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Global Vertex Operators on Riemann Surfaces

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Abstract: We develop an approach towards construction of conformal field theory starting from the basic axioms of vertex operator algebras.

1. Introduction

The notion of vertex algebras was introduced in [Bo1]; the variant of this that we call "vertex operator algebras" defined in [FLM2] and [FHL] can be regarded as a mathematical reformulation of "chiral algebras" or "conformal algebras" in conformal field theory. The basic ingredients in the definition of vertex operator algebras are a space of states and the vertex operators associated with the states. One of the two main axioms, the Jacobi identity, involves the properties of vertex operators on a complex disc; the other main axiom is about the Virasoro algebra which is supposed to encode the information of infinitesimal deformations of Riemann surfaces with local coordinates. It is expected that these axioms and certain finiteness conditions are sufficient to formulate and verify the theorems on all Riemann surfaces. The present work discusses this problem. We introduce the notions of the global vertex operators and the space of vacua on a Riemann surface with punctures, and prove some initial results. And we will discuss the relation of our approach with the modular functors defined in [Se].

For a given vertex operator algebra V and a given data

$$(\Sigma; Q_1, \dots, Q_N; z_1, \dots, z_N)\,,$$

where Σ is a compact Riemann surface, $Q_1, \dots, Q_N$ are N distinct points on Σ and z_i is a local coordinate at Q_i satisfying $z_i(Q_i)=0$, a global vertex operator on such data is defined to be the sum of the residues of an operator valued differential form associated to a primary vertex operator $Y(a, z)$ and a meromorphic differential f with the dual degree on Σ. In this language, the operators of Virasoro type and Kac–Moody type on a two-punctured Riemann surface defined in [KN] are essentially the global vertex operators associated to the Virasoro algebra and primary fields of degree one on a two-pointed Riemann surface, respectively.

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Suppose further we assign at each point Q_i a representation W_i of V, so we have the data

$$\tilde{\Sigma}=(\Sigma; Q_1, \ldots, Q_n; z_1, \ldots, z_N; W_1, \ldots, W_N),$$

the space of vacua on $\tilde{\Sigma}$ is defined to be the subspace of $(W_1 \otimes \cdots \otimes W_N)^*$ whose elements are annihilated by all the global vertex operators. This definition is motivated by physical literatures (e.g., [DVV, GGMV, W]).

In the case that the vertex operator algebra under consideration is generated by a Kac–Moody affine Lie algebra, the notion of the space of vacua is defined in [TUY], which appears to differ with our definition, but one can prove that the two definitions are essentially equivalent (see remarks in Sect. 7). After we introduce the notions of the global vertex operators and the space of vacua, we prove some results similar to the results in [TUY]. One of these results (Theorem 6.1) is that if the data

$$\tilde{\Sigma}'=(\Sigma; Q_1, \ldots, Q_N, Q_{N+1}; z_1, \ldots, z_N, z_{N+1}; W_1, \ldots, W_N, V)$$

is obtained by adding (Q_{N+1}, z_{N+1}) to the data $\tilde{\Sigma}$ and assigning the adjoint module V (or 0 sector) at Q_{N+1}, then the space of vacua on $\tilde{\Sigma}'$ and the space of vacua on $\tilde{\Sigma}$ are canonically isomorphic. This result is used to define the correlation functions associated to every vector in the space of the vacua on the Riemann surface (see Theorem 6.2). Some of the arguments used in proving these results is similar to the argument used in [TUY]. However, since there is no affine Lie algebra structure in a general vertex operator algebra, we cannot appeal to the representation theory of the affine Lie algebras as in [TUY]. To overcome this difficulty we are forced to define the quasi-global vertex operators on a Riemann surface with projective structure, and we prove that the space of quasi-global vertex operators forms a Lie algebra (Proposition 4.2) and that the space of vacua is annihilated by quasi-global vertex operators (Theorem 5.1). These results are used as technical tools in the proof of our main theorems (Theorem 6.1 and Theorem 6.2). In the end we discuss a conjectured procedure to construct the space of vacua on higher genus Riemann surfaces by gluing lower genus Riemann surfaces. This gluing construction relates to the modular functor defined in [Se].

The paper is organized as follows. Sect. 2 gives a brief review of definitions of vertex operator algebras and the results needed later in order to make this paper self-contained. Section 3 sets up the notations and gives the definition of global vertex operators and the space of vacua on a n-pointed Riemann surface. In Sect. 4 we define the space of quasi-global vertex operators on a n-pointed Riemann surface with a projective structure and prove that it is closed under the Lie bracket. In Sect. 5, we prove that the space of vacua on a Riemann surface with a projective structure is annihilated by the quasi-global vertex operators. In Sect. 6, we prove that there is a system of correlation functions corresponding to the each vector of the space of vacua. In Sect. 7, we discuss the examples of the space of vacua for various situations. In particular, we give the relations of the space of vacua on 2-pointed and 3-pointed spheres with the notion of dual representations and interwining operators defined in [FHL]. In Sect. 8, we give a conjecture on constructing the space of vacua on higher genus Riemann surfaces by gluing lower genus Riemann surfaces and discuss its relation with the modular functors.

We will denote by $\mathbf{C}$ and $\mathbf{Z}$ the set of complex numbers and the set of rational integers respectively. And we denote by $\oint_C f(z)dz$ the contour integral so normalized that $\oint_C \frac{1}{z}dz=1$ for a contour C surrounding 0.

2. Definitions of Vertex Operator Algebras and Representations

We recall the basic definitions of vertex operator algebras and representations, and give a summary of the results used later and sketch their proofs. For more details, see [FLM2] and [FHL]. And see [FLM2, FK, FZ, Li] for various examples.

Definition 2.1. *A vertex operator algebra is a graded vector space* $V=\bigoplus_{n=0}^{\infty} V_n$ *equipped with a linear map*

$$V \to (\mathrm{End}\, V)[[z, z^{-1}]]\,,$$

$$a \mapsto Y(a, z)=\sum_{n\in\mathbb{Z}} a(n)z^{-n-1}(a(n)\in \mathrm{End}\, V)$$

(we call $Y(a, z)$ *the vertex operator of* a*) and with two distinguished vectors* $1\in V_0$, $\omega\in V_2$ *satisfying the following conditions for* $a, b\in V$:

$$a(n)b=0 \quad \text{for } n \text{ sufficiently large}\,; \tag{2.1}$$

$$Y(1, z)=1\,; \tag{2.2}$$

$$Y(a, z)1\in V[[z]] \text{ and } \lim_{z\to 0} Y(a, z)1=a\,; \tag{2.3}$$

the vertex operator $Y(\omega, z)=\sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ *generates a copy of the Virasoro algebra:*

$$[L_m, L_n]=(m-n)L_{m+n}+\delta_{m+n,0}\frac{m^3-m}{12}c\,, \tag{2.4}$$

where c *is a constant which is called the rank of* V; *and*

$$L_0 a=na=\deg a\, a \quad \text{for } a\in V_n\,, \tag{2.5}$$

$$Y(L_{-1}a, z)=\frac{d}{dz}Y(a, z)\,; \tag{2.6}$$

and the following Jacobi identity holds for every $m, n, l\in\mathbf{Z}$:

$$\begin{aligned}&\mathrm{Res}_{z-w}(Y(Y(a, z-w)b, w)\iota_{w,z-w}F(z, w))\\ &=\mathrm{Res}_z(Y(a, z)Y(b, w)\iota_{z,w}F(z, w))-\mathrm{Res}_z(Y(b, w)Y(a, z)\iota_{w,z}F(z, w))\,,\end{aligned} \tag{2.7}$$

where $F(z, w)=z^m w^n(z-w)^l$. *This completes the definition.*

Identity (2.7) needs some explanation. Expressions $\iota_{w,z-w}F(z, w)$, $\iota_{z,w}F(z, w)$ and $\iota_{w,z}F(z, w)$ mean the power series expansions of the rational function $F(z, w)$ on the domains $|w|>|z-w|$, $|z|>|w|$, $|w|>|z|$ respectively, i.e.,

$$\iota_{w,z-w}(z^m w^n(z-w)^l)=\sum_{i=0}^{\infty}\binom{m}{i}w^{m+n-i}(z-w)^{l+i}\,,$$

$$\iota_{z,w}(z^m w^n(z-w)^l)=\sum_{i=0}^{\infty}(-1)^i\binom{l}{i}z^{m+l-i}w^{n+i}\,,$$

$$\iota_{w,z}(z^m w^n(z-w)^l)=\sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}z^{m+i}w^{n+l-i}\,.$$

And $\mathrm{Res}_{z-w}(\ldots)$ in (2.7) means the coefficient of the $(z-w)^{-1}$ of the formal power series in $(\ldots)$. $\mathrm{Res}_z(\ldots)$ and $\mathrm{Res}_w(\ldots)$ have the similar meanings. And we will use the similar notations later. So (2.7) is equivalent to

$$\sum_{i=0}^{\infty}\binom{m}{i}Y(a(l+i)b,w)w^{m+n-i}=\sum_{i=0}^{\infty}(-1)^i\binom{l}{i}a(m+l-i)Y(b,w)w^{n+i}$$
$$-\sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}Y(b,w)a(m+i)w^{n+l-i}\,. \quad (2.8)$$

And taking coefficient of w^{-1} in both sides of (2.8), we have

$$\sum_{i=0}^{\infty}\binom{m}{i}(a(l+i)b)(m+n-i)=\sum_{i=0}^{\infty}(-1)^i\binom{l}{i}a(m+l-i)b(n+i)$$
$$-\sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}b(n+l-i)a(m+i)\,. \quad (2.9)$$

By (2.1), for a fixed $v\in V$, there are finitely many vectors among $a(l+i)b$, $b(n+i)v$ and $a(m+i)v$ $(i\geqq 0)$ are non-zero, so all the three terms in (2.9) are well-defined linear operators on V. The Jacobi identity (2.7) are equivalent to (2.9) for every $m, n, l\in\mathbf{Z}$.

We give some immediate consequences of the definition. We have relations:

$$Y(a,z)=0 \quad \text{iff } a=0\,, \quad (2.10)$$

$$[a(m),Y(b,w)]=\sum_{i=0}^{\infty}\binom{m}{i}Y(a(i)b,w)w^{m-i}\,, \quad (2.11)$$

$$[L_{-1},Y(a,z)]=\frac{d}{dz}Y(a,z)\,, \quad (2.12)$$

$$[L_0,Y(a,z)]=\frac{d}{dz}Y(a,z)z+Y(L_0a,z)\,, \quad (2.13)$$

$$a(n)V_m\subset V_{m+\deg a-n-1} \quad \text{for } a \text{ homogeneous}\,, \quad (2.14)$$

$$a(n)1=0 \quad \text{for } n\geqq 0\,, \quad (2.15)$$

$$Y(a,z)1=\exp(zL_{-1})a\,, \quad (2.16)$$

$$a(-k-1)1=\frac{1}{k!}(L_{-1})^k a \quad \text{for } k\geqq 0\,. \quad (2.17)$$

We sketch the proofs of the above relations. Equation (2.10) follows directly from (2.3). Equation (2.11) is obtained by specifying $l=0$, $n=0$ in (2.8). Equations (2.12) and (2.13) is proved by using (2.11), (2.5) and (2.6). Equation (2.14) follows from (2.13) directly. Equation (2.15) follows from (2.3). Equations (2.16) and (2.17) are equivalent. To prove (2.17), using (2.12), we have $(adL_{-1})^kY(a,z)=\left(\frac{d}{dz}\right)^kY(a,z)$, apply this identity to 1 and take $\lim_{z\to 0}$, using $L_{-1}1=\omega(0)1=0$, we obtain $(L_{-1})^ka=k!a(-k-1)$.

From (2.10) we see that the operators $a(n)$ ($a \in V$, $n \in \mathbf{Z}$) are closed under the Lie bracket. And from (2.14), we see that for a homogeneous element a, the operator $a(n)$ ($n \in \mathbf{Z}$) maps a homogeneous subspace into a homogeneous subspace, and $a(n)$ has degree $\deg a - 1 - n$, we write $\deg(a(n)) = \deg a - 1 - n$.

Definition 2.2. *A representation of V or a V-module is a graded vector space $M = \bigoplus_{n=0}^{\infty} M_n$, such that there is a linear map*

$$V \to (\operatorname{End} M)[[z, z^{-1}]] ,$$

$$a \mapsto Y_M(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$$

and the following properties are satisfied:

$$a(n) M_m \subset M_{m + \deg a - n - 1} \text{ for every homogeneous } a \tag{2.18}$$

and (2.2) (2.4) *and* (2.6) *and the following Jacobi identity holds:*

$$\begin{aligned} &\operatorname{Res}_{z-w}(Y_M(Y(a, z-w)b, w)\iota_{w, z-w} F(z, w)) \\ &\quad = \operatorname{Res}_z(Y_M(a, z) Y_M(b, w)\iota_{z, w} F(z, w)) - \operatorname{Res}_z(Y_M(b, w) Y_M(a, z)\iota_{w, z} F(z, w)) \end{aligned} \tag{2.19}$$

for every rational function $F(z, w) = z^m w^n (z-w)^l m, n, l \in \mathbf{Z}$.

Equation (2.19) has the same interpretation as (2.7). And relations (2.11)–(2.13) hold also for representations of V. We will write $Y(a, z)$ for $Y_M(a, z)$. Note that by (2.18), for fixed $a \in V$ and $v \in M$, $a(n)v = 0$ for n sufficiently large. Thus, for a Laurent power series $f(z) = \sum_{i \geqq N}^{\infty} l_i z^i$, the operator

$$\operatorname{Res}_z(Y(a, z) f(z)) = \sum_{i \geqq N}^{\infty} l_i a(i)$$

is a well-defined operator on M.

It is clear that V itself is a representation of V; we call it the adjoint module or 0-sector.

Subrepresentations, direct sums of representations, irreducible representations, etc., are defined as expected.

An important class of vertex operator algebras are rational vertex operator algebras, which is defined as follows:

Definition 2.3. *A vertex operator algebra is rational if it has only finitely many irreducible representations, and each irreducible representation $M = \bigoplus_{n \in \mathrm{N}} M_n$ satisfies $\dim(M_n) < \infty$, and moreover every representation is a direct sum of the irreducible ones.*

We quote a result of [FLH] on correlation functions on the Riemann sphere which is not used later but will be compared with our results on correlation functions on general Riemann surfaces.

Theorem 2.1. *Let $M = \sum_{i=0}^{\infty} M_i$ be a representation of V, let $M' = \sum_{i=0}^{\infty} M_i^*$ (M_i^* is the dual space of M_i). Then for every $v' \in M'$, $v \in M$ and a_i ($i = 1, \ldots, n$), the formal power series*

$$\langle v', Y(a_1, z_1) \ldots Y(a_n, z_n) v \rangle$$

converges on the domain $|z_1|>\cdots|z_n|>0$ *to a rational function* $(v', Y(a_1, z_1)\dots Y(a_n, z_n)v)$ *with only possible poles at* $z_i=z_j$ $(i\neq j)$ *and* $z_i=0$. *For every permutation* $i_1, \dots, i_n$ *of* $1, \dots, n$, *we have the identity of the rational functions*

$$(v', Y(a_{i_1}, z_{i_1})\dots Y(a_{i_n}, z_{i_n})v)=(v', Y(a_1, z_1)\dots Y(a_n, z_n)v)\,.$$

And if C *is a contour of* z_1 *which surrounds* z_2 *and* $0, z_3, \dots, z_n$ *are outside of* C, *then*

$$\oint_C (v', Y(a_1, z_1)Y(a_2, z_2)\dots Y(a_n, z_n)v)(z_1-z_2)^k\,dz_1$$
$$=(v', Y(a_1(k)a_2, z_2)Y(a_3, z_3)\dots Y(a_n, z_n)v)\,. \tag{2.20}$$

Proof. We first derive a formula which expresses the $(n+1)$-point function

$$\langle v', Y(a_1, z_1)\dots Y(a_k, z_k)Y(b, w)Y(a_{k+1}, z_{k+1})\dots Y(a_n, z_n)v\rangle \tag{2.21}$$

in terms of the n-point function. Write $Y(b, w)=\sum_{m=0}^{\infty} b(m)w^{-m-1}+\sum_{m=1}^{\infty} b(-m)w^{m-1}$, and move the term $\sum_{m=0}^{\infty} b(m)w^{-m-1}$ across the terms $Y(a_{k+1}, z_{k+1}), \dots, Y(a_n, z_n)$ to the right, and move $\sum_{m=1}^{\infty} b(-m)w^{m-1}$ across the terms $Y(a_k, z_k), \dots, Y(a_1, z_1)$ to the left, and using (2.11) to compute the Lie bracket, we obtain

$$(2.21)=\langle v', Y(a_1, z_1)\dots Y(a_n, z_n)\sum_{m=0}^{\infty} b(m)w^{-m-1}v\rangle$$
$$+\left\langle v', \sum_{m=1}^{\infty} b(-m)w^{m-1}Y(a_1, z_1)\dots Y(a_n, z_n)v\right\rangle$$
$$+\sum_{j\geqq k+1}\sum_{i=0}^{\infty} \iota_{w, z_j}((w-z_j)^{-i-1})\langle v', Y(a_1, z_1)\dots Y(b(i)a_j, z_j)\dots Y(a_n, z_n)v\rangle$$
$$+\sum_{j\leqq k}\sum_{i=0}^{\infty} \iota_{z_j, w}((w-z_j)^{-i-1})\langle v', Y(a_1, z_1)\dots Y(b(i)a_j, z_j)\dots Y(a_n, z_n)v\rangle\,. \tag{2.22}$$

Note that all the four terms of the right-hand side of (2.22) are actually finite sums since $b(i)v=0$, $b(i)a_j=0$ and $\langle v', b(-i)x\rangle=0$ ($x\in V$ is arbitrary) for i sufficiently large. From (2.22), we see by induction that

$$\langle v', Y(a_1, z_1)\dots Y(a_n, z_n)v\rangle$$

converges on the domain $|z_1|>\cdots>|z_n|$ to some rational functions with poles at $z_i=z_j$ $(i\neq j)$ and $z_i=0$, and the fact that the limit rational function is independent of the ordering of the product of $Y(a_i, z_i)$ also follows from (2.21). To prove (2.20), let C_1 be a contour of z_1 which contains 0 while z_i $(i=2, \dots, z_n)$ are outside C_1, and C_2 be a contour of z_1 which contains 0 and z_2 while z_i $(i=3, \dots, n)$ are outside C_2. By the Cauchy Theorem for contour integrals, we have

$$\oint_C (v', Y(a_1, z_1)Y(a_2, z_2)\dots Y(a_n, z_n)v)(z_1-z_2)^k\,dz_1$$
$$=\oint_{C_2} (v', Y(a_1, z_1)Y(a_2, z_2)\dots Y(a_n, z_n)v)(z_1-z_2)^k\,dz_1$$
$$-\oint_{C_1} (v', Y(a_1, z_1)Y(a_2, z_2)\dots Y(a_n, z_n)v)(z_1-z_2)^k\,dz_1\,.$$

The both integrals over C_1 and C_2 are meromorphic functions of variables $z_2, z_3, \ldots, z_n$ with singularities at $z_i = z_j$ and $z_i = 0$. The integral on C_2 has a power series expansion on the domain $|z_3| > \cdots > |z_n| > |z_1|$ as

$$\langle v', Y(a_3, z_3) \ldots Y(a_n, z_n) \operatorname{Res}_{z_1}(\iota_{z_1, z_2}(z_1 - z_2)^k Y(a_1, z_1) Y(a_2, z_2)) v \rangle .$$

The integral on C_1 has a power series expansion on the domain $|z_3| > \cdots > |z_n| > |z_1|$ as

$$\langle v', Y(a_3, z_3) \ldots Y(a_n, z_n) \operatorname{Res}_{z_1}(\iota_{z_2, z_1}(z_1 - z_2)^k Y(a_2, z_2) Y(a_1, z_1)) v \rangle .$$

By the Jacobi identity, we have

$$\operatorname{Res}_{z_1}(\iota_{z_1, z_2}(z_1 - z_2)^k Y(a_1, z_1) Y(a_2, z_2)) - \operatorname{Res}_{z_1}(\iota_{z_2, z_1}(z_1 - z_2)^k Y(a_2, z_2) Y(a_1, z_1))$$
$$= Y(a_1(k) a_2, z_1) .$$

Thus we have proved that both sides of (2.20) have the same power series expansion on the domain $|z_3| > \cdots > |z_n| > |z_2|$, so they are the same meromorphic function. This concludes the proof. □

The rational functions $(v', Y(a_1, z_1) \ldots Y(a_n, z_n) v)$ are called correlation functions on the sphere. It can be proved that a certain converse of Theorem 2.1 is true: the convergence of the products of vertex operators and the properties of the limit as in the theorem implies the Jacobi identity.

To describe a generalization of this theorem to an arbitrary Riemann surface, we write $\langle v', Y(a_1, z_1) \ldots Y(a_n, z_n) v \rangle$ in a different way. Assume $\dim(M_i) < \infty$ for every i, let $\{e_i, i = 1, 2, \ldots\}$ be a basis of M, and $\{e_i', i = 1, 2, \ldots\}$ be its dual basis (i.e., $\langle e_i', e_j \rangle = \delta_{i,j}$). Then $x = \sum_{i=1}^{\infty} e_i \otimes e_i'$, viewed as a vector in $(M' \otimes M)^*$, has the property:

$$\langle x, v' \otimes v \rangle = \langle v', v \rangle .$$

So

$$\langle v', Y(a_1, z_1) \ldots Y(a_n, z_n) v \rangle = \langle x, v' \otimes Y(a_1, z_1) \ldots Y(a_n, z_n) v \rangle .$$

As we will see later x is a vector in the *space of vacua* of the two-punctured Riemann sphere with punctures ∞ and 0. Our generalization of Theorem 2.1 to a compact Riemann Σ surface with N punctures can be roughly described as follows (see Theorem 6.2 for detail): if x is in the space of vacua on Σ, then

$$\langle x, v_1 \otimes v_2 \otimes \cdots \otimes v_{N-1} \otimes Y(a_1, z_1) \ldots Y(a_n, z_n) v_N \rangle$$

converges on the domain $|z_1| > \cdots > |z_n| > 0$ in a coordinate neighborhood of the N-th point, and the limit can be extended to a global meromorphic section of a certain line bundle over Σ^n, and this meromorphic section is independent of the ordering of the product of the vertex operators.

By the definition, every representation of V is in particular a representation of the Virasoro algebra. We will frequently assume that V is a sum of highest weight representations of its Virasoro algebra. And we assume all the representations of V in this paper satisfy the condition that L_0 acts semi-simply.

3. Global Vertex Operators and Space of Vacua on Riemann Surfaces

We will give the definitions of global vertex operators and the space of vacua for a labeled Riemann surface. And associated to each vector in the space of vacua we define 1-point correlation functions on the underlying Riemann surface.

We first fix some notations. Let Σ be a compact Riemann surface, $Q_1, \ldots, Q_N$ be N distinct points on Σ, z_i be the local coordinate near Q_i satisfying $z_i(Q_i)=0$. We will denote this data by

$$(\Sigma; Q_1, \ldots, Q_N; z_1, \ldots, z_N) \tag{3.1}$$

and call it an *N-pointed Riemann surface*. Let κ be the canonical line bundle over Σ; we denote by

$$\Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^n) \tag{3.2}$$

the space of global meromorphic sections of the line bundle κ^n holomorphic away from points $Q_1, \ldots, Q_N$, or equivalently, (3.2) is the space of n-meromorphic differentials on Σ with possible poles at Q_i $(i=1, \ldots, N)$. κ^n has a local section $(dz_i)^n$ near the point Q_i, for a f in (3.2). Write $f=f_i(z_i)(dz_i)^n$. We denote $\iota_{z_i}f$ the Laurent series expansion of $f_i(z_i)$ in z_i; we call it the expansion of f at (Q_i, z_i). So we have a linear map ι_{z_i} from (3.2) to $\mathbf{C}((z_i))$. The following standard lemma will be used later.

