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New Identities Between Unitary Minimal Virasoro Characters

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Abstract: Two sets of identities between unitary minimal Virasoro characters at levels $m=3, 4, 5$ are presented and proven. The first identity suggests a connection between the Ising and the tricritical Ising models since the $m=3$ Virasoro characters are obtained as bilinears of $m=4$ Virasoro characters. The second identity gives the tricritical Ising model characters as bilinears in the Ising model characters and the six combinations of $m=5$ Virasoro characters which do not appear in the spectrum of the three state Potts model. The implication of these identities on the study of the branching rules of $N=4$ superconformal characters into $\widehat{SU(2)} \times \widehat{SU(2)}$ characters is discussed.

1. Introduction

The theory of unitary highest weight state representations of N-extended superconformal algebras ($N=0, 1, 2, 4$) is by now very well understood [20, 10, 11, 3, 2, 4, 5, 15, 6, 22, 7, 8, 12, 16, 17], and is of considerable interest in the analysis of the spectrum of string based models. Unlike the $N=0, 1, 2$ extended superconformal algebras, the $N=4$ algebras both with $\widehat{SU(2)}$ or $\widehat{SU(2)} \times \widehat{SU(2)}$ Kac–Moody subalgebras have no representations falling in a minimal series [7, 8, 12]. As a consequence, there is no value of the central charge c allowed by unitarity for which the characters corresponding to unitary highest weight state representations carry a finite representation of the modular group. Generically, there exists a finite number of massless characters and an inifinite tower of massive characters, corresponding to representations with non-zero and zero Witten index respectively. The modular transformations of massless characters mix massless and massive characters in a complicated way, which has only been analysed in a very limited number of cases. In the $\widehat{SU(2)}_k$ extended $N=4$ superconformal algebra, the modular transformations involve a Mordell integral when the level of $\widehat{SU(2)}$ is $k=1$ [9]. For

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the $\widehat{SU(2)}_{\tilde{k}^-} \times \widehat{SU(2)}_{\tilde{k}^+}$ extended $N=4$ algebra, certain combinations of massless characters play a particular role and transform among themselves under the modular group when $\tilde{k}^+ = \tilde{k}^- = 1$ [18]. In attempting to generalise this result for $\tilde{k}^+ > 1$ ($\tilde{k}^- = 1$), we have found new relations between unitary minimal Virasoro characters when $\tilde{k}^+ = 2$ and 3. In the following section, we present these identities, which will be proven in the Appendix, and we discuss in Sect. 3 their relevance in the study of $N=4$ superconformal algebras. Some comments on how the identities between unitary Virasoro characters at low levels could be generalised to higher levels are given in the conclusions. We also briefly discuss the potential consequences of these identities in 2-d conformal field theory.

2. The Identities

The first set of identities relates the unitary Virasoro characters at levels $m=3$ and $m=4$ in such a way that the three Ising model characters are given by the vector product of two 3-vectors whose components are the six tricritical Ising model characters,

$$\chi_{1,i}^{\mathrm{Vir}(3)}(q) = \varepsilon_{ijk}(-1)^{j+k} \chi_{j,4}^{\mathrm{Vir}(4)}(q)\, \chi_{k,2}^{\mathrm{Vir}(4)}(q) \,. \tag{2.1}$$

Our definition of the unitary Virasoro characters at level m is,

$$\chi_{r,s}^{\mathrm{Vir}(m)}(q) = \eta^{-1}(q)\left[\theta_{r(m+1)-sm,\, m(m+1)}(q) - \theta_{r(m+1)+sm,\, m(m+1)}(q)\right], \tag{2.2}$$

where the integers m, r and s have the following ranges,

$$m=2,3,\ldots; \quad r=1,2,\ldots,m-1; \quad s=1,\ldots,r\,.$$

The Dedekind function, $\eta(q)$, is defined by,

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=0}^{\infty} (1-q^{n+1})\,, \tag{2.3}$$

