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Tau Functions for the Dirac Operator on the Poincaré Disk

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Abstract: In this paper we define tau functions for holonomic fields associated with the Dirac operator on the Poincaré disk. The deformation analysis of the tau functions is worked out and in the case of the two point function, the tau function is expressed in terms of a Painlevé function of type VI.

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Introduction

In this paper we introduce τ-functions for the Dirac operator on the Poincaré disk based on the formalism in [12]. The formalism of the first author in [11, 12] is in turn a geometric reworking of the analysis of Sato, Miwa, and Jimbo [15] (SMJ henceforth). The prototypical examples of the holonomic fields that are the central objects of the SMJ theory are the scaling limits of the two dimensional Ising model from above and below the critical point. The SMJ theory of holonomic fields provides a beautiful setting for the earlier result of Wu, McCoy, Tracy, and Barouch [1, 8, 17, 20] that the scaled two point functions of the two dimensional Ising model can be expressed in terms of Painlevé functions of the third kind. Our work also owes something to the papers of [18, 6] on monodromy preserving deformation theory in the Poincaré disk. In the course of explaining our work we will point out differences with [18].

Our principal goal in this paper is to define and analyze the correlation functions for a family of quantum field theories on the Poincaré disk that are analogues of the holonomic quantum field theories defined in the Euclidean plane. Below we will sketch how the theory goes in the Euclidean version in [12] and we will point the reader to the parallel developments in the hyperbolic case.

Our starting point is a formula for the Schwinger functions of holonomic fields that can be written

$$\tau(a_1, a_2, \ldots, a_n) = \det(D^{a,\lambda}) . \tag{1}$$

where $\tau(a_1, a_2, \ldots, a_n)$ is the vacuum expectation for a product of quantum fields $\Phi_{\lambda_j}(a_j)$ [11]. In this formula the operator $D^{a,\lambda}$ is a singular Dirac operator (on Euclidean space) with a domain that incorporates functions with prescribed branching at the point a_j. The functions in the domain of $D^{a,\lambda}$ are to have monodromy multiplier $e^{i2\pi\lambda_j}$ in a counterclockwise circuit of a_j. In [12] formula (1) is given rigorous mathematical sense by first localizing $D^{a,\lambda}$ away from branch cuts emerging from the point a_j. The localization for $D^{a,\lambda}$ takes place in the exterior of a set S which is the union of strips containing the branch cuts. The localization of $D^{a,\lambda}$ is completely characterized by a family of subspaces $W^{a,\lambda} \subset H^{\frac{1}{2}}(\partial S)$ of the Sobolev space on the boundary, ∂S, of S. These subspaces are the restriction to ∂S of functions which are "locally" in the null space of $D^{a,\lambda}$. The subspaces $W^{a,\lambda}$ are contained in a restricted Grassmannian of subspaces of $H^{\frac{1}{2}}(\partial S)$ over which there sits a determinant line bundle [16]. As is explained in more detail in [11, 12], defining a determinant for the family $D^{a,\lambda}$ is morally equivalent to finding a trivialization δ for the det* bundle over the family of subspaces $W^{a,\lambda}$. Comparing the canonical section σ of det* with δ gives us a formula for the determinant,

$$\det(D^{a,\lambda}) = \frac{\sigma(W^{a,\lambda})}{\delta(W^{a,\lambda})} . \tag{2}$$

The analogue of this formula on the Poincaré disk is (6.7) below. Both the canonical section, σ, and the trivialization, δ, are determined by (different) projections on the subspaces $W^{a,\lambda}$. An explicit representation of these projections is obtained in terms of the Green function $G^{a,\lambda}$ for $D^{a,\lambda}$. The analogue of this projection formula on the Poincaré disk is (6.2) below. In order to analyze (2) it is useful to compute the logarithmic derivative

$$d_a \log \tau \,.$$

The resulting formula simplifies dramatically because the derivative of the appropriate projection in the a variables is *finite rank.* This in turn is a consequence of the analogous result for the Green function, $G^{a,\lambda}$, namely,

$$\begin{aligned} \partial_{a_\nu} G^{a,\lambda}(x,y) &= c_\nu \overline{W_\nu^*(x)} \otimes W_\nu(y)\,, \\ \bar{\partial}_{a_\nu} G^{a,\lambda}(x,y) &= c_\nu \overline{W_\nu(x)} \otimes W_\nu^*(y)\,. \end{aligned} \tag{3}$$

For simplicity we have taken some liberties in displaying only a schematic version of (3). The precise version of (3) in the Poincaré disk is (4.50) below. The functions $W_\nu(x)$ and $W_\nu^*(x)$ are multivalued solutions to the Dirac equation which are branched at the points a_j, $j=1,2,\ldots,n$ with monodromy multipliers $e^{2\pi i\lambda_j}$. Because of their role in (3) we refer to the wave functions $W_\nu(x)$ and $W_\nu^*(x)$ *as response functions.* The formula for $d\log\tau$ which one obtains is

$$d\log\tau = \sum_{\nu=1}^{n} \{A_\nu da_\nu + B_\nu d\bar{a}_\nu\}\,, \tag{4}$$

where A_ν and B_ν are low order Fourier coefficients of the response functions W_ν *and* $W_\nu^*(x)$ in the local expansions of these functions about the points a_ν. The version of this formula in the Poincaré disk is (6.8) below.

We assemble the $2n$ response functions into a single column vector

$$\mathbf{W}(x) := \begin{bmatrix} W(x) \\ W^*(x) \end{bmatrix},$$

with

$$W(x) = [W_1(x), W_2(x), \ldots, W_n(x)]^\tau\,,$$

and

$$W^*(x) = [W_1^*(x), W_2^*(x), \ldots, W_n^*(x)]^\tau\,,$$

where $[\cdot]^\tau$ is the transpose. Then $\mathbf{W}(x)$ satisfies a holonomic system

$$d_{x,a}\mathbf{W}(x) = \Omega(x,a)\,\mathbf{W}(x)\,. \tag{5}$$

in the variables (x, a), where Ω is a matrix valued one-form whose entries are determined by the low order local expansion coefficients for W_ν and W_ν^* near the branch points a_j. The analogue of (5) in the Poincaré disk is contained in (5.3) and (5.20) below. The consistency condition for the holonomic system (5) is the zero curvature condition

$$d\Omega = \Omega \wedge \Omega\,. \tag{6}$$

These are called the *deformation equations*. The analogue in the Poincaré disk is (5.26) below. These deformation equations characterize certain of the low order expansion coefficients of W_ν and W_ν^*. In particular it is possible to express the coefficients A_ν and B_ν that occur in the formula for the logarithmic derivative of the τ-function in terms of the solution to (6). The version of this relationship in hyperbolic space can be found in (6.16) and (6.17) below. The existence of the holonomic system and the resultant deformation equations is, incidentally, the one part of this theory that depends strongly on the fact that the Poincaré disk is a symmetric space. The global symmetries of Dirac operator on the Poincaré disk give rise to the holonomic system for $\mathbf{W}$.

In conclusion one sees that it is possible to characterize the logarithmic derivative of the τ-function in terms of certain solutions to the deformation equations. In the special case of the two point function in the Euclidean problem SMJ showed that it is possible to integrate the deformation equations explicitly in terms of Painlevé transcendents of the fifth kind. In the special Ising case a reduction to Painlevé III is possible (see [8]). In Sects. 5 and 6 of this paper we carry out the analysis of the two point case in the hyperbolic setting. We find that the two point deformation equations can be integrated in terms of Painlevé transcendents of type VI (Theorem 5.0) and that the logrithmic derivative of the τ-function is a rational function of the solution to P_{VI} and its derivative (see (6.19)).

We can now explain the principal differences between this paper and [18]. First, because the response functions are fundamental in this approach to holonomic fields, the basic finite dimensional space of wave functions for us is the $2n$ dimensional space spanned by W_ν and W_ν^*. In [18], as in SMJ [15], the basic finite dimensional space is the n dimensional space spanned by the wave functions that are globally in L^2. Formulating the holonomic system in this setting leads to extra complications, and the deformation equations are not explicitly worked out in [18]. Because of the difficulties with a direct formulation of the deformation equations, one of us introduced a hyperbolic Laplace transform of the associated holonomic system and found that the result was a Fuchsian system of ordinary differential equations in the complex plane. Tau functions for this transformed system were introduced by adopting the τ-functions for the associated Schlesinger theory (see also [15]). In [6] the behavior of these τ-functions was examined in the limit $R \to \infty$ and shown to coincide with the τ-functions for the Euclidean theory. We have yet to explore, the relation between the τ-functions introduced in [6, 18] and the ones we consider in this paper. The algebraic complexity of both sets of deformation equations makes comparisons difficult.

This last point suggests an important question. Why should one believe that the τ-functions we have introduced in this paper are the Schwinger functions for some quantum field theory on the Poincaré disk? The reason we believe that there is such a relation is to be found in [12]. In that paper a transfer matrix formalism was established that allowed one to connect formula (2) above with the scaling limit of some lattice field theories defined in [10]. It is also this connection that suggested the "minimal singularity" definition for the domain of $D^{a,\lambda}$ that we adopt here. We do not think that it would be economical to try to produce a lattice theory which would scale to the theory we present here. Because of the connection with the Ising model, however, this is an interesting problem. We should remark at this point that the analogue of the Ising model on the Poicaré disk is not considered in this paper. There is such an analogue and the deformation theory in Sect. 5 is relevant to its analysis, but the singular Dirac operator whose determinant gives the correlation functions (for the $T<T_c$ scaling limit) is different from any of the minimal singularity theories for $\lambda_j=\pm 1/2$. We expect to discuss this analogue in another place. To return to the quantum field theory connection; there is a preprint of Segal on conformal field theory which sketches a "vertex operator" formalism that appears well suited to address such questions as Osterwalder–Schrader positivity. One of us is currently investigating this formalism.

Finally we would like to mention that this paper is organized in almost precisely the reverse of the logical progression (1)–(6) above. The reason for this is that the existence theory for the response functions and the Green function proceeds most simply in such reverse order. We have included a table of contents to help the reader obtain some perspective.

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1. The Dirac Operator on the Hyperbolic Disk

A covering of the frame bundle. We will begin by identifying a Dirac operator on the rhyperbolic disk, $\mathbf{D}_R$, of radius R. The Dirac operator will be defined in the usual fashion except that we will use the simply connected covering of SO(2) rather than the two fold covering in our construction. This is done principally because the extra parameter k associated with the representation theory of the reals does not significantly complicate any of our analysis and we believe its inclusion might prove illuminating. Incidentally, the radius R is carried along in all the subsequent calculations precisely because the zero curvature limit $R\to\infty$ is interesting [6].

The additive group of real numbers, $\mathbf{R}$, is the simply connected covering group of the special orthogonal group, SO(2), in two dimensions with the familiar covering map

$$\mathbf{R}\quad \lambda\to R(\lambda):=\begin{bmatrix}\cos\lambda & -\sin\lambda\\ \sin\lambda & \cos\lambda\end{bmatrix}\in \mathrm{SO}(2)\ .$$

To emphasize the role $\mathbf{R}$ plays as a covering group for SO(2) we will sometimes write

$$\mathbf{R}=\widetilde{\mathrm{SO}}(2)\ .$$

Next we define a principal $\widetilde{\mathrm{SO}}(2)$ bundle on $\mathbf{D}_R$, which we call $S(\mathbf{D}_R)$ together with a bundle map,

$$\pi\colon S(\mathbf{D}_R)\to F(\mathbf{D}_R)\ ,$$

onto the frame bundle, $F(\mathbf{D}_R)$, that is compatible with the map R. If $g\in\widetilde{\mathrm{SO}}(2)$ and $s\to sg$ is the right action of $\widetilde{\mathrm{SO}}(2)$ on the bundle $S(\mathbf{D}_R)$ then the compatibility we refer to is,

$$\pi(sg)=\pi(s)R(g)\ , \tag{1.1}$$

where $R(g) \in \mathrm{SO}(2)$ acts on the frame bundle in the usual fashion. By fixing a trivialization of the frame bundle for $\mathbf{D}_R$ we will be able to use (1.1) to define the bundle $\mathrm{S}(\mathbf{D}_R)$. The metric we wish to consider on $\mathbf{D}_R$ is,

$$ds^2 := \frac{R^4(dx_1^2 + dx_2^2)}{(R^2 - x_1^2 - x_2^2)^2}, \tag{1.2}$$

with the associated invariant measure

$$d\mu := \frac{R^4 dx_1 dx_2}{(R^2 - x_1^2 - x_2^2)^2}. \tag{1.3}$$

We fix an orthonormal frame,

$$e_1 := \left(1 - \frac{x_1^2 + x_2^2}{R^2}\right) \frac{\partial}{\partial x_1},$$

$$e_2 := \left(1 - \frac{x_1^2 + x_2^2}{R^2}\right) \frac{\partial}{\partial x_2},$$

which identifies the frame bundle, $F(\mathbf{D}_R)$ with

$$\mathbf{D}_R \times \mathrm{SO}(2).$$

Relative to this trivialization of the frame bundle we identify $S(\mathbf{D}_R) = \mathbf{D}_R \times \widetilde{\mathrm{SO}}(2)$ and the map π with the covering given by

$$\mathbf{D}_R \times \widetilde{\mathrm{SO}}(2) \quad (p, \lambda) \to \pi(p, \lambda) = (p, R(\lambda)) \in \mathbf{D}_R \times \mathrm{SO}(2)$$

for $p \in \mathbf{D}_R$. There is a covariance of this construction with respect to global rigid motions of $\mathbf{D}_R$ that it will be useful for us to understand. Since the (orientation preserving) rigid motions of $\mathbf{D}_R$ are simplest to understand as fractional linear transformations we will introduce the complex notation $z = x_1 + ix_2$. Consider the fractional linear transformation

$$\frac{w}{R} = \frac{A\frac{z}{R} + B}{\bar{B}\frac{z}{R} + \bar{A}}. \tag{1.4}$$

This will be an orientation preserving rigid motion of $\mathbf{D}_R$ provided that

$$\begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix} \in \mathrm{SU}(1,1),$$

where $g \in \mathrm{SU}(1,1)$ if and only if

$$g^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $\det(g) = 1$. The action of a transformation (1.4) on the frame $[e_1, e_2]$ is simplest in the complexified tangent space and so we also introduce the complex vector fields

$$e := \tfrac{1}{2}(e_1 - ie_2)$$

and

$$\bar{e} := \tfrac{1}{2}(e_1 - ie_2).$$

For $g=\begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix}$ write

$$g(z)=R\,\frac{A\frac{z}{R}+B}{\bar{B}\frac{z}{R}+\bar{A}}\,.$$

For $g\in\mathrm{SU}(1,1)$ the pair (g, dg) induces a transformation of the frame bundle which we calculate in the trivialization $\{e_1, e_2\}$ by first determining the action of dg on the complex vector fields e and $\bar{e}$. One finds

$$dg_z(e(z))=u_g(z)\,e(g(z))\,, \tag{1.5}$$

and

$$dg_z(\bar{e}(z))=\overline{u_g(z)}\,\bar{e}(g(z))\,, \tag{1.6}$$

where

$$u_g(z):=\frac{B\frac{\bar{z}}{R}+A}{\bar{B}\frac{z}{R}+\bar{A}} \tag{1.7}$$

is a number of absolute value 1. Now write

$$g'(z)=\frac{1}{2}\begin{bmatrix} u_g(z)+\overline{u_g(z)} & i(u_g(z)-\overline{u_g(z)}) \\ -i(u_g(z)-\overline{u_g(z)}) & u_g(z)+\overline{u_g(z)} \end{bmatrix}.$$

Then without difficulty one sees that (1.5) and (1.6) imply

$$(dg_z\,e_1(z), dg_z e_2(z))=(e_1(g(z)),\, e_2(g(z)))g'(z)\,. \tag{1.8}$$

So that in the trivialization of the frame bundle determined by $\{e_1, e_2\}$ the map induced by (g, dg) is

$$(z, r)\to(g(z), g'(z)r)\,, \tag{1.9}$$

where $r\in\mathrm{SO}(2)$. Next we will show that the map (1.9) is covered by a map on $S(\mathbf{D}_R)$. For fixed $g\in\mathrm{SU}(1,1)$ this clearly comes down to defining a continuous logarithm for $\mathbf{D}_R\ni z\to u_g(z)$. Since the map $u_g(z)$ is continuous ($|A|^2-|B|^2=1$ for $g\in\mathrm{SU}(1,1)$ implies that $\bar{B}\frac{z}{R}+\bar{A}$ is never 0 for $z\in\mathbf{D}_R$) and never 0 it follows that one can define a continuous logarithm $z\to\lambda_g(z)$ so that

$$u_g(z)=e^{i\lambda_g(z)}\,. \tag{1.10}$$

The bundle map,

$$(z, \lambda)\to(g(z), \lambda+\lambda_g(z))\,, \tag{1.11}$$

on $\mathbf{D}_R\times\widetilde{SO}(2)$ then clearly covers the transformation (1.9) on the frame bundle. Of course, the map $\lambda_g(z)$ is not unique; for any integer n the map $\lambda_g(z)+2n\pi$ would serve as well. The fundamental group of SU(1, 1) is $\mathbf{Z}$ and so we do not expect to make a canonical choice of such a logarithm for all elements of SU(1, 1). We do expect that the group of bundle transformations of $S(\mathbf{D}_R)$ which covers the action of SU(1, 1) on $F(\mathbf{D}_R)$ is the simply connected covering group $\widetilde{\mathrm{SU}}(1,1)$. However, since there is no simple (i.e. matrix) model for $\widetilde{\mathrm{SU}}(1,1)$ and since we are more interested in the lifts of individual elements and of one parameter subgroups in SU(1, 1) we will not pursue this matter. However, it will be convenient for descriptive purposes to refer to a pair $(g, \lambda_g(z))$, where λ_g satisfies (1.10) as an element of $\widetilde{\mathrm{SU}}(1,1)$.

The Dirac operator. We will now introduce a Dirac operator and observe that the lift of the action of an element in SU(1, 1) to the bundle $S(\mathbf{D}_R)$ also acts on the solution space of the Dirac equation.

The Dirac operator we define will act in the space of sections of an associated bundle for $S(\mathbf{D}_R)$. The bundle we have in mind is one associated with a complex unitary representation of $\widetilde{\mathrm{SO}}(2)$ given by

$$\widetilde{\mathrm{SO}}(2)\ni\lambda \to r_k(\lambda) := \begin{bmatrix} e^{-i(k+1/2)\lambda} & 0 \\ 0 & e^{-i(k-1/2)\lambda} \end{bmatrix}, \tag{1.12}$$

where k is an arbitrary real number. We write

$$S(\mathbf{D}_R)\times_k \mathbf{C}^2$$

for this associated bundle. One can view the elements of this bundle in the usual fashion as equivalence classes of pairs (s, v), where $s\in S(\mathbf{D}_R)$ and $v\in\mathbf{C}^2$ and (s, v) is equivalent to $(s\lambda, r_k(\lambda)v)$ for any $\lambda\in\widetilde{\mathrm{SO}}(2)$. Because we are working with a fixed trivialization of the bundle $S(\mathbf{D}_R)$, the associated bundle $S(\mathbf{D}_R)\times_k\mathbf{C}^2$ can be trivialized by the map

$$(z, \lambda, v) \to (z, r_k(\lambda)^{-1}v)\in\mathbf{D}_R\times\mathbf{C}^2 .$$

To avoid confusion we will always work in this trivialization of the associated bundle.

The reason for choosing the representation (1.12) to define a bundle of spinors for the Dirac operator has to do with the existence of a complex representation of the Clifford algebra of $\mathbf{R}^2$ on $\mathbf{C}^2$ which behaves well with respect to the representation (1.12). The representation is generated by

$$\gamma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\gamma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

If we define

$$\gamma(v) := \gamma_1 v_1 + \gamma_2 v_2 ,$$

then one can easily confirm that

$$r_k(\lambda)\gamma(v)r_k(\lambda)^{-1} = \gamma(R(\lambda)v) . \tag{1.13}$$

This is the relation that allows one to define a Dirac operator that will have the appropriate covariance with respect to $\widetilde{\mathrm{SU}}(1, 1)$.

In order to write down the Dirac operator we first recall some facts about connections. On a principal bundle, once a local frame has been picked, a connection is represented by a Lie algebra valued one form over the domain of the trivialization in the base. In our case we start with the global frame $\{e_1, e_2\}$. The Levi–Civita connection form ω for this frame (which is determined by the torsion 0 condition $\nabla_{e_1}e_2 - \nabla_{e_1} - [e_1, e_2] = 0$) is

$$\omega(e_1) = -\frac{2x_2}{R^2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and

$$\omega(e_2)=\frac{2x_1}{R^2}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$

Since the map $\pi: S(\mathbf{D}_R) \to F(\mathbf{D}_R)$ is a covering map on the fiber one can pull back the connection on $F(\mathbf{D}_R)$ to a connection, $\hat{\omega}$, on $S(\mathbf{D}_R)$. One finds the connection one form relative to our standard trivialization of $S(\mathbf{D}_R)$ is

$$\hat{\omega}(e_1)=-\frac{2x_2}{R^2},$$

$$\hat{\omega}(e_2)=\frac{2x_1}{R^2},$$

where the Lie algebra of $\widetilde{\mathrm{SO}}(2)$ is identified with $\mathbf{R}$ in the usual fashion. The connection $\hat{\omega}$ on the principal bundle $S(\mathbf{D}_R)$ induces a connection on the associated bundle $S(\mathbf{D}_R)\times_k \mathbf{C}^2$. In the trivialization of the associated bundle defined above one finds the connection one form

$$dr_k\,\hat{\omega}(e_1)=\begin{bmatrix}i(k+\frac{1}{2}) & 0\\ 0 & i(k-\frac{1}{2})\end{bmatrix}\frac{2x_2}{R^2},$$

$$dr_k\,\hat{\omega}(e_2)=-\begin{bmatrix}i(k+\frac{1}{2}) & 0\\ 0 & i(k-\frac{1}{2})\end{bmatrix}\frac{2x_1}{R^2}.$$

The Dirac operator (in the trivialization $\mathbf{D}_R\times\mathbf{C}^2$ defined above for the associated bundle) is then

$$D_k=\sum_{j=1}^{2}\gamma_j(e_j+dr_k\,\hat{\omega}(e_j)) \tag{1.14}$$

or in complex notation

$$D_k=\begin{bmatrix}0 & K_{k-\frac{1}{2}}\\ -K^*_{k-\frac{1}{2}} & 0\end{bmatrix},$$

where

$$K_{k-\frac{1}{2}}:=2\left\{\left(1-\frac{|z|^2}{R^2}\right)\partial_z-\frac{(k-\frac{1}{2})}{R^2}\,\bar{z}\right\} \tag{1.15}$$

and

$$K^*_{k-\frac{1}{2}}:=-2\left\{\left(1-\frac{|z|^2}{R^2}\right)\bar{\partial}_z+\frac{(k+\frac{1}{2})}{R^2}\,z\right\}. \tag{1.16}$$

We have written K^*_k for the formal adjoint of K_k with respect to the inner product,

$$\int_{\mathbf{D}_R}\overline{f(x)}\,g(x)\,d\mu(x),$$

on $\mathbf{D}_R\times\mathbf{C}$. Thus the Dirac operator (1.14) is formally skew symmetric with respect to the inner product,

$$\int_{\mathbf{D}_R}\overline{f(x)}\cdot g(x)\,d\mu(x), \tag{1.17}$$

on $\mathbf{D}_R \times \mathbf{C}^2$, where $u \cdot v = u_1 v_1 + u_2 v_2$. The indexing by k in (1.14) and (1.15) has been done so that $k=0$ corresponds to the Dirac operator that one would obtain from the usual two fold covering of SO(2) by itself given by

$$g \to g^2 .$$

We remark that K_k is the familar Maass operator (see, e.g., [2]).

Covariance of the Dirac operator. One of the principal objects of study for us will be multivalued solutions to the Dirac equation $D_k \psi = 0$ (or more precisely, solutions to the massive version of the Dirac equation $(mI - D_k)\psi = 0$) that are branched at points $a_j \in \mathbf{D}_R$ for $j = 1, 2, \ldots n$. In order to understand these solutions to the Dirac equation in a neighborhood of the point a_j it will be convenient to employ coordinates for $\mathbf{D}_R$ which are centered at a_j. Of course, we can use a fractional linear map,

$$w = R \frac{z - a_j}{\bar{a}_j z - R},$$

to make such a change of coordinates. However, the function $w \to \psi(z(w))$ no longer satisfies the Dirac equation in the w variables. This inconvenience can be overcome by using the covariance of solutions to the Dirac equation under $\widetilde{SU}(1, 1)$ that is a consequence of the fact that the Dirac operator is a natural operator on the associated bundle of spinors which depends on a choice of spin structure and on the representation of the spin group r_k but not on the choice of local frame or base coordinates used to realize (1.14) as a differential operator. We will now explain this covariance from the passive point of view in which fractional linear maps on $\mathbf{D}_R$ are regarded as a change of coordinates on $\mathbf{D}_R$. Let

$$w = g(z) = R \frac{Az + BR}{\bar{B}z + \bar{A}R}$$

denote a fractional linear map with

$$\begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix} \in \mathrm{SU}(1, 1) .$$

The inverse coordinate change is

$$z(w) = R \frac{-\bar{A}w + BR}{\bar{B}w - AR} .$$

Also

$$e(z) := \left(1 - \frac{|z|^2}{R^2}\right) \partial_z = v(w) \left(1 - \frac{|w|^2}{R^2}\right) \partial_w = v(w)\, e(w) ,$$

where

$$v(w) := \frac{\bar{B}w - AR}{B\bar{w} - \bar{A}R} .$$

Taking complex conjugates one has

$$\bar{e}(z) = \overline{v(w)}\, \bar{e}(w)$$

as well. Thus the coordinate representation of the Dirac operator in the w coordinates is

$$D_k^{(w)} = \begin{bmatrix} 0 & v(w)\,e(w) - \dfrac{(k-\frac{1}{2})}{R^2}\bar{z}(w) \\ \overline{v(w)}\,\bar{e}(w) + \dfrac{(k+\frac{1}{2})}{R^2} z(w) & 0 \end{bmatrix} .$$

Now choose a continuous logarithm, $i\lambda(w)$, for the function $w \to v(w)$ so that

$$v(w) = e^{i\lambda(w)} ,$$

and define

$$v^l := e^{il\lambda} ,$$

for $l \in \mathbf{R}$. Let

$$M_k := \begin{bmatrix} v^{k+\frac{1}{2}} & 0 \\ 0 & v^{k-\frac{1}{2}} \end{bmatrix} .$$

Then the covariance property of the Dirac operator under the coordinate change $z \leftarrow w$ is simply expressed by

$$M_k^{-1}\, D_k^{(w)}\, M_k = \begin{bmatrix} 0 & e(w) - \dfrac{(k-\frac{1}{2})}{R^2}\bar{w} \\ \bar{e}(w) + \dfrac{(k+\frac{1}{2})}{R^2} w & 0 \end{bmatrix} . \tag{1.18}$$

Observe that the right-hand side of (1.18) has precisely the same form in the w coordinates that the Dirac operator has in the z coordinates. Thus if $\psi(z)$ is solution to the Dirac equation $D_k\psi = 0$ in the z coordinates, then $M_k^{-1}(w)\,\psi(z(w))$ will be a solution to exactly the same equation in the w coordinates. An observation which is useful in checking (1.18) with a minimum of computation is

$$e(w)\,(v^l) = lv^l\, e(w)(i\lambda) = lv^l v^{-1}\, e(w)(v) ,$$

with

$$e(w)\,(v) = \left(1 - \frac{|w|^2}{R^2}\right) \partial_w \left(\frac{\bar{B}w - AR}{B\bar{w} - \bar{A}R}\right) = \left(1 - \frac{|w|^2}{R^2}\right) \left(\frac{\bar{B}}{B\bar{w} - \bar{A}R}\right) ,$$

and

$$\bar{e}(w)\,(v^l) = lv^l\, \bar{e}(w)(i\lambda) = -lv^l v\, \bar{e}(w)(v^{-1}) ,$$

with

$$\bar{e}(w)\,(v^{-1}) = \left(1 - \frac{|w|^2}{R^2}\right) \partial_{\bar{w}} \left(\frac{B\bar{w} - \bar{A}R}{\bar{B}w - AR}\right) = \left(1 - \frac{|w|^2}{R^2}\right) \left(\frac{B}{\bar{B}w - AR}\right) .$$

We can rephrase this observation in a slightly different form that will be useful for us:

Proposition 1.0. *Suppose that $\Psi(z)$ is a solution to the Dirac equation*

$$(m - D_k)\,\Psi = 0 .$$

If $g=\begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix} \in SU(1,1)$ *and* $z \to \lambda_g(z)$ *is a continuous function of* z *defined so that*

$$v(g,z) := \frac{B\bar{z}-\bar{A}R}{\bar{B}z-AR} = e^{i\lambda_g(z)} ,$$

then the function,

$$z \to \tilde{\Psi}(z) = \begin{bmatrix} v(g,z)^{k+\frac{1}{2}} & 0 \\ 0 & v(g,z)^{k-\frac{1}{2}} \end{bmatrix} \Psi(g^{-1}z) , \tag{1.20}$$

remains a solution to the same equation $(m-D_k)\tilde{\psi}=0$.

It will also be useful to identify the infinitesimal symmetries of the $\underset{\sim}{\text{D}}$irac operator associated with lifts of the one parameter subgroups of SU(1, 1) into $\widetilde{\mathrm{SU}}(1,1)$. It will be convenient to introduce the notation,

$$\operatorname{ch} t = \cosh t ,$$

$$\operatorname{sh} t = \sinh t ,$$

$$\operatorname{th} t = \tanh t .$$

One choice of three one parameter subgroups of SU(1, 1) that generate the Lie algebra su(1, 1) is given by

$$t \to \begin{bmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{bmatrix}$$

with generator

$$X_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ,$$

and

$$t \to \begin{bmatrix} \operatorname{ch} t & i\operatorname{sh} t \\ -i\operatorname{sh} t & \operatorname{ch} t \end{bmatrix}$$

with generator,

$$X_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} ,$$

and

$$t \to \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$$

with generator,

$$X_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} .$$

For

$$g_j(t) := e^{-tX_j}$$

one finds

$$v(g_1(t), z) = \frac{R - \mathrm{th}(t)\bar{z}}{R - \mathrm{th}(t)z},$$

$$v(g_2(t), z) = \frac{R - i\mathrm{th}(t)\bar{z}}{R + i\mathrm{th}(t)z},$$

and

$$v(g_3(t), z) = e^{2it}.$$

For $j = 1, 2$ we choose the function $\lambda_{g_j(t)}(z)$ by normalizing it so that $\lambda_{g_j(t)}(0) = 0$. For $j = 3$ we choose $\lambda_{g_3(t)}(z) = 2t$. Define

$$F_j \Psi := \frac{d}{dt}\left\{\begin{bmatrix} v(g_j(t), z)^{k+\frac{1}{2}} & 0 \\ 0 & v(g_j(t), z)^{k-\frac{1}{2}} \end{bmatrix} \Psi(g_j(-t)z)\right\}_{t=0}. \tag{1.21}$$

Then the F_j are infinitesimal symmetries for the Dirac operator . It is very convenient to introduce the complexified infinitesimal symmetries,

$$M_1 := \tfrac{1}{2}(F_1 - iF_2), \tag{1.22}$$

$$M_2 := \tfrac{1}{2}(F_1 + iF_2), \tag{1.23}$$

and

$$M_3 := -\tfrac{i}{2} F_3 \tag{1.24}$$

The reason this is convenient is that the commutation relations for M_j are

$$[M_1, M_2] = -2M_3,$$

$$[M_1, M_3] = M_1,$$

$$[M_2, M_3] = -M_2.$$

The last two show that M_1 and M_2 are ladder operators on the eigenspaces of M_3 and this will allow us to parametrize local expansions of solutions to the Dirac equation in a sensible fashion.

