Commun. Math. Phys. 166, 255–277 (1994)

Communications in
Mathematical
Physics

Nonintegrability of Some Hamiltonian systems, Scattering and Analytic Continuation

C. Grotta Ragazzo★

Instituto de Matemática e Estatística, Universidade de São Paulo, CP 20570, 01498, São Paulo, SP, Brasil and Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, 10012, New York, NY, USA★★

Received: 2 April 1993/in revised form: 22 February 1994

Abstract: We consider canonical two degrees of freedom analytic Hamiltonian systems with Hamiltonian function $H=\frac{1}{2}[p_1^2+p_2^2]+U(q_1,q_2)$, where $U(q_1,q_2)=\frac{1}{2}[-\nu^2q_1^2+\omega^2q_2^2]+\mathcal{O}(q_1^2+q_2^2)^{3/2})$ and $\partial_{q_2}U(q_1,0)=0$. Under some additional, not so restrictive hypothesis, we present explicit conditions for the existence of transversal homoclinic orbits to some periodic orbits of these systems. We use a theorem of Lerman (1991) and an analogy between one of its conditions with the usual one dimensional quantum scattering problem. The study of the scattering equation leads us to an analytic continuation problem for the solutions of a linear second order differential equation. We apply our results to some classical problems.

1. Introduction

Hamiltonian systems are usually classified as integrable or nonintegrable. *In this article we restrict our attention to two degrees of freedom real analytic Hamiltonian systems and say that a system is integrable if it has an analytic first integral independent of its Hamiltonian function.* If the system is integrable then its dynamics is essentially almost periodic and we can say that it is "well-behaved." Integrability is a very strong property and integrable systems are the exception. So, why do we care about integrable systems? Besides the fact that integrable systems appear in some physical models, a possible answer is because this is the unique situation where we have a more or less complete description of the global dynamics. In the nonintegrable case the dynamics is much more complicated, and a usual approach is to consider the nonintegrable system as some perturbation of an integrable one. Due to the "practical" importance of the integrable systems it is crucial to decide if a given system is or is not integrable.

★ Partially supported by FAPESP (Fundação de Amparo a Pesquisa do Estado de São Paulo), "Projeto Temático Transição de Fase em Sistemas Evolutivos", Grant 90/3918-5.
★★ Visiting scholar (academic year 92/93) financially supported by CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior), Brasil, e-mail: ragazzo@cims.nyu.edu

In general it is very difficult to show that a system is integrable without finding its first integrals and there is no effective method to do this. Nevertheless, there are some ways of showing that a system is nonintegrable by studying some local properties near particular solutions of it (see, for instance, Arnold et al. (1988)). We shall use in the present work the result due to Poincaré that if a system has a periodic orbit of saddle type and the stable and unstable manifold of this orbit intersect transversally (i.e., the system has a transversal homoclinic orbit) then the system is nonintegrable. The presence of transversal homoclinic orbits implies the existence of invariant sets ("topological horseshoe") with very complicated dynamics and this complexity excludes the presence of another analytic first integral besides the Hamiltonian function itself (Moser (1973)). In this work, we shall restrict our attention to systems that have transversal homoclinic orbits, so *all the nonintegrable systems considered here do have a transversal homoclinic orbit.*

The main goal of this article is to study the integrability question associated to the following class of Hamiltonian systems

$$\dot{p}_k = -\partial_{q_k} H, \quad \dot{q}_k = \partial_{p_k} H, \quad p_k \in \mathbb{R}, \quad k = 1, 2\,;$$

$$H = \frac{1}{2}[p_1^2 + p_2^2] + U(q_1, q_2)\,, \tag{1}$$

where:

a) U is analytic,
b) $U(q_1, q_2) = \frac{1}{2}[-\nu^2 q_1^2 + \omega^2 q_2^2] + \mathcal{O}((q_1^2 + q_2^2)^{3/2})$,
   with $\omega^2 > 0$, $\nu^2 > 0$,
c) $\partial_{q_2} U(q_1, 0) = 0$, for any $q_1 \in \mathbb{R}$,
d) equation $U(q_1, 0) = 0$ has a nontrivial nondegenerated solution q_{1c} and no solutions in $(0, q_{1c})$.

These systems have two important properties:

- the origin is an equilibrium point of saddle-center type, namely it is associated to a pair of real $\pm \nu$ and a pair of imaginary $\pm i\omega$ eigenvalues,
- there is a homoclinic orbit to the origin (it is contained in the plane $q_2 = \dot{q}_2 = 0$).

These systems are good candidates to admit an explicit proof of nonintegrability in the way mentioned above. First, if we suppose that near the equilibrium the flow is approximately the linear one, then in each energy level the system has a periodic orbit of saddle type. Second, the existence of a homoclinic orbit to the origin suggests that the stable and the unstable manifold of these periodic orbits can intersect and by a genericity argument it may occur in a transversal way. In fact, this picture is correct. In a more general context this argument was made rigorous by Lerman (1991) (it first appeared in Russian in 1987) and also by Mielke et al. (1992). In both references, besides the requirement of having a saddle-center equilibrium and a homoclinic orbit to it, a new hypothesis appears. It requires that the monodromy operator associated to this homoclinic orbit satisfies a certain condition. As we could expect from the generic character of the nonintegrable systems this last hypothesis is generically verified. Our work begins at this point.

In Sect. 2 we present a proof of Lerman's theorem. The condition that appears in it depends on a special system of coordinates. Our first result is to rewrite this condition in a way that it is independent of the coordinate system. Then we restrict our attention to systems (1) and present a theorem relating the new condition to a well-known one dimensional quantum scattering problem. We can briefly state the theorem as:

Let $\Gamma:\mathbb{R}\to\mathbb{R}$ be the q_1-component of the solution of system (1) *homoclinic to the origin. If the "reflection coefficient" B associated to the following scattering problem:*

$$\ddot{y} = -(\partial_{q_2q_2}U(\Gamma,0))y$$

is different from zero then the system is nonintegrable.

See Sect. 2 for a more precise statement.

In Sect. 3 we present some immediate applications of the theorem above. Solving the scattering problem we can prove that:

Consider system (1) *with U satisfying*

$$U(q_1,0) = -\frac{1}{2}\nu^2 q_1^2 + \frac{\alpha}{n+1}q_1^{n+1},$$

$$\partial_{q_2q_2}U(q_1,0) = \omega^2 + \beta q_1^{n-1},$$

where $n\in\{2,3,\ldots\}$ *and* $\alpha>0$. *If*

$$\frac{\beta}{\alpha} \neq \frac{(n-1)^2}{2(n+1)}l(l-1), \quad \text{for} \quad l\in\mathbb{N},$$

then the system is nonintegrable.

This result was already applied to a two degrees of freedom Hamiltonian system related to a nonlinear wave equation (Grotta Ragazzo (1994)). It has many other applications to physical problems as, for instance, to the generalized Hénon–Heiles systems (Hénon and Heiles (1964)). For most of these systems this is probably the first rigorous proof of nonintegrability. In Sect. 3 we also present a theorem that uses the classical "reflectionless potentials" of the "inverse scattering theory."

Our thorem of Sect. 2 reduces the integrability question for systems (1) to a scattering problem. As mentioned above, the scattering condition $B\neq 0$ is "generically" true but checking it is far from being an easy problem. This leads us to alternative formulations of the condition $B\neq 0$. In Sect. 4 we introduce a new independent variable x, defined by $x=\Gamma(t)$, in the scattering equation. Considering x as a complex variable the scattering equation is transformed into the linear second order equation

$$-2U(x,0)\frac{d^2y}{dx^2} - \partial_{q_1}U(x,0)\frac{dy}{dx} + \partial_{q_2q_2}U(x,0) = 0\,.$$

This equation has at least two regular singular points at $x=0$ and $x=q_{1c}$. The scattering condition becomes an analytic continuation problem for the solutions of this new equation. Two theorems are presented in Sect. 4. In the first one, to check the scattering condition, we have to verify that two solutions of the above equation, defined by their Taylor series in neighbourhoods of $x=0$ and $x=1$, respectively, can be analytically continued one into the other. In the second theorem we present a geometric characterization of the scattering condition. In a few words it says, if we make the analytic continuation of any solution of the above equation through a closed arc that winds twice the interval $(0, q_{1c})$ and does not contain in its interior any singularity of the equation, but $x = 0$ and $x = q_{1c}$, then we should return to the "original solution" if, and only if, the reflection coefficient of the associated scattering problem is zero. We finish Sect. 4 using this theorem to prove the result presented in Sect. 3 in a very simple way. The proof illustrates well some advantages of the geometric formulation of the problem.

Section 5 is a short conclusion where we make some comments on possible extensions of the results of this article and briefly discuss the relation of this work with Ziglin's theorem (Ziglin (1983)) on the integrability of complex analytic Hamiltonian systems.

2. Saddle–Center Loops and Scattering

In this section we present the relation between the integrability question of some four dimensional Hamiltonian systems and the usual one dimensional quantum scattering problem. The first theorem, which is presented without proof, is essentially due to Moser (1958) with a supplement of Rüssmann (1964). It says that near a saddle-center equilibrium point the Hamiltonian vector field is integrable. It plays an important role in the proof of the next theorem.