while the generalised level k theta functions, $\theta_{m,k}(q)$, are given by,

$$\theta_{m,k}(q) = \sum_{n \in Z} q^{k(n+\frac{m}{2k})^2}\,, \tag{2.4}$$

with the properties,

$$\theta_{m,k}(q) = \theta_{-m,k}(q) = \theta_{2k-m,k}(q)\,.$$

This definition, (2.2), of the Virasoro characters coincides with the definition given in [20] up to a factor $q^{-\frac{c}{24}}$ where c is the central charge,

$$c = 1 - \frac{6}{m(m+1)}\,.$$

These identities (2.1) can be rewritten in a slightly different notation where the characters are labelled by their conformal dimension $h_{r,s}$,

$$h_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}\,,$$

and where the equivalence between the tricritical Ising model characters and the characters of the first theory in superconformal $N=1$, minimal series is implemented. Since the latter are given by,

$$\begin{aligned}
\chi_0^{NS}(q) &= \chi_0^{Vir(4)}(q) + \chi_{3/2}^{Vir(4)}(q)\,,\\
\tilde{\chi}_0^{NS}(q) &= \chi_0^{Vir(4)}(q) - \chi_{3/2}^{Vir(4)}(q)\,,\\
\chi_{1/10}^{NS}(q) &= \chi_{1/10}^{Vir(4)}(q) + \chi_{3/5}^{Vir(4)}(q)\,,\\
\tilde{\chi}_{1/10}^{NS}(q) &= \chi_{1/10}^{Vir(4)}(q) - \chi_{3/5}^{Vir(4)}(q)\,,\\
\chi_{3/80}^{R}(q) &= \chi_{3/80}^{Vir(4)}(q)\,,\\
\tilde{\chi}_{7/16}^{R}(q) &= \chi_{7/16}^{Vir(4)}(q)\,,
\end{aligned} \tag{2.5}$$

the identities (2.1) take the form,

$$\begin{aligned}
\chi_0^{Vir(3)}(q) + \chi_{1/2}^{Vir(3)}(q) &= \chi_{3/80}^{R}(q)\,\tilde{\chi}_0^{NS}(q) + \chi_{7/16}^{R}(q)\,\tilde{\chi}_{1/10}^{NS}(q)\,,\\
\chi_0^{Vir(3)}(q) - \chi_{1/2}^{Vir(3)}(q) &= \chi_{3/80}^{R}(q)\,\chi_0^{NS}(q) - \chi_{7/16}^{R}(q)\,\chi_{1/10}^{NS}(q)\,,\\
\chi_{1/16}^{Vir(3)}(q) &= \frac{1}{2}(\chi_0^{NS}(q)\,\tilde{\chi}_{1/10}^{NS}(q) + \chi_{1/10}^{NS}(q)\,\tilde{\chi}_0^{NS}(q))\,.
\end{aligned} \tag{2.6}$$

It is interesting to note that the vector $\chi_{1,i}^{Vir(3)}(q)$ $(i=1,2,3)$ being orthogonal to the vectors $(-1)^j\,\chi_{j,4}^{Vir(4)}(q)$ and $(-1)^k\,\chi_{k,2}^{Vir(4)}(q)$ (no summation on j and k) is a trivial consequence of the identity (2.1). It produces relations which can be derived from repeated use of the Goddard–Kent–Olive (GKO) sum-rules [11],

$$\chi_{2\ell}^{k}(q,z)\,\chi_{2\ell'}^{1}(q,z) = \sum_{2\ell''=0}^{k+1} \chi_{2\ell''}^{k+1}(q,z)\,\chi_{2\ell+1,2\ell''+1}^{Vir(k+2)}(q)\,, \tag{2.7}$$

where $2\ell'' \equiv 2\ell + 2\ell'$ (mod 2) and where the $SU(2)_k$ characters for isospin ℓ $(2\ell = 0, 1, \ldots, k)$ are defined by,

$$\begin{aligned}
\chi_{2\ell}^{k}(q,z) = q^{-1/8}z^{-1} &\prod_{n=1}^{\infty}(1-q^n)^{-1}(1-q^n z^2)^{-1}(1-q^{n-1}z^{-2})^{-1}\\
\times &\sum_{m\in Z+\frac{\ell+1/2}{k+2}} q^{(k+2)m^2}\left[z^{2(k+2)m} - z^{-2(k+2)m}\right]\,.
\end{aligned} \tag{2.8}$$

