Quantum Groups on Fibre Bundles

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Abstract: It is shown that the principle of locality and noncommutative geometry can be connected by a sheaf theoretical method. In this framework quantum spaces are introduced and examples in mathematical physics are given. Within the language of quantum spaces noncommutative principal and vector bundles are defined and their properties are studied. Important constructions in the classical theory of principal fibre bundles like associated bundles and differential calculi are carried over to the quantum case. At the end q-deformed instanton models are introduced for every integral index.

Contents

Introduction

There are two essential principles in quantum field theory, namely symmetry and locality. Especially in every gauge theory it must be explained what the symmetry objects are and what locality means.

Since some time many theorists hope for noncommutative geometry [5] becoming the right tool to formulate quantum field theory rigorously. But now we are in the dilemma that the language of noncommutative geometry provides very general and powerful symmetry objects, the quantum groups, but not an appropriate method to study local aspects. This fundamental problem was the starting point for the present paper.

Let me explain in more detail what the principle of locality in physics says. In the algebraic framework [11] a net of C^*-algebras on Minkowski space describes the local structure of observables such that observable algebras defined on spatially separated double cones commute. Alternatively one can require local commutation relations [20, 13]. In any case we need mathematical methods to compare observables or fields at different space time points or neighbourhoods. But in noncommutative geometry there are no points, respectively there is no topology, on a base space where the fields and observables are defined. The problem becomes even more serious if we want to quantise gauge theories. A field in a classical gauge theory is a (global) section in a vector bundle. Usually those vector bundles are described in local charts or in other words in local coordinates. Now gauge transformations of the second kind change the field locally such that the observable effects of the field stay the same. The principle of local coordinates and gauge transformations is mathematically well-defined in classical geometry and physics. The appropriate language is the theory of principal fiber bundles. But up to now it hasn't been possible to translate it into noncommutative geometry or rigorous quantum field theory [11].

Mathematics also provides arguments to connect local aspects and noncommutative methods. Furthermore these arguments even give a hint how to solve the above problem. Certain structures on a locally compact topological space M like differentiable or analytical ones are not characterised by the single algebra $\mathscr{C}_0(M)$ but by an appropriate sheaf on M. Additionally it is well known from complex geometry [15] that local function algebras are in general not determined by the global one. So it is quite natural to assume that we need a sheaf structure in the noncommutative setting as well. This would be very helpful also in the case where commutative function algebras are deformed. Then one can keep track of what happens to local algebras of continuous, differentiable and analytical functions or sections in a fibre bundle.

Because of these considerations we give a sheaf theoretical method to connect locality and noncommutative geometry. Furthermore within this method it is possible to define noncommutative principal fibre bundles which have quantum groups as their "structure groups."

In the first section we will find an equivalent description of principal fibre bundles in the language of sheaves. Then a definition of very general noncommutative spaces is given in Sect. 2. In the following noncommutative principal bundles are defined and important objects like local coordinates, transition functions and gauge transformations are carried over to the noncommutative case. By the same concept quantum vector bundles and associated quantum vector bundles are introduced in Sect. 4. Addtionally we study differential calculi on quantum

vector bundles. Finally q-deformed instanton models give interesting examples where local gauge transformations are set up in a noncommutative language.

This paper grew out of my diploma thesis [19] under the supervision of Prof. J. Wess at the "Sektion Physik der Universität München."

1. Principal Fibre Bundles

Let us repeat the well known definition of a principal fibre bundle (see for example [6, 9, 14]).

Definition 1.1. *Let P, M be topological spaces, G a topological group and $\pi : P \to M$ a continuous mapping. (P, M, π, G) is called a principal fibre bundle with total space P, basis M and structure group G, if the following conditions are satisfied.*

   (i) *π is surjective.*
  (ii) *G acts freely from the right on P.*
 (iii) *The equation $\pi(u_1) = \pi(u_2)$ for $u_1, u_2 \in P$ is valid if and only if $u_1 a = u_2$ for an $a \in G$.*
 (iv) *P is locally trivial over M, i.e. there exists an open covering $\mathcal{U} = (U_\iota)_{\iota \in I}$ of M and homeomorphisms $\psi_\iota : \pi^{-1}(U_\iota) \to U_\iota \times G$, $u \mapsto (\pi(u), \eta_\iota(u))$ such that*

$$\psi_\iota(ua) = (\pi(u), \eta_\iota(u)a), \quad u \in \pi^{-1}(U_\iota), \quad a \in G. \tag{1}$$

Remark 1.2. The homeomorphisms ψ_ι with $\iota \in I$ are the local trivializations of the principal bundle.

The conditions (i) to (iv) in the above definition are not independent from each other. In the next theorem we give a characterisation of principal bundles where the defining axioms are independent. The obvious proof of the theorem is skipped.

Theorem 1.3. Let P, M be topological spaces, G a topological group and $\pi : P \to M$ continuous. Assume the following two conditions to be true.

   (i) *π is surjective.*
  (ii) *P is locally trivial over M, i.e. there exists an open covering $\mathcal{U} = (U_\iota)_{\iota \in I}$ of M and homeomorphisms $\psi_\iota : \pi^{-1}(U_\iota) \to U_\iota \times G$, such that*

$$pr_1 \circ \psi_\iota = \pi|_{\pi^{-1}(U_\iota)} =: \pi|_{U_\iota}, \tag{2}$$

and

$$\psi_\iota \circ \psi_\kappa^{-1}(x, ab)$$
$$= (x, pr_2(\psi_\iota \circ \psi_\kappa^{-1}(x, a))b), \quad x \in U_\iota \cap U_\kappa, \quad a, b \in G. \tag{3}$$

Then by

$$u \cdot a := \psi_\iota^{-1}(pr_1(\psi_\iota(u)), pr_2(\psi_\iota(u))a), \quad u \in \pi^{-1}(U_\iota), \quad a \in G, \tag{4}$$

a right G-action on P is defined. Furthermore (P, M, π, G) is a principal fibre bundle over M with trivialisations ψ_ι, $\iota \in I$. On the other hand given any principal bundle (P, M, π, G) with trivialisations ψ_ι, $\iota \in I$ the above conditions (i) and (ii) hold, and the right G-action is given locally by Eq. (4).

We introduce some notation. Let $\mathcal{M}$ (resp. $\mathcal{F}$) be the sheaf of complex and bounded continuous functions on M (resp. G). Now $\mathcal{M}$ and $\mathcal{F}$ are sheaves of Banach-$*$-algebras. Let U, V be open in M. Then $\mathcal{P}(U)$ is the algebra of complex

and bounded continuous functions on $\pi^{-1}(U)$. The canonical injections

$$i_U^V : U \to V, \quad u \mapsto u,$$

where $U \subset V \subset M$ open define restrictions $\mathscr{P}(i_U^V) = r_U^V : \mathscr{P}(V) \mapsto \mathscr{P}(U)$, $f \mapsto f|_{\pi^{-1}(U)}$. Now $\mathscr{P}$ is a sheaf of algebras on M, or more clearly a sheaf of Banach-$*$-algebras. The continuous function $\pi : P \to M$ induces a sheaf morphism

$$\varrho = \pi^* : \mathscr{M} \to \mathscr{P}, \quad \pi^*(f) = f \circ \pi|_U, \quad f \in \mathscr{M}(U),$$

and the G-action on P a sheaf morphism $\phi : \mathscr{P} \to \mathscr{P} \otimes \mathscr{F}(G)$ by

$$\phi_U(h)(u, a) = h(ua), \quad h \in \mathscr{P}, \quad u \in \pi^{-1}(U), \quad a \in G. \tag{5}$$

Finally the local trivialisations give sheaf morphisms

$$\Omega_\iota : \mathscr{M}|_{U_\iota} \otimes \mathscr{F}(G) \to \mathscr{P}|_{U_\iota}$$

$$f \otimes g \mapsto (f \otimes g) \circ \psi_\iota|_U, \quad g \in \mathscr{F}_G, \quad f \in \mathscr{M}(U), \quad U \subset U_\iota.$$

Proposition 1.4. *Let P, M be locally compact topological spaces, G a locally compact topological group and $\pi : P \to M$ continuous. With the above definition of $\mathscr{P}, \mathscr{M}$, ϱ the quadrupel (P, M, π, G) is a principal fibre bundle if and only if the following conditions (i) and (ii) are satisfied.*

(i) *The sequence $0 \to \mathscr{M} \xrightarrow{\varrho} \mathscr{P}$ is exact.*

(ii) *One can find an open covering $\mathscr{U} = (U_\iota)_{\iota \in I}$ of M and continuous mappings $\psi_\iota : \pi^{-1}(U_\iota) \to U_\iota \times G$ with the following property. The sheaf morphisms $\Omega_\iota : \mathscr{M}|_{U_\iota} \otimes \mathscr{F}_G \to \mathscr{P}|_{U_\iota}, \iota \in I$ are isomorphisms and define sheaf morphisms $\Omega_{\kappa,\iota} : \mathscr{M}|_{U_\kappa} \otimes \mathscr{F}_G \to \mathscr{M}|_{U_\iota} \otimes \mathscr{F}_G, \iota, \kappa \in I$ by $(\Omega_{\kappa,\iota})_U = (\Omega_\kappa)_U^{-1} \circ (\Omega_\iota)_U$. These sheaf morphisms satisfy the equations*

$$(\Omega_\iota)_U(f \otimes 1) = \varrho_U(f), \quad f \in \mathscr{M}, \, U \subset U_\iota, \tag{6}$$

$$((\Omega_{\kappa,\iota})_U \otimes id) \circ (id \otimes \Delta) = (id \otimes \Delta) \circ (\Omega_{\kappa,\iota})_U, \quad U \subset U_\iota \cap U_\kappa, \tag{7}$$

where m (resp. Δ) is the multiplication (resp. comultiplication) in $\mathscr{F}(G)$.

Proof. For $U \subset U_\iota$ open, $f \in \mathscr{M}(U)$ and $u \in \pi^{-1}(U)$ we have the equations

$$(\Omega_\iota)_U(f \otimes 1)(u) = f(pr_1 \circ \psi_\iota(u)), \tag{8}$$

$$\varrho_U (f)(u) = f(\pi(u)), \tag{9}$$

and for $U \subset U_\iota \cap U_\kappa$ open, $f \in \mathscr{M}(U) \otimes \mathscr{F}(G)$, $x \in U$ and $a, b \in G$ the equations

$$(\Omega_{\kappa,\iota})_U(f)(x, a) = f(\psi_\iota \circ \psi_\kappa^{-1}(x, a)), \tag{10}$$

$$((id \otimes \Delta) \circ (\Omega_{\kappa,\iota})_U)(f)(x, a, b) = f(\psi_\iota \circ \psi_\kappa(x, ab)), \tag{11}$$

$$(((\Omega_{\kappa,\iota})_U \otimes id) \circ (id \otimes \Delta))(f)(x, a, b) = f(x, pr_2(\psi_\iota \circ \psi_\kappa^{-1}(x, a))b). \tag{12}$$

First suppose (P, M, π, G) to define a principal fibre bundle with trivialisations $\psi_\iota : \pi^{-1}(U_\iota) \to U_\iota \times G$. As $\pi : P \to M$ is surjective, the sequence $0 \to \mathscr{M} \xrightarrow{\varrho} \mathscr{P}$ is exact and (i) holds. Equations (8), (9) and (2) imply (6). Furthermore Eqs. (11), (12) and (3) give (7). Altogether this proves (ii).

Now we have to show the other implication. Assume (i) and (ii) being true for (P, M, π, G). Then the relations (8), (9) and (6) entail

$$f(pr_1 \circ \psi_\iota(u)) = f(\pi(u)), \quad f \in \mathscr{M}(U), \quad u \in \pi^{-1}(U).$$

As the continuous functions on U are separating, we get (2). Similarly (11), (12) and (7) give the equation

$$f(\psi_\iota \circ \psi_\kappa^{-1}(x, ab)) = f(x, pr_2(\psi_\iota \circ \psi_\kappa^{-1}(x, a)) b)$$

for $f \in \mathcal{M}(U) \otimes \mathcal{F}(G)$, $x \in U$ and $a, b \in G$. Now we have shown (3) and condition (ii) in Theorem 1.3. As $pr_1 \circ \psi_\iota = \pi|_{U_\iota}$ and $\mathcal{U} = (U_\iota)_{\iota \in I}$ covers M, π is surjective. That is all. $\qquad\square$

Corollary 1.5. *Assume to be given a tupel* $(\mathcal{P}, \mathcal{M}, \varrho, H)$, *where* $\mathcal{M}$ *is the commutative sheaf of complex bounded continuous functions on a locally compact topological space* M, $\mathcal{P}$ *is a sheaf of commutative C*-algebras over* M, $\varrho : \mathcal{M} \to \mathcal{P}$ *a sheaf morphism and* H *a commutative (topological) Hopf algebra.* $(\mathcal{P}, \mathcal{M}, M, \varrho, H)$ *can be identified with a principal fibre bundle if the following conditions hold.*

(i) *The sequence* $0 \to \mathcal{M} \overset{\varrho}{\to} \mathcal{P}$ *is exact.*
(ii) *There exists an open covering* $\mathcal{U} = (U_\iota)_{\iota \in I}$ *of* M *and sheaf isomorphisms* $\Omega_\iota : \mathcal{M}|_{U_\iota} \otimes H \to \mathcal{P}|_{U_\iota}$, $\iota \in I$ *such that* Ω_ι *and* $\Omega_{\kappa, \iota}$ *with* $(\Omega_{\kappa, \iota})_U = (\Omega_\kappa)_U^{-1} \circ (\Omega_\iota)_U$, $U \subset U_\iota \cap U_\kappa$ *satisfy Eqs. (6) and (7).*

Proof. One can construct the locally compact topological spaces P, M, G and the continuous mappings π, ψ_ι by the Gelfand transformation. Then the assumptions of Proposition 1.4 hold and the corollary is shown. $\qquad\square$

2. Quantum Spaces

In this section we define the frame in which the principle of locality and noncommutative geometry can be connected. We use a sheaf theoretical language which is already well known in the commutative setting of algebraic geometry and complex analysis. See Appendix A for the definition of sheaves and the literature [21, 16] for further details on sheaves.

Definition 2.1. *Let* **A** *be a subcategory of the category* **Alg** *of all associative algebras. An* **A**-*quantum space over a topological space* M *is a sheaf* $\mathcal{M}$ *over* M *with objects in* **A**. *The category of* **A**-*quantum spaces is dual to the category of sheaves over topological spaces and will be denoted by* **A-Qs**.

Let $\mathcal{M}$ be a sheaf over M. If we consider $\mathcal{M}$ as an object of **A-Qs**, we sometimes write $\mathcal{M}_Q$. Now let $\mathcal{N}$ be a sheaf over N and $f : M \to N$ a continuous mapping. A morphism $\mathcal{F} : \mathcal{N} \to \mathcal{M} \cdot f^{-1}$ of sheaves over f will be written $\mathcal{F}_Q : \mathcal{M}_Q \to \mathcal{N}_Q$ if regarded as a morphism in **A-Qs**. The **A**-quantum spaces over a topological space M with morphisms over the identity id_M form a subcategory **A-Qs$_M$** of **A-Qs**.

The following examples of quantum spaces show that it is possible to include a concept of locality in noncommutative geometry. They also comprise important objects of commutative and noncommutative geometry.

Example 2.2. *Let* M *be a topological space,* A *an object in* **A** *and* $\mathcal{L}$ *the locally constant sheaf on* M *with objects in* A. $\mathcal{L}$ *is an* **A**-*quantum space.*

Ringed spaces are important tools of complex analysis and algebraic geometry [12, 15]. A ringed space is simply a pair $(M, \mathcal{O}_M)$, where M is a topological space and $\mathcal{O}_M$ a sheaf of commutative rings.

Example 2.3. *Ringed spaces* $(M, \mathcal{O}_M)$ *are commutative quantum spaces.*

Manifolds, complex spaces and schemes can also be considered as ringed spaces or commutative quantum spaces. To explain that let us first write down some special ringed spaces:

$(\mathbb{R}^n, \mathscr{C})$, where $\mathscr{C}$ — sheaf of continuous functions on $\mathbb{R}^n$, $n \in \mathbb{N}$,

$(\mathbb{R}^n, \mathscr{C}^r)$, where $\mathscr{C}^r$ — sheaf of r-times continuously differentiable functions on $\mathbb{R}^n$, $n \in \mathbb{N}$, $r \in \mathbb{N}^* \cup \{\infty\}$,

$(\mathbb{R}^n, \mathscr{C}^\omega)$, where $\mathscr{C}^\omega$ — sheaf of real analytic functions on $\mathbb{R}^n$, $n \in \mathbb{N}$,

$(\mathbb{C}^n, \mathcal{O}_n)$, where $\mathcal{O}_n$ — sheaf of holomorphic functions on $\mathbb{C}^n$, $n \in \mathbb{N}$.

Example 2.4. *Let* $n \in \mathbb{N}$ *and* $r \in \mathbb{N}^* \cup \{\infty\}$. *A ringed space* $(X, \mathcal{O}_X)$ *is called a*

   (i) *topological manifold of dimension* n, *if* $(X, \mathcal{O}_X)$ *is locally isomorphic to* $(\mathbb{R}^n, \mathscr{C})$,
  (ii) *differentiable* r-*manifold of dimension* n, *if* $(X, \mathcal{O}_X)$ *is locally isomorphic to* $(\mathbb{R}^n, \mathscr{C}^r)$,
 (iii) *real analytic manifold of dimension* n, *if* $(X, \mathcal{O}_X)$ *is locally isomorphic to* $(\mathbb{R}^n, \mathscr{C}^\omega)$,
  (iv) *complex manifold of dimension* n, *if* $(X, \mathcal{O}_X)$ *is locally isomorphic to* $(\mathbb{C}^n, \mathcal{O}_n)$,
   (v) *scheme, if for every* $x \in X$ *there exists an open neighbourhood* U *of* x, *such that* $(U, \mathcal{O}_X|_U)$ *is isomorphic to an affine scheme.*

All those spaces are quantum spaces.

