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Dual Isomonodromic Deformations and Moment Maps to Loop Algebras

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Abstract: The Hamiltonian structure of the monodromy preserving deformation equations of Jimbo *et al* [JMMS] is explained in terms of parameter dependent pairs of moment maps from a symplectic vector space to the dual spaces of two different loop algebras. The nonautonomous Hamiltonian systems generating the deformations are obtained by pulling back spectral invariants on Poisson subspaces consisting of elements that are rational in the loop parameter and identifying the deformation parameters with those determining the moment maps. This construction is shown to lead to "dual" pairs of matrix differential operators whose monodromy is preserved under the same family of deformations. As illustrative examples, involving discrete and continuous reductions, a higher rank generalization of the Hamiltonian equations governing the correlation functions for an impenetrable Bose gas is obtained, as well as dual pairs of isomonodromy representations for the equations of the Painlevé transcendents P_V and P_{VI}.

1. Monodromy Preserving Hamiltonian Systems

The following integrable Pfaffian system was studied by Jimbo, Miwa, Môri and Sato in [JMMS]:

$$dN_i = -\sum_{\substack{j=1\\ j\neq i}}^{n} [N_i, N_j]\, d\log(\alpha_i - \alpha_j) - [N_i, d(\alpha_i Y) + \Theta]\,. \tag{1.1}$$

Here $\{N_i(\alpha_1, \dots, \alpha_n, y_1, \dots, y_r)\}_{i=1,\dots,n}$ is a set of $r \times r$ matrix functions of $n + r$ (real or complex) variables $\{\alpha_i, y_a\}_{\substack{i=1,\dots,n\\ a=1,\dots,r}}$, Y is the diagonal $r \times r$ matrix

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$Y = \mathrm{diag}(y_1, \ldots, y_r)$ and the matrix differential form Θ is defined by:

$$\Theta_{ab} = (1 - \delta_{ab})\left(\sum_{i=1}^{n} N_i\right)_{ab} d\log(y_a - y_b)\,. \tag{1.2}$$

This system determines deformations of the differential operator:

$$\mathscr{D}_\lambda := \frac{\partial}{\partial\lambda} - \mathscr{N}(\lambda)\,, \tag{1.3}$$

where

$$\mathscr{N}(\lambda) := Y + \sum_{i=1}^{n} \frac{N_i}{\lambda - \alpha_i}\,, \tag{1.4}$$

that preserve its monodromy around the regular singular points $\{\lambda = \alpha_i\}_{i=1,\ldots,n}$ and at $\lambda = \infty$.

It was observed in [JMMS] (Appendix 5) that, expressing the N_i's as

$$N_i = G_i^T F_i\,, \tag{1.5}$$

where $\{F_i, G_i \in M^{k_i \times r}\}_{i=1,\ldots,n}$ are pairs of maximal rank $k_i \times r$ rectangular matrices ($k_i \leqq r$), with $\{F_i G_i^T = L_i \in gl(k_i)\}_{i=1,\ldots,n}$ constant matrices related to the monodromy of $\mathscr{D}_\lambda$ at $\{\alpha_i\}_{i=1,\ldots,n}$, Eq. (1.1) may be expressed as a set of compatible nonautonomous Hamiltonian systems:

$$dF_i = \{F_i, \omega\}, \tag{1.6a}$$

$$dG_i = \{G_i, \omega\}. \tag{1.6b}$$

Here d denotes the total differential with respect to the variables $\{\alpha_i, y_a\}_{\substack{i=1,\ldots,n\\ a=1,\ldots,r}}$, the 1-form ω is defined as:

$$\omega := \sum_{i=1}^{n} H_i d\alpha_i + \sum_{a=1}^{r} K_a dy_a\,, \tag{1.7}$$

with

$$H_i := \mathrm{tr}(YN_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\mathrm{tr}(N_i N_j)}{\alpha_i - \alpha_j}, \quad i = 1, \ldots, n\,, \tag{1.8a}$$

$$K_a := \sum_{i=1}^{n} \alpha_i (N_i)_{aa} + \sum_{\substack{b=1\\ b\neq a}}^{r} \frac{\left(\sum_{i=1}^{n} N_i\right)_{ab}\left(\sum_{j=1}^{n} N_j\right)_{ba}}{y_a - y_b}, \quad a = 1, \ldots, r\,, \tag{1.8b}$$

and the Poisson brackets in the space of (F_i, G_i)'s are defined to be such that the matrix elements of $\{F_i, G_i\}_{i=1,\ldots,n}$ are canonically conjugate:

$$\{(F_i)_{a_i a}, (G_j)_{b_j b}\} = \delta_{ij}\delta_{a_i b_j}\delta_{ab}\,,$$

$$i, j = 1, \ldots, n, \quad a, b = 1, \ldots, r, \quad a_i, b_i = 1, \ldots, k_i\,. \tag{1.9}$$

The Frobenius integrability of the differential system (1.1) follows from the fact that the Hamiltonians $\{H_i, K_a\}_{\substack{i=1,\dots,n\\ a=1,\dots,r}}$ Poisson commute. It was also noted in [JMMS] that, with respect to the Poisson brackets (1.9), the matrices $\{N_i\}_{i=1,\dots,n}$ satisfy:

$$\{(N_i)_{ab}, (N_j)_{cd}\} = \delta_{ij}\,(\delta_{bc}(N_i)_{ad} - \delta_{ad}(N_i)_{cd})\,, \tag{1.10}$$

which is the Lie Poisson bracket on the space $(\bigoplus_{i=1}^n gl(r))^*$ dual to the direct sum of n copies of the Lie algebra $gl(r)$ with itself. The 1-form ω is exact on the parameter space and may be interpreted as the logarithmic differential of the τ-function,

$$\omega = d\log\tau\,. \tag{1.11}$$

Numerous applications of such systems exist; in particular, in the calculation of correlation functions for integrable models in statistical mechanics and quantum field theory [JMMS, IIKS], in matrix models of two dimensional quantum gravity [M] and in the computation of level spacing distribution functions for random matrix ensembles [TW1, TW2].

In the next section these systems will be examined within the context of loop algebras, using an approach originally developed for the autonomous case, involving isospectral flows, in [AHP, AHH1]. This is based on "dual" pairs of parameter dependent moment maps from symplectic vector spaces to two different loop algebras $\widetilde{gl}(r)$ and $\widetilde{gl}(N)$, where $N = \sum_{i=1}^n k_i$. The nonautonomous Hamiltonian systems (1.6)–(1.8) will be generated by pulling back certain spectral invariants, viewed as polynomial functions on rational coadjoint orbits, under these moment maps, and identifying the parameters determining the maps with the deformation parameters of the system. This construction leads to a pair of "dual" first order matrix differential operators with regular singular points at finite values of the spectral parameter, both of whose monodromy data are invariant under the deformations generated by these Hamiltonian systems. In Sect. 3, the generic systems so obtained will be reduced under various discrete and continuous Hamiltonian symmetry groups. A rank $r = 2s$ generalization of the systems determining the correlation functions for an impenetrable Bose gas (or equivalently, the generating function for the level spacing distribution functions for random matrix ensembles [TW1]) will be derived by reduction to the symplectic loop algebra $\widetilde{sp}(2s)$. The "dual" isomonodromy representations of the equations for the Painlevé transcendents P_V and P_{VI} will also be derived, and their Hamiltonian structure deduced through reductions under continuous groups. A brief discussion of generalizations to systems with irregular singular points is given in Sect. 4.

2. Loop Algebra Moment Maps and Spectral Invariants

In [AHP, AHH1], an approach to the embedding of finite dimensional integrable systems into rational coadjoint orbits of loop algebras was developed, based on a parametric family of equivariant moment maps $\tilde{J}_A: M \to \widetilde{gl}(r)^*_+$ from the space

$M = \{F, G \in M^{N\times r}\}$ of pairs of $N \times r$ rectangular matrices, with canonical symplectic structure

$$\omega = \operatorname{tr}(dF \wedge dG^T) \tag{2.1}$$

to the dual of the positive half of the loop algebra $\widetilde{gl}(r)$. The maps $\tilde{J}_A$, which are parametrized by the choice of an $N \times N$ matrix $A \in M^{N\times N}$ with eigenvalues $\{\alpha_i\}_{i=1,\dots,n}$ and generalized eigenspaces of dimension $\{k_i\}_{i=1,\dots,n}$, $\sum_{i=1}^n k_i = N$, are defined by:

$$\tilde{J}_A : (F, G) \mapsto G^T(A - \lambda I_N)^{-1} F, \tag{2.2}$$

where I_N denotes the $N \times N$ identity matrix. The conventions here are such that all the eigenvalues $\{\alpha_i\}_{i=1,\dots,n}$ are interior to a circle S^1 in the complex λ-plane on which the loop algebra elements $X(\lambda) \in \widetilde{gl}(r)$ are defined. The two subalgebras $\widetilde{gl}(r)_+$, $\widetilde{gl}(r)_-$ consist of elements $X_+ \in \widetilde{gl}(r)_+$, $X_- \in \widetilde{gl}(r)_-$ that admit holomorphic extensions, respectively, to the interior and exterior regions, with $X_-(\infty) = 0$. The space $\widetilde{gl}(r)$ is identified as a dense subspace of its dual space $\widetilde{gl}(r)^*$, through the pairing

$$\langle X_1, X_2 \rangle = \oint_{S^1} \operatorname{tr}(X_1(\lambda) X_2(\lambda)) d\lambda\,,$$

$$X_1 \in \widetilde{gl}(r)^*,\ X_2 \in \widetilde{gl}(r)\,. \tag{2.3}$$

This also gives identifications of the dual spaces $\widetilde{gl}(r)^*_\pm$ with the opposite subalgebras $\widetilde{gl}(r)_\mp$.

Taking the simplest case, when A is diagonal, the image of the moment map is

$$\mathcal{N}_0(\lambda) = G^T(A - \lambda I_N)^{-1} F = \sum_{i=1}^n \frac{N_i}{\lambda - \alpha_i}\,, \tag{2.4a}$$

$$N_i := -G_i^T F_i\,, \tag{2.4b}$$

where (F_i, G_i) are the $k_i \times r$ blocks in (F, G) corresponding to the eigenspaces of A with eigenvalues $\{\alpha_i\}_{i=1,\dots,n}$. The set of all $\mathcal{N}_0 \in \widetilde{gl}(r)_-$ having the pole structure given in Eq. (2.4a) forms a Poisson subspace of $\widetilde{gl}(r)_-$, which we denote $\mathbf{g}_A$. The coadjoint action of the loop group $\widetilde{Gl}(r)_+$ corresponding to the algebra $\widetilde{gl}(r)_+$, restricted to the subspace $\mathbf{g}_A$, is given by:

$$\operatorname{Ad}^*(\widetilde{Gl}(r)_+) : \mathbf{g}_A \to \mathbf{g}_A\,,$$

$$\operatorname{Ad}^*(g) : \sum_{i=1}^n \frac{N_i}{\lambda - \alpha_i} \to \sum_{i=1}^n \frac{g_i N_i g_i^{-1}}{\lambda - \alpha_i}\,,$$

$$g_i := g(\alpha_i),\quad i = 1, \dots, n\,. \tag{2.5}$$

We see that $\mathbf{g}_A$ could equally have been identified with the dual space $(\bigoplus_{j=1}^n gl(r))^*$ of the direct sum of n copies of $gl(r)$ with itself, and the Lie Poisson bracket on $\widetilde{gl}(r)^*_+ \sim \widetilde{gl}(r)_-$:

$$\{f, g\}|_{\mathcal{N}_0} = \langle \mathcal{N}_0, [df, dg]|_{\mathcal{N}_0} \rangle\,, \tag{2.6}$$

reduces on the Poisson subspace $\mathbf{g}_A$ to that for $(\bigoplus_{j=1}^n gl(r))^*$, as given in Eq. (1.10).

