exp(i\varphi s) \exp(-t \exp(i\varphi)) dt$$

for $\mathrm{Re}\,s > 0$ and for an arbitrary $\varphi \in \mathbb{R}$ such that $\pi/2 > |\varphi|$. So the estimate holds for any ε_1, $0 < \varepsilon_1 \leqq \pi/2$ and for $\mathrm{Re}\,s > 0$:

$$|\Gamma(s)| \leqq (\sin \varepsilon_1)^{-\mathrm{Re}\,s} \Gamma(\mathrm{Re}\,s) \exp\left(-\left(\frac{\pi}{2} - \varepsilon_1 \right) |\mathrm{Im}\,s| \right) \ . \tag{3.28}$$

The kernel of $(T_{-s}(v))_{x_1,x_2}$ is continuous in $(x_1, x_2) \in \overline{M}_{j_1} \times \overline{M}_{j_2}$ for $\mathrm{Re}\,s > n/2$ (according to Theorem 3.1). The equality (3.27) holds also for $c = \mathrm{Re}\,s > n/2$. For such s the integral $\int_{\mathrm{Re}\,s=c} \Gamma(s) t^{-s} (T_{-s}(v))_{x_1,x_2} ds$ is absolutely convergent (by Proposition 3.5 below and by (3.28)). Hence it defines a continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ kernel. So the kernel $E_{t,x_1,x_2}^\bullet(v)$ of $\exp(-t\Delta_v^\bullet)$ is continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ because we have

$$E_{t,x_1,x_2}^\bullet(v) = \mathcal{H}^\bullet(v)_{x_1,x_2} + (2\pi i)^{-1} \int\limits_{\mathrm{Re}\,s=c} \Gamma(s) t^{-s} (T_{-s}(v))_{x_1,x_2} dt \ , \tag{3.29}$$

where $c > n/2$. (The integral in (3.29) converges uniformly with respect to x_1, x_2 for any fixed $c > n/2$ by Proposition 3.5.)

From the functional equation $\Gamma(s) = s^{-1}(s+1)^{-1} \cdots (s+l-1)^{-1} \Gamma(s+l)$ it follows that $|\Gamma(s)|$ for $\mathrm{Re}\,s > -l$ is also estimated by $\exp(-(\pi/2 - \varepsilon_1)|\mathrm{Im}\,s|)$ as $|\mathrm{Im}\,s| \to \infty$ (with any fixed ε_1, $0 < \varepsilon_1 \leqq \pi/2$). The operator $\exp(-t\Delta_v^\bullet)$ for $\mathrm{Re}\,t > 0$ is a trace class operator. Namely its kernel is continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ (as it follows from (3.29)). Hence it is a trace class operator and its trace is equal to the sum of the integrals over the diagonals $\overline{M}_j$ of the corresponding densities (according to Proposition 3.8 below).

The *theta-function* $\theta_{v,\bullet}(t)$ for $\Delta_v^\bullet$ is defined as the trace of $\exp(-t\Delta_v^\bullet)$ for $\mathrm{Re}\,t > 0$. The analogous theta-function $\theta_{v,\bullet}(t; p_j)$ is defined as the trace $\mathrm{Tr}(p_j \exp(-t\Delta_v^\bullet))$ for $\mathrm{Re}\,t > 0$ (where $p_j: (DR^\bullet(M))_2 \to (DH^\bullet(M_j))_2 \hookrightarrow (DR^\bullet(M))_2$ is the composition of the natural restriction and of the prolongation by zero). Proposition 3.8 claims that $\theta_{v,\bullet}(t; p_j)$ is equal to the integral over $\overline{M}_j$ of the density $\mathrm{tr}(*_{x_2} i_{M_j}^* E_{t,x_1,x_2}^\bullet(v))$.

The zeta-function $\zeta_{v,\bullet}(s)$ is defined by (2.8) for $\mathrm{Re}\,s > n/2$ ($n := \dim M$). It is equal to $\mathrm{Tr}\,T_{-s}(v)$ for $\mathrm{Re}\,s > n/2$ (accroding to Theorem 3.1 and to Proposition 3.9).

The zeta-function $\zeta(s; p_j) := \mathrm{Tr}(p_j T_{-s}(v))$ is equal for such s to the integral over the diagonal $i(\overline{M}_j) \hookrightarrow \overline{M}_j \times \overline{M}_j$ of the density, corresponding to the restriction of the kernel $T_{-s}(v)$ to $i(\overline{M}_j)$. The integral of this density over $\overline{M}_j$ can be represented as the sum of the integrals of densities on $\overline{M}_j$ and on ∂M_j (they are defined by the parametrix (3.3) and can be continued to meromorphic functions on the whole complex plane $\mathbb{C} \ni s$) and of a density on $\overline{M}_j$, which is holomorphic for $\mathrm{Re}\, s > (n - m)/2$. (This assertion follows from the proof of Theorem 3.1.) The contour of the integration in (3.27) can be moved to $\mathrm{Re}\, s = a$ for an arbitrary a such that $(-2a) \notin \mathbb{Z}_+ \cup 0$ (according to the estimates of $|\Gamma(s)|$ as $|\mathrm{Im}\, s| \to +\infty$ and to Proposition 3.4 below). Then it follows from (3.27) that

$$\theta_{v,\bullet}(t; p_j) = \sum t^{-(n-k)/2} \mathrm{res}_{s=s_k}(\Gamma(s)\zeta_{v,\bullet}(s; p_j))$$
$$+ (2\pi i)^{-1} \int\limits_{\mathrm{Re}\, s = a} t^{-s}\Gamma(s)\zeta_{v,\bullet}(s; p_j)ds + \mathrm{Tr}(p_j \mathcal{H}^\bullet(v)), \quad (3.30)$$

where the sum is over k such that $s_k := (n - k)/2 > a$. The estimate of the integral over $\mathrm{Re}\, s = a$ in (3.30) is obtained with the help of (3.28) and (3.40) as follows. For $\mathrm{Re}\, t > 0$, $|\arg t| < \pi/2 - \varepsilon$ ($\varepsilon, 0 < \varepsilon < \pi/2$, is fixed) and for $\mathrm{Re}\, s = a$ the estimate is satisfied:

$$\left| t^{-s}\Gamma(s)\zeta_{v,\bullet}(s; p_j) \right| < \left(\sin\frac{\varepsilon}{4} \right)^{-(a-[a])} |\Gamma(a)| \, |t|^{-a} \exp\left(-\frac{\varepsilon}{2}|\mathrm{Im}\, s| \right) C\left(a, \frac{\varepsilon}{4} \right)$$
$$\times \left\{ c_{\varepsilon/4}^{2(a-1)}\Gamma(2(1 - a), c_{\varepsilon/4}\rho^{1/2}) + \max(\rho^{-a}, 1)(1 + \sum|s_k - a|^{-1}) \right\}, \quad (3.31)$$

where the sum is over $k < n - 2a$. The constants $C(a, \varepsilon/4), c_{\varepsilon/4}$ in (3.31) and the function $\Gamma(u; x)$ are as in Proposition 3.4, (3.40). The latter estimate is a consequence of (3.28) and (3.40)), where ε_1 and ε are replaced by $\varepsilon/4$. We see that

$$\left| \int\limits_{\mathrm{Re}\, s = a} t^{-s}\Gamma(s)\zeta_{v,\bullet}(s; p_j)ds \right| < C_1(\varepsilon, a)|t|^{-a}, \quad (3.32)$$

where $\mathrm{Re}\, t > 0$, $|\arg t| < \pi/2 - \varepsilon$, $\pi/2 > \varepsilon > 0$, $a < 0$, and $(-2a) \notin \mathbb{Z}_+$. The assertions of Theorem 3.2 about the asymptotic expansion (3.1) for $\theta_{v,\bullet}(t; p_j)$ (relative to $t \to +0$ when $|\arg t| < \pi/2 - \varepsilon$) follow from the equality (3.30) and from the estimate (3.32). The estimates analogous to (3.40) below and to (3.32) are satisfied also by the analytic continuation to $\mathbb{C} \ni s$ of the densities (on $\overline{M}_j$ and on ∂M_j) defined by the parametrix $P_\lambda^m(v)$ (as in Proposition 3.5 below). Thus we see that the equalities between the densities in the integral representation for the coefficients of the expansion (3.1) and the corresponding densities for the residues and the values of $\zeta_{v,\bullet}(s; p_j)$ are satisfied.

The uniform with respect to $v \in \mathbb{R}^2 \setminus (0,0)$ estimate (3.2) for the traces of $\exp(-t\Delta_v^\bullet)$ (for a fixed t, $\mathrm{Re}\, t > 0$) follows from (3.30) and (3.31) because for $a = -m - 1/4$, $m \in \mathbb{Z}_+$, $m \gg 1$, the integral over $\mathrm{Re}\, s = a$ on the right in (3.30) is absolutely convergent. The estimate [66] (3.31) and the equality (3.30) provide us with the uniform in v upper estimate for $\mathrm{Tr}(p_j\exp(-t\Delta_v^\bullet)), v \in \mathbb{R}^2 \setminus (0,0)$. Indeed, the estimate [67] $\dim \mathrm{Ker}\, \Delta_v^\bullet < C$ is satisfied uniformly with respect to v. The formulas

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[66] For $a < 0$ the function $\Gamma(2(1 - a), c_{v/4}\rho^{1/2})$ tends to $\Gamma(2(1 - a))$ as $\rho \to +0$ and so it is bounded for $0 < \rho < 1$.