Supersymmetric structures (see Wess, Bagger [22]) are our first examples of noncommutative quantum spaces. Most easily this can be seen with the definition of superspaces according to Manin [18].

Definition 2.5. *A superspace consists of a pair* $(M, \mathcal{O}_M)$, *where* M *is a topological space and* $\mathcal{O}_M$ *a sheaf of supercommutative rings, such that all stalks* $\mathcal{O}_{M,x}$, $x \in M$ *are local.*

Supermanifolds are superspaces which locally split into an even and odd part such that the splitting is differentiable and the odd part is a locally free module sheaf over the even part.

Example 2.6. *Superspaces and supermanifolds are noncommutative quantum spaces.*

We already cite here an example of a quantum space we are going to construct in Sect. 5.

Example 2.7. *The* q-*deformed space time over the background* S^4 *is a noncommutative quantum space.*

3. Quantum Principal Bundles

3.1. The Category of Quantum Principal Bundles

Corollary 1.5 characterises principal bundles in the language of sheaves of commutative algebras. If we simply leave out the requirement for the commutativity of the local algebras we almost get the definition of noncommutative principal bundles or quantum principal bundles. One further generalisation compared with the commutative case has to be made. The reason lies in the fact that the tensor

product of a noncommutative algebra and a Hopf algebra possesses "more multiplications" than the tensor product of the corresponding commutative objects. So we have to specify the chosen multiplication on the local tensor products by the methods of Appendix B.

Let M be a topological space and $\mathbf{A}$ a subcategory of the category of all associative $\mathbb{C}$-algebras. Suppose we are given the following objects:

(i) a sheaf $\mathscr{M}$ over M with objects in $\mathbf{A}$ called the base quantum space,
(ii) a sheaf $\mathscr{P}$ over M with objects in $\mathbf{A}$ called the total quantum space,
(iii) a sheaf morphism $\varrho : \mathscr{M} \to \mathscr{P}$ called the projection,
(iv) a Hopf algebra H called the structure quantum group,
(v) a family of sheaf morphisms $(\Omega_\iota)_{\iota \in I}$, $\Omega_\iota : \mathscr{M}|_{U_\iota} \,\#_\iota H \to \mathscr{P}|_{U_\iota}$, where $\mathscr{U} = (U_z)_{z \in I}$ is an open covering of M and $\#_\iota$ is a crossed product defined according to Theorem B.2 by a weak action $\alpha_\iota : H \times \mathscr{M}|_{U_\iota} \to \mathscr{M}|_{U_\iota}$ and a normal cocycle $\iota : H \times H \to \mathscr{M}(U_\iota)$ fulfilling the twisted module condition.

The tupel $(\mathscr{P}, \mathscr{M}, \varrho, H, (\Omega_\iota)_{\iota \in I})$ gives the data of an $\mathbf{A}$-quantum principal bundle over M. Its entries can be regarded as the noncommutative generalisations of respectively the total space, base space, projection, structure group and trivialisation of a classical principal bundle.

Definition 3.1. $(\mathscr{P}, \mathscr{M}, \varrho, H, (\Omega_\iota)_{\iota \in I})$ is said to be an $\mathbf{A}$-quantum principal bundle over M with coordinate system $(\Omega_\iota)_{\iota \in I}$, if the following conditions are fulfilled.

(i) The sequence $0 \to \mathscr{M} \overset{\varrho}{\to} \mathscr{P}$ is exact.
(ii) The algebras $\mathscr{M}(U)$ and $\mathscr{P}(U)$ are unitary for $U \subset U_\iota$ open.
(iii) Let the sheaf morphisms $\Omega_{\kappa,\iota} : \mathscr{M}|_{U_\iota \cap U_\kappa} \,\#_\iota H \to \mathscr{M}|_{U_\iota \cap U_\kappa} \,\#_\kappa H$ be defined by $(\Omega_{\kappa,\iota})_U = (\Omega_\kappa)_U^{-1} \circ (\Omega_\iota)_U$, where $U \subset U_\iota \cap U_\kappa$ open. Then the following equations are valid:

$$(\Omega_\iota)_U(f \#_\iota 1) = \varrho_U(f), \quad f \in \mathscr{M}(U), \quad U \subset U_\iota, \tag{13}$$

$$((\Omega_{\kappa,\iota})_U \otimes id) \circ (id \otimes \Delta) = (id \otimes \Delta) \circ (\Omega_{\kappa,\iota})_U, \quad U \subset U_\iota \cap U_\kappa. \tag{14}$$

Suppose we are given a second $\mathbf{A}$-quantum principal bundle $(\mathscr{P}, \mathscr{M}, \varrho, H, (\tilde{\Omega}_\kappa)_{\kappa \in K})$ over M with coordinate system $(\tilde{\Omega}_\kappa)_{\kappa \in K}$ being defined on the open covering $(V_\kappa)_{\kappa \in K}$ of M. The two $\mathbf{A}$-quantum principal bundles with coordinate system are equivalent, if for $U \subset U_\iota \cap U_\kappa$ open, $\iota \in I$, $\kappa \in J$ the sheaf morphisms

$$\tilde{\Omega}_{\kappa,\iota} : \mathscr{M}|_{U_\iota \cap U_\kappa} \,\#_\iota H \to \mathscr{M}|_{U_\iota \cap U_\kappa} \,\#_\kappa H, \quad (\tilde{\Omega}_{\kappa,\iota})_U = (\tilde{\Omega}_\kappa)_U^{-1} \circ (\Omega_\iota)_U, \tag{15}$$

satisfy the equation

$$((\tilde{\Omega}_{\kappa,\iota})_U \otimes id) \circ (id \otimes \Delta) = (id \otimes \Delta) \circ (\tilde{\Omega}_{\kappa,\iota})_U. \tag{16}$$

This relation is an equivalence relation in the class of all $\mathbf{A}$-quantum principal bundles over M with coordinate system.

Definition 3.2. An equivalence class of $\mathbf{A}$-quantum principal bundles over M with coordinate system is called an $\mathbf{A}$-quantum principal bundle over M.

Remark 3.3. If no misunderstandings can arise, we will not distinguish between quantum principal bundles with coordinate system and their equivalence classes, the quantum principal bundles.

Remark 3.4. The algebras $\mathcal{M}|_{U_i \cap U_\kappa} \#_i H$ are examples of Hopf Galois extensions (see [3] for further details). Therefore one can interpret quantum principal bundles as sheaves which locally look like appropriate Hopf Galois extensions.
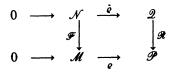
We would like to regard the quantum bundles as objects of a certain category. The following definition provides the necessary morphisms of this category.

Definition 3.5. *Let* $(P, \mathcal{M}, \varrho, H, (\tilde{\Omega}_i)_{i \in J})$ *(resp.* $(\mathcal{N}, \mathcal{Q}, \tilde{\varrho}, \tilde{H}, (\tilde{\Omega}_\kappa)_{\kappa \in K}))$ *be an* **A**-*QPB over* M *(resp. over* N*), where the coordinate system* $(\Omega_i)_{i \in J}$ *(resp.* $(\tilde{\Omega}_\kappa)_{\kappa \in K})$ *is defined on the open covering* $(U_i)_{i \in J}$ *(resp.* $(V_\kappa)_{\kappa \in K})$ *of* M *(resp.* N*). A morphism of* **A**-*quantum principal bundles*

$$(\mathcal{P}, \mathcal{M}, \varrho, H, (U_i)_{i \in J}) \to (\mathcal{Q}, N, \tilde{\varrho}, \tilde{H}, (V_\kappa)_{\kappa \in K})$$

consists of a tupel $(\mathcal{R}, \mathcal{F}, f, h)$ *such that the relations* (i) *to* (iv) *are satisfied.*

  (i) $f : M \to N$ *is a continuous mapping.*
 (ii) $\mathcal{R} : \mathcal{Q} \to \mathcal{P}$ *and* $\mathcal{F} : \mathcal{N} \to \mathcal{M}$ *are morphisms of* **A**-*sheaves over* f *such that the diagram*

$$
\begin{array}{ccc}
0 \longrightarrow \mathcal{N} & \xrightarrow{\ \tilde{\varrho}\ } & \mathcal{Q} \\
{\scriptstyle \mathcal{F}} \downarrow & & \downarrow {\scriptstyle \mathcal{R}} \\
0 \longrightarrow \mathcal{M} & \xrightarrow{\ \varrho\ } & \mathcal{P}
\end{array}
$$

commutes.
(iii) $h : \tilde{H} \to H$ *is a morphism of Hopf algebras.*
 (iv) *Let the mapping* $\mathcal{T}_{\iota,\kappa,V}$ *with* $\iota \in I, \kappa \in J, V \subset V_\kappa$ *open and* $U = f^{-1}(V) \cap U_\iota$ *be defined by*

$$\mathcal{T}_{\iota,\kappa,V} : \mathcal{N}(V) \#_\kappa \tilde{H} \xrightarrow{(\tilde{\Omega}_\iota)_V} \mathcal{Q}(V) \xrightarrow{\tilde{\mathcal{R}}_V}$$

$$\xrightarrow{\tilde{\mathcal{R}}_V} \mathcal{P}(f^{-1}(V)) \xrightarrow{r_U^{f^{-1}(V)}} \mathcal{P}(U) \xrightarrow{(\tilde{\Omega}_\iota)_V^{-1}} \mathcal{M}(U) \#_\iota H.$$

Then we have

$$(id \otimes \Delta_H) \circ \mathcal{T}_{\iota,\kappa,V} = (\mathcal{T}_{\iota,\kappa,V} \otimes h) \circ (id \otimes \Delta_{\tilde{H}}). \tag{17}$$

By a standard calculation we get the following theorem.

Theorem 3.6. *The* **A**-*quantum principal bundles and their morphisms (Definition 3.5) form a category* **A**-**Qpb***. The* **A**-*quantum principal bundles over* M *(resp. the* **A**-*quantum principal bundles over* M *with basis* $\mathcal{M}$*) together with the morphisms* $(f, \mathcal{R}, \mathcal{F}, h)$ *of the form* $f = id_M$ *(resp.* $f = id_M$ *and* $\mathcal{F} = id_\mathcal{M}$*) form a subcategory being denoted by* **A**-**Qpb**$_M$ *(resp.* **A**-**Qpb**$_\mathcal{M}$*).*

3.2. Coaction of the Structure Quantum Group

The structure quantum group H can be regarded as a gauge quantum group. In analogy with the commutative case H should (co-)act on the quantum bundle or in other words should provide noncommutative gauge transformations of the first kind. Starting from the example of commutative principal bundles we will show how to define this coaction and derive some fundamental results about it.

Let the sheaves $\mathcal{M}, \mathcal{P}$ and the Hopf algebra H be given by a principal bundle (P, M, π, G). The next question is what kind of H-coaction the G-action on P induces. To give an answer define for all $U \subset M$ open a homomorphism $\phi_U : \mathcal{P}(U) \to \mathcal{P}(U) \otimes H$ by

$$\phi_U(f)(u, a) = f(ua), \quad f \in \mathcal{P}(U), \quad u \in \pi^{-1}(U), \quad a \in G. \tag{18}$$

As $u(ab) = (ua)b$ for $u \in \pi^{-1}(U)$, $a, b \in G$, (18) entails

$$((id \otimes \Delta) \circ \phi_U)(f)(u, a, b) = f(u(ab))$$

$$= f((ua)b)$$

$$= ((\phi_U \otimes id) \circ \phi_U)(f)(u, a, b), \tag{19}$$

that is

$$(id \otimes \Delta) \circ \phi_U = (\phi_U \otimes id) \circ \phi_U. \tag{20}$$

A similar consideration using $ue = u$ for $u \in \pi^{-1}(U)$ proves

$$(id \otimes \varepsilon) \circ \phi_U = id. \tag{21}$$

Therefore ϕ_U gives $\mathcal{P}(U)$ the structure of an H-right-comodule and provides for a sheaf-morphism $\phi : \mathcal{P} \to \mathcal{P} \otimes H$. If $U \subset U_\iota$, we can express ϕ_U directly by the local trivialisations Ω_ι of the sheaf $\mathcal{P}$. Using (4) in Theorem 1.3 as well as (18) one gets the relation

$$\phi_U(f)(u, a) = f(\psi_\iota^{-1}(pr_1(\psi_\iota(u)), pr_2(\psi_\iota(u)) a))$$

$$= ((\Omega_\iota \otimes id) \circ (id \otimes \Delta) \circ \Omega_\iota^{-1}(f))(u, a), \tag{22}$$

that is

$$\phi_U = (\Omega_\iota \otimes id) \circ (id \otimes \Delta) \circ \Omega_\iota^{-1}. \tag{23}$$

Equation (23) will now be used to define a sheaf-morphism

$$\phi : \mathcal{P} \to \mathcal{P} \otimes H$$

in the general case of an arbitrary quantum principal bundle $(\mathcal{P}, \mathcal{M}, \varrho, H, (\Omega_\iota)_{\iota \in I})$. Let us show that by Eq. (23) ϕ is well-defined even in the noncommutative setting. We first have to prove

$$((\Omega_\iota \otimes id) \circ (id \otimes \Delta) \circ \Omega_\iota^{-1})(f) = ((\Omega_\kappa \otimes id) \circ (id \otimes \Delta) \circ \Omega_\kappa^{-1})(f) \tag{24}$$

for all $U \subset U_\iota \cap U_\kappa$ open and $f \in \mathcal{P}(U)$. But this is a consequence from

$$((\Omega_\iota \otimes id) \circ (id \otimes \Delta) \circ \Omega_\iota^{-1})(f) = (((\Omega_\kappa \circ \Omega_{\kappa,\iota}) \otimes id) \circ (id \otimes \Delta) \circ (\Omega_{\iota,\kappa} \circ \Omega_\kappa^{-1}))(f)$$

$$= ((\Omega_\kappa \otimes id) \circ (id \otimes \Delta) \circ \Omega_{\kappa,\iota} \circ \Omega_{\iota,\kappa} \circ \Omega_\kappa^{-1})(f)$$

$$= ((\Omega_\kappa \otimes id) \circ (id \otimes \Delta) \circ \Omega_\kappa^{-1})(f). \tag{25}$$

In the second step let $U \subset M$ be open and $f \in \mathcal{P}(U)$. Then for all $\iota \in I$ the homorphism $\phi_{U_\iota \cap U}(r^U_{U_\iota \cap U}(f))$ is defined, and for all $\iota, \kappa \in I$,

$$r^{U_\iota \cap U}_{U_\iota \cap U_\kappa \cap U} \circ \phi_{U_\iota \cap U}(r^U_{U_\iota \cap U}(f)) = \phi_{U_\iota \cap U_\kappa \cap U}(r^U_{U_\iota \cap U_\kappa \cap U}(f))$$

$$= r^{U_\kappa \cap U}_{U_\iota \cap U_\kappa \cap U} \circ \phi_{U_\kappa \cap U}(r^U_{U_\kappa \cap U}(f)) \tag{26}$$

is true. As $\mathscr{P} \otimes H$ is a sheaf, one gets a $\phi_U(f) \in \mathscr{P}(U) \otimes H$ with

$$r^U_{U_\iota \cap U} \circ \phi_U(f) = ((\Omega_\iota \otimes id) \circ (id \otimes \Delta) \circ \Omega_\iota^{-1})(r^U_{U_\iota \cap U}(f)). \tag{27}$$

Now the next theorem is evident.

Theorem 3.7. *Let* $(\mathscr{P}, \mathscr{M}, \varrho, H, (\Omega_\iota)_{\iota \in I})$ *be an* **A***-quantum principal bundle. Then there exists a uniquely defined sheaf-morphism* $\phi : \mathscr{P} \to \mathscr{P} \otimes H$ *fulfilling*

$$\phi_U(f) = ((\Omega_\iota \otimes id) \circ (id \otimes \Delta) \circ \Omega_\iota^{-1})(f), \quad f \in \mathscr{P}(U), \quad U \subset U_\iota. \tag{28}$$

If $(\mathscr{P}, \mathscr{M}, \varrho, H, (\Omega_\iota)_{\iota \in I}))$ *is given by a commutative principal bundle* (P, M, π, G), *the relation*

$$\phi_U(f)(u, a) = f(ua), \quad f \in \mathscr{P}(U), \quad u \in \pi^{-1}(U), \quad a \in G \tag{29}$$

is true.

Corollary 3.8. $\mathscr{P}(U)$ *is an* H-*right-comodule with coaction* ϕ_U, *that is the following relations hold*:

$$(id \otimes \Delta) \circ \phi_U = (\phi_U \otimes id) \circ \phi_U, \tag{30}$$

$$(id \otimes \varepsilon) \circ \phi_U = id. \tag{31}$$

Proof. As ϕ is a sheaf-morphism, it suffices to show (30) and (31) only locally for $U \subset U_\iota$ open. We have:

$$\begin{aligned}
(\phi_U \otimes id) \circ \phi_U &= (\Omega_\iota \otimes id \otimes id) \circ (id \otimes \Delta \otimes id) \circ (\Omega_\iota^{-1} \otimes id) \\
&\quad \circ (\Omega_\iota \otimes id) \circ (id \otimes \Delta) \circ \Omega_\iota^{-1} \\
&= (\Omega_\iota \otimes id \otimes id) \circ (id \otimes \Delta \otimes id) \circ (id \otimes \Delta) \circ \Omega_\iota^{-1} \\
&= (\Omega_\iota \otimes id \otimes id) \circ (id \otimes id \otimes \Delta) \circ (id \otimes \Delta) \circ \Omega_\iota^{-1} \\
&= (id \otimes \Delta) \circ (\Omega_\iota \otimes id) \circ (id \otimes \Delta) \circ \Omega_\iota^{-1} \\
&= (id \otimes \Delta) \circ \phi_U, \tag{32}
\end{aligned}$$

$$\begin{aligned}
(id \otimes \varepsilon) \circ \phi_U &= (id \otimes \varepsilon) \circ (\Omega_\iota \otimes id) \circ (id \otimes \Delta) \circ \Omega_\iota^{-1} \\
&= (\Omega_\iota \otimes id) \circ (id \otimes id \otimes \varepsilon) \circ (id \otimes \Delta) \circ \Omega_\iota^{-1} \\
&= \Omega_\iota \circ \Omega_\iota^{-1} \\
&= id. \tag{33}
\end{aligned}$$

Quod erat demonstrandum.   □

Remark 3.9. The last corollary justifies to call ϕ a noncommutative gauge transformation of the first kind.