In the approach developed in [AHP, AHH1], one studies commuting Hamiltonian flows on spaces of type $\mathbf{g}_A$ (in general, rational Poisson subspaces involving higher order poles if the matrix A is nondiagonalizable), generated by elements of the Poisson commuting spectral ring $\mathscr{I}_A^Y$ of polynomials on $\widetilde{gl}(r)^*$ invariant under the $\mathrm{Ad}^*\widetilde{Gl}(r)$-action (conjugation by loop group elements), restricted to the affine subspace $Y + \mathbf{g}_A$, where $Y \in gl(r)$ is some fixed $r \times r$ matrix. The pullback of such Hamiltonians under $\tilde{J}_A$ generates commuting flows in M that project to the quotient of M by the Hamiltonian action of the stability subgroup $G_A := \mathrm{Stab}(A) \subset Gl(N)$. The Adler–Kostant–Symes (AKS) theorem then tells us that:

(i) Any two elements of $\mathscr{I}_A^Y$ Poisson commute (and hence, so do their pullbacks under the Poisson map $\tilde{J}_A$).
(ii) Hamilton's equations for $H \in \mathscr{I}_A^Y$ have the Lax pair form:

$$\frac{d\mathscr{N}}{dt} = [(dH)_+, \mathscr{N}] = -[(dH)_-, \mathscr{N}], \tag{2.7}$$

where

$$\mathscr{N}(\lambda, t) := Y + \mathscr{N}_0(\lambda, t), \tag{2.8}$$

with $\mathscr{N}_0 \in \widetilde{gl}(r)_-$ of the form (2.4a), $dH|_{\mathscr{N}}$ viewed as an element of $(\widetilde{gl}(r)^*)^* \sim \widetilde{gl}(r)$, and the subscripts $\pm$ denoting projections to the subspace $\widetilde{gl}(r)_\pm$.

The coefficients of the spectral curve of $\mathscr{N}(\lambda)$, determined by the characteristic equation

$$\det(Y + G^T(A - \lambda I_N)^{-1} - zI_r) = 0, \tag{2.9}$$

are the generators of the ring $\mathscr{I}_A^Y$.

In particular, choosing

$$Y := \mathrm{diag}(y_1, \ldots, y_r) \tag{2.10}$$

and defining $\{H_i \in \mathscr{I}_A^Y\}_{i=1,\ldots n}$ by:

$$H_i(\mathscr{N}) := \frac{1}{4\pi i} \oint_{\lambda=\alpha_i} \mathrm{tr}(\mathscr{N}(\lambda))^2 d\lambda = \mathrm{tr}(YN_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{tr(N_iN_j)}{\alpha_i - \alpha_j} \tag{2.11}$$

(where $\oint_{\lambda=\alpha_i}$ denotes integration around a small loop containing only this pole), we see that these coincide with the H_i's defined in Eq. (1.8a). Thus, the Poisson commutativity of the H_i's follows from the AKS theorem. The Lax form of Hamilton's equations is:

$$\frac{\partial\mathscr{N}}{\partial t_i} = -[(dH_i)_-, \mathscr{N}] \tag{2.12}$$

where

$$(dH_i)_- = \frac{N_i}{\lambda - \alpha_i} \in \widetilde{gl}(r)_- . \tag{2.13}$$

Evaluating residues at each $\lambda = \alpha_i$, we see that this is equivalent to:

$$\frac{\partial N_j}{\partial t_i} = \frac{[N_j, N_i]}{\alpha_j - \alpha_i}, \quad j \neq i, \quad i, j = 1, \ldots n . \tag{2.14a}$$

$$\frac{\partial N_i}{\partial t_i} = \left[Y + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{N_j}{\alpha_i - \alpha_j}, N_i \right] . \tag{2.14b}$$

If we now identify the flow parameters $\{t_i\}_{i=1,\ldots,n}$ with the eigenvalues $\{\alpha_i\}_{i=1,\ldots,n}$, we obtain the nonautonomous Hamiltonian systems

$$\frac{\partial N_j}{\partial \alpha_i} = \frac{[N_j, N_i]}{\alpha_j - \alpha_i}, \quad j \neq i, \quad i, j = 1, \ldots, n , \tag{2.15a}$$

$$\frac{\partial N_i}{\partial \alpha_i} = \left[Y + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{N_j}{\alpha_i - \alpha_j}, N_i \right] , \tag{2.15b}$$

which are the α_i components of the system (1.1). Viewing the N_i's as functions on the fixed phase space M, Eqs. (2.15a, b) are induced by the nonautonomous Hamiltonian systems generated by the pullback of the H_i's under the parameter dependent moment map $\tilde{J}_A$. Equations (2.15a, b) are equivalent to replacing the Lax equations (2.12) by the system:

$$\frac{\partial \mathcal{N}}{\partial \alpha_i} = -[(dH_i)_-, \mathcal{N}] - \frac{\partial (dH_i)_-}{\partial \lambda} , \tag{2.16}$$

which is just the condition of commutativity of the system of operators $\{\mathcal{D}_\lambda, \mathcal{D}_i\}_{i=1,\ldots,n}$, where $\mathcal{D}_\lambda$ is given by (1.3) and

$$\mathcal{D}_i := \frac{\partial}{\partial \alpha_i} + (dH_i)_-(\lambda) = \frac{\partial}{\partial \alpha_i} + \frac{N_i}{\lambda - \alpha_i} . \tag{2.17}$$

Remark. The system (2.16) could also be viewed as a Lax equation defined on the dual of the centrally extended loop algebra $\widetilde{gl}(r)^\wedge$, in which the Ad* action is given by gauge transformations rather than conjugation [RS]. The analogue of the spectral ring $\mathscr{I}^Y_A$ is the ring of monodromy invariants, restricted to a suitable Poisson subspace with respect to a modified (R-matrix) Lie Poisson bracket structure. This viewpoint will not be developed here, but is essential to deriving such systems through reductions of autonomous Hamiltonian systems of PDE's.

The fact that the matrices $\{F_i G_i^T = L_i \in gl(k_i)\}_{i=1,\ldots,n}$ are constant under the deformations generated by Eqs. (2.15a, b), (2.16) follows from the fact that

$$J_{G_A}(F, G) := \mathrm{diag}(F_1 G_1^T, \ldots, F_n G_n^T) \in gl(N) \tag{2.18}$$

is the moment map generating the Hamiltonian action of the stabilizer of A in $Gl(N)$:

$$G_A := \prod_{i=1}^{N} Gl(k_i) = \mathrm{Stab}(A) \subset Gl(N) , \tag{2.19}$$

this action being given by

$$G_A : M \to M\,,$$
$$K : (F, G) \mapsto (KF, (K^T)^{-1} G)\,,$$
$$K = \mathrm{diag}(K_1, \ldots, K_n) \in G_A, \quad K_i \in Gl(k_i)\,. \tag{2.20}$$

The orbits of G_A are just the fibres of $\tilde{J}_A$, so $(\tilde{J}_A, J_{G_A})$ form a "dual pair" of moment maps [W]. Evidently, the pullback $\tilde{J}_A^*(H)$ is G_A- invariant for all $H \in \mathscr{I}_A^Y$, and hence J_{G_A} is constant under the H_i flows.

So far, we have only considered the part of the system (1.1) relating to the parameters $\{\alpha_i\}_{i=1,\ldots,n}$. What about the Hamiltonians $\{K_a\}_{a=1,\ldots,r}$ that generate the y_a components? As shown in [AHH1], besides $\tilde{J}_A$ there is, for each $Y \in gl(r)$, another moment map

$$\tilde{J}_Y : M \to \widetilde{gl}(N)_+^* \sim \widetilde{gl}(N)_-$$
$$\tilde{J}_Y : (F, G) \mapsto -F(Y - zI_r)^{-1} G^T\,, \tag{2.21}$$

where z denotes the loop parameter for the loop algebra $\widetilde{gl}(N)$, whose elements are defined on a circle S^1 in the complex z-plane containing the eigenvalues of Y in its interior. The pairing identifying $\widetilde{gl}(N)$ as a dense subspace of $\widetilde{gl}(N)^*$ is defined similarly to (2.3), for elements $X_1 \in \widetilde{gl}(N)^*$, $X_2 \in \widetilde{gl}(N)$. The subalgebras $\widetilde{gl}(N)_\pm$ are similarly defined with respect to this circle, and their dual spaces $\widetilde{gl}(N)_\pm^*$ are identified analogously with $\widetilde{gl}(N)_\mp$.

The moment map $\tilde{J}_Y$ is also "dual" to $\tilde{J}_A$, but in a different sense than J_{G_A} – one that is relevant for the construction of the remaining Hamiltonians $\{K_a\}_{a=1,\ldots,r}$. The space $\mathbf{g}_A$ may be identified with the quotient Poisson manifold M/G_A, with symplectic leaves given by the level sets of the symmetric invariants formed from each $F_i G_i^T$, since these are the Casimir invariants on $\mathbf{g}_A$. Since the Hamiltonians in $\mathscr{I}_A^Y$ are all also invariant under the action of the stabilizer $G_Y = \mathrm{Stab}(Y) \subset Gl(r)$, where $Gl(r) \subset \widetilde{Gl}(r)$ is the subgroup of constant loops, we may also quotient by this action to obtain $\mathbf{g}_A/G_Y \simeq M/(G_Y \times G_A)$. Doing this in the opposite order, we may define $\mathbf{g}_Y \subset \widetilde{gl}(N)_+^* \sim \widetilde{gl}(N)_-$ as the Poisson subspace consisting of elements of the form:

$$\mathscr{M}_0(z) = -F(Y - zI_N)^{-1} G^T = \sum_{a=1}^{r} \frac{M_a}{z - y_a}\,, \tag{2.22a}$$

$$(M_a)_{ij} := F_{ia} G_{ja}, \quad i, j = 1, \ldots, n, \quad a = 1, \ldots, r \tag{2.22b}$$

(where, if the $\{y_a\}_{a=1,\ldots,r}$ are distinct, the residue matrices are all of rank 1), and identify $\mathbf{g}_Y$ with M/G_Y. Denoting by $\mathscr{I}_Y^A$ the ring of Ad*-invariant polynomials on $\widetilde{gl}(N)^*$, restricted to the affine subspace $-A + \mathbf{g}_Y$ consisting of elements of the form

$$\mathscr{M} = -A + \mathscr{M}_0, \ \mathscr{M}_0 \in \mathbf{g}_Y, \tag{2.23}$$

the pullback of the ring $\mathscr{I}_Y^A$ under the moment map $\tilde{J}_Y$ also gives a Poisson commuting ring whose elements are both G_Y and G_A invariant, and hence project to $M/(G_Y \times G_A)$. In fact, the two rings $\tilde{J}_A^*(\mathscr{I}_A^Y)$ and $\tilde{J}_Y^*(\mathscr{I}_Y^A)$ *coincide* (cf. [AHH1]), because of the identity:

$$\det(A - \lambda I_N)\det(Y + G^T(A - \lambda I_N)^{-1}F - zI_r) = \det(Y - zI_r)\det(A + F(Y - zI_r)^{-1}G^T - \lambda I_N)\,, \tag{2.24}$$

which shows that the spectral curve of $\mathcal{M}(z)$, defined by

$$\det(A + F(Y - zI_r)^{-1}G^T - \lambda I_N) = 0 \tag{2.25}$$

and that of $\mathcal{N}(\lambda)$, given by Eq. (2.9), are birationally equivalent (after reducing out the trivial factors $\det(A - \lambda I_N)$ and $\det(Y - zI_r)$).