In this instance, one considers the coset $[SU(2)_1 \times SU(2)_1 \times SU(2)_1]/SU(2)_3$ and applies the GKO sumrules twice on the following trilinear in $SU(2)_1$ characters,

$$[\chi_0^1(q,z)\,\chi_0^1(q,z)]\,\chi_1^1(q,z) = \chi_0^1(q,z)[\chi_0^1(q,z)\,\chi_1^1(q,z)]\,.$$

In the second set of identities, the tricritical Ising model characters are obtained as the product of unitary Virasoro characters at levels $m=3$ and $m=5$ in the following way,

$$\begin{aligned}
\chi_{2,1}^{Vir(4)}(q) &= \chi_{2,2}^{Vir(3)}(q)\,[\chi_{2,1}^{Vir(5)}(q) - \chi_{3,1}^{Vir(5)}(q)]\,,\\
\chi_{2,2}^{Vir(4)}(q) &= \chi_{2,2}^{Vir(3)}(q)\,[\chi_{1,1}^{Vir(5)}(q) - \chi_{4,1}^{Vir(5)}(q)]\,,\\
\chi_{1,1}^{Vir(4)}(q) \pm \chi_{3,1}^{Vir(4)}(q) &= [\chi_{1,1}^{Vir(3)}(q) \pm \chi_{2,1}^{Vir(3)}(q)]\,[\chi_{2,2}^{Vir(5)}(q) \mp \chi_{3,2}^{Vir(5)}(q)]\,,\\
\chi_{1,2}^{Vir(4)}(q) \pm \chi_{3,2}^{Vir(4)}(q) &= [\chi_{1,1}^{Vir(3)}(q) \pm \chi_{2,1}^{Vir(3)}(q)]\,[\chi_{1,2}^{Vir(5)}(q) \mp \chi_{4,2}^{Vir(5)}(q)]\,.
\end{aligned} \tag{2.9}$$

It is remarkable that the six combinations of level $m=5$ Virasoro characters involved are precisely those which do *not* appear in the spectrum of the three state Potts model. The identities (2.9) are consistent with the weaker identities,

$$\chi_{2,2}^{Vir(4)}(q)\,[\chi_{2,1}^{Vir(5)}(q) - \chi_{3,1}^{Vir(5)}(q)] = \chi_{2,1}^{Vir(4)}(q)\,[\chi_{1,1}^{Vir(5)}(q) - \chi_{4,1}^{Vir(5)}(q)]\,, \tag{2.10}$$

and,

$$[\chi_{1,1}^{\mathrm{Vir}(4)}(q) \pm \chi_{3,1}^{\mathrm{Vir}(4)}(q)]\,[\chi_{1,2}^{\mathrm{Vir}(5)}(q) \mp \chi_{4,2}^{\mathrm{Vir}(5)}(q)]$$

$$= [\chi_{1,2}^{\mathrm{Vir}(4)}(q) \pm \chi_{3,2}^{\mathrm{Vir}(4)}(q)]\,[\chi_{2,2}^{\mathrm{Vir}(5)}(q) \mp \chi_{3,2}^{\mathrm{Vir}(5)}(q)]\,, \quad (2.11)$$

which can be obtained from the GKO character sumrules for the coset $[SU(2)_1 \times SU(2)_2 \times SU(2)_1]/SU(2)_4$ when considering the following trilinears in $\widehat{SU(2)}$ characters,

$$[\chi_0^1(q,z)\,\chi_0^2(q,z)]\,\chi_1^1(q,z) = \chi_0^1(q,z)[\chi_0^2(q,z)\,\chi_1^1(q,z)]\,,$$

and,

$$[\chi_0^1(q,z)\,\chi_1^2(q,z)]\,\chi_1^1(q,z) = \chi_0^1(q,z)[\chi_1^2(q,z)\,\chi_1^1(q,z)]\,.$$

A proof of the identities (2.1) and (2.9) involving the Jacobi triple product identity and standard properties of the generalised theta functions (2.4) is given in the Appendix.