Lemma 3.1. *For each $n\in\mathbf{Z}$, the linear map*

$$\iota=\bigoplus_{i=1}^{N} \iota_{z_i}: \Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^n)\to\bigoplus_{i=1}^{N} \mathbf{C}((z_i)),$$

$$f\mapsto(\iota_{z_i}f, \ldots, \iota_{z_N}f)$$

is injective. And if $\langle\,,\rangle$ is the bilinear form

$$\bigoplus_{i=1}^{N} \mathbf{C}((z_i))\times\bigoplus_{i=1}^{N} \mathbf{C}((z_i))\to\mathbf{C}$$

given by

$$\langle(f_1(z_1), \ldots, f_N(z_N)), (g_1(z_1), \ldots, g_N(z_N))\rangle=\sum_{i=1}^{N} \operatorname{Res}_{z_i}(f(z_i)g(z_i)),$$

then $(g_1(z_1), \ldots, g_N(z_N))$ is in the image $\iota(\Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^n))$ if and only if

$$\langle(g_1(z_1), \ldots, g_N(z_N)), (f_1(z_1), \ldots, f_N(z_N))\rangle=0$$

for every $(f_1(z_1), \ldots, f_N(z_N))\in\iota(\Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^{-n+1}))$.

The following consequence of the Riemann–Roch theorem is used in the proofs of Theorem 5.1, Theorem 6.1 and Theorem 6.2 without mentioning it. For every integer k, m, there exist a $f\in\Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^n)$ such that

$$\iota_{z_N}f\equiv z_N^k \quad \operatorname{Mod} z_N^m,$$

$$\text{and} \quad \iota_{z_i}f\equiv 0 \quad \operatorname{Mod} z_i^m \quad \text{for} \quad 2\leqq i<N.$$

Recall that a projective structure on Σ is a covering of coordinates chart $\{U_\alpha, z_\alpha\}$ such that every coordinate transition function is a Mobius transformation. Given a projective structure $\{U_\alpha, z_\alpha\}$ and points $Q_1, \ldots, Q_N$ on Σ, for each Q_i, we choose a local chart $(U_{\alpha_i}, z_{\alpha_i})$ such that U_{α_i} contains Q_i and take $z_i = z_{\alpha_i} - z_{\alpha_i}(Q_i)$ as the local coordinate at Q_i. So the obtained N-pointed Riemann surface is said to be *projective.*

By assigning a representation W_i of V at each point Q_i of (3.1), we have the data

$$\tilde{\Sigma} = (\Sigma; Q_1, \ldots, Q_N; z_1, \ldots, z_N; W_1, \ldots, W_N) . \tag{3.3}$$

This is the main object in our investigation, and we call it a *N-labeled Riemann surface with labels as representations of V* as simply the N-labeled Riemann surface. If (3.1) is projective, we call (3.3) a *projective N-labeled Riemann surface.*

Recall that the *space $\mathscr{P}_n(V)$ of primary fields of degree n* is defined as

$$\mathscr{P}_n(V) = \{a \in V \mid L_i a = 0 \text{ for } i > 0, L_0 a = na\} .$$

For $a \in \mathscr{P}_n(V)$, using (2.11) and (2.6), we have the commutation relation:

$$[L_m, Y(a, z)] = \left(n(m+1)z^m + z^{m+1}\frac{d}{dz} \right) Y(a, z) . \tag{3.4}$$

The commutation relation of L_m and $Y(\omega, z)$ is close to (3.4) with $n = 2$ except for a central term:

$$[L_m, Y(\omega, z)] = \left(2(m+1)z^m + z^{m+1}\frac{d}{dz} \right) Y(\omega, z) + \frac{(m^3 - m)c}{12} z^{m-2} . \tag{3.5}$$

Note that (3.4) is similar to the formula of the Lie derivative of a local n-differential $f(z)(dz)^n$ on a Riemann surface with respect to the holomorphic vector field $z^{m+1}\frac{d}{dz}$:

$$\nabla_{z^{m+1}\frac{d}{dz}} f(z)(dz)^n = \left((m+1)nz^m + z^{m+1}\frac{d}{dz} \right) f(z)(dz)^n .$$

So $Y(a, z)$ has the similar covariance property as a n-differential which is formulated in the first part of Lemma 3.2 below.

Let z and w be local coordinates near $Q \in \Sigma$ such that $z(Q) = w(Q) = 0$, $w = \phi(z) = \sum_{i=1}^{\infty} c_i z^i$ be the transition function. Write $\phi(z) = \exp(\sum_{i=0}^{\infty} l_i z^{i+1}\frac{d}{dz})z$; such an expression is unique by requiring $0 \leq \operatorname{Im} l_0 < 2\pi$. Following [H and TUY], the operator $T(\Phi)$ associated to the transition function $\phi(z)$ is defined as

$$T(\phi) = \exp\left(\sum_{i=0}^{\infty} l_i L_i \right) . \tag{3.6}$$

The following lemma is easy to prove.

Lemma 3.2. *For $\phi(z)$ and $T(\phi)$ as above, we have relations*

$$T(\phi) Y(a, z) T(\phi)^{-1} = Y(a, \phi(z))(\phi'(z))^n \quad \text{for } a \in \mathscr{P}_n(V) , \tag{3.7}$$

$$T(\phi) Y(\omega, z) T(\phi)^{-1} = Y(\omega, \phi(z))(\phi'(z))^2 + \tfrac{1}{12}\{\phi(z), z\} c , \tag{3.8}$$

where $\{\phi(z), z\} = \dfrac{\phi'''(z)}{\phi'(z)} - \dfrac{3}{2}\left(\dfrac{\phi''(z)}{\phi'(z)} \right)^2$ *is the Schwarzian derivative of* $\phi(z)$.

The additional term $\frac{1}{12}\{\phi(z), z\}c$ in (3.8) is caused by the central term in (3.5). From (3.7) we may view heuristically $Y(a,z)(a\in\mathscr{P}_n(V))$ as an operator valued n-differential on a Riemann surface Σ; and from (3.8), we view $Y(\omega,z)$ as an operator valued quadratic differential on a Riemann surface Σ with z as a local coordinate in a projective structure since the Schwarzian derivative $\{\phi(z), z\}=0$ for $\phi(z)$ a Mobius transformation. Thus we may view $Y(a,z)f(z)(dz)^{-n+1}$ ($Y(\omega,z)f(z)(dz)^2$ resp.) for f being a global $(-n+1)$-meromorphic differential on Σ (meromorphic vector field, resp.) as a global operator valued 1-differential. Motivated by this point of view, we will define global vertex operator on a N-labeled Riemann surface (3.3) by taking the "sum of residues."

We first set some notation which is used for the rest of the paper. For a tensor product $W_1\otimes\cdots\otimes W_N$ of vector spaces W_i and an operator A on W_i, we write

$$A_i=1\otimes\cdots\otimes 1\otimes A(i\text{-th place})\otimes\cdots\otimes 1\,, \tag{3.9}$$

so A_i is an operator on $W_1\otimes\cdots\otimes W_N$. And for an operator A on a vector space W, A acts on the dual space W^* from the right by the rule $\langle v'A, v\rangle=\langle v', Av\rangle$ for every $v'\in W^*$ and $v\in W$.

Definition 3.1. *For a N-labeled Riemann surface $\tilde{\Sigma}$ as* (3.3), *$a\in\mathscr{P}_n(V)$ and $f\in\Gamma(\Sigma; Q_1,\ldots,Q_N;\kappa^{-n+1})$, the global vertex operator associated to a and f on $\tilde{\Sigma}$ is defined as the operator*

$$a(f,\tilde{\Sigma})=\sum_{i=1}^{N}(\mathrm{Res}_{z_i}(Y(a,z_i)\iota_{z_i}f))_i$$

which acts on $W_1\otimes\cdots\otimes W_N$. The dependence of $a(f,\tilde{\Sigma})$ on the local coordinates z_i's can be easily derived using (3.7). *If*

$$\tilde{\Sigma}_w=(\Sigma; Q_1,\ldots,Q_N; w_1,\ldots,w_N; W_1,\ldots,W_N) \tag{3.10}$$

is the N-labeled Riemann surface obtained from (3.3) *by changing z_i to w_i. Let $w_i=\phi_i(z_i)$ be the local coordinates transition function, $T(\phi_i)$ be the associated operator. Write $f_i(z_i)=\iota_{z_i}f$ and $g_i(w_i)=\iota_{w_i}f$, then $g_i(\phi_i(z_i))\phi_i'(z_i)^{-n+1}=f_i(z_i)$. This fact together with* (3.7) *implies that*

$$\mathrm{Res}_{w_i}(Y(a,w_i)\iota_{w_i}f)=T(\phi_i)\,\mathrm{Res}_{z_i}(Y(a,z_i)\iota_{z_i}f)T(\phi_i)^{-1}\,.$$

Therefore we have

$$a(f,\tilde{\Sigma}_w)=(\Pi_{i=1}^{N}T(\phi_i)_i)a(f,\tilde{\Sigma})(\Pi_{i=1}^{N}T(\phi_i)_i^{-1})\,. \tag{3.11}$$

The following lemma makes it possible to define global vertex operators associated to ω.

Lemma 3.3. *Let $\tilde{\Sigma}$ as* (3.3) *and $\tilde{\Sigma}_w$ as* (3.10) *be projective N-labeled Riemann surfaces. Let $w_i=\phi_i(z_i)$ be the coordinate transition function at Q_i, $T(\phi_i)$ be the associated operator. Let $f\in\Gamma(\Sigma; Q_1,\ldots,Q_N;\kappa^{-1})$. Then*

$$\sum_{i=1}^{N}(\mathrm{Res}_{w_i}(Y(\omega,w_i)\iota_{w_i}f))_i$$

$$=(\Pi_{i=1}^{N}T(\phi_i)_i)\sum_{i=1}^{N}(\mathrm{Res}_{z_i}(Y(\omega,z_i)\iota_{z_i}f))_i(\Pi_{i=1}^{N}T(\phi_i)_i^{-1})\,. \tag{3.12}$$

Proof. Put $f_i(z_i)=\iota_{z_i}f$ and $g_i(w_i)=\iota_{w_i}f$. Then $g_i(\phi_i(z_i))\phi_i'(z_i)^{-1}=f_i(z_i)$. This fact together with (3.8) implies that

$$\mathrm{Res}_{w_i}(Y(\omega, w_i)\iota_{w_i}f)=T(\phi_i)\mathrm{Res}_{z_i}(Y(\omega, z_i)\iota_{z_i}f)T(\phi_i)^{-1}+\frac{c}{12}\mathrm{Res}_{z_i}(\iota_{z_i}f\{\phi_i(z_i), z_i\}) .$$

So it is sufficient to prove that

$$\sum_{i=1}^{N} \mathrm{Res}_{z_i}(\iota_{z_i}f\{\phi_i(z_i), z_i\})=0 . \tag{3.13}$$

Let $\{U_\alpha, z_\alpha\}$ and $\{V_\beta, w_\beta\}$ be the projective structure which gives the local coordinates of $\tilde{\Sigma}$ and $\tilde{\Sigma}_w$ respectively, let $w_\beta=\Phi_{\beta\alpha}(z_\alpha)$ be the local coordinate transition function. Considering $\{\Phi_{\beta\alpha}(z_\alpha), z_\alpha\}(dz_\alpha)^2$, which defines a holomorphic quadratic differential on $V_\beta \cap U_\alpha$, recall the pseudogroup property of the Schwarzian derivative (e.g. [G] p. 164): for $\phi(w)$, $\psi(z)$ and $h(z)=\phi(\psi(z))$,

$$\{h(z), z\}=\{\phi(w), w\}\cdot\psi'(z)^2+\{\psi(z), z\} . \tag{3.14}$$

Using this property, it is easy to prove that $\{\Phi_{\beta\alpha}(z_\alpha), z_\alpha\}(dz_\alpha)^2$, as α and β run over the index sets, defines a global holomorphic quadratic differential g on Σ. Then (3.13) is the sum of residues of the meromorphic differential fg, so it is 0. □

Definition 3.2. *For $\tilde{\Sigma}$ as in* (3.3), $f\in\Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^{-1})$. *Choose a projective N-labeled Riemann surface $\tilde{\Sigma}_w$ as* (3.10), *let $z_i=\phi_i(w_i)$ be the local coordinates transition function at point Q_i, $T(\phi_i)$ be the associated operator. The global vertex operator associated to ω and f on $\tilde{\Sigma}$ is the operator*

$$\omega(f, \tilde{\Sigma})=\sum_{i=1}^{N} T(\phi_i)_i^{-1}(\mathrm{Res}_{w_i}(Y(\omega, w_i)\iota_{w_i}f))_i T(\phi_i)_i ,$$

which acts on $W_1\otimes\cdots\otimes W_N$.

The independence of the choice of a projective N-labeled Riemann surface (3.10) follows from Lemma 3.3. And it is clear from Definition 3.2 that the dependence of the global vertex operator associated to ω on the local coordinates is similar to (3.11):

$$\omega(f, \widetilde{\Sigma_w})=(\Pi_{i=1}^{N}T(\phi_i))\omega(f, \tilde{\Sigma})(\Pi_{i=1}^{N}T(\phi_i)^{-1}) . \tag{3.15}$$

We denote the space spanned by the global operators associated to $a\in\mathscr{P}_n(V)$ ($n\in\mathbf{Z}$) and ω on $\tilde{\Sigma}$ by $\mathscr{G}(\tilde{\Sigma})$. Operators in $\mathscr{G}(\tilde{\Sigma})$ act on $\bigotimes_{i=1}^{N} W_i$, so they act on the dual space from the right. The space of vacua is defined by the principle that "sum of the residues of a 1-differential is 0":

Definition 3.3. *For a n-labeled Riemann surface $\tilde{\Sigma}$ as in* (3.2), *we associated $\tilde{\Sigma}$ a linear space $N(\tilde{\Sigma})$ by*

$$N(\tilde{\Sigma})=\{x\in(W_1\otimes\cdots\otimes W_N)^* \mid xA=0 \quad \text{for every } A\in\mathscr{G}(\tilde{\Sigma})\} .$$

We call $N(\tilde{\Sigma})$ the space of vacua associated to $\tilde{\Sigma}$.

The dependence of the space of vacua on local coordinates is as follows. Let $\tilde{\Sigma}'=(\Sigma; Q_1, \ldots, Q_N; z_1', \ldots, z_N'; W_1, \ldots, W_N)$ be another N-labeled Riemann surface, let $z_i'=\phi_i(z_i)$ be the transition function and $T(\phi_i)$ be its associated operator,

$T(\phi_1)\otimes\cdots\otimes T(\phi_N)$ acts from the right on space $(W_1\otimes\cdots\otimes W_N)^*$, then by (3.11) and (3.15), we have $T(\phi_1)\otimes\cdots\otimes T(\phi_N)$ map isomorphically from $N(\tilde{\Sigma}')$ to $N(\tilde{\Sigma})$.

For each $x\in N(\tilde{\Sigma})$, we can define a system of correlation functions associated to x, we first define 1-point correlation functions, the general n-pointed functions will be defined in Theorem 6.2.

Proposition 3.4. *For $x\in N(\tilde{\Sigma})$, $v_1\otimes\cdots\otimes v_N\in W_1\otimes\cdots\otimes W_N$ and $a\in\mathscr{P}_n(V)$, let*

$$g(z_i)=\langle x, v_1\otimes\cdots\otimes Y(a,z_i)v_i\otimes\cdots\otimes v_N\rangle ,$$

then there exist a unique $g\in\Gamma(\Sigma; Q_1,\ldots,Q_N;\kappa^n)$ such that $g(z_i)(dz_i)^n$ is the Laurent series expansion of g at (Q_i, z_i) for each i.

Proof. For every $f\in\Gamma(\Sigma; Q_1,\ldots,Q_N;(\kappa)^{-n+1})$, we have

$$\sum_{i=1}^{N}\operatorname{Res}((\iota_{z_i}f)g(z_i))=\langle xa(f,\tilde{\Sigma}), v_1\otimes\cdots\otimes v_N\rangle=0 .$$

Using Lemma 3.1 proves the lemma. □

The global meromorphic n-differential in Proposition 3.4 is called a 1-point correlation function for $x\in N(\tilde{\Sigma})$, $v_1\otimes\cdots\otimes v_n$ and $a\in\mathscr{P}_n(V)$. In Sect. 6, we define for each $x\in N(\tilde{\Sigma})$, $v_1\otimes\cdots\otimes v_N$ of $W_1\otimes\cdots\otimes W_N$ and $a_i\in\mathscr{P}_{l_i}(V)$ $(i=1,\ldots,n)$, a meromorphic section of the bundle

$$\pi_1^{-1}\kappa^{l_1}\otimes\cdots\otimes\pi_n^{-1}\kappa^{l_n}$$

over Σ^n, where π_i is the projection of Σ^n into the i-th component, $\pi_i^{-1}\kappa^{l_i}$ is the pull back of the line bundle κ^{l_i} under π_i. This meromorphic section is called a n-pointed correlation function.

4. Quasi-Global Vertex Operators

In this section, we define quasi-global vertex operators on a projective labeled Riemann surface which will serve as a necessary technical tool in the proof of Theorem 6.1.

We assume in this section that V is a sum of highest weight representations of the Virasoro algebra and $\dim(V_0)=1$, i.e., every element of V_0 is a multiple of 1. Recall that the *space of quasi-primary fields* $\mathscr{Q}(V)=\sum_{i=0}^{\infty}\mathscr{Q}_n(V)$ is defined as

$$\mathscr{Q}_n(V)=\{a\in V\,|\,L_1a=0 \quad and \quad L_0a=na\} . \tag{4.1}$$

It is clear that $\mathscr{P}_n(V)\subset\mathscr{Q}_n(V)$ and $\omega\in\mathscr{Q}_2(V)$. For $a\in\mathscr{Q}_n(V)$, $Y(a,z)$ transforms as a n-differential under the Mobius transformation. To be more precise, for $\phi(z)=\dfrac{k_1z}{k_2z+k_3}$, we write it as $\phi(z)=\exp\left(l_0z\dfrac{d}{dz}+l_1z^2\dfrac{d}{dz}\right)z$. The associated transition operator is $T(\phi)=\exp(l_0L_0+l_1L_1)$. Using the relation

$$[L_1, Y(a,z)]=\left(2nz+z^2\frac{d}{dz}\right)Y(a,z) ,$$

which is a corollary of the conditions in (4.1), we have

$$T(\phi)Y(a,z)T(\phi)^{-1}=Y(a,\phi(z))(\phi'(z))^n\,. \tag{4.2}$$

Let

$$\tilde{\Sigma}=(\Sigma;Q_1,\ldots,Q_N;z_1,\ldots,z_N;W_1,\ldots,W_N) \tag{4.3}$$

be a projective labeled Riemann surface. For a quasi-primary state $a\in\mathcal{Q}_n(V)$, and a global meromorphic differential $f\in\Gamma(\Sigma,Q_1,\ldots,Q_N,\kappa^{-n+1})$, we define the *quasi-global vertex operator* associated to such a and f on $\tilde{\Sigma}$ as

$$a(f,\tilde{\Sigma})=\sum_{i=1}^{N}(\mathrm{Res}_{z_i}(Y(a,z_i)\iota_{z_i}f))_i\,, \tag{4.4}$$

which acts on $\bigotimes_{i=1}^{N}W_i$. We denote the linear span of the quasi-global operators by $\mathcal{QG}(\tilde{\Sigma})$. It follows from the definitions that $\mathcal{G}(\tilde{\Sigma})\subset\mathcal{QG}(\tilde{\Sigma})$. The main result of this section is the following proposition.

Proposition 4.1. *If* V *is a sum of highest weight representations of the Virasoro algebra and* $\dim(V_0)=1$, $\tilde{\Sigma}$ *as in* (4.3) *is a projective labeled Riemann surface, then* $\mathcal{QG}(\tilde{\Sigma})$ *is a Lie algebra.*

From the formula

$$[\mathrm{Res}_z(Y(a,z)f(z)),\mathrm{Res}_z(Y(b,z)g(z))]=\sum_{l=0}^{\infty}\frac{1}{l!}\mathrm{Res}_z(Y(a(l)b,z)f^{(l)}(z)g(z))\,, \tag{4.5}$$

which is a direct corollary of (2.11), we see that in order to prove $\mathcal{QG}(\tilde{\Sigma})$ is closed under the Lie bracket, we first need to represent each $a(l)b$ $(l\geqq 0)$ as a sum $\sum_{s,t}(L_{-1})^s v_t$ for v_t quasi-primary.

Let a,b be a homogeneous quasi-primary state with degree $|a|>0$, $|b|>0$ respectively. A simple degree argument shows that $a(l)b=0$ for $l>|a|+|b|-1$, the first possible non-zero $a(l)b$ is $a(|a|+|b|-1)b$, which has degree 0 by (2.14), it is a multiple of 1, in particular it is quasi-primary. For $0\leqq n\leqq|a|+|b|-2$, we will define quasi-primary fields $x_{a,b;n}$ and represent $a(l)b$ in terms of $x_{a,b;n}$. For this purpose, we set for integers l,n satisfying $0\leqq l\leqq n\leqq|a|+|b|-2$ the constants

$$C_{a,b;l,n}=\frac{1}{(n-l)!}\,\frac{(-2|a|+n+1)!\,(-2|a|-2|b|+n+l+2)!}{(-2|a|+l+1)!\,(-2|a|-2|b|+n+2)!}\,, \tag{4.6}$$

where $x!=\Gamma(x+1)$, $\Gamma(x)$ is the usual Gamma function. (Note that since $\Gamma(x+1)=x\Gamma(x)$, the expression $\dfrac{\Gamma(x+k)}{\Gamma(x)}$ for k a non-negative integer makes sense.) With the above notation, $x_{a,b;n}$ $(0\leqq n\leqq|a|+|b|-2)$ are defined by the following linear equations:

$$a(l)b=\sum_{n=l}^{|a|+|b|-2}C_{a,b;l,n}(L_{-1})^{n-l}x_{a,b;n},\quad 0\leqq l\leqq|a|+|b|-2\,. \tag{4.7}$$

Since $a(l)b$ $(0\leqq l\leqq|a|+|b|-2)$ and $x_{a,b;l}$ $(0\leqq l\leqq|a|+|b|-2)$ are related by a triangular matrix with diagonals $C_{a,b;l,l}\neq 0$, so $x_{a,b;l}$ $(0\leqq l\leqq|a|+|b|-2)$ are uniquely fixed.

Lemma 4.2. *Under the assumption in Proposition* 4.1, $x_{a,b;l}\in\mathcal{Q}_{|a|+|b|-l-1}(V)$.