Now, similarly to the definition of the elements $\{H_i \in \mathcal{I}^Y_A\}_{i=1,\ldots,n}$, we may define $\{K_a \in \mathcal{I}^A_Y\}_{a=1,\ldots,r}$ as:

$$K_a := \frac{1}{4\pi i}\oint_{z=y_a} \operatorname{tr}(\mathcal{M}(z))^2\,dz = -\operatorname{tr}(AM_a) + \sum_{\substack{b=1\\ b\neq a}}^{r} \frac{\operatorname{tr}(M_aM_b)}{y_a - y_b}\,. \tag{2.26}$$

To verify that these coincide with the K_a's defined in Eq. (1.8b), we use Eqs. (2.4a), (2.8), (2.22a) and (2.23) to express K_a as:

$$K_a = \frac{1}{4\pi i}\oint_{z=y_a} dz \frac{1}{2\pi i}\oint_{S^1} d\lambda\lambda \operatorname{tr}(\mathcal{M}(z)(A - \lambda I_N)^{-1})^2 \tag{2.27a}$$

$$= \frac{1}{4\pi i}\oint_{S^1} d\lambda\lambda \frac{1}{2\pi i}\oint_{z=y_a} dz[\operatorname{tr}((Y - zI_r)^{-1}\mathcal{N}(\lambda))^2 - 2\operatorname{tr}((Y - zI_r)^{-1}\mathcal{N}(\lambda))]\,. \tag{2.27b}$$

Evaluating residues at $\{y_a\}_{a=1,\ldots,r}$ in z and at ∞ in λ gives (1.8b). The Poisson commutativity of the K_a's again follows from the AKS theorem, and the commutativity with the H_i's follows from the equality of the two rings $\tilde{J}^*_A\mathcal{I}^Y_A = \tilde{J}^*_Y\mathcal{I}^A_Y$. To compute the Lax form of the equations of motion generated by the K_a's, we evaluate their differentials, viewing them as functions of $\mathcal{N}(\lambda)$ defined by Eq. (2.27b). Evaluating the z integral, this gives:

$$dK_a(\lambda) = \lambda E_a + \sum_{\substack{b=1\\ b\neq a}}^{r} \frac{E_a\mathcal{N}(\lambda)E_b + E_b\mathcal{N}(\lambda)E_a}{y_a - y_b} \in \widetilde{gl}(r)\,, \tag{2.28}$$

where E_a denotes the elementary diagonal $r \times r$ matrix with (aa) entry equal to 1 and zeros elsewhere. Taking the projection to $\widetilde{gl}(r)_+$ gives:

$$(dK_a)_+(\lambda) = \lambda E_a + \sum_{\substack{b=1\\ b\neq a}}^{r}\sum_{i=1}^{n} \frac{E_aN_iE_b + E_bN_iE_a}{y_a - y_b} \in \widetilde{gl}(r)_+\,, \tag{2.29}$$

and hence

$$\sum_{a=1}^{r} (dK_a)_+ dy_a = \lambda Y + \Theta\,, \tag{2.30}$$

where Θ is defined in Eq. (1.2). By the AKS theorem, the autonomous form of the equations of motion is

$$\frac{\partial \mathcal{N}}{\partial \tau_a} = [(dK_a)_+, \mathcal{N}]\,, \tag{2.31}$$

while the nonautonomous version is

$$\frac{\partial \mathcal{N}}{\partial y_a} = [(dK_a)_+, \mathcal{N}] + \frac{\partial (dK_a)_+}{\partial \lambda} = [(dK_a)_+, \mathcal{N}] + E_a . \tag{2.32}$$

Evaluating residues at $\lambda = \alpha_i$ gives the equations

$$\frac{\partial N_i}{\partial \tau_a} = [(dK_a)_+(\alpha_i), N_i], \quad i = 1, \ldots, n \tag{2.33}$$

for the autonomous case and

$$\frac{\partial N_i}{\partial y_a} = [(dK_a)_+(\alpha_i), N_i], \quad i = 1, \ldots, n \tag{2.34}$$

for the nonautonomous one. Equations (2.34) are just the y_a components of the system (1.1). Equation (2.32) is equivalent to the commutativity of the operators $\{\mathscr{D}_\lambda, \mathscr{D}_a^*\}_{a=1,\ldots,r}$, where

$$\mathscr{D}_a^* := \frac{\partial}{\partial y_a} - (dK_a)_+(\lambda) , \tag{2.35}$$

and implies that the monodromy of $\mathscr{D}_\lambda$ is invariant under the y_a deformations. In fact, it follows from the AKS theorem that the complete system of operators $\{\mathscr{D}_\lambda, \mathscr{D}_i, \mathscr{D}_a^*\}_{i=1,\ldots,n, a=1,\ldots,r}$ commutes.

Turning now to the dual system, it follows from the AKS theorem that the Lax form of the equations of motion induced by the K_a's on $\widetilde{gl}(N)_-$, viewed now as functions of $\mathscr{M}$, in the autonomous case is

$$\frac{\partial \mathscr{M}}{\partial \tau_a} = -[(dK_a)_-, \mathscr{M}] , \tag{2.36}$$

where

$$(dK_a)_-(z) = \frac{M_a}{z - y_a} \in \widetilde{gl}(N)_- . \tag{2.37}$$

(Note that the differential dK_a entering in Eqs. (2.36), (2.37) and below has a different significance from that appearing in Eqs. (2.28)–(2.35).) Evaluating residues at $z = y_a$ shows that (2.36) is equivalent to the system

$$\frac{\partial M_b}{\partial \tau_a} = \frac{[M_b, M_a]}{y_b - y_a}, \quad b \neq a, \quad a, b = 1, \ldots r , \tag{2.38a}$$

$$\frac{\partial M_a}{\partial \tau_a} = \left[-A + \sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{M_b}{y_a - y_b}, M_a \right] . \tag{2.38b}$$

Identifying the flow parameters $\{\tau_a\}_{a=1,\ldots,r}$ now with the eigenvalues $\{y_a\}_{a=1,\ldots,r}$ of Y gives the nonautonomous Hamiltonian system

$$\frac{\partial M_b}{\partial y_a} = \frac{[M_b, M_a]}{y_b - y_a}, \quad b \neq a, \quad a, b = 1, \ldots r, \tag{2.39a}$$

$$\frac{\partial M_a}{\partial y_a} = \left[-A + \sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{M_b}{y_a - y_b}, M_a \right], \tag{2.39b}$$

or, equivalently,

$$\frac{\partial \mathcal{M}}{\partial y_a} = -[(dK_a)_-, \mathcal{M}] - \frac{\partial (dK_a)_-}{\partial z}. \tag{2.40}$$

Equations (2.39a, b), (2.40) are equivalent to the commutativity of the system of operators $\{\mathscr{D}_z, \mathscr{D}_a\}_{a=1,\ldots,r}$ defined by:

$$\mathscr{D}_z := \frac{\partial}{\partial z} - \mathcal{M}(z), \tag{2.41a}$$

$$\mathscr{D}_a := \frac{\partial}{\partial y_a} + (dK_a)_-(z) = \frac{\partial}{\partial y_a} + \frac{M_a}{z - y_a}, \quad a = 1, \ldots, r. \tag{2.41b}$$

Thus the monodromy of the "dual" operator $\mathscr{D}_z$ is also preserved under the y_a deformations.

Finally, by similar computations to the above, with $(A, \lambda, \mathcal{N})$ and $(Y, z, \mathcal{M})$ interchanged, we can express the H_i's as functions of $\mathcal{M}$:

$$H_i = \frac{1}{4\pi i} \oint_{S^1} dz z \frac{1}{2\pi i} \oint_{\lambda = \alpha_i} d\lambda [\mathrm{tr}((A - \lambda I_N)^{-1} \mathcal{M}(z))^2 - 2\mathrm{tr}((A - \lambda I_N)^{-1} \mathcal{M}(z))]. \tag{2.42}$$

Evaluating residues at $\{\alpha_j\}_{j=1,\ldots,n}$ in λ and at ∞ in z gives

$$H_i = -\sum_{a=1}^{r} (M_a)_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(\sum_{a=1}^{r} (M_a)_{ij})(\sum_{b=1}^{r} (M_b)_{ji})}{\alpha_i - \alpha_j}, \quad i = 1, \ldots, n. \tag{2.43}$$

To compute the Lax equations for $\mathcal{M}(z)$ generated by the H_i's, we evaluate their differentials when viewed as functions on $\widetilde{gl}(N)_-$ defined by Eq. (2.42). Evaluating the λ integral, this gives:

$$dH_i(z) = -zE_i + \sum_{\substack{j=1 \\ j \neq i}}^{r} \frac{E_i \mathcal{M}(z) E_j + E_j \mathcal{M}(z) E_i}{\alpha_i - \alpha_j} \in \widetilde{gl}(N), \tag{2.44}$$

where E_i now denotes the elementary diagonal $N \times N$ matrix with (ii) entry equal to 1 and zeros elsewhere. Taking the projection to $\widetilde{gl}(N)_+$ gives:

$$(dH_i)_+(z) = -zE_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{a=1}^{r} \frac{E_i M_a E_j + E_j M_a E_i}{\alpha_i - \alpha_j} \in \widetilde{gl}(N)_+, \tag{2.45}$$

and hence

$$\sum_{i=1}^{n} (dH_i)_+(z)d\alpha_i = -zdA + \Phi \,, \tag{2.46}$$

where

$$\Phi_{ij} = (1 - \delta_{ij})\left(\sum_{a=1}^{r} (M_a)_{ij} d \log(\alpha_i - \alpha_j) \right). \tag{2.47}$$

By the AKS theorem, the autonomous form of the equations of motion is

$$\frac{\partial \mathcal{M}}{\partial t_i} = [(dH_i)_+, \mathcal{M}] \,, \tag{2.48}$$

while the nonautonomous system is

$$\frac{\partial \mathcal{M}}{\partial \alpha_i} = [(dH_i)_+, \mathcal{M}] + \frac{\partial (dH_i)_+}{\partial z} = [(dH_i)_+, \mathcal{M}] - E_i \,. \tag{2.49}$$

Evaluating residues at $\lambda = \alpha_i$ gives the equations

$$\frac{\partial M_a}{\partial t_i} = [(dH_i)_+(y_a), M_a] \tag{2.50}$$

for the autonomous case and

$$\frac{\partial M_a}{\partial \alpha_i} = [(dH_i)_+(y_a), M_a] \tag{2.51}$$

for the nonautonomous one. Equation (2.49) is equivalent to the commutativity of the operators $\{\mathcal{D}_z, \mathcal{D}_i^*\}_{i=1,\ldots,n}$, where

$$\mathcal{D}_i^* := \frac{\partial}{\partial \alpha_i} - (dH_i)_+(z) \,, \tag{2.52}$$

and implies that the monodromy of $\mathcal{D}_z$ is invariant under the α_i deformations. Again, it follows from the AKS theorem that the complete system of operators $\{\mathcal{D}_z, \mathcal{D}_a, \mathcal{D}_i^*\}_{a=1,\ldots,r,\, i=1,\ldots,n}$ commutes.