[67] It follows from the exact sequence (1.14) (where $Z_j := Z \cap \partial M_j$) and from Lemma 1.1 that $\dim \mathrm{Ker}\, \Delta_v^\bullet = \dim H^\bullet(M_v, Z) \leqq \sum_j \dim H^\bullet(M_j, N \cup Z_j) + \dim H^\bullet(N)$.

$q_{-n+k} = \text{res}_{s=s_k}\ (\Gamma(s)\zeta_{v,\bullet}(s;p_j)) + \delta_{n,k}\text{Tr}(p_j\mathscr{H}^\bullet(v))$ for the coefficients q_{-n+k} of the asymptotic expansion (3.1) are consequences of (3.30) for $a = -m - 1/4$, where $m \in \mathbb{Z}_+$. For $a = -m - 1/4$ the absolute value of the integral over $\text{Re}\,s = a$ in (3.30) is estimated (with the help of (3.31)) by $C|t|^{m+1/4}$ uniformly with respect to $v \in \mathbb{R}^2 \setminus (0,0)$ (where $\text{Re}\,t > 0$, $|\arg t| < \pi/2 - \varepsilon$, $\pi/2 > \varepsilon > 0$ and ε is arbitrary but fixed). So it holds

$$\theta_{v,\bullet}(t;p_j) = \sum_{k=0}^{n+2m-1} q_{-n+k}t^{-(n-k)/2} + \{q_{-n+2m}t^m + O(|t|^{m+1/4})\}\,. \qquad (3.33)$$

The latter two terms in (3.33) are $O(|t|^m)$ relative to $t \to +\infty$ uniformly with respect to $v \neq (0,0)$ (for $|\arg t| < \pi/2 - \varepsilon$). The statements about the structure of the values and the residues of $\zeta_{v,\bullet}(s;p_j)$ (Theorem 3.1) provide us with the desired information about coefficients q_{-n+k} in (3.1). These values and residues (up to $\delta_{n,k}\text{Tr}(p_j\mathscr{H}^\bullet(v))$) are the sums of the integrals over M_j, ∂M, and N of the densities which are defined by the absolutely convergent integrals of the components a_{-2-k} and d_{-2-k+1} ([Se2], Theorem 2, and the proof of Theorem 3.1 above). The latter symbols are defined by (3.4), (3.11), and (3.10). These integrals are smooth in $v \in \mathbb{R}^2 \setminus (0,0)$. Hence the coefficients q_{-n+k} in (3.1) are smooth in $v \neq (0,0)$ (and are invariant under $v \to cv$, $c \neq 0$). Theorem 3.2 is proved. $\square$

Remark 3.2. The coefficients q_{-n+k} of (3.1) for $0 \leq k \leq m$ are completely defined (according to (3.33) and to Remark 3.1) by an arbitrary parametrix $P_\lambda^m(v)$ (3.3) for $(\Delta_v^\bullet - \lambda)^{-1}$.

Proof of Proposition 3.2. The parametrix [68] $P_{t,x,y}^{\bullet(m)}(v)$ for $E_{t,x,y}^\bullet(v)$ (defined by (2.126)) is such that it is smooth in $(x,y) \in \overline{M}_{j_1} \times \overline{M}_{j_2}$ and in $v \in \mathbb{R}^2 \setminus (0,0)$. The v-transmission boundary conditions (1.27) are satisfied for $(\Delta_{v,x}^\bullet)^k P_{t,x,y}^{\bullet(m)}(v)$ (i.e., the image of $(DR^\bullet(M))_2$ under the action of the operator with the kernel $P_{t,x,y}^{\bullet(m)}(v)$ belongs to $D((\Delta_v^\bullet)^k)$ for an arbitrary $k \in \mathbb{Z}_+$). The uniform with respect to $v \in \mathbb{R}^2 \setminus (0,0)$ estimates (2.127), (2.128) are satisfied and for x from an appropriate neighborhood U of $N \subset M$ we have $(\partial_t + \Delta_{v,x}^\bullet)P_{t,x,y}^{\bullet(m)}(v) \equiv 0$ (where U is independent of v).

Set $r_{t,x,y}^{(m)}(v) := (\partial_t + \Delta_{v,x}^\bullet)\, P_{t,x,y}^\bullet(v)$. Then the estimates are satisfied for any $k \in \mathbb{Z}_+ \cup 0$,

$$|\Delta_{v,x}^k r_{t,x,y}^{(m)}(v)| < C_{m,k}t^{-n/2+m-k}\,, \qquad (3.34)$$

where $C_{m,k}$ is independent of $v \in \mathbb{R}^2 \setminus (0,0)$ and of $t \in (0,T)(n := \dim M)$.

The kernel $E_{t,x,y}^\bullet(v)$ can be represented as the Volterra series [69]

$$E_{t,x_1,x_2}^\bullet(v) = \sum_{k\geq 0}(-t)^k \int_{\Delta_k}\int_{(y_1,\dots,y_k)\in(\overline{M}_1\cup\overline{M}_2)^k} P_{\sigma_0 t,x_1,y_1}^\bullet(v)r_{\sigma_1 t,y_1,y_2}^{(m)}(v)\cdots r_{\sigma_k t,y_k,x_2}^{(m)}(v)\,,$$

$$(3.35)$$

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[68] The properties of such a parametrix are summarized in Proposition 2.21. For the sake of brevity the proof of Proposition 3.2 is given in the case of $\partial M = \emptyset$. The terms of $P_t^{(m)}(v)$, connected with the Dirichlet and the Neumann boundary conditions on the components of ∂M, are independent of v and the proof in the case of $\partial M \neq \emptyset$ does not contain any additional difficulties.

[69] This series was used in the case of a closed manifold M in [BGV], 2.4, 2.7. See also the formula (2.137) above.

where $\Delta_k = \{(\sigma_0,\ldots,\sigma_k) : 0 \leq \sigma_i \leq 1, \sum \sigma_i = 1\}$ (and the scalar product $\operatorname{tr}(\omega_1 \wedge *\omega_2)$ with the values in densities on M is assumed in (3.35)). The proof of (3.35) (or of (2.137)) is given in the proof of Proposition 2.21.

Let $\varphi : I \to \mathbb{R}^2 \setminus (0,0)$, $v = \varphi(\gamma)$, be a C^∞-map (where $\gamma \in [-a,a] =: I$). Then the only term in $P^{\bullet(m)}_{t,(x_1,x'),(y_1,y')}(v)$ depending on γ is

$$E_{N,t} \otimes \psi(x_1) E_{I,t}(v) \varphi(y_1) =: E^\bullet_{N,I}(v) , \qquad (3.36)$$

but it does not depend on m. (Here $E_{I,t}$ corresponds to Δ_I with the Dirichlet boundary conditions on ∂I, $\varphi, \psi \in C_0^\infty(I \setminus \partial I)$, $\psi \equiv 1$ in a neighbourhood of $\operatorname{supp} \varphi \subset I \setminus \partial I$, $\varphi(x_1) \equiv 1$ for $x_1 \in [-1/2, 1/2]$ and φ, ψ are even: $\varphi(-x_1) = \varphi(x_1), \psi(-x_1) = \psi(x_1)$.) So, as it follows from the explicit formulas (2.40) for $(G_I(v))_{x_1,y_1}$ (and from the analogous formulas (2.54) and (2.55) for $(E_{I,t}(v))_{x_1,y_1}$), the uniform with respect to γ estimates are satisfied for any $k, q \in \mathbb{Z}_+$

$$|\partial^k_\gamma r^{(m)}_{t,x,y}(v)| < c_{(q,k)} t^{-n/2+q} , \qquad (3.37)$$

because they are true for $(\partial_t + \Delta_{x'} + \Delta_{x_1})\{(E_{N,t})_{x',y'} \otimes \psi(x_1)(E_{I,t}(v))_{x_1,y_1} \varphi(y_1)\}$ and for $(\partial_t + \Delta_{x'} + \Delta_{x_1})\{(E_{N,t})_{x',y'} \otimes \psi(x_1)(\sigma_1^* E_{I,t}(v))_{x_1,y_1} \varphi(y_1)\}$ (where σ_1 is the reflection of $I = [-1,1]$ with respect to $0 \in I$ which acts on the variable x_1). The kernels $\partial^k_\gamma E^\bullet_{N,I}(v)$ are the linear combinations of these two kernels with the coefficients independent of x and y. (These coefficients are smooth in γ). The estimate (3.37) is satisfied for $t \in (0,T)$ and for an arbitrary $q \in \mathbb{Z}_+$ uniformly with respect to γ, because if $\partial_{x_1}\psi(x_1) \neq 0$ then $\rho(x_1, \operatorname{supp}\varphi) > \delta > 0$.