2. Local Expansions

Eigenfunctions for infinitesimal rotations. We next wish to consider multivalued solutions to the Dirac equation,

$$(m - D_k)\Psi = 0,$$

which are branched at a point $a \in \mathbf{D}_R$ with fixed monodromy about a. Following SMJ [15] our principal tool in understanding such solutions will be to look at expansions in eigenfunctions for the infinitesimal "k-rotations" about the point a. To begin we consider the case $a = 0$. At $a = 0$ the infinitesimal generator of k-rotations is M_3. In geodesic polar coordinates (r, θ) centered at 0 we have

$$M_3 = \begin{bmatrix} -i\partial_\theta + k + \frac{1}{2} & 0 \\ 0 & -i\partial_\theta + k - \frac{1}{2} \end{bmatrix},$$

where the relation between the complex Cartesian coordinate $z=x_1+ix_2$ and geodesic polar coordinates is

$$z=\mathrm{Re}^{i\theta}\,\mathrm{th}\,\frac{r}{2}\,. \tag{2.0}$$

Of course, geodesic polar coordinates are not uniquely determined at $a=0$. One must also specify the geodesic $\theta=0$. Thus when we speak of geodesic polar coordinates we will suppose that a choice of the geodesic $\theta=0$ has been made. This will be important for us since the local constructions in this section will give rise to multivalued functions. These multivalued functions will have different branches and in this section we will pick out such branches by using the geodesic ray $\theta=0$ as a branch cut.

We turn now to the eigenvalue equation,

$$M_3\,\Psi(r,\theta)=(k+\lambda)\,\Psi(r,\theta)\,, \tag{2.1}$$

whose solutions will have a monodromy multiplier,

$$e^{2\pi i(\lambda\pm\frac{1}{2})}=-e^{2\pi i\lambda}\,,$$

after a counterclockwise circuit of 0. We will now analyse the mutlivalued solutions of the Dirac equation $(m-D_k)\,\Psi=0$ which are also eigenfunctions (2.1) of M_3. In studying solutions to the Dirac equation $(m-D_k)\,\Psi=0$ it is very convenient to introduce the associated Helmholtz equation

$$(m-D_k)^*\,(m-D_k)\,\Psi=(m^2-D_k^2)\,\Psi=0\,. \tag{2.2}$$

In the notation of the previous section this is

$$\begin{bmatrix} m^2+K_{k-\frac{1}{2}}\,K^*_{k-\frac{1}{2}} & 0 \\ 0 & m^2+K^*_{k-\frac{1}{2}}\,K_{k-\frac{1}{2}} \end{bmatrix}\Psi=0\,.$$

We introduce

$$\Delta_k=-K^*_\kappa\,K_\kappa+\frac{4\kappa(\kappa+1)}{R^2} \tag{2.3}$$

and a little calculation shows

$$\Delta_{\kappa+1}=-K_\kappa K^*_\kappa+\frac{4\kappa(\kappa+1)}{R^2}\,. \tag{2.4}$$

For reasons that will be clear later on we introduce a parameter s by

$$(Rs-1)^2=m^2R^2+4k^2\,. \tag{2.5}$$

There are two roots to (2.5) and for definiteness we will in the future restrict our attention to the solution s of (2.5) with $Rs>1$. Another form of (2.5) which is useful for us is

$$s\left(s-\frac{2}{R}\right)=m^2+\frac{4k^2-1}{R^2}\,.$$

If one rewrites the Helmholtz equation using (2.3)–(2.5) the result is

$$\begin{bmatrix} \Delta_{k+\frac{1}{2}} & 0 \\ 0 & \Delta_{k-\frac{1}{2}} \end{bmatrix} \Psi = s\left(s-\frac{2}{R}\right)\Psi . \tag{2.6}$$

To analyze the solutions of the Dirac equation which are eigenfunctions of M_3 above we will first consider the problem of finding solutions to (see, e.g., [2])

$$\Delta_\kappa f = s\left(s-\frac{2}{R}\right)f , \tag{2.7}$$

where

$$f(r,\theta) = e^{il\theta}\,\Phi(r) . \tag{2.8}$$

Note that we will have to set $\kappa = k \pm \frac{1}{2}$ and $l = \lambda \pm \frac{1}{2}$ in appropriate places to make use of (2.7) and (2.8) in the Dirac problem. In geodesic polar coordinates one finds

$$K_\kappa = \frac{2}{R}\, e^{-i\theta}\left\{\partial_r - \frac{i}{\operatorname{sh} r}\,\partial_\theta - \kappa \operatorname{th}\frac{r}{2}\right\} , \tag{2.9}$$

$$K_\kappa^* = -\frac{2}{R}\, e^{i\theta}\left\{\partial_r + \frac{i}{\operatorname{sh} r}\,\partial_\theta + (\kappa+1)\operatorname{th}\frac{r}{2}\right\} , \tag{2.10}$$

and

$$\Delta_\kappa = \frac{4}{R^2}\left\{\partial_r^2 + \frac{\operatorname{ch} r}{\operatorname{sh} r}\,\partial_r + \frac{1}{\operatorname{sh}^2 r}\,\partial_\theta^2 - \frac{2i\kappa}{1+\operatorname{ch} r}\,\partial_\theta + \frac{2\kappa^2}{1+\operatorname{ch} r}\right\} . \tag{2.11}$$

We make a substitution to deal with (2.7) and the side condition (2.8) [2, 6]. Introduce the change of variables,

$$\Phi(r) = t^{l/2}(1-t)^{Rs/2}\,F(t) , \tag{2.12}$$

with

$$t = \frac{\operatorname{ch} r - 1}{\operatorname{ch} r + 1} = \frac{|z|^2}{R^2} \tag{2.13}$$

in (2.7) using (2.11). One finds that $F(t)$ satisfies the hypergeometric equation

$$t(1-t)\,F'' + (c-(a+b+1)t)\,F' - abF = 0 \tag{2.14}$$

with

$$a = \frac{Rs}{2} - \kappa, \quad b = \frac{Rs}{2} + \kappa + l, \quad c = l+1 . \tag{2.15}$$

Note that it is precisely to simplify the description of these parameters that s was introduced in (2.5) and (2.6) above. For $c \neq$ integer, two linearly independent solutions to the hypergeometric equations are

$$F(a,b;c;t) \text{ and } t^{1-c}F(a-c+1,\, b-c+1;\, 2-c;t) ,$$

where $F(a,b;c;t)$ is the hypergeometric function of Gauss. Putting this together with (2.8) and (2.12) we see that two linearly independent solutions to (2.7) satisfying (2.8) are given by

$$\begin{aligned} v_1(r,\theta,l,\kappa,s) &:= e^{il\theta}\, P^{-l}_{Rs/2,\,-\kappa}(r) , \\ v_2(r,\theta,l,\kappa,s) &:= e^{il\theta}\, P^{l}_{Rs/2,\,\kappa}(r) , \end{aligned} \tag{2.16}$$

where

$$P^l_{\alpha,\kappa}(r) := \frac{t^{-\frac{l}{2}}(1-t)^\alpha}{\Gamma(1-l)} F(\alpha+\kappa,\ \alpha-\kappa-l;\ 1-l; t)\ .$$

It will also be useful at this point to record the solution of (2.7) of the form (2.8) which tends to 0 as $|z| \to R$ (or $r \to \infty$). Define

$$Q^l_{\alpha,\kappa}(r) := \frac{e^{il\pi}}{4\pi} \frac{\Gamma(\alpha-\kappa)\Gamma(\alpha+\kappa+l)}{\Gamma(2\alpha)} t^{-\frac{l}{2}}(l-t)^\alpha F(\alpha+\kappa,\ \alpha-\kappa-l;\ 2\alpha; 1-t)\ ,$$

and write

$$\hat{v}(r,\theta,l,\kappa,s) = e^{il\theta} Q^l_{Rs/2,\kappa}(r)\ . \tag{2.17}$$

Then $\hat{v}(r,\theta,l,\kappa,s)$ is a solution to (2.7) which tends to zero as $r \to \infty$. From the connection formula for the hypergeometric equation one finds

$$\hat{v}(l,\kappa,s) = \frac{e^{i\pi l}}{4\sin(\pi l)} \left\{ v_2(l,\kappa,s) - \frac{\Gamma(\alpha-\kappa)\Gamma(\alpha+\kappa-l)}{\Gamma(\alpha+\kappa)\Gamma(\alpha-\kappa-l)} v_1(l,\kappa,s) \right\},$$

where $\alpha = \frac{Rs}{2}$ and we have written $\hat{v}(l,\kappa,s)$ for $\hat{v}(r,\theta,l,\kappa,s)$.

Because we are interested in using the functions v_j and $\hat{v}$ to construct solutions to the Dirac equation, it will be useful to compute the action of the operators K_κ and K^*_κ on the functions v_j and $\hat{v}$. Writing $v_j(l,\kappa,s)$ for the function $v_j(r,\theta,l,\kappa,s)$ we now summarize the action of K_κ and K^*_κ on these functions:

$$K_\kappa v_1(l,\kappa,s) = \frac{2}{R} v_1(l-1,\ \kappa+1,\ s)\ ,$$

$$K_\kappa v_2(l,\kappa,s) = \frac{2}{R}\left(\frac{Rs}{2}+\kappa\right)\left(\frac{Rs}{2}-\kappa-1\right) v_2(l-1,\ \kappa+1,\ s)\ ,$$

$$K_\kappa \hat{v}(l,\kappa,s) = \frac{2}{R}\left(\frac{Rs}{2}+\kappa\right)\left(\frac{Rs}{2}-\kappa-1\right) \hat{v}(l-1,\ \kappa+1,\ s)\ ,$$

$$K^*_\kappa v_1(l,\kappa+1,s) = -\frac{2}{R}\left(\frac{Rs}{2}+\kappa\right)\left(\frac{Rs}{2}-\kappa-1\right) v_1(l+1,\ \kappa,\ s)\ ,$$

$$K^*_\kappa v_2(l,\kappa+1,s) = -\frac{2}{R} v_2(l+1,\ \kappa,\ s)\ ,$$

$$K^*_\kappa \hat{v}(l,\kappa+1,s) = -\frac{2}{R} \hat{v}(l+1,\ \kappa,\ s)\ .$$

Finally we introduce the basic wave functions that we will use to analyze the solutions to the Dirac equation:

$$w_{l,k} := \begin{bmatrix} \frac{2}{mR} v_1\left(l-\frac{1}{2},\ k+\frac{1}{2},\ s\right) \\ v_1\left(l+\frac{1}{2},\ k-\frac{1}{2},\ s\right) \end{bmatrix},$$

$$w_{l,k}^{*} := \begin{bmatrix} v_2\left(-l-\frac{1}{2}, k+\frac{1}{2}, s\right) \\ \frac{2}{mR} v_2\left(-l+\frac{1}{2}, k-\frac{1}{2}, s\right) \end{bmatrix}, \tag{2.18}$$

and

$$\hat{w}_{l,k} := \begin{bmatrix} \hat{v}\left(-l-\frac{1}{2}, k+\frac{1}{2}, s\right) \\ \frac{2}{mR} \hat{v}\left(-l+\frac{1}{2}, k-\frac{1}{2}, s\right) \end{bmatrix},$$

where s is the solution,

$$s = R^{-1}(1+\sqrt{m^2R^2+4k^2}), \tag{2.19}$$

to (2.5). The solution $\hat{w}_{l,k}$ which is well behaved at infinity is

$$\hat{w}_{l,k} = \frac{ie^{-i\pi l}}{4\cos(\pi l)} \left\{ w_{l,k}^{*} - \frac{\Gamma(\alpha-k+\frac{1}{2})\Gamma(\alpha+k-l)}{\Gamma(\alpha+k-\frac{1}{2})\Gamma(\alpha-k+l)} \frac{2}{mR} w_{-l,k} \right\}.$$

In order to see that these are indeed solutions to the Dirac equation it is useful to recompute the action of K_κ and K_κ^* on v_j and $\hat{v}$ in the case where s is given by (2.5). Using

$$\frac{2}{R}\left(\frac{Rs}{2}-\frac{1}{2}+k\right)\left(\frac{Rs}{2}-\frac{1}{2}-k\right) = \frac{m^2R}{2},$$

one finds

$$K_{k-\frac{1}{2}} v_1\left(l+\frac{1}{2}, k-\frac{1}{2}\right) = \frac{2}{R} v_1\left(l-\frac{1}{2}, k+\frac{1}{2}\right),$$

$$K_{k-\frac{1}{2}} v_2\left(-l+\frac{1}{2}, k-\frac{1}{2}\right) = \frac{m^2R}{2} v_2\left(-l-\frac{1}{2}, k+\frac{1}{2}\right),$$

$$K_{k-\frac{1}{2}} \hat{v}\left(-l+\frac{1}{2}, k-\frac{1}{2}\right) = \frac{m^2R}{2} \hat{v}\left(-l-\frac{1}{2}, k+\frac{1}{2}\right),$$

$$K_{k-\frac{1}{2}}^{*} v_1\left(l-\frac{1}{2}, k+\frac{1}{2}\right) = -\frac{m^2R}{2} v_1\left(l+\frac{1}{2}, k-\frac{1}{2}\right),$$

$$K_{k-\frac{1}{2}}^{*} v_2\left(-l-\frac{1}{2}, k+\frac{1}{2}\right) = -\frac{2}{R} v_2\left(-l+\frac{1}{2}, k-\frac{1}{2}\right),$$

$$K_{k-\frac{1}{2}}^{*} \hat{v}\left(-l-\frac{1}{2}, k+\frac{1}{2}\right) = -\frac{2}{R} \hat{v}\left(-l+\frac{1}{2}, k-\frac{1}{2}\right).$$

We have dropped the explicit dependence on s, it being understood that s is given by (2.19).

As mentioned in Sect. 1 the operator M_1 and M_2 are ladder operators on the eigenvectors for M_3. Next we record the action of these infinitesimal symmetries of the Dirac operator on the wave functions $w_{l,k}$ and $w_{l,k}^{*}$:

$$M_3 w_{l,k} = (k+l) w_{l,k},$$

$$M_3 w_{l,k}^{*} = (k-l) w_{l,k}^{*},$$

$$
\begin{aligned}
M_1 w_{l,k} &= w_{l-1,k} , \\
M_2 w^*_{l,k} &= w^*_{l-1,k} , \\
M_1 w^*_{l,k} &= m_1(l,k) w^*_{l+1,k} , \\
M_2 w_{l,k} &= m_2(l,k) w_{l+1,k} ,
\end{aligned}
\tag{2.20}
$$

where

$$
m_1(l,k) := \left(\frac{m^2 R^2}{4} + k^2 - \left(k - l - \frac{1}{2} \right)^2 \right) ,
$$

$$
m_2(l,k) := \left(\frac{m^2 R^2}{4} + k^2 - \left(k + l + \frac{1}{2} \right)^2 \right) .
$$

These results provide the rationale for the normalization we have chosen for the wave functions. The wave functions $w_{l,k}$ and $w^*_{l,k}$ are parametrized so that each becomes less singular at $z=0$ as l increases. This is the reason for changing the sign of l in the definition of $w^*_{l,k}$. The action of those infinitesimal symmetries that increase the local singularity of the wave functions will play an important role in what follows and the normalization we have adopted make the action of M_1 on $w_{l,k}$ and of M_2 on $w^*_{l,k}$ as simple as possible. Finally we have chosen the normalization for $w^*_{l,k}$ so that the following conjugation symmetry is valid,

$$
w^*_{l,k} = C \bar{w}_{l,-k} ,
$$

where $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This conjugation will play a role later on and it is especially important in the case $k=0$, where it is the conjugation with respect to which the Dirac operator is a *real* operator.

We will now make use of the covariance of the Dirac operator under the action of $\widetilde{SU}(1,1)$ to center these wave functions at the point $a \in \mathbf{D}_R$. Before we do this we will establish a convention for dealing with the branches of the mutivalued functions $w_{l,k}$ and $w^*_{l,k}$. Suppose that one has chosen a geodesic ray, ℓ, at the point $a=0$ in $\mathbf{D}_R$. Then we fix a geodesic polar coordinate system with $\{\theta=0\}=\ell$ and on $\mathbf{D}_R \backslash \ell$ we choose the branches of $w_{l,k}$ and $w^*_{l,k}$ that are associated with the determinations (2.16) and (2.17) for v_1 and v_2 with $0 \leq \theta < 2\pi$.

Now consider the translation to the point a by the rigid motion $T[a]$ on $\mathbf{D}_R$ given by

$$
T[a]z = R \frac{\frac{z}{R} + \frac{a}{R}}{\frac{\bar{a}}{R}\frac{z}{R} + 1} = R \frac{Rz + Ra}{\bar{a}z + R^2} .
\tag{2.21}
$$

Then one easily computes that

$$
v(T[a], z) = \frac{R^2 - a\bar{z}}{R^2 - \bar{a}z} .
$$

As a function of z this clearly does not wind around the origin and we can thus uniquely specify a branch of the logarithm by

$$
\log v(T[a], 0) = 0
$$

for all $a \in \mathbf{D}_R$. Using this choice of the logarithm to determine the fractional powers of v in the following formulas, we now define the local wave functions centered at a by

$$w_{l,k}(z,a) := \begin{bmatrix} v(T[a],z)^{k+\frac{1}{2}} & 0 \\ 0 & v(T[a],z)^{k-\frac{1}{2}} \end{bmatrix} w_{l,k}(T[-a]z) . \tag{2.22}$$

and

$$w^*_{l,k}(z,a) := \begin{bmatrix} v(T[a],z)^{k+\frac{1}{2}} & 0 \\ 0 & v(T[a],z)^{k-\frac{1}{2}} \end{bmatrix} w^*_{l,k}(T[-a]z) . \tag{2.23}$$

Note that $T[-a]\,z$ occurs in this formula not $T[a]\,z$. Because the functions $w_{l,k}$ and $w^*_{l,k}$ are multivalued the formulas (2.22) and (2.23) do not make sense as they stand. We make sense of them in the following manner. Suppose the ℓ is a geodesic ray at a. Then $T[-a]^{-1}\ell = T[a]\ell$ is a geodesic ray at 0 and hence determines a branch for $w_{l,k}$ and $w^*_{l,k}$.With these branches for $w_{l,k}$ and $w^*_{l,k}$ understood in (2.22) and (2.23) the functions $w_{l,k}(z,a)$ and $w^*_{l,k}(z,a)$ are well defined for $z \notin \ell$ and they are branched along ℓ. We will think of $w_{l,k}(z,a)$ and $w^*_{l,k}(z,a)$ as multivalued functions with specific branches determined by the additional specification of geodesic ray at $z=a$.

The importance of these wave functions is that the multivalued solutions to the Dirac equation branched at "a" that are of interest to us will have local expansions,

$$W(z) = \sum_{n \in \mathbf{Z}+\frac{1}{2}} \{a_n w_{n+\lambda}(z,a) + b_n w^*_{n-\lambda}(z,a)\} .$$

Differentiating local expansions with respect to the branch points. In this subsection the dependence of the wave functions $w_{l,k}$ and $w^*_{l,k}$ on the parameter k will not be very important and so we will abbreviate $w_{l,k} = w_l$ and $w^*_{l,k} = w^*_l$. In order to differentiate local expansions in the a parameter it will be useful to develop *reciprocity relations* for the behavior of $w_l(z,a)$ and $w^*_l(z,a)$ as functions of z and a. Let $a(\varepsilon) = T[\varepsilon R]a$ and $z(-\varepsilon) = T[-\varepsilon R]z$. One easily computes that

$$v(T[a(\varepsilon)],z) = \frac{R-\varepsilon\bar{z}}{R-\bar{\varepsilon}z}\,\frac{R+\varepsilon\bar{a}}{R+\bar{\varepsilon}a}\, v(T[a],\, z(-\varepsilon))$$

and

$$T[-a(\varepsilon)]\,z = \frac{R+\varepsilon\bar{a}}{R+\bar{\varepsilon}a}\, T[-a]\; z(-\varepsilon) .$$

Observe also that under rotations w_l and w^*_l transform as follows,

$$\begin{bmatrix} e^{i(k+\frac{1}{2})\theta} & 0 \\ 0 & e^{i(k-\frac{1}{2})\theta} \end{bmatrix} w_l(e^{i\theta}z) = e^{i(k+l)\theta} w_l(z)$$

and

$$\begin{bmatrix} e^{i(k+\frac{1}{2})\theta} & 0 \\ 0 & e^{i(k-\frac{1}{2})\theta} \end{bmatrix} w^*_l(e^{i\theta}z) = e^{i(k-l)\theta} w^*_l(z) .$$

Of course, neither of these equations is correct without the proper interpretation of the multivalued functions w_l and w^*_l. It will suffice for our purposes to note that the equations are correct provided that w^l and w_l^* are branched along the geodesic ray ℓ at 0, we suppose that $z \notin \ell$, and we suppose that θ is restricted to a sufficiently small neighborhood of $0 \in \mathbf{R}$ so that the curve $\theta \to e^{i\theta} z$ does not intersect ℓ.

Now we write

$$w_l(z, a(\varepsilon)) = \begin{bmatrix} v(T[a(\varepsilon)], z)^{k+\frac{1}{2}} & 0 \\ 0 & v(T[a], z)^{k-\frac{1}{2}} \end{bmatrix} w_l(T[-a(\varepsilon)]z)$$

and make use of the expressions for $v(T[-a(\varepsilon)], z)$ and $T[-a(\varepsilon)]z$ above. In the formula for $w_l(z, a(\varepsilon))$ that results, the transformation of w_l under rotations implies

$$\begin{bmatrix} \lambda^{k+\frac{1}{2}} & 0 \\ 0 & \lambda^{k-\frac{1}{2}} \end{bmatrix} w_l(\lambda T[-a]\, z(-\varepsilon)) = \lambda^{k+l} w_l(T[-a]\, a(-\varepsilon)),$$

where

$$\lambda = \left(\frac{R + \varepsilon\bar{a}}{R + \bar{\varepsilon} a}\right).$$

Furthermore one has

$$\begin{bmatrix} v(T[a], z(-\varepsilon))^{k+\frac{1}{2}} & 0 \\ 0 & v(T[a], z(-\varepsilon))^{k-\frac{1}{2}} \end{bmatrix} w_l(T[-a]\, z(-\varepsilon)) = w_l(z(-\varepsilon), a).$$

The reader should understand these formulas in the following manner. A geodesic ray ℓ at a has been chosen so that $z \notin \ell$ and ε is chosen in a sufficiently small neighborhood, U, of $0 \in \mathbf{C}$ so that the range of

$$U \ni \varepsilon \to \frac{R + \varepsilon\bar{a}}{R + \bar{\varepsilon} a} z(-\varepsilon)$$

does not intersect ℓ. The functions $w_l(z, a)$ *and* $w_l^*(z, a)$ are understood to be branched along ℓ and the functions w_l and w_l^* are understood to be branched along $T[-a]\ell$.

Putting the preceding results together one gets the reciprocity formula,

$$w_l(z, a(\varepsilon)) = \left(\frac{R + \varepsilon\bar{a}}{R + \bar{\varepsilon} a}\right)^{k+1} \begin{bmatrix} v(T[\varepsilon R], z)^{k+\frac{1}{2}} & 0 \\ 0 & v(T[\varepsilon R], z)^{k-\frac{1}{2}} \end{bmatrix} w_l(T[-\varepsilon R]z, a), \tag{2.24}$$

where we made use of the fact that

$$v(T[\varepsilon R],\ z) = \frac{R - \varepsilon\bar{z}}{R - \bar{\varepsilon} z}.$$

In a precisely similar fashion one finds that

$$w_l^*(z, a(\varepsilon)) = \left(\frac{R + \varepsilon\bar{a}}{R + \bar{\varepsilon} a}\right)^{k-1} \begin{bmatrix} v(T[\varepsilon R], z)^{k+\frac{1}{2}} & 0 \\ 0 & v(T[\varepsilon R], z)^{k-\frac{1}{2}} \end{bmatrix} w_l^*(T[-\varepsilon R]z, a). \tag{2.25}$$

These formulae will allow us to compute the local expansions for the derivatives of the wave functions $w_l(z, a)$ and $w_l^*(z, a)$ in the a variables.

One reason for the utility of these reciprocity relations for differentiating the wave functions $w_l(z,a)$ and $w_l^*(z,a)$ are the following easily confirmed formulae,

$$\frac{\partial}{\partial\varepsilon}\begin{bmatrix} v(T[\varepsilon R],z)^{k+\frac{1}{2}} & 0 \\ 0 & v(T[\varepsilon R],z)^{k-\frac{1}{2}} \end{bmatrix} F(T[-\varepsilon R]z)|_{\varepsilon=0} = -M_1 F(z)\,. \quad (2.26)$$

and

$$\frac{\partial}{\partial\bar{\varepsilon}}\begin{bmatrix} v(T[\varepsilon R],z)^{k+\frac{1}{2}} & 0 \\ 0 & v(T[\varepsilon R],z)^{k-\frac{1}{2}} \end{bmatrix} F(T[-\varepsilon R]z)|_{\varepsilon=0} = -M_2 F(z)\,. \quad (2.27)$$

We will now use the reciprocity formulae above to provide local expansions for the derivatives of the wave functions $w_l(z,a)$ and $w_l^*(z,a)$ in the "a" variable. The chain rule gives

$$\partial_\varepsilon w_l(z,a(\varepsilon))|_{\varepsilon=0} = R\left(\partial_a w_l(z,a) - \frac{\bar{a}^2}{R^2}\,\bar{\partial}_a w_l(z,a)\right), \quad (2.28)$$

and

$$\bar{\partial}_\varepsilon w_l(z,a(\varepsilon))|_{\varepsilon=0} = R\left(\bar{\partial}_a w_l(z,a) - \frac{a^2}{R^2}\,\partial_a w_l(z,a)\right), \quad (2.29)$$

with completely analogous formulas for $w_l^*(z,a)$. Differentiating the right-hand side of the reciprocity formulas (2.24) and (2.25) with the use of (2.26) and (2.27) one finds,

$$\partial_\varepsilon w_l(z,a(\varepsilon))|_{\varepsilon=0} = (k+l)\,\frac{\bar{a}}{R}\,w_l(z,a) - M_1\,w_l(z,a)\,, \quad (2.30)$$

and

$$\bar{\partial}_\varepsilon w_l(z,a(\varepsilon))|_{\varepsilon=0} = -(k+l)\,\frac{a}{R}\,w_l(z,a) - M_2\,w_l(z,a)\,, \quad (2.31)$$

with analogous results for $w_l^*(z,a)$. To compute the action of M_1 and M_2 on $w_l(z,a)$ it is useful to represent M_j for $j=1$, 2, 3 in terms of the centered operators $M_j^{(a)}$ obtained by conjugate M_j with the action (2.23) of $T[-a]$. One finds that,

$$M_1 = \frac{1}{1-|b|^2}\{M_1^{(a)} - \bar{b}^2 M_2^{(a)} + 2\bar{b} M_3^{(a)}\}\,,$$

$$M_2 = \frac{1}{1-|b|^2}\{M_2^{(a)} - b^2 M_1^{(a)} - 2b\, M_3^{(a)}\}\,,$$

$$M_3 = \frac{1}{1-|b|^2}\{bM_1^{(a)} - \bar{b}\, M_2^{(a)} + (1+|b|^2) M_3^{(a)}\}\,, \quad (2.32)$$

where we have written

$$b = \frac{a}{R}$$

for the scaled variable. Now substitute these results in (2.30) and (2.31) and equate the results with (2.28) and (2.29); one can solve for $\partial_b w_l(z,a)$ and $\bar{\partial}_b w_l(z,a)$ to find

$$\partial_b w_l(z,a) = \frac{(k+l)\bar{b}}{1+|b|^2}\,w_l(z,a) - \frac{1}{1-|b|^2}\left(M_1^{(a)} w_l(z,a) + \frac{2\bar{b}}{1+|b|^2}\,M_3^{(a)} w_l(z,a)\right)$$

and

$$\bar{\partial}_b w_l(z,a) = -\frac{(k+l)b}{1+|b|^2} w_l(z,a) - \frac{1}{1-|b|^2}\left(M_2^{(a)} w_l(z,a) - \frac{2b}{1+|b|^2} M_3^{(a)} w_l(z,a)\right).$$

The analogous results for $w_l^*(z,a)$ are

$$\partial_b w_l^*(z,a) = \frac{(k-l)\bar{b}}{1+|b|^2} w_l^*(z,a) - \frac{1}{1-|b|^2}\left(M_1^{(a)} w_l^*(z,a) + \frac{2\bar{b}}{1+|b|^2} M_3^{(a)} w_l^*(z,a)\right)$$

and

$$\bar{\partial}_b w_l^*(z,a) = -\frac{(k-l)b}{1+|b|^2} w_l^*(z,a) - \frac{1}{1-|b|^2}\left(M_2^{(a)} w_l^*(z,a) - \frac{2b}{1+|b|^2} M_3^{(a)} w_l^*(z,a)\right).$$

It is now a simple matter to use the results (2.20) for the wave functions to get the local expansions,

$$\begin{aligned}
\partial_b w_l(z,a) &= -\frac{(k+l)\bar{b}}{1-|b|^2} w_l(z,a) - \frac{1}{1-|b|^2} w_{l-1}(z,a)\,,\\
\bar{\partial}_b w_l(z,a) &= \frac{(k+l)b}{1-|b|^2} w_l(z,a) - \frac{m_2(l,k)}{1-|b|^2} w_{l+1}(z,a)\,,\\
\partial_b w_l^*(z,a) &= -\frac{(k-l)\bar{b}}{1-|b|^2} w_l^*(z,a) - \frac{m_1(l,k)}{1-|b|^2} w_{l+1}^*(z,a)\,,\\
\bar{\partial}_b w_l^*(z,a) &= \frac{(k-l)b}{1-|b|^2} w_l^*(z,a) - \frac{1}{1-|b|^2} w_{l-1}^*(z,a)\,.
\end{aligned} \tag{2.33}$$

Estimates at infinity. We conclude this section with results that we will use to justify some later applications of Stokes' theorem in the non-compact domain $\mathbf{D}_R$.