In order to gain some insight in the way one might generalise this type of relations between unitary Virasoro characters, we now turn to $N=4$ superconformal algebras whose study has prompted the identities presented here.

3. Properties of $N=4$ Superconformal Characters

The $\widehat{SU(2)}_{\tilde{k}^+} \times \widehat{SU(2)}_{\tilde{k}^-}$ extended $N=4$ algebra we consider is a non-linear superconformal algebra which, together with a dimension 2 Virasoro generator $L(z)$ and 4 dimension 3/2 supercurrents $G^a(z)$ $(a=1,2,3,4)$, contains 6 currents $T^{\pm i}(z)$ $(i=1,2,3)$ which are dimension 1 primaries with respect to $L(z)$ and generate two Kac-Moody $\widehat{SU(2)}$ algebras at levels $\tilde{k}^+$ and $\tilde{k}^-$ [21]. The representation theory and corresponding characters were given in [12, 16, 17] for unitary highest weight state representations. These are labelled by the two isospin quantum numbers ℓ^+ and ℓ^- and have conformal dimension $\tilde{h}$ whose lower bound $\tilde{h}_0$ is a function of ℓ^+, ℓ^-, $\tilde{k}^+$ and $\tilde{k}^-$. An irreducible representation with conformal dimension $\tilde{h}_0$ is called massless or chiral (in the Ramond sector, it corresponds to a representation with non-zero Witten index), while any representation with conformal dimension $\tilde{h} > \tilde{h}^0$ is called massive. For fixed $\tilde{k}^+$, $\tilde{k}^-$, there is a finite number of massless and an infinite number of massive representations and corresponding characters. In the Ramond sector, the massless characters are labelled by,

$$Ch_0^{\mathrm{R}}(\tilde{k}^+, \tilde{k}^-, \ell^+, \ell^-, \tilde{h}_0^{\mathrm{R}}; q, z_+, z_-)\,,$$

with $\ell^\pm = 0, 1/2, \ldots, \tilde{k}^\pm/2$ and,

$$\tilde{h}_0^{\mathrm{R}} = \frac{\tilde{k}^+\tilde{k}^- + 4(\ell^+ + \ell^-)(\ell^+ + \ell^- + 1)}{4(\tilde{k}^+ + \tilde{k}^- + 2)}\,,$$

while massive characters are denoted by,

$$Ch_m^{\mathrm{R}}(\tilde{k}^+, \tilde{k}^-, \ell^+, \ell^-, \tilde{h}^{\mathrm{R}}; q, z_+, z_-)\,,$$

with $\ell^\pm = 1/2, \ldots, \tilde{k}^\pm/2$ and $\tilde{h}^{\mathrm{R}} > \tilde{h}_0^{\mathrm{R}}$.

Two essential properties of the characters are used in the following. First, the massive characters split into two massless ones as $\tilde{h}^{\mathrm{R}}$ reaches the lower bound $\tilde{h}_0^{\mathrm{R}}$,

$$Ch_m^{\mathrm{R}}\left(\tilde{k}^+,\tilde{k}^-,\ell^+,\ell^-+\frac{1}{2},\tilde{h}^{\mathrm{R}};q,z_+,z_-\right)$$
$$=q^{\tilde{h}^{\mathrm{R}}-\tilde{h}_0^{\mathrm{R}}}\times\left[Ch_0^{\mathrm{R}}(\tilde{k}^+,\tilde{k}^-,\ell^+,\ell^-,\tilde{h}_0^{\mathrm{R}};q,z_+,z_-)\right.$$
$$\left.+Ch_0^{\mathrm{R}}\left(\tilde{k}^+,\tilde{k}^-,\ell^+-\frac{1}{2},\ell^-+\frac{1}{2},\tilde{h}_0^{\mathrm{R}};q,z_+,z_-\right)\right].\tag{3.1}$$