For the moment let us suppose again $\mathscr{P}, \mathscr{M}, \varrho, H$ being defined by a classical principal bundle (P, M, π, G). The relation $\pi(ua) = \pi(u)$ for $u \in P$, $a \in G$ implies

$$\phi_U \circ \varrho_U(f) = \varrho_U(f) \otimes 1, \quad f \in \mathscr{M}(U), \tag{34}$$

because the equations

$$\phi_U \circ \varrho_U(f)(u, a) = \phi_U(f \circ \pi)(u, a) = f(\pi(ua)), \tag{35}$$

$$(\varrho_U(f) \otimes 1)(u, a) = f \circ \pi(u) \tag{36}$$

are true. An analogous result holds in the noncommutative case.

Theorem 3.10. *Let* $U \subset M$ *be open. Then* $h \in \mathscr{P}(U)$ *satisfies the equation*

$$\phi_U(h) = h \otimes 1 \tag{37}$$

if and only if $h = \varrho_U(f)$ *for a* $f \in \mathscr{M}(U)$.

Proof. As ϱ and ϕ are sheaf-morphism it suffices to assume $U \subset U_\iota$. Let us first suppose that $h = \varrho_U(f)$ for a $f \in \mathscr{M}(U)$. Then we get with (13):

$$\begin{aligned}
\phi_U \circ \varrho(f) &= ((\Omega_\iota \otimes id) \circ (id \otimes \varDelta) \circ \Omega_\iota^{-1})(\varrho_U(f)) \\
&= ((\Omega_\iota \otimes id) \circ (id \otimes \varDelta))(f \#_\iota 1) \\
&= (\Omega_\iota \otimes id)(f \#_\iota 1 \otimes 1) \\
&= \varrho_U(f) \otimes 1. \tag{38}
\end{aligned}$$

This gives one direction of the assertion. Now assume $\phi_U(h) = h \otimes 1$ for an $h \in \mathscr{P}(U)$. Then Eq. (28) implies:

$$(id \otimes \varepsilon \otimes id)(\Omega_\iota^{-1}(h) \otimes 1) = ((id \otimes \varepsilon \otimes id) \circ (id \otimes \varDelta) \circ \Omega_\iota^{-1})(h) = \Omega_\iota^{-1}(h). \tag{39}$$

As $(id \otimes \varepsilon \otimes id)$ $(\Omega_\iota^{-1}(h) \otimes 1) \in \mathscr{M}(U)$ $\#_\iota$ H, the relations (13) and $f := (id \otimes \varepsilon) \circ \Omega_\iota^{-1}(h)$ entail $h = \varrho_U(f)$, which gives the other direction. $\qquad\square$

3.3. Transition Functions

In the following we will derive some basic properties of the local coordinate changes $\Omega_{\kappa,\iota}$. Define the linear mappings $\tau_{\iota,\kappa} : H \to \mathscr{M}(U_\iota \cap U_\kappa)$, $\iota, \kappa \in I$ by:

$$\tau_{\iota,\kappa}(g) := (id \otimes \varepsilon) \circ \Omega_{\kappa,\iota}(1 \#_\iota g), \quad g \in H. \tag{40}$$

Then the $\Omega_{\kappa,\iota}$ can be written in the form:

$$\begin{aligned}
\Omega_{\kappa,\iota}(f \#_\iota g) &= \Omega_{\kappa,\iota}(f \#_\iota 1) \cdot \Omega_{\kappa,\iota}(1 \#_\iota g) \\
&= (f \#_\kappa 1) \cdot ((id \otimes \varepsilon \otimes id) \circ (id \otimes \varDelta) \circ \Omega_{\kappa,\iota})(1 \#_\iota g) \\
&= (f \#_\kappa 1) \cdot ((id \otimes \varepsilon \otimes id) \circ (\Omega_{\kappa,\iota} \otimes id) \circ (id \otimes \varDelta))(1 \#_\iota g) \\
&= \sum_{(g)} (f \#_\kappa 1) \cdot ((r_U^{U_\iota \cap U_\kappa} \circ \tau_{\iota,\kappa})(g_{(1)}) \#_\kappa g_{(2)}) \\
&= \sum_{(g)} (f \cdot (r_U^{U_\iota \cap U_\kappa} \circ \tau_{\iota,\kappa})(g_{(1)})) \#_\kappa g_{(2)}, \tag{41}
\end{aligned}$$

where $f \in \mathscr{M}(U)$, $U \subset U_\iota \cap U_\kappa$, and $g \in H$, or in the form:

$$\Omega_{\kappa,\iota} = ((m \otimes id) \circ (id \otimes (r_U^{U_\iota \cap U_\kappa} \circ \tau_{\iota,\kappa}) \otimes id) \circ (id \otimes \varDelta)). \tag{42}$$

Let us show that the linear mappings $\tau_{\iota,\kappa}$ can be considered as a generalisation of the transition functions in classical geometry. Suppose the quantum principal bundle $(\mathscr{P}, \mathscr{M}, \varrho, H, (\Omega_\iota)_{\iota \in I})$ is given by a principal bundle (P, M, π, G) with trivialisations $\psi_\iota : \pi^{-1}(U_\iota) \to U_\iota \times G$. By the definition of the $\tau_{\iota,\kappa}$ it is obvious that they have the form:

$$\tau_{\iota,\kappa} : H \to \mathscr{M}(U_\iota \cap U_\kappa), \quad g \mapsto g \circ \eta_{\iota,\kappa},$$

where the $\eta_{\iota,\kappa}$ are the classical transition functions defined by:

$$\psi_\iota \circ \psi_\kappa^{-1}(x, a) = (x, \eta_{\iota,\kappa}(x)a). \tag{43}$$

In the commutative case the $\tau_{\iota,\kappa}$ are morphisms of algebras, whereas in general they are only linear mappings between algebras.

The $\tau_{\iota,\kappa}$ are not independent from each other but fulfill certain conditions we will derive in the sequel. Let us first give an important definition.

Definition 3.11. *Let* $\mathcal{U} = (U_\iota)_{\iota \in I}$ *be an open covering of* M, H *a Hopf algebra and* $\mathcal{M}$ *an* **A**-*quantum space over* M. *Further let* $(\tau_{\iota,\kappa})_{(\iota,\kappa) \in I \times I}$ *be a family of linear mappings* $\tau_{\iota,\kappa} : H \to \mathcal{M}(U_\iota \cap U_\kappa)$ *satisfying the following conditions.*

(i) $\tau_{\iota,\iota}(1) = 1$,

(ii) $(r_U^{U_\iota \cap U_\lambda} \circ \tau_{\iota,\lambda}) = (r_U^{U_\iota \cap U_\kappa} \circ \tau_{\iota,\kappa}) * (r_U^{U_\kappa \cap U_\lambda} \circ \tau_{\kappa,\lambda})$,

(iii) $\tau_{\iota,\iota}(g) = \varepsilon(g) \cdot 1, \quad g \in H$,

where $U = U_\iota \cap U_\kappa \cap U_\lambda$ *and the convolution product* $*$ *is to be formed in the convolution algebra* $\mathrm{Hom}(H, \mathcal{M}(U))$. *Then* $(\tau_{\iota,\kappa})_{(\iota,\kappa) \in I \times I}$ *is called an* H-*cocycle in* $\mathcal{M}$.[1]

Now the defining equation (40) implies:

$$\tau_{\iota,\kappa}(1) = 1. \tag{44}$$

Then we have for $U \subset U_\iota \cap U_\kappa \cap U_\lambda$ open because of (42) and the definition of the $\Omega_{\iota,\kappa}$:

$$(m \otimes id) \circ (id \otimes (r_U^{U_\iota \cap U_\lambda} \circ \tau_{\iota,\lambda}) \otimes id) \circ (id \otimes \Delta) = \Omega_{\lambda,\iota}$$

$$= \Omega_{\lambda,\kappa} \circ \Omega_{\kappa,\iota}$$

$$= (m \otimes \mathrm{id}) \circ (id \otimes (r_U^{U_\kappa \cap U_\lambda} \circ \tau_{\kappa,\lambda}) \otimes id) \circ (id \otimes \Delta)$$

$$\circ (m \otimes id) \circ (id \otimes (r_U^{U_\iota \cap U_\kappa} \circ \tau_{\iota,\kappa}) \otimes id) \circ (id \otimes \Delta)$$

$$= (m \otimes id) \circ (m \otimes id \otimes id)$$

$$\circ (id \otimes (r_U^{U_\iota \cap U_\kappa} \circ \tau_{\iota,\kappa}) \otimes (r_U^{U_\kappa \cap U_\lambda} \circ \tau_{\kappa,\lambda}) \otimes id)$$

$$\circ (id \otimes id \otimes \Delta) \circ (id \otimes \Delta)$$

$$= (m \otimes id) \circ (id \otimes m \otimes id)$$

$$\circ (id \otimes (r_U^{U_\iota \cap U_\kappa} \circ \tau_{\iota,\kappa}) \otimes (r_U^{U_\kappa \cap U_\lambda} \circ \tau_{\kappa,\lambda}) \otimes id)$$

$$\circ (id \otimes \Delta \otimes id) \circ (id \otimes \Delta). \tag{45}$$

This relation entails for $g \in H$:

$$(r_U^{U_\iota \cap U_\lambda} \circ \tau_{\iota,\lambda})(g) = ((id \otimes \varepsilon) \circ (m \otimes id) \circ (id \otimes (r_U^{U_\iota \cap U_\lambda} \circ \tau_{\iota,\lambda}) \otimes id) \circ (id \otimes \Delta))(1 \otimes g)$$

$$= ((id \otimes \varepsilon) \circ (m \otimes id) \circ (id \otimes m \otimes id)$$

$$\circ (id \otimes (r_U^{U_\iota \cap U_\kappa} \circ \tau_{\iota,\kappa}) \otimes (r_U^{U_\kappa \cap U_\lambda} \circ \tau_{\kappa,\lambda}) \otimes id)$$

$$\circ (id \otimes \Delta \otimes id) \circ (id \otimes \Delta))(1 \otimes g)$$

$$= (m \circ ((r_U^{U_\iota \cap U_\kappa} \circ \tau_{\iota,\kappa}) \otimes (r_U^{U_\kappa \cap U_\lambda} \circ \tau_{\kappa,\lambda})) \circ \Delta)(g). \tag{46}$$

Finally (14) gives for $\iota = \kappa$ and $g \in H$ the equation:

$$\tau_{\iota,\iota}(g) = \varepsilon(g) \cdot 1. \tag{47}$$

Let us subsume the last results in a proposition.

---

[1] The cocycles defined here are different from the ones in Theorem B.2. The context always makes clear which kind of cocycles are meant so that no confusions can arise.

Proposition 3.12. *The transition functions* $\tau_{\iota,\kappa}: H \to \mathcal{M}(U_\iota \cap U_\kappa)$ *of a quantum principal bundle* $(\mathcal{P}, \mathcal{M}, \varrho, H, (\Omega_\iota)_{\iota \in I})$ *form an H-cocycle in $\mathcal{M}$.*

The transition function $\tau_{\iota,\kappa}$ characterise the (quantum) principal bundle in the commutative as well as in the noncommutative case. We will show how to construct a quantum principal bundle out of a family $(\tau_{\iota,\kappa})_{(\iota,\kappa) \in I \times I}$ of transition functions fulfilling the cocycle conditions in Definition 3.11. Let us further suppose we are given a family $(\alpha_\iota)_{\iota \in I}$ of weak actions $\alpha_\iota: \mathcal{M}(U_\iota) \times H \to \mathcal{M}(U_\iota)$ and a family $(\iota)_{\iota \in I}$ of normal cocycles $\iota: H \times H \to \mathcal{M}(U_\iota)$ fulfilling the twisted module condition[2] According to Appendix B, Theorem B.2 the crossed products $\mathcal{M}(U)\#_\iota H$ exist for all $U \subset U_\iota$ open. It is assumed now that the linear mappings

$$(m \otimes id) \circ (id \otimes \tau_{\iota,\kappa} \otimes id) \circ (id \otimes \Delta): \mathcal{M}(U_\iota \cap U_\kappa)\#_\iota H \to \mathcal{M}(U_\iota \cap U_\kappa)\#_\kappa H$$

are morphisms of algebras with unit.

To construct the desired quantum principal bundle we consider the algebras

$$\mathcal{P}_0(U) = \bigoplus_{\iota \in I} \mathcal{M}(U_\iota \cap U)\#_\iota H \tag{48}$$

for all $U \subset M$ open and their subalgebras

$$\mathcal{P}(U) = \left\{ \sum_{\iota \in I} f_\iota \in \mathcal{P}_0(U): \forall \iota, \kappa \in I \ (r^{U_\iota \cap U}_{U_\iota \cap U_\kappa \cap U} \circ id)(f_\iota) \right.$$
$$= ((m \otimes id) \circ (id \otimes (r^{U_\iota \cap U_\kappa}_{U_\iota \cap U_\kappa \cap U} \circ \tau_{\iota,\kappa}) \otimes id)$$
$$\left. \circ (id \otimes \Delta) \circ (r^{U_\kappa \cap U}_{U_\iota \cap U_\kappa \cap U} \otimes id))(f_\kappa) \right\}. \tag{49}$$

Obviously $U \to \mathcal{P}_0(U)$ defines a sheaf $\mathcal{P}_0$ on M and $U \to \mathcal{P}(U)$ a subsheaf $\mathcal{P}$. The next lemma helps to characterise the sheaf $\mathcal{P}$.

Lemma 3.13. *Suppose* $\sum_{\iota \in I} f_\iota \in \mathcal{P}_0(U)$ *and* $U \subset U_\kappa$ *open. Then* $\sum_{\iota \in I} f_\iota \in \mathcal{P}(U)$ *if and only if for all* $\iota \in I$ *the equation*

$$(r^{U_\iota \cap U}_{U_\iota \cap U_\kappa \cap U} \otimes id)(f_\iota) = ((m \otimes id) \circ (id \otimes (r^{U_\iota \cap U_\kappa}_{U_\iota \cap U_\kappa \cap U} \circ \tau_{\iota,\kappa}) \otimes id)$$
$$\circ (id \otimes \Delta) \circ (r^{U_\kappa \cap U}_{U_\iota \cap U_\kappa \cap U} \otimes id))(f_\kappa) \tag{50}$$

is satisfied.

Proof. Let $\sum_{\iota \in I} f_\iota \in \mathcal{P}_0(U)$ fulfill the relation (50). According to Definition 3.11 (ii) we have for all $\iota, \lambda, \mu \in I$:

$$((m \otimes id) \circ (id \otimes (r^{U_\iota \cap U_\lambda}_{U_\iota \cap U_\lambda \cap U} \circ \tau_{\iota,\lambda}) \otimes id) \circ (id \otimes \Delta))$$
$$\circ ((m \otimes id) \circ (id \otimes (r^{U_\lambda \cap U_\mu}_{U_\lambda \cap U_\mu \cap U} \circ \tau_{\lambda,\mu}) \otimes id) \circ (id \otimes \Delta))$$
$$= ((m \otimes id) \circ (id \otimes m \otimes id)$$
$$\circ (id \otimes (r^{U_\iota \cap U_\lambda}_{U_\iota \cap U_\lambda \cap U} \circ \tau_{\iota,\lambda}) \otimes (r^{U_\lambda \cap U_\mu}_{U_\lambda \cap U_\mu \cap U} \circ \tau_{\lambda,\mu}) \otimes id)$$
$$\circ (id \otimes \Delta \otimes id) \circ (id \otimes \Delta))$$
$$= ((m \otimes id) \circ (id \otimes (r^{U_\iota \cap U_\mu}_{U_\iota \cap U_\mu \cap U} \circ \tau_{\iota,\mu}) \otimes id) \circ (id \otimes \Delta)). \tag{51}$$

---

[2] See the appendix for further details on weak actions and normal cocycles.