If one considers (2.9) and (2.10) or (2.11) then one finds

$$K_\kappa^* K_\kappa = -\frac{4}{R^2}\left\{\partial_r^2 + \frac{\operatorname{ch} r}{\operatorname{sh} r}\partial_r + \frac{l}{\operatorname{sh}^2 r}\partial_\theta^2 - \frac{i\kappa}{\operatorname{ch}^2(\frac{r}{2})}\partial_\theta + \frac{\kappa^2}{\operatorname{ch}^2(\frac{r}{2})} - \kappa(\kappa+1)\right\}.$$

Next we fix s to the value given by (2.19). Then we introduce

$$\Phi_{l,\kappa,s}(r) = e^{-i\pi l} Q_{Rs/2,\kappa}^l(r)$$

for brevity and also because $\Phi_{l,k,s}(r)$ is *real valued.* Then

$$\hat{v}(r,\theta,l,k,s) = e^{il(\theta+\pi)}\Phi_{l,\kappa,s}(r)\,. \tag{2.34}$$

The differential equation $(K_\kappa^* K_\kappa + m^2)\hat{v} = 0$ becomes the ordinary differential equation

$$\Phi_{l,\kappa,s}'' + \frac{\operatorname{ch} r}{\operatorname{sh} r}\Phi_{l,\kappa,s}' - \left\{\frac{l^2}{\operatorname{sh}^2 r} - \frac{\kappa l}{\operatorname{ch}^2(\frac{r}{2})} - \frac{\kappa^2}{\operatorname{ch}^2(\frac{r}{2})} - \kappa(\kappa+1) + \frac{m^2 R^2}{4}\right\}\Phi_{l,\kappa,s} = 0$$

for the radial wave function $\Phi_{l,\kappa,s}$. We eliminate the first derivative term in this last equation with the substitution

$$u_{l,\kappa,s}(r) = \sqrt{\operatorname{sh} r}\,\Phi_{l,\kappa,s}(r)\,.$$

After a little computation one finds that

$$u''_{l,\kappa,s}-\left(\frac{m^2R^2}{4}+\kappa(\kappa+1)+\frac{1}{4}\right)u_{l,\kappa,s}+\frac{\kappa(l+\kappa)}{\mathrm{ch}^2\left(\frac{r}{2}\right)}u_{l,\kappa,s}-\frac{l^2-\frac{1}{4}}{\mathrm{sh}^2 r}u_{l,\kappa,s}=0\,. \qquad (2.35)$$

As $r\to\infty$ the coefficients of $u_{l,\kappa,s}$ in the third and fourth terms on the left-hand side of (2.35) tend to 0 exponentially fast. It is not surprising then that the behavior of the solutions to (2.35) is governed by the first two terms. When

$$\frac{m^2R^2}{4}+\kappa(\kappa+1)+\frac{1}{4}>0$$

there is one solution to (2.35) which exponentially small at ∞ (in Hartmann's terminology [3] this is the *principal solution*). Suppose now that $\kappa=k-\frac{1}{2}$. Then

$$M^2=\frac{m^2R^2}{4}+\kappa(\kappa+1)+\frac{1}{4}=\frac{m^2R^2}{4}+k^2 \qquad (2.36)$$

is clearly positive and it follows from Theorem 8.1 in Hartmann [3] that the principal solution $u_{l,k-\frac{1}{2},s}$ to (2.35) has the asymptotics

$$u_{l,k-\frac{1}{2},s}(r)=O(e^{-Mr})\quad \text{as } r\to\infty\,.$$

Thus

$$\Phi_{l,k-\frac{1}{2},s}(r)=O\left(\frac{e^{-Mr}}{\sqrt{\mathrm{sh}\,r}}\right)\quad \text{as } r\to\infty\,. \qquad (2.37)$$

We are mostly interested in the consequences of (2.37) for the solutions to the Dirac equation which are well behaved at ∞. Since the second component of $\hat{w}_{l,k}$, for which we write $(\hat{w}_{l,k})_2$, is just $\hat{v}(-l+\frac{1}{2},k-\frac{1}{2},s)$ it follows that

$$|(\hat{w}_{l,k}(r))_2|=|\Phi_{-l+\frac{1}{2},k-\frac{1}{2},s}(r)|=O\left(\frac{e^{-Mr}}{\sqrt{\mathrm{sh}\,r}}\right)\quad \text{as } r\to\infty\,. \qquad (2.38)$$

We would like to have something similar for the first component $(\hat{w}_{l,k})_1$. This component is a solution to the Helmholtz equation

$$(K_{k-\frac{1}{2}}K^*_{k-\frac{1}{2}}+m^2)\,w=0\,. \qquad (2.39)$$

Since,

$$K_{k-\frac{1}{2}}K^*_{k-\frac{1}{2}}=K^*_{k+\frac{1}{2}}K_{k+\frac{1}{2}}-\frac{8(k+\frac{1}{2})}{R^2}\,, \qquad (2.40)$$

it follows from (2.35) that the leading behavior (at ∞) of the second order ODE that governs the radial part of the wave function $\sqrt{\mathrm{sh}\,r}\,(\hat{w}_{l,k})_1$ is

$$u''-M^2u=0\,.$$

Then one has the following asymptotic estimate for the first component of $\hat{w}_{l,k}$,

$$|(\hat{w}_{l,k}(r))_1|=\frac{mR}{2}|\Phi_{-l-\frac{1}{2},k+\frac{1}{2},s}(r)|=O\left(\frac{e^{-Mr}}{\sqrt{\mathrm{sh}\,r}}\right)\quad \text{as } r\to\infty\,. \qquad (2.41)$$

We will now use the estimates (2.38) and (2.41) to establish the following representation for the L^2 norm,

$$\int_{D_{r_0,\infty}} |\hat{w}_{l,k}(z)|^2 \, d\mu(z) = c_{l,k}(r_0) \,, \tag{2.42}$$

where

$$c_{l,k}(r) = 2\pi R^2 \operatorname{sh}(r) \, \Phi_{-l-\frac{1}{2},k+\frac{1}{2},s}(r) \; \Phi_{-l+\frac{1}{2},k-\frac{1}{2},s}(r) \,, \tag{2.43}$$

and we write

$$D_{a,b} = \left\{ z : R \operatorname{th} \frac{a}{2} \leq |z| < R \operatorname{th} \frac{b}{2} \right\}.$$

To prove (2.42) suppose first that f and g are multivalued solutions to the Dirac equation in $\mathbf{D}_R \backslash \{0\}$ with the same unitary monodromy about 0. Then a simple calculation shows that

$$m(\bar{f} \cdot g) \, d\mu(z) = 2d \left\{ \frac{\bar{f}_1 g_2}{1 - \dfrac{|z|^2}{R^2}} \, i d\bar{z} \right\}.$$

Since $\bar{f}_i g_j$ is a smooth single valued function away from the origin Stokes' theorem implies

$$\frac{m}{2} \int_{D_{r_0,r_1}} \bar{f} \cdot g(z) \, d\mu(z) = \int_{C_{r_1}} \frac{\bar{f}_1 g_2}{1 - \dfrac{|z|^2}{R^2}} \, i d\bar{z} - \int_{C_{r_0}} \frac{\bar{f}_1 g_2}{1 - \dfrac{|z|^2}{R^2}} \, i d\bar{z} \,, \tag{2.44}$$

where we write

$$C_r = \left\{ z : z = \mathrm{R} e^{i\theta} \operatorname{th} \frac{r}{2} \right\}.$$

In (2.44) let $f = g = \hat{w}_{l,k}$. Then substitute geodesic polar coordinates (2.0) in (2.44). Finally use the asymptotics (2.38) and (2.41) in the resulting equation to evaluate the limit $r_1 \to \infty$. One obtains (2.42). The reader should note that the sign change due to the factor $e^{i\pi l}$ by which $Q^l_{Rs/2,\kappa}$ differs from $\Phi_{l,\kappa,s}$ is important in getting the signs to work out properly in (2.42).

Now suppose that in the exterior of the disk $|z| < r_0 - \varepsilon$ the function f is a multivalued solution to the Dirac equation with monodromy multiplier $e^{2\pi i \lambda}$. If f is square integrable in the exterior of $|z| < r_0 - \varepsilon$ with respect to the measure $d\mu$ then it has an expansion

$$f = \sum_{n \in \mathbf{Z} + \frac{1}{2}} c_n(f) \, \hat{w}_{n+\lambda,k} \,. \tag{2.45}$$

This expansion gives the Fourier series representation for f in the angular variable θ. If one makes use of this to do the circuit integrals that appear on the right-hand side (2.44) then one finds,

$$\frac{m}{2} \int_{C_r} \frac{\bar{f}_1 f_2}{1 - \dfrac{|z|^2}{R^2}} \, i d\bar{z} = \sum_{n \in \mathbf{Z} + \frac{1}{2}} |c_n(f)|^2 \, c_{n+\lambda,k}(r) \,. \tag{2.46}$$

From (2.42) it is clear that $c_{n+\lambda,k}(r)$ is a positive, monotone decreasing function of r which tends to 0 as $r \to \infty$. Thus the monotone convergence theorem (for sums) and (2.46) imply that

$$\lim_{r\to\infty} \int_{C_r} \frac{\bar{f}_1 f_2}{1-\frac{|z|^2}{R^2}} \, id\bar{z} = 0 \tag{2.47}$$

when f is a multivalued solution to the Dirac equation which is square integrable in a neighborhood of infinity. Finally, we want to know that something similar is true if f and g are two multivalued solutions to the Dirac equation both square integrable in a neighborhood of ∞. Consider the Hermitian form,

$$(f, g)_r = \int_{C_r} \frac{\bar{f}_1 g_2}{1-\frac{|z|^2}{R^2}} \, id\bar{z} \, .$$

Then (2.46) shows that this is positive definite and so the Schwarz inequality,

$$|(f, g)_r|^2 \leqq (f, f)_r (g, g)_r \, ,$$

is valid. It follows immediately from (2.47) that

$$\lim_{r\to\infty} \int_{C_r} \frac{\bar{f}_1 g_2}{1-\frac{|z|^2}{R^2}} \, id\bar{z} = 0 \, . \tag{2.48}$$

We summarize these developments in the following proposition:

Proposition 2.0. *Suppose that f and g are two solutions to the Dirac equation which have expansions of type* (2.45) *in the region $D_{r_0,\infty}$. Then*

$$\lim_{r\to\infty} \int_{C_r} \frac{\bar{f}_1 g_2}{1-\frac{|z|^2}{R^2}} \, id\bar{z} = 0 \, .$$

3. Existence Results

A model for the simply connected covering $\tilde{\mathbf{D}}_R(a)$. To understand the asymptotics of the Green function for a Dirac operator with vertex insertions it will be important to have an understanding of certain spaces of multivalued solutions to the Dirac equation. We will follow SMJ [15] here in establishing an L^2 existence theory for wave functions with specified branching and restricted singularities. Let $\{a_1, a_2, \ldots, a_n\}$ denote n distinct points in the hyperbolic disk $\mathbf{D}_R$. It will be useful for us to use complex variables writing $a_j = \alpha_j + i\beta_j$ when $a_j = (\alpha_j, \beta_j)$. We will also write

$$a = \{a_1, a_2, \ldots, a_n\} \, .$$

Following SMJ [15] we will now show that the space of solutions, w, to the Dirac equation,

$$(m - D_k)w = 0 \, ,$$

which are branched at $\{a_1, a_2, \ldots, a_n\}$ with monodromy $e^{2\pi i}\hat{\lambda}_j$ at a_j (with $0<|\lambda_j| <\frac{1}{2}$) and *globally* in $L^2(\mathbf{D}_R, d\mu)$ is an n dimensional space.

It will be convenient (particularly in discussing smoothness questions) to work in the simply connected covering space of $\mathbf{D}_R \backslash \{a_1, a_2, \ldots, a_n\}$ and we will now specify our conventions for such considerations. Write $\mathbf{D}_R(a)$ for $\mathbf{D}_R \backslash \{a_1, a_2, \ldots, a_n\}$ and $\tilde{\mathbf{D}}_R(a)$ for the simply connected covering space of the punctured disk $\mathbf{D}_R(a)$. Multivalued functions on $\mathbf{D}_R(a)$ can, of course, be regarded as functions on $\tilde{\mathbf{D}}_R(a)$. Introducing appropriate branch cuts, ℓ_j, at a_j a multivalued function is also specified by a particular branch defined on $\mathbf{D}_R \backslash \{\ell_1, \ell_2, \ldots, \ell_n\}$. Both descriptions will be important for us and to pass easily from one to the other we will always work in a specific model for $\tilde{\mathbf{D}}_R(a)$. Fix a base point a_0 in $\mathbf{D}_R$ differernt from any of the points $a_1, a_2, \ldots, a_n$. We take our models of $\tilde{\mathbf{D}}_R(a)$ to be the homotopy classes of paths in $\mathbf{D}_R(a)$ which start at a_0. The projection, pr, from $\tilde{\mathbf{D}}_R(a)$ onto $\mathbf{D}_R(a)$ maps a path γ onto its endpoint $\gamma(1)$. This model of $\tilde{\mathbf{D}}_R(a)$ comes with a distinguished point, $\tilde{a}_0$, in the fiber over a_0 which is the class of the constant path $t \to a_0$.

For simplicity we fix the convention that our branch cuts ℓ_j are always geodesic rays joining the point a_j to a point on the boundary of $\mathbf{D}_R$. If $\{\ell_1, \ell_2, \ldots, \ell_n\}$ is a pairwise disjoint collection of branch cuts then the inverse image of

$$\mathbf{D}_R \backslash \{\ell_1, \ell_2, \ldots, \ell_n\}$$

under the projection,

$$\mathrm{pr}\colon \tilde{\mathbf{D}}_R(a) \to \mathbf{D}_R ,$$

splits into path components on which the projection is a diffeomorphism. Let $C_0(\ell)$ denote the path component of this inverse image which contains the point $\tilde{a}_0$. If f is a map defined on $\tilde{\mathbf{D}}_R(a)$ then we will refer to the restriction of f to the path component $C_0(\ell)$ as the principal branch of f. The principal branch of such a function f can also be regarded as a function defined on $\mathbf{D}_R \backslash \{\ell_1, \ell_2, \ldots, \ell_n\}$ and we will do this without further comment. Conversely if f is a function defined on $\mathbf{D}_R \backslash \{\ell_1, \ell_2, \ldots, \ell_n\}$ we can regard it as a function on $C_0(\ell)$. For multivalued functions that have a natural extension to maps on $\tilde{\mathbf{D}}_R(a)$ this gives a unique way to go from a branch for such a function to a function defined on $\tilde{\mathbf{D}}_R(a)$. For example, we have already described how to fix a branch of the multivalued function $w_l(z, a_j)$ when we are given a branch cut, ℓ_j, at a_j. We can regard this as a function on $C_0(\ell)$ which has an obvious extension to a smooth function defined on $\tilde{\mathbf{D}}_R(a)$. When we wish to regard $w_l(\cdot, a_j)$ as a function on $\tilde{\mathbf{D}}_R(a)$ we will write $w_l(\tilde{z}, a_j)$ with the understanding that $\tilde{z}$ is a point in $\tilde{\mathbf{D}}_R(a)$ which projects onto $z \in \mathbf{D}_R(a)$. It is understood that a branch cut ℓ_j must be chosen before the function $w_l(\tilde{z}, a_j)$ is uniquely determined.

Let $[\alpha_j]$ denote the homotopy class of the simple a_0 based loop, α_j, about a_j in $\mathbf{D}_R(a)$ which circles a_j in a counter-clockwise fashion and does not wind around any of the other points a_i with $i \neq j$. Let R_j denote the deck transformation on $\tilde{\mathbf{D}}_R(a)$ which maps $\tilde{a}_0$ to the point in $\tilde{\mathbf{D}}_R(a)$ which corresponds to the homotopy class $[\alpha_j]$. To simplify notation in what follows we will, as above, write $\tilde{z}$ for points in $\tilde{\mathbf{D}}_R(a)$ and $z = \mathrm{pr}(\tilde{z})$ for their projections in $\mathbf{D}_R(a)$. Suppose that f is a smooth function on $\tilde{\mathbf{D}}_R(a)$ for which

$$f(R_j \tilde{z}) = e^{2\pi i \lambda_j} f(\tilde{z}) .$$

If F is the principal branch of such a function f for some choice of branch cuts $\{\ell_1, \ell_2, \ldots, \ell_n\}$ with $a_0 \notin \ell_j$ for $j=1,2,\ldots,n$ then we will say that *F is branched at a_j with monodromy multiplier $e^{2\pi i\lambda_j}$ for $j=1,2,\ldots,n$.*

Multivalued solutions with specified branching. We now turn to the consideration of multivalued solutions, w, to the Dirac equation with specified monodromy at the branch points a_j. We can locally expand such functions near the branch points in Fourier series in the angular variables θ of geodesic polar coordinates suitably modified to reflect the appropriate monodromy. As in SMJ [15] these Fourier series translate into local expansions

$$w(z) = \sum_{n \in \mathbf{Z}+\frac{1}{2}} a_n^j(w)\, w_{n+\lambda_j}(z, a_j) + b_n^j(w)\, w^*_{n-\lambda_j}(z, a_j) \text{ for } z \text{ near } a_j\,, \tag{3.0}$$

where (3.0) is given a precise meaning when the choice of a pairwise disjoint collection of branch cuts $\{\ell_1, \ell_2, \ldots, \ell_n\}$ has been made with $a_0 \notin \ell_j$ for $j=1,2,\ldots,n$.

The following local asymptotics for the wave functions may be deduced from the formulas for $w_l(z)$ and $w_l^*(z)$ in the preceding section:

$$w_l(z) = \begin{bmatrix} \dfrac{2e^{i(l-\frac{1}{2})\theta}\left(\frac{r}{2}\right)^{l-\frac{1}{2}}}{mR\Gamma(l+\frac{1}{2})} + O(r^{l+\frac{1}{2}}) \\ \dfrac{e^{i(l+\frac{1}{2})\theta}\left(\frac{r}{2}\right)^{l+\frac{1}{2}}}{\Gamma(l+\frac{3}{2})} + O(r^{l+\frac{3}{2}}) \end{bmatrix}, \tag{3.1}$$

and

$$w_l^*(z) = \begin{bmatrix} \dfrac{e^{-i(l+\frac{1}{2})\theta}\left(\frac{r}{2}\right)^{l+\frac{1}{2}}}{\Gamma(l+\frac{3}{2})} + O(r^{l+\frac{3}{2}}) \\ \dfrac{2e^{-i(l-\frac{1}{2})\theta}\left(\frac{r}{2}\right)^{l-\frac{1}{2}}}{mR\Gamma(l+\frac{1}{2})} + O(r^{l+\frac{1}{2}}) \end{bmatrix}. \tag{3.2}$$

From these asymptotics we deduce that w_l and w_l^* are locally in $L^2(d\mu)$ provided that $l > -\frac{1}{2}$. Thus in the expansion (3.0) of an L^2 wave function we expect that the coefficients $a_m^j(w)$ will vanish unless $n+\lambda_j > -\frac{1}{2}$ and the coefficient $b_n^j(w)$ will vanish unless $n-\lambda_j > -\frac{1}{2}$. This leads us to define

$$c_j(w) = \begin{cases} a_{\frac{1}{2}}^j(w) & \text{if } \lambda_j < 0 \\ a_{-\frac{1}{2}}^j(w) & \text{if } \lambda_j > 0 \end{cases} \tag{3.3}$$

and

$$c_j^*(w) = \begin{cases} b_{\frac{1}{2}}^j(w) & \text{if } \lambda_j > 0 \\ b_{-\frac{1}{2}}^j(w) & \text{if } \lambda_j < 0\,. \end{cases} \tag{3.4}$$

The coefficients $c_j(w)$ and $c_j^*(w)$ are the lowest order coefficients that can occur (i.e., be non-zero) in an expansion of type (3.0) for a function which is locally in $L^2(d\mu)$. We will now make this more precise with a formula for the L^2 inner product of two multivalued solutions to the Dirac equation with the same monodromy. The infinitesimal version of formal skew symmetry for D_k is

$$(\bar{f}\cdot D_k g+\overline{D_k f}\cdot g)\,d\mu(z)=2d\left\{\frac{\bar{f}_1 g_2}{1-\frac{|z|^2}{R^2}}\,id\bar{z}-\frac{\bar{f}_2 g_1}{1-\frac{|z|^2}{R^2}}\,idz\right\}, \tag{3.5}$$

The d on the right-hand side is the usual exterior derivative. This is true if f and g are smooth functions on $\mathbf{D}_R$ and it will remain true for functions f and g that are smooth branched functions with the same *unitary* monodromy at the points a_j for $j=1,2,\ldots,n$ – at least if one stays away from the branch points. In this case each side of (3.5) will be single valued on $\mathbf{D}_R\backslash\{a_1,a_2,\ldots,a_n\}$.

Theorem 3.0. *Suppose that f and g are multivalued solutions to the Dirac equation,*

$$(m-D_k)f=(m-D_k)g=0\,,$$

which are branched at $a_1,a_2,\ldots,a_n$ with monodromy $e^{2\pi i\lambda_j}$ at a_j. If f and g are in $L^2(\mathbf{D}_R,d\mu)$ then

$$\frac{m^3R}{16}\int_{\mathbf{D}_R}\overline{f(z)}\cdot g(z)\,d\mu(z)=-\sum_{j=1}^{n}|s_j|\,\overline{c_j(f)}\,c_j^*(g)=-\sum_{j=1}^{n}|s_j|\,\overline{c_j^*(f)}\,c_j(g)\,, \tag{3.6}$$

where $s_j=\sin\pi\lambda_j$.

Proof. Substituting $D_kf=mf$ and $D_kg=mg$ in Eq. (3.5) one finds

$$m(\bar{f}\cdot g)\,d\mu(z)=d\left\{\frac{\bar{f}_1 g_2}{1-\frac{|z|^2}{R^2}}\,id\bar{z}-\frac{\bar{f}_2 g_1}{1-\frac{|z|^2}{R^2}}\,idz\right\}.$$

In fact, calculating the exterior derivatives of each of the terms on the right-hand side of this last equation one finds that more is true in case f and g satisfy the Dirac equation, namely,

$$m(\bar{f}\cdot g)\,d\mu(z)=2d\left\{\frac{\bar{f}_1 g_2}{1-\frac{|z|^2}{R^2}}\,id\bar{z}\right\}=-2d\left\{\frac{\bar{f}_2 g_1}{1-\frac{|z|^2}{R^2}}\,idz\right\}. \tag{3.7}$$

Now we wish to calculate the L^2 inner product of f and g by first integrating (3.7) over the complement of the union of the disks of radius ε about a_j for $j=1,\ldots,n$ inside a disk of radius ρ. Then we use Stokes' theorem to reduce the integrals on the right to the sum of integrals over the circles, $C_j(\varepsilon)$, of radius ε about the points a_j with the standard counter-clockwise orientation (this will account for some minus signs showing up below) and an integral over the circle of radius ρ. It follows from Proposition 2.0 that the boundary integral over the circle C_ρ is 0 in the limit $\rho\to\infty$. Finally we recover the L^2 inner product from the remaining circuit integrals in the limit $\varepsilon \to 0$. The result is

$$\frac{m}{2} \int_{\mathbf{D}_R} \bar{f} \cdot g \, d\mu(z) = -\lim_{\varepsilon \to \infty} \sum_{j=1}^{n} \int_{C_j(\varepsilon)} \frac{\bar{f}_1 g_2}{1 - \frac{|z|^2}{R^2}} \, id\bar{z}$$

$$= \lim_{\varepsilon \to 0} \sum_{j=1}^{n} \int_{C_j(\varepsilon)} \frac{\bar{f}_2 g_1}{1 - \frac{|z|^2}{R^2}} \, idz \, .$$

Now introduce the local variables $u_j = T[-a_j]z$ in the $C_j(\varepsilon)$ integrals. The change of variables gives

$$id\bar{z} = \frac{1 - \frac{|a_j|^2}{R^2}}{\left(1 + \frac{\bar{a}_j u_j}{R^2}\right)} \, id\bar{u}_j$$

and

$$idz = \frac{1 - \frac{|a_j|^2}{R^2}}{\left(1 + \frac{a_j \bar{u}_j}{R^2}\right)} \, idu_j \, .$$

Substituting these results in the appropriate contour integrals above and taking limits $\varepsilon \to 0$ of some of the multiplicative factors that appear inside the integrands one finds.

$$\frac{m}{2} \int_{\mathbf{D}_R} \bar{f} \cdot g \, d\mu(z) = -\lim_{\varepsilon \to 0} \sum_{j=1}^{n} \int_{C_j(\varepsilon)} \bar{f}_1 g_2(u_j) \, id\bar{u}_j$$

$$= \lim_{\varepsilon \to 0} \sum_{j=1}^{n} \int_{C_j(\varepsilon)} \bar{f}_2 g_1(u_j) \, idu_j \, ,$$

where the notation $\bar{f}_j g_k(u)$ is meant to remind the reader that it is the product $\bar{f}_j g_k$ that is a function on $\mathbf{D}_R(a)$. Geodesic polar coordinates (r_j, θ_j) about a_j (determined by the branch cut ℓ_j) give

$$u_j = \mathrm{R} e^{i\theta_j} \operatorname{th} \frac{r_j}{2} \sim \frac{R r_j}{2} e^{i\theta_j} \, .$$

We can use this asymptotic formula in the contour integrals to get

$$\frac{m}{2} \int_{\mathbf{D}_R} \bar{f} \cdot g \, d\mu(z) = -\lim_{\varepsilon \to 0} \sum_{j=1}^{n} \frac{\varepsilon R}{2} \int_0^{2\pi} \bar{f}_1 g_2(u_j) e^{-i\theta_j} d\theta_j$$

$$= -\lim_{\varepsilon \to 0} \sum_{j=1}^{n} \frac{\varepsilon R}{2} \int_0^{2\pi} \bar{f}_2 g_1(u_j) e^{i\theta_j} d\theta_j \, .$$

Finally if we use the asymptotics of the local wave functions, given by (3.1) and (3.2), in these last two formulas and the identity,

$$\frac{\pi}{\Gamma(\frac{1}{2} - l)\Gamma(\frac{1}{2} + l)} = \cos \pi l \, ,$$

then one obtains the two local expansion results for the L^2 inner product given in (3.6) above. To see this it is also useful to note that the multiplicative factor $v(T[a_j], z)$, which appears in the wave functions, $w_l(z, a_j)$ and $w_l^*(z, a_j)$ tends to 1 as $z \to a_j$. QED

Definition. *Let l denote the $n \times n$ diagonal matrix with jj entry l_j with l_j chosen so that $0 < l_j < 1$. Define $\mathscr{W}(a, l)$ to be the space of multivalued solutions, w, to the Dirac equation $(m - D_k)w = 0$ which are branched at $\{a_1, a_2, \ldots, a_n\}$ with monodromy $e^{2\pi i l_j}$ in a counter-clockwise circuit of a_j and which are globally in $L^2(\mathbf{D}_R, d\mu)$.*

Remark. We parametrize the monodromy in this definition by l_j chosen between 0 and 1 rather than λ_j chosen between $-\frac{1}{2}$ and $\frac{1}{2}$ since the local expansions of functions in $\mathscr{W}(a, l)$ will be slightly simpler to characterize in terms of the parameters l_j.

Theorem 3.0 shows that the elements, w, in $\mathscr{W}(a, l)$ are completely determined by the coefficients $c_j(w)$ for $j = 1, 2, \ldots, n$ or by the coefficients $c_j^*(w)$ for $j = 1, 2, \ldots, n$. Thus the dimension of $\mathscr{W}(a, l)$ is less than or equal to n. Following SMJ [15] we will now give a functional analytic proof that the dimension of $\mathscr{W}(a, l)$ is exactly n by constructing a canonical basis. The functional analytic part of the proof requires that we first consider an associated problem for the Helmholtz operator $m^2 + K_\kappa K_\kappa^*$, where $\kappa = k - \frac{1}{2}$. Recall from (2.2) that

$$(m - D_k)^* (m - D_k) = \begin{bmatrix} m^2 + K_\kappa K_\kappa^* & 0 \\ 0 & m^2 + K_\kappa^* K_\kappa \end{bmatrix}.$$

As a start towards constructing solutions to the Dirac equation with prescribed branching at $\{a_1, a_2, \ldots, a_n\}$ we will first consider the problem of finding multi-valued solutions, v, to the Helmholtz equation

$$(m^2 + K_\kappa K_\kappa^*)v = 0 . \tag{3.8}$$

We will construct solutions of the desired type by fixing the leading singularity of v at one branch point. Then we will get a solution to the Dirac equation of the appropriate type by considering

$$w = \begin{bmatrix} v \\ m^{-1} K_{k-\frac{1}{2}}^* v \end{bmatrix}. \tag{3.9}$$

The reader might observe in what follows that using (3.8) instead of

$$(m^2 + K_\kappa^* K_\kappa)v = 0 ,$$

and (3.9) instead of

$$w = \begin{bmatrix} -m^{-1} K_{k-\frac{1}{2}} v \\ v \end{bmatrix},$$

is dictated by the desire to produce canonical solutions satisfying $c_j(W_i) = \delta_{ij}$ rather than the dual canonical basis $c_j^*(W_i) = \delta_{ij}$.

To formulate local expansions for the Helmholtz equation in a convenient form we wish to introduce here the analogue of the wave functions $w_l(z, a_j)$ and $w_l^*(z, a_j)$ for $v_j(l, \kappa, s)$ introduced in (2.16) above. Recall from Sect. 2 that (3.8) is equivalent to,

$$\left\{ s\left(s - \frac{2}{R}\right) - \Delta_{\kappa+1} \right\} v = 0 ,$$

with $\kappa = k - \frac{1}{2}$ and s given by (2.19). Now we write

$$v_{l,\kappa}(z) = v_l(z, l, \kappa, s)\ , \tag{3.10}$$

and

$$v^*_{l,\kappa}(z) = v_2(z, -l, \kappa, s)\ , \tag{3.11}$$

with s given by (2.19). Then $v_{l,\kappa+1}$ and $v^*_{l,\kappa+1}$ are both multivalued solutions to (3.8) above with monodromy $e^{2\pi il}$ and $e^{-2\pi il}$ respectively, in a counter-clockwise circuit of 0. We translate $v_{l,k}$ and $v^*_{l,k}$ so that they are "centered at $a \in \mathbf{D}_R$" in the following manner,

$$v_{l,\kappa}(z, a) := v(T[a], z)^{\kappa}\, v_{l,\kappa}(T[-a]\, z)\ , \tag{3.12}$$

and

$$v^*_{l,\kappa}(z, a) := v(T[a], z)^{\kappa}\, v^*_{l,\kappa}(T[-a]\, z)\ . \tag{3.13}$$

Precisely as in Sect. 2 a branch cut ℓ at a is used to fix a unique branch for the functions $v_{l,k}(z, a)$ and $v^*_{l,k}(z, a)$. The local singularity in both $v_{l,k}(z)$ and $v^*_{l,k}(z)$ goes like r^l, where r is the hyperbolic distance between 0 and z. To make use of the formula (3.9) to get an L^2 solution to the Dirac equation we will require that the solutions we consider for (3.8) have both v and $K^*_\kappa v$ in L^2. It is now a simple matter to check (using the formulas for the action of K^*_κ on v_l and v^*_l given in the preceding section) that if v is a multivalued solution to (3.8) with monodromy multiplier $e^{2\pi il_j}$ in a counter-clockwise circuit of a_j and if v and $K^*_\kappa v$ are in L^2 then provided l_j is chosen so that $0 < l_j < 1$ the function v will have restricted local expansions

$$v(z) = \sum_{n=-1}^{\infty} \alpha^j_n(v)\, v_{n+l_j,\kappa+1}(z, a_j) + \sum_{n=1}^{\infty} \beta^j_n(v)\, v^*_{n-l_j,\kappa+1}(z, a_j) \tag{3.14}$$

for z near a_j. In each case the sum is now over integer values of n rather than half integers. Our strategy in finding such solutions will be to seek a canonical basis $\{V_i\}$ with

$$\alpha^j_{-1}(V_i) = \delta_{ij}\ . \tag{3.15}$$

Existence for a canonical L^2 basis. Let $\varphi_j(z)$ be a C^∞_0 function which is identically 1 in a neighborhood of the point $z = a_j$ and which vanishes in the complement of an open ball about $z = a_j$. This open ball should be small enough so that φ_j vanishes in a neighborhood of each of the points a_i with $i \neq j$. Observe next that if V_j is a solution to (3.8) with local expansions of type (3.14) and which further satisfies (3.15) then,

$$\begin{aligned}(m^2 + K_\kappa K^*_\kappa)\ &(V_j - \varphi_j v_{l_j-1,\kappa+1}(z, a_j)) \\ &= -(m^2 + K_\kappa K^*_\kappa)\ \varphi_j v_{l_j-1,\kappa+1}(z, a_j) := f_j\end{aligned} \tag{3.16}$$

where f_j is a smooth multivalued function which vanishes in a neighborhood of the point a_j (and in a neighborhood of all the other branch points as well). For simplicity we will refer to branched functions that have smooth extensions (from some $C_0(\ell)$) to functions on $\tilde{\mathbf{D}}_R(a)$ *as smooth multivalued functions*. We will construct V_j by finding a suitably regular solution to (3.16) – one whose local expansion coefficients α^i_{-1} vanish for $i = 1, 2, \ldots, n$ and which is in the Sobolev space H^1.