Let $DR^\bullet_M(l)$ be the space of forms on M of a class C^l (i.e., of forms with l continuous derivatives on M) equipped with a C^l-norm. [70] Let $DR^\bullet_{M,N}(l) := DR^\bullet_{(l)}(\overline{M}_1) \oplus DR^\bullet_{(l)}(\overline{M}_2)$ be the space of pairs (ω_1, ω_2) of forms ω_j of a class C^l on $\overline{M}_j$ with a C^l-norm. The operators with the kernels $P^{\bullet(m)}_{t,x,y}(v)$ for $v \in \mathbb{R}^2 \setminus (0,0)$ and the operators corresponding to $\partial^k_\gamma P^{\bullet(m)}_{t,x,y}(v(\gamma))$ (for a fixed $k \in \mathbb{Z}_+ \cup 0$) are families of uniformly (with respect to v and to $t \in (0,T)$) bounded operators acting from $DR^\bullet_M(l)$ into $DR^\bullet_{M,N}(l)$. For the operators corresponding to the interior terms in $P^{\bullet(m)}_{t,x,y}(v)$ this assertion is proved in [BGV], Theorem 2.29, Lemma 2.49. This proof uses that this statement is local in $x \in M$ (for a closed M) and it uses also the explicit definition of $P^{(m)}_{\mathrm{int}}$ over a geodesic ball $\exp_x B \subset M$ (where B is a ball $\|v\| \leq c$ in $T_x M$ and $\exp_x$ is the exponential map for (M, g_M) from $T_x M$).

The kernel (3.36) (i.e., the term of $P^{\bullet(m)}_{t,x,y}(v)$ corresponding to the interior boundary N) is equal (up to the factor $\psi(x_1)\varphi(y_1)$) to a linear combination given by (2.54) and (2.55) of the kernels $E^\bullet_t$ for $N \times I$ and $\sigma_1^* E^\bullet_t$ (σ_1 is the mirror symmetry with respect to $N \times 0$). [71] Its coefficients depend on (j_1, j_2, v), where $(x,y) \in \overline{M}_{j_1} \times \overline{M}_{j_2}$. These coefficients and their derivatives of a fixed order on γ are uniformly bounded.

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[70] This norm corresponds to a smooth partition of unity $\{\varphi_i\}$ subordinate to a finite cover $\{U_i\}$ of $\overline{M}_1$ and of $\overline{M}_2$ by coordinate charts (i.e., $\varphi_i \in C_0^\infty(U_i)$). For $v \in DR^\bullet_{M,N}(l)$ its C^l-norm $\|v\|_l$ is equal to $\sum \sup_{i \in U} \sup_{|\alpha| \leq l} |D_1^\alpha(\varphi_i v)|$ i.e., to the sum of the suprema of partial derivatives of orders $\leq l$. The C^l-norm for an arbitrary smooth finite cover $\{U'_i\}$ and for a partition of unity $\{\varphi'_i\}$ subordinate to $\{U'_i\}$ is equivalent to the one defined by $\{U_i\}$ and by $\{\varphi_i\}$.

[71] The operators $\partial^k_\gamma P^{(m)}(v) = \partial^k_\gamma E_{N,I}(v)$ for $k \in \mathbb{Z}_+$ are expressed similarly.

For a closed N the operators defined by $(P_N^{(m)})_t := (P_{\mathrm{int},N}^{(m)})_t$ are uniformly bounded for $0 < t \leqq T$ with respect to a C^l-norm in the space $DR_N^\bullet(l)$ of C^l-smooth forms on N. The equality (3.35) is satisfied by $E_{N,t}^\bullet, P_{N,t}^{\bullet(m)}$ and $r_{N,t}^{(m)}$. Since the estimate (3.38) below (as well as the analogous estimate (3.34)) is satisfied by $r_{N,t}^{(m)}$. (where n is replaced by $n-1$) we see that the series of operators on the right in (3.35) is convergent for $m \geqq (n+l-1)/2$ with respect to a C^l-norm in $DR_N^\bullet(l)$. Hence the sum of this series defines a family of the operators $E_{N,t}^\bullet$ in $DR_N^\bullet(l)$ bounded uniformly with respect to $t \in (0, T]$. The analogous assertion is also true for a family of operators defined by the kernels $\psi(x_1)(E_{I,t}^\bullet)_{x_1,y_1}\varphi(y_1)$ acting on smooth forms with compact supports on $(I, \partial I)$ (with respect to a C^l-norm). So the kernels $E_{N,I}^\bullet(v)_{t,(x_1,x'),(y_1,y')}$ and $\partial_\gamma^q E_{N,I}^\bullet(v(\gamma))_{t,(x_1,x'),(y_1,y')}$ (for a fixed $q \in \mathbb{Z}_+$) define collections of uniformly (in $t \in (0,T]$ and v or in t and γ) bounded with respect to a C^l-norm operators from $DR_M^\bullet(l)$ into $DR_{M,N}^\bullet(l)$.

The kernel $r_{t,x,y}^{(m)}(v)$ is smooth on $M \times \overline{M}_j$ according to the definition (2.126) of $P_{t,x,y}^{\bullet(m)}(v)$. The C^l-norm of $r_{t,x,y}^{(m)}(v)$ on each $M \times \overline{M}_j$ satisfies the estimate (analogous to (3.34))

$$\|r_{t,x,y}^{(m)}(v)\|_l \leqq C_l t^{-n/2+m-1/2} \tag{3.38}$$

uniformly with respect to $v \in \mathbb{R}^2 \setminus (0,0)$ and to $t \in (0, T]$. The estimates analogous to (3.37) an to (2.129) are satisfied uniformly with respect to v and to $t \in (0, T]$ also by the C^l-norms of $\partial_\gamma^k r_{t,x,y}^{(m)}(v)$ on each $M \times \overline{M}_j$ for any $k, q \in \mathbb{Z}_+$:

$$\|\partial_\gamma^k r_{t,x,y}^{(m)}(v)\|_l \leqq c_{(q,k,l)} t^{-n/2+q} . \tag{3.39}$$

Leibnitz's rule claims that $P_{t,x,y}^{(m)}(v(\gamma))$ is a bounded operator from the space of C^p-maps $\omega : [-a,a] \to DR_M^\bullet(l)$ (equipped with the norm $\sum_{i=0}^p \sup_{\gamma \in [-a,a]} \|\partial_\gamma^i \omega\|_l$) into $C^p([-a,a], DR_{M,N}^\bullet(l))$. Indeed, the kernel $\partial_\gamma^k P_{t,x,y}^{(m)}(v(\gamma))$ depends smoothly on $\gamma \in [-a,a]$ on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ for $k \in \mathbb{Z}_+ \cup 0$, since $P_{t,x,y}^{(m)}(v) := (P_{\mathrm{int}}^{(m)})_{t,x,y} + E_{N,I}^\bullet(v)$ and since $E_{N,I}^\bullet(v)$ is smooth in $v \in \mathbb{R}^2 \setminus (0,0)$. For $0 < t \leqq T$ the operators $(P_t^{(m)}(v(\gamma)))$ from $C^p([-a,a], DR_M^\bullet(l))$ into $C^p([-a,a], DR_{M,N}^\bullet(l))$ are uniformly bounded because $\partial_\gamma^k P_t^{(m)}$ (for a fixed $k \in \mathbb{Z}_+ \cup 0$) are the operators from $DR_M^\bullet(l)$ into $DR_{M,N}^\bullet(l)$ bounded uniformly in γ and t, $0 < t \leqq T$, with respect to a C^l-norm. Hence, according to (3.38), (3.39), and to the fact that the volume of Δ_k is equal to $(k!)^{-1}$, the series (3.35) for the derivative $\partial_\gamma^p E_{t,x_1,x_2}(v(\gamma))$ is convergent in the C^l-norm on $\cup (\overline{M}_{J_1} \times \overline{M}_{J_2})$ for $m > (n+l)/2$. (The number m in the definition $P^{(m)}$ is greater than $(n+l)/2$.)

This proves that $\partial_\gamma^k E_{t,x_1,x_2}^\bullet(v(\gamma))$ is C^∞-smooth on $\overline{M}_{J_1} \times \overline{M}_{J_2}$. (For instance, for $k = 0$ this proves that $E_{t,x_1,x_2}^\bullet(v)$ is C^∞-smooth on $\overline{M}_{J_1} \times \overline{M}_{J_2}$.)

