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Finite-Dimensional Representations of the Quantum Superalgebra $U_q[gl(n/m)]$ and Related q-Identities

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Abstract: Explicit expressions for the generators of the quantum superalgebra $U_q[gl(n/m)]$ acting on a class of irreducible representations are given. The class under consideration consists of all essentially typical representations: for these a Gel'fand-Zetlin basis is known. The verification of the quantum superalgebra relations to be satisfied is shown to reduce to a set of q-number identities.

1. Introduction

This paper is devoted to the study of a class of finite-dimensional irreducible representations of the quantum superalgebra $U_q[gl(n/m)]$. The main goal is to present explicit actions of the $U_q[gl(n/m)]$ generating elements acting on a Gel'fand-Zetlin-like basis, and to discuss some of the q-number identities related to these representations.

Quantum groups [5], finding their origin in the quantum inverse problem method [6] and in investigations related to the Yang-Baxter equation [15], have now become an important and widely used concept in various branches of physics and mathematics. A quantum (super)algebra $U_q[G]$ associated with a (simple) Lie (super)algebra G is a deformation of the universal enveloping algebra of G endowed with a Hopf algebra structure. The first example was given in [19, 30], and soon followed the generalization to any Kac-Moody Lie algebra with symmetrizable Cartan matrix [4, 12]. For the deformation of the enveloping algebra of a Lie superalgebra we mention the case of $osp(1/2)$ [20, 21], later to be extended to Lie superalgebras with a symmetrizable Cartan matrix [32] including the basic [16] Lie superalgebras [1, 2].

Representations of quantum algebras have been studied extensively, particularly for generic q-values (i.e. q not a root of unity). In this case, finite-dimensional irreducible representations of $sl(n)$ can be deformed into irreducible representations of $U_q[sl(n)]$ [13], and it was shown that one obtains all finite-dimensional irreducible

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modules of $U_q[sl(n)]$ in this way [27]. In [14], explicit expressions for the generators of $U_q[sl(n)]$ acting on the "undeformed" Gel'fand-Zetlin basis were given. It is in the spirit of this work that our present paper should be seen. Here, we study a class of irreducible representations of the quantum superalgebra $U_q[gl(n/m)]$. The class consists of so-called essentially typical representations; one can interpret these as irreducible representations for which a Gel'fand-Zetlin basis can be given. Just as in the case of $sl(n)$, the basis will remain undeformed, and the deformation will arise in the action of the quantum superalgebra generators on the basis vectors.

In the literature, some results have already appeared for representations of quantum superalgebras of type $U_q[gl(n/m)]$: representations of $U_q[gl(n/1)]$ (both typical and atypical) were examined in [25] following the study of [22, 23]; a generic example, $U_q[gl(3/2)]$, was treated in [26]; and the induced module construction of Kac [17] was generalized to $U_q[gl(n/m)]$ [38]. On the other hand, oscillator representations have been constructed [2, 7, 9] not only for $U_q[gl(n/m)]$ but also for other quantum superalgebras.

The structure of the present paper is as follows. In Sect. 2 we recall the definition of the Lie superalgebra $gl(n/m)$ and fix the notation. We also remind the reader of some representation theory of $gl(n/m)$ which will be needed in the case of $U_q[gl(n/m)]$, in particular of the concept of typical, atypical, and essentially typical representations. For the last class of representations, the Gel'fand-Zetlin basis introduced in [24] is written in explicit form. In the next section, we briefly recall the definition of the quantum superalgebra $U_q[gl(n/m)]$. Section 4 contains our main results. We present the actions of the $U_q[gl(n/m)]$ generators on the Gel'fand-Zetlin basis introduced, and we give some indications of how the relations were proved in these representations. Some of the relations actually reduce to interesting q-number identities, which can be proved using the Residue theorem of complex analysis. We conclude the paper with some comments and further outlook.

2. The Lie Superalgebra $gl(n/m)$ and Gel'fand-Zetlin Patterns

The Lie superalgebra $G = gl(n/m)$ can be defined [16, 28] through its natural matrix realization

$$gl(n/m) = \left\{ x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| A \in M_{n\times n}, B \in M_{n\times m}, C \in M_{m\times n}, D \in M_{m\times m} \right\}, \quad (1)$$

where $M_{p\times q}$ is the space of all $p\times q$ complex matrices. The even subalgebra $gl(n/m)_{\bar 0}$ has $B=0$ and $C=0$; the odd subspace $gl(n/m)_{\bar 1}$ has $A=0$ and $D=0$. The bracket is determined by

$$[a,b] = ab - (-1)^{\alpha\beta} ba, \qquad \forall a \in G_\alpha \text{ and } \forall b \in G_\beta, \quad (2)$$

where $\alpha, \beta \in \{\bar 0, \bar 1\} \equiv \mathbf{Z}_2$. If $a \in G_\alpha$ then $\alpha = \deg(a)$ is called the degree of a, and an element of $G = G_{\bar 0} \oplus G_{\bar 1}$ is called homogeneous if it belongs to either $G_{\bar 0}$ or else $G_{\bar 1}$. We denote by $gl(n/m)_{+1}$ the space of matrices $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ and by $gl(n/m)_{-1}$ the space of matrices $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$. Then $G = gl(n/m)$ has a $\mathbf{Z}$-grading which is consistent with the $\mathbf{Z}_2$-grading [28], namely $G = G_{-1} \oplus G_0 \oplus G_{+1}$ with $G_{\bar 0} = G_0$ and

