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More on Quantum Groups from the Quantization Point of View

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Abstract: Star products on the classical double group of a simple Lie group and on corresponding symplectic groupoids are given so that the quantum double and the "quantized tangent bundle" are obtained in the deformation description. "Complex" quantum groups and bicovariant quantum Lie algebras are discussed from this point of view. Further we discuss the quantization of the Poisson structure on the symmetric algebra $S(g)$ leading to the quantized enveloping algebra $U_h(g)$ as an example of biquantization in the sense of Turaev. Description of $U_h(g)$ in terms of the generators of the bicovariant differential calculus on $F(G_q)$ is very convenient for this purpose. Finaly we interpret in the deformation framework some well known properties of compact quantum groups as simple consequences of corresponding properties of classical compact Lie groups. An analogue of the classical Kirillov's universal character formula is given for the unitary irreducible representation in the compact case.

1. Introduction

Let g be a complex simple finite-dimensional Lie algebra. According to Drinfeld's theorem [11] (Proposition 3.16) there exists a special element $\mathscr{F} \in (U(g) \otimes U(g))[[h]]$ such that the linear space $U(g)[[h]]$ can be equipped with the structure of the quasitriangular Hopf algebra, with the standard multiplication and counit induced from $U(g)$ and with the twisted comultiplication Δ_h and antipode S_h given by formulas

$$\Delta_h = \mathscr{F}^{-1} \Delta \mathscr{F}, \quad S_h = u(S)u^{-1}, \tag{1}$$

with

$$u = \sum \mathscr{F}^{-(1)}(S\mathscr{F}^{-(2)}), \quad u^{-1} = c^{-1} \sum (S\mathscr{F}^{(1)})\mathscr{F}^{(2)}. \tag{2}$$

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Δ and S in the above formulae are the standard comultiplication and antipode in $U(g)$, c is a central element in $U(g)[[h]]$. The formulas for the antipode together with the shorthanded notation $\mathscr{F} = \sum \mathscr{F}^{(1)} \otimes \mathscr{F}^{(2)}$ and $\mathscr{F}^{-1} = \sum \mathscr{F}^{-(1)} \otimes \mathscr{F}^{-(2)}$ are taken from [12].

The universal $\mathscr{R}$-matrix making the above Hopf algebra a quasitriangular one is expressed with help of the symmetric g-invariant element $t \in g \otimes g$ (defined via the inverse of the Killing matrix) as

$$\mathscr{R} = \mathscr{F}_{21}^{-1} \exp(ht) \mathscr{F} . \tag{3}$$

Moreover Drinfeld's theorem claims that this quasitriangular Hopf algebra is isomorphic to the famous quantum group corresponding to the Lie algebra g as it was introduced by Drinfeld and Jimbo [1, 17]. Let us denote by φ this isomorphism $\varphi \colon U_h(g) \to U(g)[[h]]$.

Comparing (1) with the explicit formula for the antipode from [3] and remembering that the φ is identity on the Cartan subalgebra we realize that the u should be proportional up to some central element to the $\gamma = \exp(-h\rho)$, where ρ is the element of the Cartan subalgebra equal to the half-sum of the Cartan generators corresponding to the positive roots.

In the compact case $(h \in R)\varphi$ can be taken to be a $*$-homomorphism

$$\varphi(x^*) = (\varphi(x))^* , \tag{4}$$

where the $*$ on the left is the usual one in $U_h(g)$ and the $*$ on the right is the usual one in the $U(g)[[h]]$.

Unfortunately there is no explicit formula for the F.

In this paper we will also be interested in the dual (in the sense of the Hopf algebras) situation, which is nicely described in [21]. Roughly speaking in this situation we have on the vector space $C^\infty(G)[[h]]$ with the standard comultiplication and counit of the function Hopf algebra $F(G)$ on the corresponding Lie group G, but the deformed multiplication $\star_h$ (star product) and antipode S_h. The corresponding formulas expressing those with help of undeformed ones m and S are:

$$a \star_h b = m(\mathscr{F} * (a \otimes b) * \mathscr{F}^{-1}), \quad S_h a = S(u^{-1} * a * u) , \tag{5}$$

for any $a, b \in F(G)$. Here the $*$ have been used to denote the actions of $U(g)$ on $F(G)$ via left and right invariant differential operators.

In the compact case we have also

$$a^* = \overline{\gamma * a * \gamma^{-1}} = \gamma^{-1} * \bar{a} * \gamma . \tag{6}$$

The rising Hopf algebra should be of course isomorphic to the Hopf algebra of quantized functions $F(G_q)$ [3] on the group G under the φ^* dual to the φ. The reader can find an explicit example of $SL(2)$ in [21]. We will in the following often not distinguish between isomorphic objects under these isomorphisms. We hope it will be clear from the context what we have in mind.

In the above situation the Hopf algebras $F(G_q)$ are deformation quantizations $(\frac{1}{h}[\cdot,\cdot] \to i\{\cdot,\cdot\}$ if $h \to 0)$ of the corresponding Poisson–Lie Groups [1], with the Poisson bracket

$$i\{a, b\} = -m(r * (a \otimes b) - (a \otimes b) * r) , \tag{7}$$

with $r \in g \otimes g$ being the classical r-matrix.

In this paper we will use this idea of Drinfeld to introduce the star product on the classical double group of a simple Lie group and on the corresponding symplectic groupoids [13, 30, 14] to obtain the quantum double and the "quantized tangent bundle" in the deformation framework described above. We will relate our results with some recent papers on "complex" quantum groups [2, 7, 10, 15] and bicovariant quantum Lie algebras [18].

Further we will discuss the quantization of the Poisson structure on $S(g)$ leading to the $U_h(g)$ as an example of biquantization in the sense of Turaev [26, 23] which is dual to this of [23]. It turns out the description of $U_h(g)$ in terms of the generators of the bicovariant differential calculus on $F(G_q)$ [6, 5, 19] is very convenient for this purpose.