So the restrictions $i_j^* E_{t,x,x}^\bullet(v(\gamma))$ to the diagonals $i_j : \overline{M}_j \hookrightarrow \overline{M}_j \times \overline{M}_j$ are C^∞-smooth double forms on $\overline{M}_j$ which are C^∞-smooth in γ. Since $r_m(t,v)$ in (3.1) are $O(t^{(m+1)/2})$ uniformly with respect to $v \in \mathbb{R}^2 \setminus (0,0)$ and since q_i are C^∞-smooth in v we see that the asymptotic series (3.1) can be differentiated on γ. Actually, the equality (3.35) holds for $E_t^\bullet(v), P_t^{\bullet(m)}(v)$, and $r_t^{(m)}(v)$. The kernel $r_t^{(m)}(v)$ satisfies the estimates (3.34), (3.37), and (3.39) and the kernel $P_t^{\bullet(m)}(v(\gamma))$ defines a family of uniformly with respect to $t \in (0, T]$ and to γ bounded operators from $C^p([-a,a], DR_M^\bullet(l))$ into $C^p([-a,a], DR_{M,N}^\bullet(l))$. Hence the power terms

$t^{(-n+j)/2}$, $0 \leq j \leq 2m$, in the asymptotic expansion of $\int_{M_j} \mathrm{tr}\, i_j^* \partial_\gamma E_t^\bullet(v)$ as $t \to +0$ are equal to the appropriate terms in the asymptotic expansion of $\int_{M_j} i_j^* \partial_\gamma P_t^{\bullet(m)}(v)$. (The kernel $\partial_\gamma(E_t^\bullet(v) - P_t^{\bullet(m)}(v))_{x_1,x_2}$ is $O(t^{-n/2+m+1})$, according to (3.35).) But the coefficients q_i, $0 \leq i \leq 2m$, in (3.1) are completely defined by $i_j^* P_t^{\bullet(m)}(v)$, because the kernel $(E_t^\bullet(v) - P_t^{\bullet(m)}(v))_{x_1,x_2}$ is $O(t^{-n/2+m+1})$ uniformly with respect to $(x_1, x_2) \in \overline{M}_{j_1} \times \overline{M}_{j_2}$ and to $t \in (0, T]$, according to (3.35). Thus Proposition 3.2 is proved. $\square$

3.4. Estimates for Zeta-Functions and for the Corresponding Kernels in Vertical Strips in the Complex Plane.

Proposition 3.4. *The meromorphic continuation of the zeta-function $\zeta_{v,\bullet}(s; p_j) := \mathrm{Tr}(p_j T_{-s}(v))$ for $\mathrm{Re}\, s > n/2$ is estimated by $C(\varepsilon)\exp(\varepsilon|\mathrm{Im}\, s|)$ as $|\mathrm{Im}\, s| \to +\infty$ for any fixed $\varepsilon > 0$. Namely for any $\varepsilon > 0$ and for an arbitrary $a \in \mathbb{R}$ the following estimate is satisfied if $\mathrm{Re}\, s \geq a$:*

$$
\begin{aligned}
|\zeta_{v,\bullet}(s; p_j)| \leq\, & C(a,\varepsilon)\exp(\varepsilon|\mathrm{Im}\, s|) \\
& \times (c_\varepsilon^{2(\mathrm{Re}\, s - 1)}\Gamma(2(1 - \mathrm{Re}\, s), c_\varepsilon \rho^{1/2}) \\
& + \max(\rho^{-\mathrm{Re}\, s}, 1)(1 + \textstyle\sum|s - s_j|^{-1})),
\end{aligned}
\tag{3.40}
$$

where $\rho > 0$ is such that $\mathrm{Spec}(\Delta_v^\bullet) \cap (0, \rho] = \emptyset$ and the sum is over $s_j := (n - j)/2$, $-s_j \notin \mathbb{Z}_+ \cup 0$, $s_j \geq a$. The constants $C(a, \varepsilon)$ and c_ε are positive and independent of $v \in \mathbb{R}^2 \setminus (0, 0)$, and $\Gamma(u, x) := \int_x^\infty t^{u-1}\exp(-t)dt$ for $x > 0$.

Proposition 3.5. *For $\mathrm{Re}\, s > n/2$ ($n := \dim M$) and for any $\varepsilon > 0$ the following estimate is satisfied (where $\rho > 0$ is such that $\mathrm{Spec}(\Delta_v^\bullet) \cap (0, \rho] = \emptyset$):*

$$
|(T_{-s}(v))_{x_1,x_2}| \leq C_\varepsilon \rho^{-\mathrm{Re}\, s}\exp(\varepsilon|\mathrm{Im}\, s|)\left(\left(\mathrm{Re}\, s - \frac{n}{2}\right)^{-1} + 1\right).
\tag{3.41}
$$

Proof of Proposition 3.4. It is proved in Theorem 3.1 that the operator norm $\|G_\lambda^\bullet(v)\|_2$ in $DR^\bullet(M))_2$ of the Green function $G_\lambda^\bullet(v)$ for the Laplacian $\Delta_v^\bullet$ is estimated by $C_\varepsilon|\lambda|^{-1}$ for $\lambda \in \Lambda_\varepsilon := \{\lambda \in \mathbb{C}, \varepsilon \leq \arg\lambda \leq 2\pi - \varepsilon\}$, where ε, $0 < \varepsilon \leq \pi$ is fixed. The spectrum $\mathrm{Spec}(\Delta_v^\bullet)$ is a discrete subset of $\mathbb{R}_+ \cup 0$ by Theorem 3.1. So the operator $T_{-s}(v)$ defined by the integral (3.18) is equal to the same integral with the contour Γ replaced by $\Gamma_{(\varepsilon)} := \Gamma_{1,\varepsilon} \cup \Gamma_\rho^\varepsilon \cup \Gamma_{2,\varepsilon}$,

$$
\Gamma_{1,\varepsilon} = \{\lambda = x\exp(i\varepsilon), \infty > x \geq \rho\}, \quad \Gamma_\rho^\varepsilon = \{\lambda = \rho\exp(i\varphi), \varepsilon > \varphi > -\varepsilon\},
$$
$$
\Gamma_{2,\varepsilon} = \{\lambda = x\exp(-i\varepsilon), \rho \leq x < \infty\}.
\tag{3.42}
$$

There is a constant $c > 0$ such that the principal symbol $(b_2(x, \xi) - \lambda)\mathrm{id}$ of $\Delta^\bullet - \lambda\mathrm{id}$ on M is invertible for

$$
|\xi|^2 > c|\lambda|
\tag{3.43}
$$

in the coordinate charts U_l (of the same finite cover $\{U_l\}$ of M as in (3.3)). The integral (3.18) over the contour $\Gamma_{(\varepsilon)}$ (the latter one is defined by (3.42)) does not

depend on ρ for all $\rho > 0$ such that $(0, \rho] \cap \mathrm{Spec}(\Delta_\nu^\bullet) = \emptyset$. We suppose from now on that

$$0 < \rho < (2c + 1)^{-1} \tag{3.44}$$

and that $(0, \rho] \cap \mathrm{Spec}(\Delta_\nu^\bullet) = \emptyset$.

The kernel $(r_\lambda^m)_{x_1, x_2}$ of the operator $r_\lambda^m := G_\lambda^\bullet(\nu) - P_\lambda^m$ for $m \geqq n$ is continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2})$ for $\lambda \in \Lambda_\varepsilon \setminus 0$. (The parametrix P_λ^m is defined by (3.3).) It is estimated for $|\lambda| \geqq \rho, \lambda \in \Gamma_\varepsilon$ (according to (3.19)) uniformly with respect to $\nu \in \mathbb{R}^2 \setminus (0, 0)$ by

$$|(r_\lambda^m)_{x_1, x_2}| < C_{\varepsilon, \varepsilon_1}(1 + |\lambda|^{1/2})^{-(2+m)+n+\varepsilon_1} \tag{3.45}$$

for any $\varepsilon_1 > 0$. Since (3.45) is satisfied for all $\lambda \in \Gamma_{(\varepsilon)}$, we have for $\mathrm{Re}\, s > 2^{-1}(-m + n + \varepsilon_1)$

$$\left| \frac{i}{2\pi} \int_{\Gamma_i} \lambda^{-s}(r_\lambda^m)_{x_1, x_2} d\lambda \right| = \left| \frac{i}{2\pi} \left(\sum_j \int_{\Gamma_{j, \varepsilon}} + \int_{\Gamma_\rho^\varepsilon} \right) \lambda^{-s}(r_\lambda^m)_{x, x} d\lambda \right|$$

$$\leqq 2C_{\varepsilon, \varepsilon_1} \exp(\varepsilon|\mathrm{Im}\, s|) \left\{ \int_\rho^\infty |\lambda|^{-\mathrm{Re}\, s}(1 + |\lambda|^{1/2})^{-(2+m)+n+\varepsilon_1} d|\lambda| \right.$$

$$\left. + \pi\rho \max(\rho^{-\mathrm{Re}\, s}, 1) \right\}. \tag{3.46}$$

The estimate (3.46) claims that for the proof of (3.40) in the domains $\mathrm{Re}\, s \geqq a$ it is enough to prove the analogous estimate for the analytical continuation of the densities on $\overline{M}_j, N$, and on ∂M, corresponding (for $\mathrm{Re}\, s > n/2$) to the kernel of

$$\frac{i}{2\pi} \int_{\Gamma_{(i)}} \lambda^{-s} P_\lambda^m d\lambda, \tag{3.47}$$

where $m = m(a) \in \mathbb{Z}_+$ is sufficiently large. (These densities were introduced in the proof of Theorem 3.1, and the sum of their integrals is equal to the trace of (3.47).)

Let $\mathrm{Re}\, s > n/2$ and $p_{\mathrm{int}}^{s, \varepsilon}(x)$ be the density on M_j, corresponding to the restriction to the diagonal $i_j : M_j \hookrightarrow M_j \times M_j$ of the kernel $P_{\mathrm{int}}^{s, \varepsilon}(x_1, x_2)$ of the operator $\int_{\Gamma_{(i)}} \lambda^{-s}(p_j \sum_i \psi_i P_{\lambda, \mathrm{int}, U_i}^m \varphi_i) d\lambda$ (where $P_{\lambda, \mathrm{int}}^m$ is defined by (3.13) and by (3.4)). Then $p_{\mathrm{int}}^{s, \varepsilon}(x)$ can be continued to a whole complex plane $\mathbb{C} \ni s$ as a meromorphic density ([Se2], Lemma 1, or [Sh], Theorem 12.1).