$G_{\bar{1}} = G_{-1} \oplus G_{+1}$. Note that $gl(n/m)_0 = gl(n) \oplus gl(m)$. For elements x of $gl(n/m)$ given by (1), one defines the supertrace as $\mathrm{str}(x) = \mathrm{tr}(A) - tr(D)$. The Lie superalgebra $gl(n/m)$ is not simple, and (for $n \neq m$) one can define the simple superalgebra $sl(n/m)$ as the subalgebra consisting of elements with supertrace 0. However, the representation theory of $gl(n/m)$ or $sl(n/m)$ is essentially the same (the situation is similar as for the classical Lie algebras $gl(n)$ and $sl(n)$), and hence we prefer to work with $gl(n/m)$ and in the following section with its Hopf superalgebra deformation $U_q[gl(n/m)]$.

A basis for $G = gl(n/m)$ consists of matrices E_{ij} $(i,j = 1,2,\ldots,r \equiv m+n)$ with entry 1 at position (i,j) and 0 elsewhere. A Cartan subalgebra H of G is spanned by the elements $h_j = E_{jj}$ $(j = 1,2,\ldots,r)$, and a set of generators of $gl(n/m)$ is given by the h_j $(j = 1,\ldots,r)$ and the elements $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$ $(i = 1,\ldots,r-1)$. The space dual to H is H^* and is described by the forms ε_i $(i = 1,\ldots,r)$, where $\varepsilon_j : x \rightarrow A_{jj}$ for $1 \leqq j \leqq n$ and $\varepsilon_{n+j} : x \rightarrow D_{jj}$ for $1 \leqq j \leqq m$, and where x is given as in (1). On H^* there is a bilinear form defined, deduced from the supertrace on G, and explicitly given by [34]:

$$\begin{aligned} \langle \varepsilon_i | \varepsilon_j \rangle &= \delta_{ij}, && \text{for } 1 \leqq i,j \leqq n; \\ \langle \varepsilon_i | \varepsilon_p \rangle &= 0, && \text{for } 1 \leqq i \leqq n \text{ and } n+1 \leqq p \leqq r; \\ \langle \varepsilon_p | \varepsilon_q \rangle &= -\delta_{pq}, && \text{for } n+1 \leqq p,q \leqq r, \end{aligned} \tag{3}$$

where δ_{ij} is the Kronecker-δ. The components of an element $\Lambda \in H^*$ will be written as $[m] = [m_{1r}, m_{2r}, \ldots, m_{rr}]$, where $\Lambda = \sum_{i=1}^{r} m_{ir}\varepsilon_i$ and m_{ir} are complex numbers. The elements of H^* are called the weights. The roots of $gl(n/m)$ are the non-zero weights of the adjoint representation, and take the form $\varepsilon_i - \varepsilon_j$ $(i \neq j)$ in this basis; the positive roots are those with $1 \leqq i < j \leqq r$, and of importance are the nm odd positive roots

$$\beta_{ip} = \varepsilon_i - \varepsilon_p, \qquad \text{with } 1 \leqq i \leqq n \text{ and } n+1 \leqq p \leqq r. \tag{4}$$

For an element $\Lambda \in H^*$ with components $[m]$, the Kac-Dynkin labels $(a_1, \ldots, a_{n-1}; a_n; a_{n+1}, \ldots, a_{r-1})$ are given by $a_i = m_{ir} - m_{i+1,r}$ for $i \neq n$ and $a_n = m_{nr} + m_{n+1,r}$. Hence, Λ with components $[m]$ will be called an integral dominant weight if $m_{ir} - m_{i+1,r} \in \mathbf{Z}_+ = \{0,1,2,\ldots\}$ for all $i \neq n$ $(1 \leqq i \leqq r-1)$. For every integral dominant weight $\Lambda \equiv [m]$ we denote by $V^0(\Lambda)$ the simple G_0 module with highest weight Λ; this is simply the finite-dimensional $gl(n) \oplus gl(m)$ module with $gl(n)$ labels $\{m_{1,r}, \ldots m_{nr}\}$ and with $gl(m)$ labels $\{m_{n+1,r}, \ldots, m_{rr}\}$. The module $V^0(\Lambda)$ can be extended to a $G_0 \oplus G_{+1}$ module by the requirement that $G_{+1}V^0(\Lambda) = 0$. The induced G module $\overline{V}(\Lambda)$, first introduced by Kac [17] and usually referred to as the Kac-module, is defined by