Theorem 1 (Moser, Rüssman). *Let (M, Ω, H) be a Hamiltonian system defined on a 4-dimensional symplectic analytic manifold M, with symplectic form Ω and analytic Hamiltonian function H. Assume that:*

i) *p is a saddle-center equilibrium point of (M, Ω, H), namely p is associated to two nonzero real, $\pm\nu$, and two nonzero imaginary, $\pm\omega i$, eigenvalues.*

Then there exists a neighborhood U of p with conjugate canonical coordinates $(x_1, x_2; x_3, x_4) \stackrel{\text{def}}{=} \underset{\sim}{x}$, symplectic form $\Omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ and the Hamiltonian function is given by (possibly after time reversing)

$$H(\underset{\sim}{x}) = h(\xi, n) = \nu\xi + \omega\eta + \mathscr{R}(\xi, \eta), \quad \mathscr{R}(\xi, \eta) = \mathcal{O}(\xi^2 + \eta^2)\,,$$

$$(\xi = x_1 x_2; \quad \eta = (x_3^2 + x_4^2)/2; \quad \nu, \omega > 0)\,.$$

Proof. See Moser (1958), Rüssmann (1964).

The following theorem plays a fundamental role in this work. It gives a sufficient condition for the existence of transversal homoclinic orbits to periodic orbits, near a saddle-center equilibrium point. It is due to Lerman, and it first appeared in Russian in 1987. Its first English-translation appeared in Lerman (1991). It is important to say that Mielke et al. (1992) have related results.

Theorem 2 (Lerman) *Consider the same hypothesis and notation of Theorem 1. Assume that*

(ii) *there is a homoclinic orbit Γ to p.*

Let us fix some time parameterization and denote by $\psi(t, t_0)\colon T_{\Gamma(t_0)}M \to T_{\Gamma(t)}$ the linearized flow operator associated to Γ, where $T_{\Gamma(t)}M$ denotes the tangent space to M at $\Gamma(t)$. Let $t', t'' \in \mathbb{R}$ be such that $\Gamma(t') \in U$, and, in the coordinates $\underset{\sim}{x}$, $\Gamma(t') = (0, d', 0, 0)$, $\Gamma(t'') = (d'', 0, 0, 0)$; $d', d'' > 0$ (the case $d'' < 0$ is analogous). Let

$\psi_*(t'', t')$ be the matrix representation of $\psi(t'', t')$ in the coordinates $\underline{x}$, and a_{jk}, $j, k = 1, \ldots, 4$, be the elements of $\psi_*(t'', t')$. Suppose, in addition, that

iii) *for all $\delta \in [0, 2\pi]$*

$$C \stackrel{\text{def}}{=} \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} \neq \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix} \stackrel{\text{def}}{=} R_\delta .$$

Under these conditions there is an $E^* > 0$ such that in each energy level E, $0 < E \leqq E^*$, (M, Ω, H) has an unstable periodic orbit L_E ($L_E \to p$ as $E \to 0$) and four transversal homoclinic trajectories to L_E. Moreover the system does not have an analytic integral independent of H.

Proof. Let us suppose that $\nu = 1$ (this is obtained with a simple time re-scaling and a redefinition of ω). In the local coordinate system of Theorem 1 the local flow of the system is given by:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} e^{-th_\xi^0} & 0 & 0 & 0 \\ 0 & e^{th_\xi^0} & 0 & 0 \\ 0 & 0 & \cos(th_\eta^0) & -\sin(th_\eta^0) \\ 0 & 0 & \sin(th_\eta^0) & \cos(th_\eta^0) \end{pmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ x_4^0 \end{pmatrix}, \tag{2}$$

where

$$h_\xi^0 = h_\xi(\xi^0, \eta^0), \quad h_\eta^0 = h_\eta(\xi^0, \eta^0), \quad \xi^0 = x_1^0 x_2^0, \quad \eta^0 = ((x_3^0)^2 + (x_4^0)^2)/2 .$$

From (2) it is immediate that the one parameter family of periodic orbits L_E, $0 < E < E_*$, is given by

$$L_E = \{\underline{x} = (0, 0, \sqrt{2\eta}\cos\theta, \sqrt{2\eta}\sin\theta) | 0 \leqq \theta < 2\pi, E = h(0, \eta)\} .$$

It is also immediate that the local unstable and local stable manifolds of L_E are given by:

$$W^u_{\text{loc}}(L_E) = \{\underline{x} = (0, 1 \pm e^t, \sqrt{2\eta}\cos\theta, \sqrt{2\eta}\sin\theta) | 0 \leqq \theta < 2\pi, -\infty \leqq t < t_*, E = h(0, \eta)\},$$

$$W^s_{\text{loc}}(L_E) = \{\underline{x} = (\pm e^{-t}, 0, \sqrt{2\eta}\cos\theta, \sqrt{2\eta}\sin\theta) | 0 \leqq \theta < 2\pi, t_* < t \leqq \infty, E = h(0, \eta)\},$$

where t_* *is a sufficiently small number.*

Let us define the three dimensional disks

$$\Sigma^u = \{(x_1, d', x_3, x_4) | x_1^2 + x_3^2 + x_4^2 < \varepsilon'\} ,$$

$$\Sigma^s = \{(d'', x_2, x_3, x_4) | x_2^2 + x_3^2 + x_4^2 < \varepsilon'\} ,$$

both contained in U (see Fig. 1). Using that Σ^u and Σ^s are transversal to Γ and choosing ε' sufficiently small we can define the Poincaré map $\Phi: \Sigma^u \to \Sigma^s$ induced by the flow near Γ (in the coordinates $\underline{x}$ $\Phi: (x_1, d', x_3, x_4) \to (d'', \Phi_2, \Phi_3, \Phi_4)$). Let us denote the intersection between Σ^s and the energy level E (determined by the equation $h(d'', x_2, \eta) = E$) by Σ^s_E. Since $h(0, 0) = 0$ and $\partial_\xi h(0, 0) = 1$ the implicit function theorem implies that Σ^s_E is given by the graph of a function $x_2 = x_2(E, \eta)$, namely Σ^s_E is a manifold parametrized by (x_3, x_4). From the form of $W^s_{\text{loc}}(L_E)$ we have that $W^s_{\text{loc}}(L_E) \cap \Sigma^s_E \stackrel{\text{def}}{=} \gamma^s_E$ is the circle

$$\gamma^s_E = \{\underline{x} = (d'', 0, \sqrt{2\eta}\cos\theta, \sqrt{2\eta}\sin\theta) | 0 \leqq \theta < 2\pi, E = h(0, \eta)\} ,$$

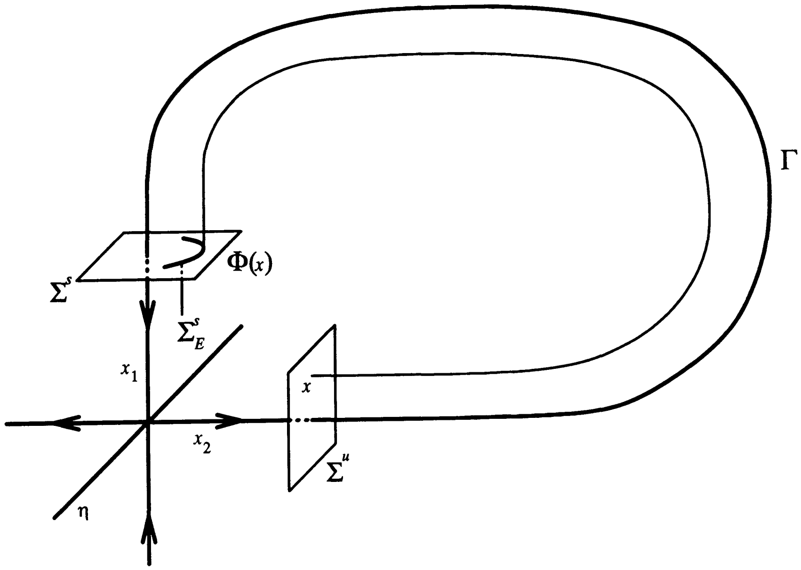

Fig. 1. Diagram showing the geometrical construction described in the text. The x_1, x_2 and η axis represent $W^s_{\mathrm{loc}}(p)$, $W^u_{\mathrm{loc}}(p)$ and $W^c_{\mathrm{loc}}(p)$, respectively. We point out that the η axis is representing the "two dimensional" center-manifold

and analogously $W^u_{\mathrm{loc}}(L_E) = \Sigma^u \overset{\mathrm{def}}{=} \gamma^u_E$ is the circle

$$\gamma^u_E = \{\underset{\sim}{x} = (0, d'', \sqrt{2\eta}\cos\theta, \sqrt{2\eta}\sin\theta) | 0 \leqq \theta < 2\pi, E = h(0, \eta)\} ,$$