Motivated by (3.16) we are interested in the existence question for solutions to

$$(m^2 + K_\kappa K^*_\kappa)\, u_j = f_j \tag{3.17}$$

for u_j in a completion of

$$C_0^\infty(\tilde{\mathbf{D}}_R(a))$$

with specified branching at a_i for $i=1,2,\ldots,n$. Here $C_0^\infty(\tilde{\mathbf{D}}_R(a))$ denotes the space of C^∞ functions on $\tilde{\mathbf{D}}_R(a)$ whose supports project onto compact subsets of $\mathbf{D}_R(a)$. In particular such functions vanish in a neighborhood of the points a_j and in a neighborhood of the boundary of $\mathbf{D}_R$. Let $C_{a,l}^\infty$ denote the subset of $C_0^\infty(\tilde{\mathbf{D}}_R(a))$ consisting of those functions f which transform under the deck transformation R_j by,

$$f(R_j\,\tilde{z})=e^{2\pi i l_j}f(\tilde{z})\,.$$

where $0\leqq l<1$ for $j=1,2,\ldots,n$ and we write

$$l=(l_1,l_2,\ldots,l_n)\,.$$

Now let $H_{a,l}$ denote the Hilbert space completion of $C_{a,l}^\infty$ with respect to the norm derived from the inner product,

$$(F,G)=\int_{\mathbf{D}_R}\{\overline{K_\kappa^*\,F(\tilde{z})}\,K_\kappa^*\,G(\tilde{z})+m^2\,\overline{F(\tilde{z})}\,G(\tilde{z})\}\,d\mu(z)\,.$$

Note that the integrand descends to a function on $\mathbf{D}_R$ because the monodromy multipliers of $\bar{F}$ and G cancel. Now consider the element $f_j\in C_{a,l}$ defined above in (3.16). For $v\in C_{a,l}^\infty$ consider the linear functional,

$$v\rightarrow\int_{\mathbf{D}_R}\overline{f_j(\tilde{z})}\,v(\tilde{z})\,d\mu(z)=\langle f_j,v\rangle\,. \tag{3.18}$$

Since $(h,h)\geqq m^2\langle h,h\rangle$, for all $h\in C_{a,l}^\infty$, it is clear that the linear functional (3.18) is continuous on the Hilbert space $H_{a,l}$. Thus by the Riesz representation theorem there exists an element $u_j\in H_{a,l}$ so that

$$\langle f_j,v\rangle=(u_j,v)\,.$$

Since the formal adjoint of K_κ^* is K_κ one sees from this last equation that u_j is a distribution solution to (3.17). We wish to be a little more explicit about the local regularity of this solution near the branch point points a_j for $j=1,\ldots,n$ and for this purpose it is useful to consider in more detail functions $F\in H_{a,l}$. Such a function F, is the limit a sequence of functions $F_m\in C_{a,l}^\infty$ in the $H_{a,l}$ norm. Recall that functions in $C_{a,l}^\infty$ have "compact supports" so that integration by parts in the formula for the $H_{a,l}$ norm of F_m is justified and one finds that (F_m,F_m) is,

$$\int_{\mathbf{D}_R}\overline{F_m(\tilde{z})}\,K_\kappa\,K_\kappa^*F_m(\tilde{z})\,d\mu(z)+m^2\int_{\mathbf{D}_R}\overline{F_m(\tilde{z})}\,F_m(\tilde{z})\,d\mu(z)\,. \tag{3.19}$$

Now a simple computation shows that

$$K_{\kappa+1}^*\,K_{\kappa+1}-K_\kappa\,K_\kappa^*=\frac{8(\kappa+1)}{R^2}\,.$$

Substituting $K_{\kappa+1}^*\,K_{\kappa+1}-\dfrac{8(\kappa+{}^1)}{R^2}$ for $K_\kappa\;K_\kappa^*$ in (3.19) and integrating by parts one finds that the $L^2\,(d\mu)$ norm of $K_{\kappa+1}\,F_m$ can be expressed in terms of the $L^2(d\mu)$ norm $K_\kappa^*\,F_m$ and the $L^2(d\mu)$ norm of F_m. The analogous result for the difference $F_m-F_{m'}$ shows that $K_{\kappa+1}\,F_m$ is a Cauchy sequence in $L^2(d\mu)$. Thus we find that $K_{\kappa+1}\,F$ is in $L^2(d\mu)$.

Next observe that if F, $K_\kappa^* F$, and $K_{\kappa+1} F$ are in $L^2(d\mu)$ it follows that F is locally in the Sobolev space H^1 (at least away from the branch points). Returning now to the distribution solution $u_j \in H_{a,l}$ of (3.17) above one can now conclude from the standard regularity results for solutions to elliptic equations that since u_j is locally in the Sobolev space H^1 it must actually be C^∞ away from the branch points a_i for $i=1,2,\ldots,n$.

Using the fact that u_j, $K_\kappa^* u_j$ and $K_{\kappa+1} u_j$ must all be locally in L^2 it is not difficult to see that the local expansions for u_j must have the form

$$u_j(\tilde{z}) = \sum_{n=0}^{\infty} \alpha_n^i(u_j) v_{n+l_i}(\tilde{z}, a_i) + \sum_{n=1}^{\infty} \beta_n^i(u_j) v_{n-l_i}^*(\tilde{z}, a_i)$$

for z near to a_i. The reader should note that it is precisely the additional condition that $K_{\kappa+1} u_j$ must be in L^2 that forces $\alpha_{-1}^i(u_j)=0$ for all i. Thus if we define,

$$V_i(\tilde{z}) = u_j(\tilde{z}) + \varphi_j(z) v_{l_j-1,\kappa+1}(\tilde{z}, a_j) ,$$

then it is clear that V_j is a solution to (3.8) which is appropriately branched at $a_1, a_2, \ldots, a_n$, and it is in $L^2(\mathbf{D}_R, d\mu)$ together with $K_\kappa^* V_j$ by construction, since $K_\kappa^* v_{l_j-1}$ is locally less singular than v_{l_j-1}. Finally

$$\alpha_{-1}^i(V_j) = \delta_{ij} ,$$

since u_j does not contribute to the local expansion coefficients $\alpha_{-1}^i(V_j)$. To exhibit the dependence of V_j on the various parameters we will write

$$V_j = V_j(l, \kappa) ,$$

where $l=(l_1, l_2, \ldots, l_n)$.

We now exhibit the canonical L^2 basis of Dirac wave functions. Define,

$$w_j(l, k) = \begin{bmatrix} \dfrac{2}{mR} V_j(l, k-\tfrac{1}{2}) \\ \dfrac{2}{m^2 R} K_{\kappa-\frac{1}{2}}^* V_j(l, k-\tfrac{1}{2}) \end{bmatrix} , \tag{3.20}$$

where $0 < l_j < 1$ for each $j=1,2,\ldots,n$. Then $w_j(l,k)$ is an L^2 wave function for the Dirac equation with monodromy $e^{2\pi i l_j}$ about the point a_j and with

$$c_j(w_j(l,k)) = \delta_{ij} . \tag{3.21}$$

The response functions. The canonical wave functions are very useful but they are not the wave functions that are fundamental in our treatment of τ-functions. We turn now to the consideration of the Dirac wave functions $W_j(\tilde{z}, \lambda, k)$ and $W_j^*(\tilde{z}, \lambda, k)$ that will be central for us. Because of the role they play in the formula (4.50) for the derivative of the Green function we will refer to these wave functions as the response functions. We start with a characterization of the desired response function W_j and then we use the L^2 wave functions to prove the existence of W_j and W_j^*.

The response function W_j whose existence we wish to establish is characterized by the following three conditions:

(1) $W_j(\tilde{z}, \lambda, k)$ is a multivalued solution to the Dirac equation which is in $L^2(d\mu)$ in the complement of some compact neighborhood of $\{a_1, \ldots, a_n\}$.
(2) $W_j(\tilde{z}, \lambda, k)$ has monodromy multiplier $e^{2\pi i \lambda_j}$ in a counter-clockwise circuit of a_j (with λ_j now chosen so that $0 < |\lambda_j| < \frac{1}{2}$).

(3) $W_j(\tilde{z}, \lambda, k)$ has local expansions at each of the points a_j of the following type,

$$W_j(\tilde{z}, \lambda, k) = \delta_{ij} w_{-\frac{1}{2}+\lambda_i} + \sum_{n>0} \{a^i_{nj}(\lambda, k) w_{n+\lambda_i} + b^i_{nj}(\lambda, k) w^*_{n-\lambda_i}\} , \tag{3.22}$$

where $n \in \mathbf{Z} + \frac{1}{2}$ and we abbreviate $w_{n+\lambda_i,k}(\tilde{z}, a_i)$ by $w_{n+\lambda_i}$ and $w^*_{n-\lambda_i,k}(\tilde{z}, a_i)$ by $w^*_{n-\lambda_i}$. The first thing to observe about (3.22) is that it characterizes $W_j(\tilde{z}, \lambda, k)$. Any two functions with local expansions of type (3.22) differ by a wave function whose local expansion has only terms involving $w_{n+\lambda_i}$ and $w^*_{n-\lambda_i}$ for $n \geqq \frac{1}{2}$. A little computation using Theorem 3.0 shows that any such wave function, w, has L^2 norm 0 since either $c_i(w) = 0$ or $c^*_i(w) = 0$ for each $i = 1, 2, \ldots, n$.

Remark. Because $W_j(\tilde{z}, \lambda, k)$ depends on many parameters we will often omit those which are of less interest in the current claculations and rely on the characteristic names for the variables to save the reader from confusion. Thus in what follows we will write $W_j(\lambda, k)$ for the functions $W_j(\tilde{z}, \lambda, k)$.

To prove the existence of $W_j(\lambda, k)$ we will consider two cases. Let $\mathbf{P}$ denote the subset of $\{1, 2, \ldots, n\}$ which consists of those j for which $0 < \lambda_j < \frac{1}{2}$ and let $\mathbf{P}'$ denote the complementary subset of $\{1, 2, \ldots, n\}$ consisting of those j for which $-\frac{1}{2} < \lambda_j < 0$. If $j \in \mathbf{P}$ then it is not hard to see that if $W_j(\lambda, k)$ satisfies (3.22) and $j \in \mathbf{P}$ then

$$c_i(W_j(\lambda, k)) = \delta_{ij} \quad \text{for } i, j \in \mathbf{P} \tag{3.23}$$

and

$$c^*_i(W_j(\lambda, k)) = 0 \quad \text{for } i \in \mathbf{P}' . \tag{3.24}$$

But Theorem 3.0 shows that if $w \in \mathscr{W}(a, l)$ and $c_i(w) = 0$ for $i \in \mathbf{P}$ and $c^*_j(w) = 0$ for $i \in \mathbf{P}'$ then $w = 0$. Since the dimension of $\mathscr{W}(a, l)$ is n it follows that $\{c_i\}_{i \in \mathbf{P}} \cup \{c^*_i\}_{i \in \mathbf{P}'}$ is a collection of independent linear functionals on $\mathscr{W}(a, l)$. It follows that there exists an element $W_j(\lambda, k)$ of $\mathscr{W}(a, l)$ which satisfie the conditions (3.23) and (3.24). Without difficulty one can see that this implies $W_j(\lambda, k)$ has local expansions of type (3.22) and finishes the existence proof when $j \in \mathbf{P}$.

When $j \in \mathbf{P}'$ the leading singularity in (3.22) at a_j is no longer locally in L^2. To prove the existence of $W_j(\lambda, k)$ in this case it is probably simplest to return to the construction of the function V_j. Instead of forcing the singularity of V_j to agree with $v_{l_j-1,\kappa+1}(\tilde{z}, a_j)$ near $z = a_j$ one can construct an analogous function $\tilde{V}_j$ whose singularity at $z = a_j$ is matched with $v_{\lambda_j-1,\kappa+1}(\tilde{z}, a_j)$, where $\lambda_j = l_j - 1$. From $\tilde{V}_j$ we construct

$$\tilde{W}_j = \begin{bmatrix} \dfrac{2}{mR} \tilde{V}_j \\ \dfrac{2}{m^2 R} K^*_{k-\frac{1}{2}} \tilde{V}_j \end{bmatrix}$$

as before. Again using the independence of the collection of c_j for $i \in \mathbf{P}$ and the collection of c^*_i for $i \in \mathbf{P}'$ we see that there exists an element $G_j \in \mathscr{W}(a, l)$ so that

$$c_i(G_j) = c_i(\tilde{W}_j) \quad \text{for } i \in \mathbf{P}$$

and

$$c^*_i(G_j) = c^*_i(\tilde{W}_j) \quad \text{for } i \in \mathbf{P} .$$

If we now define

$$W_j(\lambda, k) = \tilde{W}_j - G_j \quad \text{for } \kappa = k - \tfrac{1}{2} ,$$

then it is easy to check that $W_j(\lambda, k)$ has local expansions of type (3.22). This finishes our existence proof for the response functions satisfying (3.22).

The response functions $W_j^*(\tilde{z}, \lambda, k)$ are characterized by conditions precisely analogous to 1–3 above. In particular W_j^* is a multivalued solution to the Dirac equation with monodromy multiplier $e^{2\pi i\lambda_j}$ in a counter-clockwise circuit about a_j. Instead of (3.22), however, the local expansion for W_j^* is,

$$W_j^*(\tilde{z}, \lambda, k) = \delta_{ij} w^*_{-\frac{1}{2}-\lambda_i} + \sum_{n>0} \{c^i_{nj}(\lambda, k) w_{n+\lambda_i} + d^i_{nj}(\lambda, k) w^*_{n-\lambda_i}\} . \tag{3.25}$$

The proof of the existence of the response functions W_j^* is precisely analogous to the existence proof for W_j and we leave this to the reader. The definition of W_j^* we have given here differs by the change $\lambda \to -\lambda$ compared to the analogous response function in [12]. In the present context the response functions W_j and W_j^* are both solutions to the same Dirac equation with the *same monodromy*.

Remark. We would like to point out one possible source of confusion for readers of this paper and the original papers of SMJ. In [12] there are two different bases for the same space $\mathscr{W}(a, l)$ referred to as the canonical basis w_j and the dual canonical basis w_j^*. The response functions W_j^* above should not be confused with a dual basis of any sort. As the reader can easily see, the span of the response functions W_j and W_j^* for $j=1,2,\ldots,n$ is $2n$ dimensional as the local expansions (3.22) and (3.25) show that these functions are linearly independent.

4. The Green Function $G^{a,\lambda}$ for the Hyperbolic Dirac Operator

The goal of this section is to construct a Green function for the Dirac operator in the presence of branch type singularities at the points a_j for $j=1,2,\ldots,n$. Before we attempt this there are a number of preliminary considerations.

Green functions in the absence of branch points. We begin by identifying the Green functions for the Helmholtz and Dirac operators in the absence of any singularities. To start we require Green's identity. First observe that the formula which identifies K_κ^* with the formal adjoint of K_κ may be written,

$$(\overline{K_\kappa f(z)}\, g(z) - \overline{f(z)}\, K_\kappa^* g(z))\, d\mu(z) = d\varphi(\bar{f}, g) , \tag{4.1}$$

where the one-form φ is,

$$\varphi(f, g) = -2i \left(\frac{f(z)\, g(z)}{1-\dfrac{|z|^2}{R^2}}\, dz \right) , \tag{4.2}$$

and d denotes the exterior derivative. Taking complex conjugates of (4.1) one finds

$$(\overline{K_\kappa^* f g} - \bar{f} K_\kappa g)\, d\mu = -\overline{d\varphi(f, \bar{g})} . \tag{4.3}$$

From (4.1) and (4.3) one may deduce

$$(\overline{K_\kappa f} K_\kappa g - \bar{f} K_\kappa^* K_\kappa g)\, d\mu = d\varphi(\bar{f}, K_\kappa g) \tag{4.4}$$

and

$$(\overline{K_\kappa^* K_\kappa f g} - \overline{K_\kappa f} K_\kappa g)\, d\mu = -\overline{d\varphi(K_\kappa f, \bar{g})} . \tag{4.5}$$

Adding (4.4) and (4.5) we find the infinitesimal version of Green's theorem

$$\overline{(\Delta_\kappa f g - \bar{f}\Delta_\kappa g)}\, d\mu = d\overline{(\varphi(K_\kappa f, \bar{g}) - \varphi(\bar{f}, K_\kappa g))}\,, \tag{4.6}$$

where we recall (2.3). Finally if we write,

$$\lambda(s) = s\left(s - \frac{2}{R}\right) = m^2 + \frac{4\kappa(\kappa+1)}{R^2}\,,$$

then we can rewrite (4.6) as

$$\overline{((\Delta_\kappa - \lambda(s)) f g - \bar{f}(\Delta_\kappa - \lambda(s))\, g)}\, d\mu = d\overline{(\varphi(K_\kappa f, \bar{g}) - \varphi(\bar{f}, K_\kappa g))}\,. \tag{4.7}$$

We are interested in the Green function for $\lambda(s) - \Delta_\kappa$ and so we will look for radially symmetric solutions to

$$(\lambda(s) - \Delta_\kappa) g = 0$$

which have the appropriate singularity at $z=0$ and which are in $L^2(d\mu)$ in a neighborhood of $|z| = R$. Recall now from (2.7)–(2.15) that a solution to the Helmholtz equation can be written in the form,

$$g_{\kappa,s}(z) = \left(1 - \frac{|z|^2}{R^2}\right)^{\frac{Rs}{2}} F_{\kappa,s}\left(\frac{|z|^2}{R^2}\right)\,, \tag{4.8}$$

where $F_{\kappa,s}(t)$ satisfies the hypergeometric equation (2.14) with,

$$a = \frac{Rs}{2} - \kappa, \quad b = \frac{Rs}{2} + \kappa, \quad c = 1\,,$$

and where s is again given by (2.19).

The case $c=1$ is special for the hypergeometric equation. One solution, $F(a,b;l;t)$ is regular at $t=0$, while the second independent solution has a logarthmic singularity at $t=0$. A local analysis near $t=1$ shows that when $Rs = a + b > 1$ only one solution, $F_{\kappa,s}(t)$, determines a solution, $g_{\kappa,s}(z)$, via (4.8) which is square integrable near $|z| = R$ (square integrable with respect to $d\mu$ that is). Of course this solution is only determined up to a constant multiple by this requirement. We fix the normalization of $F_{\kappa,s}(t)$ by requiring that

$$g_{\kappa,s}(z) \sim -\frac{1}{4\pi} \log|z| + O(|z| \log|z|) \tag{4.9}$$

for z near 0. For this to work we must know that the solution $F_{\kappa,s}(t)$ which produces the solution $g_{\kappa,s}(z)$ which is square integrable near $|z| = R$ is not also the solution of the hypergeometric equation which is analytic at $t=0$. We will sketch how one can see this. Suppose that v is the radial solution of

$$(K_\kappa^* K_\kappa + m^2) v = 0$$

which tends to 0 as $|z| \to R$, then for $\kappa = k - \frac{1}{2}$,

$$w = \begin{bmatrix} -K_{k-\frac{1}{2}} v \\ mv \end{bmatrix}$$

is a solution to the Dirac equation which is square integrable at ∞. If v is also well behaved near $|z|=0$, then w will be square integrable in a disk,

$$D_\rho=\left\{z:|z|\leqq R\operatorname{th}\frac{\rho}{2}\right\},$$

of radius $\rho>0$ about 0. A calculation similar to the calculation that produced (2.42) and (2.43) shows that the L^2 norm of w on D_ρ is expressible in a form precisely analogous to (2.43). The estimates (2.38) and (2.41) for w show that this expression for the L^2 norm tends to 0 as $\rho\to\infty$. This contradiction shows that the solutions well behaved at ∞ and at 0 cannot match up.

We next consider the sense in which $g_{\kappa,s}$ is a Green function for $\lambda(s)-\Delta_\kappa$. Let f denote a smooth function of compact support in $\mathbf{D}_R$. Integrate both sides of (4.7) with $f=f$ and $g=g_{\kappa,s}$ over the complement of a disk of radius ε about $z=0$. Use Stokes' theorem to reduce the result to a boundary integral over C_ε, the circle of radius ε about 0, and then use the asymptotics (4.9) to evaluate the boundary integrals in the limit $\varepsilon\to 0$. We find

$$\int_{\mathbf{D}_R}(\lambda(s)-\Delta_\kappa)f(z)\overline{g_{\kappa,s}(z)}\,d\mu(z)=f(0)\,. \tag{4.11}$$

This shows that $g_{\kappa,s}(z)$ is the solution appropriate to determine the Green function for $\lambda(s)-\Delta_\kappa$. To obtain this Green function we must translate $g_{\kappa,s}$ in the appropriate fashion. Introduce the variable w,

$$w:=T[-a]z=R\,\frac{Rz-Ra}{-\bar{a}z+R^2}\,,$$

and define

$$v(w):=\frac{R^2+\bar{a}w}{R^2+a\bar{w}}\,.$$

Let $v(w)^\kappa$ denote the fractional power of $v(w)$ normalized so that $v(0)^\kappa=1$. Then the transformation property of Δ_κ which is the analogue of the transformation property (1.18) of the Dirac operator is

$$v(w)^{-\kappa}\Delta_\kappa^{(w)}v(w)^\kappa=-K_\kappa^*(w)K_\kappa(w)+\frac{4\kappa(\kappa+1)}{R^2}:=\Delta_\kappa(w)\,. \tag{4.12}$$

This should be understood in the following fashion. The operator $\Delta_\kappa^{(w)}$ is the expression for Δ_κ that one obtains by the change of variables $z=z(w)$. The operators $K_\kappa(w)$ and $K_\kappa^*(w)$ are the analogues of K_κ and K_κ^* in the w coordinates,

$$K_\kappa(w)=2\left\{\left(1-\frac{|w|^2}{R^2}\right)\partial_w-\frac{\kappa}{R^2}\bar{w}\right\},$$

and

$$K_\kappa^*(w)=-2\left\{\left(1-\frac{|w|^2}{R^2}\right)\bar{\partial}_w+\frac{\kappa+1}{R^2}w\right\},$$

so that $\Delta_\kappa(w)$ is the same operator in the w-coordinates that Δ_κ is in the z coordinates. Now we introduce

$$g_{\kappa,s}(y,z)=v(T[-y]z)^\kappa g_{\kappa,s}(T[-y]z)=\left(\frac{R^2-y\bar{z}}{R^2-\bar{y}z}\right)^\kappa g_{\kappa,s}(T[-y]z)\,. \tag{4.13}$$

To see that this is the appropriate Green function suppose that f is a smooth function of compact support in $\mathbf{D}_R$ as above and consider the integral

$$\begin{aligned}
&\int_{\mathbf{D}_R} (\lambda(s)-\Delta_\kappa) f(z) \overline{g_{\kappa,s}(y,z)}\, d\mu(z) \\
&\quad = \int_{\mathbf{D}_R} (\lambda(s)-\Delta_\kappa^{(w)}) f(T[y]w) \overline{v(w)^\kappa g_{\kappa,s}(w)}\, d\mu(w) \\
&\quad = \int_{\mathbf{D}_R} (\lambda(s)-\Delta_\kappa(w))\, v(w)^{-\kappa} f(T[y]w) \overline{g_{\kappa,s}(w)}\, d\mu(w) \\
&\quad = v(w)^{-\kappa} f(T[y]w)_{w=0} = f(y)\,,
\end{aligned} \tag{4.14}$$

where we made the substitution $z \leftarrow w$ to obtain the transformed integral in the second line then used (4.13), (4.12) and finally (4.11) to obtain the third and fourth lines.

Next we construct the Green function for the Dirac operator, $m-D_k$, using the Green function $g_{\kappa,s}(y,z)$. Recall now (2.2) and (2.6) from which it follows that

$$(m+D_k)(m-D_k) = \begin{bmatrix} \lambda(s)-\Delta_{k+\frac{1}{2}} & 0 \\ 0 & \lambda(s)-\Delta_{k-\frac{1}{2}} \end{bmatrix}. \tag{4.15}$$

Now let $f(z)$ denote a smooth function of compact support on $\mathbf{D}_R$ with values in $\mathbf{C}^2$. Then from (4.15) and (4.14) we have

$$\int_{\mathbf{D}_R} (m+D_k)(m-D_k) f(z) \cdot \overline{\begin{bmatrix} g_{k+\frac{1}{2},s}(y,z) \\ 0 \end{bmatrix}}\, d\mu(z) = f_1(y)\,. \tag{4.16}$$

If we exclude an ε disk about $z=y$ in (4.16) and use Stokes' theorem to "integrate by parts" once in (4.16) then we find

$$\int_{\mathbf{D}_R} (m-D_k) f(z) \cdot \overline{G_1(y,z)}\, d\mu(z) = f_1(y)\,, \tag{4.17}$$

where

$$G_1(y,z) = (m+D_k^*) \begin{bmatrix} g_{k+\frac{1}{2},s}(y,z) \\ 0 \end{bmatrix} = (m-D_k) \begin{bmatrix} g_{k+\frac{1}{2},s}(y,z) \\ 0 \end{bmatrix}. \tag{4.18}$$

The boundary of the ε disk about $z=y$ makes a contribution to the Stokes' theorem calculation that vanishes in the limit $\varepsilon \to 0$ due to the weak logarithmic singularity in $g_{k+\frac{1}{2},s}(y,z)$ at $z=y$. In a precisely similar fashion one finds that for

$$G_2(y,z) := (m-D_k) \begin{bmatrix} 0 \\ g_{k-\frac{1}{2},s}(y,z) \end{bmatrix}, \tag{4.19}$$

one has

$$\int_{\mathbf{D}_R} (m-D_k) f(z) \cdot \overline{G_2(y,z)}\, d\mu(z) = f_2(y)\,. \tag{4.20}$$

The Green function for the Helmholtz operator in the presence of branch points. Before tackling the existence question for a Green function associated with the branched version of $m-D$ we will first consider the analogous problem for the "Helmholtz" operator

$$\lambda(s)-\Delta_\kappa = m^2 + K_\kappa^* K_\kappa\,,$$

with

$$\lambda(s)=m^2+\frac{4\kappa(\kappa+1)}{R^2}.$$

We wish to consider the action of Δ_κ on functions on the unit disk with specified branching at a finite collection of points $\{a_1,a_2,\ldots,a_n\}$ in the disk $\mathbf{D}_R$. It will be useful for us to recall the conventions of Sect. 3. Thus we identify points $a_j=(\alpha_j,\beta_j)$ in the disk with their standard representation as complex numbers $a_j=\alpha_j+i\beta_j$. Write $\mathbf{D}_R(a)$ for $\mathbf{D}_R\backslash\{a_1,a_2,\ldots,a_n\}$ and $\tilde{\mathbf{D}}_R(a)$ for the simply connected covering space of the punctured disk $\mathbf{D}_R(a)$. Let $C_0^\infty\tilde{\mathbf{D}}_R(a))$ denote the smooth functions on $\tilde{\mathbf{D}}_R(a)$ whose supports project onto compact subsets of $\mathbf{D}_R(a)$. Fix a base point a_0 in $\mathbf{D}_R$ different from any of the points $a_1,a_2,\ldots,a_n$ and let $[\alpha_j]$ denote the homotopy class of the simple a_0 based loop, α_j, about a_j in $\mathbf{D}_R(a)$ which circles a_j in a counterclockwise fashion and does not wind around any of the other points a_i with $i\neq j$. Points in $\tilde{\mathbf{D}}_R(a)$ over a_0 correspond to homotopy classes of a_0 based loops in $\mathbf{D}_R(a)$ and as above we fix our model of $\tilde{\mathbf{D}}_R(a)$ to be the collection of homotopy classes of paths in $\mathbf{D}_R(a)$ which start at a_0. The canonical projection,

$$\mathrm{pr}\colon \tilde{\mathbf{D}}_R(a)\to\mathbf{D}_R(a),$$

maps each class $[\gamma]$ of paths that start at a_0 into the endpoint $\gamma(1)$. This model of $\tilde{\mathbf{D}}_R(a)$ comes with a distinguished base point. Let $\tilde{a}_0$ denote the point in $\tilde{\mathbf{D}}_R(a)$ which corresponds to the constant path $[0,1]\ni t\to a_0$. Let R_j denote the deck transformation on $\tilde{\mathbf{D}}_R(a)$ which maps $\tilde{a}_0$ to the point in $\tilde{\mathbf{D}}_R(a)$ which corresponds to the homotopy class $[\alpha_j]$. Although it is not completely precise it will be convenient to adopt the following notation to avoid clumsy expressions. We will write $\tilde{z}$ for a typical point in $\tilde{\mathbf{D}}_R(a)$ and

$$\mathrm{pr}(\tilde{z})=z$$

for the projection on $\mathbf{D}_R$.

We wish to consider the action of Δ_κ on a subset of $C_0^\infty(\tilde{\mathbf{D}}_R(a))$ with specified branching at a_j for $j=1,2,\ldots,n$. Let $\bar{C}_{a,l}^\infty$ denote the subset of $C_0^\infty(\tilde{\mathbf{D}}_R(a))$ consisting of those functions f which transform under R_j by

$$f(R_j\tilde{z})=e^{-2\pi i l_j}f(\tilde{z}),$$

where $0\leq l_j<1$ for $j=1,2,\ldots,n$. Note that since $\bar{C}_{a,l}^\infty$ is a module for the multiplicative action of smooth functions on $\mathbf{D}_R$ and since the differential operators ∂_z and $\partial_{\bar{z}}$ lift in an obvious way to differential operators on $\bar{C}_{a,l}^\infty$ there is no difficulty in defining the action of Δ_κ on $\bar{C}_{a,l}^\infty$. We will now construct a Green function for $\lambda(s)-\Delta_\kappa$ acting on $\bar{C}_{a,l}^\infty$. This is not completely accurate since we will implicitly limit the strength of the singularities at the points a_j in a completion of $\bar{C}_{a,l}^\infty$ but it will prove simpler to construct the Green function than it will to give a complete discussion of the domain of the corresponding operator. By analogy with (4.14) above we seek a function $g^{a,l}(\tilde{y},\tilde{z})$ so that for any $f\in C_{a,l}^\infty$ we have

$$\int_{\mathbf{D}_R}(\lambda(s)-\Delta_\kappa)f(\tilde{z})\overline{g^{a,l}(\tilde{y},\tilde{z})}\,d\mu(z)=f(\tilde{y}). \tag{4.21}$$

Of course, for the this integral to make sense it is necessary for the integrand,

$$\tilde{\mathbf{D}}_R(a)\ni\tilde{z}\to(\lambda(s)-\Delta_\kappa)f(\tilde{z})\overline{g^{a,l}(\tilde{y},\tilde{z})},$$

to descend to a function on $\mathbf{D}_R$. This will happen if $g^{a,l}(\tilde{y}, \tilde{z})$ transforms as follows,

$$g^{a,l}(\tilde{y}, R_j\tilde{z}) = e^{-2\pi i l_j} g^{a,l}(\tilde{a}, \tilde{z}) , \tag{4.22}$$

under the deck transformations R_j. Equation (4.21) suggests that $g^{a,l}(\tilde{y}, \tilde{z})$ should transform in the first variable

$$g^{a,l}(R_j\tilde{y}, \tilde{z}) = e^{2\pi i l_j} g^{a,l}(\tilde{y}, \tilde{z}) . \tag{4.23}$$

We will now give a functional analytic proof for the existence of a suitable version of $g^{a,l}(\tilde{y}, \tilde{z})$. To begin we will first concentrate on the behavior of $g^{a,l}(\tilde{y}, \tilde{z})$ as a function of the second variable $\tilde{z}$. To obtain a Green function $g^{a,l}(\tilde{y}, \tilde{z})$ for $\lambda(s) - \Delta_\kappa$ satisfying (4.21) above we seek a function

$$\tilde{\mathbf{D}}_R(a) \ni \tilde{z} \to g^{a,l}(\tilde{y}, \tilde{z})$$

which is a solution to the Helmholtz equation,

$$(\lambda(s) - \Delta_\kappa) g^{a,l}(\cdot, \tilde{z}) = 0 ,$$

defined for $\tilde{z} \in \tilde{\mathbf{D}}_R(a)$ with $z \in \mathbf{D}_R(a) \backslash \{y\}$ (recall that $\mathrm{pr}(\tilde{z}) = z$ and $\mathrm{pr}(\tilde{y}) = y$), with monodromy multiplier $e^{-2\pi i l_j}$ in a counterclockwise circuit of the branch point a_j. Also, to obtain a Green function we want $g^{a,l}(\tilde{y}, \tilde{z})$ to differ from $g_{\kappa,s}(y, z)$ by a smooth function when $\tilde{y}$ is close to $\tilde{z}$. Note that, because of the transformation properties we want for $g^{a,l}(\tilde{y}, \tilde{z})$ it will have singularities on the set

$$\mathscr{D} := \{\mathrm{pr}^{-1}(x) \times \mathrm{pr}^{-1}(x) : x \in \mathbf{D}_R(a)\}$$

as well as on the diagonal of $\tilde{\mathbf{D}}_R(a) \times \tilde{\mathbf{D}}_R(a)$.