Furthermore, when the conformal dimension is,

$$\tilde{h}_m^{\mathrm{R}}\equiv\frac{(\tilde{k}^++1)(\tilde{k}^-+1)}{4(\tilde{k}^++\tilde{k}^-+2)}\left[\frac{2\ell^+}{\tilde{k}^++1}-\frac{2\ell^-}{\tilde{k}^-+1}\right]^2+\frac{[(2\ell^++2\ell^-)^2+\tilde{k}^+\tilde{k}^--1]}{4(\tilde{k}^++\tilde{k}^-+2)},$$

one has,

$$\eta(q)\,Ch_m^{\mathrm{R}}(\tilde{k}^+,\tilde{k}^-,\ell^+,\ell^-,\tilde{h}_m^{\mathrm{R}};q,z_+,z_-)$$
$$=\sum_{2\lambda^+=0}^{\tilde{k}^+}\sum_{2\lambda^-=0}^{\tilde{k}^-}\chi_{2\lambda^+}^{\tilde{k}^+}(q,z_+)\,\chi_{2\lambda^-}^{\tilde{k}^-}(q,z_-)\,\chi_{2\ell^+,2\lambda^++1}^{\mathrm{Vir}(\tilde{k}^++1)}(q)\,\chi_{\tilde{k}^--2\ell^-+1,\tilde{k}-2\lambda^-+1}^{\mathrm{Vir}(\tilde{k}^-+1)}(q)\,,\tag{3.2}$$

where $2\lambda^+-2\lambda^-\equiv2\ell^+-2\ell^-+1$ (mod 2), so that the branching functions for *massive* characters into $\widehat{SU(2)}_{\tilde{k}^+}\times\widehat{SU(2)}_{\tilde{k}^-}$ characters $\chi_{2\lambda^+}^{\tilde{k}^+}(q,z_+)\;\chi_{2\lambda^-}^{\tilde{k}^-}(q,z_-)$ are products of unitary Virasoro characters at levels $m=\tilde{k}^++1$ and $m=\tilde{k}^-+1$. For $\tilde{k}^\pm=1$, we recall that $\chi_{2\ell,2\ell'}^{\mathrm{Vir}(\tilde{k}^++1)}(q)=1$.

The second property of interest is that when the angular variables $z_\pm$ are related by $z_-=-z_+^{-1}$, the massive characters in the Ramond sector vanish while the massless characters reduce to $\widehat{SU(2)}_{\tilde{k}^++\tilde{k}^-}$ characters [13],

$$Ch_m^{\mathrm{R}}(\tilde{k}^+,\tilde{k}^-,\ell^+,\ell^-,\tilde{h}_m^{\mathrm{R}};q,z_+,-z_+^{-1})=0\,,$$
$$Ch_0^{\mathrm{R}}(\tilde{k}^+,\tilde{k}^-,\ell^+,\ell^-,\tilde{h}_0^{\mathrm{R}};q,z_+,-z_+^{-1})=(-1)^{2\ell^-}\chi_{2(\ell^++\ell^-)}^{\tilde{k}^++\tilde{k}^-}(q,z_+)\,.\tag{3.3}$$

Let us now introduce the following combinations of massless characters for $\tilde{k}^-=1$, which is the only case discussed here (for simplicity the $\tilde{k}^+$, $\tilde{k}^-$ and $\tilde{h}_0^{\mathrm{R}}$ arguments have been suppressed),