Proposition 3.6. *The density $p_{\mathrm{int}}^{s, \varepsilon}(x)$ satisfies the following estimate for any $\varepsilon > 0$:*

$$|p_{\mathrm{int}}^{s, \varepsilon}(x)| < C_\varepsilon \max(\rho^{-\mathrm{Re}\, s}, 1) \exp(\varepsilon|\mathrm{Im}\, s|) \sum |s - s_k|^{-1}, \tag{3.48}$$

where the sum is over $0 \leqq k \leqq m$ such that $(-s_k) \notin \mathbb{Z}_+ \cup 0$.

Proof. The density $p_{\mathrm{int}}^{s, \varepsilon}(x)$ for $\mathrm{Re}\, s > n/2$ corresponds to the sum of the integrals

$$(2\pi)^{-n} \sum_j \varphi_j(x) \int d\xi \frac{i}{2\pi} \int_{\Gamma_{(i)}} \lambda^{-s} \theta(\xi, \lambda) \sum_{j=0}^m a_{-2-j}(x, \xi, \lambda) d\lambda, \tag{3.49}$$

where a_{-k} is a positive homogeneous of degree $(-k)$ in $(\xi, \lambda^{1/2})$ component of the symbol $s((\Delta^\bullet - \lambda)^{-1})$ in the coordinate chart U_i defined by (3.4). The integral

(3.49) is the sum of the integrals $J_{0,\varepsilon}^s(x) + J_{1,\varepsilon}^s(x) + J_{2,\varepsilon}^s(x)$ over the three corresponding domains:

$$K_0 := \{(\xi,\lambda) : |\xi|^2 \leq 1 - \rho, \lambda \in \Gamma_{(\varepsilon)}, |\lambda| \leq (1 - |\xi|^2)\},$$

$$I_1 := \{(\xi,\lambda) : |\xi|^2 \leq 1 - \rho, \lambda \in \Gamma_{(\varepsilon)}, |\lambda| > (1 - |\xi|^2)\},$$

$$I_2 := \{(\xi,\lambda) : |\xi|^2 > 1 - \rho, \lambda \in \Gamma_{(\varepsilon)}\}. \tag{3.50}$$

Since K_0 is compact and since $\lambda^{-s}\theta\sum_0^m a_{-2-j}(x,\xi,\lambda)$ is continuous on $M_j \times K_0$, the density $J_{0,\varepsilon}^s(x)$ is holomorphic in $s \in \mathbb{C}$ and it is estimated by

$$|J_{0,\varepsilon}^s(x)| \leq C_\varepsilon \exp(\varepsilon|\mathrm{Im}s|)\max(\rho^{-\mathrm{Re}s}, 1)\max_{M_j \times K_0}\sum_i\Big|\varphi_i(x)\theta\sum_0^m a_{-2-j}(x,\xi,\lambda)\Big|. \tag{3.51}$$

The latter factor on the right in (3.51) does not depend on s and ρ. Hence it is estimated by a constant.

Set $J_k^s(x) := J_{k,\varepsilon}^s(x)$ from now on. For $\mathrm{Re}\,s > n/2$ the density $J_2^s(x)$ does not change if the interior integral in (3.49) is replaced by the integral over

$$\Gamma_{(\varepsilon),|\xi|} := |\xi|^2(1 - \rho)^{-1}\Gamma_{(\varepsilon)}, \tag{3.52}$$

because $\theta\sum_0^m a_{-2-j}$ is holomorphic in λ in the domain between the contours $\Gamma_{(\varepsilon)}$ and $\Gamma_{(\varepsilon),|\xi|}$. Indeed, this symbol is holomorphic in λ for (ξ,λ) such that $|\xi|^2 + |\lambda| > 1$ and $|\xi|^2 > c|\lambda|$ (where $c > 0$ is the same as in (3.43) and (3.44)). Since $0 < \rho < (2c + 1)^{-1}$ we have for λ between $\Gamma_{(\varepsilon)}$ and $\Gamma_{(\varepsilon),|\xi|}$,

$$\rho \leq |\lambda| \leq |\xi|^2\rho(1 - \rho)^{-1}, \quad c|\lambda| \leq c|\xi|^2\rho(1 - \rho)^{-1} < 2^{-1}|\xi|^2 < |\xi|^2,$$

$$\text{and} \quad |\xi|^2 + |\lambda| > 1 \quad \text{for } (\xi,\lambda) \in I_2.$$

The density $J_2^s(x)$ is represented as the sum $J_{2,\rho}^s(x) + \sum_{j=1,2}J_{2,j}^s(x)$, where J_{2j}^s and $J_{2,\rho}^s$ correspond to (3.49), with the interior integral replaced by the integral over $\Gamma_{j,\varepsilon}^\varepsilon := |\xi|^2(1 - \rho)^{-1}\Gamma_{j,\varepsilon}$ and over $\Gamma_{\rho,|\xi|}^\varepsilon := |\xi|^2(1 - \rho)^{-1}\Gamma_\rho^\varepsilon$. The density $J_1^s(x)$ is equal to the sum $\sum_{j=1,2}J_{1,j}^s(x)$, where the interior integral in (3.49) for the term $J_{1,j}^s$ is over the contour $\Gamma_{j,\varepsilon} \setminus D_{1-|\xi|^2}$ ($D_r := \{\lambda, |\lambda| < r\}$).

Set $\lambda := \exp(i\varepsilon(-1)^{j+1})t^2$ on $\Gamma_{j,|\xi|}^\varepsilon$ (for $|\xi|^2 > 1 - \rho$) and on $\Gamma_{j,\varepsilon} \setminus D_{1-|\xi|^2}$ (for $|\xi|^2 \leq 1 - \rho$), where $t > 0$ is a new variable. Then we have

$$(J_{1,j}^s + J_{2,j}^s)(x) = 2(2\pi)^{-(n+1)}i\exp((-1)^j i\varepsilon(s - 1))$$

$$\times \sum_l \varphi_l(x)\int_F t^{-2s+1}\mathrm{tr}(\sum_0^m a_{-2-k}(x,\xi,\lambda_\varepsilon(t))dt\,d\xi, \tag{3.53}$$

where $\lambda_\varepsilon(t) := \exp((-1)^{j+1}i\varepsilon)t^2$, $\mathrm{Re}\,s > n/2$, and F is the domain

$$\{(\xi,t) : |\xi|^2 + t^2 \geq 1, |\xi|^2 \leq 1 - \rho, t \geq \rho^{1/2}\} \cup \{(\xi,t) : |\xi|^2 \geq 1 - \rho,$$

$$t \geq |\xi|\rho^{1/2}/(1 - \rho)^{1/2}\}.$$

Since $a_{-k}(x, \xi, \lambda)$ are nonsingular for $\lambda \in \Gamma_{j,\varepsilon}$ and since $r \cdot (\xi, t) \in F$ for $(\xi, t) \in F$ and $r \geq 1$, we see that (3.53) can be written as follows:

$$(J_{1,j}^s + J_{2,j}^s)(x) = 2(2\pi)^{-(n+1)} i \exp((-1)^j i \varepsilon (s - 1))$$

$$\times \sum_l \sum_{k=0}^m (2s + k - n)^{-1} \varphi_l(x) \int_{F_1} t^{-2s+1} \mathrm{tr} a_{-2-k}(x, \xi, \lambda_\varepsilon(t)) dt \omega_{n+1} ,$$

$$(3.54)$$

where $F_1 = F \cap \{(\xi, t) : \xi^2 + t^2 = 1\}$ and ω_{n+1} is the volume form on the unit sphere in $\mathbb{R}_{\xi,t}^{n+1}$. The integral over the compact F_1 in (3.54) is an entire function of $s \in \mathbb{C}$. So the right side of (3.54) realizes the analytic continuation of the density $(J_{1,j}^s + J_{2,j}^s)(x)$ to a meromorphic in $s \in \mathbb{C}$ density with no more than simple poles at the points $s_k = (n - k)/2$, $0 \leq k \leq m$.