Proof. It is clear that $x_{a,b;l}$ has degree $(|a|+|b|-l-1)$. The assumption that V is a sum of highest weight representations of the Virasoro algebra implies that every element in V with degree 1 is primary, in particular, $x_{a,b;|a|+|b|-2}$ is quasi-primary. Assume $x_{a,b;n}\in\mathscr{Q}_{|a|+|b|-n-l}(V)$ for $n\geqq l$, we want to prove based on this assumption $x_{a,b;l-1}\in\mathscr{Q}_{|a|+|b|-l-2}(V)$,

$$\begin{aligned}
&L_1 C_{a,b;l-1,l-1}x_{a,b;l-1}\\
&=L_1\left(a(l-1)b-\sum_{n=l}^{|a|+|b|-2}C_{a,b;l-1,n}(L_{-1})^{n-l+1}x_{a,b;n}\right)\\
&=(2|a|-l-1)a(l)b-\sum_{n=l}^{|a|+|b|-2}(n-l+1)\\
&\quad\times(2|a|+2|b|-n-l-2)C_{a,b;l-1,n}(L_{-1})^{n-l}x_{a,b;n}\\
&=\sum_{n=l}^{|a|+|b|-2}((2|a|-l-1)C_{a,b;l,n}-(n-l+1)\\
&\quad\times(2|a|+2|b|-n-l-2)C_{a,b;l-1,n})(L_{-1})^{n-l}x_{a,b;n}\,.
\end{aligned}$$

It is easy to see from (4.6) that

$$(2|a|-l-1)C_{a,b;l,n}-(n-l+1)(2|a|+2|b|-n-l-2)C_{a,b;l-1,n}=0\,,$$

so $L_1x_{a,b;l-1}=0$, as was to be shown. □

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. If one of a and b has degree 0, then by our assumption it is a multiple of 1, then the Lie bracket of quasi-global operators $a(f,\tilde{\Sigma})$ and $b(g,\tilde{\Sigma})$ are 0. For homogeneous quasi-primary fields a, b with degrees $|a|\geqq 1$, $|b|\geqq 1$ and Laurent series $f(z)$, $g(z)$, write $L=|a|+|b|-1$, we have

$$\begin{aligned}
&[\operatorname{Res}(Y(a,z)f(z)),\operatorname{Res}(Y(b,z)g(z))]\\
&=\sum_{l=0}^{|a|+|b|-1}\frac{1}{l!}\operatorname{Res}(Y(a(l)b,z)f^{(l)}(z)g(z))\\
&=\sum_{l=0}^{|a|+|b|-2}\frac{1}{l!}\operatorname{Res}(Y(a(l)b,z)f^{(l)}(z)g(z))+\frac{1}{L!}\operatorname{Res}(Y(a(L)b,z)f^{(L)}(z)g(z))\\
&=\sum_{l=0}^{|a|+|b|-2}\sum_{n=l}^{|a|+|b|-2}\frac{C_{a,b;l,n}}{l!}\operatorname{Res}(Y(L_{-1}^{n-l}x_{a,b;n},z)f^{(l)}(z)g(z))\\
&\quad+\frac{1}{L!}\operatorname{Res}(Y(a(L)b,z)f^{(L)}(z)g(z))\\
&=\sum_{l=0}^{|a|+|b|-2}\sum_{n=l}^{|a|+|b|-2}\frac{(-1)^{n-l}C_{a,b;l,n}}{l!}\operatorname{Res}(Y(x_{a,b;n}z)(f^{(l)}(z)g(z))^{n-l})\\
&\quad+\frac{1}{L!}\operatorname{Res}(Y(a(L)b,z)f^{(L)}(z)g(z))\\
&=\sum_{n=0}^{|a|+|b|-2}\operatorname{Res}(Y(x_{a,b;n},z)G_{a,b;f,g;n}(z))\\
&\quad+\frac{1}{L!}\operatorname{Res}(Y(a(L)b,z)f^{(L)}(z)g(z))\,,
\end{aligned}\tag{4.8}$$

where we have set for $0 \leqq n \leqq |a|+|b|-2$,

$$G_{a, b; f, g; n}(z)=\sum_{l=0}^{n} \frac{(-1)^{n-l}}{l!} C_{a, b; l, n}(f^{(l)}(z) g(z))^{n-l} . \tag{4.9}$$

For $f \in \Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^{-|a|+1})$ and $g \in \Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^{-|b|+1})$, we need to prove that

$$[a(f, \tilde{\Sigma}), b(g, \tilde{\Sigma})] \in \mathscr{Q}\mathscr{G}(\tilde{\Sigma}) .$$

Write $\iota_{z_i} f=f_i(z_i)$ and $\iota_{z_i} g=g_i(z_i)$. By (4.8), we have

$$\begin{aligned}
&[\operatorname{Res}_{z_i}(Y(a, z_i) f_i(z_i)), \operatorname{Res}_{z_i}(Y(b, z_i) g_i(z_i))] \\
&= \sum_{n=0}^{|a|+|b|-2} \operatorname{Res}_{z_i}(Y(x_{a, b; n}, z_i) G_{a, b; f_i, g_i, n}(z_i)) \\
&\quad + \frac{1}{L!} \operatorname{Res}(Y(a(L) b, z_i) f_i^{(L)}(z_i) g_i(z_i)) .
\end{aligned} \tag{4.10}$$

It suffices to prove the following two claims.

Claim 1. For each $0 \leqq n \leqq |a|+|b|-2$, $G_{a, b; f_i, g_i, n}(z_i)$ $(i=1, \ldots, N)$ are Laurent series expansions of $F_n(f, g) \in \Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^{n-|a|-|b|+2})$ at points Q_i under z_i respectively.

Claim 2.

$$\sum_{i=1}^{N} \operatorname{Res}(Y(a(L) b, z_i) f_i^{(L)}(z_i) g_i(z_i))_i=0 .$$

To prove Claim 1, let $\{U_\alpha, z_\alpha\}$ be the projective structure of $\tilde{\Sigma}$, for f and g as above, write $f=f_\alpha(z_\alpha)(dz_\alpha)^{-|a|+1}$, $g=g_\alpha(z_\alpha)(dz_\alpha)^{-|b|+1}$ on U_α. We define a meromorphic differential on U_α by

$$F_{n, U_\alpha}=G_{a, b; f_\alpha, g_\alpha, n}(z_\alpha)(dz_\alpha)^{n-|a|-|b|+2} . \tag{4.11}$$

Using Lemma 4.3 below ($k_1=-2|a|+2$, $k_2=-2|b|+2$), one can check tat $F_{n, U_{\alpha_1}}= F_{n, U_{\alpha_2}}$ on $U_{\alpha_1} \cap U_{\alpha_2}$. So (4.11) defines a global meromorphic differential $F_n(f, g)$ in $\Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^{n-|a|-|b|+2})$. This proves Claim 1.

To prove Claim 2, we first note that $a(L)b$ (recall that $L=|a|+|b|-1$) is a multiple of 1, and it can be proved that $a(L)b=0$ unless $|a|=|b|$. So it suffices to prove that when $|a|=|b|$,

$$\sum_{i=1}^{N} \operatorname{Res}_{z_i}(f_i^{(L)}(z_i) g_i(z_i))=0 . \tag{4.12}$$

Again we will use Lemma 4.3 below. We define a meromorphic differential on U_α:

$$F_{L, U_\alpha}(f, g)=\sum_{l=0}^{L}(-1)^l \binom{L}{l} \frac{(-4|a|+2L-l+2)!}{(-2|a|+L-l+1)!} \partial_{z_\alpha}^l (g_\alpha(z_\alpha) \times \partial_{z_\alpha}^{L-l} f_\alpha(z_\alpha)) \, dz_\alpha .$$

Using Lemma 4.3 ($k_1=k_2=-2|a|+2$), one checks that $F_{L, U_{\alpha_1}}(f, g)=F_{L, U_{\alpha_2}}(f, g)$ on $U_{\alpha_1} \cap U_{\alpha_2}$, so we have a global differential $F_L(f, g) \in \Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa)$. It is easy to see that (4.12) reduces to the fact that the sum of residues of $F_L(f, g)$ is 0. □

Lemma 4.3 used above is a modified version of Theorem 7.1 in [Co], which is used in [Co] to prove that certain bilinear forms in the derivatives of modular forms are modular forms.

Lemma 4.3 [Co]. *Let $f_1(z), f_2(z)$ be two meromorphic functions on an open set of* **C**. *For given real numbers k_1, k_2, set:*

$$F_n(f_1, f_2) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{(k_1+n-1)!\,(k_2+n-1)!}{(k_1+n-l-1)!\,(k_2+l-1)!} \partial_z^{n-l} f_1 \times \partial_z^l f_2 ,$$

where $x! = \Gamma(x+1)$ and $\partial_z = \frac{d}{dz}$. Then

(a) *For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{C})$ we have*

$$F_n(f_1|_{k_1}\gamma, f_2|_{k_2}\gamma) = F_n(f_1, f_2)|_{k_1+k_2+2n}\gamma ,$$

where $(f|_k\gamma)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$.

(b) *One has the identity:*

$$F_n(f_1, f_2) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{(k_1+n-1)!\,(k_1+k_2+2n-l-2)!}{(k_1+n-l-1)!\,(k_1+k_2+n-2)!} \partial_z^l (f_2 \times \partial_z^{n-l} f_1) .$$

Though the statement of Lemma 4.3 is not the same as Theorem 7.1 in [Co], the proofs are essentially the same.

5. Space of Vacua on Projective Labeled Riemann Surfaces

The purpose of this section is to prove that the space of vacua on a projective labeled Riemann surface is annihilated by quasi-global vertex operators. Only the statement of the following theorem is used later.

Theorem 5.1. *If $\tilde{\Sigma} = (\Sigma; Q_1, \ldots, Q_N; z_1, \ldots, z_N; W_1, \ldots, W_N)$ is a projective labeled Riemann surface, $x \in N(\tilde{\Sigma})$, b is a highest weight vector for the Virasoro algebra with degree $|b|$, $L(b)$ is the highest weight representation of the Virasoro algebra generated by b, $a \in \mathcal{Q}_n(V) \cap L(b)$ and $f \in \Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^{-n+1})$, then $xa(f, \tilde{\Sigma}) = 0$.*

The essential reason for $xa(f, \tilde{\Sigma}) = 0$ is that $a(f, \tilde{\Sigma})$ is generated by the global vertex operators associated to b and ω in a certain way. However we did not succeed in proving there exists an expression for $a(f, \tilde{\Sigma})$ in terms of the global vertex operators associated to b and ω. A part of the following indirect proof is in the same spirit as the proof of the Proposition 2.2.3 of [TUY].

We need to fix some notations used in the proof. Let $\{U_\alpha, z_\alpha\}$ be the projective structure of Σ which gives the local coordinates in $\tilde{\Sigma}$. Let $|b|$ be the degree of b. Let $\mathcal{U}(b)$ be the Verma module of the Virasoro algebra with a highest weight vector $\bar{b}$ such that $L_0 \bar{b} = |b| \bar{b}$, and $\mathcal{U}(1)$ be the Verma module of the Virasoro algebra with a highest weight vector $\bar{1}$ such that $L_0 \bar{1} = 0$. There are obvious morphisms of modules of the Virasoro algebra:

$$p\colon \mathscr{U}(b)\to V\colon L_{-i_1}\dots L_{-i_m}\bar{b}\mapsto L_{-i_1}\dots L_{-i_m}b\,,$$

$$p\colon \mathscr{U}(1)\to V\colon L_{-i_1}\dots L_{-i_m}\bar{1}\mapsto L_{-i_1}\dots L_{-i_m}1\,.$$

We write $\bar{\omega}=L_{-2}\bar{1}$; it is clear that $p(\bar{\omega})=\omega$. For each positive integer k and k points $Q_{N+1},\dots,Q_{N+k}$ on Σ such that $Q_i\neq Q_j$ when $i\neq j$ $(i,j=1,\dots,N+k)$, and we choose a open set $U_{\alpha_i}\ni Q_i$ for each $i=N+1,\dots,N+k$ and take $z_i=z_{\alpha_i}-z_{\alpha_i}(Q_i)$ as a local coordinate at point Q_i. So we have a projective $(M+N)$-pointed Riemann surface

$$(\Sigma;Q_1,\dots,Q_{M+N};z_1,\dots,z_{N+k})\,. \tag{5.1}$$

We assign the Verma module $\mathscr{U}(b)$ or $\mathscr{U}(1)$ at the point Q_{N+1}, and assign the Verma module $\mathscr{U}(1)$ at points $Q_{N+2},\dots,Q_{N+k}$. So we have the data

$$(\Sigma;Q_1,\dots,Q_{N+k};z_1,\dots,z_{N+k};W_1,\dots,W_N,\mathscr{U}(b),\mathscr{U}(1),\dots,\mathscr{U}(1))\,, \tag{5.2}$$

$$(\Sigma;Q_1,\dots,Q_{N+k};z_1,\dots,z_{N+k};W_1,\dots,W_N,\mathscr{U}(1),\mathscr{U}(1),\dots,\mathscr{U}(1))\,. \tag{5.3}$$

Set

$$B_k=W_1\otimes\dots\otimes W_N\otimes\mathscr{U}(b)\otimes\mathscr{U}(1)^{\otimes(k-1)}\,,$$

$$A_k=W_1\otimes\dots\otimes W_N\otimes\mathscr{U}(1)^{\otimes k}\,.$$

It is clear that the operators associated to coordinate transformations $T(\phi)$ as defined in (3.6) and operators $\mathrm{Res}_z(Y(\omega,z)f(z))$ for $Y(\omega,z)=\sum_{i=-\infty}^{\infty}L_iz^{-i-2}$ and $f(z)$ a Laurent series act on $\mathscr{U}(b)$ and $\mathscr{U}(1)$. For $f\in\Gamma(\Sigma;Q_1,\dots,Q_{N+k};\kappa^{-1})$, we write $\omega(f,z_i)=\mathrm{Res}_{z_i}(Y(\omega,z_i)\iota_{z_i}f)$. For $f,g\in\Gamma(\Sigma;Q_1,\dots,Q_{N+k};\kappa^{-1})$, write $f_i(z_i)=\iota_{z_i}f$ and $g_i(z_i)=\iota_{z_i}g$, we have the commutation relation

$$\sum_{i=1}^{N+k}[\omega(f,z_i)_i,\omega(g,z_i)_i]=\sum_{i=1}^{N+k}\mathrm{Res}_{z_i}(Y(\omega,z_i)(f_i'(z_i)g_i(z_i)-f_i(z_i)g_i'(z_i)))_i\,. \tag{5.4}$$

Note that $f_i'(z_i)g_i(z_i)-f_i(z_i)g_i'(z_i)$ $(i=1,\dots,N+k))$ is the Laurent series expansion of $-[f,g]\in\Gamma(\Sigma;Q_1,\dots,Q_k;\kappa^{-1})$ at point (Q_i,z_i). For $f\in\Gamma(\Sigma;Q_1,\dots,Q_{N+k};\kappa^{-1})$ and $g\in\Gamma(\Sigma;Q_1,\dots,Q_{N+k};\kappa^{-|b|-1})$, write $f_i(z_i)=\iota_{z_i}f$ and $g_i(z_i)=\iota_{z_i}g$, we have the commutation relation

$$\sum_{i=1}^{N+k}[\omega(f,z_i)_i,b(g,z_i)_i]=\sum_{i=1}^{N+k}\mathrm{Res}_{z_i}(Y(b,z_i)((|b|-1)f_i'(z_i)g_i(z_i)-f_i(z_i)g_i'(z_i)))_i\,, \tag{5.5}$$

where $b(g,z_i)$ denotes $\mathrm{Res}_{z_i}(Y(b,z_i)\iota_{z_i}g)$. Note that $(|b|-1)f_i'(z_i)g_i(z_i)-f_i(z_i)g_i'(z_i)$ $(i=1,\dots,N+k))$ is the Laurent series expansion of the Lie derivative of $-g$ with respect to a meromorphic vector field f at points (Q_i,z_i).

Proof of Theorem 5.1. We divide the proof into several steps.

Step 1. We will first construct vectors

$$y(Q_{N+1},\dots,Q_{N+k};z_{N+1},\dots,z_{N+k})\in B_k^*$$
$$=(W_1\otimes\dots\otimes W_N\otimes\mathscr{U}(b)\otimes\mathscr{U}(1)^{\otimes(k-1)})^*\,,$$

$$x(Q_{N+1},\dots,Q_{N+k};z_{N+1},\dots,z_{N+k})\in A_k^*=(W_1\otimes\dots\otimes W_N\otimes\mathscr{U}(1)^{\otimes k})^*\,,$$

based on x such that the following properties are satisfied for the sequence $\{y_k\}$ $(k=1,2,\ldots)$:

(1) For

$$f\in\Gamma(\Sigma;Q_1,\ldots,Q_{N+k};\kappa^{-1}),$$

$$\sum_{i=1}^{N+k} y(Q_{N+1},\ldots,Q_{N+k};z_{N+1},\ldots,z_{N+k})\omega(f,z_i)_i=0 .$$

(2) For $v'\in W_1\otimes\cdots\otimes W_N\otimes\mathscr{U}(b)\otimes\mathscr{U}(1)^{j-2}$, $v''\in\mathscr{U}(1)^{k-j}$ (so $v'\otimes\bar{1}\otimes v''\in B_k$ and $v'\otimes v''\in B_{k-1}$), we have

$$\begin{aligned}&\langle y(Q_{N+1},\ldots,Q_{N+k};z_{N+1},\ldots,z_{N+k}),v'\otimes\bar{1}\otimes v''\rangle\\ &=\langle y(Q_{N+1},\ldots,\widehat{Q_{N+j}},\ldots,Q_{N+k};z_{N+1},\ldots,\widehat{z_{N+j}},\ldots,z_{N+k}),v'\otimes v''\rangle ,\end{aligned}$$

where $\widehat{\ }$ indicates the missing terms.
(3) For v' and v'' as in (2), we have

$$\langle y(Q_{N+1},\ldots,Q_{N+k};z_{N+1},\ldots,z_{N+k}),v'\otimes uL_{-1}\bar{1}\otimes v''\rangle=0 ,$$

where u is in the universal enveloping algebra of the Virasoro algebra.
(4) For $v\in W_1\otimes\cdots\otimes W_N$, $f\in\Gamma(\Sigma;Q_1,\ldots,Q_{N+1};\kappa^{-|b|+1})$ such that

$$f|_{z_{N+1}}=z_{N+1}^{-1}+\text{higher terms} ,$$

we have

$$\langle y(Q_{N+1},z_{N+1}),v\otimes\bar{b}\rangle=-\sum_{i=1}^{N}\langle x,b(f,z_i)_iv\rangle .$$

Similarly, the sequence $\{x_k\}$ $\{k=1,2,\ldots\}$ satisfies the properties:

(1′) For

$$f\in\Gamma(\Sigma;Q_1,\ldots,Q_{N+k};\kappa^{-1}),$$

$$\sum_{i=1}^{N+k} x(Q_{N+1},\ldots,Q_{N+k};z_{N+1},\ldots,z_{N+k})\omega(f,z_i)_i=0 .$$

(2′) For $v'\in W_1\otimes\cdots\otimes W_N\otimes\mathscr{U}(1)^{j-1}$, $v''\in\mathscr{U}(1)^{k-j}$ (so $v'\otimes\bar{1}\otimes v''\in A_k$ and $v'\otimes v''\in A_{k-1}$), we have

$$\begin{aligned}&\langle x(Q_{N+1},\ldots,Q_{N+k};z_{N+1},\ldots,z_{N+k}),v'\otimes\bar{1}\otimes v''\rangle\\ &=\langle x(Q_{N+1},\ldots,\widehat{Q_{N+j}},\ldots,Q_{N+k};z_{N+1},\ldots,\widehat{z_{N+j}},\ldots,z_{N+k}),v'\otimes v''\rangle ,\end{aligned}$$

where $\widehat{\ }$ indicates the missing terms as before.
(3′) For v' and v'' as in (2′), we have

$$\langle x(Q_{N+1},\ldots,Q_{N+k};z_{N+1},\ldots,z_{N+k}),v'\otimes L_{-i_1}\ldots L_{-i_m}L_{-1}\bar{1}\otimes v''\rangle=0 .$$

(4′)

$$\langle x(Q_{N+1},z_{N+1}),v\otimes\bar{1}\rangle=\langle x,v\rangle .$$

$y(Q_{N+1},\ldots,Q_{N+k};z_{N+1},\ldots,z_{N+k})$ and $x(Q_{N+1},\ldots,Q_{N+k};z_{N+1},\ldots,z_{N+k})$ will depend on the data $(Q_{N+1},\ldots,Q_{N+k};z_{N+1},\ldots,z_{N+k})$, when no confusion will arise, we write them as y_k and x_k respectively.

Let $\bar{\mathscr{U}}(b)$ be the vector space with basis of formal symbols $\bar{L}_{-i_1}\dots\bar{L}_{-i_m}\bar{b}$ $(-i_j<0)$, $\bar{\mathscr{U}}(b)$ has two gradations given by

$$\deg_1(\bar{L}_{-i_1}\dots\bar{L}_{-i_m}\bar{b})=\sum_{j=1}^{m} i_j\,,$$

$$\deg_2(\bar{L}_{-i_1}\dots\bar{L}_{-i_m}\bar{b})=m\,.$$

There is an obvious surjective map from $\bar{\mathscr{U}}(b)$ to $\mathscr{U}(b)$.

We will construct $\{y_k\}$ by induction on k. To construct y_1, we first construct $\bar{y}_1\in(W_1\otimes\dots\otimes W_N\otimes\bar{\mathscr{U}}(b))^*$, then prove that $\bar{y}_1$ reduces to $y_1\in(W_1\otimes\dots\otimes W_N\otimes\mathscr{U}(b))^*$. Constructing $\bar{y}_1$ is equivalent to defining the numbers

$$\langle\bar{y}_1, v\otimes\bar{L}_{-i_1}\dots\bar{L}_{-i_k}\bar{b}\rangle\,,$$

we do it inductively on $\deg_2$. Choose $f\in\Gamma(\Sigma; Q_1,\dots,Q_{N+1};\kappa^{-|b|+1})$ such that

$$\iota_{z_{N+1}}f=z_{N+1}^{-1}+\text{higher terms}\,,$$

we define

$$\langle\bar{y}_1, v\otimes\bar{b}\rangle=-\sum_{i=1}^{N}\langle x, b(f,z_i)_i v\rangle\,. \tag{5.6}$$

(Note that this definition is forced by Property (4).) It follows from the conditions satisfied by x that (5.6) is independent of the choice of f. And using (5.5), we can prove that

$$\sum_{i=1}^{N}\langle\bar{y}_1,\omega(g,z_i)_i v\otimes\bar{b}\rangle=0 \tag{5.7}$$

for every $g\in\Gamma(\Sigma, Q_1,\dots,Q_N;\kappa^{-1})$ satisfying

$$\iota_{z_{N+1}}g=\text{sum of terms higher than } z_{N+1}^{1}\,.$$

Assume we have defined $\langle\bar{y}_1, v\otimes b'\rangle$ for every v and every $b'\in\bar{\mathscr{U}}(b)$ with $\deg_2 b'\leqq k$, and the property

$$\sum_{i=1}^{N}\langle\bar{y}_1,\omega(g,z_i)_i v\otimes b'\rangle=0 \tag{5.8}$$

for every $g\in\Gamma(\Sigma; Q_1,\dots,Q_N,\kappa^{-1})$ satisfying

$$\iota_{z_{N+1}}g=\text{sum of terms higher than } z_{N+1}^{1+\deg_1 b'}$$

holds. Based on this induction assumption, we define $\langle\bar{y}_1, v\otimes L_{-i}b'\rangle$ as follows. Choose $f\in\Gamma(\Sigma; Q_1,\dots,Q_{N+1};\kappa^{-1})$ such that

$$\iota_{z_{N+1}}f=z_{N+1}^{-i+1}+\text{terms higher than } z_{N+1}^{1+\deg_1 b'}\,,$$

we define

$$\langle\bar{y}_1, v\otimes L_{-i}b'\rangle=-\sum_{i=1}^{N}\langle\bar{y}_1,\omega(f,z_i)_i v\otimes b'\rangle\,,$$

it follows from (5.8) that it is independent of the choice of f. And it can be proved that property (5.8) is again satisfied. So we have completed the construction of $\bar{y}_1$.