This equation as well as (50) and 3.11 (iii) imply

$$(r^{U_\lambda \cap U}_{U_\lambda \cap U_\lambda \cap U} \otimes id)(f_\kappa) = ((m \otimes id) \circ (id \otimes (r^{U_\lambda \cap U_\lambda}_{U_\lambda \cap U_\lambda \cap U} \circ \tau_{\kappa,\lambda}) \otimes id) \circ (id \otimes \Delta)$$
$$\circ (r^{U_\lambda \cap U}_{U_\lambda \cap U_\lambda \cap U} \otimes id))(f_\lambda). \tag{52}$$

With (51) one derives the relation

$$(r^{U_\iota \cap U}_{U_\iota \cap U_\lambda \cap U} \otimes id)(f_\iota) = ((m \otimes id) \circ (id \otimes (r^{U_\iota \cap U_\kappa}_{U_\iota \cap U_\lambda \cap U} \circ \tau_{\iota,\kappa}) \otimes id) \circ (id \otimes \Delta)$$
$$\circ (r^{U_\kappa \cap U}_{U_\iota \cap U_\lambda \cap U} \otimes id))(f_\kappa)$$

$$= ((m \otimes id) \circ (id \otimes (r^{U_\iota \cap U_\kappa}_{U_\iota \cap U_\lambda \cap U} \circ \tau_{\iota,\kappa}) \otimes id) \circ (id \otimes \Delta)$$
$$\circ (m \otimes id) \circ (id \otimes (\tau^{U_\kappa \cap U_\lambda}_{U_z \cap U_\lambda \cap U} \circ \tau_{\lambda,\lambda}) \otimes id) \circ (id \otimes \Delta)$$
$$\circ (r^{U_\lambda \cap U}_{U_\iota \cap U_\lambda \cap U} \otimes id))(f_\lambda)$$

$$= ((m \otimes id) \circ (id \otimes (r^{U_\iota \cap U_\lambda}_{U_\iota \cap U_\lambda \cap U} \circ \tau_{\iota,\lambda}) \otimes id) \circ (id \otimes \Delta)$$
$$\circ (r^{U_\lambda \cap U}_{U_\iota \cap U_\lambda \cap U} \otimes id))(f_\lambda). \tag{53}$$

Therefore $\sum_{\iota \in I} f_\iota \in \mathscr{P}(U)$ and one part of the assertion is proven. The other one is trivial.   □

We have to supply sheaf-morphisms $\varrho : \mathscr{M} \to \mathscr{P}$ and $\Omega_\iota : \mathscr{M}|_{U_\iota} \#_\iota \to \mathscr{P}|_{U_\iota}$. As for all $f \in \mathscr{M}(U)$, $U \in M$ open, the sum $\sum_{\iota \in I} r^U_{U_\iota \cap U}(f) \#_\iota 1$ lies in $\mathscr{P}(U)$, we can set

$$\varrho : \mathscr{M} \to \mathscr{P}, \quad \varrho_U : \mathscr{M}(U) \to \mathscr{P}(U), \quad f \mapsto \sum_{\iota \in I} r^U_{U_\iota \cap U}(f) \#_\iota 1.$$

The mapping $\Omega_\iota : \mathscr{M}|_{U_\iota} \#_\iota \to \mathscr{P}|_{U_\iota}$ shall be given by

$$\Omega_\iota : \mathscr{M}|_{U_\iota} \#_\iota \to \mathscr{P}|_{U_\iota},$$
$$(\Omega_\iota)_U : \mathscr{M}(U) \#_\iota \to \mathscr{P}(U), \quad f \mapsto \sum_{\iota \in I} f_\iota, \quad U \subset U_\iota,$$

where

$$f_\kappa = ((m \otimes id) \circ (id \otimes (r^{U_\iota \cap U_\kappa}_{U_\iota \cap U_\lambda \cap U} \circ \tau_{\iota,\kappa}) \otimes id)$$
$$\circ (id \otimes \Delta) \circ (r^{U_\iota \cap U}_{U_\iota \cap U_\lambda \cap U} \otimes id))(f). \tag{54}$$

We have in particular $f_\iota = f$. Now Lemma 3.13 shows that Ω_ι is well-defined. By the definition of ϱ and Ω_ι it is clear that Eq. (13) $(\Omega_\iota)_U(f \otimes 1) = \varrho_U(f)$ for $f \in \mathscr{M}(U)$, $U \subset U_\iota$ is satisfied. If we can prove Ω_ι being bijective, our considerations show that ϱ_U is injective for $U \subset U_\iota$. This will give the exactness of the sequence

$$0 \to \mathscr{M} \xrightarrow{\varrho} \mathscr{P}.$$

Therefore it has to be proven that Ω_ι is an isomorphism which satisfies (14). Define for $U \subset U_\iota$ open:

$$\Xi_\iota : \mathscr{P}|_{U_\iota} \to \mathscr{M}|_{U_\iota} \#_\iota H,$$
$$(\Xi_\iota)_U : \mathscr{P}(U) \to \mathscr{M}(U) \#_\iota H, \quad \sum_{\kappa \in I} f_\kappa \mapsto f_\iota.$$

Then it is easy to see

$$\Xi_\iota \circ \Omega_\iota = id \quad \text{und} \quad \Omega_\iota \circ \Xi_\iota = id, \tag{55}$$

that is Ω_ι is a sheaf-isomorphism with inverse Ξ_ι. Further we get (14)

$$(\Omega_\iota \otimes id) \circ (id \otimes \Delta) = (id \otimes \Delta) \circ \Omega_\iota, \tag{56}$$

which follows from the definition of Ω_i and the coassociativity in H. Now we can state the final theorem.

Theorem 3.14. *Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of M, H a Hopf algebra, $\mathcal{M}$ an A-quantum space over M, and $(\tau_{i,\kappa})_{(i,\kappa) \in I \times I}$ an H-cocycle in $\mathcal{M}$. Further let $\alpha_i : \mathcal{M}(U_i) \times H \to \mathcal{M}(U_i)$ be weak actions and $\iota : H \times H \to \mathcal{M}(U_i)$ normal cocycles fulfilling the twisted module condition. The linear mappings*

$$(m \otimes id) \circ (id \otimes \tau_{i,\kappa} \otimes id) \circ (id \otimes \Delta) : \mathcal{M}(U_i \cap U_\kappa) \#_i H \to \mathcal{M}(U_i \cap U_\kappa) \#_\kappa H$$

are supposed to be morphisms of algebras with unity. Then there exists an A-quantum principal bundle $(\mathcal{P}, \mathcal{M}, \varrho, H, (\Omega_i)_{i \in I})$ over M uniquely defined up to isomorphism such that the $\tau_{i,\kappa}$ are its transition functions or in other words such that

$$(\Omega_{\kappa,i})_U = (m \otimes id) \circ (id \otimes (r^{U_i \cap U_\kappa}_{U_i \cap U_\kappa \cap U} \circ \tau_{i,\kappa}) \otimes id)$$

$$\circ (id \otimes \Delta) \circ (r^{U_i \cap U}_{U_i \cap U_\kappa \cap U} \otimes id) \tag{57}$$

is satisfied for $U \subset U_i \cap U_\kappa$ open.

Proof. Most of the theorem has been proven above, but we still have to show (57). Let $U \subset U_i \cap U_\kappa$ be open, $f \in \mathcal{M}(U) \#_i H$, $g \in \mathcal{M}(U) \#_\kappa H$ and $g = ((\Omega_{\kappa,i})_U) (f)$. Then (54) entails

$$g = g_\kappa = pr_\kappa((\Omega_\kappa)_U(g)) = pr_\kappa((\Omega_i)_U(f)) = f_\kappa = ((m \otimes id)$$

$$\circ (id \otimes (r^{U_i \cap U_\kappa}_{U_i \cap U_\kappa \cap U} \circ \tau_{i,\kappa}) \otimes id) \circ (id \otimes \Delta) \circ (r^{U_i \cap U}_{U_i \cap U_\kappa \cap U} \otimes id))(f), \tag{58}$$

which gives the desired equation. The statement about the uniqueness of the quantum principal bundle up to isomorphism is clear by definition. ☐

4. Quantum Vector Bundles

4.1. Definition and Examples

We can also translate vector bundles in the language of quantum spaces. As typical fibres we use quadratic algebras which according to Manin [17] are considered as the noncommutative linear spaces. Like in the case of quantum groups the multiplication on the tensor products serving as the local trivialisations has to be defined by the method in Appendix B.

Suppose we are given the following objects:

(i) a sheaf $\mathcal{M}$ over a topological space M with objects in the category **A** called the base quantum space,

(ii) a sheaf $\mathcal{V}$ over M with objects in **A** called the total quantum space,

(iii) a sheaf morphism $\varrho : \mathcal{M} \mapsto \mathcal{V}$ called the projection,

(iv) a quadratic algebra A called the typical fibre,

(v) a Hopf-algebra H called the structure quantum group,

(vi) a coaction $\varphi : A \mapsto H \otimes A$,

(vii) a family $(\Gamma_i)_{i \in I}$ of sheaf morphisms $\Gamma_i : \mathcal{M}|_{U_i} \#_i A \to \mathcal{V}|_{U_i}$, where $\mathcal{U} = (U_i)_{i \in I}$ is an open covering of M and $\#_i$ is a crossed product which is given according to Theorem B.6 by a weak action $\alpha_i : \mathcal{M}|_{U_i} \to H \times \mathcal{M}|_{U_i}$, a normal cocycle $\iota : H \times H \to \mathcal{M}(U_i)$ fulfilling the twisted module condition and the coaction φ.

$(\mathcal{V}, \mathcal{M}, \varrho, A, H, \varphi, (\Gamma_i)_{i \in I})$ gives the data of an **A**-quantum vector bundle over M.

Definition 4.1. *The tupel* $(\mathscr{V}, \mathscr{M}, \varrho, A, H, \varphi, (\Gamma_\iota)_{\iota \in I})$ *is said to be an* **A**-*quantum vector bundle with coordinate system* $(\Gamma_\iota)_{\iota \in I}$, *if the following conditions hold:*

(i) *The sequence* $0 \to \mathscr{M} \xrightarrow{\varrho} \mathscr{V}$ *is exact.*

(ii) *The algebras* $\mathscr{M}(U)$ *and* $\mathscr{V}(U)$ *are unitary for* $U \subset U_\iota$ *open.*

(iii) *Let the sheaf morphisms* $\Gamma_{\kappa,\iota} : \mathscr{M}|_{U_\iota \cap U_\kappa} \#_\iota A \to \mathscr{M}|_{U_\iota \cap U_\kappa} \#_\kappa A$ *be defined by* $(\Gamma_{\kappa,\iota})_U = (\Gamma_\kappa)_U^{-1} \circ (\Gamma_\iota)_U$, *where* $U \subset U_\iota \cap U_\kappa$ *open. Then one can find linear mappings* $\tau_{\iota,\kappa} : H \to \mathscr{M}(U_\iota \cap U_\kappa)$, $\iota, \kappa \in I$ *such that the following equations hold:*

$$(\Gamma_\iota)_U (f \#_\iota 1) = \varrho_U(f), \quad f \in \mathscr{M}, \ U \subset U_\iota, \tag{59}$$

$$(\Gamma_{\kappa,\iota})_U = (m \otimes id) \circ (id \otimes (r_U^{U_\iota \cap U_\kappa} \circ \tau_{\iota,\kappa}) \otimes id) \circ (id \otimes \varphi),$$

$$U \subset U_\iota \cap U_\kappa. \tag{60}$$

Remark 4.2. Equation (60) can also be written in the form

$$\Gamma_{\kappa,\iota}(f \#_\iota g) = \sum_{(g)} f \tau_{\iota,\kappa}(g_{(-1)}) \otimes g_{(0)}, \tag{61}$$

where $f \in \mathscr{M}(U)$, $g \in A$ and $U \subset U_\iota \cap U_\kappa \cap U_\lambda$.

Remark 4.3. The tupel $(\mathscr{V}, \mathscr{M}, \varrho, A, H, \varphi, (\Gamma_\iota)_{\iota \in I})$ should better be defined as a quantum vector bundle with coordinate system similarly like in the case of quantum principal bundles (see Definition 3.1). Quantum vector bundles were equivalence classes of quantum vector bundles with coordinate system. But this procedure does not give new aspects, and the technical details are analogous to the ones in the definition of quantum principal bundles.

The transition function $\tau_{\iota,\kappa}$ are not independent from each other.

Proposition 4.4. *The linear mappings*

$$\tau_{\iota,\kappa} : H \to \mathscr{M}(U_\iota \cap U_\kappa), \quad \iota, \kappa \in I$$

form an H-cocycle in $\mathscr{M}$ *over* $(U_\iota)_{\iota \in I}$.

Proof. This can be shown exactly like in Proposition 3.12. $\square$

We state the next theorem but postpone the proof till we introduce noncommutative associated bundles.

Theorem 4.5. *Let* $(\tau_{(\iota,\kappa)})_{(\iota,\kappa) \in I \times I}$ *be an H-cocycle in* $\mathscr{M}$, *and* $\varphi : A \to H \otimes A$ *a left coaction on the quadratic algebra* A. *Further suppose that the mappings* $\alpha_\iota : H \times \mathscr{M}|_{U_\iota} \to \mathscr{M}|_{U_\iota}$ *are actions and the cocycles* $\iota : H \times H \to \mathscr{M}(U_\iota)$ *are trivial, that means* $\iota(h, l) = \varepsilon(h) \varepsilon(l)$ *for* $1, h, l \in H$. *Then there exists a quantum vector bundle which has* A *as its typical fibre,* H *as its structure group and the* $\tau_{\iota,\kappa}$ *as transition functions.*

Classical vector bundles are natural examples of quantum vector bundles as will be shown in the following.

Let $\pi : E \to M$ be a real vector bundle of dimension n over the topological space M, where the structure group $G \subset Gl(n, \mathbb{R})$ is compact. Now define the

following objects[3]:

(i) $\mathscr{M}$ is the sheaf of continuous bounded $\mathbb{C}$-functions on M.

(ii) $\mathscr{V}$ is the sheaf on M defined by

$$U \to \mathscr{V}(U), \quad U \subset M \text{ open},$$

$$i_V^U \to \mathscr{V}(i_V^U), \quad V \subset U \subset M \text{ open}.$$

Here $\mathscr{V}(U)$ is the algebra of complex continuous bounded functions on $\pi^{-1}(U)$ and $\mathscr{V}(i_V^U)$ the restriction from $\pi^{-1}(U)$ to $\pi^{-1}(V)$.

(iii) ϱ is the sheaf morphism

$$\pi^* : \mathscr{M} \to \mathscr{V},$$

$$\pi_U^* : \mathscr{M}(U) \to \mathscr{V}(U), \quad f \mapsto f \circ \pi|_U, \quad U \subset M.$$

(iv) A is the quadratic algebra of complex polynomials in n variables $x_1, \ldots, x_n$, where the x_i are the coordinate projections of $\mathbb{R}^n$. Further let $\tilde{A}$ be the $*$-Fréchet algebra of complex continuous functions on $\mathbb{R}^n$. Then A lies densely in $\tilde{A}$.

(v) G gives the (topological) Hopf algebra H of continuous functions on G.

(vi) φ is dual to the action $G \times \mathbb{R}^n \to \mathbb{R}^n$, that means φ is the coaction

$$\tilde{A} \to H \otimes \tilde{A},$$

$$f \mapsto \sum_{(f)} f_{(-1)} \otimes f_{(0)} = ((a, v) \mapsto f(av)).$$

(vii) Let $(U_i)_{i \in I}$ be an open covering of M such that trivialisations $\psi_i : E|_{U_i} \to U_i \times \mathbb{R}^n$ exist. These induce sheaf isomorphisms

$$\Gamma_i : \mathscr{M}|_{U_i} \otimes \tilde{A} \to \mathscr{V}|_{U_i},$$

$$f \otimes g \mapsto (f \otimes g) \circ \psi_i|_{U_i},$$

where $g \in \tilde{A}, f \in \mathscr{M}(U)$, and $U \subset U_i$ open.

Obviously the above defined objects give rise to the following example.

Example 4.6. *The tupel* $(\mathscr{V}, \mathscr{M}, \varrho, A, H, \varphi, (\Gamma_i)_{i \in I})$ *is a quantum vector bundle over* M.

Example 4.7. *Let* $\mathscr{M}$ *be an arbitrary quantum space over* M, H *a Hopf algebra, and* A *a quadratic* H-*left comodule algebra with coaction* φ. *Then*

$$\mathscr{V} = \mathscr{M} \otimes A,$$

$$\varrho : \mathscr{M} \to \mathscr{V}, \quad f \mapsto f \otimes 1,$$

gives a trivial quantum vector bundle $(\mathscr{V}, \mathscr{M}, \varrho, A, H, \varphi, (\mathrm{id}))$.

4.2. Associated Quantum Vector Bundles

One of the most important tools in the geometry of fibre bundles are the associated vector bundles. They are used in physics as well. More precisely do material fields live

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[3] Here we use a slight topological generalisation of our concept, but don't want to go deeper in the subject at the moment.

in vector bundles which are associated to a principal bundle describing the gauge transformations. Because of their importance in geometry and physics we want to translate associated vector bundles to the quantum language. To find the right definition we will first examine the classical analogon and then dualise the classical objects and relations.

In classical geometry one first forms the cartesian product $P \times V$, where P is a principal bundle over a topological space M and V a vector space on which the structure group G of the principal bundle acts from the left. Now

$$(P \times V) \times G \rightarrow P \times V$$

$$((a, v), g) \mapsto (ag, g^{-1} v), \quad a \in P, \quad v \in V, \quad g \in G \tag{62}$$

defines a right G-action on $P \times V$.

In the noncommutative case we have a quantum principal bundle $\mathscr{P}$ over $\mathscr{M}$ with coaction $\phi : \mathscr{P} \rightarrow \mathscr{P} \otimes H$, a quadratic algebra A and a left coaction $\varphi : A \rightarrow H \otimes A$. The left coaction is supposed to be a morphism of algebras. Now one can construct a morphism of sheaves of complex vector spaces $\psi : A \otimes \mathscr{P} \rightarrow A \otimes \mathscr{P} \otimes H$ by

$$f \otimes g \mapsto \psi_U(f \otimes g) = \sum_{(f), (g)} f_{(0)} \otimes g_{(0)} \otimes (S^{-1} f_{(-1)}) g_{(1)}, \tag{63}$$

where $f \in A$, $g \in \mathscr{P}(U)$ and $U \subset M$ open. Furthermore we used the notation

$$\phi(g) = \sum_{(g)} g_{(0)} \otimes g_{(1)} \quad \text{and} \quad \varphi(f) = \sum_{(f)} f_{(-1)} \otimes f_{(0)}. \tag{64}$$

ψ is the noncommutative analogon to the above G-action on $P \times V$.