Since $\lambda = \rho(1 - \rho)^{-1} |\xi|^2 \exp(i\varphi)$ on $\Gamma_{\rho,|\xi|}^\varepsilon$ (where $\varepsilon \geq \varphi \geq -\varepsilon$), we have

$$J_\rho^s(x) = -i(2\pi)^{-(n+1)} \sum_l \sum_{k=0}^m \varphi_l(x) \int I_{l,k}^{\varepsilon,\rho}(s, \xi) d\xi , \qquad (3.55)$$

where the integral is over $\{\xi : |\xi|^2 \geq 1 - \rho\}$ and

$$I_{l,k}^{\varepsilon,\rho}(s, \xi) := \int_{-\varepsilon}^{\varepsilon} \exp(-i\varphi(s - 1)) a_{-2-k}(x, \xi, \lambda(\varphi, |\xi|)) d\varphi |\xi|^{-2(s-1)} \rho_+^{-(s-1)} ,$$

$$\lambda(\varphi, |\xi|) := |\xi|^2 \rho_+ \exp(i\varphi), \qquad \rho_+ := \rho(1 - \rho)^{-1} . \qquad (3.56)$$

The symbol (3.56) is positive homogeneous of degree $(-2s - k)$ in ξ. It is analytic in $s \in \mathbb{C}$ and nonsingular. So (3.55) realizes a meromorphic continuation of $J_\rho^s(x)$ to the whole complex plane $\mathbb{C} \ni x$. Namely

$$J_\rho^s(x) = 2^{-1}(2\pi)^{-(n+1)} \sum_l \sum_{k=0}^m \varphi_l(x) \left(\int_{|\xi|=1} I_{l,k}^{\varepsilon,\rho}(s, \xi)_x \omega_n \right) (s - s_k)^{-1} (1 - \rho)^{-(s-s_k)} ,$$

$$(3.57)$$

where ω_n is the volume form on the unit sphere in $\mathbb{R}^n$. The formulas (3.54) and (3.57) provide us with a meromorphic continuation of the density $p_{\text{int}}^{s,\varepsilon}(x)$. Together with the estimate (3.51) they provide us with the estimate (3.48). (However, with the sum in it over all s_k, $0 \leq k \leq m$.) The analytic continuation of the density defined by the sum of the integrals (3.49) with the interior integral over the contour $\Gamma_{(\pi)}$ (i.e., with $\varepsilon = \pi$) is regular in $s = s_k$ for $(-s_k) \in \mathbb{Z}_+ \cup 0$. (This assertion is obtained in the proof of Theorem 3.1.) For $|\xi|^2 > 1 - \rho$ the interior integral over $\Gamma_{(\varepsilon)}$ in (3.49) is equal to the integral over $\Gamma_{(\pi)}$. So the estimate (3.48) is satisfied, where the sum is over s_k, $0 \leq k \leq m$, such that $(-s_k) \notin \mathbb{Z}_+ \cup 0$. $\square$

Let $\mathrm{Re}\, s > n/2$ and let $p_{j,\partial}^{s,\varepsilon}(x)$ be the density on M_j corresponding to the term in (3.47) determined by the ν-transmission interior boundary conditions. [72] It is

---

[72] From now on we'll suppose that $\partial M = \emptyset$. Estimates of the contributions into $\zeta_{\nu,\bullet}(s; p_j)$ from the Dirichlet and the Neumann boundary conditions on the components of $\partial M_j \setminus N$ are analogous to the estimates for the contributions from the ν-transmission interior boundary conditions.

defined by the restrictions to the diagonal $\overline{M}_j \hookrightarrow \overline{M}_j \times \overline{M}_j$ of the kernel for the operator

$$\frac{i}{2\pi} \int_{\Gamma_{(\varepsilon)}} \lambda^{-s} d\lambda \, p_j \sum_l \psi_l \mathscr{D}_{m,U_l} \varphi_l \, , \tag{3.58}$$

where $\mathscr{D}_m = \mathscr{D}_m(\lambda, \nu)$ is defined by (3.9), (3.10), and (3.12).

The operator $\mathscr{D}_m$ is defined by the symbols $\sum d_{-2-k}$, $0 \leqq k \leqq m$, in a co-ordinate chart (x', t), $U_i \subset \mathbb{R}^{n-1} \times \mathbb{R}^1$ (where $N \cap U_i = (\mathbb{R}^{n-1} \times 0) \cap U_i$ and the structure $\mathbb{R}^{n-1} \times \mathbb{R}^1$ corresponds to the direct product structure of the metric g_M near N). Its action on f, $f \in DR_c^\bullet(\mathbb{R}^{n-1} \times (\mathbb{R}_+ \setminus 0))$, can be represented for $t > 0$, $t_1 > 0$, as follows ([Se1], (26)–(28)):

$$\sum_{k=0}^m \mathrm{Op}(\theta_1 d_{-2-k}) f(x', t) := \sum_{k=0}^m (2\pi)^{-n} \int \int \exp(i(\xi', x')) \theta_1 d_{-2-k}^- f^-(\xi', t_1) dt_1 d\xi' \, ,$$

$$d_{-2-k}^-(x', t, \xi', t_1, \lambda) := -\int_{\Gamma_-} \exp(-i\tau t_1) d_{-2-k}(x', t, \xi', \tau, \lambda) \, ,$$

$$f^-(\xi', t_1) = \int \exp(-i(\xi', y)) f(y, t_1) dy \, ,$$

where $\Gamma_- = \Gamma_-(\xi', \lambda)$ is a simple contour in the half-plane $\mathrm{Im}\,\tau < 0$ which once goes round (in the direction opposite to the clockwise) the only zero of the principal symbol $(b_2(x', \xi', \tau) - \lambda)\mathrm{id}$ of the Laplacian $(\Delta^\bullet - \lambda\mathrm{id})$. [73] Lemmas 2 and 3 of [Se2] and Lemma 2 of [Se1] claim that the integral over $\mathbb{R}_+$ (where $U_i \cap (\mathbb{R}^{n-1} \times \mathbb{R}_+) = U_i \cap M_j$) $\int d_{-2-k}^-(x', t, \xi', t, \lambda) dt$ is a symbol of (x', ξ', λ) positive homogeneous of degree $(-2 - k)$ in $(\xi', \lambda^{1/2})$. They claim also that the kernel on $\mathbb{R}^{n-1}$ defined by the integral over $[T, \infty) \subset \mathbb{R}_+$ (for an arbitrary $T > 0$)

$$\int_\Gamma \exp(i(\xi', x' - y')) d\xi' \int_\Gamma d\lambda \int_T^\infty dt \, \theta_1(\xi', \lambda) \lambda^{-s} d_{-2-k}(x', t, \xi', t, \lambda)$$

is an entire function of $s \in \mathbb{C}$ smooth in x', y', s and vanishing at s for $(-s) \in \mathbb{Z}_+ \cup 0$. The latter assertion is an immediate consequence of the estimate ([Se1],(29)) for d_{-2-k}^-:

$$|D_{x'}^\gamma D_{\xi'}^q t^r t_1^l D_t^m D_{t_1}^n D_\lambda^p d_{-2-k}^-(x', t, \xi', t_1, \lambda)|$$

$$\leqq C_1 \exp(-c_\varepsilon(|t| + |t_1|)(|\xi| + |\lambda|^{1/2}))(1 + |\xi| + |\lambda|^{1/2})^{-1-k(|q|+2p)-(r+l)+m+n} \tag{3.59}$$

with positive constants $C_1(\varepsilon)$ and c_ε independent of $\nu \in \mathbb{R}^2 \setminus (0,0)$.

Proposition 3.7. *The analytic continuation to the whole complex plane $\mathbb{C} \ni s$ of the integral over M_j of $p_{j,\partial}^{s,\varepsilon}(x)$ (which is defined for $\mathrm{Re}\,s > n/2$) is estimated by*

$$\left| \int_{M_j} p_{j,\partial}^{s,\varepsilon}(x) \right| < c_{1,\varepsilon} \max(\rho^{-\mathrm{Re}\,s}, 1) \exp(\varepsilon|\mathrm{Im}\,s|) \sum |s - s_k|^{-1}$$

$$+ c_{2,\varepsilon} \exp(\varepsilon|\mathrm{Im}\,s|)(2c_\varepsilon^{2(\mathrm{Re}\,s-1)} \Gamma(2(1 - \mathrm{Re}\,s), c_\varepsilon \rho^{1/2})$$

$$+ \pi\rho \max(\rho^{-\mathrm{Re}\,s}, 1)) \, , \tag{3.60}$$

where the sum is over $1 \leqq k \leqq m + 1$ such that $(-s_k) \notin \mathbb{Z}_+ \cup 0$.

---

[73] The whole symbol of $\Delta^\bullet$ does not depend on t in the neighborhood of N.

Proof. The trace of the operator (3.58) for $\mathrm{Re}\,s > n/2$ is given by the integral

$$(2\pi)^{-n}\sum\int dx'dt\varphi_l(x',t)\int d\xi'\left(\frac{i}{2\pi}\int_{\Gamma_L}\lambda^{-s}d\lambda\theta_1(\xi',\lambda)\sum_{k=0}^{m}d_{-2-k}^{-}(x',t,\xi',t,\lambda)\right).$$

(We suppose that $\varphi_l(x',t)$ are independent of t for $0 \leq |t| \leq 1$). It follows from the estimate (3.59) that the density on N corresponding to the integral

$$\int_{\mathbb{R}_+} dt(\varphi_l(x',t) - \varphi_l(x',0))\int d\xi'\mathrm{tr}\left(\frac{i}{2\pi}\int_{\Gamma_{(\varepsilon)}}\lambda^{-s}d\lambda\,\theta_1\sum d_{-2-k}^{-}\right)$$

has the analytic continuation which is an entire function of $s \in \mathbb{C}$ and which satisfies the estimate

$$C_{\varepsilon,m,n_1}\exp(\varepsilon|\mathrm{Im}\,s|)h_\rho(c_\varepsilon,\mathrm{Re}\,s)\int(1 + |\xi|)^{-n_1}\exp(-c_\varepsilon|\xi'|)d\xi', \qquad (3.61)$$

$$h_\rho(c_\varepsilon,\mathrm{Re}\,s) := \int_\rho^\infty\exp(-c_\varepsilon t^{1/2})t^{-\mathrm{Re}\,s}dt + \pi\rho\max(\rho^{-\mathrm{Re}\,s},1)$$

$$= 2c_\varepsilon^{2(\mathrm{Re}\,s-1)}\Gamma(2(1 - \mathrm{Re}\,s),c_\varepsilon\rho^{1/2}) + \pi\rho\max(\rho^{-\mathrm{Re}\,s},1),$$

where $n_1 \in \mathbb{Z}_+$ is sufficiently large.