Finally we hope to show that the deformation reinterpretation of the group algebra of the compact quantum group of Podles and Woronowicz [2] lead us to some kind of universal formula for character generalizing the classical situation [36]. Here it means that a very simple change in the classical formula allows us to express the trace of a functional $b \to h_c(ab)$ (h_c is the Haar measure on $F(G_q)$, $a, b \in F(G_q)$) in the unitary irreducible representation of $U_h(g) \sim U(g)[[h]]$ in terms of an integral on the classical coadjoint orbit with the standard Kirillov–Souriau–Kostant symplectic structure (which is isomorphic as a symplectic manifold to the dressing orbit with the symplectic structure induced from the Poisson–Lie structure on the dual group G_r [20]). It supports the recent idea of Xu and Weinstein [25] to construct an symplectic counterpart of the Reshetikhin–Turaev construction of link invariants [35, 34].

Through the paper we assume $q = e^h$ with generic values of q.

2. Double

Here we will be interested in the local double Lie group corresponding to the connected complex simple Lie group G. As it follows from [13] this can be described as $D = G \times G$. The starting group G and its connected dual group G_r [13] are identified with the subgroups of D via its Iwasawa decomposition [24], which leads to the identification $D = G \times G_r$, where now G is assumed as the diagonal subgroup $\{(x, x), x \in G\}$ and $G_r = AU^+$, where $A = \{(x, x^{-1}), x \in H\}$ and $U^+ = \{(x_+, x_-), x_+ \in N^+, x_- \in N^-\}$, with H being the Cartan subgroup and $N^\pm$ the connected nilpotent subgroups corresponding to the positive and negative roots respectively. Let us remember that the double D as well as the groups G and G_r are Poisson–Lie groups and that the above factorization based on the Iwasawa can for more detailed exposition consult [13, 14, 30]). Here we will only write down the corresponding Poisson brackets using the classical r-matrix.

The Poisson structure on G have been already given by (7). We will take the classical universal r-matrix $\in g \otimes g$ of the form

$$r = -(P_- - P_+ + t)\,, \tag{8}$$

where $P_\pm$ are projectors on the nilpotent subalgebras correspoinding to the positive and negative roots respectively. Here the correspondence between the elements of $g \otimes g$ and maps $g \to g$ via the dualization of the first factor in $g \otimes g$ with help of the Killing form has been used.

We will not distinguish between the universal r-matrix and its representative in the fundamental representation. The Poisson bracket (7) then becomes on the matrix elements of the fundamental representation τ

$$i\{\tau \otimes, \tau\} = [r, \tau \otimes \tau] . \tag{9}$$

The Poisson bracket in $G_r = \{(g_+, g_-)\}$ then can be given as [16] (we take a slightly different convention)

$$\begin{aligned} i\{g_\pm \otimes, g_\pm\} &= [r, g_\pm \otimes g_\pm] , \\ i\{g_- \otimes, g_+\} &= [r, g_- \otimes g_+] . \end{aligned} \tag{10}$$

Because of the factorizability of simple Lie algebras in the sense of [8] the above Poisson Lie group G_r can be as a manifold identified with G taken with a new Poisson bracket. If we denote now as $y = g_+ g_-^{-1}$ the corresponding element of G we have

$$\begin{aligned} i\{y \otimes, y\} = &-r_{21}(y \otimes y) + (y \otimes y)r_{12} - (y \otimes 1)r_{12}(1 \otimes y) \\ &+ (1 \otimes y)r_{21}(y \otimes 1) . \end{aligned} \tag{11}$$

The double D as a Poisson manifold is a direct product of Poisson manifolds G and G_r. The Poisson structure on the D if this is described as a manifold $G \times G$ can be also easily described for $(\tau, \hat{\tau}) \in G \times G$,

$$\begin{aligned} i\{\tau \otimes, \tau\} &= [r, \tau \otimes \tau] , \\ i\{\hat{\tau} \otimes, \hat{\tau}\} &= [r, \hat{\tau} \otimes \hat{\tau}] , \\ i\{\hat{\tau} \otimes, \tau\} &= [r, \hat{\tau} \otimes \tau] . \end{aligned} \tag{12}$$

From these explicit formulas it is seen immediately that the above mentioned Iwasawa decomposition

$$(\tau, \hat{\tau}) = (\check{\tau}, \check{\tau})(g_+, g_-) \tag{13}$$

is a Poisson mapping.

Let us now assume the function algebra $F(D) \sim F(G) \otimes F(G)$ with the Lie–Poisson structure given above. If $\mathscr{F} \in (U(g) \otimes U(g))[[h]]$ is this one introducing on $F(G)$ the structure of the quantum group $F(G_q)$ with $\mathscr{R}$ the corresponding $\mathscr{R}$-matrix, then the $\mathscr{F}^D \in (U(g \otimes g) \otimes U(g \otimes g))[[h]]$ given by

$$\mathscr{F}^D_{1234} = \mathscr{F}_{13}\mathscr{F}_{24}\mathscr{R}_{23} \tag{14}$$

plays the same role for $F(D)$. The corresponding $\mathscr{R}^D$ is

$$\mathscr{R}^D_{1234} = (\mathscr{F}^D_{3412})^{-1} \exp(h(t_{13} - t_{24}))\mathscr{F}^D_{1234} = \mathscr{R}^{-1}_{41}\mathscr{R}^{-1}_{42}\mathscr{R}_{13}\mathscr{R}_{23} . \tag{15}$$

Here we have used the fact [13] that in the situation of the double the t from Drinfeld's construction should be taken as $(t_{13} - t_{24})$.