$$\overline{V}(\Lambda) = \mathrm{Ind}_{G_0 \oplus G_{+1}}^{G} V^0(\Lambda) \cong U(G_{-1}) \otimes V^0(\Lambda), \tag{5}$$

where $U(G_{-1})$ is the universal enveloping algebra of G_{-1}. It follows that $\dim \overline{V}(\Lambda) = 2^{nm} \dim V^0(\Lambda)$. By definition, $\overline{V}(\Lambda)$ is a highest weight module; unfortunately, $\overline{V}(\Lambda)$ is not always a simple G module. It contains a unique maximal (proper) submodule $M(\Lambda)$, and the quotient module

$$V(\Lambda) = \overline{V}(\Lambda)/M(\Lambda) \tag{6}$$

is a finite-dimensional simple module with highest weight Λ. In fact, Kac [17] proved the following:

Theorem 1. *Every finite-dimensional simple G module is isomorphic to a module of type* (6), *where* $\Lambda \equiv [m]$ *is integral dominant. Moreover, every finite-dimensional simple G module is uniquely characterized by its integral dominant highest weight* Λ.

An integral dominant weight $\Lambda = [m]$ (resp. $\overline{V}(\Lambda)$, resp. $V(\Lambda)$) is called a typical weight (resp. a typical Kac module, resp. a typical simple module) if and only if $\langle \Lambda + \rho | \beta_{ip} \rangle \neq 0$ for all odd positive roots β_{ip} of (4), where 2ρ is the sum of all positive even roots of G minus the sum of all positive odd roots of G. Otherwise Λ, $\overline{V}(\Lambda)$ and $V(\Lambda)$ are called atypical. The importance of these definitions follows from another theorem of Kac [17]:

Theorem 2. *The Kac-module* $\overline{V}(\Lambda)$ *is a simple G module if and only if* Λ *is typical.*

For an integral dominant highest weight $\Lambda = [m]$ it is convenient to introduce the following labels [24]:

$$l_{ir} = m_{ir} - i + n + 1, \quad (1 \leqq i \leqq n); \quad l_{pr} = -m_{pr} + p - n, \quad (n+1 \leqq p \leqq r). \tag{7}$$

In terms of these, one can deduce that $\langle \Lambda + \rho | \beta_{ip} \rangle = l_{ir} - l_{pr}$, and hence the conditions for typicality take a simple form.

For typical modules or representations one can say that they are well understood, and a character formula was given by Kac [17]. A character formula for all atypical modules has not been proven so far, but there are several breakthroughs in this area: for singly atypical modules (for which the highest weight Λ is atypical with respect to one single odd root β_{ip}) a formula has been constructed [34]; for all atypical modules a formula has been conjectured [33]; for atypical Kac-modules the composition series has been conjectured [11] and partially shown to be correct [31]. On the other hand, the modules for which an explicit action of generators on basis vectors can be given, similar to the action of generators of $gl(n)$ on basis vectors with Gel'fand-Zetlin labels, is only a subclass of the typical modules, namely the so-called essentially typical modules [24], the definition of which shall be recalled here.

For simple $gl(n)$ modules the Gel'fand-Zetlin basis vectors [10] and their labels – with the conditions ("in-betweenness conditions") – are reflecting the decomposition of the module with respect to the chain of subalgebras $gl(n) \supset gl(n-1) \supset \cdots \supset gl(1)$. In trying to construct a similar basis for the finite-dimensional modules of the Lie superalgebra $gl(n/m)$ it was natural to consider the decomposition with respect to the chain of subalgebras $gl(n/m) \supset gl(n/m-1) \supset \cdots \supset gl(n/1) \supset gl(n) \supset gl(n-1) \supset \cdots \supset gl(1)$. However, in order to be able to define appropriate actions of the $gl(n/m)$ generators on basis vectors with respect to this decomposition, it was necessary that at every step in this reduction the corresponding modules are completely reducible with respect to the submodule under consideration. A sufficient condition is that for every step in the above reduction the modules are typical, i.e. a typical $gl(n/m)$ module V must decompose into typical $gl(n/m-1)$ modules, each of which must decompose into typical $gl(n/m-2)$ modules, etc. Such modules are called essentially typical [24], and a Gel'fand-Zetlin-like basis can be constructed with an action of the $gl(m/n)$ generators. In terms of the above quantities l_{ir}, a module with highest weight $\Lambda \equiv [m]$ is essentially typical if and only if

$$\{l_{1r}, l_{2r}, \ldots, l_{nr}\} \cap \{l_{n+1,r}, l_{n+1,r}+1, l_{n+1,r}+2, \ldots, l_{rr}\} = \emptyset . \tag{8}$$

The explicit form of the action [23,24] will not be repeated here, but the reader interested can deduce it from relations (24–30) of the present paper by taking the limit $q \to 1$ (in fact, the limit of our present relations also improve some minor misprints in the transformations of the GZ basis as given in [23, 24]). It is necessary, however, to recall the labelling of the basis vectors for these modules, since the labelling of basis vectors of representations of the quantum algebra $U_q[gl(n/m)]$ is exactly the same (note that also for the quantum algebra $U_q[gl(n)]$, the finite-dimensional representations can be labelled by the same Gel'fand-Zetlin patterns as in the non-deformed case of $gl(n)$, when q is not a root of unity [14]).