Notice that by the conservation of energy $\Phi(\gamma^u_E)$ is contained in Σ^s_E. Moreover, since γ^u_E belongs to the global unstable manifold of L_E (denoted by $W^u(L_E)$), if there is a point q such that $q \in \Phi(\gamma^u_E) \cap \gamma^s_E$ then there is a homoclinic orbit to L_E passing through q. This homoclinic orbit will be transversal if the two one-dimensional tangent spaces $T_q\gamma^s_E$ and $T_q\Phi(\gamma^u_E)$ span the two-dimensional tangent space $T_q\Sigma^s_E$ (or if $T_q\gamma^s_E$ does not coincide with $T_q\Phi(\gamma^u_E)$) (see Fig. 2). In the polar coordinates $x_3 = \sqrt{2\eta}\,\cos\theta, x_4 = \sqrt{2\eta}\,\sin\theta$ the condition for the existence of these homoclinic orbits is the equation

$$\eta = \Phi_\eta(0, d', \eta, \theta) \overset{\mathrm{def}}{=} [\Phi_3^2(0, d', x_3, x_4) + \Phi_4^2(0, d', x_3, x_4)]/2 \tag{3}$$

to have solutions η_*, θ_* and the transversality condition is

$$\frac{1}{\sqrt{2\eta_*}}\partial_\theta\sqrt{2\Phi_\eta(0, d', \eta_*, \theta_*)} = \frac{\partial_\theta\Phi_\eta(0, d', \eta_*, \theta_*)}{\sqrt{\eta_*\Phi_\eta(0, d', \eta_*, \theta_*)}} \neq 0 \tag{4}$$

(see Fig. 2). Since Φ_3 and Φ_4 are equal to zero for $x_3 = x_4 = 0$ we can expand Φ_η in Taylor series around $\eta = 0$ and using that $a_{ij} = \partial_{x_j}\Phi_i(0, d', 0, 0)$, $i, j = 3, 4$, we can write Eq. (3) as

$$\begin{aligned} 1 &= \partial_\eta\Phi_\eta(0, d', \eta, \theta) + \mathcal{O}(\eta^{1/2}) \\ &= (a_{33}\cos\theta + a_{34}\sin\theta)^2 + (a_{43}\cos\theta + a_{44}\sin\theta)^2 + \mathcal{O}(\eta^{1/2}) . \end{aligned}$$

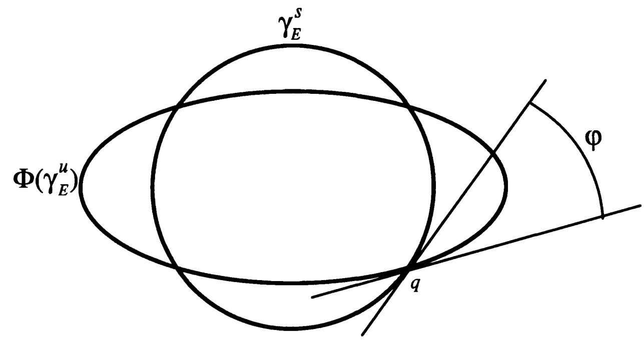

Fig. 2. Diagram representing the intersection between γ_E^s and $\Phi(\gamma_E^u)$. In this figure φ is the angle between $T_q\gamma_E^s$ and $T_q\Phi(\gamma_E^u)$ in the (x_3, x_4) coordinate system. It is given by

$$\varphi = \frac{1}{\sqrt{2\eta_*}}\partial_\theta\sqrt{2\Phi_\eta(0, d', \eta_*, \theta_*)} = \frac{\partial_\theta\Phi_\eta(0, d', \eta_*, \theta_*)}{\sqrt{\eta_*\Phi_\eta(0, d', \eta_*, \theta_*)}},$$

that is the expression appearing in condition (4)

In order to show that this equation has some solution for $\eta > 0$ we will apply the implicit function theorem using $\eta = 0$ as a basis point. This requires the equation

$$1 = \partial_\eta\Phi_\eta(0, d', 0, \theta) = (a_{33}\cos\theta + a_{34}\sin\theta)^2 + (a_{43}\cos\theta + a_{44}\sin\theta)^2 \quad (5)$$

to have some solution θ_0 and that $\partial_\theta\partial_\eta\Phi_\eta(0, d', 0, \theta_0) \neq 0$. This last condition also implies the transversality one (4) because of the limit $\partial_\theta\Phi_\eta(0, d', \eta_*, \theta_*) \to \eta_*\partial_\theta\partial_\eta\Phi_\eta(0, d', 0, \theta_0)$ as $\eta_* \to 0$. Equation (5) is equivalent to find the points of intersection between the unit circle (S^1) and its image (CS^1) by the linear transformation defined by the matrix C. The condition $\partial_\theta\partial_\eta\Phi_\eta(0, d', 0, \theta_0) \neq 0$ is equivalent to these intersections being transversal. If C is nonsingular then CS^1 is an ellipses with semi-axis λ_1, λ_2 such that $\lambda_1\lambda_2 = \det(C)$ (this can be easily seen if we use that C can be represented as the product of a positive symmetric matrix by an orthogonal one). Let us assume that $\det(C) = 1$. Then CS^1 is either an ellipsis with semi-axis $\lambda > 1, 1/\lambda < 1$, which implies that Eq. (5) has four solutions that verify the transversality condition, or it coincides with S^1 which implies that Eq. (5) has a continuum of solutions and the transversality condition is not verified anymore. This last case implies that C is orthogonal and, since $\det(C) = 1$, it is a rotation matrix which contradicts hypothesis iii). So Eq. (3) has four transversal solutions for η sufficiently small.

In order to finish the proof we have to show that $\det(C)$ is one. More than this, we will present some properties of the matrix $\psi_*(t'', t')$, that will be useful in the proof of the next theorem. Notice that:

- if (0, 1, 0, 0) are the coordinates of a vector in $T_{\Gamma(t')}M$, then $\psi_*(t'', t')$ maps this vector into a vector in $T_{\Gamma(t'')}M$ with coordinates $(\beta, 0, 0, 0)$, $\beta \in \mathbb{R}$, since $\dot\Gamma(t) = \psi(t, t_0)\dot\Gamma(t_0)$, $\dot\Gamma(t)$ being the velocity vector associated to Γ at the point $\Gamma(t)$;
- $\psi_*(t'', t')$ maps the linear space $\{\underline{x} | x_1 = 0\}$ contained in $T_{\Gamma(t')}M$ into the linear space $\{\underline{x} | x_2 = 0\}$ contained in $T_{\Gamma(t'')}M$, since the energy H of the system is conserved and $\psi(t'', t')$ takes tangent spaces to energy level onto tangent spaces to the same energy level.

- $\psi_*(t'', t')$ is a symplectic linear transformation, that is, it preserves the symplectic form Ω.

These remarks imply that $\psi_*(t'', t')$ can be written as

$$\begin{pmatrix} a_{11} & \beta & a_{13} & a_{14} \\ -1/\beta & 0 & 0 & 0 \\ a_{31} & 0 & \multicolumn{2}{c}{\multirow{2}{*}{C}} \\ a_{41} & 0 & & \end{pmatrix}, \tag{6}$$

where $\det(C) = 1$.

We remark that the nonexistence of another analytic integral of motion, besides H, is a well-known consequence of the existence of transversal homoclinic orbits (see Moser (1973). □

In order to apply the previous theorem to a given system we have to verify that hypothesis iii) is true using a very special coordinate system. In the following theorem we present a hypothesis equivalent to iii) that does not depend on the coordinate system. This greatly facilitates any explicit calculation. The way we handle the linearized flow operator in the proof suggests an algorithm for numerical estimation of the required limits as t goes to infinity.

Theorem 3. *Using the same notation of Theorems* 1 *and* 2, *let us consider a Hamiltonian system that satisfies hypothesis* i) *and* ii) *of them. Then, the following limit is well defined*:

$$\Lambda(r) = \lim_{n \to \infty} \psi(nT + r, -nT), \quad r \in [0, T), \quad n \in \mathbb{N},$$

where $T = 2\pi/\omega$ and $\Lambda(r)\colon T_pM \to T_pM$. Moreover, the tangent space to the center manifold at p, T_pW^c, is invariant by $\Lambda(r)$ and hypothesis iii) *of Theorem* 2 *is equivalent to*

iv) *If $\Lambda_c(r)$ is the restriction of $\Lambda(r)$ to T_pW^c then there is a value of r such that* $\mathrm{Sp}(\Lambda_c(r))$ *(the spectrum of Λ_c) does not contain points with modulus one.*

Proof. Let n be such that $nT + r > t''$, $-nT < t'$, where t', t'' were defined in Theorem 2. Using the matrix representation (6) and the local flow (2) we obtain:

$$\psi_*(nT + r, -nT) = \psi_*(nT + r, t'')\psi_*(t'', t')\psi_*(t', -nT)$$

$$= \begin{pmatrix} a_{11}e^{-2nT-r+\tau} & \beta e^{t'+''-r} & b_{13}e^{-nT} & b_{14}e^{-nT} \\ -e^{-t'-t''+r}/\beta & 0 & 0 & 0 \\ b_{31}e^{-nT} & 0 & \multicolumn{2}{c}{\multirow{2}{*}{R_{\omega r}D}} \\ b_{41}e^{-nT} & 0 & & \end{pmatrix},$$

where: $b_{31}, b_{41}, b_{13}, b_{14}$ are independent of n and $D = R_{-\omega t''}\, C R_{-\omega t'}$. Therefore the limit

$$\Lambda_*(r) = \lim_{n \to \infty} \psi_*(nT + r, -nT)$$

exits and is given by

$$\begin{pmatrix} 0 & \beta e^{t'+t''-r} & 0 & 0 \\ -e^{-t'-t''+r}/\beta & 0 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{R_{\omega r} D}} \\ 0 & 0 & & \end{pmatrix},$$

which implies that $\Lambda(r) = \lim_{n\to\infty} \psi(nT + r, -nT)$ also exists. Using that $\lim_{n\to\infty} \Gamma(\pm nT) = p$ we conclude that $\Lambda(r)$ can be defined as a mapping of T_pM into T_pM. Moreover, since W^c is given by $x_1 = x_2 = 0$ in the $\underset{\sim}{x}$ coordinates, we have that T_pW^c is invariant by $\Lambda(r)$ and the representation matrix $\Lambda_{c_*}(r)$ of $\Lambda_c(r)$ in these coordinates is $R_{\omega r}D$. In order to finish the proof we use the following proposition

Proposition 1. *If* C *is a* 2×2 *real matrix with* $\det(C) = 1$ *then the following assertions are equivalent:*

$$\mathrm{Sp}(R_\theta C) \cap \{\lambda \,|\, |\lambda| = 1\} \neq \emptyset \quad \textit{for any value of } \theta, \quad \theta \in [0, 2\pi)\,, \tag{7}$$

$$|\mathrm{Tr}(R_\theta C)| \leqq 2, \quad \textit{for any value of } 0,\ \theta \in [0, 2\pi)\,, \tag{8}$$

$$C \textit{ is a rotation matrix}\,, \tag{9}$$

where $\mathrm{Tr}(R_\theta C)$ *denotes the trace of* $R_\theta C$.