The corresponding u in the formula for the antipode can be taken as

$$u^D_{12} = \mathscr{R}_{21}(\gamma \otimes \gamma) . \tag{16}$$

In the above formulas we assume of course that the multiplication, comultiplication, antipode and counit of the classical double are the standard one of the tensor product.

For the proof of these facts it is enough to note that the above formulae are nothing else but a direct application of Theorem 2.9 of [8] to the $\varphi(U_h(g))$.

The deformed multiplication $\star_D$ can be written as

$$
\begin{aligned}
(a \otimes 1) \star_D (b \otimes 1) &= (a \star_h b) \otimes 1 , \\
(1 \otimes a) \star_D (1 \otimes b) &= 1 \otimes (a \star_h b) , \\
(a \otimes 1) \star_D (1 \otimes b) &= a \otimes b , \\
(1 \otimes a) \star_d (b \otimes 1) &= \sigma(\mathscr{R} * (a \otimes b) * (\mathscr{R}^{-1}) .
\end{aligned} \tag{17}
$$

Here σ denotes transposition.

The * is given simply by

$$(a \otimes b)^* = b^* \otimes a^* ,$$

where * on the components on the right is this one given by (6).

Comparing (17) with the Poisson structure on the classical double given by (12) we see that this product is really a deformation product along this Poisson bracket. We denote the obtained quantum double as D_q.

It is also immediately seen that the mapping $p_G = \star_h \colon F(D_q) \to F(G_q)$ is a Hopf algebra homomorphism as it should be. Similarly there is a Hopf algebra homomorphism $p_U \colon F(D_q) \to U_h(g)^{\mathrm{op}} \sim U(g)[[h]]^{\mathrm{op}} \colon (a \otimes b) \to \mathscr{R}_{21}^{-1} \cdot, b)\mathscr{R}(\cdot, a)$, where "op" means the opposite multiplication. So factorizing out the kernels of these surjective homomorphisms we obtain the $F(G_q)$ and $U_h(g)^{\mathrm{op}}$ as Hopf-subalgebras.

It is instructive to write out the commutation relations resulting from (17) for the representations of the double $\mathscr{T} = (\tau \otimes 1)$ and $\hat{\mathscr{T}} = (1 \otimes \tau)$, where τ is the fundamental representation of G. We have of course the familiar relations of [7, 10, 15],

$$
\begin{aligned}
R\mathscr{T}_1\mathscr{T}_2 &= \mathscr{T}_2\mathscr{T}_1 R , \\
R\hat{\mathscr{T}}_1\mathscr{T}_2 &= \mathscr{T}_2\hat{\mathscr{T}}_1 R , \\
R\hat{\mathscr{T}}_1\hat{\mathscr{T}}_2 &= \hat{\mathscr{T}}_2\hat{\mathscr{T}}_1 R .
\end{aligned} \tag{18}
$$

Here R is the universal $\mathscr{R}$-matrix for g in the fundamental representation. We have also omitted the $\star_D$ as a sign for the multiplication.

To define the quantum double with the help of generators $\mathscr{T}$ and $\hat{\mathscr{T}}$ as it was done in the above mentioned papers we should suppose that the matrices $\mathscr{T}$ and $\hat{\mathscr{T}}$ the quantum determinant conditions in the A_n case and quantum orthogonality conditions in the remaining cases. They as well as the antipode can be obtained also from the deformation formulas. The comultiplication and counit on $\mathscr{T}$ and $\hat{\mathscr{T}}$ are clearly the standard one [3].

Let us now briefly discuss the quantum Iwasawa decomposition. Let T denote the matrix of generators of $F(G_q)$ (the fundamental representation). So we have the famous relations of [3]

$$RT_1T_2 = T_2T_1R . \tag{19}$$

Let $\Lambda^{\pm}$ be the same as $L^{\pm}$ of [3] but now taken with the opposite multiplication

$$R\Lambda_1^{\pm}\Lambda_2^{\pm} = \Lambda_2^{\pm}\Lambda_1^{\pm}R, \quad R_1^{-}\Lambda_2^{+} = \Lambda_2^{+}\Lambda_1^{-}R . \tag{20}$$

With the help of generators T, $\Lambda^{\pm}$ (entries of T are supposed to commute with those of $\Lambda^{\pm}$) there can be introduced according to [2, 8, 9] another description of the quantum double.

A slight generalization of Proposition 4.5 of [9] shows that this Hopf algebra can be assumed as a completion of the Hopf algebra generated by $\mathscr{T}$ and $\hat{\mathscr{T}}$. The explicit formulas

$$\mathscr{T} = T\Lambda^{+}, \quad \hat{\mathscr{T}} = T\Lambda^{-} \tag{21}$$

describe an injective Hopf algebra morphism $(p_G \otimes p_U)\Delta$ of the algebra generated by $\mathscr{T}, \hat{\mathscr{T}}$ into the algebra generated by T and $\Lambda^{\pm}$. So they give the quantum generalization of the Iwasawa decomposition (13). There is also the opposite Iwasawa decomposition

$$\mathscr{T} = \Lambda^{+} T, \quad \hat{\mathscr{T}} = \Lambda^{-} T . \tag{22}$$

Moreover the Poisson–Lie structure on G_r given by Eqs. (10) can be understood as the Poisson limit from the Hopf algebra structure of the Hopf algebra $U_h(g)^{\mathrm{op}}$ generated by $\Lambda^{\pm}$. For later convenience we will discuss this now for the Poisson bracket (10) taken with the minus sign. Taking on G_r the coordinates y we can identify G and G_r as manifolds (locally). Now using results of [27] we can introduce on the function algebra $F(G)$ a new structure of an associative noncommutative algebra expressed in terms of the $\star_h$-product (5) as

$$a \star_r b = a_2 \star_h b_3 \mathscr{R}(a_1, b_2) \tilde{\mathscr{R}}(a_3, b_1) . \tag{23}$$

Here we used the notation $\Delta a = a_1 \otimes a_2$, etc. $\tilde{\mathscr{R}}$ denotes an element of $(U(g) \otimes U(g))[[h]]$ inverse to the $\mathscr{R}$ but now in the algebra taken with the opposite multiplication in the first factor. It follows from [27] that such an algebra structure is isomorphic to the usual one of $U_h(g)$. Taking the classical limit we arrive at the Poisson structure (10) with the minus sign.