Thus, at the level of the reduced spaces $\mathbf{g}_A/G_Y \sim \mathbf{g}_Y/G_A$, we have two equivalent "dual" isomonodromy representations of the Hamiltonian systems generated by the spectral invariants $\{H_i, K_a\}_{i=1,\ldots,n,\, a=1,\ldots,r}$-systems (2.15a, b), (2.24) in the $\mathcal{N}(\lambda) \in \mathbf{g}_A$ representation and (2.39a, b), (2.51) in the $\mathcal{M}(z) \in \mathbf{g}_Y$ representation.

3. Reductions

The general scheme of [AHP, AHH1] may be combined with continuous or discrete Hamiltonian symmetry reductions to deduce systems corresponding to subalgebras of $\widetilde{gl}(r)$ and $\widetilde{gl}(N)$ or, more generally, to invariant submanifolds. In particular, the Marsden–Weinstein reduction method may be applied to the symmetry groups G_A and G_Y, or to other invariants of the system.

The discrete reduction method (see [AHP, HHM] for further details) may be summarized as follows. Suppose $\tilde{\sigma}_{r_*}: \widetilde{gl}(r)_+ \to \widetilde{gl}(r)_+$ is a Lie algebra homomorphism that is semisimple, of finite order and induced by the group homomorphism

$\tilde{\sigma}_r : \widetilde{Gl}(r)_+ \to \widetilde{Gl}(r)_+$. Let $\tilde{\sigma}_r^* : \widetilde{gl}(r)_- \to \widetilde{gl}(r)_-$ denote the dual map, which is a Poisson homomorphism, and let $h_\sigma \subset \widetilde{gl}(r)_+$, $h_\sigma^* \subset \widetilde{gl}(r)_-$ denote the subspaces consisting of the fixed point sets under $\tilde{\sigma}_{r_*}$ and $\tilde{\sigma}_r^*$, respectively. (These are naturally dual to each other, since h_σ^* may be identified with the annihilator of the complement of h_σ under the decomposition of $\widetilde{gl}(r)_+$ into eigenspaces of $\tilde{\sigma}_{r_*}$.) Then $h_\sigma \subset \widetilde{gl}(r)_+$ is a subalgebra, and its dual space h_σ^* has the corresponding Lie Poisson structure. Suppose there also exists a finite order symplectomorphism $\sigma_M : M \to M$ such that the moment map $\tilde{J}_A$ satisfies the intertwining property:

$$\tilde{J}_A \circ \sigma_M = \tilde{\sigma}_r^* \circ \tilde{J}_A . \tag{3.1}$$

The fixed point set $M_\sigma \subset M$ is, generally, a symplectic submanifold, invariant under the flows generated by σ_M-invariant Hamiltonians. The restriction $\tilde{J}_A|_{M_\sigma} := \tilde{J}_{A\sigma}$ takes its values in h_σ^*, defining a Poisson map:

$$\tilde{J}_{A\sigma} : M_\sigma \to h_\sigma^* , \tag{3.2}$$

which is the equivariant moment map generating the action of the subgroup $H_\sigma \subset \widetilde{Gl}(r)_+$ consisting of the fixed points under $\tilde{\sigma}_r : \widetilde{Gl}(r)_+ \to \widetilde{Gl}(r)_+$. Such reductions, when applied to the spectral invariants on $Y + \mathbf{g}_A$ generate systems satisfying the criteria of the AKS theorem, provided the matrices Y and A are appropriately chosen to be compatible with the reduction. The same procedure may be applied to the dual systems on $-A + \mathbf{g}_Y$ if a similar homomorphism $\tilde{\sigma}_{N*} : \widetilde{gl}(N)_+ \to \widetilde{gl}(N)_+$ exists, satisfying the intertwining property:

$$\tilde{J}_Y \circ \sigma_M = \tilde{\sigma}_N^* \circ \tilde{J}_Y . \tag{3.3}$$

The corresponding moment map,

$$\tilde{J}_{Y\sigma} : M_\sigma \to k_\sigma^* , \tag{3.4}$$

obtained by restriction $\tilde{J}_{Y\sigma} := \tilde{J}_Y|_{M_\sigma}$ takes its values in the Poisson subspace $k_\sigma^* \subset \widetilde{gl}(N)_-$ consisting of the fixed point set under the dual map $\tilde{\sigma}_N^* : \widetilde{gl}(N)_- \to \widetilde{gl}(N)_-$, and $k_\sigma \subset \widetilde{gl}(N)_+$ is the corresponding subalgebra consisting of fixed points under $\tilde{\sigma}_{N*}$.

The following examples illustrate both the discrete and continuous reduction procedures.

3a. Symplectic Reduction (discrete). Let $r = 2s$ and define $\tilde{\sigma}_r^*$, $\tilde{\sigma}_N^*$ and σ_M by:

$$\tilde{\sigma}_r^* : X(\lambda) \to JX^T(\lambda)J , \tag{3.5a}$$

$$\tilde{\sigma}_N^* : \xi(z) \to \xi^T(-z) , \tag{3.5b}$$

$$\sigma_M : (F, G) \to J(GJ, -FJ) , \tag{3.5c}$$

$$X \in \widetilde{gl}(r)_- , \quad \xi \in \widetilde{gl}(N)_- , \quad (F, G) \in M ,$$

where

$$J = \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix} \in M^{2s \times 2s} . \tag{3.6}$$

Then $M_\sigma \subset M$ consists of pairs (F, G) of the form:

$$F = \frac{1}{\sqrt{2}}(Q, P), \quad G = \frac{1}{\sqrt{2}}(P, -Q), \tag{3.7}$$

where $Q, P \in M^{N\times s}$ are $N \times s$ matrices. The blocks $\{(F_i, G_i)\}_{i=1,\dots,n}$ corresponding to the eigenvalues $\{\alpha_i\}_{i=1,\dots,n}$ are similarly of the form

$$F_i = \frac{1}{\sqrt{2}}(Q_i, P_i), \quad G_i = \frac{1}{\sqrt{2}}(P_i, -Q_i), \tag{3.8}$$

where $Q_i, P_i \in M^{k_i\times s}$. The reduced symplectic form on M_σ is

$$\omega = \mathrm{tr}(dQ \wedge dP^T). \tag{3.9}$$

The subalgebra $h_\sigma \subset \widetilde{gl}(r)_+$ is just the positive part $\widetilde{sp}(2s)_+$ of the symplectic loop algebra $\widetilde{sp}(2s)$, and the dual space h_σ^* is similarly identified with $\widetilde{sp}(2s)_-$. The subalgebra $k_\sigma \subset \widetilde{gl}(N)_+$ is the positive part $\widetilde{gl}^{(2)}(N)_+$ of the "twisted" loop algebra $\widetilde{gl}^{(2)}(N)$, with dual space $k_s^* \sim \widetilde{gl}^{(2)}(N)_-$. The image of the moment map $\tilde{J}_{A\sigma}$ has the form

$$\mathcal{N}_0(\lambda) = \tilde{J}_{A\sigma}(Q, P) = \sum_{i=1}^n \frac{\begin{pmatrix} -P_i^T Q_i & -P_i^T P_i \\ Q_i^T Q_i & Q_i^T P_i \end{pmatrix}}{\lambda - \alpha_i}. \tag{3.10}$$

In order that the pullback of the elements of the ring $\mathcal{I}_A^Y$ under $\tilde{J}_A$ be σ_M-invariant, and that the intertwining property (3.3) for the moment map $\tilde{J}_Y$ be satisfied, the matrix Y must be in $sp(2s)$. For diagonal Y, this means

$$Y = \begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix},$$

$$y = \mathrm{diag}(y_1, \dots, y_s) \in gl(s). \tag{3.11}$$

The image of $\tilde{J}_{Y\sigma}$ then has the block form

$$\mathcal{M}_0(z) = \tilde{J}_{Y\sigma}(Q, P) = \frac{1}{2}\sum_{a=1}^r \left(\frac{q_a p_a^T}{z - y_a} - \frac{p_a q_a^T}{z + y_a}\right), \tag{3.12}$$

where $\{q_a\}_{a=1,\dots,r}$ and $\{p_a\}_{a=1,\dots,r}$ denote the a^{th} columns of Q and P, respectively.