It is immediate that (7) implies (8). If (8) is true then

$$[\mathrm{Tr}(R_\theta C)]^2 = [(a_{33} + a_{44})^2 + (a_{34} - a_{43})^2]\sin^2(\theta + \theta') \leqq 4\,,$$

for some $\theta' \in [0, 2\pi)$ and any $\theta \in [0, 2\pi)$, which implies

$$a_{33}^2 + a_{34}^2 + a_{43}^2 + a_{44}^2 + 2(a_{33}a_{44} - a_{34}a_{43}) \leqq 4\,.$$

Subtracting $4\det(C) = 4$ from this inequality we obtain

$$(a_{33} - a_{44})^2 + (a_{34} + a_{43})^2 \leqq 0\,,$$

which implies (9). If (9) is true, then (7) is immediate. Using this proposition, and that $D = R_{-\omega t''}C R_{\omega t'}$ is a rotation matrix if, and only if, C is a rotation matrix, we conclude that condition iii) of Theorem (2) is satisfied if, and only if,

$$\mathrm{Sp}(\Lambda_{c^*}(r)) \cap \{\lambda \,|\, |\lambda| = 1\} = \emptyset \quad \text{for some value of } r, \quad r \in [0, \tau)\,.$$

Since $\mathrm{Sp}(\Lambda_c(r))$ does not depend on coordinates we obtain condition iv) of the theorem. □

In all the theorems above we have considered generic four-dimensional Hamiltonian systems containing saddle-center loops. In the rest of this article we will restrict our attention to the class of Hamiltonian systems (1) presented in Sect. 1. It is convenient to introduce the following notation:

$$P(q_1) = \nu^2 q_1^2 + \mathcal{O}(q_1^3) = -2U(q_1, 0)\,,$$

$$\omega^2 + Q(q_1) = \omega^2 + \mathcal{O}(q_1) = \partial_{q_2q_2} U(q_1, 0)\,. \tag{10}$$

Notice that

$$P(q_1) > 0 \quad \text{for} \quad q_1 \in (0, q_{1c})\,,$$
$$\text{and } P(q_{1c}) = 0, \quad \frac{dP}{dq_1}(q_{1c}) < 0, \tag{11}$$

where q_{1c} was introduced in hypothesis d) in Sect. 1. It is easy to see that systems (1) satisfy hypothesis i) and ii) of Theorems 1 and 2. Our main goal in this article is to establish conditions on H such that they also satisfy hypothesis iii) of Theorem 2. The program that we develop below certainly works for systems more general than (1), but, as a first approach, we prefer to work with this simple and relevant case. In the following theorem we relate condition iv) of Theorem 3 to the well-known one dimensional quantum scattering problem.

Theorem 4. *Consider a Hamiltonian system of class* (1). *It has a homoclinic solution to the origin given by* $q_1 = \Gamma(t)$, $p_1 = \dot{\Gamma}(t)$, $q_2(t) = p_2(t) = 0$, *where* Γ *satisfies the equation*

$$\dot{x}^2 = P(x)\,. \tag{12}$$

Consider the (q_2, p_2) *components of the first variational equation associated to this homoclinic solution*

$$\ddot{y} = -(\omega^2 + Q(\Gamma))\,y\,. \tag{13}$$

It has a complex-valued solution ϕ *with the following asymptotic behavior*

$$\phi(t) \to Ae^{i\omega t} + Be^{-i\omega t} \quad \text{as } t \to -\infty\,,$$
$$\phi(t) \to e^{i\omega t} \quad \text{as } t \to \infty, \quad A, B \in \mathbb{C}\,. \tag{14}$$

The system verifies all hypothesis of Theorem 2 *if, and only if,*

v) $B \neq 0$.

Proof. The Hamiltonian system (1) satisfies hypothesis i) and ii) of Theorems 1 and 2, with the equilibrium point p being the origin. It will satisfy hypothesis iii) of Theorem 2, if and only if, it satisfies hypothesis iv) of Theorem 3. In the following we show that, in this case, hypothesis iv) is equivalent to v).

First, since $|Q(\Gamma(t))|$ decays exponentially fast with t, Eq. (13) has a solution with asymptotic behavior given by (14). For systems (1) T_pW^c is given by $p_1 = q_1 = 0$, which implies that the operator $\Lambda_c(r)$ defined in Theorem 3 can be written as

$$\Lambda_c(r) = \lim_{n\to\infty} \psi_c(nT + r, -nT), \quad r \in [0, T), \quad T \overset{\text{def}}{=} 2\pi/\omega\,,$$

where ψ_c is the linearized flow operator associated to Eq. (13). Using the solution $\phi(t) \overset{\text{def}}{=} \phi_1(t) + i\phi_2(t)$ and its asymptotics (14) we can write the matrix representation of $\psi_c(t, t_0)$ in the $(y, \dot{y})$ coordinates as:

$$\psi_c(t, t_0) = \frac{1}{\omega}\begin{pmatrix} \phi_1(t)\dot{\phi}_2(\mathrm{t}_0) - \phi_2(t)\dot{\phi}^1(t_0)_1 & -\phi_1(t)\phi_2(t_0) + \phi_2(t)\phi_1(t_0) \\ \dot{\phi}_1(t)\dot{\phi}_2(\mathrm{t}_0) - \dot{\phi}_2(t)\dot{\phi}^1(t_0)_1 & -\dot{\phi}_1(t)\phi_2(t_0) + \dot{\phi}_2(t)\phi_1(t_0) \end{pmatrix}.$$

Making $t = t_0, t_0 \to \infty$ and using the asymptotics (14) we obtain the following relation

$$1 = \lim_{t_0 \to -\infty} \frac{1}{\omega}(\phi_1(t_0)\dot{\phi}_2(t_0) - \phi_2(t_0)\dot{\phi}_1(t_0)) = |A|^2 - |B|^2 . \tag{15}$$

Now, from Proposition 1, hypothesis iv) is not satisfied if, and only if, $|\mathrm{Tr}(\Lambda_c(r))| \leqq 2$ for any $r \in [0, T)$, with

$$\begin{aligned}\mathrm{Tr}(\Lambda_c(r)) &= \frac{1}{\omega}\lim_{n\to\infty}\{\phi_1(nT + r)\dot{\phi}_2(-nT) - \phi_2(nT + r)\dot{\phi}_1(-nT) \\ &\quad - \dot{\phi}_1(nT + r)\phi_2(-nT) + \dot{\phi}_2(nT + r)\phi_1(-nT)\} \\ &= \frac{1}{\omega}\lim_{n\to\infty}\{\mathrm{Im}[\bar{\phi}(nT + r)\dot{\phi}(-nT) + \dot{\phi}(nT + r)\bar{\phi}(-nT)]\} \\ &= 2|A|\cos(r\omega - \theta_A) ,\end{aligned}$$

where $\mathrm{Im}[\phi]$ is the imaginary part of ϕ, $\bar{\phi}$ is the complex conjugate of ϕ, and θ_A is the argument of A. Thus, using identity (15) we get

$$|\mathrm{Tr}(\Lambda_c(r))| = 2\sqrt{|B|^2 + 1}|\cos(\omega r - \theta_A)| ,$$

which is smaller than or equal to two for any $r \in [0, T)$ if, and only if, $B = 0$. □

Sometimes we may refer to hypothesis v) as the "nonresonance" condition and to $|B|^2/|A|^2$ as the reflection coefficient. Both expressions are borrowed from quantum mechanics.

3. Applications

In this section we present two applications of Theorem 4. Consider a Hamiltonian system of class (1) with U satisfying

$$\begin{aligned}P(q_1) &= v^2 q_1^2 - \frac{2\alpha}{n+1} q_1^{n+1} , \\ Q(q_1) &= \beta q_1^{n-1} ,\end{aligned} \tag{16}$$

where P, Q are defined in (10), $n \in \{2, 3, \ldots\}$ and $\alpha > 0$. In this case it is possible to solve Eqs. (12), (13) and prove that system (1) is nonintegrable for almost all values of β/α.