3. Quantization of $\pi_{\pm}$

Let us remember [13] that having a general Lie–Poisson group G with the Poisson structure given by (7) we can introduce on the manifold G two new Poisson structures denoted by $\pi_{\pm}$. They are given by

$$\pi_{\pm}(da, db) = i\{a, b\}_{\pm} = \pm\, m(r * (a \otimes b) + (a \otimes b) * r\} . \tag{24}$$

Here r should be taken as antisymmetric solution of the modified classical Yang–Baxter equation. Of course G equipped with these Poisson brackets is no more a Lie–Poisson group. Nevertheless the natural left (for the $+$ sign) and the right (for the $-$ sign) group actions of G (G is assumed to be equipped with the Lie–Poisson structure (7)) are Poisson mappings.

In the case of $G = D$ these Poisson structures are nondegenerated [13] and the manifold D with the above Poisson structures contains all ingredients to be a symplectic groupoid over G (in the $+$ sign case) or over G_r (in the $-$ sign case) [30]. We should warn the reader that we use different notation than what is used in this reference.

Now making the necessary (but straightforward changes) in the proofs of corresponding Propositions of [21] concerning the quantization of the

Poisson–Lie structure (7) leading to the (5) we can easily state the following:

1. Formulas

$$a \star_+ b = \mathscr{F}_{21} * (a \otimes b) * \mathscr{F}^{-1} \tag{25}$$

and

$$a \star_- b = \mathscr{F} * (a \otimes b) * \mathscr{F}_{21}^{-1} \tag{26}$$

define associative products on $F(G)$, which are quantizations of (24).

2. If we denote as $F(G, \star_h)$ and $F(G, \star_\pm)$ the function algebras equipped with corresponding products, $F(G, \star_h)$ having the standard comultiplication, then the following $F(G, \star_h)$-coactions

$$\delta_+ : F(G, \star_+) \to F(G, \star_h) \otimes F(G, \star_+) : (\delta_+ a)(g, h) = a(gh) , \tag{27}$$

$$\delta_- : F(G, \star_-) \to F(G, \star_-) \otimes F(G, \star_h) : (\delta_- a)(g, h) = a(gh) \tag{28}$$

are algebra morphisms.

We can now apply all the above to the double D, with $\mathscr{F}^D$ given by (14). The resulting algebra structure on $\mathscr{T} = \tau \otimes 1$ and $\hat{\mathscr{T}} = 1 \otimes \tau$ (τ is again the fundamental representation of G) gives e.g. in the "+" case

$$\begin{aligned} R\mathscr{T}_1\mathscr{T}_2 &= \mathscr{T}_2\mathscr{T}_1R_{21} , \\ R\hat{\mathscr{T}}_1\mathscr{T}_2 &= \mathscr{T}_2\hat{\mathscr{T}}_1R^{-1} , \\ R\hat{\mathscr{T}}_1\hat{\mathscr{T}}_2 &= \hat{\mathscr{T}}_2\hat{\mathscr{T}}_1R_{21} . \end{aligned} \tag{29}$$

Here we have also omitted $\star_+$ as a sign of the multiplication.

The coaction δ_+ is given as

$$\delta_+(\mathscr{T}) = \mathscr{T} \otimes .\mathscr{T}, \quad \delta_+(\hat{\mathscr{T}}) = \hat{\mathscr{T}} \otimes .\hat{\mathscr{T}} , \tag{30}$$

with a proper understanding of the algebra structure of the factors in the tensor products (we did not graphically distinguish between doubles taken with different algebra structures).

The "−" case can be treated similarly.

In the same sense as in the previous section we have the Iwasawa decomposition

$$\mathscr{T} = TL^+, \quad \hat{\mathscr{T}} = TL^- , \tag{31}$$

which leads us to the following well known commutation relations of [19]:

$$\begin{aligned} RT_1T_2 &= T_2T_1R , \\ R_{21}L_1^\pm L_2^\pm &= L_2^\pm L_1^\pm R_{21} , \\ R_{21}L_1^+ L_2^- &= L_2^- L_1^+ R_{21} , \\ L_1^+ T_2 = T_2R_{21}L_1^+, &\quad L_1^- T_2 = T_2R_{12}^{-1}L_1^- . \end{aligned} \tag{32}$$

As shown also in [19] the relations in the last line are equivalent to the standard pairing between $F(G_q)$ and $U_h(g)$, which is in this way implicitly contained in the quantization of π_+. The action of $U_h(g)$ on $F(G_q)$ resulting from these relations is easily recognized as the left action $X * a = X(a_2)a_1$ of $x \in U_h(g)$ on the $a \in F(G_q)$. Similar algebra was also introduced and investigated in [33].