The Hamiltonians $\{H_i\}_{i=1,\dots,n}$ reduce in this case to:

$$H_i = -\mathrm{tr}(yP_i^T Q_i) + \frac{1}{4}\sum_{\substack{j=1 \\ j\neq i}}^n \frac{\mathrm{tr}((Q_i P_j^T - P_i Q_j^T)(P_j Q_i^T - Q_j P_i^T))}{\alpha_j - \alpha_i}, \tag{3.13}$$

and generate the equations of motion:

$$\frac{\partial Q_i}{\partial \alpha_j} = \frac{P_i Q_j^T Q_j - Q_i P_j^T Q_j}{2(\alpha_i - \alpha_j)}, \quad j \neq i \tag{3.14a}$$

$$\frac{\partial P_i}{\partial \alpha_j} = \frac{P_i Q_j^T P_j - Q_i P_j^T P_j}{2(\alpha_i - \alpha_j)}, \quad j \neq i \tag{3.14b}$$

$$\frac{\partial Q_i}{\partial \alpha_i} = -Q_i y + \sum_{\substack{j=1 \\ j \neq i}}^{r} \frac{Q_i P_j^T Q_j - P_i Q_j^T Q_j}{2(\alpha_i - \alpha_j)} \tag{3.14c}$$

$$\frac{\partial P_i}{\partial \alpha_i} = P_i y + \sum_{\substack{j=1 \\ j \neq i}}^{r} \frac{Q_i P_j^T P_j - P_i Q_j^T P_j}{2(\alpha_i - \alpha_j)}. \tag{3.14d}$$

The Hamiltonians $\{K_a = -K_{a+s})_{a=1,\dots,s}$ reduce to:

$$K_a = \frac{1}{2} p_a^T A q_a + \frac{1}{4} \sum_{\substack{b=1 \\ b \neq a}}^{r} \frac{\operatorname{tr}(q_a p_a^T q_b p_b^T)}{y_a - y_b} - \frac{1}{4} \sum_{b=1}^{r} \frac{\operatorname{tr}(q_a p_a^T p_b q_b^T)}{y_a + y_b} \tag{3.15}$$

and generate the equations

$$\frac{\partial q_a}{\partial y_b} = \frac{1}{4} \frac{q_b p_b^T q_a}{y_b - y_a} + \frac{1}{4} \frac{p_b q_a^T q_b}{y_b + y_a}, \quad b \neq a, \tag{3.16a}$$

$$\frac{\partial p_a}{\partial y_b} = -\frac{1}{4} \frac{p_b p_a^T p_b}{y_b - y_a} - \frac{1}{4} \frac{q_b p_b^T p_a}{y_b + y_a}, \quad b \neq a, \tag{3.16b}$$

$$\frac{\partial q_a}{\partial y_a} = \frac{1}{2} A q_a + \frac{1}{4} \sum_{\substack{b=1 \\ b \neq a}}^{r} \frac{q_b p_b^T q_a}{y_a - y_b} + \frac{1}{4} \sum_{\substack{b=1 \\ b \neq a}}^{r} \frac{p_b q_b^T q_a}{y_b + y_a}, \tag{3.16c}$$

$$\frac{\partial p_a}{\partial y_a} = -\frac{1}{2} A p_a - \frac{1}{4} \sum_{\substack{b=1 \\ b \neq a}}^{r} \frac{p_b p_a^T q_b}{y_a - y_b} + \frac{1}{4} \sum_{\substack{b=1 \\ b \neq a}}^{r} \frac{q_b p_a^T p_b}{y_b + y_a}. \tag{3.16d}$$

The particular case $s = 1$, $\{k_i = 1\}_{i=1,\dots,n}$ of (3.13), (3.14a–d) reduces, up to a simple change of basis, to the system of Theorem 7.5, [JMMS]. The corresponding τ-function gives the n-particle correlation function for an impenetrable Bose gas or, equivalently, the level spacing distribution function for a set of random matrices having no eigenvalues in the intervals $\{[\alpha_{2i-1}, \alpha_{2i}]\}_{i=1,\dots,m}$, $n = 2m$, in the scaling limit [TW1].

Hamiltonian Structure of Painlevé Equations. The following two examples show how the Painlevé transcendents P_V and P_{VI} may be derived from the generic systems (2.15a, b), (2.34) or (2.39a, b), (2.51) through Hamiltonian reduction under continuous symmetry groups. Our derivation will be guided by the formulation of Painlevé transcendents as monodromy preserving deformation equations given in [JM], but the emphasis here will be on the loop algebra content, the Hamiltonian reductions and the associated "dual" systems. For previous work on the Hamiltonian structure of the Painlevé transcendents, see [OK] and references therein.

3b. Painlevé V. Choose $N = 2, r = 2$, and

$$Y = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.17}$$

Then F and G are 2×2 matrices

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \tag{3.18}$$

with row vectors $\{F_i = (F_{i1} F_{i2}), G_i = (G_{i1} G_{i2})\}_{i=1,2}$. The stabilizer $G_A \subset Gl(2)$ of A is the diagonal subgroup generated by the moment map

$$J_{G_A}(F, G) = (F_1 G_1^T, F_2 G_2^T) := (\mu_1, \mu_2). \tag{3.19}$$

Fixing a level set, we parametrize the quotient under this abelian Hamiltonian group action by choosing the symplectic section $M_A \subset M$ defined (on a suitable connected component) by

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} x_1 & y_1 - \dfrac{\mu_1}{x_1} \\ x_2 & y_2 - \dfrac{\mu_2}{x_2} \end{pmatrix}, \quad G = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1 + \dfrac{\mu_1}{x_1} & -x_1 \\ y_2 + \dfrac{\mu_2}{x_2} & -x_2 \end{pmatrix}. \tag{3.20}$$

The reduced manifold $M_{\text{red}} = J_{G_A}^{-1}(\mu_1, \mu_2)/G_A$ is identified with $\mathbf{R}^2 \times \mathbf{R}^2$ minus the coordinate axes $\{x_1 = 0, x_2 = 0\}$, quotiented by the group of reflections in these axes. The reduced symplectic form is

$$\omega_{\text{red}} = \sum_{i=1}^{2} dx_i \wedge dy_i . \tag{3.21}$$

The image of the reduced moment map $\tilde{J}_A : M_{\text{red}} \to \widetilde{gl}(2)_-$ translated by Y is

$$\mathcal{N}(\lambda) = Y + \tilde{J}_A(F, G) = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} + \frac{\begin{pmatrix} -x_1 y_1 - \mu_1 & -y_1^2 + \dfrac{\mu_1^2}{x_1^2} \\ x_1^2 & x_1 y_1 - \mu_1 \end{pmatrix}}{2\lambda} + \frac{\begin{pmatrix} -x_2 y_2 - \mu_2 & -y_2^2 + \dfrac{\mu_2^2}{x_2^2} \\ x_2^2 & x_2 y_2 - \mu_2 \end{pmatrix}}{2(\lambda - 1)} . \tag{3.22}$$

The stabilizer $G_Y \subset Gl(2)$ of Y is the diagonal subgroup, acting by conjugation on $\mathcal{M}(\lambda)$, which corresponds to scaling transformations

$$(x_1, x_2, y_1, y_2) \to (e^{\tau} x_1, e^{\tau} x_2, e^{-\tau} y_1, e^{-\tau} y_2), \tag{3.23}$$

and is generated by

$$a := \tfrac{1}{2}(x_1 y_1 + x_2 y_2). \tag{3.24}$$

(The trace part of $gl(2)$ acts trivially, since it coincides with the Casimir $\mu_1 + \mu_2$.)

The G_Y-invariant Hamiltonians $H_1, H_2 \in \mathscr{I}^Y_A$ may be expressed

$$H_1 = \frac{1}{4\pi i} \oint_{\lambda=0} \operatorname{tr}(\mathscr{N}(\lambda))^2 d\lambda = \operatorname{tr}(YN_1) - \operatorname{tr}(N_1 N_2) = -tH - \frac{\mu_1\mu_2}{2} - a^2 - 2at , \tag{3.25a}$$

$$H_2 = \frac{1}{4\pi i} \oint_{\lambda=1} \operatorname{tr}(\mathscr{N}(\lambda))^2 d\lambda = \operatorname{tr}(YN_2) + \operatorname{tr}(N_1 N_2) = tH + \frac{\mu_1\mu_2}{2} + a^2 , \tag{3.25b}$$

where

$$H = -\frac{1}{4t}(x_1^2 + x_2^2)(y_1^2 + y_2^2) + \frac{1}{4t}\left(\mu_1^2 \frac{x_2^2}{x_1^2} + \mu_2^2 \frac{x_1^2}{x_2^2} \right) - x_2 y_2 . \tag{3.26}$$

The dual system is determined by the moment map $\tilde{J}_Y : M \to \widetilde{gl}(2)$ defined by Eq. (2.21) which, when restricted to the symplectic section M_A defined by Eq. (3.20), gives

$$\mathscr{M}(z) = -A - F(Y - zI_2)^{-1}G^T = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$
$$+ \frac{\begin{pmatrix} x_1 y_1 + \mu_1 & x_1 y_2 + \mu_2 \frac{x_1}{x_2} \\ x_2 y_1 + \mu_1 \frac{x_2}{x_1} & x_2 y_2 + \mu_2 \end{pmatrix}}{2(z-t)} + \frac{\begin{pmatrix} -x_1 y_1 + \mu_1 & -x_2 y_1 + \mu_1 \frac{x_2}{x_1} \\ -x_1 y_2 + \mu_2 \frac{x_1}{x_2} & -x_2 y_2 + \mu_2 \end{pmatrix}}{2(z+t)} . \tag{3.27}$$

Here, the quantities a and $\mu_1 + \mu_2$ are interpreted as Casimir invariants, whereas μ_1 and μ_2 individually are conserved quantities because they belong to the spectral ring $\mathscr{I}^A_Y$. In terms of the dual system, we may express the Hamiltonians $K_1, K_2 \in \mathscr{I}^A_Y$ as:

$$\begin{aligned} K_1 &= \frac{1}{4\pi i} \oint_{\lambda=t} \operatorname{tr}(\mathscr{M}(z))^2 dz = -\operatorname{tr}(AM_1) + \frac{1}{2t}\operatorname{tr}(M_1 M_2) \\ &= \frac{H}{2} + \frac{\mu_1^2 + \mu_2^2}{8t} - \frac{\mu_2}{2} \end{aligned} \tag{3.28a}$$

$$\begin{aligned} K_2 &= -\frac{1}{4\pi i} \oint_{\lambda=t} \operatorname{tr}(\mathscr{M}(z))^2 dz = -\operatorname{tr}(AM_2) - \frac{1}{2t}\operatorname{tr}(M_1 M_2) \\ &= -\frac{H}{2} - \frac{\mu_1^2 + \mu_2^2}{8t} - \frac{\mu_2}{2} . \end{aligned} \tag{3.28b}$$

The relations

$$H_1 = -H_2 - 2at \tag{3.29a}$$

$$= -2tK_1 + \frac{(\mu_1 - \mu_2)^2}{4} - a^2 - 2at - t\mu_2 \tag{3.29b}$$

$$= 2tK_2 + \frac{(\mu_1 - \mu_2)^2}{4} - a^2 - 2at + t\mu_2 \tag{3.29c}$$

can also be derived from the identity (following from Eq. (2.24)):

$$\frac{z^2 - z\mathrm{tr}\mathscr{N}(\lambda) + \frac{1}{2}((\mathrm{tr}\mathscr{N}(\lambda))^2 - \mathrm{tr}\mathscr{N}^2(\lambda))}{z^2 - t^2}$$

$$= \frac{\lambda^2 + \lambda\mathrm{tr}\mathscr{M}(z) + \frac{1}{2}((\mathrm{tr}\mathscr{M}(z))^2 - \mathrm{tr}\mathscr{M}^2(z))}{\lambda(\lambda - 1)}. \tag{3.30}$$

Integrating both sides around contours in the z-plane containing either the pole at $z = t$ or the one at $z = -t$ and contours in the λ-plane containing either the pole at $\lambda = 0$ or the one at 1, we obtain Eqs. (3.29a–c).