Theorem 5. *A Hamiltonian system of class* (1) *with* P, Q *(defined in* (10)*) is nonintegrable if*

$$\frac{\beta}{\alpha} \neq \frac{(n-1)^2}{2(n+1)} l(l-1), \quad \text{for} \quad l \in \mathbb{N} . \tag{17}$$

Proof. In order to prove this theorem we show that hypothesis v) of Theorem 4 is verified. Using (16) Eq. (12) can be written as

$$\dot{x}^2 = v^2 x^2 - \frac{2\alpha}{n+1} x^{n+1} .$$

The nontrivial even solution Γ of this equation is given by

$$\Gamma(t) = C \operatorname{sech}^{2/(n-1)}\left(\frac{tv(n-1)}{2}\right), \quad \text{where } C = \left(\frac{(n+1)v^2}{2\alpha}\right)^{1/(n-1)} .$$

Using it we write Eq. (13) as

$$\ddot{y} = -\left(\omega^2 + \frac{\beta(n+1)v^2}{2\alpha}\operatorname{sech}^2\left(\frac{tv(n-1)}{2}\right)\right)y . \tag{18}$$

This equation represents a well-known one dimensional quantum mechanics scattering problem and can be solved using hypergeometric functions. The modulus of B is given in Landau and Lifshitz (1965), Sect. 25, Problem 4, as

$$|B|^2 = \frac{\cos^2((\pi/2)\sqrt{\gamma})}{\sinh^2(\pi\varepsilon)}, \quad \text{if } \gamma > 0 ,$$

$$|B|^2 = \frac{\cosh^2((\pi/2)\sqrt{-\gamma})}{\sinh^2(\pi\varepsilon)}, \quad \text{if } \gamma < 0 ,$$

where

$$\varepsilon = \frac{2\omega}{(n-1)v}, \quad \gamma = \frac{\beta}{\alpha}\frac{8(n+1)}{(n-1)^2} + 1 .$$

These relations and hypothesis (17) ensures that $B \neq 0$. □

It is important to point out that the class of Hamiltonian systems presented in Theorem 5 contains some classical problems of the physical literature as the Hénon–Heiles system. The generalized Hénon–Heiles system is very well studied and its nonintegrability was previously proved for some parameter values (Churchill and Rod (1980), Holmes (1982)) but, besides this special case, we believe that Theorem 5 is the first rigorous proof of nonintegrability for most of the remaining cases.

The very convenience of working with condition v) of Theorem 4 is the existence of well-developed one dimensional scattering and inverse scattering theories. The following theorem is an example of an immediate result obtained using the classical "reflectionless" potentials of the inverse scattering theory (also associated to the soliton solutions of the KdV equation).

Theorem 6. *Consider a one parameter family of Hamiltonian systems* (1) *parameterized by* ω^2, $\omega^2 > 0$. *Let* Γ *be the solution of Eq.* (12). *Then a necessary condition for the integrability of all systems of such family is that* $Q(\Gamma(t))$ *can be written in the following form*:

$$Q(\Gamma(t)) = 2\frac{d^2}{dt^2}\log[\det(\mathrm{I} + \mathscr{A}(t))] , \tag{19}$$

where: Q *is defined in* (10), I *is the* $N \times N$ *identity matrix,* $\mathscr{A}$ *is an* $N \times N$ *matrix with elements given by*

$$A_{mn}(t) = \frac{a_n a_m e^{(k_m + k_n)t}}{k_m + k_n}, \quad n, m \in \{1, 2, \ldots N\} ,$$

$N \in \mathbb{N}$, *a_n and k_n are some positive numbers, k_n are distinct.*

Proof. This theorem is a consequence of Theorem 4 and of a theorem in the inverse scattering theory (Kay and Moses (1956)), saying that condition (19) is necessary and sufficient for $B = 0$ in (14) for any value of $\omega^2, \omega^2 > 0$. □

Notice that the function on the right-hand side of (19) is positive and decays exponentially fast as $t \to \pm\infty$. If $N = 1$ we can write $k_1 = v$, $a_1^2 = 2ve^{-2vt_0}$ and the right-hand side of (19) as

$$2v^2 \operatorname{sech}^2(v(t - t_0)) .$$

This is exactly the function that appears in the right-hand side of Eq. (18) when $\alpha = \beta, n = 3$ and $t_0 = 0$. In this case, as imposed by Theorem 5, condition (17) is not satisfied.

In Grotta Ragazzo (1993) we present a particular function U, satisfying (16), such that when condition (17) is not satisfied then the system is integrable. The determination of the general algebraic form of P and Q such that condition (19) is satisfied seems to be a very interesting question. Among other things, it can suggest possible families of integrable Hamiltonian systems.

4. Analytic Continuation

In Sect. 2 we saw that the integrability question for the class of Hamiltonian systems (1) reduces to the verification of the nonresonance condition v) of Theorem 4. In the way the problem was posed it involves two steps, first to integrate Eq. (12) and second to analyze the scattering problem. This is far from being easy for most systems. In this section we present new formulations of Theorem 4 so that we can reduce condition v) to the problem of analytic continuation into the complex domain.

Let us fix a convenient scale for q_1 such that the value of q_{1c}, defined in Sect. 1 hypothesis d), is one. Let us also normalize the solution Γ of Eq. (12) such that $\Gamma(0) = 1$. The central idea is to consider the following change of independent variable in Eq. (13):

$$x = \Gamma(t) ,$$

for $t < 0$ or $t > 0$. In the new variable x, Eq. (13) becomes

$$y'' = -\frac{P'}{2P} y' - \frac{\omega^2 + Q}{P} y , \tag{20}$$

where the prime means differentiation with respect to x. This equation has no singularities in the interval (0, 1) and has two regular singular points on its extremes. Let us consider Eq. (20) in the complex domain, that is $y: \mathbb{C} \to \mathbb{C}$. From the theory of second order linear equations (Morse and Feshbach (1953), Hille (1976)) we know that Eq. (20) has two independent analytic solutions in a neighborhood of each one of its regular points. In a punctured neighborhood of each regular singular point ζ it has a pair of independent singular solutions that can be written as $(x - \zeta)^s g(x)$ where g is analytic and $s \in \mathbb{C}$. The value of s is determined by the indicial equation

$$s(s - 1) - as - b = 0 , \tag{21}$$

where a and b are such that

$$-\frac{P'}{2P} = \frac{a}{(x-\zeta)} + \mathcal{O}(1)\,,$$

$$-\frac{\omega^2 + Q}{P} = \frac{b}{(x-\zeta)^2} + \mathcal{O}\left(\frac{1}{|x-\zeta|}\right).$$

We denote by G_1, G_2 and G_3, G_4 the pairs of local solutions of Eq. (20) defined in the neighborhoods of $x = 0$ and $x = 1$, respectively. These local solutions are explicitly given by

$$\begin{aligned} G_1(x) &= x^{\alpha_1} g_1(x), \quad G_3(x) = (x-1)^{\alpha_3} g_3(x)\,, \\ G_2(x) &= x^{\alpha_2} g_1(x), \quad G_4(x) = (x-1)^{\alpha_4} g_4(x)\,, \end{aligned} \tag{22}$$

where

$$\begin{aligned} g_1(x) &= 1 + A_1 x + \cdots\,, \\ g_2(x) &= 1 + B_1 x + \cdots\,, \\ g_3(x) &= 1 + C_1(x-1) + \cdots\,, \\ g_4(x) &= 1 + D_1(x-1) + \cdots\,. \end{aligned}$$

The indicial equation at $x = 0$ and $x = 1$ implies

$$\alpha_1 = i\frac{\omega}{v} \stackrel{\text{def}}{=} i\eta, \quad \alpha_2 = -i\eta, \quad \alpha_3 = 0, \quad \alpha_4 = 1/2\,.$$

Using that $P(x)$ and $Q(x)$ are real when x is real we conclude that C_k, D_k are real and $A_k = \bar{B}_k$, $k \in \mathbb{N}$. The values of A_k, C_k, D_k are given by some recursion relations obtained from the substitution of the respective series into Eq. (20).

At this point it is easy to see an advantage of working with the independent variable x instead of t, we can locally compute the solutions of Eq. (20) by convergent power series expansions. As usual in the theory of linear equations the problem that arises is how to "connect" these local solutions. In this case the problem is the following.

Let us define the functions ψ_- and ψ_+ as

$$\psi_-(x) = \psi_-(\Gamma(t)) = \phi(t), \quad x \in (0,1), \quad t < 0\,,$$

$$\psi_+(x) = \psi_+(\Gamma(t)) = \phi(t), \quad x \in (0,1), \quad t > 0\,,$$

where ϕ is the solution of Eq. (13) with asymptotic behavior (14). Near $x = 1$ the solutions ψ_- and ψ_+ can be written as

$$\begin{aligned} \psi_-(x) &= \alpha_- g_3(x) + \beta_-(x-1)^{1/2} g_4(x)\,, \\ \psi_+(x) &= \alpha_+ g_3(x) + \beta_+(x-1)^{1/2} g_4(x)\,, \end{aligned} \tag{23}$$

where $\alpha_-, \beta_-, \alpha_+, \beta_+ \in \mathbb{C}$. In order to relate α_-, β_- with α_+ and β_+ we use the continuity and the differentiability of ϕ at $t = 0$. We get

$$\lim_{x\to 1-} \psi_-(x) = \lim_{t\to 0-} \phi(t) = \lim_{t\to 0+} \phi(t) = \lim_{x\to 1-} \psi_+(x)\,, \tag{24}$$

and

$$\lim_{x\to 1-} \psi'_-(x)\sqrt{P(x)} = \lim_{t\to 0-} \psi'_-(\Gamma(t))\dot{\Gamma}(t) = \lim_{x\to 0-} \dot{\phi}(t) =$$

$$\lim_{t\to 0+} \dot{\phi}(t) = \lim_{t\to 0+} \psi'_+(\Gamma(t))\dot{\Gamma}(t) = \lim_{x\to 1-} \psi'_+(x)(-\sqrt{P(x)})\,. \tag{25}$$

From relations (23) and (24) we get $\alpha_- = \alpha_+ \overset{\text{def}}{=} \alpha$. From relations (23) and (25) we get $\beta_- = -\beta_+ \overset{\text{def}}{=} \beta$. Therefore we can rewrite relations (23) as

$$\psi_-(x) = \alpha g_3(x) + \beta(x-1)^{1/2} g_4(x)\,,$$
$$\psi_+(x) = \alpha g_3(x) - \beta(x-1)^{1/2} g_4(x)\,. \tag{26}$$

These relations say that to "connect" ψ_- and ψ_+ at some point ζ, sufficiently near $x=1$, we should make the analytic continuation of ψ_- along a small circle of radius $1-\zeta$ centered at $x=1$. In fact, since g_3 and g_4 are single valued, after winding once $x=1$, they return with same value, while $(1-\zeta)^{1/2}$ returns with value $e^{i\pi}(1-\zeta)^{1/2}$.