By a direct computation using (5.4), we can prove that

$$\langle \bar{y}_1, v \otimes \bar{L}_{-m}\bar{L}_{-n}b' \rangle - \langle \bar{y}_1, v \otimes \bar{L}_{-n}\bar{L}_{-m}b' \rangle = (n-m)\langle \bar{y}_1, v \otimes \bar{L}_{-(m+n)}b' \rangle .$$

So $\bar{y}_1$ reduces to a vector $y_1 \in (W_1 \otimes \cdots \otimes W_N \otimes \mathcal{U}(b))^*$, and with some effort we can prove that y_1 satisfies the condition (1) for $k=1$. By the same method as constructing y_1 based on x, we can construct y_{k+1} based on y_k. For example, $\langle y_{k+1}, v \otimes \bar{1} \rangle$ for $v \in W_1 \otimes \cdots \otimes W_N \otimes \mathcal{U}(b) \otimes (\mathcal{U}(1))^{k-1}$ is defined to be $\langle y_k, v \rangle$. Then it can be proved that $y_k = y(Q_{N+1}, \ldots, Q_{N+k}; z_{N+1}, \ldots, z_{N+k})$ satisfies the properties (1), (2), (3), (4). It is not hard to see that $\{y_k\}$ satisfying (1)–(4) are unique. Taking $b=1$, we then get the sequence $\{x_k\}$ $(k=1, \ldots)$ $x_k = y(Q_{N+1}, \ldots, Q_{N+k}; z_{N+1}, \ldots, z_{N+k})$ satisfies (1′)–(4′) above, and they are characterized by (1′)–(4′).

Step 2. We prove the following claim in Step 2.

Claim A. If (U, z) is a coordinate chart in $\{U_\alpha, z_\alpha\}$, and $Q_{N+1}, \ldots, Q_{N+M}$ are in the domain $\{|z|<\varepsilon\} \subset U$ while $Q_1, \ldots, Q_N$ are outside $\{|z|<\varepsilon\}$, let $\xi_{N+i} = z(Q_{N+i})$ the z-coordinates of the points Q_{N+i}, take $z_{N+i} = z - \xi_{N+i}$ as the local coordinates at the point Q_{N+i}, then for $v \in W_1 \otimes \cdots \otimes W_N$ and $\bar{a} \in \mathcal{U}(b)$, the function

$$\begin{aligned} &Y(\xi_{N+1}, \ldots, \xi_{N+M}) \\ &\quad = \langle y(Q_1, \ldots, Q_{N+M}; z_{N+1}, \ldots, z_{N+M}), v \otimes \bar{a} \otimes (\bar{\omega})^{\otimes (M-1)} \rangle \end{aligned} \tag{5.9}$$

and

$$X(\xi_{N+1}, \ldots, \xi_{N+M}) = \langle y(Q_1, \ldots, Q_{N+M}; z_{N+1}, \ldots, z_{N+M}), v \otimes (\bar{\omega})^{\otimes M} \rangle$$

are meromorphic functions on $\{|\xi_{N+i}|<\varepsilon;\ (i=1, \ldots, M)\}$ with singularities at $\xi_{N+i} = \xi_{N+i}$ $(i \neq j)$. And for fixed $\xi_{N+1}, \ldots, \xi_{N+M-1}$, $Y(\xi_{N+1}, \ldots, \xi_{N+M})$ has the Laurent series expansion

$$\langle y_{M-1}, Y(\omega, \xi_{N+M} - \xi_{N+1})_{N+1} v \otimes \bar{a} \otimes (\bar{\omega})^{\otimes (M-2)} \rangle \tag{5.10}$$

for the variable ξ_{N+M} at the ξ_{N+1}. (When we want to emphasize v and $\bar{a}$ in (5.9), we write the left-hand side of (5.9) as $Y(v, a; \xi_{N+1}, \ldots, \xi_{N+M})$.)

Proof of Claim A. By Hartog's Theorem, to prove $Y(\xi_{N+1}, \ldots, \xi_{N+M})$ is meromorphic, it is sufficient to prove that for each k, $Y(\xi_{N+1}, \ldots, \xi_{N+M})$ is meromorphic for ξ_k when the rest of ξ_i are fixed. To prove that $Y(\xi_{N+1}, \ldots, \xi_{N+M})$ is meromorphic with respect to ξ_{N+M}, we consider the Laurent series

$$g_i(z_i) = \langle y_{M-1}, Y(\omega, z_i)_i v \otimes \bar{a} \otimes \bar{\omega}^{\otimes (M-2)} \rangle$$

$(i=1, \ldots, N+M-1)$, by the condition (1), g_i's satisfy the condition of the second part of Lemma 3.1, so there exists a meromorphic 2-differential $g \in \Gamma(\Sigma, Q_1, \ldots, Q_{N+M-1}; \kappa^2)$, such that $g_i(z_i) = \iota_{z_i} g$ $(i=1, \ldots, N+M-1)$. Write $g = g(z)(dz)^{-1}$ on $\{|z|<\varepsilon\} \subset U$, so $g(z)$ is a meromorphic function on z with poles at $\xi_{N+1}, \ldots, \xi_{N+M-1}$, we want to prove that

$$g(\xi_{N+M}) = Y(\xi_1, \ldots, \xi_{N+M-1}, \xi_{N+M}) . \tag{5.11}$$

For this purpose, we choose $f \in \Gamma(\Sigma, Q_1, \ldots, Q_{N+M}; \kappa^{-1})$ such that

$$f|_{z_{N+M}} = z_{N+M}^{-1} + \text{higher terms} .$$

Then we have

$$g(\xi_{N+M})=\mathrm{Res}_{Q_{N+M}}(gf)=-\sum_{i=1}^{N+M-1}\mathrm{Res}_{Q_i}(gf)=-\sum_{i=1}^{N+M-1}\mathrm{Res}_{z_i}(g_i(z_i)\iota_{z_i}f)$$
$$=-\sum_{i=1}^{N+M-1}\langle y_{M-1},\omega(f,z_i)_i v\otimes\bar{a}\otimes\bar{\omega}^{\otimes(M-2)}\rangle$$
$$=\langle y_M, v\otimes\bar{a}\otimes(\bar{\omega})^{\otimes(M-1)}\rangle$$
$$=Y(\xi_{N+1},\ldots,\xi_{N+M}).$$

This proves (5.11). So $Y(\xi_{N+1},\ldots,\xi_{N+M})$ is meromorphic for ξ_{N+M} with poles at points ξ_{N+i} $(i\neq M)$ and has Laurent series expansion (5.10). The same argument proves the same statement for the variables $\xi_{N+2},\ldots,\xi_{N+M}$. It remains to prove that $Y(\xi_{N+1},\ldots,\xi_{N+M})$ is meromorphic with respect to ξ_{N+1}. For $\bar{a}=\bar{b}$ and $M=1$, considering Laurent series

$$g_i(z_i)=\langle x, Y(b,z_i)_i v\rangle \quad i=1,\ldots,N,$$

then by Proposition 3.4, there exists a $g\in\Gamma(\Sigma,Q_1,\ldots,Q_N;\kappa^{|b|})$ such that $\iota_{z_i}g=g_i(z_i)$. Write g as $g=g(z)(dz)^{|b|}$ on $\{|z|<\varepsilon\}\subset U$. By the similar argument as in the proof of (5.11), one can prove that $g(\xi_{N+1})=Y(\xi_{N+1})$. This sets down the case $\bar{a}=\bar{b}$ and $M=1$. For the case $\bar{a}=\bar{b}$ and $M=2$, for fixed Q_{N+2}, choose a $f\in\Gamma(\Sigma;Q_1,\ldots,Q_N,Q_{N+2};\kappa^{-1})$ such that

$$\iota_{z_{N+2}}f=z_{N+2}^{-1}+\text{higher terms},$$

then we have

$$Y(\xi_{N+1},\xi_{N+2})=\langle y(Q_{N+1},Q_{N+2};z_{N+1},z_{N+2}),\bar{v}\otimes\bar{b}\otimes\bar{\omega}\rangle$$
$$=-\sum_{i=1}^{N+1}\langle y(Q_{N+1},z_{N+1}),\omega(f,z_i)_i v\otimes\bar{b}\rangle$$
$$=-\sum_{i=1}^{N}\langle y(Q_{N+1},z_{N+1}),\omega(f,z_i)_i v\otimes\bar{b}\rangle$$
$$-\langle y(Q_{N+1},z_{N+1}),\omega(f,z_{N+1})_{N+1}v\otimes\bar{b}\rangle. \tag{5.12}$$

The first term of the right-hand side of (5.12) is meromorphic for ξ_{N+1} by the case $M=1$. For the second term of the right-hand side of (5.12), if f has the expansion

$$\iota_{z_{N+1}}f=\sum_{i=0}^{\infty}l_i z_{N+1}^i,$$

then

$$\omega(f,z_{N+1})=\sum_{i=0}^{\infty}l_i L_{i-1}.$$

Thus the second term of the right-hand side of (5.12) is

$$\langle y(Q_{N+1},z_{N+1}),l_0 v\otimes L_{-1}\bar{b}+l_1 v\otimes L_0\bar{b}\rangle.$$

Using Property (1), it is easy to prove that it can be written as

$$\sum_{i=0}^{1} l_i \langle y(Q_{N+1}, z_{N+1}), v_i \otimes \bar{b} \rangle , \tag{5.13}$$

for some v_i's in $W_1 \otimes \cdots \otimes W_N$. By the case $M=1$ and since l_i depends on ξ_{N+1} meromorphically, so (5.13) is meromorphic for ξ_{N+1}. This proves that for $\bar{a}=\bar{b}$ and $M=2$, $Y(\xi_{N+1}, \xi_{N+2})$ is meromorphic for ξ_{N+1}. The argument generalizes to the cases $\bar{a}=\bar{b}$ and $M \geqq 3$. Thus we have proved that Claim A for $\bar{a}=\bar{b}$. Assume that Claim A is true for $\bar{a}=\bar{a}_1$, based on this assumption, we want to prove that Claim A is true for $\bar{a}=L_{-k}\bar{a}_1$. By (5.10) in the induction assumption, we have

$$\begin{aligned} &Y(v, \bar{a}; \xi_{N+1}, \ldots, \xi_{N+M}) \\ &\quad = \oint_C Y(v, \bar{a}_1; \xi_{N+1}, \ldots, \xi_{N+M+1})(\xi_{N+M+1}-\xi_{N+1})^{-i+1} d\xi_{N+M+1} , \end{aligned} \tag{5.14}$$

where C is a contour of ξ_{N+M} surrounding ξ_{N+1}. Since $Y(\bar{a}_1; \xi_{N+1}, \ldots, \xi_{N+M+1})$ is a meromorphic function for the variables ξ_{N+i} $(i=1, \ldots, N+M+1)$, (5.14) implies that $Y(\bar{a}; \xi_{N+1}, \ldots, \xi_{N+M})$ is a meromorphic function for the variable ξ_{N+1}. So we have proved the assertions about $Y(\xi_{N+1}, \ldots, \xi_{N+M})$ in Claim A. The assertions about $X(\xi_{N+1}, \ldots, \xi_{N+M})$ is proved by setting $\bar{b}=1$. This completes the proof of Claim A.

Step 3. We prove Claim B in this step.

Claim B. Let (U, z) be a coordinate chart in $\{U_\alpha, z_\alpha\}$ such that the domain $\{|z|<\varepsilon\} \subset U$ contains the points $Q_k, Q_{N+1}, \ldots, Q_{N+M}$ while Q_i $(1 \leqq i \leqq N, i \neq k)$ are outside $\{|z|<\varepsilon\}$, and the local coordinate of Q_k is given by $z_k = z - z(Q_k) = z - \xi_k$. Write $\xi_{M+i} = z(Q_{M+i})$, take $z_{N+i} = z - \xi_{N+i}$ as the local coordinates at the point Q_{N+i}. For $v \in W_1 \otimes \cdots \otimes W_N$ and $\bar{a} \in \mathscr{U}(b)$, let $Y(\xi_{N+1}, \ldots, \xi_{N+M})$ be the function defined as

$$\begin{aligned} Y(\xi_{N+1}, \ldots, \xi_{N+M}) &= \langle y_M, v \otimes \bar{a} \otimes \bar{\omega}^{\otimes(M-1)} \rangle . \\ &= \langle Y(Q_1, \ldots, Q_{N+M}; z_{N+1}, \ldots, z_{N+M}), v \otimes \bar{a} \otimes \bar{\omega}^{\otimes(M-1)} \rangle . \end{aligned} \tag{5.15}$$

(When we want to emphasize of v and $\bar{a}$ in (5.15), we write it as $Y(v, \bar{a}; \xi_{N+1}, \ldots, \xi_{N+M})$.) Then we have the following

(B1) $Y(\xi_{N+1}, \ldots, \xi_{N+M})$ is a meromorphic function on $\{|\xi_{N+i}|<\varepsilon;$ $(i=1, \ldots, N)\}$ with singularities at $\xi_{N+i}=\xi_{N+j}$ $(i \neq j)$ and $\xi_{N+i}=\xi_k$.

(B2) For fixed $\xi_{N+1}, \ldots, \xi_{N+M-1}$, $Y(\xi_{N+1}, \ldots, \xi_{N+M})$ has the Laurent series expansion

$$\langle y_{M-1}, Y(\omega, \xi_{N+M}-\xi_{N+1})_{N+1} v \otimes \bar{a} \otimes \bar{\omega}^{\otimes(M-2)} \rangle$$

for the variable ξ_{N+M} at ξ_{N+1}.

(B3) For fixed $\xi_{N+2}, \ldots, \xi_{N+M}$, $Y(\xi_{N+1}, \ldots, \xi_{N+M})$ has the Laurent series expansion

$$\langle x(Q_{N+2}, \ldots, Q_{N+M}; z_{N+2}, \ldots, z_{N+M}), Y(p(\bar{a}), \xi_{N+1}-\xi_k)_k v \otimes \bar{\omega}^{\otimes(M-1)} \rangle$$

for the variable ξ_{N+1} at ξ_k.

Proof of Claim B. The statements (B1) and (B2) can be proved using exactly the same method used in the proof of Claim A. We only need to prove (B3). We may assume $k=N$. We prove the statement (B3) for the case $\bar{a}=\bar{b}$ first. We use the induction on M. If $M=1$, considering for $i=1,\ldots,N$ the Laurent series

$$g_i(z_i)=\langle x, Y(b, z_i)_i v\rangle ,$$

by Proposition 3.1, there exists a $g\in\Gamma(\Sigma; Q_1,\ldots,Q_N; \kappa^{|b|})$ such that $\iota_{z_i}g=g_i(z_i)$. Let $g=g(z)(dz)^{|b|}$ on the domain $\{|z|<\varepsilon\}\subset U$, then by the same argument as used in the proof of (5.11), we can prove that $g(\xi_{N+1})=Y(\xi_{N+1})$. This proves (B3) in the case $\bar{a}=\bar{b}$ and $M=1$.

Now assume that (B3) is true for $\bar{a}=\bar{b}$ and $M=S-1$, we want to prove (B3) for the case $\bar{a}=\bar{b}$ and $M=S$. For fixed $Q_{N+2},\ldots,Q_{N+S}$, we choose a $f\in\Gamma(\Sigma; Q_1,\ldots,Q_N, Q_{N+S}; \kappa^{-1})$ such that

$$\iota_{z_{N+S}}f=z_{N+S}^{-1}+\text{higher terms} ;$$

$$\iota_{z_{N+i}}f\equiv 0 \operatorname{Mod} z_{N+i}^4 \quad \text{for } 2\leqq i<S .$$

Write $x(Q_{N+2},\ldots,Q_{N+S})$ for $x(Q_{N+2},\ldots,Q_{N+S}; z_{N+2},\ldots,z_{N+S})$ for simplicity. Then we have

$$\begin{aligned}
&\langle x(Q_{N+2},\ldots,Q_{N+S}), Y(b, z_N)_N v\otimes\bar{\omega}^{\otimes(S-1)}\rangle\\
&=-\sum_{i=1}^{N+S-1}\langle x(Q_{N+2},\ldots,Q_{N+S-1}), \omega(f, z_i)_i Y(b, z_N)_N v\otimes\bar{\omega}^{\otimes(S-2)}\rangle\\
&=-\sum_{i=1}^{N}\langle x(Q_{N+2},\ldots,Q_{N+S-1}), \omega(f, z_i)_i Y(b, z_N)_N v\otimes\bar{\omega}^{\otimes(S-2)}\rangle\\
&=-\sum_{i=1}^{N}\langle x(Q_{N+2},\ldots,Q_{N+S-1}), Y(b, z_N)_N(\omega, f, z_i)_i v\otimes\bar{\omega}^{\otimes(S-2)}\rangle\\
&\quad-\langle x(Q_{N+2},\ldots,Q_{N+S-1}), [\omega(f, z_N)_N, Y(b, z_N)_N]v\otimes\bar{\omega}^{\otimes(S-2)}\rangle . \qquad (5.16)
\end{aligned}$$

The first term of the right side of (5.16) converges on a domain $\{0<|z_N|<\varepsilon_1\}$, since it is a Laurent series expansion of some meromorphic function on $\{|z_N|<\varepsilon\}$ by the induction assumption. For the second term, a direct computation shows that

$$[\omega(f, z_N)_N, Y(b, z_N)_N]=f(z_N)\frac{d}{dz_N}Y(b, z_N)_N+|b|\frac{d}{dz_N}f(z_N)Y(b, z_N)_N ,$$

where $f(z_N)=\iota_{z_N}f$, so the second term of (5.16) also converges on the domain $\{0<|z_N|<\varepsilon_1\}$ by the induction assumption. This proves that the Laurent series

$$\langle x(Q_{N+2},\ldots,Q_{N+S}), Y(b, z_N)_N v\otimes\bar{\omega}^{\otimes(S-1)}\rangle .$$

converges on $\{0<|z_N|<\varepsilon_1\}$.

To complete the induction step, it remains to prove that

$$\langle x(Q_{N+2},\ldots,Q_{N+S}), Y(b, z_N)_N v\otimes\bar{\omega}^{\otimes(S-1)}\rangle|_{z_N=\xi_{N+1}-\xi_N}=Y(\xi_{N+1},\ldots,\xi_{N+S})$$

when $|\xi_{N+S}-\xi_N|<\varepsilon_1$. To achieve this, we choose a $f\in\Gamma(\Sigma; Q_1, \ldots, Q_N, Q_{N+S}; \kappa^{-1})$ such that

$$\begin{aligned} &\iota_{z_{N+S}}f = z_{N+S}^{-1} + \text{higher terms} , \\ &\iota_{z_{N+i}}f \equiv 0 \operatorname{Mod} z_{N+i}^4 \quad \text{for } 2\leqq i<S , \\ &\iota_{z_{N+1}}f \equiv 0 \operatorname{Mod} z_{N+1}^2 . \end{aligned} \tag{5.17}$$

Using (5.16), we have

$$\begin{aligned} &\langle x(Q_{N+2}, \ldots, Q_{N+S}), Y(b, z_N)_N v \otimes \bar{\omega}^{\otimes(S-1)}\rangle \\ &= -\sum_{i=1}^{N} \langle x(Q_{N+2}, \ldots, Q_{N+S-1}), Y(b, z_N)_N \omega(f, z_i)_i v \otimes \bar{\omega}^{\otimes(S-2)}\rangle \\ &\quad -\langle x(Q_{N+2}, \ldots, Q_{N+S-1}), [\omega(f, z_N)_N, Y(b, z_N)_N] v \otimes \bar{\omega}^{\otimes(S-2)}\rangle . \end{aligned} \tag{5.18}$$

Denote by $T_1(z_N)$ and $T_2(z_N)$ the first term and the second term of the right-hand side of (5.18) respectively. Using the induction assumption, we have

$$\begin{aligned} &T_1(z_N)|_{z_N=\xi_{N+1}-\xi_N} \\ &= \sum_{i=1}^{N} Y(\omega(f, z_i)_i v, \bar{b}; \xi_{N+1}, \ldots, \xi_{N+S-1}) \\ &= \sum_{i=1}^{N} \langle y(Q_{N+1}, \ldots, Q_{N+S-1}; z_{N+1}, \ldots, z_{N+S-1}), \omega(f, z_i)_i v \otimes \bar{b} \otimes \bar{\omega}^{\otimes(S-2)}\rangle \\ &= \sum_{i=1}^{N+S-1} \langle y(Q_{N+1}, \ldots, Q_{N+S-1}; z_{N+1}, \ldots, z_{N+S-1}), \omega(f, z_i)_i v \otimes \bar{b} \otimes \bar{\omega}^{\otimes(S-2)}\rangle \\ &= \langle y(Q_{N+1}, \ldots, Q_{N+S}; z_{N+1}, \ldots, z_{N+S}), v \otimes \bar{b} \otimes \bar{\omega}^{\otimes(S-1)}\rangle \\ &= Y(v, \bar{b}; \xi_{N+1}, \ldots, \xi_{N+S}) . \end{aligned}$$

It remains to prove that

$$T_2(z_N)|_{z_N=\xi_{N+1}-\xi_N}=0 .$$

Using the identity

$$[\omega(f, z_N)_N, Y(b, z_N)_N] = f(z_N)\frac{d}{dz_N} Y(b, z_N)_N + |b|\frac{d}{dz_N} f(z_N) Y(b, z_N)_N ,$$

(where $f(z_N)=\iota_{z_N}f$) we have

$$\begin{aligned} T_2 = &-\left(f(z_N)\frac{d}{dz_N} + |b|\frac{d}{dz_N} f(z_N)\right) \\ &\times \langle x(Q_{N+2}, \ldots, Q_{N+S-2}), Y(b, z_N)_N v \otimes \bar{\omega}^{\otimes(S-2)}\rangle . \end{aligned}$$

Using the induction assumption and the fact

$$f(z_N)|_{z_N=\xi_{N+1}-\xi_N} = \frac{d}{dz_N} f(z_N)|_{z_N=\xi_{N+1}-\xi_N} = 0$$

(this follows by the third property of f in (5.17)), we have $T_2(z_N)|_{z_N=\xi_{N+1}-\xi_N}=0$. This completes the proof of (B3) for $\bar{a}=\bar{b}$.

Assume (B3) is true for $\bar{a}_1$ and every v. Based on this assumption, we are going to prove that (B3) is true for $\bar{L}_{-k}\bar{a}_1$. Since for fixed ξ_{N+i} $(i=2,\dots,M)$, $Y(v, L_{-k}\bar{a}_1; \xi_{N+1},\dots,\xi_{N+M})$ is a meromorphic function of $\xi_{N+1}\in\{|\xi_{N+1}|<\varepsilon\}$ with poles at $\xi_N, \xi_{N+2},\dots,\xi_{N+M}$. The Laurent series expansion of $Y(v, L_{-k}\bar{a}_1; \xi_{N+1},\dots,\xi_{N+M})$ for variable ξ_{N+1} at the point ξ_N is

$$\sum_{i=-\infty}^{\infty}\left(\oint_{C_1} Y(v, L_{-k}\bar{a}_1; \xi_{N+1},\dots,\xi_{N+M})(\xi_{N+1}-\xi_N)^n d\xi_{N+1}\right)(\xi_{N+1}-\xi_N)^{-n-1},$$

where C_1 is a small contour of ξ_{N+1} surrounding the ξ_N. It suffices to prove that

$$\oint_{C_1} Y(v, L_{-k}\bar{a}_1; \xi_{N+1},\dots,\xi_{N+M})(\xi_{N+1}-\xi_N)^n d\xi_{N+1}$$
$$=\langle x(Q_{N+2},\dots,Q_{N+M}),(p(L_k\bar{a}_1)(n))_N v\otimes\bar{\omega}^{\otimes(M-1)}\rangle\,. \qquad (5.19)$$

If C_2 is a contour of ξ_{N+M+1} surrounding ξ_{N+1}, we have

$$\begin{aligned}
&\oint_{C_1} Y(v, L_{-k}\bar{a}_1; \xi_{N+1},\dots,\xi_{N+M})(\xi_{N+1}-\xi_N)^n d\xi_{N+1}\\
&=\oint_{C_1}\oint_{C_2} Y(v, \bar{a}_1; \xi_{N+1},\dots,\xi_{N+M+1})(\xi_{N+M+1}-\xi_{N+1})^{-k+1}\\
&\quad\times(\xi_{N+1}-\xi_N)^n d\xi_{N+1}\, d\xi_{N+M+1}\\
&=\oint_{C_2''}\oint_{C_1} Y(v, \bar{a}_1; \xi_{N+1},\dots,\xi_{N+M+1})(\xi_{N+M+1}-\xi_{N+1})^{-k+1}\\
&\quad\times(\xi_{N+1}-\xi_N)^n d\xi_{N+M+1}\, d\xi_{N+1}\\
&\quad-\oint_{C_1}\oint_{C_2'} Y(v, \bar{a}_1; \xi_{N+1},\dots,\xi_{N+M+1})(\xi_{N+M+1}-\xi_{N+1})^{-k+1}\\
&\quad\times(\xi_{N+1}-\xi_N)^n d\xi_{N+1}\, d\xi_{N+M+1}\\
&=I-II\,,
\end{aligned}$$

where C_2'' is a contour of ξ_{N+M+1} which is outside C_1, and C_2' is a contour of ξ_{N+M+1} which is inside C_1. The first equality follows from (B2) and the second equality follows from the Cauchy theorem for contour integrals. By (B2) and the induction assumption, we have

$$I=\langle x(Q_{N+2},\dots,Q_{N+M}),(I)_N v\otimes(\bar{\omega})^{\otimes(M-1)}\rangle\,,$$

and

$$II=\langle x(Q_{N+2},\dots,Q_{N+M}),(II)_N v\otimes(\bar{\omega})^{\otimes(M-1)}\rangle\,,$$

where

$$(I)=\mathrm{Res}_{w_2}\mathrm{Res}_{w_1}(Y(\omega, w_2)Y(p(\bar{a}_1), w_1)\iota_{w_2,w_1}((w_2-w_1)^{-k+1}w_1^n))$$

and

$$(II)=\mathrm{Res}_{w_1}\mathrm{Res}_{w_2}(Y(p(\bar{a}_1), w_1)Y(\omega, w_2)\iota_{w_1,w_2}((w_2-w_1)^{-k+1}w_1^n))\,.$$

Using the Jacobi identity, we have

$$(I)+(II)=p(L_{-k}\bar{a}_1)(n)\,.$$

This proves (B3).