$$Ch_0^{\mathrm{R}}(L=0;q,z_+,z_-)\equiv-Ch_0^{\mathrm{R}}(\ell^+=0,\ell^-=0;q,z_+,z_-)\,,$$
$$Ch_0^{\mathrm{R}}(L=1,\ldots,\tilde{k}^+;q,z_+,z_-)\equiv\frac{1}{2}\left[Ch_0^{\mathrm{R}}\left(\frac{1}{2}(L-1),\frac{1}{2};q,z_+,z_-\right)\right.$$
$$\left.-Ch_0^{\mathrm{R}}\left(\frac{1}{2}L,0;q,z_+,z_-\right)\right],$$
$$Ch_0^{\mathrm{R}}(L=\tilde{k}^++1;q,z_+,z_-)\equiv Ch_0^{\mathrm{R}}\left(\frac{1}{2}\tilde{k}^+,\frac{1}{2};q,z_+,z_-\right),\tag{3.4}$$

where $2(\ell^++\ell^-)=L$. The combinations for $L=1,\ldots,\tilde{k}^+$ are orthogonal to the combinations present in (3.1).

In terms of $\widehat{SU(2)}_{\tilde{k}^+} \times \widehat{SU(2)}_1$ characters, the above combinations are given by,

$$Ch_0^{\mathrm{R}}(L;\, q, z_+, z_-) = \sum_{2\lambda^+=0}^{\tilde{k}^+} \sum_{2\lambda^-=0}^{1} (-1)^{2\lambda^+ + L + 1}\, Y_{2\lambda^+ + 1, L+1; 2\lambda^-}^{(\tilde{k}^+ + 2)}(q)\, \chi_{2\lambda^+}^{\tilde{k}^+}(q, z_+)\, \chi_{2\lambda^-}^{1}(q, z_-)\,, \tag{3.5}$$

where $L \equiv 2\lambda^+ + 2\lambda^-$ mod 2 and the branching functions $Y_{2\lambda^+ + 1, L+1, 2\lambda^-}^{\tilde{k}^+ + 2}(q)$ are not known for $\tilde{k}^+ > 1$. It turns out that,

$$Y_{2\lambda^+ + 1, L+1; 2\lambda^-}^{(\tilde{k}^+ + 2)}(q) = Y_{\tilde{k}^+ + 1 - 2\lambda^+, \tilde{k}^+ + 2 - L; 1 - 2\lambda^-}^{(\tilde{k}^+ + 2)}(q)\,, \tag{3.6}$$

so that there are as many such functions as unitary Virasoro characters at level $m = \tilde{k}^+ + 2$. We find it convenient to introduce the following notation ($\tilde{k}^- = 1$),

$$Y_{2\lambda^+ + 1, L+1; 2\lambda^-}^{(\tilde{k}^+ + 2)}(q) = q^{\frac{(2\lambda^+ - L)^2 - (2\lambda^-)^2}{4}}\, Y_{2\lambda^+ + 1, L+1}^{(\tilde{k}^+ + 2)}(q)\,, \tag{3.7}$$

where the functions $Y_{2\lambda^+ + 1, L+1}^{(\tilde{k}^+ + 2)}(q)$ have a q expansion,

$$Y_{2\lambda^+ + 1, L+1}^{(\tilde{k}^+ + 2)}(q) \sim q^{\frac{c}{24} - h_{2\lambda^+ + 1, L+1}} \sum_{n=0}^{\infty} a_n q^n, \quad a_n \in Z\,,$$

characterised by a prefactor which is the inverse of the corresponding $m = \tilde{k}^+ + 2$ Virasoro character prefactor. We emphasize that the symmetry property (3.6) does not necessarily imply a similar symmetry property on the functions $Y_{2\lambda^+ + 1, L+1}^{(\tilde{k}^+ + 2)}(q)$. For instance, when $\tilde{k}^+ = 1$, $Y_{2,1}^{(3)}(q) = q\, Y_{1,3}^{(3)}(q)$.