To make this precise we let $\phi(z)$ denote a C_0^∞ function of $\mathbf{R}^2$ with

$$\phi(z) = 1 \text{ for } |z| \leqq 1$$

and

$$\phi(z) = 0 \quad \text{for } |z| \geqq 2 ,$$

and define

$$\phi_\varepsilon(z) := \phi\left(\frac{z}{\varepsilon}\right) .$$

Then for y fixed and different from a_j for $j = 1, \ldots, n$ and ε small enough the support of $z \to \phi_\varepsilon(z - y)$ will not contain any of the branch points $\{a_1, a_2, \ldots, a_n\}$. In such circumstances it is easy to see that the function,

$$z \to \phi_\varepsilon(z - y) g_{\kappa,s}(y, z) ,$$

has support in an *elementary neighborhood of the point* $z = y$. An elementary neighborhood, U, of a point $y \in \mathbf{D}_R(a)$ is one for which the inverse image of U under the projection,

$$\mathrm{pr}\colon \tilde{\mathbf{D}}_R(a) \to \mathbf{D}_R(a) ,$$

splits into components V_α with the property that

$$\mathrm{pr}\colon V_\alpha \to U$$

is a diffeomorphism. The sets of V_α are indexed by homotopy classes, α, of paths that join the base point a_0 to y. Using this property we will now indicate how to obtain a lift of the function,

$$U\times U\ni(x,y)\to\phi_\varepsilon(y-x)g_{\kappa,s}(x,y)\,,$$

to a function defined on

$$\mathrm{pr}^{-1}(U)\times\mathrm{pr}^{-1}(U)\backslash\mathscr{D}$$

with prescribed multipliers. Let a_0 denote the base point in $\mathbf{D}_R(a)$ as above. Let α_0 denote a path joining a_0 to y representing fixed homotopy class $[\alpha_0]$. Once $[\alpha_0]$ is fixed we can index the components of $\mathrm{pr}^{-1}(U)$ by homotopy classes of loops based at y. Furthermore the choice of α_0 allows us to identify the homotopy group of loops based at $a_0\in\mathbf{D}_R(a)$ with the homotopy group of loops based at $y\in\mathbf{D}_R(a)$. Thus if we let π denote the representation of the fundamental group of $(\mathbf{D}_R(a),a_0)$ which results from assigning the number $e^{-2\pi i l_j}$ to the generator which starts at a_0 and makes a simple counterclockwise circuit of a_j not winding around any of the other points a_k for $k\neq j$, then this naturally gives a representation of the fundamental group of $(\mathbf{D}_R(a),y)$ as well; namely,

$$\pi[\gamma]:=\pi[\alpha_0^{-1}\gamma\alpha_0]\ \text{for}\quad[\gamma]\in\pi_1(\mathbf{D}_R(a),y\,.$$

Now we define a lift of

$$\tilde{g}^\varepsilon_{\kappa,s}(\tilde{x},\tilde{y}):=\phi_\varepsilon(y-x)g_{\kappa,s}(x,y)$$

which is defined on $U\times U\backslash\{(x,x)\colon x\in\mathbf{D}_R(a)\}$ by

$$\tilde{g}^\varepsilon_{\kappa,s}(\tilde{x},\tilde{y}):=\pi[\gamma_2\gamma_1^{-1}]g^\varepsilon_{\kappa,s}(x,y)\quad\text{for}\quad\tilde{x}\in V_{[\gamma_1,\alpha_0]}\quad\text{and}\quad\tilde{y}\in V_{[\gamma_2,\alpha_0]}\,.$$

The function $\tilde{g}^\varepsilon_{\kappa,s}(\tilde{x},\tilde{y})$ is defined on $\mathrm{pr}^{-1}(U)\times\mathrm{pr}^{-1}(U)\backslash\mathscr{D}$, and $[\gamma_k]\in\pi_1(\mathbf{D}_R(a),y)$ for $k=1,2$. It is not hard to see that this extension is independent of the choice of $[\alpha_0]$, for if α_0' is another path from a_0 to y one has

$$\pi(\gamma_2\gamma_1^{-1}]=\pi[\alpha_0^{-1}\gamma_2\gamma_1^{-1}\alpha_0]=\pi[\alpha_0^{-1}\alpha_0']\pi[\alpha_0'^{-1}\gamma_2\gamma_1^{-1}\alpha_0']\pi[\alpha_0'^{-1}\alpha_0]$$

and

$$\pi[\alpha_0^{-1}\alpha_0']\pi[\alpha_0'^{-1}\alpha_0]=1\,.$$

It is now natural to extend $\tilde{g}^\varepsilon_{\kappa,s}(\tilde{y},\tilde{z})$ to all of $\tilde{\mathbf{D}}_R(a)\times\tilde{\mathbf{D}}_R(a)\backslash\mathscr{D}$ by setting it equal to 0 on the complement of $\mathrm{pr}^{-1}(U)\times\mathrm{pr}^{-1}(U)\backslash\mathscr{D}$. Now fix $\tilde{y}\in\tilde{\mathbf{D}}_R(a)$. We will construct $g^{a,l}(\tilde{y},\tilde{z})$ whenever z, the projection of $\tilde{z}$, is close to y, the projection of $\tilde{y}$. More precisely, we require that

$$(\lambda(s)-\Delta_\kappa)\{g^{a,l}(\tilde{y},\tilde{z})-\tilde{g}^\varepsilon_{\kappa,s}(\tilde{y},\tilde{z})\}\tag{4.24}$$

be free of singularities as a function of $\tilde{z}\in\tilde{D}_R(a)$. When $\tilde{z}$ is not in the fiber over $y=\mathrm{pr}(\tilde{y})$ we want $(\lambda(s)-\Delta_\kappa)g^{a,l}(\tilde{y},\tilde{z})=0$. Thus with the formal cancellation of the delta functions in (4.24) the only remaining contribution comes when the differential operator part of Δ_κ hits the function $\phi_\varepsilon(z-y)$ in $\tilde{g}^\varepsilon_{\kappa,s}(\tilde{y},\tilde{z})$ (note that the multiplicative part of Δ_κ which does not contain any derivatives is absorbed in the delta function that arises when Δ_k hits $g_{\kappa,s}(y,z)$). Now choose ε small enough so that 2ε will produce an elementary neighborhood U above. Then $\phi_{2\varepsilon}(z-y)$ is identically 1 on the support of $\phi_\varepsilon(z-y)$. Using this one can easily see that there is an explicit second order differential operator, $p(\partial_z,\bar{\partial}_z)$, whose lowest order terms are first order in ∂_z and $\bar{\partial}_z$ and for which we want

$$(\lambda(s)-\Delta_\kappa)\{g^{a,l}(\tilde{y},\tilde{z})-\tilde{g}^\varepsilon_{\kappa,s}(\tilde{y},\tilde{z})\}=\{p(\partial_x,\bar{\partial}_z)\phi_\varepsilon(z-y))\}\tilde{g}^{2\varepsilon}_{\kappa,s}(\tilde{y},\tilde{z})\,.\tag{4.25}$$

Since $p(\partial_z, \bar{\partial}_z)$ kills $\phi_\varepsilon(z-y)$ near $z=y$ it follows that the right-hand side of (4.25) is a smooth function of $\tilde{z}$ in $\bar{C}^\infty_{a,l}$. We will now show that for any $f \in \bar{C}^\infty_{a,l}$ whose support projects onto an elementary neighborhood it is possible to find a multivalued function F, transforming as f does under deck transformations, so that

$$(\lambda(s) - \Delta_\kappa) F = f . \tag{4.26}$$

Once we have this existence result we can then recover $g^{a,l}(\tilde{y}, \tilde{z})$ from the solution to (4.25) above any adding in $\tilde{g}^\varepsilon_{\kappa,s}(\tilde{y}, \tilde{z})$.

Now we will recall the Hilbert space technique of Sect. 3 for finding a solution to (4.26). Let $\bar{H}_{a,l}$ denote the Hilbert space completion of $\bar{C}^\infty_{a,l}$ with respect to the norm derived from the inner product

$$(F, G) = \int_{\mathbf{D}_R} \{\overline{K_k F(\tilde{z})} K_k G(\tilde{z}) + m^2 \overline{F(\tilde{z})} G(\tilde{z})\} \, d\mu(z) .$$

Note that the integrand descends to a function on $\mathbf{D}_R$ because the monodromy multipliers of $\bar{F}$ and G cancel. Now suppose that $f \in \bar{C}^\infty_{a,l}$ and the support of f projects onto an elementary neighborhood. For $v \in \bar{C}^\infty_{a,l}$ consider the linear functional

$$v \to \int_{\mathbf{D}_R} \overline{f(\tilde{z})} v(\tilde{z}) d\mu(z) := \langle f, v \rangle . \tag{4.27}$$

Since $(h, h) \geqq m^2 \langle h, h \rangle$, for all $h \in \bar{C}^\infty_{a,l}$, it is clear that the linear functional (4.27) is continuous on the Hilbert space $\bar{H}_{a,l}$. Thus by the Riesz representation theorem there exists an element $F \in \bar{H}_{a,l}$ so that

$$\langle f, v \rangle = (F, v) .$$

Since the formal adjoint of K_k is K_k^* one sees from this last equation that F is a distribution solution to (4.26). Recall from the discussion in Sect. 3 that F will be locally in the Sobolev space H^1 (at least away from the branch points) and so by standard regularity results for elliptic equations, one can conclude that F must actually be C^∞ away from the branch points a_j for $j = 1, 2, \ldots, n$.

We now use this result to solve the equation

$$(\lambda(s) - \Delta_\kappa) F = \{p(\partial_z, \bar{\partial}_z) \phi_\varepsilon(z-y)\} \tilde{g}^{2\varepsilon}_{\kappa,s}(\tilde{y}, \tilde{z}) . \tag{4.28}$$

Let $g^{a,l}(\tilde{y}, \tilde{z})$ denote the sum of the solution $F(\tilde{y}, \tilde{z})$ to (4.28) and $\tilde{g}^\varepsilon_{\kappa,s}(\tilde{y}, \tilde{z})$. Then the reader can check that $g^{a,l}$ satisfies the following three conditions:

G1: The map $\tilde{z} \to g^{a,l}(\tilde{y}, \tilde{z})$ is a $C^\infty(\tilde{\mathbf{D}}_R(a))$ solution of the Helmholtz equation,

$$(\lambda(s) - \Delta_\kappa) g^{a,l}(\tilde{y}, \tilde{z}) = 0 ,$$

for $z \neq y$ and $z \neq a_j$ for $j = 1, 2, \ldots, n$ which transforms under the deck transformations R_j as follows,

$$g^{a,l}(\tilde{y}, R_j \tilde{z}) = e^{-2\pi i l_j} g^{a,l}(\tilde{y}, \tilde{z}) .$$

G2: If U is an elementary neighborhood of the point y, then in the complement $\mathbf{D}_R(a) \backslash \mathrm{pr}^{-1}(U)$ the functions,

$$\tilde{z} \to g^{a,l}(\tilde{y}, \tilde{z}) ,$$

$$\tilde{z} \to K_k g^{a,l}(\tilde{y}, \tilde{z}) ,$$

$$\tilde{z} \to K^*_{k-1} g^{a,l}(\tilde{y}, \tilde{z}) ,$$

have absolute squares that are integrable with respect to the natural hyperbolic measure, $d\mu(z)$, on $\mathbf{D}_R$.

G3: For some elementary neighborhood, U, of the point y the difference

$$g^{a,l}(\tilde{y}, \tilde{z}) - \tilde{g}^{\varepsilon}_{\kappa,s}(\tilde{y}, \tilde{z})$$

is smooth as a function of $\tilde{z} \in \mathrm{pr}^{-1}(U)$.

Next we want to recall the consequences of **G2** for the local expansions of $\tilde{z} \to g^{a,l}(\tilde{y}, \tilde{z})$ about the points a_j for $j=1, 2, \ldots, n$. Because $g^{a,l}(\tilde{y}, \tilde{z})$ is a multi-valued solution to the Helmholtz equation with monodromy $e^{-2\pi i l_j}$ about the point a_j it follows that $g^{a,l}(\tilde{y}, \tilde{z})$ has a local expansion,

$$g^{a,l}(\tilde{y}, \tilde{z}) = \sum_{n \in \mathbf{Z}} \{\alpha^j_n(\tilde{y}) v_{n-l_j,\kappa}(\tilde{z}, a_j) + \beta^j_n(\tilde{y}) v^*_{n+l_j,\kappa}(\tilde{z}, a_j)\} ,$$

valid for z near a_j. Recall now that l_j is chosen so that $0 \leqq l_j < 1$ and that the leading singularity in $v_{l,\kappa}(\tilde{z}, a)$ and $v^*_{l,\kappa}(\tilde{z}, a)$ goes like r^l in both cases, where r is the distance from z to a. It is now a straightforward exercise to verify that the condition **G2** implies the following restriction on the local expansion for $g^{a,l}(\tilde{y}, \tilde{z})$,

$$g^{a,l}(\tilde{y}, \tilde{z}) = \sum_{n=1}^{\infty} \{\alpha^j_n(\tilde{y}) v_{n-l_j,\kappa}(\tilde{z}, a_j) + \sum_{n=0}^{\infty} \beta^j_n(\tilde{y}) v^*_{n+l_j,\kappa}(\tilde{z}, a_j)\} , \tag{4.29}$$

for z near a_j. We will see that this restriction on the local expansions of $g^{a,l}(\tilde{y}, \tilde{z})$ together with **G1** and **G3** and the fact that $\tilde{z} \to g^{a,l}(\tilde{y}, \tilde{z})$ is square integrable on $\mathbf{D}_R$ suffice to characterize $g^{a,l}(\tilde{y}, \tilde{z})$. We also wish to characterize $g^{a,l}(\tilde{y}, \tilde{z})$ as a function of the first variable $\tilde{y}$. Since Δ_κ is Hermitian symmetric one might expect that

$$g^{a,l}(\tilde{y}, \tilde{z}) = \overline{g^{a,l}(\tilde{z}, \tilde{y})} . \tag{4.30}$$

Observe that the right-hand side of (4.30) transforms under $\tilde{y} \to R_j \tilde{y}$ by the multiplier $e^{2\pi i l_j}$ which is appropriate for $g^{a,l}(\tilde{y}, \tilde{z})$. Also since $g_{\kappa,s}(y, z) = \overline{g_{\kappa,s}(z, y)}$ it follows that the right-hand side of (4.30) differs from the Green function $g_{\kappa,s}(y, z)$ along the diagonal by a smooth function – this is additional evidence that the right-hand side of (4.30) should agree with $g^{a,l}(\tilde{x}, \tilde{z})$. We will now confirm (4.30) by a Stokes' theorem calculation along the lines of the standard argument for such symmetry. Now choose $\tilde{x}$ and $\tilde{y}$ two *distinct* points in $\tilde{\mathbf{D}}_R(a)$ and let

$$g(\tilde{z}) = g^{a,l}(\tilde{y}, \tilde{z})$$

and

$$f(\tilde{z}) = g^{a,l}(\tilde{x}, \tilde{z}) .$$

Substitute these last two functions in (4.7) and integrate the resulting equality over the complement of the union of the balls of radius ε about each of the point a_j and the balls of radius ε about both x and y. The left-hand side gives 0 since

$$(\lambda(s) - \Delta_\kappa) f = (\lambda(s) - \Delta_\kappa) g = 0$$

on this set and Stokes' theorem reduces the integral on the right to an integral over the circles of radius ε about the points a_j for $j=1, \ldots, n$ and x and y. Using the results for the local expansion of $g^{a,l}(\tilde{z}, \tilde{x})$ given in (4.28) above one finds that each of the integrals,

$$\int\limits_{\partial B_\varepsilon(a_j)} \varphi(\bar{f}, K_\kappa g) - \overline{\varphi(K_\kappa f, \bar{g})},$$

vanishes in the limit $\varepsilon \to 0$. Finally, using the fact that $g^{a,l}(\tilde{x}, \tilde{y})$ differs from the Green function for $\lambda(s) - \Delta_\kappa$ by a smooth function near the diagonal one finds that the contribution to the integral of

$$\varphi(\bar{f}, K_\kappa g) - \overline{\varphi(K_\kappa f, \bar{g})}$$

over the circles of radius ε about x and y is just

$$g(\tilde{x}) - \overline{f(\tilde{y})} = g^{a,l}(\tilde{y}, \tilde{x}) - \overline{g^{a,l}(\tilde{x}, \tilde{y})}$$

in the limit $\varepsilon \to 0$. This last quantity must vanish and this establishes the Hermitian symmetry of the Green function we are considering.

The Green function for the Dirac operator in the presence of branch points. Next we turn to the construction of a Green function for the Dirac operator acting on $\bar{C}^\infty_{a,l}$ by using formulas (4.18) and (4.19) with $g^\varepsilon_{\kappa,s}$ replaced by $g^{a,l}$. This will not give us the Green function we want because the local expansions will not have the "minimal singularity" type that we desire. However, by subtracting suitable wave functions from the results we are able to pass to the desired Green function in much the same fashion that we constructed the wave functions W_j from the canonical L^2 wave functions w_j. In analogy with (4.18) and (4.19) we now define $\mathcal{G}^{a,l}$ by

$$\mathcal{G}^{a,l}_1(\tilde{y}, \tilde{z}) = (m - D_k)\begin{bmatrix} g^{a,l}_{k+\frac{1}{2},s}(\tilde{y}, \tilde{z}) \\ 0 \end{bmatrix} = \begin{bmatrix} m g^{a,l}_{k+\frac{1}{2},s}(\tilde{y}, \tilde{z}) \\ K^*_{k-\frac{1}{2}} g_{k+\frac{1}{2},s}(\tilde{y}, \tilde{z}) \end{bmatrix}, \tag{4.31}$$

$$\mathcal{G}^{a,l}_2(\tilde{y}, \tilde{z}) = (m - D_k)\begin{bmatrix} 0 \\ g^{a,l}_{k-\frac{1}{2},s}(\tilde{y}, \tilde{z}) \end{bmatrix} = \begin{bmatrix} -K_{k-\frac{1}{2}} g_{k-\frac{1}{2},s}(\tilde{y}, \tilde{z}) \\ m g^{a,l}_{k-\frac{1}{2},s}(\tilde{y}, \tilde{z}) \end{bmatrix}, \tag{4.32}$$

where we have explicitly noted the dependence of $g^{a,l}_{\kappa,s}$ on (κ, s) to avoid confusion. We now record the local expansion results for (4.31) and (4.32) which are obtained from (4.29). There are two different cases to consider depending on whether $0 < l_j < \frac{1}{2}$ or $\frac{1}{2} \leqq l_j < 1$. In case $0 < l_j < \frac{1}{2}$ we write $\lambda_j = l_j$ and the local expansions for $\mathcal{G}^{a,l}$ about $z = a_j$ are

$$\mathcal{G}^{a,l}_1(\tilde{y}, \tilde{z}) = \sum_{n \geqq \frac{3}{2}} \alpha^j_{n,1}\, w_{n-\lambda_j,k}(\tilde{z}, a_j; -m) + \sum_{n \geqq -\frac{1}{2}} \beta^j_{n,1}\, w^*_{n+\lambda_j,k}(\tilde{z}, a_j; -m),$$

$$\mathcal{G}^{a,l}_2(\tilde{y}, \tilde{z}) = \sum_{n \geqq \frac{1}{2}} \alpha^j_{n,2}\, w_{n-\lambda_j,k}(\tilde{z}, a_j; -m) + \sum_{n \geqq \frac{1}{2}} \beta^j_{n,2}\, w^*_{n+\lambda_j,k}(\tilde{z}, a_j; -m), \tag{4.33}$$

where all the sums are over elements of $\mathbf{Z} + \frac{1}{2}$ and we have written $w_{l,k}(\tilde{z}, a; m)$ and $w^{*l,k}(\tilde{z}, a; m)$ for (2.22) and (2.23) in order to recognize explicitly the dependence on the parameters k and m (it is $-m$ which occurs in (4.33)).

In case $\frac{1}{2} \leqq l_j < 1$ we write $\lambda_j = l_j - 1$ and the local expansion results about $z = a_j$ are

$$\mathscr{G}_1^{a,l}(\tilde{y}, \tilde{z}) = \sum_{n \geqq \frac{1}{2}} \alpha_{n,1}^j \, w_{n-\lambda_j,k}(\tilde{z}, a_j; -m) + \sum_{n \geqq \frac{1}{2}} \beta_{n,1}^j \, w_{n+\lambda_j,k}^*(\tilde{z}, a_j; -m) \,,$$

$$\mathscr{G}_2^{a,l}(\tilde{y}, \tilde{z}) = \sum_{n \geqq -\frac{1}{2}} \alpha_{n,2}^j \, w_{n-\lambda_j,k}(\tilde{z}, a_j; -m) + \sum_{n \geqq \frac{3}{2}} \beta_{n,2}^j \, w_{n+\lambda_j,k}^*(\tilde{z}, a_j; -m) \,. \tag{4.34}$$

In (4.33) and (4.34) we have supressed the dependence on parameters $\tilde{y}$, k, m, and λ_j for the coefficients $\alpha_{n,i}^j$ and $\beta_{n,i}^j$ to avoid further complicating expressions that are already notationally overburdened.

To obtain the Green function we desire our object will be to eliminate those terms in the local expansions (4.33) and (4.34) with $n < \frac{1}{2}$. We can do this by subtracting suitable multiples of the wave functions $W_j(-m)$ and $W_j^*(-m)$ without introducing additional singularities on the "diagonal" $y = z$. As is the case in (4.33) and (4.34) it is now convenient to explicitly identify the m-dependence of the wave functions W_j and W_j^* since the Green functions $\mathscr{G}_j^{a,l}(\tilde{y}, \tilde{z})$ are solutions to the equation $(m + D_k)\psi = 0$ in the $\tilde{z}$ variable instead of the original Dirac equation $(m - D_k)\psi = 0$. As always our notation for the response functions will suppress dependence on parameters that do not play a role in the current calculations and will rely on characteristic names for the parameters to make our intentions clear to the reader. Thus $W_j(\tilde{z}, -m)$ is our short-hand notation for $W_j(\tilde{z}, k, \lambda, -m)$. Define

$$G_1^{a,\lambda}(\tilde{y}, \tilde{z}) = \mathscr{G}_1^{a,l}(\tilde{y}, \tilde{z}) - \sum_{j:\lambda_j > 0} \beta_{-\frac{1}{2},1}^j \, W_j^*(\tilde{z}, -m) \,,$$

$$G_2^{a,\lambda}(\tilde{y}, \tilde{z}) = \mathscr{G}_2^{a,l}(\tilde{y}, \tilde{z}) - \sum_{j:\lambda_j < 0} \alpha_{-\frac{1}{2},2}^j \, W_j(\tilde{z}, -m) \,. \tag{4.37}$$

As the reader can easily verify the subtractions in (4.37), remove all terms in the local expansions on the right-hand side with $n < \frac{1}{2}$ and we find

$$G_i^{a,\lambda}(\tilde{y}, \tilde{z}) = \sum_{n \geqq \frac{1}{2}} \{ e_{n,i}^j(\tilde{y}) \, w_{n-\lambda_j}(\tilde{z}, a_j; -m) + f_{n,i}^j(\tilde{y}) \, w_{n+\lambda_j}^*(\tilde{z}, a_j; -m) \} \,, \tag{4.38}$$

where we have once again omitted the k dependence of $w_{l,k}$ and $w_{l,k}^*$ for brevity.

We turn next to a characterization of $G_i^{a,\lambda}(\tilde{y}, \tilde{z})$ along the lines of the characterization **G1**, **G2**, and **G3** of $g^{a,l}(\tilde{y}, \tilde{z})$.

Proposition 4.0. *The components $G_i^{a,\lambda}(\tilde{y}, \tilde{z})$ of the Green function constructed above for $(m - D_k)$ acting on $\bar{C}_{a,l}^\infty$ are uniquely characterized by the following three properties:*

(1) *For $i = 1, 2$ the vector valued functions $\tilde{z} \to G_i^{a,\lambda}(\tilde{y}, \tilde{z})$ are solutions to the equation,*

$$(m + D_k) G_i^{a,\lambda}(\tilde{y}, \tilde{z}) = 0 \,,$$

when z is not equal to a_j for $j = 1, \ldots, n$ and not equal to y. Under the deck transformation R_j, $G_i^{a,\lambda}(\tilde{y}, \tilde{z})$ transforms as follows,

$$G_i^{a,\lambda}(\tilde{y}, R_j \tilde{z}) = e^{-2\pi i \lambda_j} G_i^{a,\lambda}(\tilde{y}, \tilde{z}) \,,$$

(2) *About each of the points a_j for $j = 1, 2, \ldots, n$ the functions $G_i^{a,\lambda}(\tilde{y}, \tilde{z})$ have restricted local expansions given by* (4.38) *above. In the complement of a compact*

neighborhood of the set $\{y, a_1, a_2, \ldots, a_n\}$ *the function* $\tilde{z} \to G_i^{a,\lambda}(\tilde{y}, \tilde{z})$ *has a norm which is square integrable.*
(3) *For some elementary neighborhood* U, *of* y *the difference,*

$$G_i^{a,\lambda}(\tilde{y}, \tilde{z}) - \tilde{G}_i^{\varepsilon}(\tilde{y}, \tilde{z}) ,$$

is smooth as a function of $\tilde{z} \in \mathrm{pr}^{-1}(U)$ *and* $\tilde{G}_i^{\varepsilon}(\tilde{y}, \tilde{z})$ *is defined by,*

$$\tilde{G}_1^{\varepsilon}(\tilde{y}, \tilde{z}) = \begin{bmatrix} m\tilde{g}_{k+\frac{1}{2},s}^{\varepsilon}(\tilde{y}, \tilde{z}) \\ K_{k-\frac{1}{2}}^{*}\tilde{g}_{k+\frac{1}{2},s}^{\varepsilon}(\tilde{y}, \tilde{z}) \end{bmatrix},$$

$$\tilde{G}_2^{\varepsilon}(\tilde{y}, \tilde{z}) = \begin{bmatrix} -K_{k-\frac{1}{2}}\tilde{g}_{k-\frac{1}{2},s}^{\varepsilon}(\tilde{y}, \tilde{z}) \\ m\tilde{g}_{k-\frac{1}{2},s}^{\varepsilon}(\tilde{y}, \tilde{z}) \end{bmatrix}.$$

Proof. These three properties of $G_i^{a,\lambda}(\tilde{y}, \tilde{z})$ are simple translations of the properties **G1**, **G2**, and **G3** for the Green function $g^{a,l}(\tilde{y}, \tilde{z})$. The fact that these three conditions characterize $G_i^{a,\lambda}(\tilde{y}, \tilde{z})$ can be understood as follows. Suppose that $G_i^{a,\lambda}(\tilde{y}, \tilde{z})$ and $H_i^{a,\lambda}(\tilde{y}, \tilde{z})$ both satisfy all the conditions in Proposition 4.0 and consider the difference

$$\Delta(\tilde{y}, \tilde{z}) = G_i^{a,\lambda}(\tilde{y}, \tilde{z}) - H_i^{a,\lambda}(\tilde{y}, \tilde{z}) .$$

Then (3) of Proposition 4.0 implies that $\Delta(\tilde{y}, \tilde{z})$ is free of singularities near $z = y$ and (1) and (2) imply that $\tilde{z} \to \Delta(\tilde{y}, \tilde{z})$ is a solution to $(m + D_k)\Delta(\tilde{y}, \tilde{z}) = 0$ with monodromy multipliers $e^{-2\pi i\lambda_j}$ about a_j and which is globally in $L^2(d\mu)$. However, the restricted local expansion (4.38) implies that for each $j = 1, 2, \ldots, n$ either $c_j(\Delta) = 0$ or $c_j^*(\Delta) = 0$ and this together with Theorem 3.0 implies that $\Delta = 0$. QED.