Let $[m]$ be the labels of an integral dominant weight Λ. Associated with $[m]$ we define a pattern $|m)$ of $r(r+1)/2$ complex numbers m_{ij} $(1 \leqq i \leqq j \leqq r)$ ordered as in the usual Gel'fand-Zetlin basis for $gl(r)$:

$$|m) = \left| \begin{array}{lllllllll} m_{1r} & \cdots & m_{n-1,r} & m_{nr} & m_{n+1,r} & \cdots & m_{r-1,r} & m_{rr} \\ m_{1,r-1} & \cdots & m_{n-1,r-1} & m_{n,r-1} & m_{n+1,r-1} & \cdots & m_{r-1,r-1} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & ⋰ & & \\ m_{1,n+1} & \cdots & m_{n-1,n+1} & m_{n,n+1} & m_{n+1,n+1} & & & \\ m_{1n} & \cdots & m_{n-1,n} & m_{nn} & & & & \\ m_{1,n-1} & \cdots & m_{n-1,n-1} & & & & & \\ \vdots & ⋰ & & & & & & \\ m_{11} & & & & & & & \end{array} \right) \tag{9}$$

Such a pattern should satisfy the following set of conditions:

1. the labels m_{ir} of Λ are fixed for all patterns,
2. $m_{ip} - m_{i,p-1} \equiv \theta_{i,p-1} \in \{0, 1\}$, $\quad (1 \leqq i \leqq n;\ n+1 \leqq p \leqq r)$,
3. $m_{ip} - m_{i+1,p} \in \mathbf{Z}_+$, $\quad (1 \leqq i \leqq n-1;\ n+1 \leqq p \leqq r)$,
4. $m_{i,j+1} - m_{ij} \in \mathbf{Z}_+$ and $m_{i,j} - m_{i+1,j+1} \in \mathbf{Z}_+$,
   $(1 \leqq i \leqq j \leqq n-1$ or $n+1 \leqq i \leqq j \leqq r-1)$. (10)

The last condition corresponds to the in-betweenness condition and ensures that the triangular pattern to the right of the $m \times n$ rectangle m_{ip} $(1 \leqq i \leqq n;\ n+1 \leqq p \leqq r)$ in (9) corresponds to a classical Gel'fand-Zetlin pattern for $gl(m)$, and that the triangular pattern below this rectangle corresponds to a Gel'fand-Zetlin pattern for $gl(n)$.

The following theorem was proved [24]:

Theorem 3. *Let $\Lambda \equiv [m]$ be an essentially typical highest weight. Then the set of all patterns* (9) *satisfying* (10) *constitute a basis for the* (*typical*) *Kac-module* $\overline{V}(\Lambda) = V(\Lambda)$.

The patterns (9) are referred to as Gel'fand-Zetlin (GZ) basis vectors for $V(\Lambda)$ and an explicit action of the $gl(n/m)$ generators h_j $(1 \leqq j \leqq r)$, e_i and f_i $(1 \leqq i \leqq r-1)$ has been given in Ref. [24].

In the following section we shall recall the definition of the quantum algebra $U_q[gl(n/m)]$. We shall then define an action of the quantum algebra generators on the Gel'fand-Zetlin basis vectors $|m)$ introduced here. In other words, just as for the finite-dimensional $gl(n)$ modules, one can use the same basis vectors and only the action is deformed.

3. The Quantum Superalgebra $U_q[gl(n/m)]$

The quantum superalgebra $U_q \equiv U_q[gl(n/m)]$ is the free associative superalgebra over $\mathbf{C}$ with parameter $q \in \mathbf{C}$ and generators k_j, k_j^{-1} $(j = 1,2,\dots,r \equiv n+m)$ and e_i, f_i $(i = 1,2,\dots,r-1)$ subject to the following relations (unless stated otherwise, the indices below run over all possible values):

- The Cartan-Kac relations:

$$k_ik_j = k_jk_i, \qquad k_ik_i^{-1} = k_i^{-1}k_i = 1\,; \tag{11}$$

$$k_ie_jk_i^{-1} = q^{(\delta_{ij}-\delta_{i,j+1})/2}e_j, \qquad k_if_jk_i^{-1} = q^{-(\delta_{ij}-\delta_{i,j+1})/2}f_j\,; \tag{12}$$

$$e_if_j - f_je_i = 0 \quad \text{if } i \neq j; \tag{13}$$

$$e_if_i - f_ie_i = (k_i^2k_{i+1}^{-2} - k_{i+1}^2k_i^{-2})/(q-q^{-1}) \quad \text{if } i \neq n; \tag{14}$$

$$e_nf_n + f_ne_n = (k_n^2k_{n+1}^2 - k_n^{-2}k_{n+1}^{-2})/(q-q^{-1}); \tag{15}$$