Step 4. We are now ready to give the final touch. For $a\in\mathcal{Q}_n(V)\cap L(b)$, it is easy to prove that a has a preimage $\bar{a}\in\mathcal{U}(b)$ such that $L_1\bar{a}=0$ and $\deg\bar{a}=n$. Let κ_Q^n be the fiber of the line bundle κ^n at the point Q; it has a basis $(dz)^n$ for z a local coordinates at Q. We will first prove Claim C.

Claim C. For every $v\in W_1\otimes\cdots\otimes W_N$, the vector

$$\langle y(Q_{N+1};z_{N+1}), v\otimes\bar{a}\rangle(dz_{N+1})^n\in\kappa_Q^n \tag{5.20}$$

is independent of the local coordinates chosen from the projective structure $\{U_\alpha, z_\alpha\}$.

Proof of Claim C. If z'_{N+1} is another local coordinate obtained from the projective structure $\{U_\alpha, z_\alpha\}$, let $z'_{N+1}=\phi(z_{N+1})$ be the transition function, (note that it is a Mobius transformation), and $T(\phi)$ be the associated operator, so $T(\phi)$ has the form $T(\phi)=\exp(l_0L_0+l_1L_1)$. We define $\hat{y}(Q_{N+1};z_{N+1})\in(B_1)^*$ by

$$\langle\hat{y}(Q_{N+1};z_{N+1}), v_1\otimes\bar{a}_1\rangle=\langle y(Q_{N+1};z'_{N+1}), v_1\otimes T(\phi)\bar{a}_1\rangle\,.$$

It can be proved that $\hat{y}(Q_{N+1};z_{N+1})$ satisfies the property (1) and (4) satisfied by $y(Q_{N+1};z_{N+1})$. Since (1) and (4) uniquely determine $y(Q_{N+1};z_{N+1})$, so $\hat{y}(Q_{N+1};z_{N+1})=y(Q_{N+1};z_{N+1})$. Therefore

$$\langle\hat{y}(Q_{N+1};z_{N+1}), v_1\otimes\bar{a}_1\rangle=\langle y(Q_{N+1};z'_{N+1}), v_1\otimes T(\phi)\bar{a}_1\rangle\,. \tag{5.21}$$

Using the fact that $L_1\bar{a}=0$ and $L_0\bar{a}=n\bar{a}$, we have

$$T(\phi)\bar{a}_1=(\phi'(0))^n\bar{a}_1\,.$$

Substitute this in (5.21). We have

$$\langle y(Q_{N+1};z_{N+1})v\otimes\bar{a}\rangle=\langle y(Q_{N+1};z'_{N+1})v\otimes\bar{a}\rangle(\phi'_{N+M}(0))^n\,.$$

This means that (5.20) is independent of the local coordinate chosen at Q_{N+1}. This completes the proof of Claim C.

So (5.20) defines a section $g=g(Q_{N+1})$ of the line bundle κ^n on the domain $Q_{N+1}\neq Q_i$ $(i=1,\ldots,N)$. By Claim A and Claim B above, we know that g is meromorphic with possible poles at $Q_1,\ldots,Q_N$, and by Claim B, g has the Laurent series expansion at the point Q_i $(i=1,\ldots,N)$ as $\langle x, Y(a,z_i)_i v\rangle$. For $f\in\Gamma(\Sigma;Q_1,\ldots,Q_N;\kappa^{-n+1})$, we have

$$\begin{aligned}\langle x, a(f,\tilde{\Sigma})v\rangle&=\sum_{i=1}^{N}\langle x, \mathrm{Res}_{z_i}(Y(a,z_i)_i\iota_{z_i}f)_i v\rangle\\&=\sum_{i=1}^{N}\mathrm{Res}_{Q_i}(gf)=0\,.\end{aligned} \tag{5.22}$$

Since (5.22) is true for every $v\in W_1\otimes\cdots\otimes W_N$, we conclude that $\sum_{i=1}^{N}xa(f,\tilde{\Sigma})=0$. □

Theorem 5.2. *If V is a sum of highest weight representations of the Virasoro algebra and* $\dim(V_0)=1$ and $\tilde{\Sigma}=(\Sigma;Q_1,\ldots,Q_N;z_1,\ldots,z_N;W_1,\ldots,W_N)$ *is a projective labeled Riemann surface, $x\in N(\tilde{\Sigma})$, $a\in\mathcal{Q}_n(V)$ and $f\in\Gamma(\Sigma;Q_1,\ldots,Q_N;\kappa^{-n+1})$, then $xa(f,\tilde{\Sigma})=0$.*

Proof. To apply Theorem 5.1, we need to prove that $a \in \mathcal{Q}_n(V)$ can be written as a $\sum_{i=1}^{n} a_i$ such that $a_i \in \mathcal{Q}_n(V) \cap L(b_i)$ $(i=1, \ldots, n)$ for $L(b_i)$ a highest weight representation of the Virasoro algebra generated by the highest weight vector b_i. This is the following lemma. □

Lemma 5.3. *If V is a sum of highest weight representations of the Virasoro algebra and* $\dim(V_0)=1$, *then every* $a \in \mathcal{Q}_n(V)$ *can be written as* $a=\sum_{i=1}^{n} a_i$ *such that* $a_i \in \mathcal{Q}_n(V) \cap L(b_i)$ $(i=1, \ldots, n)$ *for* $L(b_i)$ *a highest weight representation of the Virasoro algebra generated by the highest weight vector* b_i.

Proof. We first prove that every element x in a highest weight representation $L(b)$ of the Virasoro algebra generated by the highest weight vector b can be written as a linear combination of elements $L_{-1}^{n} y$ for $n \geqq 0$ and $y \in L(b)$ homogeneous and quasi-primary. We prove the statement using induction on $\deg x$. If $\deg x=0$ or 1 or x has the lowest degree in $L(b)$, then x is primary, in particular it is quasi-primary. We assume that the statement is true for every x with $\deg x \leqq S$ $(S \geqq 1)$. If $\deg x=S+1$, since $\deg L_1 x=S$, applying the induction assumption, we have $L_1 x=\sum_{i=1}^{k} L_{-1}^{n_i} x_i$ for some non-negative integer n_i and homogeneous quasi-primary field x_i in $L(b)$. Set $y=x-\sum_{i=1}^{k} (1/(n_i+1)(n_i+2\deg x_i))L_{-1}^{n_i+1} x_i$. It is easy to check that $L_1 y=0$, so x can be written by a linear combination of $L_{-1}^{n} x$ for x homogeneous quasi-primary in $L(b)$. This completes the induction.

If $a \in \mathcal{Q}_n(V)$, by our assumption on V, $a=\sum_{j=1}^{m} x_j$ for $x_j \in L(b_j)$, $L(b_j)$ is the highest weight representation of the Virasoro algebra generated by the highest weight vector b_i. And by what we just proved, we can write each x_j as $\sum_{i=1}^{k_j} L_{-1}^{n_{j,i}} x_{j,i}$ for $x_{j,i}$ homogeneous quasi-primary fields in $L(b_j)$. We choose these x_j's and $x_{j,i}'$s so that the number $\sum_{j=1}^{m} k_j$ is minimal. Applying operator L_1 to the equality $x=\sum_{j=1}^{m} \sum_{i=1}^{k_j} L_{-1}^{n_{j,i}} x_{j,i}$, we can prove that each $n_{j,i}=0$ and $k_j=1$. This completes the proof. □

6. Correlation Functions

In this section we prove that the space of vacua on a N-labeled Riemann surface is unchanged when adding a new point and assigning the 0-sector V at the point. To be more precise, let

$$\tilde{\Sigma}=(\Sigma; Q_1, \ldots, Q_N; z_1, \ldots, z_N; W_1, \ldots, W_N)$$

be a N-labeled Riemann surface. Adding (Q_{N+1}, z_{N+1}, V) to $\tilde{\Sigma}$, we have the $(N+1)$-labeled Riemann surface

$$\tilde{\Sigma}'=(\Sigma; Q_1, \ldots, Q_{N+1}; z_1, \ldots, z_{N+1}; W_1, \ldots, W_N, V)\,.$$

We will prove $N(\tilde{\Sigma}') \simeq N(\tilde{\Sigma})$. This result leads to a definition of n-pointed correlation functions associated to a vector of $N(\tilde{\Sigma})$.

Theorem 6.1. *Assume V is a sum of highest weight representations of its Virasoro algebra and* $\dim(V_0)=1$. *Let i be the linear map*:

$$i: \bigotimes_{k=1}^{N} W_k \to \bigotimes_{k=1}^{N} W_k \otimes V, \quad v \mapsto v \otimes 1\,,$$

i^*: $(\otimes_{k=1}^N W_k \otimes V)^* \to (\otimes_{k=1}^N W_k)^*$ is the dual map. Then the restriction of i^* on $N(\tilde{\Sigma}') \in (\otimes_{k=1}^N W_k \otimes V)^*$ is an isomorphism from $N(\tilde{\Sigma}')$ to $N(\tilde{\Sigma})$. Moreover if $\tilde{\Sigma}_w$ and $\tilde{\Sigma}'_w$ are the labeled Riemann surfaces obtained by changing coordinates z_i to w_i, let T_1, T_2 be the associated isomorphisms from $N(\tilde{\Sigma}_w)$ to $N(\tilde{\Sigma})$ and from $N(\tilde{\Sigma}'_w)$ to $N(\tilde{\Sigma}')$ respectively, then the diagram of maps

$$\begin{array}{ccc} N(\tilde{\Sigma}'_w) & \xrightarrow{i^*} & N(\tilde{\Sigma}_w) \\ {\scriptstyle T_2}\downarrow & & {\scriptstyle T_1}\downarrow \\ N(\tilde{\Sigma}') & \xrightarrow{i^*} & N(\tilde{\Sigma}) \end{array}$$

is commutative.

Proof. For $a \in \mathscr{P}_n(V)$, $f \in \Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^{-n+1})$, we need to check that $(i^*x) \cdot a(f, \tilde{\Sigma}) = 0$. Since f is regular at Q_{N+1}, we have

$$\mathrm{Res}_{z_{N+1}}(Y(a, z_{N+1})\iota_{z_{N+1}} f)1 = 0 .$$

So for $v \in \otimes_{i=1}^N W_i$,

$$\begin{aligned} \langle i^*x, a(f, \tilde{\Sigma})v \rangle &= \sum_{i=1}^N \langle i^*x, \mathrm{Res}_{z_i}(Y(a, z_i)\iota_{z_i} f)_i v \rangle \\ &= \sum_{i=1}^N \langle x, \mathrm{Res}_{z_i}(Y(a, z_i)\iota_{z_i} f)_i v \otimes 1 \rangle \\ &= \sum_{i=1}^{N+1} \langle x, \mathrm{Res}_{z_i}(Y(a, z_i)\iota_{z_i} f)_i v \otimes 1 \rangle \\ &= \langle x, a(f, \tilde{\Sigma}')v \otimes 1 \rangle = 0 . \end{aligned}$$

Similarly, we can prove that i^* is annihilated by the global vertex operators associated to the Virasoro element. This proves $i^*x \in N(\tilde{\Sigma})$. So we have a map $i^*: N(\tilde{\Sigma}') \to N(\tilde{\Sigma})$. Tracing the definitions, we can prove that the above diagram is commutative.

Next we prove that $i^*: N(\tilde{\Sigma}') \to N(\tilde{\Sigma})$ is injective. By the above commutative diagram, we may assume that $\tilde{\Sigma}$ is projective. For x such that $i^*x = 0$, we need to prove $x = 0$. It suffices to prove that $\langle x, v \otimes a \rangle = 0$ for all $v \in \otimes_{i=1}^N W_i$ and $a \in V$. By our assumption on V, we know from Theorem 5.2 that x is annihilated by the quasi-global vertex operators and from Lemma 5.3 that every $a \in V$ is a linear combination of the elements of form $L_{-1}^m b$ for b homogeneous quasi-primary. So it suffices to prove $\langle x, v \otimes L_{-1}^m b \rangle = 0$. For this purpose, choose $f \in \Gamma(\Sigma, Q_1, \ldots, Q_{N+1}; \kappa^{-\deg b + 1})$ such that

$$\iota_{z_{N+1}} f = m! z_{N+1}^{-m-1} + \text{regular terms} .$$

Then we have

$$\begin{aligned} \langle x, v \otimes L_{-1}^m b \rangle &= \langle x, \mathrm{Res}_{z_{N+1}}(Y(b, z_{N+1})\iota_{z_{N+1}} f)_{N+1} v \otimes 1 \rangle \\ &= -\sum_{i=1}^N \langle x, \mathrm{Res}_{z_i}(Y(b, z_i)\iota_{z_i} f)_i v \otimes 1 \rangle \\ &= -\sum_{i=1}^N \langle i^*x, \mathrm{Res}_{z_i}(Y(b, z_i)\iota_{z_i} f)_i v \rangle = 0 . \end{aligned}$$

Thus $x = 0$. This proves the injectivity of i^*.

To prove the surjectivity of i^*, we may also assume $\tilde{\Sigma}$ is projective. Let $\{U_\alpha, z_\alpha\}$ be the projective structure of $\tilde{\Sigma}$. We choose a local coordinate chart (U, z) among $\{U_\alpha, z_\alpha\}$ such that $\{|z|<\varepsilon\}$ for some ε contains Q_N, while Q_i, $i=1,\dots,N-1$ are outside $\{|z|<\varepsilon\}$. We only need to prove the surjectivity for the case $Q_{N+1}\in\{|z|<\varepsilon\}$. Because if we prove the isomorphism of $N(\tilde{\Sigma})$ and $N(\tilde{\Sigma}')$ at the above situation, then for Q_{N+1} at arbitrary position, we may choose points $Q_{N+2}, Q_{N+3},\dots,Q_{N+k}$ such that any two successive points P, Q in the sequence $Q_N, Q_{N+2},\dots, Q_{N+k}, Q_{N+1}$ fit the above situation. Let $\tilde{\Sigma}_k$ be the labeled Riemann surface obtained by adding points $Q_{N+2},\dots,Q_{N+k}, Q_{N+1}$ to $\tilde{\Sigma}$ and assigning V at these points. By deleting points Q_{N+i} $(i=2,\dots,k)$ in the order $Q_{N+2},\dots Q_{N+k}$, we have an isomorphism of the spaces of vacua each time, thus we have $N(\tilde{\Sigma}_k)\simeq N(\tilde{\Sigma}')$. On the other hand, by deleting points Q_{N+i} $(i=1,\dots,k)$ in the order $Q_{N+1}, Q_{N+2},\dots,Q_{N+k}$, we have the isomorphism $N(\tilde{\Sigma}_k)\simeq N(\tilde{\Sigma})$. Thus $N(\tilde{\Sigma})$ is isomorphic to $N(\tilde{\Sigma}')$ and it is easy to see that this isomorphism is just i^*.

Thus we can assume that Q_N and Q_{N+1} are in $\{|z|<\varepsilon\}$. For every $x\in N(\tilde{\Sigma})$, we need to find $x'\in N(\tilde{\Sigma}')$ such that $i^*x'=x$. Our method to construct such x' is similar to the proof of Theorem 5.1 with the Lie algebra of the quasi-global vertex operators playing the role of the Viraroso algebra and the "Verma module" $\mathscr{V}$ define below playing the role of $\mathscr{U}(1)$ and $\mathscr{U}(b)$.

For a positive integer M, pick M points $Q_{N+1},\dots,Q_{N+M}$ in $\{|z|<\varepsilon\}$, set $\xi_{N+i}=z(Q_{N+i})$ $(i=0,1,\dots,M)$, take $z_{N+i}=z-\xi_{N+i}$ as the local coordinate at Q_{N+i}. In this way, we obtain a projective $(N+M)$-pointed Riemann surface

$$(\Sigma; Q_1,\dots,Q_{N+M}; z_1,\dots,z_{N+M}).$$

By assigning the "Verma module" $\mathscr{V}$ (which is defined below) at each point Q_{N+i} $(i=1,\dots,M)$, we have the data

$$\begin{aligned}&\tilde{\Sigma}(\xi_{N+1},\dots,\xi_{N+M})\\ &\quad=(\Sigma; Q_1,\dots,Q_{N+M}; z_1,\dots,z_{N+M}; W_1,\dots,W_N,\mathscr{V},\dots,\mathscr{V}).\end{aligned}\tag{6.1}$$

To construct $x'\in N(\tilde{\Sigma}')\subset(\bigotimes_{i=1}^N W_i\otimes V)^*$ such that $i^*x'=x$, we first construct $x(\xi_{N+1})\in(\bigotimes_{i=1}^N W_i\otimes\mathscr{V})^*$, then we prove that $x(\xi_{N+1})$ reduces to the needed x'. For this purpose we will construct for each M, $x(\xi_{N+1},\dots,\xi_{N+M})\in(\bigotimes_{i=1}^N W_i\otimes\mathscr{V}^{\otimes M})^*$ associated to the data (6.1).

The "Verma module" $\mathscr{V}$ is defined as follows. Let $\bar{\mathscr{A}}$ be the free associative algebra with identity $\bar{1}$ generated by the symbols $a\langle n\rangle$, where $a\in\mathscr{Q}(V)$ and $n\in\mathbf{Z}$, and $a\langle n\rangle$ is linear in a. Set $Y\langle a,z\rangle=\sum_{i=-\infty}^{\infty} a\langle n\rangle z^{-n-1}$, so $\mathrm{Res}_z(Y\langle a,z\rangle z^m)=a\langle m\rangle$. Let $\mathscr{A}$ be the quotient algebra of $\bar{\mathscr{A}}$ modulo the relations

$$\begin{aligned}[a\langle m\rangle, b\langle n\rangle]=&\sum_{i=0}^{|a|+|b|-2}\mathrm{Res}_z(Y\langle x_{a,b;i},z\rangle G_{a,b;z^m,z^n;i}(z))\\ &+\frac{1}{L!}\mathrm{Res}_z(Y\langle a(L)b,z\rangle(z^m)^{(L)}(z)z^n),\end{aligned}\tag{6.2}$$

and $1\langle -1\rangle=\bar{1}$, $1\langle i\rangle=0$ for $i\neq -1$, where a, b are homogeneous quasi-primary fields with degrees $|a|$ and $|b|$, $L=|a|+|b|-1$, $x_{a,b;i}$ are quasi-primary fields as defined by (4.7), $G_{a,b;z^m,z^n;i}(z)$ is defined in (4.9). The relation (6.2) is motivated by the formula (4.8). We continue to write elements of $\mathscr{A}$ as $a\langle m\rangle b\langle n\rangle\bar{1}$.

Let $\bar{\mathscr{V}}$ be the left regular representation of $\bar{\mathscr{A}}$. So a typical element of $\bar{\mathscr{V}}$ is $a_1\langle i_1\rangle \ldots a_n\langle i_n\rangle \bar{1}$. Let $\mathscr{V}$ be the left regular representation of $\mathscr{A}$ modulo the relations

$$a_1\langle i_1\rangle \ldots a_n\langle i_n\rangle \bar{1}=0 \quad \text{when } i_n \geqq 0 . \tag{6.3}$$

We continue to write elements in $\mathscr{V}$ as $a_1\langle i_1\rangle \ldots a_n\langle i_n\rangle \bar{1}$. And we write $\bar{a}$ for $a\langle -1\rangle \bar{1} \in \mathscr{V}$. We have the obvious surjective linear maps

$$p: \bar{\mathscr{V}} \to \mathscr{V}: a_1\langle i_1\rangle \ldots a_n\langle i_n\rangle \bar{1} \mapsto a_1\langle i_1\rangle \ldots a_n\langle i_n\rangle \bar{1} ,$$

$$p: \mathscr{V} \to V: a_1\langle i_1\rangle \ldots a_n\langle i_n\rangle \bar{1} \mapsto a_1\langle i_1\rangle \ldots a_n\langle i_n\rangle 1 . \tag{6.4}$$

Both $\bar{\mathscr{V}}$ and $\mathscr{V}$ have a gradation defined by

$$\deg(a_1\langle i_1\rangle \ldots a_n\langle i_n\rangle \bar{1})=\sum_{i=1}^{n} (\deg a_i - i_1 - 1) .$$

It is clear that both maps in (6.4) preserve the gradation. And it is easy to prove using (6.2) and (6.3) that $\mathscr{V}$ doesn't have a non-zero element with negative degree. Thus for fixed $a \in \mathscr{Q}(V)$ and $v \in \mathscr{V}$, $a\langle n\rangle v=0$ for n sufficiently large. Therefore for a Laurent series $f(z)=\sum_{i \geq k} l_i z^i$, the operator $\operatorname{Res}_z(Y\langle a, z\rangle f(z))=\sum_{i \geq k} l_i a\langle i\rangle$ acts on $\mathscr{V}$. We will write $\operatorname{Res}_z(Y(a, z) f(z))$ for $\operatorname{Res}_z(Y\langle a, z\rangle f(z))$, and for a homogeneous quasi-primary, $f \in \Gamma(\Sigma; Q_1, \ldots, Q_{N+k}; \kappa^{-|a|+1})$, we write $a(f, z_i)=\operatorname{Res}_{z_i}(Y(a, z_i)\iota_{z_i} f)$ as before.