The derivative of the Green function $G_i^{a,\lambda}$. Next we will identify the low order expansion coefficients $e_{\frac{1}{2},i}^{j}(\tilde{y})$ and $f_{\frac{1}{2},i}^{j}(\tilde{y})$ in (4.38) as the components of wave functions for the Dirac equation. Suppose that $f(\tilde{z})$ is a solution to the Dirac equation $(m - D_k) f = 0$ which transforms under the deck transformations R_j as follows,

$$f(R_j\tilde{z}) = e^{-2\pi i\lambda_j} f(\tilde{z}) ,$$

and which has restricted local expansions,

$$f(\tilde{z}) = \sum_{n \geq -\frac{1}{2}} \{a_n^j w_{n-\lambda_j}(\tilde{z}, a_j; m) + b_n^j w_{n+\lambda_j}^*(\tilde{z}, a_j; m)\} . \tag{4.39}$$

To lighten the notation for the next calculation we write

$$g(\tilde{z}) = G_i^{a,\lambda}(\tilde{y}, \tilde{z}) .$$

Now let $D_\varepsilon(y)$ denote the disk of radius ε about y, and let $C_\varepsilon(y)$ denote the circle of radius ε about y. Then for z in the complement of the union,

$$D_\varepsilon(y) \cup D_\varepsilon(a_1) \cup \cdots \cup D_\varepsilon(a_n) , \tag{4.40}$$

we have

$$D_k f(\tilde{z}) = m f(\tilde{z}) ,$$

$$D_k g(\tilde{z}) = -m g(\tilde{z}) . \tag{4.41}$$

If we now make use of (4.41) in (3.5) we find that

$$d\left\{\frac{\bar{f}_1 g_2}{1-\frac{|z|^2}{R^2}}\,id\bar{z}-\frac{\bar{f}_2 g_1}{1-\frac{|z|^2}{R^2}}\,idz\right\}=0$$

in the complement of the set (4.40). If we now integrate this last equality in the complement of the set (4.40) and use Stokes' theorem to rewrite the result as a circuit integral on the boundary, then we find

$$\sum_{j=1}^{n}\int_{C_\varepsilon(a_j)}\left\{\frac{\bar{f}_1 g_2}{1-\frac{|z|^2}{R^2}}\,id\bar{z}-\frac{\bar{f}_2 g_1}{1-\frac{|z|^2}{R^2}}\,idz\right\}=-\int_{C_\varepsilon(y)}\left\{\frac{\bar{f}_1 g_2}{1-\frac{|z|^2}{R^2}}\,id\bar{z}-\frac{\bar{f}_2 g_1}{1-\frac{|z|^2}{R^2}}\,idz\right\}. \tag{4.42}$$

Of course, this makes sense since $\bar{f}_i g_j$ is a well defined function on $\mathbf{D}_R(a)\backslash\{y\}$. We now wish to evaluate both sides of (4.42) in the limit $\varepsilon\to 0$. As in Theorem 3.0 it is useful at this point to introduce the local coordinate $u_j=T[-a_j]z$ in the $C_\varepsilon(a_j)$ integral and the local coordinate $u=T[-y]z$ in the $C_\varepsilon(y)$ integral. Then in geodesic polar coordinates (r_j,θ_j) and (r,θ) we have

$$u_j=\mathrm{Re}^{i\theta_j}\mathrm{th}\frac{r_j}{2}\sim\frac{Rr_j}{2}e^{i\theta_j},$$

$$u=\mathrm{Re}^{i\theta}\mathrm{th}\frac{r}{2}\sim\frac{Rr}{2}e^{i\theta}$$

and the asymptotics are given for small r_j and small r. Making use of this asymptotic parametrization in (4.42) and replacing some factors arising from the change of variables by their limits as $\varepsilon\to 0$, one finds

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sum_{j=1}^{n}\varepsilon R\int_0^{2\pi}\{\bar{f}_1 g_2(u_j)e^{-i\theta_j}+\bar{f}_2 g_1(u_j)e^{i\theta_j}\}\,d\theta_j\\&=-\lim_{\varepsilon\to 0}\sum_{j=1}^{n}\varepsilon R\int_0^{2\pi}\{\bar{f}_1 g_2(u)e^{-i\theta}+\bar{f}_2 g_1(u)e^{i\theta}\}\,d\theta\,.\end{aligned} \tag{4.43}$$

We have taken the liberty of scaling the result by a factor of 2. Next we use (4.38), the asymptotics of the wave functions w_l and w_l^* given in (3.1) and (3.2), and the local asymptotics of the Green function for $\tilde{z}$ near $\tilde{y}$ to evaluate both sides of (4.43) in the limit $\varepsilon\to 0$. We find

$$\sum_{j=1}^{n}\frac{8\sin(\pi\lambda_j)}{m^2R}\{\overline{b^j_{-\frac{1}{2}}}\,e^j_{\frac{1}{2},i}(\tilde{y})+\overline{a^j_{-\frac{1}{2}}}\,f^j_{\frac{1}{2},i}(\tilde{y})\}=-\overline{f_i(\tilde{y})}\,, \tag{4.44}$$

where we used

$$\frac{\pi}{\Gamma(\lambda)\Gamma(1-\lambda)}=\sin(\pi\lambda)\,.$$

We now use (4.44) to evaluate the low order expansion coefficients $e^j_{\frac{1}{2},i}(\tilde{y})$ and $f^j_{\frac{1}{2},i}(\tilde{y})$ in terms of the wave functions W_j and W_j^*. First choose

$$f(\tilde{z})=W_\nu(\tilde{z},m)\,,$$

then

$$a^j_{-\frac{1}{2}} = \delta_{j\nu} ,$$

$$b^j_{-\frac{1}{2}} = 0 ,$$

and (4.44) specializes to

$$f^\nu_{\frac{1}{2},i}(\tilde{y}) = -\frac{m^2 R}{8 s_\nu} \overline{W_\nu(\tilde{y}, m)_i} , \tag{4.45}$$

where $W_\nu(\cdots)_i$ is the i^{th} component of $W_\nu(\cdots)$. Next choose

$$f(\tilde{z}) = W^*_\nu(\tilde{z}, m) ,$$

then

$$a^j_{-\frac{1}{2}} = 0 ,$$

$$b^j_{-\frac{1}{2}} = \delta_{j\nu} ,$$

and (4.44) specializes to

$$e^\nu_{\frac{1}{2},i}(\tilde{y}) = -\frac{m^2 R}{8 s_\nu} \overline{W^*_\nu(\tilde{y}, m)_i} . \tag{4.46}$$

Now we are ready for the principal result of this section. We write $G^{a,\lambda}_{ij}(\tilde{y}, \tilde{z})$ for the j^{th} component of the vector $G^{a,\lambda}_i(\tilde{y}, \tilde{z})$. We are interested in calculating the derivative of $G^{a,\lambda}_{ij}(\tilde{y}, \tilde{z})$ with respect to the a variables. As in Sect. 2 it is convenient to introduce the scaled variables

$$b_\nu = \frac{a_\nu}{R} .$$

Taking the derivative of $G^{a,\lambda}_i(\tilde{y}, \tilde{z})$ in the parameter b_ν kills the singularity on the diagonal $y = z$. Thus the function

$$\tilde{z} \to \partial_{b_\nu} G^{a,\lambda}_i(\tilde{y}, \tilde{z})$$

will be a solution to $(m + D_k)\psi = 0$ with monodromy multiplier $e^{-2i\pi\lambda_j}$ associated with the deck transformation R_j. This function will be determined uniquely by its local expansions at a_j for $j = 1, 2, \ldots, n$. Using (2.32) and (2.33) to differentiate the local expansions (4.38) we find that

$$\partial_{b_\nu} G^{a,\lambda}_i(\tilde{y}, \tilde{z}) = -(1 - |b_\nu|^2)^{-1} e^\nu_{\frac{1}{2},i}(\tilde{y}) W_\nu(\tilde{z}, -m) . \tag{4.48}$$

Using (2.32) and (2.33) to differentiate the local expansions (4.38) we find that

$$\bar{\partial}_{b_\nu} G^{a,\lambda}_i(\tilde{y}, \tilde{z}) = -(1 - |b_\nu|^2)^{-1} f^\nu_{\frac{1}{2},i}(\tilde{y}) W^*_\nu(\tilde{z}, -m) . \tag{4.49}$$

Finally substituting (4.45) and (4.46) in (4.48) and (4.49) we find

$$\partial_{b_\nu} G^{a,\lambda}_{ij}(\tilde{y}, \tilde{z}) = \frac{m^2 R}{8 s_\nu (1 - |b_\nu|^2)} \overline{W^*_\nu(\tilde{y}, m)_i} W_\nu(\tilde{z}, -m)_j ,$$

$$\partial_{b_\nu} G^{a,\lambda}_{ij}(\tilde{y}, \tilde{z}) = \frac{m^2 R}{8 s_\nu (1 - |b_\nu|^2)} \overline{W_\nu(\tilde{y}, m)_i} W^*_\nu(\tilde{z}, -m)_j . \tag{4.50}$$

5. Deformation Equations

A holonomic system. In this section we will find deformation equations for the low order expansion coefficients of the response functions,

$$W_\nu = \delta_{j\nu} w_{-\frac{1}{2}+\lambda_j} + \sum_{n>0} \{a^j_{n\nu} w_{n+\lambda_j} + b^j_{n\nu} w^*_{n-\lambda_j}\} \tag{5.1}$$

and

$$W^*_\nu = \delta_{j\nu} w^*_{-\frac{1}{2}-\lambda_j} + \sum_{n>0} \{c^j_{n\nu} w_{n+\lambda_j} + d^j_{n\nu} w^*_{n-\lambda_j}\} \,. \tag{5.2}$$

It might be helpful for the reader to observe a parallel between the developments here and one way of understanding the *Schlesinger equations* from the theory of monodromy preserving deformations of linear equations in the complex plane. In the Schlesinger theory the fundamental object is a multivalued analytic function $z \to Y(z, a)$ which takes values in the general linear group. The matrix valued function $Y(z, a)$ has fixed (i.e., independent of the a_j) monodromy matrices M_j in simple circuits of the branch points a_j and satisfies an ordinary differential equation,

$$\frac{dY}{dz} = \sum_{j=1}^{n} \frac{A_j}{z - a_j} Y \,,$$

in the z variables. This equation is the analogue of (5.3) below. The fixed monodromy condition allows one to extend this differential equation to a differential equation,

$$d_a Y = \sum_{j=1}^{n} \frac{A_j da_j}{(z - a_j)} Y \,,$$

in the a variables. This is the analogue of (5.20) below. The compatibility conditions for the z and a equations gives the Schlesinger equations,

$$d_a A_\nu = -\sum_{\mu \neq \nu} \frac{[A_\nu, A_\mu]}{a_\nu - a_\mu} d(a_\nu - a_\mu) \,,$$

which is the analogue of (5.26) below. We derive the deformation equations (5.26) in a manner precisely analogous to this. One difference is that the internal compatibility conditions for the z and a equations do not give anything new in the Schlesinger case. In our case there is some interesting information that needs to be extracted from the internal consistency of (5.3).

Our first tool will be the following linear relations, that arise when the infinitesimal symmetries M_j are applied to W_ν and W^*_ν:

$$\begin{aligned}
M_3 W_\nu - b_\nu M_1 W_\nu &= -\frac{1}{2} W_\nu + \sum_\mu \{e_{\nu\mu} W_\mu + f_{\nu\mu} \bar{b}_\mu W^*_\mu\} \,, \\
M_3 W^*_\nu + \bar{b}_\nu M_2 W^*_\nu &= \frac{1}{2} W^*_\nu + \sum_\mu \{g_{\nu\mu} b_\mu W_\mu + h_{\nu\mu} W^*_\mu\} \,, \\
b^2_\nu M_1 W_\nu + M_2 W_\nu &= \sum_\mu \{\alpha_{\nu\mu} W_\mu + \beta_{\nu\mu} W^*_\mu\} \,, \\
M_1 W^*_\nu + \bar{b}^2_\nu M_2 W^*_\nu &= \sum_\mu \{\gamma_{\nu\mu} W_\mu + \delta_{\nu\mu} W^*_\mu\} \,.
\end{aligned} \tag{5.3}$$

In these relations the matrices $e, f, h, \alpha, \beta, \gamma$, and δ are matrices that depend on the branch points a_j for $j=1, \ldots, n$ but not on $\tilde{z}$. In the case of e, f, g and h we have also partly anticipated the form that the relations will take. This will simplify some later results and will avoid the necessity of introducing further definitions. The terms $-\frac{1}{2}W_\nu$ and $\frac{1}{2}W_\nu^*$ are stuck out in front of the first two equations instead of being combined with $e_{\nu\mu}$ and $h_{\nu\mu}$ to avoid introducing the Kronecker $\delta_{\nu\mu}$. The $\delta_{\nu\mu}$ that appears in the last equation is not the Kronecker delta – the possible confusion with this will shortly be addressed when α, β, γ, and δ are all eliminated in favor of e, f, g and h.

These relations (5.3) will be deduced using a combination of (2.20) and (2.32) – the result that describe the action of the infinitesimal symmetries M_j on local expansions and the results for centering the operators M_j at different points in the disk. As the reader can easily check using (2.20) and (2.32) the linear combinations on the left-hand side of (5.3) are all chosen so that the leading singularities in the local expansions cancel at level $n=-\frac{3}{2}$. Since W_μ and W_μ^* for $\mu=1, \ldots, n$ are a basis for the wave functions whose local expansions contain terms no lower than $n=-\frac{1}{2}$ and which are in L^2 near $|z|=R$ relations of the form (5.3) follow immediately. To find the coefficient matrices e, f, and etc. one need only compare the local expansions on both sides of (5.3) at level $n=-\frac{1}{2}$. One finds,

$$\begin{aligned}
e&=(k+\lambda)+[\mathbf{a}_1, B]\ ,\\
f&=(B\mathbf{b}_1\bar{B}-\mathbf{b}_1)\ ,\\
g&=(\mathbf{c}_1-\bar{B}\mathbf{c}_1 B)\ ,\\
h&=(k+\lambda)-[\mathbf{d}_1, \bar{B}]\ ,
\end{aligned}\tag{5.4}$$

where we used the notation

$$\begin{aligned}
(\mathbf{a}_n)_{\nu\mu}&=a^{\mu}_{n-\frac{1}{2},\nu}(1-|b_\mu|^2)^{-1}\ ,\\
(\mathbf{b}_n)_{\nu\mu}&=b^{\mu}_{n-\frac{1}{2},\nu}(1-|b_\mu|^2)^{-1}\ ,\\
(\mathbf{c}_n)_{\nu\mu}&=c^{\mu}_{n-\frac{1}{2},\nu}(1-|b_\mu|^2)^{-1}\ ,\\
(\mathbf{d}_n)_{\nu\mu}&=d^{\mu}_{n-\frac{1}{2},\nu}(1-|b_\mu|^2)^{-1}\ .
\end{aligned}\tag{5.5}$$

Also we have written B for the diagonal matrix with $\nu\nu^{\text{th}}$ entry given by b_ν and $(k+\lambda)$ for the diagonal matrix with entry $(k+\lambda_\nu)$ on the diagonal. The analogue of the display (5.4) for the coefficients α, β, etc. is

$$\begin{aligned}
\alpha&=-2B(k+\lambda-\tfrac{1}{2})-[\mathbf{a}_1, B^2]\ ,\\
\beta&=(\mathbf{b}_1-B^2\mathbf{b}_1\bar{B}^2)\ ,\\
\gamma&=(\mathbf{c}_1-\bar{B}^2\mathbf{c}_1 B^2)\ ,\\
\delta&=2\bar{B}(k+\lambda+\tfrac{1}{2})-[\mathbf{d}_1, \bar{B}^2]\ .
\end{aligned}\tag{5.6}$$

From (5.4) and (5.6) one may deduce some simple relations among the matrix coefficients,

$$\begin{aligned}
\alpha&=B-eB-Be\ ,\\
\beta&=-f-Bf\bar{B}\ ,\\
\gamma&=g+\bar{B}gB\ ,\\
\delta&=\bar{B}+h\bar{B}+\bar{B}h\ .
\end{aligned}\tag{5.7}$$

Next we will deduce some less obvious relations from the compatibility requirements of (5.3). Consider the Lie algebra relation,

$$(M_3M_2-M_2M_3-M_2)W=0\ , \tag{5.8}$$

where we have written W for the column vector.

$$W=\begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix}.$$

We now treat (5.3) as equations for M_2W and M_3W in terms of M_1W and W and W^* and as equations for M_1W^* and M_3W^* in terms of M_2W^* and W and W^*. If we now use (5.3) to eliminate the appearance of M_3W, M_2W, M_3W^*, and M_1W^* in favor of M_1W, M_2W^*, W and W^* one finds that the coefficient of M_1W in the resulting equation is,

$$[e,B^2]+[\alpha,B]\ ,$$

which vanishes as a consequence of (5.7). The coefficient of M_2W^* in (5.8) is,

$$(B^2f\bar{B}^2-f+B\beta\bar{B}-\beta)\bar{B}\ ,$$

which also vanishes as a consequence of (5.7). Since W and W^* are linearly independent we can equate the remaining coefficients of W and W^* to 0 in (5.8) and one finds,

$$\begin{gathered}\alpha+[e,\alpha]+2Be+B\beta\gamma+B^2f\bar{B}\gamma-\beta gB-B=0\ ,\\ e\beta+2Bf\bar{B}+B\beta\delta+B^2f\bar{B}\delta-\alpha f\bar{B}-\beta h=0\ .\end{gathered} \tag{5.9}$$

In a similar fashion if one starts with the Lie algebra relation,

$$(M_3M_1-M_1M_3+M_1)W^*=0\ , \tag{5.10}$$

and uses (5.3) to eliminate M_1W^*, M_3W^*, M_2W, and M_3W in favor of M_2W^*, M_1W, and W^* then one finds the coefficient of M_1W is,

$$(\bar{B}^2gB^2-g+\gamma-\bar{B}\gamma B)B\ ,$$

which vanishes as a consequence of (5.7) and the coefficient of M_2W^* is,

$$[h,\bar{B}^2]+[\bar{B},\delta]\ ,$$

which also vanishes as a consequence of (5.7). Equating the coefficients of W and W^* to 0 in what remains of (5.10) one finds,

$$\begin{gathered}h\gamma+\delta gB+\gamma e+\bar{B}\gamma\alpha-2\bar{B}gB-B^2gB\alpha=0\ ,\\ \delta-[h,\delta]+\gamma f\bar{B}+\bar{B}\gamma\beta-2\bar{B}h-\bar{B}^2gB\beta-\bar{B}=0\ .\end{gathered} \tag{5.11}$$

Both (5.9) and (5.11) simplify if one uses (5.7) to eliminate α, β, γ, and δ from these equations. Substituting (5.7) in (5.9) one finds

$$[B,e^2-fg]=0 \tag{5.12}$$

and

$$X-BX\bar{B}=0\ ,$$

where

$$X = ef - fh\ .$$

However, $X - BX\bar{B} = 0$, implies $X = 0$ so the second consequence of (5.9) can be written

$$ef - fh = 0\ . \tag{5.13}$$

Observe also that (5.12) implies that the matrix $e^2 - fg$ is diagonal. In a precisely analogous way one can substitute (5.7) in (5.11) to get

$$[h^2 - gf, \bar{B}] = 0 \tag{5.14}$$

and

$$X - \bar{B}XB = 0\ ,$$

where

$$X = ge - hg\ ,$$

and as before this implies

$$ge - hg = 0\ , \tag{5.15}$$

and (5.14) implies that the matrix $h^2 - gf$ is diagonal. The equations we have found in this fashion do not determine the diagonal parts of $e^2 - fg$ and $h^2 - gf$. However, we can determine these by looking at the coefficients of $w_{\frac{1}{2}+\lambda_j}$ in the local expansions of the first and third equations in (5.3). One finds for the first equation,

$$\frac{b_j - b_v}{1 - |b_j|^2} a^j_{\frac{3}{2}v} + \frac{b_v \bar{b}_j^2 - \bar{b}_j}{1 - |b_j|^2} m_2(-\tfrac{1}{2} + \lambda_j)\delta_{jv} + \frac{1 + |b_j|^2 - 2b_v \bar{b}_j}{1 - |b_j|^2} (k + \lambda_j + \tfrac{1}{2})\, a^j_{\frac{1}{2}v}$$
$$= \sum_{\mu=1}^{n} (e_{\mu v} - \tfrac{1}{2}\delta_{v\mu})\, a^j_{\frac{1}{2}\mu} + \sum_{\mu=1}^{n} f_{v\mu} \bar{b}_\mu c^j_{\frac{1}{2}\mu}\ .$$

If we set $v = j$ this simplifies to,

$$-\bar{b}_j m_2(-\tfrac{1}{2} + \lambda) + (k + \lambda_j + 1) a^j_{\frac{1}{2}j} = \sum_{\mu=1}^{n} e_{j\mu} a^j_{\frac{1}{2}\mu} + \sum_{\mu=1}^{n} f_{j\mu} \bar{b}_\mu c^j_{\frac{1}{2}\mu}\ , \tag{5.16}$$

where we have taken the liberty of collecting some of the terms involving $a^j_{\frac{1}{2}j}$ on the left-hand side. If we look at the coefficient of $w_{\frac{1}{2}+\lambda_j}$ in the third equation of (5.3) then set $v = j$ and collect some of the terms involving $a^j_{\frac{1}{2}j}$ on one side then we find,

$$(1 + |b_j|^2)\,(m_2(\lambda_j - \tfrac{1}{2}) - 2b_j(k + \lambda_j + 1)\, a^j_{\frac{1}{2}j}$$
$$= -\sum_{\mu=1}^{n} (b_\mu + b_j) e_{j\mu} a^j_{\frac{1}{2}\mu} - \sum_{\mu=1}^{n} (1 + b_j \bar{b}_\mu) f_{j\mu} c^j_{\frac{1}{2}\mu}\ . \tag{5.17}$$

If one now multiplies (5.16) by $2b_j$ and adds the results to (5.17) then one finds (after a little simplification),

$$(e^2 - fg)_{jj} = k^2 + \frac{m^2 R^2}{4}\ .$$

Since we already know that $e^2 - fg$ is diagonal it follows that

$$e^2 - fg = \left(k^2 + \frac{m^2 R^2}{4}\right) I\ . \tag{5.18}$$

In a precisely analogous fashion one can use the second and fourth equations in (5.3) to show that

$$h^2 - gf = \left(k^2 + \frac{m^2 R^2}{4} \right) I . \tag{5.19}$$

The holonomic Extension. Next we consider extending the differential equations (5.3) to the a variables. To do this it is convenient to use (2.32), and (2.33) to compute the local expansions for the exterior derivatives $d_b W_\nu$ and $d_b W_\nu^*$. One finds for the local expansion of $d_b W_\nu$ about the point a_j,

$$\begin{aligned}(1-|b_j|^2) d_b W_\nu = & \, w_{-\frac{3}{2}+\lambda_j}[-\delta_{j\nu} db_j] \\ & + w_{-\frac{1}{2}+\lambda_j}[\delta_{j\nu} k_-(\lambda_j, b_j) - a^j_{\frac{1}{2}\nu}\, db_j] \\ & + w^*_{-\frac{1}{2}-\lambda_j}[-b^j_{\frac{1}{2}\nu}\, d\bar{b}_j] \\ & + w_{\frac{1}{2}+\lambda_j}[da^j_{\frac{1}{2}\nu} + a^j_{\frac{1}{2}\nu} k_+(\lambda_j, b_j) - a^j_{\frac{3}{2}\nu}\, db_j - \delta_{j\nu} m_2(\lambda_j - \tfrac{1}{2}) d\bar{b}_j] \\ & + w^*_{\frac{1}{2}-\lambda_j}[db^j_{\frac{1}{2}\nu} + b^j_{\frac{1}{2}\nu} k_-(\lambda_j, b_j) - b^j_{\frac{3}{2}\nu}\, db_j] + \cdots ,\end{aligned}$$

where we have introduced the abbreviation

$$k_\pm(\lambda_j, b_j) = (k + \lambda_j \pm \tfrac{1}{2})(b_j d\bar{b}_j - \bar{b}_j db_j) .$$

For the local expansion of $d_b W_\nu^*$ about the point a_j one finds,

$$\begin{aligned}(1-|b_j|^2) d_b W_\nu^* = & \, w_{-\frac{3}{2}-\lambda_j}[-\delta_{j\nu} db_j] \\ & + w^*_{-\frac{1}{2}-\lambda_j}[\delta_{j\nu} k_+(\lambda_j, b_j) - d^j_{\frac{1}{2}\nu}\, d\bar{b}_j] \\ & + w_{-\frac{1}{2}-\lambda_j}[-c^j_{\frac{1}{2}\nu}\, db_j] \\ & + w^*_{\frac{1}{2}-\lambda_j}[d(d^j_{\frac{1}{2}\nu}) + d^j_{\frac{1}{2}\nu} k_-(\lambda_j, b_j) - d^j_{\frac{3}{2}\nu}\, d\bar{b}_j - \delta_{j\nu} m_1(-\lambda_j - \tfrac{1}{2})] \\ & + w_{\frac{1}{2}+\lambda_j}[dc^j_{\frac{1}{2}\nu} + c^j_{\frac{1}{2}\nu} k_-(\lambda_j, b_j) - c^j_{\frac{3}{2}\nu}\, db_j] + \cdots .\end{aligned}$$

Using these results one can easily check that the terms at level $n = -\frac{1}{2}$ cancel in the local expansions of

$$d_b W_\nu + db_\nu M_1 W_\nu$$

and

$$d_b W_\nu^* + d\bar{b}_\nu M_2 W_2^* .$$

Thus one has relations of the form,

$$d_b W_\nu + db_\nu M_1 W_\nu = \sum_\mu (E_{\nu\mu} W_\mu + F_{\nu\mu} W_\mu^*) ,$$

$$d_b W_\nu^* + d\bar{b}_\nu M_2 W_\nu = \sum_\mu (G_{\nu\mu} W_\mu + H_{\nu\mu} W_\mu^*) . \tag{5.20}$$

By comparing terms in the local expansions of both sides of (5.20) at level $n = \frac{1}{2}$ one finds that,

$$E = \frac{(k + \lambda - \frac{1}{2})(B d\bar{B} + \bar{B} dB)}{1 - |B|^2} + [dB, \mathbf{a}_1] ,$$

$$F = -dB \mathbf{b}_1 \bar{B}^2 - \mathbf{b}_1 d\bar{B} ,$$

$$G = -d\bar{B}\mathbf{c}_1 B^2 - \mathbf{c}_1 dB ,$$

$$H = -\frac{(k+\lambda+\frac{1}{2})(Bd\bar{B}+\bar{B}dB)}{1-|B|^2} + [d\bar{B}, \mathbf{d}_1] . \tag{5.21}$$

The deformation equations. The combination of the Dirac equation, the first two equations of (5.3) and (5.20) determine an extended holonomic system for the response functions W and W^*. We will deduce deformation equations for the coefficients e, f, g, and h by examining the compatibility conditions between (5.20) and the first two equations in (5.3). The two compatibility conditions we will examine are

$$(M_3 d_b - d_b M_3) W = 0 ,$$

$$(M_3 d_b - d_b M_3) W^* = 0 .$$

We use the equations in (5.3) and (5.20) to express all the terms in the preceding equations in terms of $M_j W$, $M_j W^*$, W and W^* for $j=1,2$ and $\mu=1,2,\ldots,n$. In a single term in each of the equations this also requires the use the commutation relations for the Lie algebra of M_1, M_2, and M_3 and one should also make the observation that the most singular terms containing $M_1^2 W$ and $M_2^2 W^*$ manifestly cancel. The condition that the lowest coefficients in the local expansions of the resulting expressions must vanish leads to the following identities:

$$[E, B] - [dB, e] = 0 ,$$

$$dB f\bar{B}^2 + f d\bar{B} + BF\bar{B} - F = 0 ,$$

$$[\bar{B}, H] - [d\bar{B}, h] = 0 ,$$

$$d\bar{B} g B^2 + g dB - \bar{B} G B + G = 0 . \tag{5.22}$$

All these identities are simple consequences of (5.21) and (5.4) but it will be useful to record them here for future use. Once one has identified these relations all the remaining terms involving $M_j W$ and $M_j W^*$ can be written as linear combinations of W and W^* using (5.3). Since W and W^* are linearly independent we can equate the coefficients of the resulting equations to 0. We find the first form of the deformation equations,

$$de = -(dBf\bar{B} + BF)\gamma + [E, e] + FgB - f\bar{B}G ,$$

$$d(f\bar{B}) = -(dBf\bar{B} + BF)\delta + (Ef\bar{B} - eF) + (Fh - f\bar{B}H) + F ,$$

$$d(gB) = (\bar{B}G - d\bar{B}gB)\alpha + (Ge - gBE) + (HgB - hG) - G ,$$

$$dh = (\bar{B}G - d\bar{B}gB)\beta + [H, h] + (Gf\bar{B} - gBF) . \tag{5.23}$$

We will first eliminate the appearance of α, β, γ, and δ from these equations using (5.7) and then we use the identities (5.22) to rework the equations as explicit equations for df and dg. We will illustrate the calculations involved for e and f and leave the analogous reductions for g and h to the reader.

First substitute $\gamma = g + \bar{B}gB$ into (5.23) to get

$$de = -(dBf\bar{B} + BF)g - (dBf\bar{B}^2 + BF\bar{B})gB + [E, e] + FgB - f\bar{B}G .$$

Now use (5.22) to substitute $F - f d\bar{B}$ for $dBf\bar{B}^2 + BF\bar{B}$ on the right-hand side of the last equation. After some cancellation one finds

$$de = f(d\bar{B}B - \bar{B}G) - (dBf\bar{B} + BF)g + [E, e] .$$

It will prove useful to introduce,

$$\hat{F} = -(dBf\bar{B}+BF) = dB\mathbf{b}_1\bar{B}+B\mathbf{b}_1 d\bar{B} \tag{5.24}$$

and

$$\hat{G} = d\bar{B}gB-\bar{B}G = d\bar{B}\mathbf{c}_1 B+\bar{B}\mathbf{c}_1 dB\,. \tag{5.25}$$

The equation for de becomes

$$de = f\hat{G}+\hat{F}f+[E, e]\,.$$

Now we work on the equation for df. Again we start by replacing δ on the right-hand side by $\bar{B}+h\bar{B}+\bar{B}h$. One finds that,

$$(df)\bar{B}+f\,d\bar{B} = \hat{F}(\bar{B}+h\bar{B}+\bar{B}h)+Ef\bar{B}-eF+Fh-f\bar{B}H+F\,.$$

Next we use (5.22) to replace $\hat{F}\bar{B}$ by $f\,d\bar{B}-F$ in this last equation and make some cancellations to find,

$$(df)\bar{B} = \hat{F}h\bar{B}+f\,d\bar{B}h+Ef\bar{B}-eF-f\bar{B}H+fhd\bar{B}\,.$$

Using (5.22) we can replace the term $f(d\bar{B}h-\bar{B}H)$ that appears on the right of this equation by

$$f(hd\bar{B}-H\bar{B}) = ef\,d\bar{B}-fH\bar{B}\,,$$

where we used $fh = ef$. One finds,

$$(df)\bar{B} = \hat{F}h\bar{B}+Ef\bar{B}-fH\bar{B}+e(f\,d\bar{B}-F)\,.$$

But using (5.22) we can replace $f\,d\bar{B}-F$ in this last result by $\hat{F}\bar{B}$ and then cancel the common factor of $\bar{B}$ on the right to obtain

$$df = \hat{F}h+e\hat{F}+Ef-fH\,.$$

Doing precisely analogous calculations for g and h one finds,

$$\begin{aligned}
de &= f\hat{G}+\hat{F}g+[E, e]\,,\\
df &= e\hat{F}+\hat{F}h+Ef-fH\,,\\
dg &= h\hat{G}+\hat{G}e+Hg-gE\,,\\
dh &= g\hat{F}+\hat{G}f+[H, h]\,.
\end{aligned} \tag{5.26}$$

Further identities. There are a number of identities among the low order expansion coefficients of the response functions W and W^* that are associated with the transformation $m \leftarrow -m$ and $(k, \lambda) \leftarrow (-k, -\lambda)$ which we will now record. First observe that,

$$\begin{aligned}
\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} w_{l,k}(\cdot, m) &= w_{l,k}(\cdot, -m)\,,\\
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} w^*_{l,k}(\cdot, m) &= w^*_{l,k}(\cdot, -m)\,.
\end{aligned} \tag{5.27}$$

Because we are interested in the dependence on the parameter m we have made it explicit in the preceding equation when we did not always do so before. From these relations it follows by comparing local expansions at level $n=\frac{1}{2}$ that,

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} W_\nu(k, \lambda, m) = W_\nu(k, \lambda, -m) ,$$

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} W_\nu^*(k, \lambda, m) = W_\nu^*(k, \lambda, -m) . \tag{5.28}$$

Now comparing local expansions in (5.28) at level $n=\frac{1}{2}$ one finds,

$$\begin{aligned} a^\mu_{\frac{1}{2}\nu}(k, \lambda, m) &= a^\mu_{\frac{1}{2}\nu}(k, \lambda, -m) , \\ b^\mu_{\frac{1}{2}\nu}(k, \lambda, m) &= -b^\mu_{\frac{1}{2}\nu}(k, \lambda, -m) , \\ c^\mu_{\frac{1}{2}\nu}(k, \lambda, m) &= -c^\mu_{\frac{1}{2}\nu}(k, \lambda, -m) , \\ d^\mu_{\frac{1}{2}\nu}(k, \lambda, m) &= d^\mu_{\frac{1}{2}\nu}(k, \lambda, -m) . \end{aligned} \tag{5.29}$$

Since the dependence of the low order coefficients on the parameter m is so simple we will suppress the explicit notational dependence on m with the convention that

$$x^\mu_{j\nu}(k, \lambda) = x^\mu_{j\nu}(k, \lambda, m)$$

for $x=a, b, c$ and d. If $b^\mu_{j\nu}(k, \lambda, -m)$ (or $c^\mu_{j\nu}(k, \lambda, -m)$) occurs we will replace it with $-b^\mu_{j\nu}(k, \lambda)$ (or $-c^\mu_{j\nu}(k, \lambda)$) without further comment.