When $\tilde{k}^+ = 1$, it is easy to calculate the three branching functions [13, 18], which can be written in terms of Ising model characters,

$$Y_{1,1;0}^{(3)}(q) \pm Y_{2,1,1}^{(3)}(q) = Y_{1,1}^{(3)}(q) \pm Y_{2,1}^{(3)}(q) = \frac{1}{\chi_{1,1}^{\mathrm{Vir}(3)}(q) \pm \chi_{2,1}^{\mathrm{Vir}(3)}(q)}\,,$$

$$Y_{2,2;0}^{(3)}(q) = Y_{2,2}^{(3)}(q) = \frac{1}{2\chi_{2,2}^{\mathrm{Vir}(3)}(q)}\,. \tag{3.8}$$

In an attempt to determine the branching functions for $\tilde{k}^+ > 1$ [19], we analyse the properties of the matrix $A^{(+)}$ of branching functions for the combinations (3.4) of massless characters $Ch_0^{\mathrm{R}}(2L)$, ($L = 0, \ldots, [\frac{1}{2}(\tilde{k}^+ + 1)]$) and the massive characters $\eta(q) Ch_m^{\mathrm{R}}(L, \frac{1}{2})$ ($L = 1, \ldots, [\frac{1}{2}\tilde{k}^+]$) as well as the properties of the matrix $A^{(-)}$ of branching functions for the combinations of massless characters $Ch_0^{\mathrm{R}}(2L+1)$ ($L = 0, \ldots, [\frac{1}{2}\tilde{k}^+]$) and massive characters $\eta(q) Ch_m^{\mathrm{R}}(L + \frac{1}{2}, \frac{1}{2})$ ($L = 0, \ldots, [\frac{1}{2}(\tilde{k}^+ - 1)]$). By $[r]$, we mean as usual the integer part of the real number r, i.e. the biggest integer smaller than or equal to r. The matrices $A^{(+)}$ and $A^{(-)}$ are both of dimension $(\tilde{k}^+ + 1) \times (\tilde{k}^+ + 1)$, and have the same determinant (up to a sign) when $\tilde{k}^+$ is even,

$$\det A^{(+)} = -\det A^{(-)} \quad \text{for } \tilde{k}^+ \in 2Z\,.$$

The inverse matrices, $[A^{(+)}]^{-1}$ and $[A^{(-)}]^{-1}$, are encoded in the relation,

$$\begin{aligned}\chi_{2\ell^+}^{\tilde{k}^+}(q, z_+)\, \chi_{2\ell^-}^{1}(q, z_-) = (-1)^{2\ell^- + 1} &\sum_{\substack{2\ell=0 \\ 2\ell \equiv 2\ell^+ + 2\ell^- \bmod 2}}^{\tilde{k}^+ + 1} Ch_0^{\mathrm{R}}(2\ell; q, z_+, z_-)\, \chi_{2\ell^+ + 1, 2\ell + 1}^{\mathrm{Vir}(\tilde{k}^+ + 2)}(q) \\ + \eta(q) &\sum_{\substack{2\ell=1 \\ 2\ell \equiv 2\ell^+ + 2\ell^- \bmod 2}}^{\tilde{k}^+} Ch_m^{\mathrm{R}}\left(\ell, \frac{1}{2}, \tilde{h}_m^{\mathrm{R}}; q, z_+, z_-\right) X_{2\ell, 2\ell^+ + 1; 2\ell^-}^{(\tilde{k}^+ + 1)}(q)\end{aligned} \tag{3.9}$$

for $L=2(\ell^+ + \ell^-)$ being even and odd respectively. The functions $X^{(\tilde{k}^+ +1)}_{2\ell,2\ell^+ +1;2\ell^-}(q)$ are unknown, except for $\tilde{k}^+ = 1$ where they are equal to 1. They obey the symmetry,

$$\chi^{(\tilde{k}^+ +1)}_{2\ell,2\ell^+ +1;2\ell^-}(q) = \chi^{(\tilde{k}^+ +1)}_{\tilde{k}^+ +1-2\ell,\tilde{k}^+ +1-2\ell^+,1-2\ell^-}(q) , \tag{3.10}$$

and there are as many such functions as Virasoro characters at level $m=\tilde{k}^+ +1$. (Recall that when $\tilde{k}^- = 1$, $2\ell^- = 0$ or 1 when $2\ell - 2\ell^+$ is even or odd respectively.) We define,