We have of course also the opposite Iwasawa decomposition

$$\mathcal{T} = \tilde{L}^+ \tilde{T}, \quad \hat{\mathcal{T}} = \tilde{L}^- \tilde{T} . \tag{33}$$

The corresponding commutation relations are

$$\begin{aligned} R_{21} \tilde{T}_1 \tilde{T}_2 &= \tilde{T}_2 \tilde{T}_1 R_{21} , \\ R \tilde{L}_1^\pm \tilde{L}_2^\pm &= \tilde{L}_2^\pm \tilde{L}_1^\pm R , \\ R \tilde{L}_1^- \tilde{L}_2^+ &= \tilde{L}_2^+ \tilde{L}_1^- R , \\ \tilde{T}_1 \tilde{L}_2^+ = \tilde{L}_2^+ R_{12}^{-1} \tilde{T}_1, &\quad \tilde{T}_1 \tilde{L}_2^- = \tilde{L}_2^- R_{21} \tilde{T}_1 . \end{aligned} \tag{34}$$

Now computing the commutation relations between T and $X = TL^+(L^-)^{-1}T^{-1}$ we get

$$T_2 X_1 = R_{21}^{-1} X_1 R_{21}^{-1} T_2 . \tag{35}$$

This is as easy to see is the same as the commutation relation between T and $S(L^+)L^-$ if they are assumed in the algebra which is obtained from $F(G_q)$ and $U_h(g)$ as a semidirect product with the help of the right action of $U_h(g)$ on $F(G_q)$ $(= a * X = X(a_1)a_2)$ [27]. Comparing two above Iwasawa decompositions gives

$$X = \mathcal{T} \hat{\mathcal{T}}^{-1} = \tilde{L}^+ (\tilde{L}^-)^{-1} ,$$

and we realize that the subalgebra generated by $\tilde{L}^\pm$ can be identified with the algebra of right-invariant maps on $F(G_q)$ (let us remember the known fact that the matrices of generators $\tilde{L}^\pm$ are uniquely obtained from the X via decomposition to the triangular parts [28]).

Let us now assume the left coaction δ of $F(G_q)$ on $F(D, \star_+)$ given by δ_+ followed with the projection $F(D, \star_h) \to F(G_q)$ in the first factor. As a product of two algebra morphisms it is again an algebra morphism. The explicit formula reads

$$\delta(T) = T \otimes . T, \quad \delta(\hat{\mathcal{T}}^{-1} \mathcal{T} = (L^-)^{-1} L^+) = 1 \otimes (L^-)^{-1} L^+ . \tag{36}$$

As it should be δ acts trivially on left-invariant maps.

Computing the coaction on right-invariant maps we have

$$\delta((\tilde{L}^+ (\tilde{L}^-)^{-1}))_{ij} = T_{ik} T_{lj}^{-1} \otimes (\tilde{L}^+ (\tilde{L}^-)^{-1})_{kl} , \tag{37}$$

which is nothing else but the left dressing action of $F(G_q)$ on $U_h(g)^{\mathrm{op}}$ corresponding to the opposite Iwasawa decomposition of the double given by (22) [9].

With slightly different conventions the above coaction δ has been investigated also in [18]. Here we hope it was introduced in a more general context.

We will now briefly discuss some facts generalizing the classical situation [13, 30]. Owing to the above described Iwasawa decompositions of $F(D, \star_+)$ we have the following natural projections (algebra homomorphisms):

$$\begin{aligned} p_1 &: F(D, \star_+) \to F(G_q) , \\ p_2 &: F(D, \star_+) \to U_h(g) , \end{aligned} \tag{38}$$

corresponding to the first Iwasawa decomposition (31) and

$$\tilde{p}_1 : F(D, \star_+) \to F(G_q)^{\mathrm{op}} ,$$
$$\tilde{p}_2 : F(D, \star_+) \to U_h(g)^{\mathrm{op}} \tag{39}$$

corresponding to the opposite one (33).

Further as easily shown by direct calculation the entries of the matrices $\mathscr{T}\hat{\mathscr{T}}^{-1}$ (right-invariant elements) and $\hat{\mathscr{T}}^{-1}\mathscr{T}$ (left-invariant elements) mutually commutes as it should. So the subalgebras of $F(D, \star_+)$ generated by $L^\pm$ and $\tilde{L}^\pm$ can be viewed as a generalization of the notion of the dual pair from the classical case. As in the classical case [31] their only common elements are their Casimirs.

It is well known [13, 14, 30], that the symplectic manifolds $(D, \pi_\pm)$ play an important role in the description of symplectic leaves of the Poisson structures on G and G_r. Their quantization presented in this section should play an analogous role in the representation theory of $F(G_q)$ and $U_h(g)$. E.g. the irreducible finite-dimensional representations of $U_h(g)$ for the generic value of q can be obtained by decomposing the representation of $U_h(g)$ on $F(G_q)$ given by the left action $X * a$, $X \in U_h(g)$, $a \in F(G_q)$ which is contained as it was shown in the algebra structure of the $F(D, \star_+)$.

We hope that also in the quantum case all ingredients to satisfy the formal definition of quantum groupoid [22] are contained in $F(D, \star_\pm)$.

4. Biquantization of $S(g)$

This section is motivated by the appendix of [23], where the dual situation has been described. Here we will use without further explanation the terminology introduced in [26, 23]. The reader is referred to the algebraic parts of these papers. We think that it is not necessary to reproduce here all details, because we hope that the presented example is enough illustrative. Minor differences from [26, 23] are insubstantial.

Let us remember that a simple Lie algebra g can be equipped in a standard way with a structure of Lie bialgebra [1]. It means that in addition to the Lie bracket $[\cdot,\cdot]$ we have also a Lie cobracket $\nu : g \to g \wedge g$ (which is according to [1] equivalent to the Poisson structure on G) and the Lie bracket and Lie cobracket are compatible. In our case the Lie cobracket is given in terms of the classical r-matrix

$$\nu(X) = [r, 1 \otimes X + X \otimes 1], \quad X \in g . \tag{40}$$

The symmetric algebra $S(g)$ is then endowed with the structure of the so-called bi-Poisson bialgebra. Roughly speaking it is equipped in addition to the commutative multiplication and the standard Poisson bracket (given by the extension of the Lie bracket on g via the Leibniz rule) with the

1. comultiplication Δ (coalgebra structure)

$$\Delta(X) = 1 \otimes X + X \otimes 1, \quad X \in g , \tag{41}$$

which is extended to the entire $S(g)$ as an algebra homomorphism.