Lemma 4.8. $\psi_U : A \otimes \mathscr{P}(U) \rightarrow A \otimes \mathscr{P}(U) \otimes H$ *is a right H-coaction for all $U \subset M$ open, that is*

$$(id \otimes \Delta) \circ \psi_U = (\psi_U \otimes id) \circ \psi_U, \tag{65}$$

$$(id \otimes id \otimes \varepsilon) \circ \psi_U = id. \tag{66}$$

Proof. Let us first show Eq. (65). On the one hand the relation

$$(\psi_U \otimes id) \circ \psi_U(f \otimes g)$$

$$= (\psi_U \otimes id) \sum_{(f), (g)} f_{(0)} \otimes g_{(0)} \otimes (S^{-1} f_{(-1)}) g_{(1)}$$

$$= \sum_{(f), (g)} f_{(0)} \otimes g_{(0)} \otimes (S^{-1} f_{(-1)}) g_{(1)} \otimes (S^{-1} f_{(-2)}) g_{(2)}, \tag{67}$$

is true. On the other hand we have with the flip operator τ

$$\Delta(S^{-1} f_{(-1)} g_{(1)}) = \Delta(S^{-1} f_{(-1)}) \cdot \Delta(g_{(1)})$$

$$= ((S^{-1} \otimes S^{-1}) \circ \tau \circ \Delta(f_{(-1)})) \cdot \Delta(g_{(1)})$$

$$= (S^{-1} f_{(-1)} \otimes S^{-1} f_{(-2)}) \cdot (g_{(1)} \otimes g_{(2)}). \tag{68}$$

Altogether this gives

$$(\psi_U \otimes id) \circ \psi_U (f \otimes g) = \sum_{(f), (g)} f_{(0)} \otimes g_{(0)} \otimes \Delta(S^{-1} f_{(-1)} g_{(1)})$$

$$= (id \otimes \Delta) \psi_U (f \otimes g). \tag{69}$$

The relation (66) is a consequence of

$$(id \otimes id \otimes \varepsilon) \circ \psi_U (f \otimes g) = (\varepsilon \circ S^{-1} \otimes id \otimes id \otimes \varepsilon) \circ (\varphi \otimes \phi)(f \otimes g)$$

$$= (\varepsilon \otimes id \otimes id \otimes \varepsilon) \circ (\varphi \otimes \phi)(f \otimes g)$$

$$= f \otimes g. \tag{70}$$

This proves the lemma.   □

In the theory of commutative fibre bundles one defines an equivalence relation $\sim$ on $P \times V$ by

$$(a, v) \sim (b, w) \Leftrightarrow (a, v) = (bg, g^{-1}w), \quad g \in G. \tag{71}$$

The equivalence classes of this equivalence relation form a vector bundle E over M, the associated vector bundle.

We want to dualise this. A function $f \in \mathscr{C}(P \times V)$ is said to be lifted by a function $\bar{f} \in \mathscr{C}(E)$, if $\bar{f} \circ p = f$ for the canonical projection $p : P \times V \to E$. f can be lifted or regarded as a function on the vector bundle E if and only if for all $a \in \mathscr{P}$, $v \in V$, and $g \in G$,

$$f(ag, g^{-1}v) = f(a, v). \tag{72}$$

In the language of quantum principal bundle this means that the element

$$f = \sum_i f_i \otimes f'_i \in A \otimes \mathscr{P}(U), \quad U \subset M \tag{73}$$

can be regarded as an element of the associated quantum principal bundle if and only if

$$\psi_U \left(\sum_i f_i \otimes f'_i \right) = f_i \otimes f'_i \otimes 1. \tag{74}$$

Therefore we define for all $U \subset M$ open,

$$\mathscr{V}(U) = \left\{ \sum_i f_i \otimes f'_i \in A \otimes \mathscr{P}(U) : \psi_U \left(\sum_i f_i \otimes f'_i \right) = f_i \otimes f'_i \otimes 1 \right\}.$$

Remark 4.9. $\mathscr{V}(U)$ is the cotensor product of A and $\mathscr{P}(U)$ over H, in signs

$$\mathscr{V}(U) = A \,\square_H\, \mathscr{P}(U). \tag{75}$$

Theorem 4.10. *Let us set*

$$\mathscr{V}(U) = A \,\square_H\, \mathscr{P}(U),$$

$$\mathscr{V}(i_V^U) = id_A \otimes r_V^U$$

for $V \subset U \subset M$ open. This gives a sheaf of associative algebras over M, where the multiplication $m : \mathscr{V}(U) \otimes \mathscr{V}(U) \to \mathscr{V}(U)$ is defined by $f \cdot g = \sum_{i,j} f_i g_j \otimes f'_i g'_i$, with $f = (\sum_i f_i \otimes f'_i)$, $g = (\sum_j g_j \otimes g'_j) \in \mathscr{V}(U)$. So $\mathscr{V}$ becomes a quantum space.

Proof. It is obvious that $\mathscr{V}$ is a subsheaf of the sheaf $A \otimes \mathscr{P}$ of complex vector spaces. Therefore we only have to show

$$\psi_U \left(\sum_{i,j} f_i g_j \otimes f'_i g'_j \right) = \sum_{i,j} f_i g_j \otimes f'_i g'_j \otimes 1. \tag{76}$$

We get with the universal property of the tensor product:

$$\psi_U\left(\sum_{i,j} f_i g_j \otimes f'_i g'_j\right) = \sum_{i,j} \sum_{(f_i),(g_j),(f'_i),(g'_j)} f_{i(0)} g_{j(0)} \otimes f'_{i(0)} g'_{j(0)}$$

$$\otimes (S^{-1} g_{j(-1)})(S^{-1} f_{i(-1)}) f'_{i(1)} g'_{j(1)}$$

$$= \sum_{j} \sum_{(g_j),(g'_j)} \left(\sum_{i} \sum_{(f_i),(f'_i)} f_{i(0)} g_{j(0)} \otimes f'_{i(0)} g'_{j(0)} \right)$$

$$\otimes (S^{-1} g_{j(-1)})((S^{-1} f_{i(-1)}) f'_{i(1)}) g'_{j(1)}$$

$$= \sum_{j} \sum_{(g_j),(g'_j)} \left(\sum_{i} f_i g_{j(0)} \otimes f'_i g'_{j(0)} \otimes (S^{-1} g_{j(-1)}) g'_{j(1)} \right)$$

$$= \sum_{i,j} f_i g_j \otimes f'_i g'_j \otimes 1. \tag{77}$$

$\square$

The quantum space $\mathscr{V}$ will turn out to be a quantum vector bundle. We are going to prove this and want to find appropriate local trivialisations. Let $\mathscr{U} = (U_i)_{i \in I}$ be an open covering of M such that the given quantum principal bundle $(\mathscr{P}, \mathscr{M}, \varrho, H, (\Omega_i)_{i \in I})$ is locally trivial over $\mathscr{U}$ and the $\tau_{i,\kappa}$ are transition functions. According to Proposition 3.12 the family $(\tau_{i,\kappa})_{i \in I}$ is an H-cocycle in $\mathscr{M}$. Now the morphisms of sheaves with values in the category of complex vector spaces

$$\Gamma_i : \mathscr{M}|_{U_i} \#_i A \to A \otimes \mathscr{P}|_{U_i},$$

$$f \otimes g \mapsto (\tau \circ (\Omega_i \otimes id) \circ (id \otimes \varphi))(f \otimes g)$$

$$= \sum_{(g)} g_{(0)} \otimes (\Omega_i(f \# g_{(-1)}))$$

can be defined, where the crossed product $\mathscr{M}|_{U_i} \#_i A$ is given by the weak action $\alpha_i : H \times \mathscr{M}|_{U_i} \to \mathscr{M}|_{U_i}$, the normal cocycle $\iota : H \times H \to \mathscr{M}(U_i)$ and the left H-coaction $\varphi : A \to H \otimes A$ in the sense of Theorem B.6. The Γ_i are the local trivialisations we are looking for. Before we can prove this, some general statements about the Γ_i have to be made.

Lemma 4.11. *For all $U \subset M$ open,*

  (i) $(\Gamma_i)_U$ *is a morphism of algebras with unit,*
  (ii) $\mathrm{Im}(\Gamma_i)_U \subset \mathscr{V}(U)$.

Proof. We only show the lemma for the case of a trivial cocycle $\iota : H \times H \to \mathscr{M}(U_i)$. The general case goes with the same argument but requires a lot more writing. Let $f, f' \in \mathscr{M}(U)$ and $g, g' \in A$. Then i) is a consequence of the following two equations:

$$\Gamma_i((f \otimes g)(f' \otimes g')) = \sum_{(g)} \Gamma_i(f g_{(-1)} f' \otimes g_{(0)} g')$$

$$= \tau\left(\sum_{(g)} (\Omega_i(f g_{(-2)} f' \# g_{(-1)} g'_{(-1)}) \otimes g_{(0)} g'_{(0)}) \right)$$

$$= \tau\left(\sum_{(g)} (\Omega_i(f \# g_{(-1)}) \cdot \Omega_i(f' \# g'_{(-1)}) \otimes g_{(0)} g'_{(0)}) \right)$$

$$= \tau\left(\sum_{(g)} (\Omega_i(f \# g_{(-1)}) \otimes g_{(0)}) \cdot (\Omega_i(f' \# g'_{(-1)}) \otimes g'_{(0)})\right)$$

$$= \Gamma_i(f \otimes g) \cdot \Gamma_i(f' \otimes g'), \tag{78}$$

$$\Gamma_i(1 \otimes 1) = (1 \otimes 1). \tag{79}$$

As the inverse S^{-1} of the antipode S satisfies the relation

$$\sum_{(g)} S^{-1}(g_{(2)}) \cdot g_{(1)} = \varepsilon(g) \, 1, \tag{80}$$

one gets

$$\psi \circ \Gamma_i(f \otimes g) = \sum_{(g)} \psi(g_{(0)} \otimes \Omega_i(f \# g_{(-1)}))$$

$$= \sum_{(g)} g_{(0)} \otimes \Omega_i(f \# g_{(-3)}) \otimes S^{-1}(g_{(-1)}) \cdot g_{(-2)}$$

$$= \sum_{(g)} g_{(0)} \otimes \Omega_i(f \# g_{(-2)}) \otimes \varepsilon(g_{(-1)}) 1$$

$$= \sum_{(g)} g_{(0)} \otimes \Omega_i(f \# g_{(-1)}) \otimes 1$$

$$= \psi(f \otimes g) \otimes 1. \tag{81}$$

This entails (ii). □

Now Γ_i can be considered as a mapping $\Gamma_i : \mathcal{M}|_{U_i} \#_i A \to \mathcal{V}|_{U_i}$. Next we need an inverse of this Γ_i. Define

$$\zeta_i : A \otimes \mathcal{P}|_{U_i} \to \mathcal{M}|_{U_i} \#_i A,$$

$$(f \otimes g) \mapsto ((id \otimes \varepsilon) \circ \Omega_i^{-1})(g) \otimes f$$

$$= (id \otimes \varepsilon \otimes id) \circ (\Omega_i^{-1} \otimes id) \circ \tau(f \otimes g).$$

Then the equation

$$\zeta_i \circ \Gamma_i(f \otimes g) = (id \otimes \varepsilon \otimes id) \circ (\Omega_i \otimes id) \circ (\Omega_i \otimes id) \circ (id \otimes \varphi)(f \otimes g)$$

$$= (id \otimes \varepsilon \otimes id) \circ (id \otimes \varphi)(f \otimes g)$$

$$= f \otimes g, \tag{82}$$

is true and therefore

$$\zeta_i \circ \Gamma_i = id. \tag{83}$$

If we further set $P_i = \Gamma_i \circ \zeta_i : A \otimes \mathcal{P}|_{U_i} \to A \otimes \mathcal{P}|_{U_i}$, Eq. (83) implies

$$P_i = P_i \circ P_i, \tag{84}$$

that means $(P_i)_U$ is a projection onto $Im(\Gamma_i)_U$. Finally only $Im(P_i)_U = \mathcal{V}(U)$ has to be shown for $U \subset U_i$ open.

From now on we must assume $\iota(h, l) = \varepsilon(h) \varepsilon(l) 1$ for all $h, l \in H$.

Lemma 4.12. $A \otimes \mathcal{P}(U)$, $U \subset U_i$ is an H-bimodule, if the left and right action are defined as follows:

(i)
$$H \times (A \otimes \mathcal{P}(U)) \to A \otimes \mathcal{P}(U),$$
$$(h, f \otimes g) \mapsto h \cdot (f \otimes g) = f \otimes (\Omega_i(1 \# h)g),$$

(ii)
$$(A \otimes \mathcal{P}(U)) \times H \to A \otimes \mathcal{P}(U),$$
$$(f \otimes g, h) \mapsto (f \otimes g) \cdot h = f \otimes (g\Omega_i(1 \# h)).$$

Proof. The proof is done by an easy calculation.   □

Now we have all the necessary tools to show the main proposition.

Proposition 4.13. *For all $\iota \in I$,*

$$\Gamma_\iota : \mathscr{M}|_{U_\iota} \#_\iota A \to \mathscr{V}|_{U_\iota},$$

$$f \otimes g \mapsto \sum_{(g)} g_{(0)} \otimes (\Omega_\iota(f \# g_{(-1)}))$$

is an isomorphism of sheaves of algebras. The restriction of the sheaf morphism

$$\zeta_\iota : A \otimes \mathscr{P}|_{U_\iota} \to \mathscr{M}|_{U_\iota} \#_\iota A,$$

$$(f \otimes g) \mapsto ((id \otimes \varepsilon) \otimes \Omega_\iota^{-1})(g) \otimes f$$

to $\mathscr{V}|_{U_\iota}$ gives the inverse of Γ_ι. Furthermore for all $\iota, \kappa \in I$,

$$(\Gamma_\kappa^{-1})_U \circ (\Gamma_\iota)_U (f \otimes g) = (\zeta_\kappa)_U \circ (\Gamma_\iota)_U (f \otimes g)$$

$$= \sum_{(g)} f \cdot \tau_{\iota,\kappa}(g_{(-1)}) \otimes g_{(0)}, \quad f \otimes g \in \mathscr{M} \#_\iota A. \tag{85}$$

Proof. Because of Eq. (84) it suffices to show

$$Im(P_\iota)_U = \mathscr{V}(U), \quad U \subset U_\iota \tag{86}$$

for the proof of the first part of the proposition. The elements $f \in A$, $g \in \mathscr{M}(U)$, $h \in H$ satisfy the equation

$$\psi(f \otimes \Omega_\iota(g \# h)) = \sum_{(f),(h)} f_{(0)} \otimes \Omega_\iota(g \# h_{(1)}) \otimes (S^{-1} f_{(-1)}) h_{(2)} \tag{87}$$

according to the definition of ψ. As also

$$P_\iota(f \otimes \Omega_\iota(g \# h))$$

$$= \Gamma_\iota(\varepsilon(h) g \otimes f)$$

$$= \sum_{(f)} f_{(0)} \otimes (\varepsilon(h)\Omega_\iota(g \# f_{-1})). \tag{88}$$

is true, Eq. (80) and the right action in Lemma 4.12 imply the following relation:

$$\sum_{(f),(h)} P_\iota(f_{(0)} \otimes \Omega_\iota(g \# h_{(1)})) \cdot ((S^{-1} f_{(-1)}) h_{(2)})$$

$$= \sum_{(f)} (f_{(0)} \otimes \Omega_\iota(g \# f_{(-1)})) \cdot ((S^{-1} f_{(-2)}) h)$$

$$= \sum_{(f)} f_{(0)} \otimes (\Omega_\iota(g \# f_{(-1)}) \Omega_\iota(1 \# (S^{-1} f_{(-2)})) \Omega_\iota(1 \# h))$$

$$= \sum_{(f)} f_{(0)} \otimes (\Omega_\iota(g \# (f_{(-1)})(S^{-1} f_{(-2)})) \Omega_\iota(1 \# h))$$

$$= \sum_{(f)} f_{(0)} \otimes (\Omega_\iota(g \# \varepsilon(f_{(-1)}) 1) \Omega_\iota(1 \# h))$$

$$= f \otimes \Omega_\iota(g \# h). \tag{89}$$

For $\sum_i f_i \otimes g_i \in A \otimes \mathscr{P}(U)$ assume $\psi(\sum_i f_i \otimes g_i) = \sum_j f_j' \otimes g_j' \otimes h_j$. As Ω_ι is a sheaf morphism (87) and (89) entail P_ι having the property

$$\sum_j P_\iota(f_j' \otimes g_j') \cdot h_j = \sum_i f_i \otimes g_i. \tag{90}$$

So if $\sum_i f_i \otimes g_i \in \mathscr{V}(U)$, Eq. (90) gives

$$\sum_i P_i(f_i \otimes g_i) = \sum_j P_i(f'_j \otimes g'_j) \cdot h_j = \sum_j f_i \otimes g_i. \tag{91}$$

Therefore $Im(P_i)_U = \mathscr{V}(U)$.

Equation (85) is a direct consequence of the definition of the Γ_i, ζ_i and the transition functions $\tau_{i,\kappa}$. Explicitly the relation (41) gives

$$(\zeta_\kappa)_U \circ (\Gamma_i)_U (f \otimes g) = \sum_{(g)} (id \otimes \varepsilon) \circ \Omega_\kappa^{-1} \circ \Omega_i (f \otimes g_{(-1)}) \otimes g_{(0)}$$

$$= \sum_{(g)} f \cdot \tau_{i,\kappa}(g_{(-1)}) \otimes g_{(0)}. \tag{92}$$

Quod erat demonstrandum.   □

Now we get the desired result.

Corollary 4.14. *Let A be a quadratic algebra, φ a left H-coaction and $\mathscr{P}$ an A-quantum principal bundle over $\mathscr{M}$, where the trivialisations are defined by actions $\alpha_i : H \times \mathscr{M}|_{U_i} \to \mathscr{M}|_{U_i}$ and trivial cocycles $\iota : H \times H \to \mathscr{M}(U_i)$. Then the tupel $(\mathscr{V}, \mathscr{M}, \tilde{\varrho}, A, H, \varphi, (\Gamma_i)_{i \in I})$, gives an A-quantum vector bundle with transition functions $\tau_{i,\kappa}$, ι, $\kappa \in I$, where $\mathscr{V}$, Γ_i are constructed like above, and $\tilde{\varrho}$ is defined by*

$$\tilde{\varrho} : \mathscr{M} \to \mathscr{V}$$

$$f \mapsto 1 \otimes \varrho(f).$$

Proof. It only has to be shown that $\tilde{\varrho}$ is well-defined. Now

$$\phi(\varrho(f)) = \varrho(f) \otimes 1, \quad f \in \mathscr{M}(U), \quad U \subset M \tag{93}$$

implies

$$\psi(1 \otimes \varrho(f)) = 1 \otimes \varrho(f) \otimes 1, \tag{94}$$

and that proofs $\tilde{\varrho}$ being welldefined.   □

Now the proof of Theorem 4.5 can be given.