Viewing K_1 and K_2 as functions of $\mathscr{N}$ we have, from Eq. (2.29),

$$(dK_1)_+ = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4t}\begin{pmatrix} 0 & -y_1^2 - y_2^2 + \frac{\mu_1^2}{x_1^2} + \frac{\mu_2^2}{x_2^2} \\ x_1^2 + x_2^2 & 0 \end{pmatrix}, \tag{3.31a}$$

$$(dK_2)_+ = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} - \frac{1}{4t}\begin{pmatrix} 0 & -y_1^2 - y_2^2 + \frac{\mu_1^2}{x_1^2} + \frac{\mu_2^2}{x_2^2} \\ x_1^2 + x_2^2 & 0 \end{pmatrix}, \tag{3.31b}$$

$$(dK_1)_+ - (dK_2)_+ = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} + \frac{1}{2t}\begin{pmatrix} 0 & -y_1^2 - y_2^2 + \frac{\mu_1^2}{x_1^2} + \frac{\mu_2^2}{x_2^2} \\ x_1^2 + x_2^2 & 0 \end{pmatrix}, \tag{3.31c}$$

and hence the monodromy preserving deformation equation for $\mathscr{N}(\lambda)$ generated by $K_1 - K_2$ is given by the commutativity of the operators $\mathscr{D}_\lambda = \frac{\partial}{\partial\lambda} - \mathscr{N}(\lambda)$ and $\mathscr{D}_t^*$, where $\mathscr{N}(\lambda)$ is given by Eq. (3.22), and

$$\mathscr{D}_t^* = \frac{\partial}{\partial t} - \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} - \frac{1}{2t}\begin{pmatrix} 0 & -y_1^2 - y_2^2 + \frac{\mu_1^2}{x_1^2} + \frac{\mu_2^2}{x_2^2} \\ x_1^2 + x_2^2 & 0 \end{pmatrix}. \tag{3.32}$$

Viewing K_1 and K_2 instead as functions of $\mathcal{M}$, from (2.37) we have

$$(dK_1)_- = \frac{\begin{pmatrix} x_1y_1+\mu_1 & x_1y_2+\mu_2\dfrac{x_1}{x_2} \\ x_2y_1+\mu_1\dfrac{x_2}{x_1} & x_2y_2+\mu_2 \end{pmatrix}}{2(z-t)} \tag{3.33a}$$

$$(dK_2)_- = \frac{\begin{pmatrix} -x_1y_1+\mu_1 & -x_2y_1+\mu_1\dfrac{x_2}{x_1} \\ -x_1y_2+\mu_2\dfrac{x_1}{x_2} & -x_2y_2+\mu_2 \end{pmatrix}}{2(z+t)}. \tag{3.33b}$$

To obtain the corresponding dual monodromy preserving deformation equation for $\mathcal{M}(z)$, we cannot simply restrict Eqs. (2.39a, b), (2.40) to the submanifold $M_A \subset M$. The image of M_A under $\tilde{J}_Y$ is 3-dimensional, since points related by the scaling transformation (3.23) have the same image. Unlike $\tilde{J}_A|_{M_A}$, the restriction $\tilde{J}_Y|_{M_A}$ is not a Poisson map, since G_A does not leave $\tilde{J}_Y$ invariant, but acts by conjugation on the image $\tilde{J}_Y(F, G)$. Therefore $\tilde{J}_Y$ does not project to define a map on the quotient space M/G_A. However, the reduced systems generated by the G_A and G_Y-invariant Hamiltonians H_1, H_2, K_1 and K_2 *are* determined by the projection of their Lax equations to the quotient manifold M/G_A. The projected (nonautonomous) Hamiltonian vector field determined by $K_1 - K_2$ has a unique lift that preserves the section M_A, obtained by adding a "vertical" term

$$dK_v := \begin{pmatrix} \dfrac{1}{2t}\left(\dfrac{\mu_1 x_2^2}{x_1^2} - \mu_2\right) & 0 \\ 0 & \dfrac{1}{2t}\left(\dfrac{\mu_2 x_1^2}{x_2^2} - \mu_1\right) \end{pmatrix} \tag{3.34}$$

to the factor $(dK_1)_- - (dK_2)_-$ entering in the Lax equation for $\mathcal{M}(z)$. Equation (3.34) is obtained by noting that, apart from the conditions that the diagonal terms μ_1, μ_2 in $M_1 + M_2$ be conserved, and the Casimirs $\det M_1$, $\det M_2$ vanish, all of which are automatically satisfied by the Lax system, the only remaining condition defining the image $\tilde{J}_Y(M_A)$ is:

$$\det(\mathcal{M}(z) - \mathcal{M}^T(-z)) = 0\,, \tag{3.35a}$$

or, equivalently,

$$\det(M_1 + M_2^T) = 0\,. \tag{3.35b}$$

Up to multiples of the identity matrix, the unique element in the diagonal subalgebra (corresponding to G_A) which, when added to $(dK_1)_- - (dK_2)_-$, preserves this condition is dK_v. Thus, the correct dual deformation operator $\mathcal{D}_t$, whose commutativity with $\mathcal{D}_z = \dfrac{\partial}{\partial z} - \mathcal{M}(z)$, gives the Hamiltonian system generated by

$K_1 - K_2$, is:

$$\mathcal{D}_t = \frac{\partial}{\partial t} + \frac{\begin{pmatrix} x_1y_1 + \mu_1 & x_1y_2 + \mu_2\dfrac{x_1}{x_2} \\ x_2y_1 + \mu_1\dfrac{x_2}{x_1} & x_2y_2 + \mu_2 \end{pmatrix}}{2(z-t)} - \frac{\begin{pmatrix} -x_1y_1 + \mu_1 & -x_2y_1 + \mu_1\dfrac{x_2}{x_1} \\ -x_1y_2 + \mu_2\dfrac{x_1}{x_2} & -x_2y_2 + \mu_2 \end{pmatrix}}{2(z+t)}$$

$$+ \begin{pmatrix} \dfrac{1}{2t}\left(\dfrac{\mu_1 x_2^2}{x_1^2} - \mu_2\right) & 0 \\ 0 & \dfrac{1}{2t}\left(\dfrac{\mu_2 x_1^2}{x_2^2} - \mu_1\right) \end{pmatrix}. \tag{3.36}$$

To reduce the system under the G_Y action generated by a, we first introduce the "spectral Darboux coordinates" (u, w) (cf. [AHH2]) defined by

$$2\mathcal{N}(\lambda)_{21} = \left(\frac{x_1^2}{\lambda} + \frac{x_2^2}{\lambda - 1}\right) := \frac{w(\lambda - u)}{\lambda(\lambda - 1)}, \tag{3.37}$$

where

$$w = x_1^2 + x_2^2, \tag{3.38a}$$

$$u = \frac{x_1^2}{x_1^2 + x_2^2}. \tag{3.38b}$$

In terms of these, the symplectic form (3.21) becomes

$$\omega_{\text{red}} = d\log w \wedge da + du \wedge dv, \tag{3.39}$$

where

$$v = \frac{1}{2}\left(\frac{x_1y_1}{u} + \frac{x_2y_2}{u-1}\right) \tag{3.40}$$

is the momentum conjugate to u and a is the invariant defined in Eq. (3.24). The operators $\mathcal{D}_\lambda$, $\mathcal{D}_z$, $\mathcal{D}_t^*$ and $\mathcal{D}_t$ may be expressed in terms of these coordinates by substituting

$$x_1^2 = uw, \quad x_2^2 = (1-u)w, \quad x_1y_1 = 2u(a - uv + v), \quad x_2y_2 = 2(1-u)(a - uv),$$

$$y_1^2 = 4\frac{u}{w}(a - uv + v)^2, \quad y_2^2 = 4\frac{(1-u)}{w}(a - uv)^2 \tag{3.41}$$

in Eqs. (3.22), (3.27), (3.32) and (3.36).

Choosing a level set $a = a_0$, the symplectic form (3.21) reduces to

$$\omega_{\text{red}}|_{a=a_0} = du \wedge dv, \tag{3.42}$$

so (u, v) provide canonical coordinates on the reduced space obtained by quotienting by the G_Y-flow. From Eqs. (3.22), (3.25a, b) it follows that we may write

$$\frac{1}{2}\operatorname{tr}\mathcal{N}^2(\lambda) = t^2 + \frac{H_1}{\lambda} + \frac{H_2}{\lambda - 1} + \frac{\mu_1^2}{2\lambda^2} + \frac{\mu_2^2}{2(\lambda-1)^2} \, . \tag{3.43}$$

From (3.37) and (3.40) it follows that

$$\frac{1}{2}\operatorname{tr}\mathcal{N}^2(u) = (v-t)^2 + \frac{1}{4}\left(\frac{\mu_1}{u} + \frac{\mu_2}{u-1}\right)^2 . \tag{3.44}$$

Evaluating the integral

$$\begin{aligned}\frac{1}{4\pi i}\oint_{\lambda=u} \frac{\lambda(\lambda-1)\operatorname{tr}\mathcal{N}^2(\lambda)}{\lambda - u}\, d\lambda &= \frac{1}{2}u(u-1)\operatorname{tr}\mathcal{N}^2(u) \\ &= t^2 u(u-1) + u(H_1 + H_2) - H_1 \\ &\quad + \frac{\mu_1^2(u-1)}{2u} + \frac{\mu_2^2 u}{2(u-1)}\end{aligned} \tag{3.45}$$

and using Eqs. (3.29a–c), (3.44), (3.45) gives

$$K_1 - K_2 = \frac{u(u-1)}{t}(v^2 - 2vt) + 2au + \frac{\mu_1^2}{4ut} - \frac{\mu_2^2}{4(u-1)t} - \frac{a^2}{t} - 2a \, . \tag{3.46}$$

(Note that the simple canonical change of coordinates $v \to v + \frac{\mu_1}{2u} + \frac{\mu_2}{2(u-1)}$ transforms this to polynomial form; cf. [Ok].) The reduced equations of motion generated by the Hamiltonian $K_1 - K_2$ are therefore

$$\frac{du}{dt} = \frac{2u(u-1)}{t}(v-1) \, , \tag{3.47a}$$

$$\frac{dv}{dt} = -\frac{2u-1}{t}(v^2 - 2vt) + \frac{\mu_1^2}{4u^2 t} - \frac{\mu_2^2}{4(u-1)^2 t} - 2a \, . \tag{3.47b}$$

Eliminating v by taking second derivatives gives:

$$\begin{aligned}\frac{d^2u}{dt^2} = &\left(\frac{1}{2u} + \frac{1}{2(u-1)}\right)\left(\frac{du}{dt}\right)^2 - \frac{1}{t}\frac{du}{dt} - \alpha\frac{u}{t^2(u-1)} - \beta\frac{u-1}{t^2 u} \, , \\ &- \gamma\frac{u(u-1)}{t} - \delta u(u-1)(2u-1) \, ,\end{aligned} \tag{3.48}$$

where

$$\alpha = \frac{\mu_2^2}{2}, \quad \beta = -\frac{\mu_1^2}{2}, \quad \gamma = 4a + 2, \quad \delta = 2 \, , \tag{3.49}$$

which is one of the equivalent forms of P_V. The more usual form is obtained by transforming to the new variable

$$w = \frac{u}{u-1} \, . \tag{3.50}$$

3c. Painlevé VI. We take $N = 3$, $r = 2$, $A = \mathrm{diag}(0, 1, t)$, $Y = 0$, so F, G are 3×2 matrices