$x(\xi_{N+1}, \ldots, \xi_{N+k}) \in (\bigotimes_{i=1}^{N} W_i \otimes \mathscr{V}^{\otimes k})^*$ $(k=1, 2, \ldots)$ to be constructed will satisfy the following properties:

(1) For every homogeneous $a \in \mathscr{Q}(V)$ with degree $|a|$, and every

$$f \in \Gamma(\Sigma; Q_1, \ldots, Q_{N+k}; \kappa^{-|a|+1}) ,$$

$$\sum_{i=1}^{N+k} x(\xi_{N+1}, \ldots, \xi_{N+k}) \operatorname{Res}_{z_i}(Y(a, z_i)\iota_{z_i} f)_i=0 .$$

(2) For $v \in \bigotimes_{i=1}^{N} W_i$,

$$\langle x(\xi_{N+1}), v \otimes \bar{1}\rangle=\langle x, v\rangle .$$

(3) For $v \in \bigotimes_{i=1}^{N} W_i \otimes \mathscr{V}^{\otimes k}$,

$$\langle x(\xi_{N+1}, \ldots, \xi_{N+k+1}), v \otimes \bar{1}\rangle=\langle x(\xi_{N+1}, \ldots, \xi_{N+k}), v\rangle .$$

(4) For $v_1 \in \bigotimes_{i=1}^{N} W_i \otimes \mathscr{V}^{\otimes i}$, $v_2 \in \mathscr{V}^{\otimes j}$, (so $v_1 \otimes \bar{1} \otimes v_2 \in \bigotimes_{i=1}^{N} W_i \otimes \mathscr{V}^{\otimes (i+j+1)}$, and $v_1 \otimes v_2 \in \bigotimes_{i=1}^{N} W_i \otimes \mathscr{V}^{\otimes (i+j)}$,)

$$\langle x(\xi_{N+1}, \ldots, \xi_{N+i+j+1}), v_1 \otimes \bar{1} \otimes v_2\rangle$$
$$=\langle x(\xi_{N+1}, \ldots, \widehat{\xi_{N+i+1}}, \ldots, \xi_{N+i+j+1}), v_1 \otimes v_2\rangle ,$$

where $\widehat{\ }$ indicate the missing terms.

To construct $x(\xi_{N+1})$, we construct $\bar{x}(\xi_{N+1}) \in (W_1 \otimes \cdots \otimes W_N \otimes \bar{\mathscr{V}})^*$ first, then prove that $\bar{x}(\xi_{N+1})$ reduces to $x_1(\xi_{N+1})) \in (W_1 \otimes \cdots \otimes W_N \otimes \mathscr{V})^*$. Defining $\bar{x}(\xi_{N+1})$ is equivalent to defining

$$\langle \bar{x}(\xi_{N+1}, v \otimes a_1\langle i_1\rangle \ldots a_k\langle i_k\rangle \bar{1}\rangle ,$$

we do it inductively on k. For $k=0$, we define

$$\langle \bar{x}(\xi_{N+1}), v \otimes \bar{1} \rangle = \langle x, v \rangle .$$

For $k=1$, choose $f \in \Gamma(\Sigma; Q_1, \ldots, Q_{N+1}; \kappa^{-|a_1|+1})$ such that

$$\iota_{z_{N+1}} f = z_{N+1}^{i_1} + \text{positive terms} ,$$

we define

$$\langle \bar{x}(\xi_{N+1}), v \otimes a_1 \langle i_1 \rangle \bar{1} \rangle = - \sum_{i=1}^{N} \langle \bar{x}(\xi_{N+1}), a_1(f, z_i)_i v \otimes \bar{1} \rangle .$$

Using (4.10) in Sect. 4, we can prove that

$$\sum_{i=1}^{N} \langle \bar{x}(\xi_{N+1}), a(g, z_i)_i v \otimes \bar{b} \rangle = 0 \tag{6.5}$$

for $b = a_1(i_1)\bar{1}$ or $\bar{1}$, $a \in \mathscr{Q}(V)$ with degree $|a|$ and $g \in \Gamma(\Sigma, Q_1, \ldots, Q_N; \kappa^{-|a|+1})$ satisfying

$$\iota_{z_{N+1}} g = \text{sum of terms higher than } z_{N+1}^{-|a|+1+\deg \bar{b}} .$$

Assume we have defined $\langle \bar{x}(\xi_{N+1}), v \otimes \bar{b})$ for every v and every $\bar{b} = a_1 \langle i_1 \rangle \ldots a_k \langle i_k \rangle \bar{1}$ and the property that

$$\sum_{i=1}^{N} \langle \bar{x}(\xi_{N+1}), a(g, z_i)_i v \otimes \bar{b} \rangle = 0 \tag{6.6}$$

for every $a \in \mathscr{Q}(V)$ with degree $|a|$ and $g \in \Gamma(\Sigma; Q_1, \ldots, Q_N; \kappa^{-|a|+1})$ satisfying

$$\iota_{z_{N+1}} g = \text{sum of terms higher than } z_{N+1}^{-|a|+1+\deg \bar{b}} .$$

Based on this induction assumption, we define $\langle \bar{x}(\xi_{N+1}), v \otimes a\langle i \rangle \bar{b} \rangle$ as follows. Choose $f \in \Gamma(\Sigma; Q_1, \ldots, Q_{N+1}; \kappa^{-|a|+1})$, such that

$$\iota_{z_{N+1}} f = z_{N+1}^{i} + \text{terms higher than } z_{N+1}^{-|a|+1+\deg \bar{b}} ,$$

we define

$$\langle \bar{x}(\xi_{N+1}), v \otimes a\langle i \rangle \bar{b} \rangle = - \sum_{i=1}^{N} \langle \bar{x}(\xi_{N+1}), a(f, z_i)_i v \otimes \bar{b}) ,$$

it follows from (6.6) that it is independent of the choice of f. And with some effort, it can be proved that the property (6.6) is again satisfied. So we have completed the construction of $\bar{x}(\xi_{N+1})$. By a direct computation, we can prove that

$$\langle \bar{x}(\xi_{N+1}), v \otimes b_1 \langle i_1 \rangle \ldots b_k \langle i_k \rangle \bar{1} \rangle = 0$$

when $i_k \geqq 0$, and

$$\begin{aligned} &\langle \bar{x}(\xi_{N+1}), v \otimes [a\langle m \rangle, b\langle n \rangle] \bar{c} \rangle \\ &= \sum_{i=0}^{|a|+|b|-2} \langle \bar{x}(\xi_{N+1}), v \otimes \operatorname{Res}_z(Y\langle x_{a,b;i}, z \rangle G_{a,b;z^m,z^n;i}(z)) \bar{c} \rangle \\ &\quad + \left\langle \bar{x}(\xi_{N+1}), v \otimes \frac{1}{L!} \operatorname{Res}_z(Y\langle a(L)b, z \rangle (z^m)^{(L)} z^n) \bar{c} \right\rangle . \end{aligned}$$

So $\bar{x}(\xi_{N+1})$ reduces to a vector $x(\xi_{N+1})\in(W_1\otimes\cdots\otimes W_N\otimes\mathscr{V})^*$. And we can prove that $x(\xi_{N+1})$ satisfies the condition (1) for $k=1$ and condition (3). By the same method, we can construct $x(\xi_{N+1},\xi_{N+2})$ based on $x(\xi_{N+1})$ (constructing $\bar{x}(\xi_{N+1},\xi_{N+2})\in(W_1\otimes\cdots\otimes W_N\otimes\mathscr{V}\otimes\bar{\mathscr{V}})^*$ first as constructing $\bar{x}(\xi_{N+1})$, then proving it reduces $x(\xi_{N+1},\xi_{N+2})\in(W_1\otimes\cdots\otimes W_N\otimes\mathscr{V}\otimes\mathscr{V})^*$). For example, $\langle\bar{x}(\xi_{N+1},\xi_{N+2}),v\otimes\bar{1}\rangle$ for $v\in W_1\otimes\cdots\otimes W_N\otimes\mathscr{V}$ is $\langle x(\xi_{N+1}),v\rangle$. And we continue this way; we can construct $x(\xi_{N+1},\ldots,\xi_{N+k})$ $(k=1,2,\ldots)$. It can be proved that $x(\xi_{N+1},\ldots,\xi_{N+k})$ satisfies the properties (1), (2), (3), (4) above. It is not hard to see that $\{x(\xi_{N+1},\ldots,\xi_{N+k})\}$ satisfying (1), (2), (3), (4) are unique.

Next we prove the following claim.

Claim A. For every $v\in W_1\otimes\cdots\otimes W_N$, $\bar{b}\in\mathscr{V}$, $a_2,\ldots,a_M\in\mathscr{Q}(V)$ homogeneous. Then the function

$$\langle x(\xi_{N+1},\ldots,\xi_{N+M}),v\otimes\bar{b}\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_M\rangle \tag{6.7}$$

is a M-variable meromorphic function on the domain $\{|\xi_{N+i}|<\varepsilon,\ i=1,\ldots,M\}$ with singularities at $\xi_{N+i}=\xi_{N+i}$ $(i\neq j)$ and $\xi_{N+i}=\xi_N$. And for fixed $\xi_{N+1},\ldots,\xi_{N+M-1}$, (6.7) has the Laurent series expansion

$$\langle x(\xi_{N+1},\ldots,\xi_{N+M-1}),Y(a_M,\xi_{N+M}-\xi_{N+1})_{N+1}v\otimes\bar{b}\otimes\bar{a}_2\otimes\cdots\otimes a_{M-1}\rangle \tag{6.8}$$

for the variable ξ_{N+M} at ξ_{N+1}.

Proof of Claim A. Consider the Laurent series given by

$$g_i(z_i)=\langle x(\xi_{N+1},\ldots,\xi_{N+M-1}),Y(a_{N+M},z_i)_i v\otimes\bar{a}_1\otimes\cdots\otimes a_{N+M-1}\rangle$$

$(i=1,\ldots,N+M-1)$. The condition (1) implies that g_i's satisfy the condition of the second part of Lemma 3.1. Apply Lemma 3.1; there exists a meromorphic ($|a_{N+M}|$)-differential $g\in\Gamma(\Sigma,Q_1,\ldots,Q_{N+M};\kappa^{|a_M|})$, such that $g_i(z_i)=\iota_{z_i}g$ $(i=1,\ldots,N+M-1)$. Write $g=g(z)(dz)^{|a_{N+M}|}$ on $\{|z|<\varepsilon\}\subset U$, so $g(z)$ is a meromorphic function on $\{|z|<\varepsilon\}$ with poles at $\xi_N,\ldots,\xi_{N+M-1}$. We want to prove that

$$g(\xi_{N+M})=\langle x(\xi_{N+1},\ldots,\xi_{N+M});v\otimes\bar{b}\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_M\rangle\,. \tag{6.9}$$

For this purpose, we choose $f\in\Gamma(\Sigma,Q_1,\ldots,Q_{N+M};\kappa^{-|a_{M+N}|+1})$ such that

$$f|_{z_{N+M}}=z_{N+M}^{-1}+\text{higher terms}\,,$$

Then we have

$$g(\xi_{N+M})=\mathrm{Res}_{Q_{N+M}}(gf)=-\sum_{i=1}^{N+M-1}\mathrm{Res}_{Q_i}(gf)=-\sum_{i=1}^{N+M-1}\mathrm{Res}_{z_i}(g_i(z_i)\iota_{z_i}f)$$

$$=-\sum_{i=1}^{N+M-1}\langle x(\xi_{N+1},\ldots,\xi_{N+M-1}),a_{M+N}(f,z_i)_i v\otimes\bar{b}\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_{M-1}\rangle$$

$$=\langle x(\xi_{N+1},\ldots,\xi_{N+M}),v\otimes\bar{b}\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_M\rangle\,.$$

This proves (6.9), so (6.7) is meromorphic for ξ_{N+M} on $\{|\xi_{N+M}|<\varepsilon\}$ with poles at points ξ_{N+i} $(i=0,1,\ldots,N+M-1)$ and has Laurent series expansion at ξ_{N+1} as (6.8). The same argument proves that (6.7) is meromorphic for ξ_{N+j} $(j=2,\ldots,M)$ on $\{|\xi_{N+j}|<\varepsilon\}$ with poles at points ξ_{N+i} $(i=0,1,\ldots,N+M,\ i\neq j,)$.

If $\bar{b}=\bar{a}_1$ for a_1 quasi-primary and homogeneous, the above argument applies to the variable ξ_{N+1}, so we have in this case (6.7) is meromorphic for ξ_{N+1} on $\{|\xi_{N+1}|<\varepsilon\}$ and has poles at ξ_{N+i} $(i=0,2,\ldots,M)$. Thus, if $\bar{b}=\bar{a}_1$ for a_1 quasi-primary and homogeneous, by the Hartog's Theorem, Claim A is true.

Suppose Claim A is true for $\bar{b}$, based on this assumption, we want to prove it is true for $\bar{b}'=a\langle i\rangle\bar{b}$, where a is quasi-primary and homogeneous. Set $a_{M+1}=a$. By induction assumption, we know that

$$\langle x(\xi_{N+1},\ldots,\xi_{N+M+1}), v\otimes\bar{b}\otimes\bar{a}_1\otimes\cdots\otimes\bar{a}_{M+1}\rangle$$

is a meromorphic function of $\xi_{N+1},\ldots,\xi_{N+M+1}$ with poles at $\xi_{N+i}=\xi_{N+j}$ $(i\neq j)$ and $\xi_{N+i}=\xi_N$, and it has Laurent series expansion for the variable ξ_{N+M+1} at ξ_{N+1} as

$$\langle x(\xi_{N+1},\ldots,\xi_{N+M}), v\otimes Y(a,\xi_{N+M+1}-\xi_{N+1})_{N+1}\bar{b}\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_M\rangle\,.$$

So

$$\begin{aligned}&\langle x(\xi_{N+1},\ldots,\xi_{N+M}); v\otimes a\langle i\rangle\bar{b}\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_M\rangle\\&=\oint_C\langle x(\xi_{N+1},\ldots,\xi_{N+M+1}), v\otimes\bar{b}\otimes\bar{a}_1\otimes\cdots\otimes\bar{a}_{M+1}\rangle\\&\quad\times(\xi_{N+M+1}-\xi_{N+1})^i\,d\xi_{N+M+1}\end{aligned}\tag{6.10}$$

for C a contour surrounding ξ_{N+1}. Since the right side of (6.10) a meromorphic function of variables $\xi_{N+1},\ldots,\xi_{N+M}$ with poles at $\xi_{N+i}=\xi_{N+j}$ $(i\neq j)$ and $\xi_{N+i}=\xi_N$, so is the left side of (6.10). The proof of the fact that the left side of (6.10) has a Laurent series expansion for ξ_{N+M} at ξ_{N+1} as

$$\langle x(\xi_{N+1},\ldots,\xi_{N+M+1}), v\otimes Y(a,\xi_{N+M}-\xi_{N+1})_{N+1}\bar{b}'\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_{M-1}\rangle$$

is already given before. This completes the proof of Claim A.

Claim B. (B1) Equation (6.7) has the Laurent series expansion for the variable ξ_{N+M} at ξ_N as

$$\langle x(\xi_{N+1},\ldots,\xi_{N+M-1}), Y(a_M,\xi_{N+M}-\xi_N)_N v\otimes\bar{b}\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_{M-1}\rangle\,.$$

(B2) Equation (6.7) has the Laurent series expansion for the variable ξ_{N+1} at ξ_N as

$$\langle x(\xi_{N+2},\ldots,\xi_{N+M}), Y(b,\xi_{N+1}-\xi_N)_N v\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_M\rangle\,,$$

where $b=p(\bar{b})$.

Proof of Claim B. (B1) is already proven in the proof of Claim A. To prove (B2), it suffices to prove the case when $\bar{b}=b_1\langle i_1\rangle\ldots b_k\langle i_k\rangle\bar{1}$ for $b_i\in\mathcal{Q}(V)$ and homogeneous. We use induction on k.

If $k=0$, then $\bar{b}=\bar{1}$, then we have

$$\begin{aligned}(6.7)&=\langle x(\xi_{N+2},\ldots,\xi_{N+M}), v\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_M\rangle\\&=\langle x(\xi_{N+2},\ldots,\xi_{N+M}), Y(1,\xi_{N+1}-\xi_N)_N v\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_M\rangle\,.\end{aligned}$$

This proves (B2) for the case $k=0$.

Assume (B2) is true for $\bar{b}=\bar{b}_1$, based on this assumption, we are going to prove that (B2) is true for $\bar{b}=a\langle i\rangle\bar{b}_1$, where a is in $\mathscr{Q}(V)$ with degree $|a|$. Set $a_{M+1}=a$, and set

$$F(\xi_{N+1},\ldots,\xi_{N+M})=\langle x(\xi_{N+1},\ldots,\xi_{N+M}), v\otimes\bar{b}\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_M\rangle\,,$$

and set

$$G(\xi_{N+1},\ldots,\xi_{N+M+1})=\langle x(\xi_{N+1},\ldots,\xi_{N+M+1}), v\otimes\bar{b}_1\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_{M+1}\rangle\,.$$

By Claim A,

$$F(\xi_{N+1},\ldots,\xi_{N+M})$$

is a M-variable meromorphic function on $\{|\xi_{N+i}|<\varepsilon\}$ with singularities at $\xi_{N+i}=\xi_{N+j}$ $(i\neq j)$ and $\xi_{N+i}=\xi_N$. The Laurent series expansion of $F(\xi_{N+1},\ldots,\xi_{N+M})$ for the variable ξ_{N+1} at the point ξ_N is

$$\sum_{i=-\infty}^{\infty}\left(\oint_{C_1}F(\xi_{N+1},\ldots,\xi_{N+M})(\xi_{N+1}-\xi_N)^n\,d\xi_{N+1}\right)(\xi_{N+1}-\xi_N)^{-n-1}\,,$$

where C_1 is a contour of ξ_{N+1} surrounding the ξ_N. It suffices to prove that

$$\oint_{C_1}F(\xi_{N+1},\ldots,\xi_{N+M})(\xi_{N+1}-\xi_N)^n\,d\xi_{N+1}$$
$$=\langle x(\xi_{N+2},\ldots,\xi_{N+M}), p(a\langle i\rangle\bar{b}_1)(n)_N v\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_M\rangle\,. \tag{6.11}$$

By the statement on the Laurent series expansion in Claim A, we have, for C_2 a contour of ξ_{N+M+1} surrounding ξ_{N+1},

$$\oint_{C_2}G(\xi_{N+1},\ldots,\xi_{N+M+1})(\xi_{N+M+1}-\xi_{N+1})^i\,d\xi_{N+M+1}=F(\xi_{N+1},\ldots,\xi_{N+M})\,.$$

So we have

$$\begin{aligned}
&\oint_{C_1}F(\xi_{N+1},\ldots,\xi_{N+M})(\xi_{N+1}-\xi_N)^n\,d\xi_{N+1}\\
&\quad=\oint_{C_1}\oint_{C_2}G(\xi_{N+1},\ldots,\xi_{N+M+1})(\xi_{N+M+1}-\xi_{N+1})^i(\xi_{N+1}-\xi_N)^n\,d\xi_{N+M+1}\,d\xi_{N+1}\\
&\quad=\oint_{C_2''}\oint_{C_1}G(\xi_{N+1},\ldots,\xi_{N+M+1})(\xi_{N+M+1}-\xi_{N+1})^i(\xi_{N+1}-\xi_N)^n\,d\xi_{N+M+1}\,d\xi_{N+1}\\
&\qquad-\oint_{C_1}\oint_{C_2'}G(\xi_{N+1},\ldots,\xi_{N+M+1})(\xi_{N+M+1}-\xi_{N+1})^i(\xi_{N+1}-\xi_N)^n\,d\xi_{N+1}\,d\xi_{N+M+1}\\
&\quad=I-II\,,
\end{aligned}$$

where C_2'' is a contour of ξ_{N+M+1} which is outside C_1, and C_2' is a contour of ξ_{N+M+1} which is inside C_1, the second equality follows from the Cauchy theorem for the contour integrals. By (B1) and the induction assumption, we have

$$I=\langle x(\xi_{N+2},\ldots,\xi_{N+M}), (I)_N v\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_M\rangle$$

and

$$II=\langle x(\xi_{N+2},\ldots,\xi_{N+M}), (II)_N v\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_M\rangle\,,$$

where

$$(I)=\mathrm{Res}_{w_2}\mathrm{Res}_{w_1}(Y(a,w_2)Y(p(\bar{b}_1),w_1)\iota_{w_2,w_1}((w_2-w_1)^i w_1^n))$$

and

$$(II)=\mathrm{Res}_{w_1}\mathrm{Res}_{w_2}(Y(p(\bar{b}),w_1)Y(a,w_2)\iota_{w_1,w_2}((w_2-w_1)^i w_1^n))\,.$$

Using the Jacobi identity, we have

$$(I)-(II)=(a(i)p(\bar{b}_1))(n)=p(a\langle i\rangle\bar{b}_1)(n)\,.$$

This proves (6.11) therefore (B2).

Now we are ready to give the final touch. By Claim A, for $v\in\bigotimes_{i=1}^N W_i$ and $\bar{b}\in\mathscr{V}$,

$$\langle x(\xi_{N+1}),v\otimes\bar{b}\rangle \tag{6.12}$$

is a meromorphic function of ξ_{N+1} on $\{|z|<\varepsilon\}$ with poles at ξ_N. By Claim B, (6.12) has Laurent series expansion at ξ_N as

$$\langle x,Y(p(\bar{b}),\xi_{N+1}-\xi_N)_N v\rangle\,.$$

This means that if $p(\bar{b})=0$, then $\langle x(\xi_{N+1}),v\otimes\bar{b}\rangle=0$. So $x(\xi_{N+1})\in(W_1\otimes\cdots\otimes W_N\otimes\mathscr{V})^*$ reduces an element $x'\in(W_1\otimes\cdots\otimes W_N\otimes V)^*$. By condition (1), it is clear that $x'\in N(\tilde{\Sigma}')$ and $i^*x'=x$. This proves the surjectivity of i^*. □

If V is a sum of highest weight representations of the Virasoro algebra and $\dim(V_0)=1$, $\tilde{\Sigma}$ is a N-labeled Riemann surface as above. Let $P=(P_1,\ldots,P_n)$ be n-different points on Σ such that $P_i\neq Q_j$, w_i be a local coordinate at P_i; write $w=(w_1,\ldots,w_n)$. Let $\tilde{\Sigma}_{P,w}$ be the $(N+n)$-labeled Riemann surface given by adding points P on $\tilde{\Sigma}$ and assigning w_i, V at P_i. For $x\in N(\tilde{\Sigma})$, let $x_{P,w}$ be the image of x in the isomorphism $N(\tilde{\Sigma})\simeq N(\tilde{\Sigma}_{P,w})$. Then for $v\in W_1\otimes\cdots\otimes W_N$, $a_i\in\mathscr{P}_{l_i}$ $(i=1,\ldots,n)$,

$$F(v,a_1,\ldots,a_n;P_1,\ldots,P_n)=\langle x_{P,w},v\otimes a_1\otimes\cdots\otimes a_n\rangle(dw_1)^{l_1}\ldots(dw_n)^{l_n} \tag{6.13}$$

(the local coordinate w_i at P_i defines a basis $(dw_i)^{l_i}$ of the fiber $\kappa_P^{l_i}$, we continue to use $(dw_i)^{l_i}$ to denote the corresponding basis in $\pi_i^{-1}\kappa_P^{l_i}$) defines a vector on $(\pi_1^{-1}\kappa^{l_1}\otimes\cdots\otimes\pi_n^{-1}\kappa^{l_n})_P$. We have the following theorem.