Proof (Theorem 4.5). For the H-cocycle $(\tau_{z,\kappa})_{(z,\kappa) \in I \times I}$ construct a quantum principal bundle according to Theorem 3.14. Corollary 4.14 provides the quantum vector bundle we are looking for.   □

4.3. Differential Calculus

In this section we want to define a differential calculus and connections on quantum principal and quantum vector bundles.

First we generalise the concept of a differential calculus over an algebra [24, 23] to quantum spaces.

Definition 4.15. *Let $\mathscr{M}$ be a quantum space, $\mathscr{D}$ a bimodule sheaf over $\mathscr{M}$ and $d : \mathscr{M} \to \mathscr{D}$ a sheaf morphism. $(\mathscr{D}, d)$ is said to be a differential calculus over $\mathscr{M}$, if for all $f, g \in \mathscr{M}(U)$, $U \subset M$ the following conditions are satisfied:*

(i) *$d(fg) = (df)g + f(dg)$.*
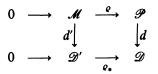(ii) *Every element $\omega \in \mathscr{D}(U)$ has the form*

$$\omega = \sum_{k=1}^{n} f_k \, dg_k,$$

where $f_k, g_k \in \mathscr{M}(U)$, $k = 1, \dots, n$, $n \in \mathbb{N}$.

If we assume to have a quantum principal bundle $0 \to \mathcal{M} \xrightarrow{\varrho} \mathcal{P}$ and differential calculi over $\mathcal{M}$ and $\mathcal{P}$, the question is in which sense the differential calculi fit together.

Definition 4.16. *Let* $0 \to \mathcal{M} \xrightarrow{\varrho} \mathcal{P}$ *be a quantum principal bundle,* $(\mathcal{D}', d')$ *a differential calculus over* $\mathcal{M}$ *and* $(\mathcal{D}, d)$ *a differential calculus over* $\mathcal{P}$. $(\mathcal{D}, d)$ *is said to induce* $(\mathcal{D}', d')$, *if there exists a sheaf morphism* $\varrho_* : \mathcal{D}' \to \mathcal{D}$ *over* ϱ *such that the diagram*

$$
\begin{array}{ccccc}
0 & \longrightarrow & \mathcal{M} & \xrightarrow{\varrho} & \mathcal{P} \\
 & & {\scriptstyle d'}\downarrow & & \downarrow{\scriptstyle d} \\
0 & \longrightarrow & \mathcal{D}' & \xrightarrow[\varrho_*]{} & \mathcal{D}
\end{array}
$$

commutes with exact horizontal lines.

The morphism ϱ determines the morphism ϱ_* in a certain sense. To see this choose $f_k, f_k' \in \mathcal{M}(U), U \subset M$ open, $k = 1, \dots, n$. Then

$$
\varrho_* \left(\sum_{k=1}^{n} f_k \, d' f_k' \right) = \sum_{k=1}^{n} \varrho(f_k) \, d\varrho(f_k'). \tag{95}
$$

This equation can be taken to define a morphism ϱ_*.

Theorem 4.17. *Let* $(\mathcal{D}, d)$ *be a differential calculus over a quantum principal bundle* $\mathcal{P}$. *If the pair* $(\mathcal{D}', d')$ *is defined by*

$$
\mathcal{D}'(U) = \left\{ \sum_{k=1}^{n} \varrho(f_k) \, d\varrho(f_k') \in \mathcal{D}(U) : f_k, f_k' \in \mathcal{M}(U), \, k = 1, \dots, n, \, n \in \mathbb{N} \right\}
$$

$$
d' = d|_{\varrho(\mathcal{M})} \circ \varrho,
$$

one receives a differential calculus $(\mathcal{D}', d')$ *over* $\mathcal{M}$, *which is induced by the morphism*

$$
\varrho_* : \mathcal{D}' \to \mathcal{D}
$$

$$
(\varrho_*)_U : \mathcal{D}'(U) \to \mathcal{D}(U)
$$

$$
\sum_{k=1}^{n} \varrho(f_k) \, d\varrho(f_k') \mapsto \sum_{k=1}^{n} \varrho(f_k) \, d\varrho(f_k').
$$

Proof. Obvious by the above considerations. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the following let $(U_i)_{i \in I}$ be a trivialisation covering of $\mathcal{P}$. Further assume the cocycles $\iota : H \times H \to \mathcal{M}(U_i)$ being trivial and the $\alpha_i : H \times \mathcal{M}|_{U_i} \to \mathcal{M}_{U_i}$ being actions.

For an open $U \subset U_i$ the equations $\mathcal{D}_i = \mathcal{D}|_{U_i}$ and $d_i(f \#_i g) = d \circ \Omega_i(f \#_i g)$, $f \in \mathcal{M}(U)$, $g \in H$ define a differential calculus on $\mathcal{M}|_{U_i} \#_i H$ such that $d_i(f \#_i 1) = d'(f)$ for $f \in \mathcal{M}(U)$ and

$$
d_i(f \#_i g) = d'(f) \cdot \Omega_i(1 \#_i g) + \varrho(f) \cdot d\Omega(1 \#_i g). \tag{96}
$$

Now we would like to give attention to the inverse problem and suppose there is a differential calculus (D_H, d) on the Hopf algebra H and a differential calculus $(\mathcal{D}'_{\mathcal{M}}, d')$ on $\mathcal{M}$. Assuming the following consistency condition we will construct a differential calculus $(\mathcal{D}_{\mathcal{P}}, d)$ on $\mathcal{P}$.

● (Consistency Condition)
The actions $\alpha_i : H \times \mathscr{M}|_{U_i} \to \mathscr{M}_{U_i}$ satisfy the equation

$$\sum_{(g),\, 1 \le k \le n} (g_{(1)} \cdot_i f_k)\, d'(g_{(2)} \cdot_i f'_k) = 0 \tag{97}$$

for all $g \in H$, if $\sum_{k=1}^{n} f_k d' f'_k = 0$ with $f_k, f'_k \in \mathscr{M}(U)$, $U \subset U_i$ open.
This consistency condition guarantees the existence of an action

$$\alpha : H \times \mathscr{D}_{\mathscr{M}}|_{U_i} \to \mathscr{D}_{\mathscr{M}}|_{U_i},$$

$$\left(g, \sum_{k=1}^{n} f_k \, d' f'_k\right) \mapsto \sum_{(g),\, 1 \le k \le n} (g_{(1)} \cdot_i f_k)\, d'(g_{(2)} \cdot_i f'_k).$$

Lemma 4.18. *Define for $U \subset M$ open*:

$$\mathscr{D}_{\mathscr{P}}^{0}(U) = \bigoplus_{i \in I} (\mathscr{D}_{\mathscr{P}}^{0}(U))_i,$$

$$(\mathscr{D}_{\mathscr{P}}^{0}(U))_i = \mathscr{D}_{\mathscr{M}}(U_i \cap U) \otimes H \oplus \mathscr{M}(U_i \cap U) \otimes D_H.$$

Then $\mathscr{D}_{\mathscr{P}}^{0}(U)$ is a bimodule over

$$\mathscr{P}_0(U) = \bigoplus_{i \in I} \mathscr{M}(U_i \cap U) \#_i H.$$

The left and right actions are given by

$$(f \otimes g)(a \otimes g' + f' \otimes b) = \sum_{(g)} f(g_{(1)} \cdot a) \otimes g_{(2)} g' + f(g_{(1)} \cdot f') \otimes (g_{(2)} \cdot b), \tag{98}$$

$$(a \otimes g + f \otimes b)(f' \otimes g') = \sum_{(g),(b)} a(g_{(1)} \cdot f') \otimes g_{(2)} g' + f(b_{(-1)} \cdot f') \otimes (b_{(0)} \cdot g'), \tag{99}$$

where $f, f' \in \mathscr{M}(U_i \cap U)$, $g, g' \in H$, $a \in \mathscr{D}_{\mathscr{M}}(U_i \cap U)$ and $b \in D_H$.

Proof. The following two equations

$$((f'' \otimes g'')(f \otimes g))(a \otimes g' + f' \otimes b) = \sum_{(g'')} (f''(g''_{(1)} \cdot f) \otimes g''_{(2)} g)(a \otimes g' + f' \otimes b)$$

$$= \sum_{(g),(g'')} f''(g''_{(1)} \cdot f)((g''_{(2)} g_{(1)}) \cdot a) \otimes g''_{(3)} g_{(2)} g'$$

$$+ f''(g''_{(1)} \cdot f)((g''_{(2)} g_{(1)}) \cdot f') \otimes ((g''_{(3)} g_{(2)}) \cdot b), \tag{100}$$

$$(f'' \otimes g'')((f \otimes g)(a \otimes g' + f' \otimes b))$$

$$= (f'' \otimes g'')\left(\sum_{(g)} f(g_{(1)} \cdot a) \otimes g_{(2)} g' + f(g_{(1)} \cdot f') \otimes (g_{(2)} \cdot b)\right)$$

$$= \sum_{(g),(g'')} f''(g''_{(1)} \cdot (f(g_{(1)} \cdot a))) \otimes g''_{(2)} g_{(2)} g'$$

$$+ f''(g''_{(1)} \cdot (f(g_{(1)} \cdot f'))) \otimes ((g''_{(2)} g_{(2)}) \cdot b)$$

$$= \sum_{(g),(g'')} f''(g''_{(1)} \cdot f)((g''_{(2)} g_{(1)}) \cdot a) \otimes g''_{(3)} g_{(2)} g'$$

$$+ f''(g''_{(1)} \cdot f)((g''_{(2)} g_{(1)}) \cdot f') \otimes ((g''_{(3)} g_{(2)}) \cdot b), \tag{101}$$

prove

$$((f'' \otimes g'')(f \otimes g))(a \otimes g' + f' \otimes b) = (f'' \otimes g'')((f \otimes g)(a \otimes g' + f' \otimes b)). \quad (102)$$

The rest is shown similarly. □

In the next step we will provide the derivation operator d.

$$d : \mathscr{P}^0 \to \mathscr{D}^0_{\mathscr{P}}(U), \quad d = \sum_{\iota \in I} d_\iota,$$

$$d_\iota : \mathscr{M}(U_\iota \cap U) \#_\iota H \to (\mathscr{D}^0_{\mathscr{P}}(U))_\iota,$$

$$f \# g \mapsto d' f \otimes g + f \otimes dg.$$

The operator d satisfies the Leibniz rule:

$$d_\iota((f \otimes g)(f' \otimes g')) = \sum_{(g)} d_\iota(f(g_{(1)} \cdot f') \otimes g_{(2)} g')$$

$$= \sum_{(g)} (d'(f(g_{(1)} \cdot f'))) \otimes g_{(2)} g' + f(g_{(1)} \cdot f') \otimes d(g_{(2)} g')$$

$$= \sum_{(g)} ((d' f)(g_{(1)} \cdot f') + f d'(g_{(1)} \cdot f')) \otimes g_{(2)} g'$$

$$+ f(g_{(1)} \cdot f') \otimes ((dg_{(2)}) g' + g_{(2)}(dg')), \quad (103)$$

$$(d_\iota(f \otimes g))(f' \otimes g') + (f \otimes g)(d_\iota(f' \otimes g'))$$

$$= \sum_{(g)} (d' f \otimes g + f \otimes dg)(f' \otimes g') + (f \otimes g)(d' f' \otimes g' + f' \otimes dg')$$

$$= \sum_{(g)} (d' f)(g_{(1)} \cdot f') \otimes g_{(2)} g' + f(g_{(1)} f') \otimes (dg_{(2)}) g'$$

$$+ f d'(g_{(1)} \cdot f') \otimes g_{(2)} g' + f(g_{(1)} \cdot f') \otimes g_{(2)}(dg'). \quad (104)$$

Now we have to show that every $e = \sum_{k=1}^n a_k d' b_k \otimes c_k + a'_k \otimes b'_k dc'_k$ with $a_k, a'_k, b_k \in \mathscr{M}(\mathscr{U}_\iota \cap U)$, $c_k, b'_k, c'_k \in H$ has the form

$$e = \sum_{l=1}^m f_l df'_l \quad (105)$$

with $f_l, f'_l \in \mathscr{M}(U \cap U_\iota) \#_\iota H$. But this is an easy consequence from

$$e = \sum_{k=1}^n \sum_{(c_k)} (a_k \otimes c_{k(2)}) d((S^{-1} c_{k(1)} \cdot b_k) \otimes 1) + (a_k \otimes b'_k) d(1 \otimes c'_k), \quad (106)$$

where we have used that the antipode is bijective.

The $\mathscr{P}(U)$-bimodule $\mathscr{D}_{\mathscr{P}}(U)$ has to be defined. According to Eq. (49) the algebra $\mathscr{P}(U)$ can be regarded as a subalgebra of $\mathscr{P}_0(U)$. Therefore we set

$$\mathscr{D}_{\mathscr{P}}(U) = {}_{\mathscr{P}} Mod \left\{ \sum_{\iota \in I} d_\iota f_\iota : \sum_\iota f_\iota \in \mathscr{P}(U)) \right\} \subset \mathscr{D}^0_{\mathscr{P}}(U), \quad (107)$$
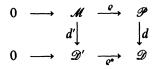
which means that $\mathscr{D}_{\mathscr{P}}(U)$ is the $\mathscr{P}(U)$-left module generated by $\sum_{\iota \in I} d_\iota f_\iota$. Because of the Leibniz rule $\mathscr{D}_{\mathscr{P}}(U)$ is a $\mathscr{P}(U)$-right module as well. Furthermore the relation

$$\varrho_* \left(\sum_{k=1}^n f_k d' f'_k \right) = \sum_{k=1}^n \sum_{\iota \in I} (r^U_{U_\iota \cap U}(f'_k) d' r^U_{U_\iota \cap U}(f_k)) \otimes 1 + 0, \quad (108)$$

with $f_k, f'_k \in \mathcal{M}(U)$, $1 \leq k \leq n$, $n \in \mathbb{N}$ gives a well-defined morphism $\varrho_* : \mathcal{D}' \to \mathcal{D}$. Then

$$\varrho_* \left(\sum_{k=1}^{n} f_k\, d' f'_k \right) = \sum_{k=1}^{n} (r^U_{U, \cap U}(f_k) \# 1)\, d(r^U_{U, \cap U}(f'_k) \# 1), \tag{109}$$

and the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{M} & \overset{\varrho}{\longrightarrow} & \mathcal{P} \\
 & & d' \downarrow & & \downarrow d \\
0 & \longrightarrow & \mathcal{D}' & \underset{\varrho^*}{\longrightarrow} & \mathcal{D}
\end{array}
$$

commutes. It still has to be shown that the lower sequence is exact. But this is obvious remembering Eq. (108) and the properties of the restriction mappings. We subsume our results in the following theorem.

Theorem 4.19. *Let H be a Hopf algebra with differential calculus (D_H, d). Further let $(\mathcal{D}'_\mathcal{M}, d')$ be a differential calculus on the base quantum space $\mathcal{M}$. Then every quantum principal bundle $\mathcal{P}$ which fulfills the consistency condition has a differential calculus $(\mathcal{D}_\mathcal{P}, d)$ induced by $(\mathcal{D}'_\mathcal{M}, d')$.*

Finally the concept of connections on a quantum principal bundle shall be explained. It is still assumed the differential calculi $(\mathcal{D}'_\mathcal{M}, d')$ and $(\mathcal{D}_\mathcal{P}, d)$ being induced by $\varrho : \mathcal{D}'_\mathcal{M} \to \mathcal{D}_\mathcal{P}$. Denote by $A_\mathcal{M}(\mathcal{P})$ (resp. $\mathcal{A}_\mathcal{M}(\mathcal{P})$) the basis subalgebra (resp. basis subsheaf) of differential forms on $\mathcal{P}$, i.e.

$$A_\mathcal{M}(\mathcal{P}) = Im((\varrho_*)_M) \quad \text{and} \quad \mathcal{A}_\mathcal{M}(\mathcal{P}) = Im(\varrho_*). \tag{110}$$

Then $\mathcal{A}_h = {}_\mathcal{P}\mathrm{Mod}(\mathcal{A}_\mathcal{M}(\mathcal{P}))$ is the sheaf of horizontal differential forms on $\mathcal{P}$ or in other words is the $\mathcal{P}$-submodule sheaf generated by $\mathcal{A}_\mathcal{M}(\mathcal{P})$.

The motivation for this comes out of commutative geometry, where the basis subalgebra of differential forms and the sheaf of horizontal forms are defined in an analogous way (compare Greub, Halperin, Vanstone [9]). Now if the sheaf $\mathcal{A}_h$ has a complementary sheaf $\mathcal{A}_v$ in $\mathcal{D}_\mathcal{P}$, we say that $\mathcal{A}_v$ defines a connection on $\mathcal{P}$.

Definition 4.20. *A quantum principal bundle is said to have a connection, if the sequence*

$$0 \to \mathcal{A}_h \to \mathcal{D}_\mathcal{P} \to \mathcal{D}_\mathcal{P}/\mathcal{A}_h \to 0$$

splits. In that case the connection is a module sheaf $\mathcal{A}_v$ such that $\mathcal{D}_\mathcal{P} = \mathcal{A}_h \oplus \mathcal{A}_v$ gives the splitting.

Remark 4.21. Locally connections always exist, but it is not obvious whether they can be glued together. In the classical case of principal bundles over paracompact manifolds this is possible as one has an appropriate partition of unity.

5. Noncommutative Instanton Models

5.1. A q-Deformed Space Time

In the following we will construct a quantum space $\mathcal{M}$, which is a deformation of the sheaf of continuous functions on the 4-sphere. This quantum space will be important for the definition of the noncommutative instanton models and can

be regarded as a noncommutative space time over the classical euclidean background S^4.