2. Lie cobracket (co-Poisson structure) given on g by (40) and extended to the entire $S(g)$ with help of the rule

$$\nu(ab) = \nu(a)\Delta(b) + \Delta(a)\nu(b)\,. \tag{42}$$

Let us note that using the classical Yang–Baxter equation we can write the Poisson bracket in coordinates $-X = (\tau \otimes id)(r + r_{21})$ (τ- the fundamental representation, r- the classical universal r-matrix) as

$$i\{X_1, X_2\} = [r + r_{21}, X_2]\,. \tag{43}$$

It is a well known fact [14, 13] that this Poisson structure is the linearization of the Poisson structure on $F(G_r)$.

Further let V be the associative algebra over $C[h]$ obtained in the following way:

It is the tensor algebra $T(C[h] \otimes g)$ over $C[h] \otimes g$ divided by a two-sided ideal generated by elements of the form

$$ab - ba - h[a, b]\,.$$

The generators of V are simply only rescaled generators of g. We have $V/hV = S(g)$ and $V/(h-1)V = U(g)$. In this same way as above we can equip V with a coalgebra and co-Poisson structures. The V as an $C[h]$-algebra is quantization of Poisson algebra $S(g)$ in the usual sense. The corresponding projection $q_h\colon V \to S(g)$ is called a quantization homomorphism. q_h is of course a surjective bialgebra homomorphism and preserves the cobracket. If r is the classical r-matrix (8), then collecting the generators of g with the help of the fundamental representation τ into the matrix $X = (\tau \otimes id)(r + r_{21})$ we can write, thanks to the classical Yang–Baxter equation, the commutation relation in the form

$$X_1 X_2 - X_2 X_1 = h[r + r_{21}, X_2]\,. \tag{44}$$

Let us now assume the $U_{h\cdot\hbar}(g)$. It means that in the definition of the quantized enveloping algebra we make simply the change $h \to h\cdot\hbar$, where $\hbar$ is a new Planck constant. Further let us introduce with the help of the standard R-matrix a new matrix $\bar{R}$,

$$\bar{R}(h\hbar) = (h\hbar)^{-1}(R(h\hbar) - I)\,, \tag{45}$$

so that we have

$$\bar{R} = -r + O(h\hbar)\,. \tag{46}$$

We will collect the generators in the matrix denoted again as X,

$$X = (\hbar)^{-1}(L^+ S(L^-) - I)\,. \tag{47}$$

We have the following commutation relations:

$$\begin{aligned}(X_1 X_2 - X_2 X_1) &+ h\hbar(\bar{R}_{21} X_1 X_2 + X_1 \bar{R}_{12} X_2 - X_2 \bar{R}_{21} X_1 - X_2 X_1 \bar{R}_{12}) \\ &+ (h\hbar)^2(\bar{R}_{21} X_1 \bar{R}_{12} X_2 - X_2 \bar{R}_{21} X_1 \bar{R}_{12}) \\ &= -h^2\hbar(\bar{R}_{21}\bar{R}_{12} X_2 - X_2 \bar{R}_{21}\bar{R}_{12}) - h[\bar{R}_{12} + \bar{R}_{21}, X_2]\,. \end{aligned} \tag{48}$$

The comultiplication is given as

$$\Delta(X_{ij}) = X_{ij} \otimes 1 + 1 \otimes X_{ij} + X_{kl} \otimes (L^{+}_{ik} S(L^{-})_{lj} - \delta_{ik}\delta_{jl}) \,. \tag{49}$$

Here it is assumed that the entries of $L^{\pm}$ in the last term are expressed as functions of X, which is possible due to the already mentioned triangular decomposition of $\hbar X + I$. The entries of the matrix X are nothing else but the properly normalized generators of the bicovariant differential calculus on $F(G_{q = \exp(h\hbar)})$ [6, 5, 19]. The $C[[h, \hbar]]$-bialgebra A generated by X (47) with relations (48–49) is essentially the $U_h(g)$, which is a completion of $A/(\hbar - 1)A$.

In the limit $\hbar \to 0$ we get the anove described co-Poisson algebra V (more precisely their completion via the inclusion $C[h] \subsetneq C[[h]]$) so that A is a co-quantization of V in thesense that for $a \in A$ holds

$$(p_\hbar \otimes p_\hbar)\hbar^{-1}(\Delta(a) - \sigma\Delta(a)) = \nu(p_\hbar(a)) \,. \tag{50}$$

The corresponding projection $p_\hbar$, which is again a surjective bialgebra homomorphism, is called co-quantization homomorphism.

The last we need in our discussion of the biquantization of $S(g)$ is the following Poisson bialgebra F. Let $y = g_+ g_-^{-1}$ be the coordinates in G_r introduced in Sect. 1. Let us introduce new coordinates collected in the matrix again denoted as X via the relation $X = \hbar^{-1}(y - I)$. The Poisson bracket in this new coordinates reads

$$i\{X_1, X_2\} = \hbar(r_{21}X_1X_2 + X_1 r X_2 - X_2 r_{21} X_1 - X_2 X_1 r) + [r + r_{21}, X_2] \,. \tag{51}$$

The comultiplication is given by

$$\Delta(X_{ij}) = X_{ij} \otimes 1 + 1 \otimes X_{ij} + X_{kl} \otimes ((g_+)_{ik}(g_-)^{-1}_{lj} - \delta_{ik}\delta_{jl}) \,. \tag{52}$$

The $g_{\pm}$ are assumed as functions of X in the last term. The Poisson bialgebra F over $C[\hbar]$ with coordinates X is nothing else but $F(G_r)$. Namely the later is a completion of $F/(\hbar - 1)F$.