Theorem 6.2. *$F(v,a_1,\ldots,a_n;P_1,\ldots,P_n)\in(\pi_1^{-1}\kappa^{l_1}\otimes\ldots\pi_n^{-1}\kappa^{l_n})_P$ defined in (6.13) is independent of the local coordinates w. And as P varying on Σ^n, it defines a global meromorphic section of the line bundle $\pi_1^{-1}\kappa^{l_1}\otimes\ldots\pi_n^{-1}\kappa^{l_n}$ over Σ^n, and the only possible singularities of this section are those P's satisfying $P_i=P_j$ for some $i,j=1,\ldots,n$ or $P_i=Q_j$ for some $i=1,\ldots,n$, $j=1,\ldots,N$.*

The meromorphic section $F(v,a_1,\ldots,a_n;P_1,\ldots,P_n)$ relates to the vertex operators $Y(a_1,z_1),\ldots,Y(a_1,z_n)$ as follows. For each Q_k $(k=1,\ldots,N)$, let $U=\{|z_k|<\varepsilon\}$ be a neighborhood of Q_k which contains no other Q_i's, let $(z_k^1,\ldots,z_k^n)$ be the coordinates on $U^n\subset\Sigma^n$ induced from z_k, write

$$F(v,a_1,\ldots,a_n;P_1,\ldots,P_n)=f(z_k^{(1)},\ldots,z_k^{(n)})(dz_k^{(1)})^{l_1}\ldots,(dz_k^{(n)})^{l_n}$$

on U^n. Then for every permutation $(i_1,\ldots,i_n)$ of $(1,\ldots,n)$, the meromorphic function $f(z_k^{(1)},\ldots,z_k^{(n)})$ has the expansion

$$\langle x,Y(a_{i_1},z_k^{(i_1)})_k\ldots Y(a_{i_n},z_k^{(i_n)})_k v\rangle$$

on the domain $\varepsilon > |z_k^{(i_1)}| > \cdots > |z_k^{(i_n)}| > 0$. (*Recall as in Sect.* 3, $Y(a_{i_s}, z_k^{(i_s)})_k$ *denotes the operator* $1 \otimes \cdots \otimes Y(a_{i_s}, z_k^{(i_s)})$ (*k-th place*) $\otimes \cdots \otimes 1$.)

Proof. To prove the right-hand side of (6.13) is independent of the local coordinates w, let $w' = (w'_1, \ldots, w'_n)$ be another choice of the local coordinates, let $w'_i = \Phi_i(w_i) = \exp\left(\sum_{j=0}^{\infty} c_{i,j} w_i^{j+1} \frac{d}{dw_i}\right) w_i$ be the transition functions, $T_i = \exp(\sum_{j=0}^{\infty} c_{i,j} L_j)$ be the associated operators, then we have $x_{P,w} = x_{P,w'} \Pi_{i=1}^n (T_i)_{N+i}$. In other words, we have

$$\langle x_{P,w}, v \otimes a_1 \otimes \cdots \otimes a_n \rangle = \langle x_{P,w'}, v \otimes T_1 a_1 \otimes \cdots \otimes T_n a_n \rangle . \tag{6.14}$$

A direct computation using the facts $L_k a_i = 0$ for $k > 0$ and $L_0 a_i = l_i a_i$ shows that $T_i a_i = \exp(l_i c_{i,0}) a_i = (\Phi'_i(w_i))^{l_i}|_{w_i=0}, a_i$. Substituting this to (6.14) and using the facts $(dw_i)^{l_i} (\Phi'_i(w_i))^{l_i}|_{w_i=0} = (dw'_i)^{l_i}$, one proves that

$$\langle x_{P,w}, v \otimes a_1 \otimes \cdots \otimes a_n \rangle (dw_1)^{l_1} \ldots (dw_n)^{l_n} = \langle x_{P,w'}, v \otimes a_1 \otimes \cdots \otimes a_n \rangle \times (dw'_1)^{l_1} \ldots (dw'_n)^{l_n} .$$

This proves (6.13) is independent of w.

As P varying on Σ^n, $F(v, a_1, \ldots, a_n; P_1, \ldots, P_n)$ is a section of $\pi_1^{-1} \kappa^{l_1} \otimes \ldots \pi_n^{-1} \kappa^{l_n}$ defined on $P_i \neq P_j$ $(i \neq j)$ and $P_i \neq Q_j$. We next prove that $F(v, a_1, \ldots, a_n; P_1, \ldots, P_n)$ is meromorphic. By Hartog's Theorem, it suffices to prove that F is meromorphic for each variable P_i. To prove that F is meromorphic with respect to P_n, let $\tilde{\Sigma}_{n-1}$ be the $(N+n-1)$-labeled Riemann surface given by deleting (P_n, w_n, V) from $\tilde{\Sigma}_{P,w}$. Let $x_{n-1} \in N(\tilde{\Sigma}_{n-1})$ correspond to x as in Theorem 6.1. Considering the Laurent power series $g_i(z_i)$ $(i = 1, \ldots, N+n-1)$ $(z_{N+i} = w_i)$ given by

$$g_i(z_i) = \langle x_{n-1}, Y(a_n, z_i)_i v \otimes a_1 \otimes \cdots \otimes a_{n-1} \rangle .$$

By Proposition 3.4, there exists a $g \in \Gamma(\Sigma; Q_1, \ldots, Q_{N+n-1}; \kappa^{l_n})$ (where $Q_{N+i} = P_i$) such that $g_i(z_i)(dz_i)^{l_n}$ is the Laurent series expansion of g. Write $g = g_{N+n}(w_n)(dw_n)^{l_n}$ near P_n; we claim that

$$g_{N+n}(0) = \langle x_{P,w}, v \otimes a_1 \otimes \cdots \otimes a_n \rangle .$$

Indeed, choose a $f \in \Gamma(\Sigma; Q_1, \ldots, Q_{N+n}; \kappa^{-l_n+1})$ such that

$$\iota_{w_n} f = w_n^{-1} + \text{higher terms} .$$

So we have

$$\begin{aligned} g_n(0) = \mathrm{Res}_{P_n}(gf) &= -\sum_{i=1}^{N+n-1} \mathrm{Res}_{Q_i}(gf) \\ &= -\sum_{i=1}^{N+n-1} \langle x_{n-1}, \mathrm{Res}_{z_i}(Y(a_n, z_i)\iota_{z_i} f)_i v \otimes a_1 \otimes \cdots \otimes a_{n-1} \rangle \\ &= \langle x_{P,w}, v \otimes a_1 \otimes \cdots \otimes a_n \rangle . \end{aligned}$$

This proves our claim. So F is meromorphic for P_n with poles at $Q_1, \ldots, Q_N$, $P_1, \ldots, P_{n-1}$. Similarly we can prove that F is meromorphic for other variables. Thus F is a meromorphic section.

It remains to prove that F has the Laurent series expansions as in the theorem. We may assume $k=1$ and that the permutation $(i_1,\ldots,i_n)$ is $(1,2,\ldots,n)$. Write

$$F(v,a_1,\ldots,a_n;P_1,\ldots,P_n)=f(z_1^{(1)},\ldots,z_1^{(n)})(dz_1^{(1)})^{l_1}\ldots,(dz_1^{(n)})^{l_n}$$

on U^n. Let $\tilde{\Sigma}_i$ $(i=1,\ldots,n)$ be the $(N+i)$-labeled Riemann surface obtained by adding $(P_1, z-z(P_1), V),\ldots,(P_i, z-z(P_i), V)$ to $\tilde{\Sigma}$, let $x_i\in N(\tilde{\Sigma}_i)$ correspond $x\in N(\tilde{\Sigma})$. We have the expansion on the domain $\{\varepsilon>|z_1^{(1)}|>|z_1^{(2)}|>\cdots>|z_1^{(n)}|>0\}$:

$$f(z_1^{(1)},\ldots,z_1^{(n)})=\sum_{(i_1,\ldots,i_n)\in\mathbf{Z}^n} c_{i_1,\ldots,i_n}(z_1^{(1)})^{-i_1-1}\ldots(z_1^{(n)})^{-i_n-1},$$

where

$$c_{i_1,\ldots,i_n}=\oint_{C_1}\cdots\oint_{C_n} f(z_1^{(1)},\ldots,z_1^{(n)})(z_1^{(1)})^{i_1}\ldots(z_1^{(n)})^{i_n}dz_1^{(1)}\ldots dz_1^{(n)}, \tag{6.15}$$

where the contour C_i is for $z_1^{(i)}$, and C_i contains C_{i+1}.

By the proof above, we know that $f(z_1^{(1)},\ldots,z_1^{(n)})$ have expansion for $z_1^{(n)}$ at ξ_1,

$$\langle x_{n-1}, Y(a_n, z_1^{(n)})_1 v\otimes a_1\otimes\cdots\otimes a_{n-1}\rangle .$$

Thus we have

$$\oint_{C_n} f(z_1^{(1)},\ldots,z_1^{(n)})(z_1^{(n)})^{i_n}dz_1^{(n)}=\langle x_{n-1}, a_n(i_n)v\otimes a_1\otimes\cdots\otimes a_{n-1}\rangle .$$

Similarly,

$$\oint_{C_{n-1}}\langle x_{n-1}, a_n(i_n)v\otimes a_1\otimes\cdots\otimes a_{n-1}\rangle(z_1^{(n-1)})^{i_{n-1}}dz_{n-1}$$
$$=\langle x_{n-2}, a_{n-1}(i_{n-1})_N a_n(i_n)_N v\otimes a_1\otimes\cdots\otimes a_{n-2}\rangle .$$

Continue this way, thus we have

$$c_{i_1,\ldots,i_n}=\langle x, a_1(i_i)_N\ldots a_n(i_n)_N v\rangle .$$

This is precisely the coefficient of $(z_1^{(1)})^{-i_1-1}\ldots(z_1^{(n)})^{-i_n-1}$ in

$$\langle x, Y(a_1, z_1^{(1)})_N\ldots Y(a_n, z_1^{(n)})_N v\rangle ,$$

as was to be shown. □

F in the theorem is called the n-pointed correlation functions associated to x, v and a_i $(i=1,\ldots,n)$.

Remark. If $\tilde{\Sigma}$ is projective with the projective structure $\{U_\alpha, z_\alpha\}$, we use $\{U_\alpha, z_\alpha\}$ to give local coordinates w_i at P_i, then Theorem 6.2 is true for $a_1,\ldots,a_n$ quasi-primary. Using this fact together with the fact that every element of V is a sum of the form $L_{-1}^k b_k$ with b_k quasi-primary, one can prove that for arbitrary $a_1,\ldots,a_n\in V$,

$$\langle x, Y(a_1, w_1)_k\ldots Y(a_n, w_n)_k v\rangle$$

converges on the domain $\varepsilon>|w_1|>\cdots>|w_n|>0$, and the limit can be extended to a meromorphic function on the domain $\varepsilon>|w_i|>0$ $(i=1,\ldots,n)$ with the only possible singularities at $w_i=w_j$, and this meromorphic function is independent of the ordering of $Y(a_i, z_i)$. This generalizes Theorem 2.1.

7. Examples

We give examples of the space of vacua for some cases. The space of the vacua for an one-pointed sphere relates to the vacuum vector $1 \in V$. The space of vacua for a two-pointed sphere relates to the dual representations. And the space of vacua for a three-pointed sphere relates to the interwining operators. The notions of dual representations and interwining operators are introduced in [FHL]. The space of vacua on a torus with one puncture relates to the q-trace of vertex operators, which is studied in [Z]. We also discuss the space of vacua for the vertex operator algebras associated to the affine Lie algebras and Heisenberg algebras. The space of vacua for vertex operator superalgebras associated to the Clliford algebra is known completely.

We assume that the vertex operator algebra V in the examples 1–4 below satisfies that V is a sum of highest weight representations of its Virasoro algebra and $\dim(V_0)-1$ and a representation $W=\bigoplus_{n=0}^{\infty} W(n)$ of V under consideration satisfies that $\dim W(n)<\infty$ and L_0 acts on $W(n)$ as $n+h$ for some constant h. Under this assumption on V, by Theorem 5.2 and Lemma 5.3, every vector in V is a linear combination of vectors $L_{-1}^k a$ for $k \in \mathbf{Z}$ and a homogeneous quasi-primary, and the quasi-global vertex operators annihilate the space of vacua on a projective N-labeled Riemann surface.

We first recall the notions of dual representations and intertwining operators for representations (see [FHL] for details). Only the basic definitions is deeded.

Let $W=\bigoplus_{n=1}^{\infty} W_n$ be a representation of V. The restricted dual $W'=\sum_{n=1}^{\infty} W_n^*$ of W admits a structure of representation of V given by

$$\langle Y(a,z)v', v\rangle = \langle v', Y(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1})v\rangle \,. \tag{7.1}$$

The identity (7.1) for $a \in \mathcal{Q}_n(V)$ is equivalent to

$$\langle a(i)v', v\rangle = (-1)^n \langle v', a(2n-2-i)v\rangle \text{ for every } i \in \mathbf{Z} \,. \tag{7.2}$$

The representation W' is called the dual representation of W.

Let $W_i=\bigoplus_{n=0}^{\infty} W_i(n)$ $(i=1,2,3)$ be representations of V such that L_0 acts on $W_i(n)$ as $n+h_i$. An intertwining operator of type $\begin{pmatrix} W_3 \\ W_2\, W_1 \end{pmatrix}$ is a linear map $I(\ ,z)$

$$I(\ ,z)\colon W_2 \to \mathrm{End}(W_1, W_3)[[z, z^{-1}]]z^{-h_1-h_2+h_3} \,,$$

$$v \mapsto I(v,z) = \sum_{i=-\infty}^{\infty} v(i) z^{-i-1-h_1-h_2+h_3}$$

such that $I(\ ,z)$ satisfies that for fixed $v \in W_2 v_1 \in W_1$, $v(i)v_2=0$ for i sufficiently large (truncation condition), and for every $f(z,w)=(z-w)^m z^n$,

$$\begin{aligned} &\mathrm{Res}_{z-w}(I(Y(a,z-w)v, w)\iota_{w,z-w} f(z,w)) \\ &\quad = \mathrm{Res}_z(Y(a,z)I(v,w)\iota_{z,w} f(z,w)) - \mathrm{Res}_z(I(v,w)Y(a,z)\iota_{w,z} f(z,w)) \,, \end{aligned} \tag{7.3}$$

and $I(L_{-1}v, z) = \frac{d}{dz} I(v,z)$.

1. *One-Pointed Sphere.* Consider $\tilde{\Sigma}=(\mathbf{C}P^1; 0; z; W)$, where z is the standard coordinate. It is clear that $\tilde{\Sigma}$ is projective, so Theorem 5.2 applies. And it is easy to see that $L_0=w(1)$ is a global operator on $\tilde{\Sigma}$. So if $L_0W=W$ (this situation happens when L_0 acts semi-simply on W without 0 eigenvalue), then $N(\mathbf{C}P^1; 0; z; W)=0$. Note that $\Gamma(\Sigma; 0; \kappa^{-n+1})$ has a basis

$$z^{2n-2}(dz)^{-n+1}, z^{2n-3}(dz)^{-n+1}, z^{2n-4}(dz)^{-n+1}, \dots,$$

so $\mathscr{Q}\mathscr{G}(\tilde{\Sigma})$, the space of quasi-global vertex operators, is spanned by

$$\{a(i) \mid a\in\mathscr{Q}_n(V), n\in\mathbf{Z}, i\leqq 2n-2\} . \tag{7.4}$$

The following Proposition is easy to prove.

Proposition 7.1. $N(\mathbf{C}P^1; 0; z; V)\simeq\mathbf{C}$. *The dual vacuum* $1'\in V^*$ *defined by* $\langle 1', a\rangle=0$ *if* $\deg a>0$ *and* $\langle 1', 1\rangle=1$ *is a basis of* $N(\mathbf{C}P^1; 0; z; V)$.

2. *Two-Pointed Sphere.* Considering $\tilde{\Sigma}=(\mathbf{C}P^1; 0, \infty; z, 1/z; W_1, W_2)$. It is clear that $\tilde{\Sigma}$ is projective, so Theorem 5.2 applies. And $\Gamma(\mathbf{C}P^1; \infty, 0; \kappa^{-n+1})$ has a basis $z^i(dz)^{-n+1}$ $(i\in\mathbf{Z})$. At the point ∞ and coordinate $w=1/z$, $z^i(dz)^{-n+1}$ is written as $(-1)^{n+1}w^{2n-2-i}(dw)^{-n+1}$, so the quasi-global vertex operator associated to $a\in\mathscr{Q}_n(V)$ and the differential $z^i(dz)^{-n+1}$ is $1\otimes(-1)^{n+1}a(2n-2-i)+a(i)\otimes 1$.

Proposition 7.2. *The vacua space* $N(\tilde{\Sigma})=N(\mathbf{C}P^1; \infty, 0; 1/z, z; W_1, W_2)$ *is isomorphic to the space* $\mathrm{Hom}_V(W_1, W_2')$, *where* W_2' *is the dual representation of* W_2.

Proof. Given a homomorphism $f\in\mathrm{Hom}_V(W_1, W_2')$, we define $N(f)\in(W_1\otimes W_2)^*$ as usual by $\langle N(f), v_1\otimes v_2\rangle=\langle f(v_1), v_2\rangle$. Using (7.2) and the fact $f(a(i)v_1)=a(i)f(v_1)$, we have $N(f)\cdot a(f,\tilde{\Sigma})=0$ for every quasi-global operator $a(f,\tilde{\Sigma})$ on $\tilde{\Sigma}$, so $N(f)\in N(\tilde{\Sigma})$. Thus $f\mapsto N(f)$ defines a linear map from $\mathrm{Hom}_V(W_1, W_2')$ to $N(\tilde{\Sigma})$. Conversely, if $x\in N(\tilde{\Sigma})\subset(W_1\otimes W_2)^*$, we define for each $v_1\in W_1$ a linear functional $f_x(v_1)\in W_2^*$ as usual by $\langle f_x(v_1), v_2\rangle=\langle x, v_1\otimes v_2\rangle$. $L_0\otimes 1-1\otimes L_0$ is a global vertex operator on $\tilde{\Sigma}$, we have $\langle x, (L_0\otimes 1-1\otimes L_0)v_1\otimes v_2\rangle=0$. This implies that $\langle f_x(L_0v_1), v_2\rangle=\langle f_x(v_1), L_0v_2\rangle$. So for $v_1\in W_1(n)$, $\langle f_x(v_1), v_2\rangle=0$ unless $L_0v_2=(n+h_1)v_2$, this implies that $f_x(v_1)\in W_2'$. So we have a linear map f_x: $W_1\to W_2'$. It remains to check that

$$f_x(a(i)v_1)=a(i)f_x(v_1) \quad \text{for every } a\in V . \tag{7.5}$$

For $a\in\mathscr{Q}_n(V)$, we have

$$\begin{aligned}\langle f_x(a(i)v_1), v_2\rangle &=\langle x, a(i)v_1\otimes v_2\rangle=\langle x, v_1\otimes(-1)^n a(2n-2-i)v_2\rangle\\ &=\langle f_x(v_1), (-1)^n a(2n-2-i)v_2\rangle=\langle a(i)f_x(v_1), v_2\rangle .\end{aligned}$$

This proves that (7.5) is true for a quasi-primary. Using the facts that every $a\in V$ is a sum of $L_{-1}^k b$ for b quasi-primary and $(L_{-1}^k b)(i)=(-1)^k i(i+1)\dots(i+k-1)b(i-k)$, it is clear that (7.5) is true for every $a\in V$. Thus f_x is a morphism of representations of V. It is clear that the maps $f\mapsto N(f)$ and $x\mapsto f_x$ are inverse maps. □

Since the points 0 and ∞ are symmetric, we also have $N(\mathbf{C}P^1; \infty, 0; 1/z, z; W_1, W_2)\simeq\mathrm{Hom}_V(W_2, W_1')$. We give some corollaries of Proposition 7.2.

Corollary 7.3. *If* W_1, W_2 *are irreducible representations,* W_2' *is the dual representa-*

tion of W_2, *then*

$$N(\mathbf{C}P^1;\infty,0;1/z,z;W_1,W_2)=\begin{cases}\mathbf{C} & \text{if } W_1=W_2',\\ 0 & \text{if } W_1\neq W_2'.\end{cases}$$

Proof. Using the Proposition 7.2 and the fact that W_2' is irreducible. □

Remark. If $W_1=W_2'$, let e_i $(i=1,2,\dots)$ be a homogeneous basis of W_1, $\{e_i'\}\subset W_2$ be a dual basis of $\{e_i\}$, then $\sum_{i=1}^{\infty} e_i'\otimes e_i$ is in $N(\mathbf{C}P^1;\infty,0;1/z,z;W_1,W_2)$, it corresponds to the identity map of $\mathrm{Hom}_V(W_1,W_1)$.

3. *Three-Point Sphere.* Every three-pointed sphere is conformally equivalent to $(\mathbf{C}P^1;0,1,\infty)$. It suffices to consider

$$\tilde{\Sigma}=(\mathbf{C}P^1;0,1,\infty;z,z-1,1/z;W_1,W_2,W_3)\,.$$

Since $\tilde{\Sigma}$ is projective, so Theorem 5.2 applies. $\Gamma(\mathbf{C}P^1;0,1,\infty;\kappa^{-n+1})$ has a basis $\{z^m(z-1)^l(dz)^{-n+1},\ m,l\in\mathbf{Z}\}$.

Proposition 7.4. *$N(\tilde{\Sigma})=N(\mathbf{C}P^1;0,1,\infty;z,z-1,1/z;W_1,W_2,W_3)$ is isomorphic to the space $I(W_1,W_2,W_3)$ of intertwining operators of type* $\begin{pmatrix} W_3' \\ W_2\,W_1\end{pmatrix}$. *$W_3'$ is the dual representation of W_3.*

Proof. Let $I(\ ,z)$ be a interwining operator of type $\begin{pmatrix} W_3' \\ W_2\,W_1\end{pmatrix}$, we define a linear functional $f_I\in(W_1\otimes W_1\otimes W_3)^*$ by $\langle F_I, v_1\otimes v_2\otimes v_3\rangle=\langle v_3, I(v_2,1)v_1\rangle$. For $a\in\mathcal{Q}_k(V)$, $f=z^m(z-1)^n(dz)^{-k+1}$, the quasi-global vertex operator $a(f,\tilde{\Sigma})$ is

$$\begin{aligned}&\mathrm{Res}_z(Y(a,z)z^m(z-1)^n)\otimes 1\otimes 1\\ &+1\otimes \mathrm{Res}_{z_1}(Y(a,z_1)(z_1+1)^m z_1^n)\otimes 1\\ &+1\otimes 1\otimes \mathrm{Res}_{z_\infty}(Y(a,z_\infty)(-1)^{k+1}z_\infty^{2k-2-m-n}(1-z_\infty)^n\,.\end{aligned}$$

Set $f(z,w)=z^m(z-w)^n$, by (7.3), we have

$$\begin{aligned}0=&\langle v_3, \mathrm{Res}_{z-w}(I(Y(a,z-w)v_2,w)\iota_{w,z-w}f(z,w))v_1\rangle\\ &-\langle v_3, \mathrm{Res}_z(Y(a,z)I(v_2,w)\iota_{z,w}f(z,w))v_1\rangle\\ &+\langle v_3, \mathrm{Res}_z(I(v,w)Y(a,z)\iota_{w,z}f(z,w))v_1\rangle\,.\end{aligned}\tag{7.6}$$

Put $w=1$ in (7.6), the right-hand side is precisely $\langle F_I, a(f,\tilde{\Sigma})v_1\otimes v_2\otimes v_3\rangle$. Thus we have $F_I\cdot a(f,\tilde{\Sigma})=0$, so $F_I\in N(\tilde{\Sigma})$. Thus we have established a linear map from $I(W_1,W_2,W_3)$ to $N(\tilde{\Sigma})$. Conversely, for $x\in N(\tilde{\Sigma})$, we define I_x by

$$\langle v_3', I_x(v_2,w)v_1\rangle=\langle x, w^{-L_0}v_1\otimes w^{-L_0}v_2\otimes w^{L_0}v_3'\rangle\,.\tag{7.7}$$

Note that for fixed v_1,v_2,v_3', (7.7) is a in $\mathbf{C}[w,w^{-1}]w^{-h_1-h_2+h_3}$. And it is easy to see that $I(v_2,w)$ is an element of the space $\mathrm{End}(W_1,W_3)[[w,w^{-1}]]w^{-h_1-h_2+h_3}$ and it satisfies the truncation condition. To prove the L_{-1} property, choose $f=z(dz)^{-1}$, the global vertex operator $\omega(f,\tilde{\Sigma})$ is

$$L_0\otimes 1\otimes 1+1\otimes(L_0+L_{-1})\otimes 1-1\otimes 1\otimes L_0\,,$$

$\langle x, \omega(f, \tilde{\Sigma})w^{-L_0}v_1 \otimes w^{-L_0}v_2 \otimes w^{L_0}v_3' \rangle = 0$ is precisely

$$\langle v_3', I_x(L_{-1}v_2, w)v_1 \rangle = \frac{d}{dw}\langle v_3', I_x(v_2, w)v_1 \rangle .$$

This proves L_{-1} property. We next prove the Jacobi identity (7.3). For $a \in \mathcal{Q}_k(V)$, $f(z, w) = (z-w)^m z^n$, set $f = (z-1)^m z^n (dz)^{-k+1}$, then we can check that the identity $\langle x, a(f, \tilde{\Sigma})w^{-L_0}v_1 \otimes w^{-L_0}v_2 \otimes w^{L_0}v_3' \rangle = 0$ is the same as (7.3). This proves (7.3) for a quasi-primary. For arbitrary $a \in V$, write a as a sum of elements of type $L_{-1}^l b$ for b quasi-primary, using L_{-1} property of vertex operators and (7.3) for quasi-primary fields, it is easy to prove (7.3) is true in general. So I_x is an interwining operator, thus we have established a linear map from $N(\tilde{\Sigma})$ to $I(W_1, W_2, W_3)$ which is clearly the inverse map of the map of $I(W_1, W_2, W_3)$ to $N(\tilde{\Sigma})$ defined earlier. □

4. *One-Painted Torus.* Set $q = e^{2\pi\sqrt{-1}\tau}$. Let L_τ be the lattice $(m\tau + n\}$, and let T_τ be the torus $\mathbf{C}/L_\tau$. Take the image $0 \in \mathbf{C}$ in T_τ to be the marked point Q and the standard cordinate z be a local coordinate at Q, and we denote Q by 0. We will consider the 1-labeled torus $(T_\tau; 0; z; V)$.