Let us first introduce some notation.

$$M = S^4 \qquad \text{(space time)},$$

$$\tilde{M} = S^3 \times [-1, 1] \qquad \text{(enlarged space time)},$$

$$NP = (0, 0, 0, 0, 1) \qquad \text{(north pole of } S^4 \text{)},$$

$$SP = (0, 0, 0, 0, -1) \qquad \text{(south pole of } S^4 \text{)},$$

$$U_1 = S^4 \backslash \{SP\} \qquad \text{(northern hemisphere of } S^4 \text{)},$$

$$U_2 = S^4 \backslash \{NP\} \qquad \text{(southern hemisphere of } S^4 \text{)}.$$

Let $\mathscr{R}$ be the locally constant sheaf on S^3 with objects in the algebra $SU_q(2)$, $\mathscr{L}$ be the sheaf of continuous, bounded and complex valued functions on the interval $[-1, 1]$ such that the condition

$$f(x) = 0 \tag{111}$$

for $f \in \mathscr{L}(U)$, $U \subset [-1, 1]$ open and $x \in \{-1, 1\}$ is fulfilled.

Before we start with the explicit and somewhat technical construction let us give some motivation and interpretation. The sheaf $\mathscr{R}$ is being regarded as the q-deformed spatial part of the classical background space time $M = S^4$, the sheaf $\mathscr{L}$ as the undeformed time part. It is not possible to directly build a sheaf $\mathscr{M}$ out of $\mathscr{R}$ and $\mathscr{L}$. First the tensor product of $\mathscr{R}$ and $\mathscr{L}$ is formed which gives a sheaf $\tilde{\mathscr{M}}$ on the cylinder $\tilde{M}$. In the next step $S^3 \times \{-1\}$ (resp. $S^3 \times \{1\}$) are glued together to the north pole (resp. south pole) of $\tilde{M}$. The gluing is carried over to the local algebras of the sheaf $\tilde{\mathscr{M}}$, where it gives the desired $\mathscr{M}$. Altogether one could say that $\mathscr{M}$ is q-deformed in the horizontal direction and undeformed in the vertical direction.

Now for the construction of $\tilde{\mathscr{M}}$ take $W, W' \subset S^3$ and $V, V' \subset [-1, 1]$ open. Define

$$\tilde{\mathscr{M}}(V \times W) := \mathscr{R}(V) \otimes \mathscr{L}(W),$$

$$\tilde{\mathscr{M}}(i_{V' \times W'}^{V \times W}) := r_{V' \times W'}^{V \times W} := r_{V'}^{V} \otimes r_{W'}^{W}. \tag{112}$$

As the $V \times W$ form a basis of the topology of $\tilde{M}$, we get a uniquely defined sheaf $\tilde{\mathscr{M}}$ on $\tilde{M} = S^3 \times [-1, 1]$. Let us now make precise what we mean by "gluing." Mathematically this is nothing other than the projection

$$\pi : S^3 \times [-1, 1] \rightarrow S^4,$$

$$((y_1, \ldots, y_4), r) \mapsto (\sqrt{1 - r^2}\, y_1, \ldots, \sqrt{1 - r^2}\, y_4, r).$$

This projection has an inverse on $S^3 \times]-1, 1[$, namely

$$\psi : S^4 \backslash \{NP, SP\} \rightarrow S^3 \times]-1, 1[,$$

$$(x_1, \ldots, x_5) \mapsto \left(\frac{1}{\sqrt{1 - x_5^2}}(x_1, \ldots, x_4), x_5 \right).$$

This mirrors the fact that the 4-sphere without the north and south pole is topologically equivalent to the cylinder $S^3 \times]-1, 1[$. Now set

$$\mathscr{M}(U) := \tilde{\mathscr{M}}(\pi^{-1}(U)) = \tilde{\mathscr{M}}(\psi(U)) \tag{113}$$

for $U \subset S^4 \setminus \{NP, SP\} = U_1 \cap U_2$. To find the right definition for $\mathcal{M}(U)$, if $NP \in U$ or $SP \in U$, let us first have a look at the sheaf of continuous bounded functions on the sphere S^4. Each such function f defined on the northern hemisphere U_1 can uniquely be written in the form

$$f = f_{\mathrm{red}} + z, \tag{114}$$

where $f_{\mathrm{red}} \in \mathscr{C}(U)$, $f(NP) = 0$ and $z \in \mathbb{C}$. Furthermore the set of continuous bounded and complex-valued functions on U_1 whose value at the north pole vanish can be identified with the (topological) tensor product $\mathscr{C}(S^3) \otimes \mathscr{L}(]-1, 1[)$. Therefore the set of complex continuous bounded functions on the northern hemisphere U_1 is isomorphic to $(\mathscr{C}(S^3) \otimes \mathscr{L}(]-1, 1[)) \oplus \mathbb{C}$. U_2 satisfies an analogous result. The above considerations suggest to set

$$\mathcal{M}(U_1) := \tilde{\mathcal{M}}(S^3 \times]-1, 1]) \oplus \mathbb{C} = \tilde{\mathcal{M}}(\pi^{-1}(U_1)) \oplus \mathbb{C},$$
$$\mathcal{M}(U_2) := \tilde{\mathcal{M}}(S^3 \times [-1, 1[) \oplus \mathbb{C} = \tilde{\mathcal{M}}(\pi^{-1}(U_2)) \oplus \mathbb{C}. \tag{115}$$

Furthermore we define

$$\mathcal{M}(U) := \tilde{\mathcal{M}}(\pi^{-1}(U)) \oplus \mathbb{C}, \tag{116}$$

where $U \subset M$ and either $NP \in U$ or $SP \in U$. The restriction morphisms $r_V^U : \mathcal{M}(U) \to \mathcal{M}(V)$, $V \subset U \subset M$ are given by the following definition.

- Let $U, V \subset M \setminus \{NP, SP\}$ and $f \in \mathcal{M}(U) = \tilde{\mathcal{M}}(\psi(U))$. Then define

$$r_V^U(f) := r_{\psi(V)}^{\psi(U)}(f). \tag{117}$$

- Let $U \subset M$ and $V \subset U$, where either $NP \in U$ or $SP \in U$, and $f \in \mathcal{M}(U)$. With the representation $f = f_{\mathrm{red}} + z$, $f_{\mathrm{red}} \in \tilde{\mathcal{M}}(\pi^{-1}(U))$, $z \in \mathbb{C}$ define

$$r_V^U(f) := r_{\psi(V)}^{\psi(U)}(f_{\mathrm{red}}) + z. \tag{118}$$

For a basis of the topology of M we have defined local algebras $\mathcal{M}(U)$ and restriction morphisms such that over the basis of the topology the sheaf axioms are satisfied. Therefore we get a sheaf $\mathcal{M}$ over M, which we call the q-deformed space time over the background S^4.

Proposition 5.1. *The q-deformed space time is a quantum space.*

5.2. The q-Deformed Instanton Models

The base quantum space of the q-deformed instanton models is the quantum space in Proposition 5.1.

Let us introduce some more notation:

$$V_a = \{x \in S^4 : x_5 > a\} \text{ with } -1 \leq a < 1,$$
$$_bV = \{x \in S^4 : x_5 < b\} \text{ with } -1 < b \leq 1.$$

Now we can define the mappings $\tau_{\iota,\kappa}^1 : SU_q(2) \to \mathcal{M}(U_\iota \cap U_\kappa)$ for $\iota, \kappa = 1, 2$ by

$$\tau_{1,1}^1(h) = \varepsilon(h)\,1,$$
$$\tau_{2,2}^1(h) = \varepsilon(h)\,1,$$
$$\tau_{1,2}^1(h) = h \otimes 1,$$
$$\tau_{2,1}^1(h) = S(h) \otimes 1,$$

where we used $h \in SU_q(2)$ and the relation $\mathcal{M}(U_1 \cap U_2) = SU_q(2) \otimes \mathcal{L}(]-1,1[)$ (see (113) and (112)). The following lemma is obvious.

Lemma 5.2. $(\tau^1_{\iota,\kappa})_{1 \leq \iota,\kappa \leq 2}$ is a $SU_q(2)$-cocycle in $\mathcal{M}$ over $(U_\iota)_{\iota=1,2}$.

Remark 5.3. If $q = 1$ (that means in the commutative case) we can write for $h \in SU_1(2) = \mathcal{F}(SU(2))$, $x \in U_1 \cap U_2$:

$$\tau^1_{1,1}(h)(x) = h(1) \qquad = h(\eta_{1,1}(x)),$$

$$\tau^1_{2,2}(h)(x) = h(1) \qquad = h(\eta_{2,2}(x)),$$

$$\tau^1_{1,2}(h)(x) = h(\eta_{1,2}(x)),$$

$$\tau^1_{2,1}(h)(x) = h(\eta_{2,1}(x)) \quad = h(\eta^{-1}_{1,2}(x)),$$

where the $\eta_{\iota,\kappa} : U_\iota \cap U_\kappa \to SU(2)$, $\iota, \kappa = 1, 2$ are classical transition functions given by

$$(x_1, \ldots, x_5) \mapsto \eta_{1,1}(x_1, \ldots, x_5) = 1,$$

$$(x_1, \ldots, x_5) \mapsto \eta_{2,2}(x_1, \ldots, x_5) = 1,$$

$$(x_1, \ldots, x_5) \overset{\psi}{\mapsto} (y_1, \ldots, y_5)$$

$$\mapsto \eta_{1,2}(x_1, \ldots, x_5) = \begin{pmatrix} y_1 + iy_2 & -y_3 + iy_4 \\ y_3 + iy_4 & y_1 - iy_2 \end{pmatrix},$$

$$(x_1, \ldots, x_5) \mapsto \eta_{2,1}(x_1, \ldots, x_5) = \eta^{-1}_{1,2}(x_1, \ldots, x_5).$$

In the following we need $SU_q(2)$-actions on the quantum space $\mathcal{M}(U_\iota)$, $\iota = 1, 2$. The quantum group $SU_q(2)$ acts trivially on $\mathcal{M}(U_2)$:

$$(\alpha^1_2)_U : SU_q(2) \times \mathcal{M}(U) \to M(U), \quad U \subset U_2 \text{ open}$$

$$(h, f) \mapsto h \cdot_2 f = \varepsilon(h) f.$$

The family $((\alpha^1_2)_U)_{U \subset U_2}$ gives a sheaf morphism $\alpha^1_2 : SU_q(2) \times \mathcal{M}|_{U_2} \to \mathcal{M}|_{U_2}$. Now define functions

$$(\alpha^1_1)_U : SU_q(2) \times \mathcal{M}(U) \to M(U), \quad U \subset U_1 \text{ open}$$

$$(h, f) \mapsto h \cdot_1 f$$

by the following procedure:

- If $U = \pi(V \times]a, b[)$ with $V \subset S^3$ open, $-1 \leq a < b \leq 1$, Eqs. (112) and (113) prove $\mathcal{M}(U) = SU_q(2) \otimes \mathcal{L}(]a, b[)$. In this case define for $h, f \in SU_q(2)$, $r \in \mathcal{L}(]a, b[)$:

$$h \cdot_1 (f \otimes r) = (\alpha^1_1)_U(h, f \otimes r) = \sum_{(h)} h_{(1)} f(S h_{(2)}) \otimes r. \tag{119}$$

- If $U = V_c$ with $c > -1$, Eq. (116) implies $\mathcal{M}(U) = SU_q(2) \otimes \mathcal{L}(]c, 1[) \oplus \mathbb{C}$. In that case define for $h, f \in SU_q(2)$, $r \in \mathcal{L}(]c, 1[)$, $z \in \mathbb{C}$:

$$h \cdot_1 (f \otimes r + z) = (\alpha^1_1)_U(h, f \otimes r + z) = \sum_{(h)} h_{(1)} f(S h_{(2)}) \otimes r + z. \tag{120}$$

The mappings $(\alpha'_1)_U$ are $SU_q(2)$-actions on $\mathcal{M}(U)$. Furthermore the diagram

$$
\begin{array}{ccc}
SU_q(2) \otimes \mathcal{M}(U) & \xrightarrow{\;(\alpha'_1)_U\;} & \mathcal{M}(U) \\[2pt]
{\scriptstyle id \,\otimes\, i^U_{\tilde{U}}} \downarrow & & \downarrow {\scriptstyle i^U_{\tilde{U}}} \\[2pt]
SU_q(2) \otimes \mathcal{M}(\tilde{U}) & \xrightarrow[\;(\alpha'_1)_U\;]{} & \mathcal{M}(\tilde{U})
\end{array}
$$

commutes for all $\tilde{U} \subset U$ open with $U, \tilde{U} \in \mathcal{B}$ and

$$\mathcal{B} = \{\pi(V \times \,]a, b[), \, V_c \in Pot(M):$$
$$V \subset S^3 \text{ open}, \; -1 \leq a < b \leq 1, c > -1\}.$$

As $\mathcal{B}$ is a basis of the open sets in U, we get a sheaf morphism $\alpha^1_1 : SU_q(2) \times \mathcal{M}|_{U_1} \to \mathcal{M}|_{U_1}$, whose components are actions.

Now we can construct the smash products $\mathcal{M}|_{U_1} \#_1 SU_q(2)$ and $\mathcal{M}|_{U_2} \#_2 SU_q(2)$ with the actions α^1_1 and α^1_2. Next we will show that the noncommutative coordinate changes

$$\Omega^1_{\kappa,\iota} = (m \otimes id) \circ (id \otimes \tau^1_{\iota,\kappa} \otimes id) \circ (id \otimes \Delta):$$

$$\mathcal{M}(U_1 \cap U_2) \#_\iota SU_q(2) \to \mathcal{M}(U_1 \cap U_2) \#_\kappa SU_q(2)$$

are morphisms of algebras. Let $\iota = 1, \kappa = 2$. Then $\Omega^1_{\kappa,\iota}$ has the form

$$\Omega^1_{\kappa,\iota} ((f \otimes r) \# h) = \sum_{(h)} (f \cdot h_{(1)} \otimes r) \# h_{(2)}, \qquad (121)$$

where $h, f \in SU_q(2)$ and $r \in \mathscr{X}(\,] - 1, 1[)$. But that is exactly the form the morphism Φ in Theorem B.5 has, if we let $u_1, u_2 \in \mathrm{Hom}(SU_q(2), \mathcal{M}(U))$ be defined by:

$$u_1(h) = h \otimes 1, \quad h \in SU_q(2),$$

$$u_2(h) = \varepsilon(h) \, 1, \quad h \in SU_q(2).$$

The quantum space $\mathcal{M}$, the $SU_q(2)$-cocycle $(\tau^1_{\iota,\kappa})_{1 \leq \iota, \kappa \leq 2}$ and the actions α_ι, $\iota = 1, 2$ fulfill the conditions of Theorem 3.14. Therefore we get a unique quantum principal bundle

$$(\mathscr{P}^1, \mathcal{M}, \varrho^1, SU_q(2), (U_\iota)_{\iota \in \{1,2\}})$$

with transition functions $(\tau^1_{\iota,\kappa})_{1 \leq \iota, \kappa \leq 2}$ and call it the instanton model for the index $k = 1$. In the case $q = 1$ we get the undeformed instanton model with index 1.

The instanton model for the index $k = 0$ is the trivial one:

$$(\mathscr{P}^0, \mathcal{M}, M, \varrho^0, SU_q(2))$$

that means $\mathscr{P}^0 = \mathcal{M} \otimes SU_q(2)$ and

$$\varrho^0 : M \to \mathscr{P}^0,$$

$$f \mapsto \varrho^0(f) = f \otimes 1, \quad f \in \mathcal{M}(U), \quad U \subset M.$$

Let us subsume the results in a theorem.

Theorem 5.4. *The q-deformed instanton models*

$$(\mathscr{P}^k, \mathcal{M}, M, \varrho^k, SU_q(2))$$

with index $k = 0, 1$ are noncommutative quantum principal bundles, which turn into the classical ones for $q = 1$.

But there exist q-deformed instanton models for all $k \in \mathbb{Z}$. To see that consider the mappings:

$$u_1^k, u_2^k \in \text{Hom}(SU_q(2), \mathcal{M}(U_1 \cap U_2)), \quad k \in \mathbb{Z},$$

which are defined for all $n \in \mathbb{N}$ by

$$u_1^n(h) = \sum_{(h)} h_{(1)} \cdot \ldots \cdot h_{(n)} \otimes 1, \quad h \in SU_q(2),$$

$$u_2^n(h) = \varepsilon(h) 1 \otimes 1, \qquad h \in SU_q(2),$$

$$u_1^{-n}(h) = u_2^n(h), \qquad h \in SU_q(2),$$

$$u_2^{-n}(h) = u_1^n \circ S(h), \qquad h \in SU_q(2).$$

Then for all $k \in \mathbb{Z}$:

$$u_1^k * u_2^{-k} = 1 = u_2^k * u_1^{-k}. \tag{122}$$

Let us construct actions $\alpha_i^k : SU_q(2) \times \mathcal{M}|_{U_i} \to \mathcal{M}|_{U_i}$ with the u_i^k and the following recipe:

- If $U = V_c, c > -1$, $\mathcal{M}(U)$ has the form $SU_q(2) \otimes \mathcal{L}(]c, 1]) \oplus \mathbb{C}$ and we define for $h, f \in SU_q(2)$, $r \in \mathcal{L}(]c, 1])$ and $z \in \mathbb{C}$:

$$h \cdot_1 (f \otimes r + z) = (\alpha_1^k)_U(h, f \otimes r + z) = \sum_{(h)} u_1^k(h_{(1)})(f \otimes r)u_2^{-k}(h_{(2)}) + z. \tag{123}$$

- If $U = {}_dV, d < 1$, $\mathcal{M}(U)$ has the form $SU_q(2) \otimes \mathcal{L}([-1, d[) \oplus \mathbb{C}$ and we define for $h, f \in SU_q(2)$, $r \in \mathcal{L}(]c, 1])$ and $z \in \mathbb{C}$:

$$h \cdot_2 (f \otimes r + z) = (\alpha_2^k)_U(h, f \otimes r + z) = \sum_{(h)} u_2^k(h_{(1)})(f \otimes r)u_1^{-k}(h_{(2)}) + z. \tag{124}$$

- But if $U = \pi(V \times]a, b[)$ with $V \subset S^3$ open, $-1 \le a < b \le 1$, $\mathcal{M}(U)$ has the form $SU_q(2) \otimes \mathcal{L}(]a, b[)$ and we set for $h, f \in SU_q(2)$ and $r \in \mathcal{L}(]a, b[)$:

$$h \cdot_1 (f \otimes r) = (\alpha_1^k)_U(h, f \otimes r) = \sum_{(h)} u_1^k(h_{(1)})(f \otimes r)u_2^{-k}(h_{(2)}), \tag{125}$$

$$h \cdot_2 (f \otimes r) = (\alpha_2^k)_U(h, f \otimes r) = \sum_{(h)} u_2^k(h_{(1)})(f \otimes r)u_1^{-k}(h_{(2)}). \tag{126}$$

Altogether this provides sheaf morphisms

$$\alpha_1^k : SU_q(2) \times \mathcal{M}|_{U_1} \to \mathcal{M}|_{U_1},$$

$$\alpha_2^k : SU_q(2) \times \mathcal{M}|_{U_2} \to \mathcal{M}|_{U_2},$$

whose components $(\alpha_i^k)_U$, $U \subset U_i$, $\iota = 1, 2$ are weak actions by Theorem B.4. Furthermore Theorem B.4 gives normal cocycles $\iota : SU_q(2) \times SU_q(2) \to \mathcal{M}(U_i)$ and crossed products $\mathcal{M}(U_i) \#_\iota SU_q(2)$.