The solution ϕ of Eq. (13) Γ of Eq. (12) have the asymptotic limits

$$\phi(t) \to Ae^{i\omega t} + Be^{-i\omega t} \quad \text{as } t \to -\infty\,,$$
$$\phi(t) \to e^{i\omega t} \quad \text{as } t \to +\infty\,,$$

and

$$\Gamma(t) \to e^{\nu t} \quad \text{as } t \to -\infty\,,$$
$$\Gamma(t) \to e^{-\nu t} \quad \text{as } t \to +\infty\,.$$

Therefore, as $x \to 0_+$,

$$\psi_-(x) = \phi(t) \to Ae^{i\omega t} + Be^{-i\omega t} = A(e^{\nu t})^{i\frac{\omega}{\nu}} + B(e^{\nu t})^{-i\frac{\omega}{\nu}} \to Ax^{i\eta} + Bx^{-i\eta}\,,$$
$$\psi_+(x) = \phi(t) \to e^{i\omega t} = (e^{-\nu t})^{-i\frac{\omega}{\nu}} \to x^{-i\eta},$$

and we conclude that, for x near zero, ψ_- and ψ_+ can be written as

$$\psi_-(x) = Ax^{i\eta} g_1(x) + Bx^{-i\eta} g_2(x)\,,$$
$$\psi_+(x) = x^{-i\eta} g_2(x)\,. \tag{27}$$

From relations (26) and (27) we see that we can determine A and B if we know how to connect the local solutions G_1, G_2 with G_3, G_4 through the interval (0, 1).

Below we present two theorems that give us necessary and sufficient conditions for $B=0$ in Theorem 4. The first one is a straightforward consequence of the argument above. In the second one we consider Eq. (20) in the whole complex plane and obtain a geometric condition on the Riemann surface associated to its solutions. The result reinforces the idea that integrable systems imply "simple" geometric structures.

Theorem 7. *A necessary and sufficient condition for $B = 0$ in Theorem 4, is that Eq. (20) has a complex solution $\psi:(0,1) \to \mathbb{C}$ such that*

$$\psi(x) = x^{i\eta} g_1(x) \quad \textit{for } x \textit{ near } x = 0 \, ,$$

$$\psi(x) = e^{i\delta}\left(|\alpha| g_3(x) - i\frac{2\omega}{|\alpha|\sqrt{-P'(1)}}\sqrt{1-x}\, g_4(x) \right) \quad \textit{for } x \textit{ near } x = 1 \, ,$$

where $\alpha \stackrel{\text{def}}{=} |\alpha| e^{-i\delta}$ is some nonnull complex number.

Proof. Relations (27), the analyticity of ψ and ψ_+ and $g_1(x) = \bar{g}_2(x)$, $x \in (0,1)$, imply that $\psi(x) = \bar{\psi}_+(x)$ for $x \in (0,1)$. Therefore the α defined above is equal to that one appearing in relations (26). Let us assume that $B = 0$. Then $\bar{\psi}_+ = A^{-1}\psi_-$. Using relations (26), and the definition

$$(x-1)^{1/2} = i\sqrt{1-x}, \quad \text{for} \quad x \in (0,1) \, ,$$

we obtain

$$\bar{\psi}_+(x) = \bar{\alpha} g_3(x) + \bar{\beta} i\sqrt{1-x}\, g_4(x) = A^{-1}\psi_-(x) = A^{-1}(\alpha g_3(x) + \beta i \sqrt{1-x}\, g_4(x)) \, ,$$

which implies $\bar{\alpha} = A^{-1}\alpha$ and $\bar{\beta} = A^{-1}\beta$, or

$$\alpha\bar{\beta} = \bar{\alpha}\beta \, . \tag{28}$$

Actually, condition (28) is also sufficient for $B = 0$. Suppose $\alpha/\bar{\alpha} = \beta/\bar{\beta} = C$. Then from relations (26) we get

$$\psi_-(x) = \alpha g_3(x) + \beta i\sqrt{1-x}\, g_4(x) = C(\bar{\alpha} g_3(x) + \bar{\beta} i \sqrt{1-x}\, g_4(x)) = C\bar{\psi}_+(x) \, .$$

This and relations (27) imply $B = 0$ and $C = A$.

In order to relate $|\alpha|$ and $|\beta|$ we use the fact that ψ satisfies the Wronskian identity

$$L \stackrel{\text{def}}{=} \sqrt{P}(\psi\bar{\psi}' - \psi'\bar{\psi})\frac{i}{2} = \text{constant} \, .$$

This identity near $x = 0$ implies

$$\begin{aligned} L &= \frac{i}{2}\lim_{x\to 0+} (\psi\bar{\psi}' - \psi'\bar{\psi})\sqrt{P(x)} \\ &= \frac{i}{2}\lim_{x\to 0+}\left[-2i\eta\frac{|g_1(x)|^2}{x} + g_1(x)\bar{g}_1'(x) - g_1'(x)\bar{g}_1(x) \right]\sqrt{P(x)} \\ &= \lim_{x\to 0+} \frac{\sqrt{P(x)}}{x}\eta |g_1(x)|^2 = \eta\nu = \omega \, . \end{aligned}$$

Doing the same thing near $x = 1$ we get

$$\omega = L = \frac{i}{2} \lim_{x\to 1-} (\psi\bar{\psi}' - \psi'\bar{\psi}) \sqrt{P(x)}$$

$$= \frac{i}{2} \lim_{x\to 1-} \left(\frac{ig_3(x)g_4(x)}{2\sqrt{1-x}}\right)(\alpha\bar{\beta} + \bar{\alpha}\beta) \sqrt{P(x)}$$

$$= -\frac{\alpha\bar{\beta} + \bar{\alpha}\beta}{4} \sqrt{-P'(1)}\,.$$

This relation and (28) imply that $B = 0$ if, and only if,

$$\alpha\bar{\beta} = \frac{-2\omega}{\sqrt{-P'(1)}}\,.$$

Using this, relations (26) and $\bar{\psi}_+ = \psi$ we finish the proof. □

Before presenting the next theorem we have to introduce some notation. If $\mathbf{G}$ is a global analytic function (or the set of all "branches" of a "multi-valued" analytic function) let us denote by $\mathbf{G}_\zeta$ a function element of $\mathbf{G}$ (or a branch of $\mathbf{G}$) defined in a neighborhood of $\zeta \in \mathbb{C}$. We say that two function elements a, b are equivalent, $a \sim b$, if their domains of definition have a nonempty intersection and they coincide on it. Let γ be an arc beginning at x_i, ending at x_f, and that does not cross any singularity of $\mathbf{G}$. We denote by

$$(\mathbf{G}_{x_i}, \gamma) \stackrel{\text{def}}{=} \mathbf{G}_{x_f}$$

the element function $\mathbf{G}_{x_f}$ obtained by the analytic continuation of $\mathbf{G}_{x_i}$ from x_i to x_f through γ. If γ_1 is an arc going from x_i to x_m and γ_2 is an arc going from x_m to x_f we represent the arc γ going from x_i to x_f, through γ_1 and γ_2, as $\gamma = \gamma_1\gamma_2$. Notice that if $\gamma = \gamma_1\gamma_2$ and $\mathbf{G}_{x_f} = (\mathbf{G}_{x_i}, \gamma)$, then $\mathbf{G}_{x_f} = ((\mathbf{G}_{x_i}, \gamma_1), \gamma_2)$. The composition of arcs define a product operation. For closed arcs we define γ^n as the product $\gamma\gamma \ldots \gamma$ with n factors. Let us define some special arcs. Let x_0 and x_1 be two points belonging to the interval (0, 1) such that the unique singular point of Eq. (20) in the region $|x| < x_0$ is $x = 0$ and in the region $|x - 1| < 1 - x_1$ is $x = 1$. We denote by γ_1 the counter clockwise positively oriented circle centered at $x = 0$ that begins and ends at x_0. We denote by γ_2 the straight line going from x_0 to x_1 and by γ_3 the counter clockwise positively oriented circle centered at $x = 1$ that begins and ends at x_1 (see Fig. 3). If $\mathbf{G}$ is a solution of Eq. (20) then any of its elements $\mathbf{G}_{x_0}$ can be written as a linear combination of G_1, G_2, the same being true for $\mathbf{G}_{x_1}$ with respect to G_3, G_4 (G_i, $i = 1, \ldots, 4$, were defined in (22)). This implies that the analytic continuation of $\mathbf{G}_{x_0}$ is determined by the analytic continuation of G_1, G_2. For instance, if

$$(G_1, \gamma_2) \sim \alpha_{11} G_3 + \alpha_{21} G_4 \quad \text{and} \quad (G_2, \gamma_2) \sim \alpha_{12} G_3 + \alpha_{22} G_4\,,$$

then $\mathbf{G}_{x_0} \sim aG_1 + bG_2$ implies $(\mathbf{G}_{x_0}, \gamma_2) \sim (a\alpha_{11} + b\alpha_{12})G_3 + (a\alpha_{21} + b\alpha_{22})G_4$. Representing the basis (G_1, G_2) at x_0 as

$$G_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad G_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

and the basis (G_3, G_4) at x_1 as

$$G_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad G_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we can write

$$\mathbf{G}_{x_0} \sim \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad (\mathbf{G}_{x_0}, \gamma_2) \sim M_{\gamma 2} \begin{pmatrix} a \\ b \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