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}, \tag{3.51}$$

with rows $\{F_i = (F_{i1} F_{i2}),\ G_i = (G_{i1} G_{i2})\}_{i=1,\ldots,3}$. If $t \neq 0, 1$ the eigenvalues of A are distinct and the stabilizer $G_A \subset Gl(3)$ is the diagonal subgroup generated by the moment map

$$J_{G_A} = (\mu_1, \mu_2, \mu_3) = (F_1 G_1^T, F_2 G_2^T, F_3 G_3^T)\,. \tag{3.52}$$

Fixing a level set, we parametrize the quotient under this abelian Hamiltonian group action by choosing the symplectic section $M_A \subset M$ defined (on a suitable connected component) by

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} x_1 & y_1 - \dfrac{\mu_1}{x_1} \\ x_2 & y_2 - \dfrac{\mu_2}{x_2} \\ x_3 & -y_3 + \dfrac{\mu_3}{x_3} \end{pmatrix}, \quad G = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1 + \dfrac{\mu_1}{x_1} & -x_1 \\ y_2 + \dfrac{\mu_2}{x_2} & -x_2 \\ y_3 + \dfrac{\mu_3}{x_3} & x_3 \end{pmatrix}. \tag{3.53}$$

(The choice of signs is made such that subsequent reductions be at nonsingular points in the real case.) The reduced symplectic form is

$$\omega_{\mathrm{red}} = \sum_{i=1}^{3} dx_i \wedge dy_i\,, \tag{3.54}$$

and the reduced manifold $M_{\mathrm{red}} = J_{G_A}^{-1}(\mu_1, \mu_2, \mu_3)/G_A$ is identified with $\mathbf{R}^3 \times \mathbf{R}^3$ minus the coordinate planes $\{x_1 = 0, x_2 = 0, x_3 = 0\}$, quotiented by the group of reflections in these planes. The image of the reduced moment map $\tilde{J}_A : M_{\mathrm{red}} \to \widetilde{gl}(2)_-$ is

$$\mathcal{N}(\lambda) = \mathcal{N}_0(\lambda) = \tilde{J}_A(F, G)$$

$$= \frac{\begin{pmatrix} -x_1 y_1 - \mu_1 & -y_1^2 + \dfrac{\mu_1^2}{x_1^2} \\ x_1^2 & x_1 y_1 - \mu_1 \end{pmatrix}}{2\lambda} + \frac{\begin{pmatrix} -x_2 y_2 - \mu_2 & -y_2^2 + \dfrac{\mu_2^2}{x_2^2} \\ x_2^2 & x_2 y_2 - \mu_2 \end{pmatrix}}{2(\lambda - 1)} + \frac{\begin{pmatrix} -x_3 y_3 - \mu_3 & y_3^2 - \dfrac{\mu_3^2}{x_3^2} \\ -x_3^2 & x_3 y_3 - \mu_3 \end{pmatrix}}{2(\lambda - t)}. \tag{3.55}$$

Choose the Hamiltonian

$$H = H_3 = \frac{1}{4\pi i} \oint_{\lambda = t} \operatorname{tr}(\mathcal{N}(\lambda))^2 d\lambda = \frac{\operatorname{tr}(N_1 N_3)}{t} + \frac{\operatorname{tr}(N_1 N_3)}{t-1}$$
$$= \frac{1}{4t}\left[(x_1 y_3 + x_3 y_1)^2 - \mu_1^2 \frac{x_3^2}{x_1^2} - \mu_3^2 \frac{x_1^2}{x_3^2} + 2\mu_1\mu_3 \right]$$
$$+ \frac{1}{4(t-1)}\left[(x_2 y_3 + x_3 y_2)^2 - \mu_2^2 \frac{x_3^2}{x_2^2} - \mu_3^2 \frac{x_2^2}{x_3^2} + 2\mu_2\mu_3 \right]. \quad (3.56)$$

Let

$$a = \frac{1}{2}\sum_{i=1}^{3} x_i y_i \,, \quad (3.57a)$$

$$b = \frac{1}{2}(y_1^2 + y_2^2 - y_3^2) - \frac{\mu_1^2}{2x_1^2} - \frac{\mu_2^2}{2x_2^2} + \frac{\mu_3^2}{2x_3^2}\,, \quad (3.57b)$$

$$c = \frac{1}{2}(x_1^2 + x_2^2 - x_3^2)\,. \quad (3.57c)$$

These are the generators of the constant $Sl(2)$ conjugation action

$$g : \mathcal{N}_0(\lambda) \mapsto g\mathcal{N}_0(\lambda)g^{-1}\,, \quad (3.58)$$

and satisfy

$$\{a, b\} = b, \quad \{c, a\} = c, \quad \{b, c\} = -2a\,. \quad (3.59)$$

(In this case, since $Y = 0$, $G_Y = Gl(2)$, but the trace term acts trivially.) The Hamiltonian (3.56) is invariant under this $Sl(2)$-action

$$\{a, H\} = \{b, H\} = \{c, H\} = 0\,, \quad (3.60)$$

since the elements of the spectral ring $\mathcal{I}_A^Y$ are G_Y invariant.

The monodromy preserving deformations generated by H are then determined by the commutativity of the operators $\mathcal{D}_\lambda = \frac{\partial}{\partial\lambda} - \mathcal{N}(\lambda)$ and $\mathcal{D}_t$, with $\mathcal{N}(\lambda)$ given by Eq. (3.55) and

$$\mathcal{D}_t = \frac{\partial}{\partial t} + \frac{\begin{pmatrix} -x_3 y_3 - \mu_3 & y_3^2 - \frac{\mu_3^2}{x_3^2} \\ -x_3^2 & x_3 y_3 - \mu_3 \end{pmatrix}}{2(\lambda - t)}\,. \quad (3.61)$$

They also preserve the monodromy of the "dual" operator $\mathcal{D}_z = \frac{\partial}{\partial z} - \mathcal{M}(z)$, where $\mathcal{M}(z)$ is determined by restricting the moment map $\tilde{J}_Y : M_{\mathrm{red}} \to \widetilde{gl}(3)_-$ to the submanifold $M_A \subset M$,

$$\mathscr{M}(z) = -A + \mathscr{M}_0(z) = -A + \tilde{J}_Y(F, G) = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}$$

$$+\frac{1}{2z}\begin{pmatrix} 2\mu_1 & x_1y_2 - x_2y_1 + \frac{\mu_1 x_2}{x_1} + \frac{\mu_2 x_1}{x_2} & x_1y_3 + x_3y_1 - \frac{\mu_1 x_3}{x_1} + \frac{\mu_3 x_1}{x_3} \\ x_2y_1 - x_1y_2 + \frac{\mu_1 x_2}{x_1} + \frac{\mu_2 x_1}{x_2} & 2\mu_2 & x_2y_3 + x_3y_2 - \frac{\mu_2 x_3}{x_2} + \frac{\mu_3 x_2}{x_3} \\ x_3y_1 + x_1y_3 + \frac{\mu_1 x_3}{x_1} - \frac{\mu_3 x_1}{x_3} & x_3y_2 + x_2y_3 + \frac{\mu_2 x_3}{x_2} - \frac{\mu_3 x_2}{x_3} & 2\mu_3 \end{pmatrix}. \tag{3.62}$$

In this $\widetilde{gl}(3)_-$ representation, the quantities

$$\operatorname{tr}(FG^T) = \mu_1 + \mu_2 + \mu_3\,, \tag{3.63a}$$

$$\operatorname{tr}(FG^T)^2 = \operatorname{tr}(G^TF)^2 = 2(a^2 - bc) + \frac{1}{2}\left(\sum_{i=1}^{3} \mu_i\right)^2, \tag{3.63b}$$

$$\det(FG^T) = 0 \tag{3.63c}$$

are the Casimir invariants, while the individual elements μ_1, μ_2, μ_3 are not Casimirs, but generators of the stabilizer $G_A \subset Gl(3)$ of A, and hence constants of motion. Thus, what appeared before as Casimirs on $\widetilde{gl}(2)_-$ become elements of the spectral ring $\mathscr{I}_Y^A$, while the element of $\mathscr{I}_A^Y$ given by Eq. (3.63b) becomes a Casimir on $\widetilde{gl}(3)_-$.

Viewing H_3 now as a function of $\mathscr{M}$, we have, from Eq. (2.45),

$$(dH_3)_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -z \end{pmatrix}$$

$$+\frac{1}{2}\begin{pmatrix} 0 & 0 & \frac{1}{t}(x_1y_3 + x_3y_1 - \frac{\mu_1 x_3}{x_1} + \frac{\mu_3 x_1}{x_3} \\ 0 & 0 & \frac{1}{t-1}\left(x_2y_3 + x_3y_2 - \frac{\mu_2 x_3}{x_2} + \frac{\mu_3 x_2}{x_3}\right) \\ \frac{1}{t}\left(x_3y_1 + x_1y_3 + \frac{\mu_1 x_3}{x_1} - \frac{\mu_3 x_1}{x_3}\right) & \frac{1}{t-1}\left(x_3y_2 + x_2y_3 + \frac{\mu_2 x_3}{x_2} - \frac{\mu_3 x_2}{x_3}\right) & 0 \end{pmatrix}. \tag{3.64}$$

The dual monodromy preserving representation is therefore given by the commutativity of the operators $\mathscr{D}_z = \frac{\partial}{\partial z} - \mathscr{M}(z)$, and $\mathscr{D}_t^*$, with $\mathscr{M}(z)$ given by

Eq. (3.62), and

$$\mathscr{D}_t^* = \frac{\partial}{\partial t} - (dH_3)_+ + dH_v$$

$$= \frac{\partial}{\partial t} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}$$

$$- \frac{1}{2} \begin{pmatrix} \frac{1}{t}\left(\mu_3 - \frac{\mu_1 x_3^2}{x_1^2}\right) & 0 & \frac{1}{t}\left(x_1 y_3 + x_3 y_1 - \frac{\mu_1 x_3}{x_1} + \frac{\mu_3 x_1}{x_3}\right) \\ 0 & \frac{1}{t-1}\left(\mu_3 - \frac{\mu_2 x_3^2}{x_2^2}\right) & \frac{1}{t-1}\left(x_2 y_3 + x_3 y_2 - \frac{\mu_2 x_3}{x_2} + \frac{\mu_3 x_2}{x_3}\right) \\ \frac{1}{t}\left(x_3 y_1 + x_1 y_3 + \frac{\mu_1 x_3}{x_1} - \frac{\mu_3 x_1}{x_3}\right) & \frac{1}{t-1}\left(x_3 y_2 + x_2 y_3 + \frac{\mu_2 x_3}{x_2} - \frac{\mu_3 x_2}{x_3}\right) & \frac{1}{t}\left(\mu_1 - \frac{\mu_3 x_1^2}{x_3^2}\right) + \frac{1}{t-1}\left(\mu_2 - \frac{\mu_3 x_2^2}{x_3^2}\right) \end{pmatrix} \tag{3.65}$$

Here

$$dH_v = \begin{pmatrix} \frac{1}{2t}\left(\frac{\mu_1 x_3^2}{x_1^2} - \mu_3\right) & 0 & 0 \\ 0 & \frac{1}{2(t-1)}\left(\frac{\mu_2 x_3^2}{x_2^2} - \mu_3\right) & 0 \\ 0 & 0 & \frac{1}{2t}\left(\frac{\mu_3 x_1^2}{x_3^2} - \mu_1\right) + \frac{1}{2(t-1)}\left(\frac{\mu_3 x_2^2}{x_3^2} - \mu_2\right) \end{pmatrix} \tag{3.66}$$

is the element of the diagonal subalgebra (corresponding to G_A) that must be added in order that the lift of the G_A-reduced Hamiltonian vector field on M_{red} be tangential to M_A.