We also immediately see that A is a quantization of F (in the limit $h \to 0$ we get the completion of F via the inclusion $C[\hbar] \subsetneq C[[\hbar]]$). F itself is a co-quantization of $S(g)$. The corresponding surjective bialgebra homomorphisms p_h and $q_\hbar$ are thus quantization and co-quantization homomorphisms respectively. Moreover the map $q_\hbar$ is an Poisson algebra homomorphism.

The collection consisting of the bialgebra A, co-Poisson bialgebra V, Poisson bialgebra F, homomorphisms p_h, $q_\hbar$, q_h, $q_\hbar$ and the surjective bialgebra homomorphism

$$p \colon A \to S(g) \,,$$

$$p = q_h \circ p_\hbar = q_\hbar \circ p_h \,, \tag{53}$$

which is simultaneously the quantization of the Poisson bracket and co-quantization of the co-Poisson bracket in $S(g)$ realize the notion of the reduced biquantization of the bi-Poisson bialgebra $S(g)$.

Now we will briefly discuss some consequences of the facts collected above in the context of the representation theory. We will be interested only in the algebraic and Poisson structures appeared (we will forget all co-algebra and co-Poisson algebra structures). Let us remember that $S(g) \sim Pol(g^*)$ and let us assume a particular integral coadjoint orbit O in g^* of the maximal dimension (we assume in the following the connected and simply connected, simple compact groups). According to the classical Borel–Weil–Bott theorem there is one to one correspondence between such orbits and irreducible unitary representations. The quantization hommorphism q_h "restricted" to the particular representation T_O which extends to be an irreducible unitary representation of V) and the corresponding orbit O (we hope that it is clear what we mean) is most conveniently described with the help of coherent states connected with T_O [37]. The introduction of the Planck constant h in the commutation relations as it was done in the case of introducing the algebraic structure of V is a common trick used for the discussion of the classical limit [39]. The corresponding modification of coherent states described there is this just what we need. We refer the reader to this article for a more detailed discussion. In our situation we know that the range of the exponential mapping is up to a set of a zero measure on the entire G. If Λ_i are the generators of g, then the corresponding coherent states are defined as

$$e^{\frac{1}{h}\vec{\lambda}\cdot\vec{\Lambda}}|\rangle\,, \tag{54}$$

where the coefficients λ_i are taken such that the exponent belongs to the definition domain of the exponential mapping and the $|\rangle$ is the highest weight vector of T_O. Now as it is easy to see the covariant symbols of rescaled generators $h\Lambda_i$ have the proper limit as $h \to 0$ and the resulting symplectic structure is the usual Kirillov–Souriau–Kostant we need.

The algebra homomorphism $p_\hbar$ is described with the help of the algebra isomorphism φ. Formula

$$\hbar^{-1}((\tau \otimes \mathrm{id})\mathscr{R}(h\hbar)_{21}\mathscr{R}(h\hbar) - I) = X$$

gives an expression of the generators X in terms of the generators of g, which are nothing else as $h^{-1} \times$ generators of V. This way this formula also gives the expression of X in terms of generators of V. The projection $p_\hbar$ is then given by taking the limit $\hbar \to 0$ in the last expression.

So the discussed representation extends to the unitary irreducible representation of A which is already discussed is essentially the quantized enveloping algebra of g. The representative operators of X_{ij} are now quantizations of their covariant symbols [38] in the coherent states representation. The limit $h \to 0$ from the symbols gives of course a Poisson map $F \to O$ and describes in such a way the symplectic leaf of F. As well known [13, 14] the last should be a dressing orbit of G in G_r. It is also known [20] that the Poisson manifolds g^* and G_r are isomorphic and that this isomorphism sends the coadjoint orbits into the dressing orbits. So we have finally arrived to a particular realization of this isomorphism. It is clear that there is a similar relation between this isomorphism and the algebra homomorphism $q_\hbar$ as it was between φ and p_h.

5. Compact quantum groups and the trace formula

Now we will discuss some consequences of introducing the $*$-structure in the way described in Sect. 1 (6).

Following the lines of proofs of Proposition 3.16 of [11] and Proposition 4.3 of [29] it is easy to see that in the compact case $\mathscr{F}\mathscr{F}^*$ is g-invariant. The * in $U(g)[[h]] \otimes U(g)[[h]]$ is the usual component-wise one. From the property of the $\mathscr{R}$-matrix (Proposition 4.2 of [29])

$$\mathscr{R}^* = \mathscr{R}_{21}$$

we conclude the symmetry of $\mathscr{F}\mathscr{F}^*$. Twisting [11] with the help of the symmetric g-invariant element $(\mathscr{F}\mathscr{F}^*)^{-1/2}$ leads to the new one $\tilde{\mathscr{F}}$ which is unitary. We will assume in the following the $\mathscr{F}$ to be unitary. As a simple consequence we have

$$u^* = S(cu^{-1}) .$$

Further if t^α denotes the unitary irreducible representation of G, then computing the $(t^\alpha_{ij})^*$ and $S_h(t^\alpha_{ji})$ according to the formulas (6) and (1) we get $t^*_{ij} = S_h(t_{ji})$. It means t^α is also a unitary irreducible representation of the G_q (corepresentation of the $F(G_q)$) in the sense of [4] (the comultiplication remains unchanged). So the Peter–Weyl theorem generalizes immediately from the classical to the quantum case. The last is of course also well known [4]. The above $\star_h$-product is in this case the same as introduced in [32] and the Weyl transformation described there is identical with isomorphism φ^*. The Haar measure h_c on the G_q of [4] reduce under this isomorphism to the usual Haar measure $\eta = \int dg$ on G as already noted in [32].