- The Serre relations for the e_i (e-Serre relations):

$$e_ie_j = e_je_i \text{ if } |i-j| \neq 1; \qquad e_n^2 = 0\,; \tag{16}$$

$$e_i^2e_{i+1} - (q+q^{-1})e_ie_{i+1}e_i + e_{i+1}e_i^2 = 0, \quad \text{for } i \in \{1,\dots,n-1\} \cup \{n+1,\dots,n+m-2\}\,; \tag{17}$$

$$e_{i+1}^2e_i - (q+q^{-1})e_{i+1}e_ie_{i+1} + e_ie_{i+1}^2 = 0, \quad \text{for } i \in \{1,\dots,n-2\} \cup \{n,\dots,n+m-2\}\,; \tag{18}$$

$$e_ne_{n-1}e_ne_{n+1} + e_{n-1}e_ne_{n+1}e_n + e_ne_{n+1}e_ne_{n-1} + e_{n+1}e_ne_{n-1}e_n - (q+q^{-1})e_ne_{n-1}e_{n+1}e_n = 0\,; \tag{19}$$

- The relations obtained from (16–19) by replacing every e_i by f_i (f-Serre relations).

Equation (19) is the so-called extra Serre relation [8, 18, 29], which can also be obtained from an R-matrix approach [35, 36, 37]. The $\mathbf{Z}_2$-grading in U_q is defined by the requirement that the only odd generators are e_n and f_n; the degree of a homogeneous element a of U_q shall be denoted by $\deg(a)$. It can be shown that U_q is a Hopf superalgebra with counit ε, comultiplication Δ and antipode S, defined by:

$$\varepsilon(e_i) = \varepsilon(f_i) = 0, \qquad \varepsilon(k_j) = 1\,; \tag{20}$$

$$\begin{aligned}
\Delta(k_j) &= k_j \otimes k_j, \\
\Delta(e_i) &= e_i \otimes k_ik_{i+1}^{-1} + k_i^{-1}k_{i+1} \otimes e_i, \quad \text{if } i \neq n, \\
\Delta(e_n) &= e_n \otimes k_nk_{n+1} + k_n^{-1}k_{n+1}^{-1} \otimes e_n, \\
\Delta(f_i) &= f_i \otimes k_ik_{i+1}^{-1} + k_i^{-1}k_{i+1} \otimes f_i, \quad \text{if } i \neq n, \\
\Delta(f_n) &= f_n \otimes k_nk_{n+1} + k_n^{-1}k_{n+1}^{-1} \otimes f_n;
\end{aligned} \tag{21}$$

$$
\begin{aligned}
S(k_j) &= k_j^{-1}, \\
S(e_i) &= -qe_i, \quad S(f_i) = -q^{-1}f_i, \qquad \text{if } i \neq n, \\
S(e_n) &= -e_n, \quad S(f_n) = -f_n.
\end{aligned} \tag{22}
$$

Remember that $\Delta : U_q \to U_q \otimes U_q$ is a morphism of *graded* algebras, and that the multiplication in $U_q \otimes U_q$ is given by

$$(a \otimes b)(c \otimes d) = (-1)^{\deg(b)\deg(c)} ac \otimes bd. \tag{23}$$

4. The $U_q[gl(n/m)]$ Representations

Let $\Lambda \equiv [m]$ be an essentially typical highest weight, and denote by $W(\Lambda)$ the vector space spanned by the basis vectors $|m)$ of the form (9) satisfying the conditions (10). On this vector space, we shall define an action of the generators of $U_q = U_q[gl(n/m)]$, thus turning $W(\Lambda)$ into a U_q module. For convenience, we introduce the following notations: $l_{ij} = m_{ij} - i + n + 1$ for $1 \leqq i \leqq n$, $l_{pj} = -m_{pj} + p - n$ for $n+1 \leqq p \leqq r$, and $|m)_{\pm ij}$ is the pattern obtained from $|m)$ by replacing the entry m_{ij} by $m_{ij} \pm 1$.