For $k=0$ the Dirac operator is a real operator. One consequence of this is that there is a natural conjugation that acts on the $k=0$ solutions to the Dirac equation. This conjugation is,

$$w \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bar{w} . \tag{5.30}$$

It is not difficult to check that this conjugation acts on the local wave functions (2.22) and (2.23) as follows,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \overline{w_{l,k}(\cdot, a)} = w^*_{l,-k}(\cdot, a) ,$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \overline{w^*_{l,k}(\cdot, a)} = w_{l,-k}(\cdot, a) . \tag{5.31}$$

Using (5.31) to compare local expansions level $n=-\frac{1}{2}$ one finds,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \overline{W_\nu(k, \lambda)} = W_\nu^*(-k, -\lambda) ,$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \overline{W_\nu^*(k, \lambda)} = W_\nu(-k, -\lambda) . \tag{5.32}$$

Comparing local expansions at level $n=\frac{1}{2}$ in (5.32) one finds,

$$\begin{aligned} d^\mu_{\frac{1}{2}\nu}(k, \lambda) &= \overline{a^\mu_{\frac{1}{2}\nu}(-k, -\lambda)} , \\ c^\mu_{\frac{1}{2}\nu}(k, \lambda) &= \overline{b^\mu_{\frac{1}{2}\nu}(-k, -\lambda)} . \end{aligned} \tag{5.33}$$

Finally we will note some identities that follow from the derivative formula for the Green function (4.50). To make use of this formula it is convenient to modify the

Green function with (5.32) in mind. We introduce

$$\hat{G}^{a,\lambda}=\frac{8}{m^2R}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}G^{a,\lambda},$$

and (4.50) translates into

$$\partial_{b_\nu}\hat{G}^{a,\lambda}=\frac{1}{s_\nu(1-|b_\nu|^2)}W_\nu(-k,-\lambda,m)\otimes W_\nu(k,\lambda,-m),$$

$$\bar{\partial}_{b_\nu}\hat{G}^{a,\lambda}=\frac{1}{s_\nu(1-|b_\nu|^2)}W_\nu^*(-k,-\lambda,m)\otimes W_\nu^*(k,\lambda,-m). \tag{5.34}$$

If we now use this equation and the local expansion formulas for d_bW_ν and $d_bW_\nu^*$ given above to compute the local expansion coefficient of

$$w_{-\frac{1}{2}-\lambda_i}(\cdot,a_i)\otimes w_{-\frac{1}{2}+\lambda_j}(\cdot,a_j)$$

in the equation,

$$(\partial_{b_\nu}\partial_{b_\mu}-\partial_{b_\mu}\partial_{b_\nu})\hat{G}^{a,\lambda}=0,$$

one finds after some simplification

$$s^{-1}a_{\frac{1}{2}}(-k,-\lambda)-a_{\frac{1}{2}}(k,\lambda)^{\tau}s^{-1}=\text{diagonal}, \tag{5.35}$$

where $a_{\frac{1}{2}}$ is the matrix with $\nu\mu$ matrix element $a^{\mu}_{\frac{1}{2}\nu}$ and X^{τ} denotes the transpose of the matrix X and s denotes the diagonal matrix with $\nu\nu$ element $s_\nu=\sin(\pi\lambda_\nu)$.

In a similar fashion if one calculates the local expansion coefficient of

$$w^*_{-\frac{1}{2}+\lambda_i}(\cdot,a_i)\otimes w_{-\frac{1}{2}+\lambda_j}(\cdot,a_j)$$

in the identity,

$$(\bar{\partial}_{b_\nu}\partial_{b_\mu}-\partial_{b_\mu}\bar{\partial}_{b_\nu})\hat{G}^{a,\lambda}=0,$$

then one finds

$$s^{-1}b_{\frac{1}{2}}(-k,-\lambda)-c_{\frac{1}{2}}(k,\lambda)^{\tau}s^{-1}=0. \tag{5.36}$$

Now the combination of (5.35) and (5.33) produces the following relation between the deformation variables h and e above (in which the parameters are understood to be (k,λ,m)),

$$h=s(1-|B|^2)e^*s^{-1}(1-|B|^2)^{-1}, \tag{5.37}$$

where e^* is the conjugate transpose of e.

The combination of (5.36) and (5.33) produces the relation,

$$(s^{-1}b_{\frac{1}{2}}(k,\lambda))^*=s^{-1}b_{\frac{1}{2}}(k,\lambda), \tag{5.38}$$

from which it follows that

$$(fs(1-|B|^2))^*=fs(1-|B|^2). \tag{5.39}$$

One also has

$$(s^{-1}c_{\frac{1}{2}}(k,\lambda))^*=s^{-1}c_{\frac{1}{2}}(k,\lambda), \tag{5.40}$$

from which it follows that

$$(s^{-1}(1-|B|^2)^{-1}g)^* = s^{-1}(1-|B|^2)^{-1}g \ . \tag{5.41}$$

We summarize these developments in the following theorem:

Theorem 5.0. *The low order expansion coefficient e, f, g, and h defined by* (5.4) *above satisfy the non-linear deformation equations,*

$$de = g\hat{G} + \hat{F}g + [E, e] \ ,$$
$$df = e\hat{F} + \hat{F}h + Ef - fH \ ,$$
$$dg = h\hat{G} + \hat{G}e + Hg - gE \ ,$$
$$dh = g\hat{F} + \hat{G}f + [H, h] \ .$$

The coefficients e, f, g, and h also satisfy the relations,

$$ef - fh = 0 \ ,$$
$$ge - hg = 0 \ ,$$
$$e^2 - fg = M^2 \ ,$$
$$h^2 - gf = M^2 \ ,$$

where $M^2 = k^2 + \dfrac{m^2R^2}{4}$. If we further introduce,

$$\mathbf{f} = fs(1-|B|^2) \ ,$$
$$\mathbf{g} = s^{-1}(1-|B|^2)^{-1}g \ ,$$

then we have the additional symmetry relations,

$$\mathbf{f}^* = \mathbf{f} \ ,$$
$$\mathbf{g}^* = \mathbf{g} \ ,$$
$$h = s(1-|B|^2)e^*s^{-1}(1-|B|^2)^{-1} \ .$$

We have one final observation to make concerning these deformation equations. If we combine the commutator equation $ef - fh = 0$ with the final expression for h we find,

$$e\mathbf{f} = \mathbf{f}e^* \ .$$

If for some reason f turns out to be invertible (this will happen in an interesting special case we consider later on) then we can use the non-linear relations above to eliminate g and h from these equations in favor of e and f. In particular one finds,

$$h = f^{-1}ef \ ,$$
$$g = f^{-1}(e^2 - M^2) \ .$$

Theorem 5.1. *If f is an invertible matrix and we define,*

$$\hat{\mathbf{F}} = \hat{F}s(1-|B|^2) \ ,$$
$$\hat{\mathbf{G}} = s^{-1}(1-|B|^2)^{-1}\hat{G} \ , \tag{5.42}$$

then the deformation equations for e and $\mathbf{f}$ *are*

$$de=\mathbf{f}\hat{\mathbf{G}}+\hat{\mathbf{F}}\mathbf{g}+[E,e]\,,$$
$$d\mathbf{f}=e\hat{\mathbf{F}}+\hat{\mathbf{F}}e^{*}+E\mathbf{f}+\mathbf{f}E^{*}\,. \tag{5.43}$$

The matrices e and $\mathbf{f}$ *satisfy the conditions,*

$$e\mathbf{f}=\mathbf{f}e^{*}\,,$$
$$\mathbf{f}^{*}=\mathbf{f}\,, \tag{5.44}$$

with $\mathbf{g}$ *determined by e and* $\mathbf{f}$,

$$\mathbf{g}=\mathbf{f}^{-1}(e^{2}-M^{2})=e\mathbf{f}^{-1}e-M^{2}\mathbf{f}^{-1}\,, \tag{5.45}$$

and $\hat{\mathbf{G}}$ *determined by* $\mathbf{g}$.

We leave it to the reader to confirm that (5.43)–(5.45) incorporate all the information about e, f, g, and h that we have found so far in the event f is invertible.

The two point deformation theory in a special case. In this subsection we will integrate the deformation equations in a special case of the two point problem. Our principal result is that it is possible to integrate the deformation equations in terms of a Painlevé transcendent of type VI.

It is useful to start by determining what the deformation equations have to say about the action of the rotational symmetry $b_j \to e^{i\theta}b_j$. This corresponds to the infinitesimal symmetry,

$$\mathrm{rot}=\frac{\partial}{\partial\theta_1}\oplus\cdots\oplus\frac{\partial}{\partial\theta_n}=i\left(b_1\frac{\partial}{\partial b_1}-\bar{b}_1\frac{\partial}{\partial\bar{b}_1}\right)\oplus\cdots\oplus i\left(b_n\frac{\partial}{\partial b_n}-\bar{b}_n\frac{\partial}{\partial\bar{b}_n}\right).$$

One sees immediately that

$$dB(\mathrm{rot})=iB\,,$$
$$\mathrm{d}\bar{B}(\mathrm{rot})=-i\bar{B}\,.$$

It is then a simple matter to calculate,

$$E(\mathrm{rot})=-ie+i(k+\lambda)\,,$$
$$F(\mathrm{rot})=if\bar{B}\,,$$
$$G(\mathrm{rot})=-ig\bar{B}\,,$$
$$H(\mathrm{rot})=-ih+i(k+\lambda)\,.$$

The deformation equations become

$$de(\mathrm{rot})=i[k+\lambda,e]=i[\lambda,e]\,,$$
$$df(\mathrm{rot})=i[\lambda,f]\,.$$

From these equations it follows that,

$$e(e^{i\theta}b_1,\ldots,e^{i\theta}b_n)=e^{i\lambda\theta}e(b_1,\ldots,b_n)e^{-i\lambda\theta}\,,$$
$$f(e^{i\theta}b_1,\ldots,e^{i\theta}b_n)=e^{i\lambda\theta}f(b_1,\ldots,b_n)e^{-i\lambda\theta}\,. \tag{5.46}$$

Next we want to consider what happens for the infinitesimal version of the SU(1, 1) one parameter group,

$$\begin{bmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{bmatrix}.$$

The vector field associated with the action of this one parameter group on the disk of radius R is

$$v = v_1 \oplus \cdots \oplus v_n$$

with

$$v_j = \frac{1}{R}(1-b_j^2)\frac{\partial}{\partial b_j} + \frac{1}{R}(1-\bar{b}_j^2)\frac{\partial}{\partial \bar{b}_j}.$$

It follows that

$$dB(v) = \frac{1}{R}(1-B^2),$$

$$d\bar{B}(v) = \frac{1}{R}(1-\bar{B}^2).$$

One computes,

$$E(v) = \frac{1}{R}\left((k+\lambda)(\bar{B}-B) - \frac{1}{2}(\bar{B}+B) + eB + Be\right),$$

$$F(v) = \frac{1}{R}(f + Bf\bar{B}),$$

$$G(v) = -\frac{1}{R}(g + \bar{B}gB),$$

$$H(v) = \frac{1}{R}\left((k+\lambda)(\bar{B}-B) - \frac{1}{2}(\bar{B}+B) - h\bar{B} - \bar{B}h\right),$$

and

$$\hat{F}(v) = -\frac{1}{R}(f\bar{B} + Bf),$$

$$\hat{G}(v) = \frac{1}{R}(gB + \bar{B}b).$$

A little computation using the relations among e, f, g, and h shows that

$$de(v) = \frac{1}{R}\left[\lambda - \frac{1}{2}(\bar{B}+B), e\right],$$

$$df(v) = \frac{1}{R}\left[\lambda - \frac{1}{2}(\bar{B}+B), f\right]. \tag{5.47}$$

Since the matrices

$$\lambda - \tfrac{1}{2}(\bar{B}+B)$$

are diagonal they commute for different values of the matrices B and it follows that the differential equations for e and f may be explicitly integrated along the orbits of the one parameter group action,

$$t \to \begin{bmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{bmatrix} \cdot p \,,$$

for $p \in \mathbf{D}_R$. We wish to parametrize the two points b_1 and b_2 in the following fashion,

$$b_1 = e^{i\theta_1} \operatorname{th} \frac{r_1}{2} \,,$$

$$b_2 = e^{i\theta_1} \frac{\operatorname{ch}\left(\frac{r_1}{2}\right)\delta + \operatorname{sh}\left(\frac{r_1}{2}\right)}{\operatorname{sh}\left(\frac{r_1}{2}\right)\delta + \operatorname{ch}\left(\frac{r_1}{2}\right)} \,,$$

where

$$\delta = e^{i\theta} \operatorname{th}\left(\frac{r}{2}\right) .$$

The parameters for b_1 and b_2 are θ, θ_1, r, and r_1. Note that (r, θ) are essentially the geodesic polar coordinates for b_2 relative b_1. The reason for this parametrization of b_1 and b_2 is as follows. First rotate both b_1 and b_2 by $-\theta_1$, then take the result of this and flow along the appropriate v—orbit by $-\frac{r_1}{2}$. The result is

$$b_1 \to 0, \quad b_2 \to \delta \,.$$

Finally rotate 0 and δ simultaneously by $-\theta$. One finds that

$$b_1 \to 0, \quad b_2 \to \operatorname{th}\left(\frac{r}{2}\right) .$$

It follows that the symmetry flows that we have determined above suffice to determine $e(b_1, b_2)$ and $f(b_1, b_2)$ for all values of (b_1, b_2) in terms of $e(0, t)$ and $f(0, t)$ for t real. Next we will use the deformation equations to write down ordinary differential equations for $e(0, t)$ and $f(0, t)$. Let

$$u = 0 \oplus \frac{\partial}{\partial t} \,.$$

Now we substitute $(0, t)$ in the deformation equations and evaluate them along the vector field u. One finds,

$$B = \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} ,$$

$$\bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} , \tag{5.48}$$

and

$$dB(u) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} ,$$

$$d\bar{B}(u) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} . \tag{5.49}$$

To make use of the symmetry conditions to reduce the number of variables it is desirable to work with e and $\mathbf{f}$ rather than e and f. Indeed at this point we will confine our attention to the case in which $\lambda_j > 0$ for $j = 1, 2$. In this case the response functions W_ν are in L^2 and the matrix $-s^{-1} b_{\frac{1}{2}}(k, \lambda)$ is the matrix of the inner products of these response functions and is consequently positive definite. Using the fact that $|b_j| < 1$ one easily sees that this implies $\mathbf{f}$ is positive definite (and hence invertible). Hence the deformation equations become

$$\begin{aligned} de &= \mathbf{f}\hat{\mathbf{G}} + \hat{\mathbf{F}}\mathbf{g} + [E, e] , \\ d\mathbf{f} &= e\hat{\mathbf{F}} + \hat{\mathbf{F}}e^* + E\mathbf{f} + \mathbf{f}E^* , \end{aligned} \tag{5.50}$$

with

$$\mathbf{g} = e^* \mathbf{f}^{-1} e - M^2 \mathbf{f}^{-1} . \tag{5.51}$$

and $\hat{\mathbf{G}}$ determined by $\mathbf{g}$. At this point it is useful to introduce

$$e = \begin{bmatrix} k+\lambda_1 & e_+ \\ \bar{e}_- & k+\lambda_2 \end{bmatrix} \tag{5.52}$$

and

$$\mathbf{f} = c \begin{bmatrix} \kappa \operatorname{ch}\psi & \varepsilon \operatorname{sh}\psi \\ \bar{\varepsilon} \operatorname{sh}\psi & \kappa^{-1}(1-t^2)\operatorname{ch}\psi \end{bmatrix} , \tag{5.53}$$

where $c > 0$, κ, and ψ are real and $\varepsilon\bar{\varepsilon} = 1$. This parametrization incorporates the known diagonal for e and the fact that $\mathbf{f}$ is a symmetric positive definite matrix which is related in a particular fashion to the matrix of the L^2 inner product of wave functions. The last algebraic relation $e\mathbf{f} = \mathbf{f}e^*$ implies,

$$\begin{aligned} (\bar{\varepsilon} e_- - \varepsilon \bar{e}_-) \operatorname{sh}\psi &= 0 , \\ (\bar{\varepsilon} e_+ - \varepsilon e_+) \operatorname{sh}\psi &= 0 , \\ (\kappa^{-1}(1-t^2) e_+ - \kappa e_-) \operatorname{ch}\psi - \lambda \varepsilon \operatorname{sh}\psi &= 0 , \end{aligned}$$

where $\lambda = \lambda_2 - \lambda_1$. Now define

$$\xi = \bar{\varepsilon} \kappa^{-1} (1-t^2) e_+ + \bar{\varepsilon} \kappa \varepsilon_- .$$

If we suppose that $\operatorname{sh}\psi \neq 0$ then we find that $\bar{\varepsilon} e_+$ and $\bar{\varepsilon} e_-$ are real and

$$\begin{aligned} e_+ &= \frac{\varepsilon\kappa}{2(1-t^2)} (\xi + \lambda \operatorname{th}\psi) , \\ e_- &= \frac{\varepsilon}{2\kappa} (\xi - \lambda \operatorname{th}\psi) , \end{aligned} \tag{5.54}$$

where $\bar{\xi} = \xi$. Now write

$$\begin{aligned} k_j &= k + \lambda_j , \\ \mu &= k_1 + k_2 = 2k + \lambda_1 + \lambda_2 , \\ \lambda &= k_2 - k_1 = \lambda_2 - \lambda_1 , \end{aligned}$$

for $j=1,2$ and

$$\mathbf{f}=\begin{bmatrix} \mathbf{f}_{11} & \mathbf{f}_{12} \\ \mathbf{f}_{21} & \mathbf{f}_{22} \end{bmatrix}$$

and

$$\mathbf{g}=\begin{bmatrix} \mathbf{g}_{11} & \mathbf{g}_{12} \\ \mathbf{g}_{21} & \mathbf{g}_{22} \end{bmatrix}.$$

The one easily calculates,

$$E(u)=\frac{2t(k_2-\frac{1}{2})}{1-t^2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}-t^{-1}\begin{bmatrix} 0 & e_+ \\ \bar{e}_- & 0 \end{bmatrix},$$

$$\mathbf{F}(u)=\begin{bmatrix} 0 & \mathbf{f}_{12} \\ 0 & \dfrac{1+t^2}{1-t^2}\mathbf{g}_{22} \end{bmatrix},$$

$$\mathbf{G}(u)=\begin{bmatrix} 0 & -\mathbf{g}_{12} \\ 0 & -\dfrac{1+t^2}{1-t^2}\mathbf{g}_{22} \end{bmatrix},$$

$$\hat{\mathbf{F}}(u)=-\frac{2t}{1-t^2}\begin{bmatrix} 0 & 0 \\ 0 & \mathbf{f}_{22} \end{bmatrix},$$

$$\hat{\mathbf{G}}(u)=\frac{2t}{1-t^2}\begin{bmatrix} 0 & 0 \\ 0 & \mathbf{g}_{22} \end{bmatrix}.$$

If we now substitute these into the equation for $d\mathbf{f}(u)$ we find

$$d\mathbf{f}(u)=N\,,$$

where

$$N_{11}=-t^{-1}(e_+\mathbf{f}_{21}+\bar{e}_+\mathbf{f}_{12})\,,$$

$$N_{12}=-\frac{t+t^{-1}}{1-t^2}e_+\mathbf{f}_{22}-t^{-1}e_-\mathbf{f}_{11}+\frac{2t(k_2-\frac{1}{2})}{1-t^2}\mathbf{f}_{12}\,,$$

$$N_{21}=-\frac{t+t^{-1}}{1-t^2}\bar{e}_+\mathbf{f}_{22}-t^{-1}\bar{e}_-\mathbf{f}_{11}+\frac{2t(k_2-\frac{1}{2})}{1-t^2}\mathbf{f}_{21}\,,$$

$$N_{22}=-t^{-1}(\bar{e}_-\mathbf{f}_{12}+e_-\mathbf{f}_{21})-\frac{2t}{1-t^2}\mathbf{f}_{22}\,.$$

Next we convert this using (5.33) and (5.54) to find,

$$N_{11}=-c\kappa\frac{t^{-1}}{1-t^2}(\xi+\lambda\operatorname{th}\psi)\operatorname{sh}\psi\,,$$

$$N_{12}=-c\varepsilon\left(\frac{t^{-1}}{1-t^2}\xi\operatorname{ch}\psi-\frac{t(\mu-1)}{1-t^2}\operatorname{sh}\psi\right),$$

$$N_{21} = -c\bar{\varepsilon}\left(\frac{t^{-1}}{1-t^2}\xi \operatorname{ch}\psi - \frac{t(\mu-1)}{1-t^2}\operatorname{sh}\psi\right),$$

$$N_{22} = -c\kappa^{-1}t^{-1}(\xi - \lambda \operatorname{th}\psi)\operatorname{sh}\psi - c\kappa^{-1}2t\operatorname{ch}\psi.$$

where $f' = \frac{df}{dt}$ and the equation $\mathbf{df}(u) = N$ for c', ε', κ' and ψ' becomes,

$$c^{-1}c'\operatorname{ch}\psi + \kappa^{-1}\kappa'\operatorname{ch}\psi + \operatorname{sh}\psi\,\psi' = c^{-1}\kappa^{-}N_{11}\,,$$

$$c^{-1}c'\operatorname{sh}\psi + \varepsilon^{-1}\varepsilon'\operatorname{sh}\psi + \operatorname{ch}\psi\,\psi' = c^{-1}\varepsilon^{-1}N_{12}\,,$$

$$c^{-1}c'\operatorname{sh}\psi - \varepsilon^{-1}\varepsilon'\operatorname{sh}\psi + \operatorname{ch}\psi\,\psi' = c^{-1}\varepsilon N_{21}\,,$$

$$-2t\operatorname{ch}\psi + (1-t^2)(c^{-1}c'\operatorname{ch}\psi - \kappa^{-1}\kappa'\operatorname{ch}\psi + \operatorname{sh}\psi\,\psi') = c^{-1}\kappa N_{22}\,.$$

The very last equation simplifies somewhat when combined with the expressions for N_{22}. One finds,

$$c^{-1}c'\operatorname{ch}\psi - \kappa^{-1}\kappa'\operatorname{ch}\psi + \operatorname{sh}\psi\,\psi' = -\frac{t^{-1}}{1-t^2}(\xi - \lambda\operatorname{th}\psi)\operatorname{sh}\psi\,.$$

Solving these equations for $c^{-1}c'$, $\kappa^{-1}\kappa'$, $\varepsilon^{-1}\varepsilon'$ and ψ' one finds

$$c^{-1}c' = -\frac{(\mu-1)t}{1-t^2}\operatorname{sh}^2\psi\,,$$

$$\varepsilon^{-1}\varepsilon' = 0\,,$$

$$\kappa^{-1}\kappa' = -\frac{\lambda t^{-1}}{1-t^2}\operatorname{th}^2\psi\,,$$

$$\psi' = -\frac{t^{-1}}{1-t^2}\xi + \frac{(\mu-1)t}{1-t^2}\operatorname{ch}\psi\operatorname{sh}\psi\,.$$

Next we work out the differential equation associated with e. Substituting the results for $\hat{F}$ and $\hat{G}$ into the differential equation for e above, one finds,

$$e' = \mathbf{f}\begin{bmatrix} 0 & 0 \\ 0 & \frac{2t}{1-t^2}\mathbf{g}_{22}\end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \frac{2t}{1-t^2}\mathbf{f}_{22}\end{bmatrix}\mathbf{g} + [E, e]\,,$$

$$= \begin{bmatrix} 0 & \frac{2t}{1-t^2}\mathbf{f}_{12}\mathbf{g}_{22} \\ -\frac{2t}{1-t^2}\mathbf{f}_{22}\mathbf{g}_{21} & 0\end{bmatrix} + [E, e]\,. \tag{5.55}$$

We also calculate

$$[E, e] = \left(\frac{\lambda + (\mu-1)t^2}{t(1-t^2)}\right)\begin{bmatrix} 0 & -e_+ \\ \bar{e}_- & 0\end{bmatrix}. \tag{5.56}$$

Combining (5.55) and (5.56) we find,

$$e'_+ = \frac{2}{1-t^2}\,\mathbf{f}_{12}\mathbf{g}_{22} - \frac{\lambda+(\mu-1)t^2}{t(1-t^2)}\,e_+ ,$$

$$\bar{e}'_- = -\frac{2t}{1-t^2}\,\mathbf{f}_{22}\mathbf{g}_{21} + \frac{\lambda+(\mu-1)t^2}{t(1-t^2)}\,\bar{e}_- . \tag{5.57}$$

Use

$$\mathbf{g} = \mathbf{f}^{-1}(e^2 - M^2)$$

to obtain,

$$(1-t^2\operatorname{ch}^2\psi)c\mathbf{g}_{21} = -\bar{\varepsilon}(k_1^2 - M^2 + \bar{e}_-e_+)\operatorname{sh}\psi + \kappa\mu e_-\operatorname{ch}\psi ,$$

$$(1-t^2\operatorname{ch}^2\psi)c\mathbf{g}_{22} = -\bar{\varepsilon}\mu\varepsilon_+\operatorname{sh}\psi + \kappa(k_2^2 - M^2 + \bar{e}_-e_+)\operatorname{ch}\psi . \tag{5.58}$$

Now substitute this last expression for $\mathbf{g}_{21}$ in the second equation in (5.57) along with the parametrization for $\mathbf{f}_{22}$ and the expression for $\bar{e}_-$. Then multiply both sides of the resulting equation by κ, make use of the equation for $\kappa^{-1}\kappa'$ given above and the observation that

$$\bar{e}_-e_+ = \frac{\xi^2 - \lambda^2\operatorname{th}^2\psi}{4(1-t^2)} ,$$

to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\xi - \lambda\operatorname{th}\psi) &= \frac{\lambda(1-\operatorname{th}^2\psi)+(\mu-1)t^2}{2t(1-t^2)}(\xi - \lambda\operatorname{th}\psi)\\ &\quad + \frac{t\operatorname{ch}\psi\operatorname{sh}\psi}{1-t^2\operatorname{ch}^2\psi}\left(k_1^2 + k_2^2 - 2M^2 + \frac{\xi^2 - \lambda^2\operatorname{th}^2\psi}{2(1-t^2)}\right)\\ &\quad - \frac{t\mu\operatorname{ch}^2\psi}{1-t^2\operatorname{ch}^2\psi}\,\xi .\end{aligned} \tag{5.59}$$

As a check on this equation we remark that the variable ψ in the Poincaré disk has the same significance as the variable ψ in [8, 12, 15]. Introducing $t = \frac{r}{R}$ and letting $R \to \infty$ one finds that the limiting form of (5.59) is

$$r\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = \lambda^2(1-\operatorname{th}^2\psi)\operatorname{th}\psi + \frac{m^2r^2}{2}\operatorname{sh}2\psi$$

which is Eq. (3.3.46) in [12]. In the special case $\lambda = 0$ this equation was first found in [8].

Next consider the substitutions

$$s = t^2,\ w = \frac{1}{\operatorname{ch}^2\psi} .$$

One finds,

$$\frac{d\psi}{ds} = \frac{1}{2t}\frac{d\psi}{dt} = -\frac{1}{2w\sqrt{1-w}}\frac{dw}{ds}$$

$$\xi = \frac{1}{w\sqrt{1-w}}\left(s(1-s)\frac{dw}{ds} + s(\mu-1)(1-w)\right) .$$

Now convert (5.59) to a differential equation in s by multiplying both sides by t^{-1} and substituting s for t^2. In the resulting equation make the substitution $w=1/\mathrm{ch}^2\psi$ and multiply both sides by $w\sqrt{1-w}$. One finds,

$$\begin{aligned}
&\left(w\sqrt{1-w}\frac{d}{ds}\frac{1}{w\sqrt{1-w}}\right)\left(s(1-s)\frac{dw}{ds}+((\mu-1)s-\lambda w)(1-w)\right)\\
&+\frac{d}{ds}\left(s(1-s)\frac{dw}{ds}+((\mu-1)s-\lambda w)(1-w)\right)\\
&=\frac{\lambda w+(\mu-1)s}{2s(1-s)}\left(s(1-s)\frac{dw}{ds}+((\mu-1)s-\lambda w)(1-w)\right)\\
&+\frac{w(1-w)}{w-s}\left(k_1^2+k_2^2-2M^2+\frac{\xi^2-\lambda^2(1-w)}{2(1-s)}\right)\\
&-\frac{\mu}{w-s}\left(s(1-s)\frac{dw}{ds}+s(\mu-1)(1-w)\right).
\end{aligned}\tag{5.60}$$

It is simple to compute,

$$w\sqrt{1-w}\frac{d}{ds}\frac{1}{w\sqrt{1-w}}=\frac{2-3w}{2w(w-1)}\frac{dw}{ds}.$$

Substituting this last expression in (5.60) and simplifying the resulting equation one finds,

$$\begin{aligned}
&w''-\frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-s}\right)(w')^2+\left(\frac{1}{s}+\frac{1}{s-1}+\frac{1}{w-s}\right)w'\\
&=\frac{w(w-1)(w-s)}{s^2(1-s)^2}\left(\frac{(1-4M^2)s(s-1)}{2(w-s)^2}-\frac{(\mu-1)^2 s}{2w^2}+\frac{\lambda^2}{2}\right).
\end{aligned}\tag{5.61}$$

This is Painlevé VI (see, e.g., [4])

$$\begin{aligned}
&w''-\frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-s}\right)(w')^2+\left(\frac{1}{s}+\frac{1}{s-1}+\frac{1}{w-s}\right)w'\\
&=\frac{w(w-1)(w-s)}{s^2(1-s)^2}\left(\frac{\delta s(s-1)}{(w-s)^2}+\frac{\gamma(s-1)}{(w-1)^2}+\frac{\beta s}{w^2}+\alpha\right),
\end{aligned}$$

with

$$\alpha=\frac{\lambda^2}{2},$$

$$\beta=-\frac{(\mu-1)^2}{2},$$

$$\gamma=0,$$

$$\delta=\frac{1-4M^2}{2}.$$

We summarize these developments in the following theorem:

Theorem 5.2. *Suppose that* $\lambda_j > 0$ *for* $j = 1, 2$. *Let* $b_j = \frac{a_j}{R}$ *and consider* $e(b_1, b_2)$ *and* $f(b_1, b_2)$ *as functions of the scaled variables* b_j. *Define* $\lambda = \lambda_2 - \lambda_1$ *and* $\mu = 2k + \lambda_1 + \lambda_2$. *Write*

$$e(0, t) = \begin{bmatrix} k+\lambda_1 & e_+(t) \\ e_-(t) & k+\lambda_2 \end{bmatrix},$$

and

$$\mathbf{f}(0, t) = c(t) \begin{bmatrix} \kappa(t)\operatorname{ch}\psi(t) & \varepsilon(t)\operatorname{sh}\psi(t) \\ \bar{\varepsilon}(t)\operatorname{sh}\psi(t) & \kappa(t)^{-1}(1-t^2)\operatorname{ch}\psi(t) \end{bmatrix}.$$

Then the algebraic relation $\mathbf{ef} = \mathbf{f}e^*$ *implies,*

$$e_+ = \frac{\varepsilon\kappa}{2(1-t^2)}(\xi + \lambda \operatorname{th}\psi),$$

$$e_- = \frac{\varepsilon}{2\kappa}(\xi - \lambda \operatorname{th}\psi),$$

for a real valued function ξ *and* ε *of absolute value* 1. *The deformation equation for* $\mathbf{f}$ *implies,*

$$c^{-1}c' = -\frac{(\mu-1)t}{1-t^2}\operatorname{sh}^2\psi,$$

$$\varepsilon^{-1}\varepsilon' = 0,$$

$$\kappa^{-1}\kappa' = -\frac{\lambda t^{-1}}{1-t^2}\operatorname{th}^2\psi,$$

$$\psi' = -\frac{t^{-1}}{1-t^2}\xi + \frac{(\mu-1)t}{1-t^2}\operatorname{ch}\psi\operatorname{sh}\psi.$$

The last equation allows us to eliminate ξ *in favor of* ψ *and* ψ'. *Once this is done and the further substitutions,*

$$s = t^2,$$

$$w = \frac{1}{\operatorname{ch}^2\psi},$$

are made, the deformation equation for e becomes the type VI Painlevé equation (5.61) *above for* w.