The density on N is defined by the integral

$$p_{l,j,\partial}^{s,\varepsilon}(x') := (2\pi)^{-n}\varphi(x',0)\int_{\mathbb{R}_+} dt\int d\xi'\mathrm{tr}\left(\frac{i}{2\pi}\int_{\Gamma_{(\varepsilon)}}\lambda^{-s}d\lambda\theta_1\sum_{k=0}^{m}d_{-2-k}^{-}\right), \qquad (3.62)$$

which is absolutely convergent for $\mathrm{Re}\,s > (n - 1)/2$. Hence it is analytic in s for such s. The integral over $t \in \mathbb{R}_+$ of d_{-2-k}^{-} is a positive homogeneous of degree $(-2 - k)$ in $(\xi',\lambda^{1/2})$ symbol, which is smooth in (x',ξ',λ) and analytic in λ for $c|\lambda| < |\xi'|^2$ (where c is the same as in (3.43) and (3.44)) and in $\lambda \in \Lambda_{\varepsilon/2} := \{\lambda : \varepsilon/2 < \arg\lambda < 2\pi - \varepsilon/2\}$ for $(\xi',\lambda)\neq(0,0)$ ([Se1], Lemma 2, [Se2], Lemma 2). So the proof of Proposition 3.6 is valid also for the density (3.62) (where (x,ξ,n) are replaced by $(x',\xi',n - 1)$). We conclude that this density has a meromorphic continuation $p_{l,j,\partial}^{s,\varepsilon}(x')$ with no more than simple poles at the points $s_1,\ldots,s_{m+1}$. This proof provides us with the estimate

$$\left|p_{l,j,\partial}^{s,\varepsilon}(x')\right| < c_{1,\varepsilon}\max\left(\rho^{-\mathrm{Re}\,s},1\right)\exp(\varepsilon|\mathrm{Im}\,s|)\sum|s - s_k|^{-1}, \qquad (3.63)$$

where the sum is over $1 \leq k \leq m + 1$. The analytic continuation of the density on N defined by the sum (over l) of the integrals (3.62) for the interior integrals over the contour $\Gamma_{(\pi)}$ (i.e., with $\varepsilon = \pi$) is regular at $s = s_k$ for $(-s_k) \in \mathbb{Z}_+ \cup 0$. (It is proved in Theorem 3.1.) For $|\xi'|^2 > 1 - \rho$ the integral over $\Gamma_{(\varepsilon)}$ in (3.62) is equal to the integral over $\Gamma_{(\pi)}$. Hence the estimate (3.63) is satisfied if the sum is over $k, 1 \leq k \leq m + 1$, such that $(-s_k) \notin \mathbb{Z}_+ \cup 0$. The estimate (3.60) follows from (3.63), (3.61). $\square$

The estimate (3.40) follows from Propositions 3.6, 3.7, and from (3.46). Thus Proposition 3.4 is proved. $\square$

Proof of Proposition 3.5. The estimate (3.46) in the proof of Proposition 3.4 is satisfied by the integral of $(r_\lambda^m)_{x_1x_2}$. So it is enough to obtain the estimate (3.41) for the kernel of

$$\frac{i}{2\pi}\int_{\Gamma_{(\varepsilon)}}\lambda^{-s}P_\lambda^m d\lambda,$$

where $\operatorname{Re}s > n/2$. The term $p_{\mathrm{int}}^{s,\varepsilon}(x_1,x_2)$ in this kernel has the same form as in (3.49) but with the addition factor $\exp(i\xi(x_1 - x_2))$ under the integral sign (where x and $\varphi_j(x)$ are replaced by x_1 and by $\psi_j(x_1)\varphi_j(x_2)$). The integration over the domains (3.50) in the integral corresponding to (3.49) represents this kernel as the sum $(J_{0,\varepsilon}^s + J_{1,\varepsilon}^s + J_{2,\varepsilon}^s)(x_1,x_2)$, where $J_{l,\varepsilon}^s$ corresponds to the integration over the appropriate domain in (3.50). The term $J_{0,\varepsilon}^s(x_1,x_2)$ satisfies the estimate (3.51) if $\operatorname{Re}s > n/2$. (In this estimate $\max(\rho^{-\operatorname{Re}s}, 1)$ can be replaced by $\rho^{-\operatorname{Re}s}$ since $\operatorname{Re}s > 0$ and since $0 < \rho < 1$.)

The contour $\Gamma_{(\varepsilon)}$ of the interior integral in (3.49) for $J_{2,\varepsilon}^s$ can be replaced by the contour $\Gamma_{(\varepsilon),|\xi|}$ defined by (3.52). The sum of the integrals over the straight line pieces of $\Gamma_{(\varepsilon)}$ and of $\Gamma_{(\varepsilon),|\xi|}$ in the kernel $(J_{1,\varepsilon}^s + J_{2,\varepsilon}^s)(x_1,x_2)$ has the same form as (3.53) (but with the factor $\exp(i(x_1 - x_2)\xi)$ under the integral sign). The integral over the circle part of $\Gamma_{(\varepsilon),|\xi|}$ for the kernel $J_{2,\varepsilon}^s(x_1,x_2)$ is also completely analogous to (3.55). This provides us with the estimate for the kernel $p_{\mathrm{int}}^{s,\varepsilon}(x_1,x_2)$ (where $\operatorname{Re}s > n/2$):

$$|p_{\mathrm{int}}^{s,\varepsilon}(x_1,x_2)| < C_\varepsilon \exp(\varepsilon|\operatorname{Im}s|)\rho^{-\operatorname{Re}s}(\operatorname{Re}s - n/2)^{-1}. \qquad (3.64)$$

The proof of (3.41) for an arbitrary closed manifold M follows from the estimate (3.64) together with the estimate (3.46) for $(r_j^m)_{x_1,x_2}$. (They also give us the proof of (3.41) for a part $p_{\mathrm{int}}^{s,\varepsilon}(x_1,x_2)$ of the kernel $(T_{-s})_{x_1,x_2}$ defined by a local parametrix $\sum_j \psi_j P_{j,\mathrm{int}}^m \varphi_j$.) If (M, g_M) is mirror symmetric with respect to (N, g_N) (and the v-transmission interior boundary conditions are given on N) then the kernel $(T_{-s})_{x_1,x_2}$ for $\operatorname{Re}s > -n/2$ can be represented by the formulas (analogous to (2.54), (2.55), and to (2.118)), where $v = (\alpha, \beta) \neq (0,0)$ and T_{-s}^M corresponds to a closed manifold (M, g_M) (or to $v_0 = (1,1)$ that is equivalent according to Proposition 1.1):

$$(T_{-s})_{x_1,x_2} = (T_{-s}^M)_{x_1,x_2} + \frac{\beta^2 - \alpha^2}{\alpha^2 + \beta^2}(\sigma_1^* T_{-s}^M)_{x_1,x_2} \quad \text{for } x_1,x_2 \in M_1 \,,$$

$$(T_{-s})_{x_1,x_2} = (T_{-s}^M)_{x_1,x_2} + \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}(\sigma_1^* T_{-s}^M)_{x_1,x_2} \quad \text{for } x_1,x_2 \in M_2 \,,$$

$$(T_{-s})_{x_1,x_2} = \frac{2\alpha,\beta}{\alpha^2 + \beta^2}(T_{-s}^M)_{x_1,x_2} \quad \text{for } x_1,x_2 \text{ from different } M_k \,, \qquad (3.65)$$

where σ_1 is the mirror symmetry on M with respect to N, acting on the variable x_1. The kernel $(T_{-s}^M)_{x_1,x_2}$ can be analytic (meromorphic) continued to the whole complex plane $\mathbb{C} \ni s$ (separately on the diagonal $x_1 = x_2$ and off the diagonal). It follows from [Se2], Theorem 1, or from the proof of Theorem 3.1. Hence (3.65) is true for all $s \in \mathbb{C}$. So the estimate (3.41) is satisfied also in the case of the v-transmission interior boundary conditions on N if (M, g_M) is mirror-symmetric with respect to N.

The boundary term [74] $\sum \psi_j \mathscr{D}_{m,U} \varphi_j$ of the parametrix P^m can be identified with the same term in the mirror-symmetric case (as it is defined in a neighborhood $N \times I$ of $N = N \times 0$, $I = [-2,2]$). The estimate (3.46) for the integral over $\Gamma_{(\varepsilon)}$ of $(r_j^m)_{x_1,x_2}$ is satisfied for the mirror-symmtric case also. So the estimate (3.41) is satisfied by the kernel $p_{x_1,x_2}^{\hat{\partial},\varepsilon}(s)$ of the operator $\sum \psi_j \mathscr{D}_{m,U} \psi_j$. This estimate for

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[74] The operator $\mathscr{D}_m = \mathscr{D}_{m,j}(v)$ is defined in the coordinate chart $\mathbb{R}^{n-1} \times \mathbb{R}^1 \ni (x',t)$ by (3.12), (3.11), (3.10).