In a similar way we define the matrices $M_{\gamma 1}$ and $M_{\gamma 3}$ related to the arcs γ_1 and γ_3, respectively. Due to this correspondence between arcs and matrices the analytic continuation operation through products of arcs is reduced to a matrix multiplication. For instance if $\mathbf{G}_{x_0} \sim aG_1 + bG_2$, then

$$(\mathbf{G}_{x_0}, \gamma_2\gamma_3) \sim M_{\gamma 3} M_{\gamma 2} \begin{pmatrix} a \\ b \end{pmatrix}, \quad (\mathbf{G}_{x_0}, \gamma_2\gamma_3^2) \sim M_{\gamma 3}^2 M_{\gamma 2} \begin{pmatrix} a \\ b \end{pmatrix},$$

and so on. When γ is a closed arc then M_γ is called the monodromy matrix associated to γ.

Theorem 8. *Let* $\mathbf{G}$ *be a nontrivial global analytic solution of Eq.* (20) *and* m, n *be any pair of integers with* $m \neq 0$. *Then* $B = 0$ *in Theorem* 4 *if, and only if,*

$$\mathbf{G}_{x_0} \sim (\mathbf{G}_{x_0}, \gamma^2),$$

where $\gamma = \gamma_2 \gamma_3^{2n+1} \gamma_2^{-1} \gamma_1^m$ *(or, equivalently, if, and only if, the monodromy matrix* M_γ *satisfies* $M_\gamma^2 =$ *identity) (see Fig.* 3).

Proof. Let us begin calculating the matrices $M_{\gamma 1}, M_{\gamma 2}$ and $M_{\gamma 3}$. From Eqs. (26) and (27) we have $(G_2, \gamma_2) \sim \alpha G_3 - \beta G_4$. Since $G_2(x) = \overline{G_1(x)}$, $G_3(x) = \overline{G_3(x)}$, $G_4(x) = -\overline{G_4(x)}$, for $x \in (0, 1)$ we get $(G_1, \gamma_2) \sim \bar{\alpha} G_3 + \bar{\beta} G_4$ which implies

$$M_{\gamma 2} = \begin{pmatrix} \bar{\alpha} & \alpha \\ \bar{\beta} & -\beta \end{pmatrix}.$$

Using that G_3 is analytic and G_4 has a square root singularity inside γ_3 we obtain $(G_3, \gamma_3) \sim G_3$ and $(G_4, \gamma_3) \sim -G_4$ which implies

$$M_{\gamma 3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

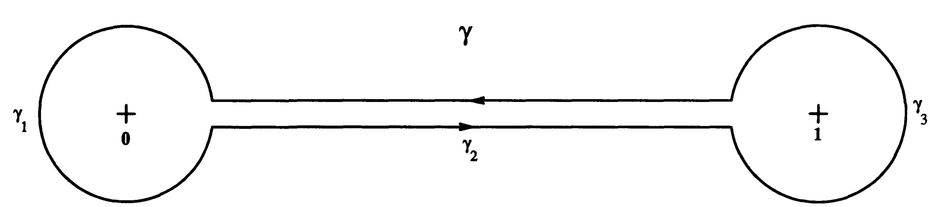

Fig. 3. The closed arc γ defined in Theorem 8 in the case $m = 1$ and $n = 0$

Using that $G_1 = x^{i\eta} g_1(x)$ and $G_2 = x^{-i\eta} g_2(x)$ with g_1, g_2 analytic inside γ_1 we get $(G_1, \gamma_1) \sim e^{-2\pi\eta} G_1$ and $(G_2, \gamma_1) \sim e^{2\pi\eta} G_2$ which implies

$$M_{\gamma 1} = \begin{pmatrix} e^{-2\pi\eta} & 0 \\ 0 & e^{2\pi\eta} \end{pmatrix}.$$

The form of these matrices imply: $M_{\gamma 3}^{2n+1} = M_{\gamma 3}$,

$$M_{\gamma 2}^{-1} = K^{-1} \begin{pmatrix} \beta & \alpha \\ \bar{\beta} & -\bar{\alpha} \end{pmatrix},$$

where $K \stackrel{\text{def}}{=} (\bar{\alpha}\beta + \alpha\bar{\beta}) = -\det(M_{\gamma 2})$, and

$$M_{\gamma 2}^{-1} M_{\gamma 3} M_{\gamma 2} = K^{-1} \begin{pmatrix} \beta\bar{\alpha} - \bar{\beta}\alpha & 2\alpha\beta \\ 2\bar{\alpha}\bar{\beta} & -\beta\bar{\alpha} + \bar{\beta}\alpha \end{pmatrix}.$$

From Eq. (27) and the remark below Eq. (26) we have that $(G_2, \gamma_2\gamma_3\gamma_2^{-1}) \sim AG_1 + BG_2$, so we can write $A = 2\beta\alpha/K$, $B = (\bar{\beta}\alpha - \bar{\alpha}\beta)/K$, and

$$M_{\gamma 2}^{-1} M_{\gamma 3} M_{\gamma 2} = M_{\gamma 2}^{-1} M_{\gamma 3}^{2n+1} M_{\gamma 2} = \begin{pmatrix} \bar{B} & A \\ \bar{A} & B \end{pmatrix}.$$

Therefore

$$M_\gamma = \begin{pmatrix} \bar{B} e^{-2\pi\eta m} & A e^{-2\pi\eta m} \\ \bar{A} e^{2\pi\eta m} & B e^{2\pi\eta m} \end{pmatrix}$$

and

$$M_\gamma^2 = \begin{pmatrix} \bar{B}^2 e^{-4\pi\eta m} + |A|^2 & A(\bar{B} e^{-4\pi\eta m} + B) \\ \bar{A}(\bar{B} + B e^{4\pi\eta m}) & B^2 e^{4\pi\eta m} + |A|^2 \end{pmatrix}.$$

Finally, using that $|A|^2 - |B|^2 = 1$ (Eq. (15)) it is easy to see that M_γ^2 is the identity if, and only if, $B = 0$. □

We end this section applying this theorem to the system considered in Theorem 5. Using the functions P and Q given in (16) we write Eq. (20) as

$$\left(v^2 x^2 - \frac{2\alpha}{n+1} x^{n+1} \right) y'' + (v^2 x - \alpha x^n) y' + (\omega^2 + \beta x^{n-1}) y = 0 . \qquad (29)$$

This equation has singularities at: $x = 0$,

$$x = C \exp\left\{ i \frac{2\pi k}{n-1} \right\}, \quad C \stackrel{\text{def}}{=} \left(\frac{(n+1)v^2}{2\alpha} \right)^{1/(n-1)} \quad k = 1, 2, \ldots, n-1,$$

and possibly at $x = \infty$. The symmetry of these singularities suggests the following change of independent variable

$$z = \left(\frac{x}{C} \right)^{n-1}, \qquad (30)$$

so that Eq. (29) becomes

$$z^2(1-z)y'' + \frac{z}{2}(2-3z)y' + (\Omega^2 + \sigma z)y = 0 \,, \tag{31}$$

where

$$\Omega^2 = \frac{\omega^2}{(n-1)^2 v^2}, \quad \sigma = \frac{\beta(n+1)}{2\alpha(n-1)^2} \,.$$

This is the hypergeometric equation and to prove Theorem 5 we implicitly used some well-known properties of its solutions. Now, we are going to prove Theorem 5 in a different way.