Using the definition of the $\star_h$-product and the property of the Haar measure η

$$\eta((x * a)b) = \eta(a(S(x) * b)), \quad \eta((a * x)b) = \eta((a(b * S(x))),$$
$$a, b \in F(G), x \in U(g) , \tag{55}$$

we have

$$\eta(a \star_h b) = \eta((S(cu^{-1}) * a * u)b) = \eta(a(((cu^{-1}) * b * S(u))) . \tag{56}$$

Applying these formulas and the definition of * (6) we can compute for the unitary irreducible representations t^α and t^β of G using the well known orthogonality relations for the compact groups [36]

$$\eta(t^\alpha_{ij} \star_h (t^\beta)^*_{kl}) = \frac{1}{\dim(\alpha)} \delta_{\alpha\beta} \delta_{ik} uS(cu^{-1})(t^\alpha_{lj}) . \tag{57}$$

Comparing this with Theorem 5.7 of [4] we see that

$$\dim(\alpha)\gamma^2 = M_\alpha uS(cu^{-1}) , \tag{58}$$

where we have made an identification between γ^2 and f_1 of [4] and where M_α denotes the trace of γ^2 in the representation t^α.

Similarly computing $\eta((t^\alpha)^*_{ij} \star_h t^\beta_{mn})$ we have

$$\dim(\alpha)\gamma^{-2} = M_\alpha cS(u)u^{-1} . \tag{59}$$

Further using the two possible expressions for the c following from the definitions of u and u^{-1} (2) and the fact that $\mathscr{F}$ is now unitary we get immediately from the above Eqs. (57) and (58)

$$cc^* = \left(\frac{\dim(\alpha)}{M_\alpha}\right)^2 , \tag{60}$$

valid for any unitary irreducible representation t^α of G. Let us denote the positive square root of the Casimir cc^* as N. We have $N = S(N)$ and we can rewrite formula (56) as

$$\eta(a \star_h b) = \eta(((N\gamma) * a * \gamma)b) = \eta(a((N\gamma^{-1}) * b * \gamma^{-1})) . \tag{61}$$

Let us now remember that according to [2] we can to any $\hat{a} \in F(G_q)$ relate a functional $\xi_{\hat{a}}$ which gives on $\hat{b} \in F(G_q)$ the value

$$\xi_{\hat{a}}(\hat{b}) = h_c(\hat{a}\hat{b}) . \tag{62}$$

In our deformation description we have, owing to (61),

$$\xi_a(b) = \eta(((N\gamma) * a * \gamma)b) = \eta(a((N\gamma^{-1}) * b * \gamma^{-1})) \tag{63}$$

for $a, b \in F(G)$. On the other hand we have also the classical functional ξ_a^{cl} given simply by

$$\xi_a^{\mathrm{cl}}(b) = \eta(ab) . \tag{64}$$

Comparing (63) and (64) we obtain

$$\xi_a = N\gamma^{-1}\xi_a^{\mathrm{cl}}\gamma^{-1} = \xi_{\tilde{a}}^{\mathrm{cl}} , \tag{65}$$

with

$$\tilde{a} = (N\gamma) * a * \gamma .$$

Let us now assume the unitary irreducible representation t_O of G (simply connected, connected, compact) corresponding to the integral coadjoint orbit of the maximal dimension (it is as already noted above a unitary irreducible representation of G_q). We can now apply the classical universal trace formula [36] to the functional ξ_a^{cl} to compute the trace of the operator ξ_a in this representation. We get

$$\begin{aligned}\mathrm{Tr}_O(\xi_a) &= \frac{\dim(O)}{M_O}\mathrm{Tr}_O(\gamma^{-2}\xi_a^{\mathrm{cl}}) = \mathrm{Tr}_O(\xi_{\tilde{a}}) \\ &= \frac{\dim(O)}{M_O}\int_O\left(\int_U a(\gamma\exp(X)\gamma)Q^{1/2}(\exp X))e^{2\pi i\langle F,X\rangle}dX\right)d\beta_O(F) . \end{aligned} \tag{66}$$

In the last formula $F \in g^*$, $\langle\cdot,\cdot\rangle$ is the dualization between g and g^*, U is the inverse image in g of an open region (of the complement of the zero measure) in G covered by the canonical system of coordinates, β_O means the canonical measure on O defined by the Kirillov–Souriau–Kostant symplectic form, dX-Lebesgue measure on g^* and Q is a universal function [36]. Here of course the ordering of terms in the argument of a is insubstantial. We have taken the most symmetric one.

So the trace of the functional $h_c(\hat{a}\cdot)$, $\hat{a} \in F(G_q)$ in the unitary irreducible representation of G_q is expressed in the following way:

1. in the Peter–Weyl expansion there are matrix elements of unitary irreducible representation of G_q replaced by those of the corresponding unitary irreducible representations of G so we get the corresponding element $a = (\varphi^*)^{-1}(\hat{a}) \in F(G)$ and

2. applying the trace formula to the so-obtained $a \in F(G)$ to express the trace with the help of integration over the correspoinding coadjoint orbit (dressing orbit) O.

Let us note the interesting fact that the trace of ξ_a is proportional to the Markov trace of ξ_a^{cl}.

We finish with the simple formulas for the left and right invariant measures h_{dL} and h_{dR} of [2] on ξ_a,

$$h_{dR}(\xi_a) = \sum_O \dim(O)\,\mathrm{Tr}_O(\xi_a^{\mathrm{cl}}) = a(e)\,, \tag{67}$$

which agrees with the formula (2.24) of [2] (here we have instead of the left invariant measure the right one, because the ρ_a of [2] differs by antipode from ξ_a). For the left invariant Haar measure we get

$$h_{dL}(\xi_a) = \sum_O \dim(O)\,\mathrm{Tr}_O(\xi_{\gamma^4 * \alpha}^{\mathrm{cl}}) = a(\gamma^4)\,. \tag{68}$$

So using the deformation formalism described above we can view many properties of the compact quantum groups in particular some of them described in [4, 2] as consequences of the well known properties of the compact Lie groups.

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