6. The Tau Function

The Grassmannian formalism. In this section we will introduce the τ-function for the Dirac operator with branched singularities whose Green function was described in Sect. 4. Suppose that S is an open subset of $\mathbf{D}_R$ with a boundary ∂S that consists of the union of smooth simple curves in $\mathbf{D}_R$ (we assume that the closure of S is contained in the open set $\mathbf{D}_R$). Suppose that f is a smooth function in the Sobolev space $H^1(\mathbf{D}_R)$ which satisfies the Dirac equation

$$(m-D_k)f(x)=0 \text{ for } x\in S\,.$$

Then since the integral operator associated with the Green function inverts the Dirac operator we have

$$f_j(x)=\int_{\mathbf{D}_R\setminus S}(m-D_k)f\cdot\overline{G_j(x,y)}\,d\mu(y)\quad\text{for } x\in S\,.$$

The integral can be confined to $\mathbf{D}_R\setminus S$ since $(m-D_k)f(y)=0$ for $y\in S$. Since the Green function $G_j(x,y)$ is a solution to

$$(m+D_k)_y G_j(x,y)=0$$

in the second variable when $x\neq y$, it follows that the expression for $f_j(x)$ can be written

$$f_j(x)=\int_{\mathbf{D}_R\setminus S}\{(m-D_k)f(y)\cdot\overline{G_j(x,y)}-f(y)\cdot\overline{(m+D_k)G_j(x,y)}\}\,d\mu(y)$$

which becomes, using (3.5) and Stokes' theorem, the following boundary integral representation for $f_j(x)$,

$$f_j(x)=2\int_{\partial S}\left\{\frac{\overline{G_{j1}(x,y)}f_2(y)}{1-\frac{|y|^2}{R^2}}\,i\,d\bar{y}-\frac{\overline{G_{j2}(x,y)}f_1(y)}{1-\frac{|y|^2}{R^2}}\,i\,dy\right\}\text{ for } x\in S\,, \tag{6.1}$$

in which each component of the boundary, ∂S, is given the standard counterclockwise orientation. If $f\in H^{\frac{1}{2}}(\partial S)$ is prescribed arbitrarily then we can also regard (7.1) as a formula for the projection f onto the space of boundary values on ∂S of $H^1(S)$ solutions to the Dirac equation in the interior of S. The complementary subspace for this projection is the space of boundary values of ∂S of solutions to the Dirac equation in $H^1(\mathbf{D}_R\setminus S)$.

Our principal tool in the discussion of the tau function for the Dirac operator is a formula analogous to (6.1) with the Green function G replaced by $G^{a,\lambda}$. To explain the significance of this formula it will be useful to describe the transfer formalism [12] and its relation to "localization" away from singularities. It is natural when considering the Dirac operator with branch type singularities at the points a_j for $j=1,2,\ldots,n$ to localize the operator away from the branch cuts. One natural way to do this in the hyperbolic disk is to transform the disk into the upper half plane by a fractional linear transformation and then to draw horizontal strips around horizontal branch cuts emanating from each of the points a_j for $j=1,2,\ldots,n$. There are many ways to transform the disk into the upper half plane and it is clear that, provided the branch points a_j are all distinct, one can choose a suitable such transform so that the *second coordinates* of the transformed branch points are distinct. Supposing this to be done we can then choose the strips S_j containing a_j to be pairwise disjoint by making them sufficiently narrow. It is certainly possible to transform the Dirac operator into upper half plane coordinates and work exclusively in the upper half plane, so that the description of the localization is geometrically simple. However, because we wish to use the many formulas which we have written down in the disk but have not written down in the upper half plane this is not economical. Instead we adopt the expedient of using the geometrically natural terminology in describing the localization via horizontal strips in the upper half plane, but when doing calculations we transform the strips S_j back into the disk where they become crescent shaped objects with the sharp ends meeting at a common point on the boundary of the disk. The reader should not find it difficult to keep in mind that any reference to horizontal branch cuts or strips implicitly assumes that one has chosen an appropriate upper half plane coordinate system.

Until now it has been convenient to work on the simply connected covering $\tilde{\mathbf{D}}_R(a)$ of $\mathbf{D}_R(a)$. However, at this point it is simpler to draw horizontal branch cuts (rays), ℓ_j, emanating to the right from each of the branch points a_j and to work with functions on $\mathbf{D}_R(a)\backslash\{\ell_1, \ell_2, \ldots, \ell_n\}$ that have appropriate branching behavior. We will now describe the localization of the singular Dirac operator, $D^{a,\lambda}$, which we wish to consider. Let $S = \cup S_j$ denote the union of the strips S_j. In the exterior of the set S the operator, $D^{a,\lambda}$, will act precisely as the ordinary Dirac operator acts. However, in order for this operator to adequately mirror the sigular Dirac operator whose domain contains functions that are branched at the points a_j with specified monodromy we restrict its domain. We will now define the subspaces of $H^{\frac{1}{2}}(\partial S)$ that are important in our description of localization of the singular Dirac operator $D^{a,\lambda}$. We will say that a function F is locally in the null space of $D^{a,\lambda}$ at a_j provided that for some $\varepsilon > 0$ the function F has a local expansion,

$$F(x) = \sum_{n \geqq \frac{1}{2}} \{a_n(F) w_{n+\lambda_j}(x, a_j) + b_n(F) w^*_{n-\lambda_j}(x, a_j)\} ,$$

valid for $x \in B_\varepsilon(a_j)$, the ball of radius ε about a_j. In this description it is understood that a branch cut ℓ_j has been chosen for a_j and that brances for F, $w_{n+\lambda_j}$, and $w^*_{n-\lambda_j}$ are fixed. Suppose now that S_j is a horizontal strip containing a_j in its interior. We define a subspace of $H^{\frac{1}{2}}(\partial S_j)$ in the following fashion,

Definition. *A function f on ∂S_j will be in the subspace $W_{\text{int}}(a_j)$ provided that it is the boundary value of a function F defined in S_j with the following properties:*

(1) *F is a branched solution to the Dirac equation in S_j with branch cut ℓ_j and monodromy $e^{2\pi i \lambda_j}$.*
(2) *F is locally in the null space of $D^{a,\lambda}$ at a_j.*
(3) *$F \in H^1(S_j \backslash B_\varepsilon(a_j) \cup \ell_j\})$ for some $\varepsilon > 0$.*

If each branch point a_j is contained in a strip S_j and the strips S_j are pairwise disjoint then we also write,

$$W_{\text{int}}(a) = W_{\text{int}}(a_1) \oplus W_{\text{int}}(a_2) \oplus \cdots \oplus W_{\text{int}}(a_n) ,$$

to define a subspace of $H^{\frac{1}{2}}(\partial S)$. Next we define a subspace of $H^{\frac{1}{2}}(\partial S_j)$ complementary to $W_{\text{int}}(a_j)$.

Definition. *We define the subspace $W_{\text{ext}}(a_j)$ to consist of functions f defined on ∂S_j which are the boundary values of functions F defined $\mathbf{D}_R \backslash S_j$ which satisy the Dirac equation $(m - D_k)F = 0$ outside S_j and such that $F \in H^1(\mathbf{D}_R \backslash S_j)$.*

Note: For both of the preceding definitions the functions of interest are in the first Sobolev space H^1 near the boundary and so the Sobolev embedding theorem implies that the boundary values are assumed in the $H^{\frac{1}{2}}$ norm on the boundary.

Finally we let W_{ext} denote the subspace of boundary values of ∂S of functions F defined in $\mathbf{D}_R \backslash S$ which satisfy the Dirac equation $(m-D_k)F=0$ in the exterior of S and such that $F \in H^1(\mathbf{D}_R \backslash S)$. Note that this subspace is certainly not the direct sum of the subspaces $W_{\text{ext}}(a_j)$. The following formula defines a projection that is fundamental for us,

$$P^{a,\lambda} f_j(x) = 2 \int_{\partial S} \left\{ \frac{\overline{G_{j1}^{a,\lambda}(x,y) f_2(y)}}{1-\frac{|y|}{R^2}} i\,d\bar{y} - \frac{\overline{G_{j2}^{a,\lambda}(x,y) f_1(y)}}{1-\frac{|y|}{R^2}} i\,dy \right\}. \tag{6.2}$$

Theorem 6.0. *The subspaces $W_{\text{int}}(a)$ and W_{ext} are transverse to one another in $H^{\frac{1}{2}}(\partial S)$. The operator $P^{a,\lambda}$ defined in* (6.2) *above is the projection on $W_{\text{int}}(a)$ along W_{ext}.*

The proof of this theorem is parallel to the calculation used to establish (4.7) in reference [12] and so we will not repeat it here.

There is another projection that will be important for us. Let P_j denote the projection of $H^{\frac{1}{2}}(\partial S_j)$ on $W_{\text{int}}(a_j)$ along $W_{\text{ext}}(a_j)$. We define

$$F(a) = P_1 \oplus P_2 \oplus \cdots \oplus P_n\,. \tag{6.3}$$

We are now prepared to discuss the determinant bundle formalism that wil allow us to define a τ-function for the singular Dirac operator $D^{a,\lambda}$. Fix a point $a^0 = (a_1^0, a_2^0, \ldots, a_n^0)$ with $a_j^0 \neq a_k^0$ if $j \neq k$. Fix a collection of pairwise disjoint open strips S_j with $a_j^0 \in S_j$. Choose $\varepsilon > 0$ so that for each $j=1,2,\ldots,n$ the ball of radius ε about a_j^0 is contained in S_j. That is,

$$B_\varepsilon(a_j^0) \subset S_j\,.$$

Let $H = H^{\frac{1}{2}}(\partial S)$ and write

$$H = W_{\text{int}}(a^0) \oplus W_{\text{ext}} \tag{6.4}$$

for a distinguished splitting of the Hilbert space H. Write P_0 for the projection of H on $W_{\text{int}}(a^0)$ along W_{ext}. We are interested in the Grassmannian, Gr_0, of subspaces of H which are close to $W_{\text{int}}(a^0)$ in the following sense. A closed subspace W is in Gr_0 provided that the map,

$$P_0\colon W \to W_{\text{int}}(a^0)\,, \tag{6.5}$$

is Fredholm with index 0, and the map,

$$(I-P_0)\colon W \to W_{\text{ext}}\,, \tag{6.6}$$

is compact. Except that we have additionally specified the index of the first projection this is essentially the definition of the Grassmannian that Segal and Wilson consider in [16] and the reader can find a detailed theory of such Grassmannians in the book [14] by Segal and Pressley. The restriction to index 0 in (6.5) means that we confine our attention to the *connected component* of the Grassmannian, Gr_0, containing $W_{\text{int}}(a^0)$. It is simpler to discuss the det* bundle over the connected component of the Grassmannian and it will suffice for our purposes. The first result we require is

Theorem 6.1. *If $a_j \in B_\varepsilon(a_j^0)$ then $W_{\text{int}}(a) \in \mathrm{Gr}_0$.*

The proof of this theorem is precisely parallel to the proof of Theorem 4.1 in [12] and so we refer the reader to this paper for details.

The localization of the singular Dirac operator we are interested in can now be described as follows. The operator, $D^{a,\lambda}$, acts as the ordinary Dirac operator in the exterior of the union of strips S and its domain is the subspace of $H^1(\mathbf{D}_R \setminus S)$ whose boundary values belong to the subspace $W_{\text{int}}(a)$. We define a determinant for this family of Dirac operators by trivializing the det* bundle over the family of subspaces $W_{\text{int}}(a) \in \text{Gr}_0$. This trivialization is then compared with the canonical section for the det* bundle and the result defines the τ-function for the Dirac operator.

Recall that an invertible linear map $F\colon W_{\text{int}}(a^0) \to W$ is an *admissible frame* for the subspace $W \in \text{Gr}_0$ provided that $P_0 F\colon W_{\text{int}}(a^0) \to W_{\text{int}}(a^0)$ is a trace class perturbation of the identity. We will now introduce two different frames for the subspace $W_{\text{int}}(a)$. The first such frame will define the canonical section of the det* bundle and the second will be used to provide the trivialization of det* over the family of subspaces

$$a \to W_{\text{int}}(a) \in \text{Gr}_0 \, .$$

First we show that the restriction $P^{a,\lambda}\colon W_{\text{int}}(a^0) \to W_{\text{int}}(a)$ inverts the projection $P_0\colon W_{\text{int}}(a) \to W_{\text{int}}(a^0)$. From this it follows that the restriction of $P^{a,\lambda}$ to $W_{\text{int}}(a^0)$ is an admissible frame for $W_{\text{int}}(a)$ which defines the *canonical section* of the det* bundle. The argument is simple. Suppose that $w \in W_{\text{ext}}$. Then $P^{a,\lambda} w = w_a$ and writing $w_a = w_a + w_e$ with $w_a \in W_{\text{int}}(a)$ and $w_e \in W_{\text{ext}}$. Then $P^{a,\lambda} w = w_a$ and writing $w_a = w - w_e$ we see that $P_0 w_a = w$. Evidently, this is just an expression of the fact that the complementary subspace W_{ext} is the same for both projections P_0 and $P^{a,\lambda}$.

The second frame arises from considering the direct sum of "one point"projections:

Proposition 6.2. *The restriction of $F(a)$ to the subspace $W_{\text{int}}(a^0)$ is an admissible frame for $W_{\text{int}}(a)$.*

Once again this is identical to Proposition 4.2 in [12] and we refer the reader to that paper for more details.

We now recall that the fiber in the det* bundle over a subspace $W \in \text{Gr}_0$ can be identified with equivalence classes of pairs (w, α) where $w\colon W_{\text{int}}(a^0) \to W$ is an admissible frame and α is a complex number. The equivalence relation which defines the fiber is $(w_1, \alpha_1) = (w_2, \alpha_2)$ if and only if

$$\alpha_1 = \alpha_2 \det(w_2^{-1} w_1) \, .$$

In this representation the canonical section of the det* bundle is given by

$$\text{Gr}_0 \ni W \to (w, \det(P_0 w)) \in \det{}^* \, ,$$

where w is any admissible frame for W. Sinc $P_0 P^{a,\lambda}$ is the identity on $W_{\text{int}}(a^0)$ it follows that we may regard

$$\text{Gr}_0 \ni W_{\text{int}}(a) \to (P^{a,\lambda}|_0, 1)$$

as a representation of the canonical section $\sigma(W_{\text{int}}(a))$. We use the notation $P^{a,\lambda}|_0$ to signify the restriction of the projection $P^{a,\lambda}$ to $W_{\text{int}}(a^0)$. We now use $F(a)$ to define a trivialization of the det* bundle over the family of subspaces $W_{\text{int}}(a)$. Define

$$\delta(W_{\text{int}}(a)) = (F(a)|_0, 1) \, .$$

We may then define a determinant $\tau(a, a^0)$ for the Dirac operator as follows,

$$\tau(a, a^0) = \frac{\sigma(W_{\text{int}}(a))}{\delta(W_{\text{int}}(a))} = \det{}_0(P^{a,\lambda}|_0^{-1} F(a)) = \det{}_0(F(a)|_0^{-1} P^{a,\lambda})^{-1} . \tag{6.7}$$

In this formula $\det_0$ refers to the determinant of an operator restricted to the subspace $W_{\text{int}}(a^0)$. The second form of the determinant as a reciprocal is included since it will be slightly simpler for us to calculate the logarithmic derivative of the τ-function in this form. We can now state one of the principal results of this paper:

Theorem 6.3. *The τ-function defined by* (6.7) *has logarithmic derivative given by,*

$$d \log \tau = \sum_{j=1}^{n} \frac{1}{1-|b_j|^2} \{a^j_{\frac{1}{2}j} db_j + d^j_{\frac{1}{2}j} d\bar{b}_j\} , \tag{6.8}$$

where the coefficients $a^j_{\frac{1}{2}j}$ and $d^j_{\frac{1}{2}j}$ are the local expansion coefficients for W_j and W_j^* found in (3.22) *and* (3.25) *above.*

Proof. For simplicity we write $P(a) = P^{a,\lambda}$ in the following calculation. We also write d for the exterior derivative with respect to $\{a_1, a_2, \ldots, a_n\}$. The rule for differentiating determinants when applied to the final term in (6.7) above yields,

$$d \log \tau(a, a^0) = -\text{Tr}_0(d(F(a)^{-1} P(a)) P(a)^{-1} F(a)) ,$$

where Tr_0 is the trace on $W_{\text{int}}(a^0)$ and it is understood that both $F(a)$ and $P(a)$ are regarded as maps from $W_{\text{int}}(a^0)$ to $W_{\text{int}}(a)$ in this last formula. However, we know that $P(a^0)$ inverts $P(a)$ restricted to $W_{\text{int}}(a^0)$. Thus

$$P(a)^{-1} F(a) = P(a^0) F(a)$$

when both sides are restricted to $W_{\text{int}}(a^0)$. In a precisely similar fashion $F(a^0)$ inverts $F(a)$ restricted $W_{\text{int}}(a^0)$. Thus we find,

$$d \log \tau(a, a^0) = -\text{Tr}_0(F(a^0) d(P(a)) P(a^0) F(a)) .$$

But $P(a)(I - P(a^0) = 0$ so that $d(P(a)) P(a^0) = dP(a)$, where $dP(a)$ is now regarded as a map on all of H. Combining this observation with the fact that the range of $F(a^0)$ is all of $W_{\text{int}}(a^0)$ we can remove the subspace restriction on the trace to find,

$$d \log \tau(a, a^0) = -\text{Tr}(F(a^0)(dP(a)) F(a)) = -\text{Tr}((dP(a)) F(a) F(a^0)) .$$

Since $F(a)(I - F(a^0)) = 0$ we find that $F(a) F(a^0) = F(a)$. Thus

$$d \log \tau(a, a^0) = -\text{Tr}((dP(a)) F(a)) . \tag{6.9}$$

The reader might note that although $\tau(a, a^0)$ depends on a^0 its logarithmic derivative does not! Since it is clear that $\tau(a^0, a^0) = 1$ from the definition, the fact that the logarithmic derivative is independent of a^0 implies that there exists a function $\tau(a)$ such that

$$\tau(a, a^0) = \frac{\tau(a)}{\tau(a^0)} .$$

The function $\tau(a)$ is the function which we would like to identify as a correlation function in a quantum field theory. This will have to await further developments. However, one should observe that

$$d \log \tau(a, a^0) = d \log \tau(a) ,$$

We will now use the derivative formula (4.50) and the formula (6.2) for the projection to evaluate the trace (6.9) in terms of the low order expansion coefficients for the wave functions W_j and W_j^*. The derivative formula (4.50) and the projection formula (6.2) together imply,

$$(\partial_{b_\nu} P(a) f)(x) = \frac{m^2 R W_\nu(x,m)}{4s_\nu(1-|b_\nu|^2)} \int_{\partial S} \frac{\overline{W_\nu^*(y,-m)_1}}{1-\frac{|y|^2}{R^2}} f_2(y) i\, d\bar{y}$$

$$- \frac{m^2 R W_\nu(x,m)}{4s_\nu(1-|b_\nu|^2)} \int_{\partial S} \frac{\overline{W_\nu^*(y,-m)_2}}{1-\frac{|y|^2}{R^2}} f_1(y) i\, dy\,, \tag{6.10}$$

and

$$(\bar{\partial}_{b_\nu} P(a) f)(x) = \frac{m^2 R W_\nu^*(x,m)}{4s_\nu(1-|b_\nu|^2)} \int_{\partial S} \frac{\overline{W_\nu(y,-m)_1}}{1-\frac{|y|^2}{R^2}} f_2(y) i\, d\bar{y}$$

$$- \frac{m^2 R W_\nu^*(x,m)}{4s_\nu(1-|b_\nu|^2)} \int_{\partial S} \frac{\overline{W_\nu(y,-m)_2}}{1-\frac{|y|^2}{R^2}} f_1(y) i\, dy\,. \tag{6.11}$$

The appropriate trace is,

$$-\mathrm{Tr}(F(a)\partial_{b_\nu} P(a)) = -\frac{m^2 R(x,m)}{4s_\nu(1-|b_\nu|^2)} \int_{\partial S} \frac{\overline{W_\nu^*(y,-m)_1}\, F(a) W_\nu(y,m)_2}{1-\frac{|y|^2}{R^2}} i\, d\bar{y}$$

$$+ \frac{m^2 R(x,m)}{4s_\nu(1-|b_\nu|^2)} \int_{\partial S} \frac{\overline{W_\nu^*(y,-m)_2}\, F(a) W_\nu(y,m)_1}{1-\frac{|y|^2}{R^2}} i\, dy\,, \tag{6.12}$$

and

$$-\mathrm{Tr}(F(a)\bar{\partial}_{b_\nu} P(a)) = -\frac{m^2 R(x,m)}{4s_\nu(1-|b_\nu|^2)} \int_{\partial S} \frac{\overline{W_\nu(y,-m)_1}\, F(a) W_\nu^*(y,m)_2}{1-\frac{|y|^2}{R^2}} i\, d\bar{y}$$

$$+ \frac{m^2 R}{4s_\nu(1-|b_\nu|^2)} \int_{\partial S} \frac{\overline{W_\nu(y,-m)_2}\, F(a) W_\nu^*(y,m)_1}{1-\frac{|y|^2}{R^2}} i\, dy\,. \tag{6.13}$$

According to (3.5) both of the integrands in (6.12) and (6.13) are exact away from the singularities at a_j for $j=1,2,\ldots,n$. The contours ∂S_j can be closed about the singularities a_j and the same asymptotic calculation that leads to (4.4) above shows that,

$$-\mathrm{Tr}(F\partial_{b_\nu} P) = \frac{1}{1-|b_\nu|^2}\, a(FW_\nu)_{\frac{1}{2}}^{\nu} \tag{6.14}$$

and

$$-\mathrm{Tr}(F\bar{\partial}_{b_\nu}P)=\frac{1}{1-|b_\nu|^2}b(FW_\nu^*)^\nu_{\frac{1}{2}}\,, \tag{6.15}$$

where we used the notation,

$$g(x)=\sum_n \{a(g)^\nu_n w_{n+\lambda_\nu}+b(g)^\nu_n w^*_{n-\lambda_\nu}\}\,,$$

for the local expansion coefficients of g about the point a_ν. Note that the contours ∂S_j with $j\neq\nu$ do not make a contribution in this calculation. To compute FW_ν and FW_ν^* on ∂S_ν we need only subtract from W_ν and W_ν^* the "one point versions" of these two functions (just the translate of the appropriate multiple of the function $\hat{w}_{-\frac{1}{2}+\lambda_\nu}$ to a_ν). Doing this we find,

$$a(FW_\nu)^\nu_{\frac{1}{2}}=a^\nu_{\frac{1}{2}\nu}$$

and

$$b(FW_\nu^*)^\nu_{\frac{1}{2}}=d^\nu_{\frac{1}{2}\nu}\,.$$

This finishes the proof of Theorem 6.3. QED

We will now provide the connection between the coefficients in the formula for the logarithmic derivative of τ given in (6.8) and the deformation theory of Sect. 5. If one collects all the terms in (5.16) that involve $a^j_{\frac{1}{2}j}$ on the left-hand side using $e_{jj}=k+\lambda_j$ then one finds,

$$a^j_{\frac{1}{2}j}=\bar{b}_j m_2(\lambda_j-\tfrac{1}{2})+\sum_{\mu\neq j}e_{j\mu}a^j_{\frac{1}{2}\mu}+\sum_\mu f_{j\mu}\bar{b}_\mu c^j_{\frac{1}{2}\mu}\,. \tag{6.16}$$

The analogous calculations for $d^j_{\frac{1}{2}j}$ yield,

$$d^j_{\frac{1}{2}j}=b_j m_1(-\lambda_j-\tfrac{1}{2})+\sum_{\mu\neq j}h_{j\mu}d^j_{\frac{1}{2}\mu}-\sum_\mu g_{j\mu}b_\mu b^j_{\frac{1}{2}\mu}\,. \tag{6.17}$$

The right-hand sides of both (6.16) and (6.17) are completely determined by the deformation variables e, f, g and h.

The log derivative of τ for the two point function. In the subsection we will write out the formula for the logarithmic derivative of the tau function in terms of the Painlevé transcendent of the sixth kind that was found in the integration of a special case of the deformation equations described at the end of Sect. 5. We suppose therefore that $\lambda_j>0$ for $j=1,2$ and that $b_1=0$ and $b_2=t$. In this special case (6.8) becomes,

$$\frac{\partial}{\partial t}\log\tau(0,t)=\frac{1}{1-t^2}\left\{a^2_{\frac{1}{2},2}+d^2_{\frac{1}{2},2}\right\}.$$

From (6.16) and (6.17) one finds,

$$a^2_{\frac{1}{2},2}=tm_2(\lambda_2-\tfrac{1}{2})+e_{21}a^2_{\frac{1}{2},1}+tf_{22}c^2_{\frac{1}{2},2}$$

and

$$d^2_{\frac{1}{2},2}=tm_1(-\lambda_2-\tfrac{1}{2})-h_{21}d^2_{\frac{1}{2},1}-tg_{22}b^2_{\frac{1}{2},2}\,.$$

Expressing the off diagonal elements of a, b, c and d in terms of the deformation parameters one finds,

$$a^2_{\frac{1}{2},1}=t^{-1}(1-t^2)e_{12}\ ,$$
$$b^2_{\frac{1}{2},2}=-f_{22}\ ,$$
$$c^2_{\frac{1}{2},2}=g_{22}\ ,$$
$$d^2_{\frac{1}{2},1}=-t^{-1}(1-t^2)h_{12}\ .$$

Thus,

$$a^2_{\frac{1}{2},2}=tm_2(\lambda_2-\tfrac{1}{2})+t^{-1}(1-t^2)e_{21}e_{12}+tf_{22}g_{22}\ ,$$
$$d^2_{\frac{1}{2},2}=tm_1(\lambda_2-\tfrac{1}{2})+t^{-1}(1-t^2)h_{21}h_{12}+tf_{22}g_{22}\ ,$$

Now we observe that

$$m_2(\lambda_2-\tfrac{1}{2})=m_1(-\lambda_2-\tfrac{1}{2})=\frac{m^2R^2}{4}+k^2-(k+\lambda_2)^2\ ,$$

and since

$$\mathbf{g}=s^{-1}(1-|B|^2)^{-1}g\ ,$$
$$\mathbf{f}=fs(1-|B|^2)\ ,$$

it follows that

$$f_{22}g_{22}=\mathbf{f}_{22}\mathbf{g}_{22}\ .$$

Eliminating h in favor of e^* one has,

$$h_{12}=\frac{s_1}{s_2}(1-t^2)^{-1}e^*_{12}\ ,$$

$$h_{21}=\frac{s_2}{s_1}(1-t^2)e^*_{21}\ ,$$

so that,

$$h_{21}h_{12}=e^*_{21}e^*_{12}=e_{21}e_{12}\ ,$$

where we have used the fact that $e_{21}e_{12}=\bar{e}_-e_+$ is real. Thus

$$\frac{\partial}{\partial t}\log\tau(0,t)=\frac{1}{1-t^2}\{2tm_2(\lambda_2-\tfrac{1}{2})+2t^{-1}(1-t^2)\bar{e}_-e_++2t\mathbf{f}_{22}\mathbf{g}_{22}\}\ . \quad (6.18)$$

Using (5.53) (5.57) and (5.58) one finds,

$$\mathbf{f}_{22}=\kappa^{-1}(1-t^2)\operatorname{ch}\psi\ ,$$

$$\mathbf{g}_{22}=\frac{-\bar{\varepsilon}\mu e_+\operatorname{sh}\psi+\kappa(k_2^2-M^2+\bar{e}_-e_+)\operatorname{ch}\psi}{1-t^2\operatorname{ch}^2\psi}\ ,$$

$$=\kappa\frac{-\mu\dfrac{\xi+\lambda\operatorname{th}\psi}{2(1-t^2)}\operatorname{sh}\psi+\left(k_2^2-M^2+\dfrac{\xi^2-\lambda^2\operatorname{th}^2\psi}{4(1-t^2)}\right)\operatorname{ch}\psi}{1-t^2\operatorname{ch}^2\psi}\ ,$$

so that,

$$\mathbf{f}_{22}\mathbf{g}_{22}$$

$$=\frac{-2\mu(\xi+\lambda\operatorname{th}\psi)\operatorname{ch}\psi\operatorname{sh}\psi+(\xi^2-\lambda^2\operatorname{th}^2\psi)\operatorname{ch}^2\psi+4(1-t^2)(k_2^2-M^2)\operatorname{ch}^2\psi}{4(1-t^2\operatorname{ch}^2\psi)}.$$

As a check on (6.18) we can, as we did for (5.59), introduce $t=\frac{r}{R}$ and then take the limit $R\to\infty$. One finds that the limit of $d\log\tau\left(0,\frac{r}{R}\right)$ as $R\to\infty$ is

$$\frac{1}{2r}\left(r^2\left(\frac{\partial\psi}{\partial r}\right)^2-\lambda^2\operatorname{th}^2\psi-m^2r^2\operatorname{sh}^2\psi\right)dr\,.$$

This should be compared with (4.5.42) in [15] and (5.39) in [12]. The overall sign difference with (4.5.42) is a consequence of the fact that (4.5.42) is a formula for the log derivative of the *Bosonic* tau function SMJ.

Next we make the substitutions, $w=1/\operatorname{ch}^2\psi$ and $s=t^2$ in the expression for $\mathbf{f}_{22}\mathbf{g}_{22}$. Substitute the result in (6.18) along with (5.54) for e_+ and $\bar{e}_-$. Eliminate ξ in the result with the substitutions

$$\xi\operatorname{th}\psi=\frac{1}{w}(s(1-s)w'+s(1-\mu)(\omega-1))\,,$$

$$\xi^2=\frac{(s(1-s)w'+s(1-\mu)(w-1))^2}{w^2(1-w)}\,.$$

Noting that

$$\frac{1}{2t}\frac{\partial}{\partial t}\log\tau(0,t)=\frac{\partial}{\partial s}\log\tau(0,\sqrt{s})\,,$$

we find after some calculation

$$\begin{aligned}\frac{\partial}{\partial s}\log\tau(0,\sqrt{s})&=\frac{s(s-1)}{4w(w-1)(w-s)}\left(\frac{dw}{ds}-\frac{w-1}{s-1}\right)^2-\frac{M^2}{w-s}\\&\quad-\frac{\mu^2}{4(s-1)w}-\frac{\lambda^2 w}{4s(s-1)}-\frac{\lambda^2}{4s}+\frac{4M^2-\mu^2-\lambda^2}{4(1-s)}\,.\end{aligned}\tag{6.19}$$

The formula on the right-hand side of this last equation bears some striking resemblance to the formula in Okamoto [7] for a Hamiltonian associated to the Painlevé VI equation. It does not, however, appear to be a Hamiltonian for P_{VI} for any simple choice of canonical coordinates.

It would be interesting to study the behavior of this tau function as $t\to 0,1$. The asymptotics for t near 0 should match onto the asymptotics of the analogous tau function in the Euclidean domain. The behavior as $t\to 1$ would follow if one knew the appropriate connection formula for P_{VI}. We hope to return to this question elsewhere.

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