The following is the main result of this paper (as usual, $[x]$ stands for $(q^x - q^{-x})/(q - q^{-1})$):

Theorem 4. *For generic values of q every essentially typical $gl(n/m)$ module $V(\Lambda)$ with highest weight Λ can be deformed into an irreducible $U_q[gl(n/m)]$ module $W(\Lambda)$ with the same underlying vector space and with the action of the generators given by:*

$$k_i|m) = q^{(\Sigma_{j=1}^{i} m_{ji} - \Sigma_{j=1}^{i-1} m_{j,i-1})/2}|m), \quad (1 \leqq i \leqq r), \tag{24}$$

$$
\begin{aligned}
e_k|m) &= \sum_{j=1}^{k} \left(-\frac{\Pi_{i=1}^{k+1}[l_{i,k+1} - l_{jk}]\Pi_{i=1}^{k-1}[l_{i,k-1} - l_{jk} - 1]}{\Pi_{i \neq j=1}^{k}[l_{ik} - l_{jk}][l_{ik} - l_{jk} - 1]} \right)^{1/2} |m)_{jk}, \\
&\quad (1 \leqq k \leqq n-1),
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
f_k|m) &= \sum_{j=1}^{k} \left(-\frac{\Pi_{i=1}^{k+1}[l_{i,k+1} - l_{jk} + 1]\Pi_{i=1}^{k-1}[l_{i,k-1} - l_{jk}]}{\Pi_{i \neq j=1}^{k}[l_{ik} - l_{jk} + 1][l_{ik} - l_{jk}]} \right)^{1/2} |m)_{-jk}, \\
&\quad (1 \leqq k \leqq n-1),
\end{aligned} \tag{26}
$$

$$e_n|m) = \sum_{i=1}^{n} \theta_{in}(-1)^{i-1}(-1)^{\theta_{1n} + \cdots + \theta_{i-1,n}} \left(\frac{\Pi_{k=1}^{n-1}[l_{k,n-1} - l_{i,n+1}]}{\Pi_{k \neq i=1}^{n}[l_{k,n+1} - l_{i,n+1}]} \right)^{1/2} |m)_{in}, \tag{27}$$

$$
\begin{aligned}
f_n|m) &= \sum_{i=1}^{n}(1-\theta_{in})(-1)^{i-1}(-1)^{\theta_{1n}+\quad+\theta_{i-1,n}}[l_{i,n+1}-l_{n+1,n+1}] \\
&\quad\times\left(\frac{\Pi_{k=1}^{n-1}[l_{k,n-1}-l_{i,n+1}]}{\Pi_{k\neq i=1}^{n}[l_{k,n+1}-l_{i,n+1}]}\right)^{1/2}|m)_{-in},
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
e_p|m) &= \sum_{i=1}^{n}\theta_{ip}(-1)^{\theta_{1p}+\quad+\theta_{i-1,p}+\theta_{i+1,p-1}+\quad+\theta_{n,p-1}}(1-\theta_{i,p-1}) \\
&\quad\times\prod_{k\neq i=1}^{n}\left(\frac{[l_{i,p+1}-l_{kp}][l_{i,p+1}-l_{kp}-1]}{[l_{i,p+1}-l_{k,p+1}][l_{i,p+1}-l_{k,p-1}-1]}\right)^{1/2}|m)_{ip} \\
&\quad+\sum_{s=n+1}^{p}\left(-\frac{\Pi_{q=n+1}^{p-1}[l_{q,p-1}-l_{sp}+1]\Pi_{q=n+1}^{p+1}[l_{q,p+1}-l_{sp}]}{\Pi_{q\neq s=n+1}^{p}[l_{qp}-l_{sp}][l_{qp}-l_{sp}+1]}\right)^{1/2} \\
&\quad\times\prod_{k=1}^{n}\frac{[l_{kp}-l_{sp}][l_{kp}-l_{sp}+1]}{[l_{k,p+1}-l_{sp}][l_{k,p-1}-l_{sp}+1]}|m)_{sp},\quad(n+1\leqq p\leqq r-1),
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
f_p|m) &= \sum_{i=1}^{n}\theta_{i,p-1}(-1)^{\theta_{1p}+\quad+\theta_{i-1,p}+\theta_{i+1,p-1}+\quad+\theta_{n,p-1}}(1-\theta_{ip}) \\
&\quad\times\prod_{k\neq i=1}^{n}\left(\frac{[l_{i,p+1}-l_{kp}][l_{i,p+1}-l_{kp}-1]}{[l_{i,p+1}-l_{k,p+1}][l_{i,p+1}-l_{k,p-1}-1]}\right)^{1/2} \\
&\quad\times\frac{\Pi_{q=n+1}^{p-1}[l_{i,p+1}-l_{q,p-1}-1]\Pi_{q=n+1}^{p+1}[l_{i,p+1}-l_{q,p+1}]}{\Pi_{q=n+1}^{p}[l_{i,p+1}-l_{qp}-1][l_{i,p+1}-l_{qp}]}|m)_{-ip} \\
&\quad+\sum_{s=n+1}^{p}\left(-\frac{\Pi_{q=n+1}^{p-1}[l_{q,p-1}-l_{sp}]\Pi_{q=n+1}^{p+1}[l_{q,p+1}-l_{sp}-1]}{\Pi_{q\neq s=n+1}^{p}[l_{qp}-l_{sp}-1][l_{qp}-l_{sp}]}\right)^{1/2}|m)_{-sp}, \\
&\quad(n+1\leqq p\leqq r-1).
\end{aligned}
\tag{30}
$$

In the above expressions, $\sum_{k\neq i=1}^{n}$ or $\prod_{k\neq i=1}^{n}$ means that k takes all values from 1 to n with $k\neq i$. If a vector from the rhs of (24–30) does not belong to the module under consideration, then the corresponding term is zero even if the coefficient in front is undefined; if an equal number of factors in numerator and denominator are simultaneously equal to zero, they should be cancelled out. The Eqs. (25, 26) are the same as in [14]; they describe the transformation of the basis under the action of the $gl(n)$ generators.