Finally we have to find transition functions which lead to quantum principal bundles. Theorem B.5 in the appendix gives a hint. If we define $\tau_{\iota,\kappa}^k : SU_q(2) \to \mathcal{M}(U_\iota \cap U_\kappa)$ for $\iota, \kappa = 1, 2$ by

$$\tau_{1,1}^k(h) = \varepsilon(h)1, \qquad h \in SU_q(2),$$

$$\tau_{2,2}^k(h) = \varepsilon(h)1, \qquad h \in SU_q(2),$$

$$\tau_{1,2}^k(h) = \sum_{(h)} u_1^k(h_{(1)})u_1^{-k}(h_{(2)}), \quad h \in SU_q(2),$$

$$\tau_{2,1}^k(h) = \sum_{(h)} u_2^k(h_{(1)})u_2^{-k}(h_{(2)}), \quad h \in SU_q(2),$$

Theorem B.5 shows the mappings

$$\Omega_{\kappa,\iota}^{k} = (m \otimes id) \circ (id \otimes \tau_{\iota,\kappa}^{k} \otimes id) \circ (id \otimes \varDelta):$$

$$\mathcal{M}(U_1 \cap U_2) \#_\iota SU_q(2) \to \mathcal{M}(U_1 \cap U_2) \#_\kappa SU_q(2)$$

being homomorphisms. Now by 3.14 the following is true.

Theorem 5.5. *For every index $k \in \mathbb{Z}$ there exists a quantum principal bundle $\mathcal{P}^k$ over the q-deformed space time $\mathcal{M}$ and with structure quantum group $SU_q(2)$ such that the above defined $\tau_{\iota,\kappa}^{k}$ are its transition functions. This quantum principal bundle is the q-deformed instanton model for the index k. If $q = 1$ the q-deformed instanton model turns into the $SU(2)$-principal fibre bundle with index k.*

A. Sheaf Theory

Sheaves provide the natural mathematical language to switch from the local to the global and vice versa. Only the definition of sheaves and their morphisms are given. For further details see the literature, for example Tennison [21] or Mac Lane, Moerdijk [16].

Every topological space M gives rise to the category $\mathcal{T}_M$ of open sets in M. Its morphisms are the inclusion maps $i_V^U : V \to U, u \mapsto u$, where $V \subset U \subset M$ open. Furthermore a continuous mapping $F : M \to N$ between topological spaces defines a contravariant functor $f^{-1} : \mathcal{T}_N \to \mathcal{T}_M$ by

$$U \mapsto f^{-1}(U), \quad U \in Obj(\mathcal{T}_N),$$

$$i_V^U \mapsto i_{f^{-1}(V)}^{f^{-1}(U)}, \quad V, U \in Obj(\mathcal{T}_N), \quad V \subset U.$$

Definition A.1. *Let M be a topological space, and $\mathcal{K}$ a subcategory of the category of sets. Then a contravariant functor*

$$\mathcal{G} : \mathcal{T}_M \to \mathcal{K}$$

is called a presheaf of $\mathcal{K}$-objects over M. The elements of the set $\mathcal{G}(U)$ with $U \subset M$ open are the sections of $\mathcal{G}$ over U. $\mathcal{G}$ is called a sheaf of $\mathcal{K}$-objects over M, if the following conditions are satisfied for each open covering $(U_\iota)_{\iota \in I}$ of an open set $U \subset M$:

(i) *If $s, s' \in \mathcal{G}(U)$ and*

$$\mathcal{G}_{U_\iota}^{U}(s) = \mathcal{G}_{U_\iota}^{U}(s'), \tag{127}$$

   for all $\iota \in I$, then $s = s'$.
(ii) *Let $(s_\iota)_{\iota \in I}$ be a family of sections $s_\iota \in \mathcal{G}(U_\iota)$, such that for all $\iota, \kappa \in I$*

$$\mathcal{G}_{U_\iota \cap U_\kappa}^{U_\iota}(s_\iota) = \mathcal{G}_{U_\iota \cap U_\kappa}^{U_\kappa}(s_\kappa). \tag{128}$$

   Then there exists a uniquely defined $s \in \mathcal{G}(U)$, which fulfills the relation

$$\mathcal{G}_{U_\iota}^{U}(s) = s_\iota$$

   for all $\iota \in I$.

The language of sheaves provides for an abstract characterisation of local function algebras. Important examples are given by the sheaf $\mathcal{C}_M$ (resp. $\mathcal{C}_M^\infty, \mathcal{C}_M^\omega, \mathcal{O}_M$) of continuous (resp. differentiable, analytical, holomorphic) functions on a topological space (resp. differentiable, real analytic, complex manifold) M.

Definition A.2. *Let $f: M \to N$ be a continuous mapping between topological spaces, and let*

$$\mathscr{G} : \mathscr{T}_M \to \mathscr{K}$$

$$\mathscr{G}' : \mathscr{T}_N \to \mathscr{K}$$

be sheaves over M resp. N with objects in the category $\mathscr{K}$. A morphism of sheaves from $\mathscr{G}$ to $\mathscr{G}'$ over f is given by a morphism of functors

$$\mathscr{F} : \mathscr{G}' \to \mathscr{G} \circ f^{-1}.$$

If $f: M \to N$ is a continuous mapping, the pull back $f^* : \mathscr{C}_N \to \mathscr{C}_M$ gives an example of a morphism of sheaves.

B. Crossed Products and Smash Products

Crossed products and smash products serve to define a multiplication on the tensor product of a Hopf algebra and an algebra. Most of the definitions and theorems are given according to Blattner, Cohen and Montgomery [2]. All algebras are supposed to have a unit.

Definition B.1. *Let H be a Hopf algebra and A an algebra over the field k. A weak H-action on A is a bilinear mapping*

$$H \times A \to A$$

$$(h, a) \mapsto h \cdot a,$$

such that for all $h \in H$ and $a, b \in A$:

 (i) $h \cdot ab = \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot b),$

 (ii) $h \cdot 1 = \varepsilon(h) \cdot 1,$

 (iii) $1 \cdot a = a.$

An H-action on A is given by a weak action fulfilling the condition

 (iv) $h \cdot (l \cdot a) = (hl) \cdot a$

for all $h, l \in H$ and $a \in A$.

Theorem B.2. *Let the Hopf algebra H weakly coact on the algebra A. Furthermore let $\sigma : H \times H \to A$ be a mapping which satisfies:*

 (i) (*normality condition*)

$$\sigma(1, h) = \sigma(h, 1) = \varepsilon(h) \quad \text{for all } h \in H.$$

 (ii) (*cocycle condition*)

$$\sum_{(h),(l),(m)} (h_{(1)} \cdot \sigma(l_{(1)}, m_{(1)})) \, \sigma(h_{(2)}, l_{(2)} m_{(2)})$$

$$= \sum_{(h),(l)} \sigma(h_{(1)}, l_{(1)}) \, \sigma(h_{(2)} l_{(2)}, m) \quad \text{for all } h, l, m \in H.$$

(iii) (*twisted module condition*)

$$\sum_{(h),(l)} (h_{(1)} \cdot (l_{(1)} \cdot a)) \, \sigma(h_{(2)}, l_{(2)})$$

$$= \sum_{(h),(l)} \sigma(h_{(1)}, l_{(1)}) \, (h_{(2)} l_{(2)} \cdot a) \quad \text{for all} \quad h, l \in H, \quad a \in A.$$

Then the equation

$$(a \otimes h)(b \otimes l) = \sum_{(h),(l)} a(h_{(1)} \cdot b) \sigma(h_{(2)}, l_{(1)}) \otimes h_{(3)} l_{(2)} \tag{129}$$

gives a multiplication on $A \otimes H$, which defines an algebra $A \#_\sigma H$ called a crossed product with unity $1 \otimes 1$. The elements $a \otimes h$ of $A \#_\sigma H$ are sometimes written in the form $a \# h$.

Proof. See Blattner et al. [2], Corollary 4.6. $\qquad\qquad\qquad\qquad\qquad\square$

Example B.3. *Let the Hopf algebra H act on the algebra A. The bilinear mapping $\sigma : H \times H \to A$ shall be trivial, which means that for all $h, l \in H$*

$$\sigma(h, l) = \varepsilon(h)\varepsilon(l)1.$$

Then the assumptions of Theorem B.2 are satisfied and the crossed product $A \#_\sigma H$ is defined. We call it the smash product of H and A and simply write $A \# H$. The multiplication on the smash product is given by

$$(a \# h)(b \# l) = \sum_{(h)} a(h_{(1)} \cdot b) \, \# h_{(2)} l, \quad (a \# h), (b \# l) \in A \# H. \tag{130}$$

Theorem B.4. *Let H weakly act on A. Suppose the weak action is defined by an element of the convolution algebra $\mathrm{Hom}(H, A)$, i.e. there exists an invertible element $u \in \mathrm{Hom}(H, A)$ with $u(1) = 1$ such that for all $h \in H$, $a \in A$*

$$h \cdot a = \sum_{(h)} u(h_{(1)}) a u^{-1}(h_{(2)}). \tag{131}$$

Define the bilinear mapping $\sigma : H \times H \to A$ by

$$\sigma(h, l) = \sum_{(h),(l)} u(h_{(1)}) u(l_{(1)}) u^{-1}(h_{(2)} l_{(2)}). \tag{132}$$

Then σ is a normal cocycle fulfilling the twisted module condition. Therefore the assumptions of Theorem B.2 are satisfied and the crossed product $A \#_\sigma H$ exists.

Proof. See Blattner et al. [2], Example 4.11. $\qquad\qquad\qquad\qquad\qquad\square$

Theorem B.5. *Let $u_1, u_2 \in \mathrm{Hom}(H, A)$ be invertible and $u_1(1) = u_2(1) = 1$. For each $i \in 1, 2$ define the weak H-action*

$$\alpha_i : H \times A \to A$$

$$(h, a) \mapsto h \cdot a = \sum_{(h)} u_i(h_{(1)}) a u_i^{-1}(h_{(2)}).$$

Further construct the bilinear mappings

$$\sigma_i : H \times H \to A$$

$$(h, l) \mapsto \sigma_i(h, l) = \sum_{(h),(l)} u_i(h_{(1)}) u_i(l_{(1)}) u_i^{-1}(h_{(2)} l_{(2)})$$

and the crossed products $A \#_{\sigma_1} H$ and $A \#_{\sigma_2} H$ according to Theorem B.4. Then the linear mapping

$$\Phi : A \#_{\sigma_1} H \rightarrow A \#_{\sigma_2} H$$

$$(a \# h) \mapsto \sum_{(h)} a\, u_1(h_{(1)}) u_2^{-1}(h_{(2)}) \# h_{(3)}$$

is a morphism of algebras with unit.

Proof. See Proposition 1.19, Theorem 5.3 and Corollary 5.4 in Blattner et al. [2] $\square$

Suppose we are given a Hopf algebra H acting on an algebra B and an H-comodule algebra A with coaction $\phi : A \rightarrow H \otimes A$. Then define a bilinear mapping

$$m : (B \otimes A) \times (B \otimes A) \rightarrow (B \otimes A)$$

$$((f \otimes g), (f' \otimes g')) \mapsto (f \otimes g) \cdot (f' \otimes g')$$

$$= \sum_{(g)} f(g_{(-1)} \cdot f') \otimes g_{(0)} g'$$

on the tensor product $B \otimes A$. An easy calculation shows m being associative. Additionally we have for $(f \otimes g), (f' \otimes g') \in (B \otimes A)$

$$(1 \otimes 1) \cdot (f \otimes g) = (f \otimes g),$$

$$(f' \otimes g') \cdot (1 \otimes 1) = (f' \otimes g'). \tag{133}$$

This proves the first part of the following theorem.

Theorem B.6. *Suppose the Hopf algebra H acts on the algebra B and A is an H-comodule algebra. Then the mapping*

$$m : (B \otimes A) \times (B \otimes A) \rightarrow (B \otimes A)$$

$$((f \otimes g), (f' \otimes g')) \mapsto (f \otimes g) \cdot (f' \otimes g')$$

$$= \sum_{(g)} f(g_{(-1)} \cdot f') \otimes g_{(0)} g'$$

turns the vector space $B \otimes A$ into an algebra with unit $1 \otimes 1$. This algebra is called the smash product of B and A and will be notated by $B \# A$. If H acts only weakly on B and $\sigma : H \times H \rightarrow B$ is a normal cocycle fulfilling the twisted module condition, the term

$$m : (B \otimes A) \times (B \otimes A) \rightarrow (B \otimes A)$$

$$((f \otimes g), (f' \otimes g')) \mapsto (f \otimes g) \cdot (f' \otimes g')$$

$$= \sum_{(g),(g')} f(g_{(-2)} \cdot f') \sigma(g_{(-1)}, g'_{(-1)}) \otimes g_{(0)} g'_{(0)}$$

defines an algebra structure on $B \otimes A$ with unit $1 \otimes 1$. The algebra defined in this way is called a crossed product and will be written $B \#_{\sigma} A$.

Proof. The first part has been shown above, the second one can be proven by an analog argument. $\square$

Note added in proof. After having submitted this paper I received a preprint of the paper [4] by T. Brzezinski and S. Majid which is concerned with a similar matter.

References

1. Abe, E.: Hopf Algebras. Cambridge Tracts in Math. N **74**. Cambridge, New York: Cambridge University Press, 1980
2. Blattner, R.J., Cohen, M., Montgomery, S: Crossed products and inner actions of Hopf algebras. Trans. Am. Math. Soc. **298**, 671–711 (1986)
3. Blattner, R.J., Montgomery, S.: Crossed Product and Galois Extensions of Hopf Algebras. Pac. J. Math. Vol. **137**, No. 1. 37–54 (1989)
4. Brzezinski, T., Majid, S.: Quantum group gauge theory on quantum spaces. Commun. Math. Phys. **157**, No. 3 (1993)
5. Connes, A.: Non-Commutative Differential Geometry. Publ. Math. IHES **62** (1985)
6. Daniel, M., Viallet, C.M.: The geometrical setting of gauge theories of the Yang-Mills type. Rev. Mod. Phys. **52**, No. 1 (1980)
7. Drinfeld, M.: Quantum Groups. Proc. of the ICM 1986, Berkeley (1986), pp 798–820
8. Greub, W., Halperin, S., Vanstone, R.: Connections, Curvature and Cohomology, Volume I. New York, London: Academic Press, 1972
9. Greub, W., Halperin, S., Vanstone, R.: Connections, Curvature and Cohomology, Volume II. New York, London: Academic Press, 1973
10. Grothendieck, A.: A General Theory of Fibre Spaces with Structure Sheaf. University of Kansas, Department of Mathematics, Report No. 4 (Lawrence, Kansas 1965[2])
11. Haag, R.: Local Quantum Physics. Texts and Monographs in Physics, Berlin, Heidelberg, New York: Springer, 1992
12. Hartshorne, R.: Algebraic Geometry. Graduate Texts in Mathematics, Berlin, Heidelberg, New York: Springer, 1977
13. Itzykson, C., Zuber, J.B.: Quantum Field Theory. New York: McGraw-Hill, 1980
14. Kasper, U.: Fibre Bundles: An Introduction to Concepts of Modern Differential Geometry. In: Geometry and Theoretical Physics, eds. Debrus, J., Hirshfeld, A.C., Berlin, Heidelberg, New York: Springer, 1991
15. Kaup, L. Kaup, R.: Holomorphic Functions of Several Variables. de Gruyter Studies in Mathematics, Berlin, New York, 1983
16. Mac Lane, S., Moerdijk, I.: Sheaves in Geometry and Logic. Berlin, Heidelberg, New York: Springer-Verlag 1992
17. Manin, Y.I.: Quantum Groups and Non-Commutative Geometry. Montreal Lecture Notes (1988)
18. Manin, Y.I.: Gauge Field Theory and Complex Geometry. Grundlehren der mathematischen Wissenschaften **289**, Berlin, Heidelberg, New York: Springer, 1988
19. Pflaum, M.J.: Quantengruppen auf Faserbündeln. Diplomarbeit an der Sektion Physik der Universität München, München, 1. November 1992
20. Streater, R.F., Wightman, A.S.: PCT, Spin and Statistics and all that. New York, Amsterdam: Academic Press, 1964
21. Tennison, B.R.: Sheaf Theory. Cambridge University Press, London Math. Soc. Lecture Note Series **20** Cambridge, 1975
22. Wess, J., Bagger, J.: Supersymmetry and supergravity. Princeton, N.J.: University Press, 1983
23. Wess, J., Zumino, B.: Covariant Differential Calculus on the Quantum Hyperplane. preprint CERN-TH-5697/90 (April 1990)
24. Woronowicz, S.L.: Differential Calculus on Compact Matrix Pseudogroups. Commun. Math. Phys. **122**, 125–170 (1989)