Consider the closed arc $\gamma^2 = (\gamma_2\gamma_3\gamma_2^{-1}\gamma_1)^2$ in the x-plane where $\gamma_1, \gamma_2, \gamma_3$ are defined as in Theorem 8 with the singularity at $x = 1$ being replaced by another one at $x = C$. We want to find the conditions on the parameters of Eq. (29) such that some solution $\mathbf{G}$ satisfies

$$\mathbf{G}_{x_0} \sim (\mathbf{G}_{x_0}, \gamma^2) \,. \tag{32}$$

The image of the arc γ^2 in the z-plane is homotopic to the arc $\gamma_{**}^2 \stackrel{\text{def}}{=} (\gamma_2\gamma_3\gamma_2^{-1}\gamma_1^{n-1})^2$ where again $\gamma_1, \gamma_2, \gamma_3$ are defined as in Theorem 8. Denoting by $\mathbf{F}$ the solution of Eq. (31), related to $\mathbf{G}$ through $z = (x/C)^{n-1}$, we obtain that condition (32) is equivalent to

$$\mathbf{F}_{z_0} \sim (\mathbf{F}_{z_0}, \gamma_{**}^2) \,, \tag{33}$$

where $z_0 = (x_0/C)^{n-1}$. Since Eq. (31) is of the same kind of Eq. (20), with

$$P(z) = z^2(1-z), \quad \omega^2 + Q(z) = \Omega^2 + \sigma z \,,$$

we conclude, using Theorem 8, that relation (33) is true if, and only if,

$$\mathbf{F}_{z_0} \sim (\mathbf{F}_{z_0}, \gamma_*^2) \,, \tag{34}$$

where $\gamma_* = \gamma_2\gamma_3\gamma_2^{-1}\gamma_1$ (if the condition of Theorem 8 is true for some pair m, n then it is true for any pair). Now, we use that Eq. (31) has only three singularities, at $z = 0$, $z = 1$ and $z = \infty$, to deform the arc γ_*^2 to an homotopic one γ_∞^2 that winds twice the singular point at infinity. Therefore we reduced the problem to the analysis of the singularity at infinity. In order to do this we make the change of independent variables

$$z = \frac{1}{u}$$

in Eq. (31) and obtain

$$y'' + \frac{1}{2}\left(\frac{1}{u} + \frac{1}{(u-1)}\right)y' + \left(-\frac{\sigma}{u^2} - \frac{\Omega^2+\sigma}{u} + \frac{\Omega^2+\sigma}{u-1}\right)y = 0 \,.$$

The indicial equation (21) at $u = 0$ becomes

$$s(s-1) + \frac{1}{2}s - \sigma = 0 \,,$$

which implies

$$s = \frac{1 \pm \sqrt{1 + 16\sigma}}{4} = \frac{1}{4}\left(1 \pm \sqrt{1 + \frac{\beta 8(n+1)}{\alpha(n-1)^2}}\right).$$

Relation (34) will be true if, and only if, both values of s are multiples of 1/2, that is

$$(2l - 1)^2 = 1 + \frac{\beta 8(n+1)}{\alpha(n-1)^2}, \quad l \in \mathbb{Z}$$

or

$$\frac{\beta}{\alpha} = \frac{(n-1)^2}{2(n+1)} l(l-1), \quad l \in \mathbb{N}.$$

This condition is the same one that we got in Theorem 5.

5. Conclusion

The results presented in this article represent a first approach to the question of integrability of two degrees of freedom Hamiltonian systems using the theorem of Lerman (1991), Mielke et al. (1992), and scattering ideas. Theorems 4, that plays a central role in this work, can be extended to some more general systems than (1). If we relax the definition of the "scattering equation", keeping its essential features, we can go further with such generalization. How far it is interesting to go depends on our ability in extracting information of the new "scattering problem."

In Sect. 3 we presented Theorem 6 as an illustration of an application of scattering results to the integrability question of systems (1). We believe that many other interesting results can be obtained in this way. For instance, we may use the "quasi-classical" approximation to the scattering problem (Landau and Lifshitz (1965), Fedoryuk (1965)) or the related theory of "adiabatic invariants" (Arnold et al. (1988)).

Finally, I would like to mention some very interesting comments pointed out by one of the referees. One can consider the integrability question in view of "complex" Hamiltonian systems. Formally, complex analytic Hamiltonian systems are defined in the same way as the real analytic ones, with the difference that we consider complex phase spaces ($(p, q) \in \mathbb{C}^{2n}$) and orbits parametrized by a complex time. The definition of integrability is also formally the same. In particular real analytic Hamiltonian systems can be extended into complex plane. For complex Hamiltonian systems an important necessary condition for integrability was obtained by Ziglin (1983) (see Kozlov (1983) for a brief discussion on Ziglin's result and also Ito (1985)). Ziglin's condition requires that we analyze the first variational equation associated to some periodic orbit of the system. If we think about the homoclinic orbit Γ of the system of Theorem (5) as being a periodic orbit with "infinite" period and consider Eq. (29) (or Eq. (18)) in the complex time domain then the condition of Theorem 8 is similar to Ziglin's condition for those periodic orbits contained in the $p_2 = q_2 = 0$ plane that approximate the homoclinic orbit Γ. In particular, Ito (1987) applied Ziglin's theorem to systems similar to those we considered here and got a necessary condition for complex integrability that formally resembles the condition of Theorem 8. Another interesting remark comes from Yoshida's applications of Ziglin's theorem (see for instance Yoshida (1988)) and further work by Churchchill and Rod (1988) concerning Yoshida's approach. Yoshida was able to verify Ziglin's condition for an equation similar to (29) using the same change of variables (30) we used. Churchill and Rod (1988) having considered Ziglin's theorem and Yoshida's approach from a geometric point of view were able to extend Yoshida's method for more general systems. It would be very interesting to apply Churchill and Rod (1988) ideas in the context of Theorem 8. Concluding, the similarities between our results and Ziglin's theorem provide many interesting questions on the relationship between complex integrability and the presence of transversal Homoclinic orbits in real systems, as well as, by mean of previous results on Ziglin's theorem and its applications, it indicates some ways the results in this paper can be extended.

Acknowledgements The author thanks: J. Koiller and W.M. Oliva for the discussions during the preparation of Grotta Ragazzo et al. (1994), that was the first motivation for the present work, to C P Malta and M.S T. Piza for their suggestions, and to one of the referees who corrected some mistakes in the first version of this work, presented many suggestions and pointed out most of the references on Ziglin's theorem mentioned in the conclusion.

References

Arnold, V.I., Kozlov, V.V., Neishtadt, A.I.: Dynamical Systems III: Mathematical Aspects of Classical and Celestial Mechanics. Encyclopaedia of Math Sciences v. **3**, Berlin: Springer-Verlag, 1988

Churchill R.C., Rod, D.L.: Pathology in dynamical systems III, Analytic Hamiltonians. J. Diff. Eqns. **37**, 351–373 (1980)

Churchill, R.C., Rod, D.L : Geometrical aspects of Ziglin's non-integrability theorem for complex Hamiltonian systems. J. Diff. Eqns. **76**, 91–114 (1988)

Fedoryuk, M.V.: One-dimensional scattering in the quasi-classical approximation. Diff. Eqns. **1**, 483–495, 1201–1210 (1965)

Grotta Ragazzo, C.: Chaos and integrability in a nonlinear wave equation. J Dyn. Diff. Eq. **6**, 227–244, 4 (1994)

Grotta Ragazzo, C., Koiller, J., Oliva, W.M.: Motion of massive vortices in two dimensions. To appear in J. Nonlinear Science (1994)

Holmes, P.: Proof of nonintegrability for the Hénon–Heiles Hamiltonian near an exceptional integrable case. Physica **5D**, 335–347 (1982)

Hénon, M., Heiles, C.: The applicability of the third integral of motion: Some numerical experiments Astron. J **69**, 73–79 (1964)

Hille, E.: Ordinary Differential Equations in the Complex Domain New York: Wiley-Interscience, 1976

Ito, H.: Non-integrability of Hénon–Heiles system and a theorem of Ziglin. Kodai Math. J. **8**, 120–138 (1985)

Ito, H.: A criterion for non-integrability of Hamiltonian systems with nonhomogeneous potentials. J. Appl. Math. Phys. (ZAMP) **38**, 459–476 (1987)

Kay, I., Moses, H.E.: Reflectionless transmission through dielectrics and scattering potentials. J. Appl. Phys. **27**, 1503–1508 (1956)

Kozlov, V.V.: Integrability and non-integrability in Hamiltonian mechanics. Russ. Math. Surv. **38**, 1–76 (1983)

Landau, L.D., Lifshitz, E.M.: Quantum mechanics, nonrelativistic theory. Oxford: Pergamon Press, 1965

Lerman, L.M.: Hamiltonian systems with loops of a separatrix of a saddle-center. Sel Math Sov **10**, 297–306 (1991); Originally published in "Metody kachestvennoĭ teorii differentsial'nykh uravneniĭ". Gorkiĭ State University, 89–103 (1987)

Mielke, A., Holmes, P., O'Reilly, O : Cascades of homoclinic orbits to, and chaos near, a Hamiltonian saddle center. J. Dyn. Diff. Eqns. **4**, 95–126 (1992)
Moser, J : On the generalization of a theorem of Liapunoff. Comm. Pure Appl. Math. **11**, 257–271 (1958)
Moser, J.: Stable and Random Motions in Dynamical Systems. Princeton: Princeton University Press, 1973
Morse, P., Feshbach, H.: Methods of Theoretical Physics **v.1.** New York: McGraw-Hill, 1953
Rüssmann, H : Uber das verhalten analytischer Hamiltonscher differentialgleichungen in der nahe einer gleichgewichtslosung. Math. Ann. **154**, 285–300 (1964)
Yoshida, H.: Non-integrability of the truncated Toda lattice Hamiltonian at any order. Commun. Math. Phys. **116**, 529–538 (1988)
Ziglin, S.L.: Branching of solutions and the nonexistence of first integrals in Hamiltonian mechanics. Funct. Anal. Appl. **16**, 181–189 (1989); **17**, 6–17 (1983)

Communicated by M. Jimbo