To conclude this section, we shall make a number of comments on the proof of this theorem. Provided that all coefficients in (24–30) are well defined (which is indeed the case under the conditions required here), it is sufficient to show that the explicit actions (24–30) satisfy the relation (11–19) (plus, of course, also the f-Serre relations). The irreducibility then follows from the results of Zhang [38] or from the observation that for generic q a deformed matrix element in the GZ basis is zero only if the corresponding non-deformed matrix element vanishes.

To show that (11), (12) and (13) are satisfied is a straightforward matter. The difficult Cartan-Kac relations to be verified are (14) and (15). We shall consider one case in more detail, namely (15). This relation, with the actions (24–30), is valid if and only if

$$\sum_{i=1}^{n}[l_{i,n+1}-l_{n+1,n+1}]\frac{\Pi_{k=1}^{n-1}[l_{k,n-1}-l_{i,n+1}]}{\Pi_{k\neq i=1}^{n}[l_{k,n+1}-l_{i,n+1}]}$$

$$=\left[\sum_{k=1}^{n-1}(l_{k,n+1}-l_{k,n-1})+l_{n,n+1}-l_{n+1,n+1}\right]. \tag{31}$$

Putting $a_i=l_{i,n+1}$ for $i=1,2,\ldots,n$, $b_i=l_{i,n-1}$ for $i=1,2,\ldots,n-1$ and $b_n=l_{n+1,n+1}$, the identity between q-numbers to be proved reduces to

$$\sum_{i=1}^{n}\frac{\Pi_{k=1}^{n}[a_i-b_k]}{\Pi_{k\neq i=1}^{n}[a_i-a_k]}=\left[\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}b_k\right]. \tag{32}$$

Using the explicit definition of a q-number, and relabelling $q^{2a_i}=A_i$ and $q^{2b_i}=B_i$, this becomes

$$\sum_{i=1}^{n}\frac{\Pi_{k=1}^{n}(A_i-B_k)}{A_i\Pi_{k\neq i=1}^{n}(A_i-A_k)}=1-\frac{B_1B_2\cdots B_n}{A_1A_2\cdots A_n}. \tag{33}$$

To prove this last identity, consider the complex function

$$f(z)=\frac{\Pi_{k=1}^{n}(z-B_k)}{z\Pi_{k=1}^{n}(z-A_k)}. \tag{34}$$

This function is holomorphic over **C** except in its singular poles $0, A_1,\ldots,A_n$ (under the present conditions, all A_k are indeed distinct). Let C be a closed curve whose interior contains all these poles. Then the Residue Theorem of complex analysis implies that $\oint_C f(z)dz=2\pi i(\mathrm{Res}(0)+\sum_{i=1}^{n}\mathrm{Res}(A_i))$. It is easy to see that $\mathrm{Res}(0)=\lim_{z\to 0}f(z)z=(B_1\cdots B_n)/(A_1\cdots A_n)$ and that $\mathrm{Res}(A_i)=\lim_{z\to A_i}f(z)(z-A_i)=\prod_{k=1}^{n}(A_i-B_k)/(A_i\prod_{k\neq i=1}^{n}(A_i-A_k))$. On the other hand,

$$\oint_C f(z)dz=-2\pi i\,\mathrm{Res}(\infty)=-2\pi i\lim_{z\to\infty}(-z)f(z)=2\pi i, \tag{35}$$

and hence the identity (33) holds.

For the other cases, the method of proof is similar and we shall no longer mention the details. For (14) with $i=k\leqq n-1$, the identity to be verified is of the following type:

$$\sum_{i=1}^{k}\left(\frac{\Pi_{j=1}^{k}[a_i-b_j][a_i-c_j-1]}{\Pi_{j\neq i=1}^{k}[a_i-a_j][a_i-a_j-1]}-\frac{\Pi_{j=1}^{k}[a_i-c_j][a_i-b_j+1]}{\Pi_{j\neq i=1}^{k}[a_i-a_j][a_i-a_j+1]}\right)$$

$$=\left[\sum_{j=1}^{k}(b_j+c_j-2a_j)\right], \tag{36}$$

or, using a similar transformation as before,

$$\sum_{i=1}^{k} \left(\frac{\Pi_{j=1}^{k}(A_i - B_j)(A_i - q^2 C_j)}{A_i \Pi_{j\neq i=1}^{k}(A_i - A_j)\Pi_{j=1}^{k}(A_i - q^2 A_j)} + \frac{\Pi_{j=1}^{k}(A_i - C_j)(A_i - q^{-2} B_j)}{A_i \Pi_{j\neq i=1}^{k}(A_i - A_j)\Pi_{j=1}^{k}(A_i - q^{-2} A_j)} \right) = 1 - \frac{\Pi_{j=1}^{k}(B_j C_j)}{\left(\Pi_{j=1}^{k} A_j\right)^2}. \tag{37}$$