$p^{\partial,\varepsilon}_{x_1,x_2}(s)$ together with the estimates (3.64), (3.46) for $p^{s,\varepsilon}_{\text{int}}(x_1,x_2)$, and with the estimate (3.46) of the integral of $(r^m_\lambda)_{x_1,x_2}$ over $\Gamma_{(\varepsilon)}$ provides us with the estimate (3.41). □

3.5. Appendix. Trace Class Operators and their Traces. A bounded linear operator A acting in a separable Hilbert space H is a *trace class* operator if the series of its singular numbers (i.e., of the arithmetic square roots of the eigenvalues for the self-adjoint operator A^*A) is absolutely convergent. If A is a trace class operator then its matrix trace exists for any orthonormal basis (e_i) in H:

$$\sum(Ae_i, e_i) =: \operatorname{Sp} A$$

and this sum is independent of the orthonormal basis ([Kr]). It is called the matrix trace of A. The Lidskii theorem ([Li]) claims that if A is a trace class operator then the series of its eigenvalues is absolutely convergent: $\sum|\lambda_j(A)| < \infty$ and its trace $\operatorname{Tr} A := \sum\lambda_j(A)$ is equal to its matrix trace: $\operatorname{Tr} A = \operatorname{Sp} A$.

Proposition 3.8. *For $t > 0$ the operators $\exp(-t\Delta_{v_0,j})$ and $p_1\exp(-t\Delta_{v_0,j})$ are trace class operators in the L_2-completion $(DR^j(M))_2$ of $DR^j(M)$ [75] and their traces are equal to the integrals of the densities defined by the restrictions to the diagonals of their kernels:*

$$\operatorname{Tr}\exp(-t\Delta_{v_0,j}) = \sum_{r=1,2} \int_{\overline{M}_I} \operatorname{tr}(*_{x_2} i^*_{M_I} E'_{t,x_1,x_2}(v_0)), \qquad (3.66)$$

$$\operatorname{Tr}(p_1\exp(-t\Delta_{v_0,j})) = \int_{\overline{M}_1} \operatorname{tr}(*_{x_2} i^*_{M_1} E^j_{t,x_1,x_2}(v_0)). \qquad (3.67)$$

Here $p_k : (DR^j(M))_2 \to (DR^j(M_k))_2 \hookrightarrow (DR^j(M))_2$ is the composition of the restriction to M_k of differential forms and of their prolongation to M by zero on another piece of M, $i_{M_r} : \overline{M}_r \hookrightarrow \overline{M}_r \times \overline{M}_r$ is an immersion of the diagonal and the exterior product of the double forms (restricted to the diagonal) is assumed.

Proof. The operator $A_t := \exp(-t\Delta_{v_0,j})$ is positive definite on $(DR^j(M))_2$ and for an arbitrary $f \in (DR^j(M))_2$, $f \neq 0$, it holds $(A_t f, f) > 0$ (where the scalar product on $(DR^j(M))_2$ corresponds to (1.23)).

The operator $B_t := p_1\exp(-t\Delta_{v_0,j})$ is positive definite on the subspace $(DR^j(M_1))_2$ of $(DR^j(M))_2$. Namely $(B_t m, m) > 0$ for $m \in (DR^j(M_1))_2$, $m \neq 0$, and it is a non-negative operator on $(DR^\bullet(M))_2 : (B_t f, f) \geq 0$ for $f \in (DR^j(M))_2$.

The operator $\exp(-t\Delta_{v_0,j})$ is self-adjoint on $(DR^j(M))_2$ by Theorem 3.2. Its kernel $A(x_1, x_2)$ is smooth on $\overline{M}_{r_1} \times \overline{M}_{r_2}$ (as it is proved in Proposition 3.2) and its trace is equal to

$$\operatorname{Tr}\exp(-t\Delta_{v_0,j}) = \sum_{k=1,2} \operatorname{Tr}(p_k\exp(-t\Delta_{v_0,j})) = \sum_{k=1,2} \operatorname{Tr}(p_k\exp(-t\Delta_{v_0,j})p_k). \qquad (3.68)$$

(The matrix trace for $\exp(-t\Delta_{v_0,j})$ in $(DR^j(M))_2 =: H$ can be computed with the help of an orthonormal basis $(e_i(1)), (e_i(2))$ in H, where $(e_j(k))$ is an orthonormal basis in $H_k := (DR^j(M_k))_2$.)

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[75] $DR^j(\overline{M}_1) \oplus DR^j(\overline{M}_2) \subset (DR^j(M))_2$

The operator $A_k := p_k \exp(-t\Delta_{v_0,j})p_k$ acting in $H = H_1 \oplus H_2$ has a continuous kernel $A_k(x_1,x_2) = E^0_{t,x_1,x_2}(v_0)$ on $\overline{M}_k \times \overline{M}_k$ (and it has the zero kernel on $\overline{M}_{k_1} \times \overline{M}_{k_2}$ for $k_1 \neq k_2$). The operator A_k is a self-adjoint operator acting in the Hilbert space H_k and it is positive definite, $(A_k f, f) > 0$ for $f \in H_k$, $f \neq 0$. So according to the Mercer theorem [GG], IV.3, [RiN], Sect. 98, the Fourier series for the kernel of A_k by the eigenforms of A_k

$$A_k(x_1,x_2) = \sum \mu_j \omega_j(x_1) \otimes \omega_j(x_2),\qquad(3.69)$$

(where $\mu_j \geqq 0$ are the eigenvalues of A_k) converges absolutely and uniformly with respect to $\overline{M}_k \times \overline{M}_k$. Hence integrating this series over the diagonals in $\overline{M}_k \times \overline{M}_k$ (for $k = 1,2$) we obtain the equality (3.66):

$$\mathrm{Tr}\exp(-t\Delta_{v_0 j}) = \sum_{k=1,2} \mathrm{Tr}\, A_k = \sum_{k=1,2}\int_{\overline{M}_k} \mathrm{tr}(i^*_{M_k} A_k(x_1,x_2))\,.$$

The equality (3.67) is obtained similarly

$$\mathrm{Tr}(p_1 \exp(-t\Delta_{v_0,t})) = \mathrm{Tr}(p_1 \exp(-t\Delta_{v_0,t})p_1) = \int_{\overline{M}_1} \mathrm{tr}(i^*_{M_1} A_1(x_1,x_2))\,.$$

The proposition is proved. □

Proposition 3.9. *For* $\mathrm{Re}\,s > n/2$ *the operator* T_{-s} *defined by the integral* [76] *(3.18) and the operators* $p_j(\Delta^\bullet_v)^{-s}$ *are trace class operators* ($n := \dim M$). *The traces of these operators for* $\mathrm{Re}\,s > n/2$ *are equal to the integrals over the diagonals of the densities, defined by the restrictions of their kernels to these diagonals.*

$$\mathrm{Tr}((\Delta^\bullet_v)^{-s} = \sum_{r=1,2}\int_{\overline{M}_l} \mathrm{tr}\,(*_{x_2}i^*_{M_l} T_{-s}(x_1,x_2))\,,\qquad(3.70)$$

$$\mathrm{Tr}(p_j(\Delta^\bullet_v)^{-s} = \int_{\overline{M}_l} \mathrm{tr}\,(*_{x_2}i^*_{M_l} T_{-s}(x_1,x_2))\qquad(3.71)$$

Proof. The kernel $T_{-s}(x_1,x_2)$ for $\mathrm{Re}\,s > n/2$ is continuous on $\overline{M}_{j_1} \times \overline{M}_{j_2}$ (Theorem 3.1). The operator T_{-s} for such s is nonnegative, $(T_{-s}f, f) \geqq 0$, and self-adjoint. It is a trace class operator (Theorem 3.1). For $\mathrm{Re}\,s > n/2$ the equality holds (analogous to (3.68))

$$\mathrm{Tr}\,T_{-s} = \sum_{j=1,2} \mathrm{Tr}(p_j T_{-s} p_j)\,.$$

The operator $p_j T_{-s} p_j$ is self-adjoint in the Hilbert subspace $DR^\bullet(M_j))_2$ of $(DR^\bullet(M))_2$ and its kernel coincides with the kernel $K_{-s,j}$ of T_{-s} on $\overline{M}_j \times \overline{M}_j$. So for $\mathrm{Re}\,s > n/2$ its kernel is continuous and, according to the Mercer theorem, the series on $\overline{M}_j \times \overline{M}_j$ for $K_{-s,j}$ expressed by the eigenforms of $p_j T_{-s} p_j$ (analogous to (3.69)) is absolutely and uniformly convergent on $\overline{M}_j \times \overline{M}_j$. Hence for such s the integral over the diagonal of the density, defined by the restriction of the kernel $K_{-s,j}$, is equal to $\mathrm{Tr}(p_j T_{-s} p_j)$. Thus the equalities (3.71) and (3.70) are proved. □

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[76] The operator T_{-s} for such s is defined on $(DR^\bullet(M))_2$ and it is equal to the direct sum of the operator $(\Delta^\bullet_v)^{-s}$ on the orthogonal complement to $\mathrm{Ker}(\Delta^\bullet_v)$ and of the zero operator on $\mathrm{Ker}(\Delta^\bullet_v)$, by Theorem 3.1.

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