The torus T_τ can be also obtained by identifying the boundaries of the annuli $\{|q| \leqq |w| \leqq 1\}$ by the relation $w \sim wq$. The point Q corresponds to the image of 1. We have another local coordinate $z' = w - 1$ at Q. z and z' are related by $z' = \exp(2\pi\sqrt{-1}z) - 1$. Let T be the associated operator with respect to the transition function $z' = \exp(2\pi\sqrt{-1}z) - 1$.

The process of gluing the boundaries of the annuli $\{|q| \leqq |w| \leqq 1\}$ corresponds the process of taking traces of the vertex operators. Let $W = \bigoplus_{i=0}^{\infty} W(i)$ be a representation of V with the action of L_0 on $W(i)$ as $i + h$ for some constant h. Consider the trace of the operator $Y(z^{L_0}a, z)q^{L_0}$ on W:

$$\mathrm{tr}|_W Y(z^{L_0}z, z)q^{L_0} . \tag{7.8}$$

It is easy to see that (7.8) is a power series in $\mathbf{C}[[q]]q^h$. It is proved in [Z] that $\mathrm{tr}|_W Y(z^{L_0}z, z)q^{L_0}$ converges on $0 < |q| < 1$ for every a under the condition that $\dim V/C_2(V) < \infty$, where $C_2(V)$ is the subspace of V spanned by the vectors of the form $b_1(-2)b_2$ for $b_1, b_2 \in V$. We define a functional $\chi(W, \tau) \in V^*$ by

$$\langle \chi(W, \tau), a \rangle = \mathrm{tr}_W o(Ta)q^{L_0} ,$$

where we put $o(Ta)$ instead of a because of the coordinates transformation. The results of [Z] about the trace $\mathrm{tr}_W o(a)q^{L_0}$ implies that $\chi(W, \tau) \in N(T_\tau; 0; z; V)$.

5. *Vertex operator algebras associated to the affine Lie algebras.* Let $\mathfrak{g}$ be a simple Lie algebra, $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}\mathbf{k}$ be the associated affine Kac–Moody Lie algebra [K]. We write $a(n)$ for $a \otimes t^n$, and $a(z) = \sum_{i=-\infty}^{\infty} a(n)z^{-n-1}$. For k a positive integer, the integrable highest weight representation of $\mathfrak{g}$ of level k has a vertex operator algebra structure. The set of irreducible presentations of L_k are the same with the set of integrable highest weight representation of $\mathfrak{g}$ of level k, and L_k is rational. See [FZ] for detail.

For a N-labeled Riemann surface

$$\tilde{\Sigma} = (\Sigma; Q_1, \ldots, Q_N; z_1, \ldots, z_n; W_1, \ldots, W_N) .$$

Let $M(\tilde{\Sigma}; Q_1, \ldots, Q_N)$ be the space of meromorphic functions on Σ with poles at most at $Q_1, \ldots, Q_N$ (so it is the same as $\Gamma(\tilde{\Sigma}; Q_1, \ldots, Q_N; \kappa^0)$). For $a \in \mathfrak{g}$, $f \in M(\tilde{\Sigma}; Q_1, \ldots, Q_N)$, put $a(f, \tilde{\Sigma}) = \sum_{i=1}^{N} \mathrm{Res}_{z_i}(a(z_i)\iota_{z_i} f)_i$ which acts on $\otimes_{i=1}^{N} W_i$.

The space of vacua defined by Tsuchiya–Ueno–Yamada is

$$\mathscr{V}(\tilde{\Sigma}) = \{x \in (W_1 \otimes \cdots \otimes W_N)^* \mid x \cdot a(f, \tilde{\Sigma}) = 0 \quad \text{for all } a \in \mathfrak{g} \text{ and } f \in M(\tilde{\Sigma}; Q_1, \ldots, Q_N)\} .$$

Since $a(z)$ is a vertex operator of a primary field of degree 1, we see that the condition for $x \in \mathscr{V}(\tilde{\Sigma})$ appears weaker than the condition for $x \in N(\tilde{\Sigma})$. Since L_k is now generated by $\hat{\mathfrak{g}}$, using the same method in proof of Theorem 5.1 with the Lie algebra $\hat{\mathfrak{g}}$ playing the role of the Virasoro algebra, one can prove that $\mathscr{V}(\tilde{\Sigma}) = N(\tilde{\Sigma})$.

6. *Vertex operator algebras associated to Heisenberg algebras.* Let $\{1, a(n)(n \in \mathbf{Z})\}$ be an infinite dimensional Heisenberg algebra; it has commutation relations $[h(m), h(n)] = m\delta_{m+n,0}1$. The polynomial ring $V = \mathbf{C}[a(-1), a(-2), \ldots]$ of variables $a(-1), a(-2), \ldots$ (V is also called the Fock space) is a representation of the Heisenberg algebra. F has a vertex operator algebra structure; the vertex operators are generated by the basic vertex operator $a(z) = \sum_{n=-\infty}^{\infty} a(n) z^{-n-1}$ [FLM2]. For a 1-labeled Riemann surface $\tilde{\Sigma} = (\Sigma, Q, z, V)$ (Q is a point on Σ, z is local coordinate at Q), with a little effort, one can prove that

$$N(\tilde{\Sigma}) = \{x \in V^* \mid x \cdot \mathrm{Res}_z(a(z) f(z)) = 0 \quad \text{for every } f \in H^0(\Sigma - Q)\} ,$$

where $H^0(\Sigma - Q)$ denotes the space of meromorphic functions with possible poles at Q. Then we can follow a method in [DVV] to compute $N(\tilde{\Sigma})$. It goes as follows (see [DVV] Sect. 6c for detail). Let $g(z)$ be a multi-valued meromorphic function with possible poles at Q and with constant shifts around the nontrivial cycles of the surface, then $dg(z)$ is a meromorphic differential with possibles at Q, so

$$[\mathrm{Res}_z(a(z)g(z)), \mathrm{Res}_z(a(z)f(z))] = \mathrm{Res}_z(f(z)dg(z)) = 0 ,$$

for every $f(z) \in H^0(\Sigma - \{Q\})$. Thus the operators $\mathrm{Res}_z(a(z)g(z))$ preserve $N(\tilde{\Sigma})$. The space of such $g(z)$ modulo $H^0(\Sigma - Q)$ is naturally dual to $H_1(\Sigma, \mathbf{C})$, hence is $2g$ dimensional (g is the genus of Σ). We take $2g$ such multi-valued meromorphic functions $g_{A_1}, \ldots, g_{A_g}; g_{B_1}, \ldots, g_{B_g}$ corresponding to cycles A_i, B_i in $H_1(\Sigma)$ such that

$$[\mathrm{Res}_z(a(z)g_{A_i}(z)), \mathrm{Res}_z(a(z)g_{A_j}(z))] = [\mathrm{Res}_z(a(z)g_{B_i}(z)), \mathrm{Res}_z(a(z)g_{B_j}(z))] = 0 ,$$

$$[\mathrm{Res}_z(a(z)g_{A_i}(z)), \mathrm{Res}_z(a(z)g_{B_j}(z))] = \delta_{ij}1 .$$

One can then prove that there is a unique $x_0 \in N(\tilde{\Sigma})$ (up to scalar) such that x_0 is an eigenvector for $\mathrm{Res}_z(a(z)g_{A_i}(z))$ $(i = 1, \ldots, g)$ and $N(\tilde{\Sigma})$ is a completion of the space spanned by $\mathrm{Res}_z(a(z)g_{B_{i_1}}(z)) \ldots \mathrm{Res}_z(a(z)g_{B_{i_k}}(z))x_0$. In particular, we see that $N(\tilde{\Sigma})$ is infinite dimensional if $g > 0$.

7. *Vertex operator superalgebras associated to Clifford algebras.* Recall that an infinite dimensional Clifford algebra generated by b_n, $c_n n \in \mathbf{Z}$ has anticommuting relations:

$$b_m b_n + b_n b_m = 0, \quad c_m c_n + c_n c_m = 0, \quad b_m c_n + c_n b_m = \delta_{m+n,0} .$$

For an integer j, the canonical Clifford module V_j is generated by a vacua vector 1 and 1 is annihilated by annihilation operators $b_{1-j}, b_{2-j}, \ldots; c_j, c_{j+1}, \ldots$. A basis of V_j is obtained by acting the creation operators $b_j, b_{-j-1}, \ldots; c_{-(1-j)}, c_{-(1-j)-1}, \ldots$ on 1. V_j has a structure of vertex operator superalgebra, the vertex operators are generated by two basic vertex operators $b(z)=\sum_{n=-\infty}^{\infty} b_n z^{-n-j}$ and $c(z)=\sum_{n=-\infty}^{\infty} c_n z^{-n+j-1}$ (see [A] Sect. 4 for a proof).

The notion of global vertex operators and the space of vacua generalize directly to vertex operator superalgebras. It is easy to see that V_j is its own unique irreducible representation, and use a similar argument as in the proof of Theorem 6.1, we can prove that the space of vacua for N-labeled Riemann surface is isomorphic to that for 1-labeled Riemann surface. Therefore is sufficient to consider the case $\tilde{\Sigma}=\Sigma, Q, z, V_j)$. Note that $b(z)$ $(c(z))$ is a primary field of degree j $(1-j)$. One can prove

$$N(\tilde{\Sigma})=\{x\in V_j^* \mid x\cdot \mathrm{Res}_z(b(z)f(z))=x\cdot \mathrm{Res}_z(c(z)g(z))=0$$
$$\text{for every } f(z)\in\Gamma(\Sigma; Q, \kappa^{-j-1}),\ g(z)\in\Gamma(\Sigma; Q, \kappa^{j})\}\ .$$

In [GGMV], $N(\tilde{\Sigma})$ is proved to be one dimensional and an explicit formula for a basis is given. This conformal field theory is also studied in [KNTY].

8. Gluing Construction of the Space of Vacua and Modular Functors

Having defined the space of vacua on labeled Riemann surfaces, an immediate question is to study the structure of these spaces. We give a conjecture on the construction of the space of vacua by gluing Riemann surfaces. This conjecture can be roughly stated as that $\tilde{\Sigma}\to N(\tilde{\Sigma})$ is a modular functor if the vertex operator algebra V satisfies a certain finiteness condition.

Let

$$\hat{\Sigma}_1=(\Sigma_1; Q_1, \ldots, Q_N, P'; z_1, \ldots, z_N, z')\ ,$$
$$\hat{\Sigma}_2=(\Sigma_2; Q_{N+1}, \ldots, Q_{N+M}, P''; z_{N+1}, \ldots, z_{N+M}, z'') \tag{8.1}$$

be two pointed Riemann surfaces. Let $D_1=\{|z'|\leqq|q_1|\}$ and $\{D_2=|z''|\leqq|q_2|\}$ the discs near P' and P'' respectively such that they contain none of Q_i's. We cut off D_1 from Σ_1 and D_2 from Σ_2 and glue the two boundaries together by the relation $z'z''=q_1q_2$, we get a $(M+N)$-pointed Riemann surface with the genus equal to the sum of genus of Σ_1 and Σ_2. Similarly, we can glue two marked points on a single pointed Riemann surface. If

$$\hat{\Sigma}'=(\Sigma_1'; Q_1, \ldots, Q_N, P', P''; z_1, \ldots, z_N, z', z'') \tag{8.2}$$

is a $(N+2)$-pointed Riemann surface, we cut off two discs $\{|z'|\leqq|q_1|\}$ and $\{|z''|\leqq|q_2|\}$ which contains none of Q_i's and glue the two boundaries by the relation $z'z''=q_1q_2$, we get a N-pointed Riemann surface with genus increase by 1. In both cases the resulting pointed Riemann surface Σ_q depends only on the product $q=q_1q_2$. Every n-pointed Riemann surface can be obtained in this way by successively gluing the 1-pointed, 2-pointed or and 3-pointed Riemann spheres.

Let V be a rational vertex operator algebra, $W_1, \ldots, W_m$ be a sequence of irreducible representations of V and assume W_s and W_t $(1\leqq s<t\leqq m)$ are dual with

each other, a linear functional $f \in (W_1 \otimes \cdots \otimes W_m)^*$ is called *contractable at the s-th argument and the t-th argument* if the following holds: let $\{e_i\}_{i=1}^{\infty}$ be a homogeneous basis of W_s, $\{e_i'\}_{i=1}^{\infty}$ be its dual of W_t (so $\langle e_i, e_j'\rangle = \delta_{ij}$), for every $x_i \in W_i$ $1 \leqq i \leqq m$, $i \neq s, t$), the series

$$\sum_{i=1}^{\infty} f(x_1 \otimes \cdots \otimes e_i \otimes \cdots \otimes e_i' \otimes \cdots \otimes x_m) \tag{8.3}$$

converges absolutely. For such f, the *contraction of f at the s-th argument and the t-th argument* is defined to be the vector in $(W_1 \otimes \ldots \hat{W}_s \ldots \hat{W}_t \ldots \otimes W_m)^*$ (where $\hat{}$ denotes the omission) whose value on $\bigotimes_{i \neq s,t} x_i$ is (8.3).

Conjecture 8.1. *Assume vertex operator algebra V satisfies the conditions*: (1) *V is rational.* (2) *V is a sum of highest weight representations of its Virasoro algebra,* $\dim(V_0) = 1$. (3) *V is an irreducible representation. Let $W_{\alpha_1}, \ldots, W_{\alpha_n}$ be a complete list of irreducible representations of V. Then*

(1) *For two labeled Riemann surfaces*:

$$\tilde{\Sigma}_{1,\alpha} = (\Sigma_1; Q_1, \ldots, Q_N, P'; z_1, \ldots, z_N, z'; W_1, \ldots, W_N, W_\alpha),$$

$$\tilde{\Sigma}_{2,\alpha'} = (\Sigma_2; P'', Q_{N+1}, \ldots, Q_{N+M}; z_{N+1}, \ldots, z_{N+M}; W_\alpha', W_{N+1}, \ldots, W_{N+M}).$$

let

$$\tilde{\Sigma}_q = (\Sigma_q; Q_1, \ldots, Q_{N+M}; z_1, \ldots, z_{N+M}; W_1, \ldots, W_{N+M})$$

be the labeled Riemann surface obtained by gluing the points P′ and P″ with the parameter q described as above. Let be $q_{N+1}^{L_0}$ be the operator which acts on $(W_1 \otimes \cdots \otimes W_N \otimes W_\alpha)^*$ as the operator q^{L_0} on the $(N+1)$-th factor W_α. For every

$$x_1 \in N(\tilde{\Sigma}_{1,\alpha}) \subset (W_1 \otimes \cdots \otimes W_N \otimes W_\alpha)^*$$

and

$$x_2 \in N(\tilde{\Sigma}_{2,\alpha'}) \subset (W_{N+1} \otimes \cdots \otimes W_{N+M} \otimes W_\alpha')^*,$$

$q_{N+1}^{L_0} x_1 \otimes x_2$ *is contractable at $(N+1)$-th argument and $(N+2)$-th argument and the contraction is in $N(\tilde{\Sigma}_q)$. This defines a linear map L_α: $N(\tilde{\Sigma}_{1,\alpha}) \otimes N(\tilde{\Sigma}_{2,\alpha'}) \to N(\tilde{\Sigma}_q)$. The linear map*

$$\bigoplus_{i=1}^{n} L_{\alpha_i} : \bigoplus_{i=1}^{n} N(\tilde{\Sigma}_{1,\alpha_i}) \otimes N(\tilde{\Sigma}_{2,\alpha_i'}) \to N(\tilde{\Sigma}_q)$$

is a linear isomorphism.

(2) *For a labeled Riemann surface*

$$\tilde{\Sigma}_\alpha' = (\Sigma'; Q_1, \ldots, Q_N, P', P''; z_1, \ldots, z_N, z', z''; W_1, \ldots, W_N, W_\alpha, W_{\alpha'}),$$

let

$$\tilde{\Sigma}_q = (\Sigma_q; Q_1, \ldots, Q_{N+M}; z_1, \ldots, z_{N+M}; W_1, \ldots, W_{N+M})$$

be the labeled Riemann surface obtained by gluing $\tilde{\Sigma}_\alpha'$ at the points P′ and P″ described as above. Then for every $x \in N(\tilde{\Sigma}_\alpha')$, $q_{N+1}^{L_0} x$ is contractable at $(N+1)$-th argument and $(N+2)$-th argument, and the contraction defines a linear map

L_α: $N(\tilde{\Sigma}'_\alpha) \to N(\tilde{\Sigma}_q)$. And the map

$$\bigoplus_{i=1}^{n} L_{\alpha_i}: \bigoplus_{i=1}^{n} N(\tilde{\Sigma}'_{\alpha_i}) \to N(\tilde{\Sigma}_q)$$

is a linear isomorphism.
(3) *$N(\tilde{\Sigma})$ is finite dimensional for every $\tilde{\Sigma}$.*

There are three infinite families of known vertex operator algebras satisfies these conditions: the vertex operator algebras associated to integrable highest weight representations of the affine Kac–Moody algebras [FZ]; the vertex operator algebras associated to the minimal modules of the Virasoro algebra [Wa]; and the vertex operator algebras associated to positive even lattices [FLM, Do1]. In the lattice case, the relation of vertex operator algebras with the path integral approach is discussed in [T]. These three families corresponds to WZW-models, Minimal Models and the torus models in conformal field theory, respectively (see e.g., [BS, TUY, KNTY] for other approaches to conformal field theory). And the Moonshine module [FLM1, Bo1, FLM2, Bo2] also satisfies these conditions, and it is proved in [Do2] that the moonshine module is rational and it has the unique irreducible representation.

This conjecture reduces to the construction of the space of vacua on any labeled Riemann surfaces to the construction of the space of vacua on 1, 2 and 3-labeled spheres with representations assigned at each marked point irreducible, these cases have been discussed in Sect. 7.

For the vertex operator algebras associated to integrable highest weight representations of the affine Lie algebras, the results in [TYU] imply the above conjecture.

The truth of Conjecture 8.1 together with Theorem 6.2 implies the corresponding gluing properties for correlation functions. Let $x_1 \in N(\tilde{\Sigma}_{1,\alpha})$ and $x_2 \in N(\tilde{\Sigma}_{2,\alpha'})$ as in the Conjecture, $x \in N(\tilde{\Sigma}_q)$ be their contraction. Let $\{e_i\}$ be a homogeneous basis of W_α and $\{e'_i\}$ be its dual basis in $W_{\alpha'}$. Let $P_1, \ldots, P_m$ be m points on Σ_q such that the first s points are in Σ_1 and the last $m-s$ points are in Σ_2. Then the correlation function on Σ_q

$$F_x(v_1, \ldots, v_{N+M}; a_1, \ldots, a_m; P_1, \ldots, P_s)$$

associated to x, $v_i \in W_i$ $(i=1, \ldots, N+M)$, $a_i \in V (i=1, \ldots, m)$ is equal to

$$\sum_{i=1}^{\infty} F_{x_1}(v_1, \ldots, v_N, e_i; a_1, \ldots, a_s; P_1, \ldots, P_s)$$
$$\cdot F_{x_2}(e'_i, v_{N+1}, \ldots, v_{N+M}; a_{s+1}, \ldots, a_m; P_{s+1}, \ldots, P_m) .$$

Recall the definition of modular functors in [Se]. Let Φ be a finite set of labels which contains 1 and has an involution $\phi \mapsto \bar{\phi}$ such that $\bar{1}=1$. Let $\mathfrak{R}_\Phi$ be the category whose objects are disjoint unions of Riemann surfaces with each boundary circle parametrized and equipped with a label from Φ. A morphism in $\mathfrak{R}_\Phi$ is several sewing operations which sew together pairs of parametrized boundaries, and we allow a pair circles to be identified only if they have the same labels. A modular functor is a holomorphic functor from $\mathfrak{R}_\Phi$ to finite dimensional complex vector spaces satisfying the certain properties [Se]. If we take the label set Φ to the set of irreducible representations of V, and the involution in Φ is given by the dual representations, the label 1 is the adjoint representation. And we modify the category $\mathfrak{N}_\Phi$ by taking the objects as disjoint unions of labeled Riemann surfaces and a morphism as the gluing operations on pairs of labeled points with dual labels described as above. For an object $O=\tilde{\Sigma}_1 \amalg \cdots \amalg \tilde{\Sigma}_k$, we define $N(O)=N(\tilde{\Sigma}_1)\otimes\cdots\otimes N(\tilde{\Sigma}_k)$. Then Conjecture 8.1 says that N defines a functor from the category $\mathfrak{N}_\Phi$ to finite dimensional vector spaces which satisfies similar properties with a modular functor. One of the conditions of a modular functor is that when $\{X_b\}_{b\in B}$ is a holomorphic family of surfaces parametrized by a complex manifold B the spaces corresponding to $\{X_b\}$ forms a holomorphic vector bundle on B. In our situation, we can define the sheaf of vacua on a local family of N-labeled Riemann surfaces as in [TUY], presumably the Virosoro algebra gives a connection of the sheaf of the vacua. However in order to generalize the results in [TUY] to arbitrary rational vertex operator algebras satisfied the assumptions given earlier in the section, we need a structure theory for rational vertex operator algebras which is not available today.

The spaces of vacua on a 1-pointed Riemann surface with 0-section assigned at the puncture for the vertex operator algebras associated to integrable highest weight representations of Kac–Moody affine Lie algebras (they are the same as the spaces of vacua defined in [TUY], see Sect. 7) can be identified with the space of global sections of certain line bundles on the moduli space of stable G-bundles on the underlining Riemann surface [Fa]. We expect similar geometric interpretations for the spaces of vacua associated to other rational vertex operator algebras, e.g., the space of vacua for the moonshine module may relate to the moduli space of M-structure (M is the Monster group) on the underlying Riemann surface. And we expect that the vertex operators $Y(a, z)$ for a primary and their correlation functions associated to a vector in the space of vacua also have interesting geometric meanings.

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