This identity is proven by taking the function $f(z) = \prod_j (z - B_j)(z - q^2 C_j)/(z \prod_j (z - A_j)(z - q^2 A_j))$ and applying the same Residue theorem. Finally, the most complicated case is (14) with $i = p > n$. The identity to prove is (with $s = p - n$):

$$-\sum_{i=1}^{s} \frac{\Pi_{j=1}^{s}[a_i - c_j][a_i - b_j + 1]}{\Pi_{j\neq i=1}^{s}[a_i - a_j][a_i - a_j + 1]} \prod_{k=1}^{n} \frac{[a_i - d_k - f_k + 1]}{[a_i - d_k + 1]}$$
$$+\sum_{i=1}^{s} \frac{\Pi_{j=1}^{s}[a_i - b_j][a_i - c_j - 1]}{\Pi_{j\neq i=1}^{s}[a_i - a_j][a_i - a_j - 1]} \prod_{k=1}^{n} \frac{[a_i - d_k - f_k]}{[a_i - d_k]}$$
$$+\sum_{k=1}^{n} (f_k) \frac{\Pi_{j=1}^{s}[d_k - c_j - 1][d_k - b_j]}{\Pi_{j=1}^{s}[d_k - a_j - 1][d_k - a_j]} \prod_{l\neq k=1}^{n} \frac{[d_k - d_l - f_l]}{[d_k - d_l]}$$
$$= \left[\sum_{k=1}^{n} f_k + \sum_{j=1}^{s} (b_j + c_j - 2a_j) \right]. \tag{38}$$

Herein, the f_k are equal to $\theta_{k,p-1} - \theta_{kp}$, and since the θ's take only the values 0 and 1, the f_k's take only the values $0, \pm 1$. To prove (38), one again has to use the same technique on a function

$$f(z) = \frac{\Pi_{j=1}^{s}(z - B_j)(z - q^2 C_j)}{z \Pi_{j=1}^{s}(z - A_j)(z - q^2 A_j)} \prod_{k=1}^{n} \frac{(z - F_k D_k)}{(z - D_k)}.$$

However, it turns out that (38) is true as a general identity only when in the third summation the factor (f_k) is replaced by $[f_k]$. In the present case, this can be done without harm since for the values $x = 0, \pm 1$ we have indeed that $[x] = x$. This completes the verification of the Cartan-Kac relations.

For the e-Serre relations, the calculations are extremely lengthy, but when collecting terms with the same Gel'fand-Zetlin basis vector and then taking apart the common factors, the remaining factor always reduces to a simple finite expression which is easily verified to be zero. These expressions always reduce to one of the following (trivial) identities:

$$[a][b + 1] - [a + 1][b] = [a - b], \tag{39}$$

$$[a + 1] + [a - 1] = [2][a], \tag{40}$$

$$\frac{1}{[a - 1][a]} + \frac{1}{[a][a + 1]} = \frac{[2]}{[a - 1][a + 1]}. \tag{41}$$

In fact, the last of these reduces to the second one, and in some sense the only identities needed to prove the e-Serre relations are (39) and (40), and combinations of them. Finally, the calculations for the f-Serre relations are of a similar nature as those for the e-Serre relations.

5. Comments

We have studied the class of essentially typical representations of the quantum superalgebra $U_q[gl(n/m)]$ and connected the relations to be satisfied for these representations with certain q-identities. At present, we do not know how to extend the present results to other finite-dimensional representations of $U_q[gl(n/m)]$. In fact, also in the non-deformed case the problem of how to modify the classical analogs of (24–30) remains an open problem. There is some indication that for a typical representation the only modification would be to simply delete those terms for which the coefficient becomes undefined; however, this is still under investigation and we hope to report results in the future. For atypical representations, the GZ basis will presumably be no longer appropriate: if one still uses the same GZ-patterns in the case that $[m]$ is atypical, it turns out that some $|m\rangle$-vectors have a non-trivial projection both on the maximal submodule and on the quotient module (of the module spanned by the GZ basis vectors with a modified action when the corresponding coefficient is undefined). This property was observed for atypical representations of $gl(2/2)$, and here again further investigations are under way. Note that we also have not examined the representation theory of $U_q[gl(n/m)]$ in the case of q being a root of unity.

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