To obtain the $Sl(2)$-reduced system, first choose the level set

$$b = c = 0\,, \tag{3.67}$$

and again define the "spectral Darboux coordinates" (u, w) by

$$2\mathscr{N}(\lambda)_{21} = \left(\frac{x_1^2}{\lambda} + \frac{x_2^2}{\lambda - 1} - \frac{x_3^2}{\lambda - t}\right) = \frac{w(u - \lambda)}{a(\lambda)} \tag{3.68}$$

on this level set, where

$$w = (1 + t)x_1^2 + t x_2^2 - x_3^2\,, \tag{3.69a}$$

$$u = \frac{t x_1^2}{w}\,, \tag{3.69b}$$

$$a(\lambda) := \lambda(\lambda - 1)(\lambda - t)\,. \tag{3.69c}$$

Thus (u, w) are elliptic-hyperbolic coordinates on the cone

$$x_1^2 + x_2^2 = x_3^2 \,. \tag{3.70}$$

On the invariant manifold defined by $b = c = 0$, the symplectic form reduces to

$$\omega_{\mathrm{red}}|_{(b,c)=(0,0)} = d(\log w) \wedge da + du \wedge dv \,, \tag{3.71}$$

where

$$v = \frac{1}{2}\left(\frac{x_1 y_1 - \mu_1}{u} + \frac{x_2 y_2 - \mu_2}{u-1} + \frac{x_3 y_3 - \mu_3}{u-t} \right) \tag{3.72}$$

is the momentum coordinate conjugate to u. (Note that the slight difference between this choice and that of Eq. (3.40) results in a polynomial form for the Hamiltonian.)

Restricting to the level set $a = a_0$, we have

$$\omega_{\mathrm{red}}|_{(a,b,c)=(a_0,0,0)} = du \wedge dv \,, \tag{3.73}$$

which is the reduction of the symplectic form under the $Sl(2)$-action generated by (a, b, c). The coordinates (u, v) project to the quotient under the action of the stability group of the image $(a_0, 0, 0)$ of this $sl(2)$ moment map, since they satisfy $\{u, a\} = \{v, a\} = 0$ on the level set $(a, b, c) = (a_0, 0, 0)$.

To compute the Hamiltonian in terms of the reduced coordinates, we write

$$\frac{1}{2}\operatorname{tr}(\mathcal{N}(\lambda))^2) = \frac{P_0 + P_1\lambda}{a(\lambda)} + \frac{\mu_1^2}{2\lambda^2} + \frac{\mu_2^2}{2(\lambda-1)^2} + \frac{\mu_3^2}{2(\lambda-t)^2} \,. \tag{3.74}$$

where

$$P_1 = a^2 + \frac{1}{4}\left(\sum_{i=1}^{3} \mu_i \right)^2 - \frac{1}{2}\sum_{i=1}^{3} \mu_i^2 \,, \tag{3.75}$$

and evaluate

$$\begin{aligned} \frac{1}{4\pi i}\oint_{\lambda=u} \frac{a(\lambda)\operatorname{tr}(\mathcal{N}(\lambda))^2}{\lambda-u}\, d\lambda &= \frac{1}{2}u(u-1)(u-t)\mathcal{N}^2(u) \\ &= P_0 + P_1 u + \frac{\mu_1^2(u-1)(u-t)}{2u} \\ &\quad + \frac{\mu_2^2 u(u-t)}{2(u-1)} + \frac{\mu_3^2 u(u-1)}{2(u-t)} \,, \end{aligned} \tag{3.76}$$

where the integral is around a circle containing only the pole at $\lambda = u$. Since

$$\frac{x_1^2}{u} + \frac{x_2^2}{u-1} - \frac{x_3^2}{u-t} = 0 \,, \tag{3.77}$$

we have

$$\frac{1}{2}\operatorname{tr}\mathcal{N}^2(u) = v^2 + v\left(\frac{\mu_1}{u} + \frac{\mu_2}{u-1} + \frac{\mu_3}{u-t}\right) + \frac{1}{2}\left(\frac{\mu_1}{u} + \frac{\mu_2}{u-1} + \frac{\mu_3}{u-t}\right)^2, \tag{3.78}$$

and hence,

$$\begin{aligned} P_0 &= u(u-1)(u-t)v^2 + v(\mu_1(u-1)(u-t) + \mu_2 u(u-t) + \mu_3 u(u-1)) \\ &\quad + \mu_1\mu_2(u-t) + \mu_2\mu_3 u + \mu_1\mu_3(u-1) - P_1 u\,. \end{aligned} \tag{3.79}$$

From (3.56), (3.79),

$$\begin{aligned} H &= \frac{P_0 + P_1 t}{t(t-1)} \\ &= \frac{1}{t(t-1)}[u(u-1)(u-t)v^2 + v(\mu_1(u-1)(u-t) + \mu_2 u(u-t) \\ &\quad + \mu_3 u(u-1)) + \mu_1\mu_2(u-t) + \mu_2\mu_3 u + \mu_1\mu_3(u-1) + (t-u)P_1]\,. \end{aligned} \tag{3.80}$$

To compute Hamilton's equations, the explicit t-dependence of the coordinates u, v implied by Eqs. (3.69a, b), (3.72) must be taken into account. The t-derivatives with respect to this t-dependence are:

$$u_t = \frac{u(u-1)}{t(t-1)}, \tag{3.81a}$$

$$v_t = \frac{-v(2u-1) + a_0 - \frac{1}{2}(\mu_1 + \mu_2 + \mu_3)}{t(t-1)}\,. \tag{3.81b}$$

The reduced form of Hamilton's equations are then

$$\frac{du}{dt} = \frac{1}{t(t-1)}[2u(u-1)(u-t)v + \mu_1(u-1)(u-t) + \mu_2 u(u-t) + (\mu_3+1)u(u-1)]\,, \tag{3.82a}$$

$$\begin{aligned} \frac{dv}{dt} &= -\frac{1}{t(t-1)}\Bigg[(u(u-1) + u(u-t) + (u-1)(u-t))v^2 \\ &\quad + v(\mu_1(2u-t-1) + \mu_2(2u-t) + (\mu_3+1)(2u-1)) \\ &\quad - a_0^2 - a_0 + \frac{1}{4}\sum_{i=1}^{3}\mu_i^2 + \frac{1}{2}\sum_{i=1}^{3}\mu_i\Bigg]\,. \end{aligned} \tag{3.82b}$$

Upon elimination of v, this gives P_{VI}:

$$\frac{d^2u}{dt^2} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\left(\frac{du}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)\frac{du}{dt}$$

$$+ \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{u^2} + \gamma\frac{t-1}{(u-1)^2} + \delta\frac{t(t-1)}{(u-t)^2}\right), \quad (3.83)$$

where

$$\alpha = 2a_0^2 + 2a_0 + \frac{1}{2}, \quad \beta = -\frac{1}{2}\mu_1^2, \quad \gamma = \frac{1}{2}\mu_2^2, \quad \delta = -\frac{1}{2}\mu_3^2 + \frac{1}{2}. \quad (3.84)$$

4. Generalizations

The approach developed in [AHP, AHH1] is equally valid for matrices A, Y that are nondiagonalizable, giving rise to isospectral deformations of matrices of the form

$$\mathcal{N}(\lambda) = Y + \sum_{i=1}^{n}\sum_{l_i=1}^{n_i}\frac{N_{i,l_i}}{(\lambda-\alpha_i)^{l_i}}, \quad (4.1a)$$

$$\mathcal{M}(z) = -A + \sum_{a=1}^{r}\sum_{m_a=1}^{r_a}\frac{M_{a,m_a}}{(z-y_a)^{m_a}}. \quad (4.1b)$$

A straightforward generalization of the moment map construction may also be made, yielding the more general forms

$$\mathcal{N}(\lambda) = \sum_{l_0=0}^{n_0} Y_{l_0}\lambda^{l_0} + \sum_{i=1}^{n}\sum_{l_i=1}^{n_i}\frac{N_{i,l_i}}{(\lambda-\alpha_i)^{l_i}}, \quad (4.2a)$$

$$\mathcal{M}(z) = \sum_{m_0=0}^{r_0} A_{m_0}z^{m_0} + \sum_{a=1}^{r}\sum_{m_a=1}^{r_a}\frac{M_{a,m_a}}{(z-y_a)^{m_a}}. \quad (4.2b)$$

Such $\mathcal{N}(\lambda)$, $\mathcal{M}(z)$ may be viewed as elements, respectively, of subspaces $\mathbf{g}_A^Y \subset \widetilde{gl}(r)^*$, $\mathbf{g}_Y^A \subset \widetilde{gl}(N)^*$ defined by the rational structure appearing in Eqs. (4.2a, b). These are Poisson subspaces with respect to the Lie Poisson bracket on $\widetilde{gl}(r)^*$ (resp. $\widetilde{gl}(N)^*$) corresponding to the Lie bracket:

$$[X, Y]_R := \frac{1}{2}[RX, Y] + \frac{1}{2}[X, RY], \quad (4.3)$$

where

$$R := P_+ - P_- \quad (4.4)$$

is the *classical R-matrix* given by the difference of the projection operators $P_\pm$ to the subalgebras $\widetilde{gl}(r)_\pm$ (resp. $\widetilde{gl}(N)_\pm$). The R-matrix version of the AKS theorem

[S] again implies that results of the type (i) and (ii) (Eq. (2.7)) hold for the autonomous systems generated by elements of the ring $\mathscr{I}^Y_A$ (resp $\mathscr{I}^A_Y$) obtained by restriction of the ring of Ad*-invariant polynomials on $\widetilde{gl}(r)^*$ (resp. $\widetilde{gl}(N)^*$) to $\mathbf{g}_A$ (resp. $\mathbf{g}_Y$). Isomonodromic deformations of operators of the type $\mathscr{D}_\lambda = \frac{\partial}{\partial \lambda} - \mathscr{N}(\lambda)$, where $\mathscr{N}(\lambda)$ is of the form (4.2a), were the subject of the series of papers [JMU, JM]. They are required, in particular, for the isomonodromy formulation of the remaining Painlevé transcendent equations $P_I - P_{IV}$ ([JM, HW]) and for the Hamiltonian dynamics governing the level-spacing distribution functions in random matrix models at the "edge of the spectrum" [TW2]. A brief discussion of the latter from the loop algebra viewpoint is given in [HTW]. The Hamiltonian formulation of more general systems of monodromy preserving deformations of operators with irregular singular points of arbitrary order within the framework of spectral invariants on loop algebras will be addressed in a subsequent work.

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