$$X^{(\tilde{k}^+ +1)}_{2\ell,2\ell^+ +1;2\ell^-}(q) = q^{(\ell-\ell^+)(\ell-\ell^+-1)+(\ell^-)^2} X^{(\tilde{k}^+ +1)}_{2\ell,2\ell^+ +1}(q) , \tag{3.11}$$

where the q expansion of the functions $X^{(\tilde{k}^+ +1)}_{2\ell,2\ell^+ +1}(q)$ is,

$$X^{(\tilde{k}^+ +1)}_{2\ell,2\ell^+ +1}(q) = q^{\frac{c}{24} - h_{2\ell,2\ell^+ +1}} \sum_{n=0}^{\infty} b_n q^n, \quad b_n \in Z ,$$

with the prefactor being the inverse of the corresponding level $m=\tilde{k}^+ +1$ Virasoro character.

Note that this relation (3.9) reduces to the GKO sumrule when $z_- = -z_+^{-1}$, as can be easily seen using the properties (3.3) and

$$\chi^k_{2\ell}(q, -z^{-1}) = (-1)^{2\ell} \chi^k_{2\ell}(q, z) .$$

The key observation which led us to the identities, (2.1) and (2.9), presented in the previous section is that the product of the determinants of the matrices $A^{(+)}$ and $A^{(-)}$ is equal to minus one when $\tilde{k}^+ = 1$. Indeed, the matrices $A^{(+)}$ and $A^{(-)}$ in this case are given by (suppressing the q dependence),

$$A^{(+)} = \begin{pmatrix} -Y^{(3)}_{1,1;0} & Y^{(3)}_{2,1;1} \\ -Y^{(3)}_{2,1;1} & Y^{(3)}_{1,1,0} \end{pmatrix}, \quad A^{(-)} = \begin{pmatrix} Y^{(3)}_{2,2;0} & -Y^{(3)}_{2,2;0} \\ 1 & 1 \end{pmatrix} , \tag{3.12}$$

and,

$$\det A^{(+)} \det A^{(-)} = \frac{1}{\chi^{\mathrm{Vir}(3)}_{2,2}(q)([\chi^{\mathrm{Vir}(3)}_{2,1}(q)]^2 - [\chi^{\mathrm{Vir}(3)}_{1,1}(q)]^2)} = -1 . \tag{3.13}$$

This can be seen by using the expressions (3.8) and the well known identity,

$$1 = \chi^{\mathrm{Vir}(3)}_{2,2}(q)(\chi^{\mathrm{Vir}(3)}_{1,1}(q) + \chi^{\mathrm{Vir}(3)}_{2,1}(q))(\chi^{\mathrm{Vir}(3)}_{1,1}(q) - \chi^{\mathrm{Vir}(3)}_{2,1}(q)) ,$$

which can be easily checked by using the infinite product representation of Ising model characters given in the Appendix. There is some evidence that the product $\det A^{(+)} \det A^{(-)}$ is a modular invariant [14] and we therefore conjecture the result,

$$\det A^{(+)} \det A^{(-)} = -1, \quad \forall \tilde{k}^+ \in N, \ \tilde{k}^- = 1 . \tag{3.14}$$

The identities of Sect. 2 enable us to prove it for $\tilde{k}^+ = 2$.

Indeed for $\tilde{k}^+ = 2$, we have, according to relations (3.5) and (3.9),

$$A^{(+)} = \begin{pmatrix} -Y^{(4)}_{1,4;1} & Y^{(4)}_{2,4;0} & -Y^{(4)}_{3,4;1} \\ -Y^{(4)}_{1,2;1} & Y^{(4)}_{2,2;0} & -Y^{(4)}_{3,2;1} \\ \chi^{\mathrm{Vir}(3)}_{1,1} & \chi^{\mathrm{Vir}(3)}_{1,2} & \chi^{\mathrm{Vir}(3)}_{1,3} \end{pmatrix} ,$$

$$[A^{(+)}]^{-1}=\begin{pmatrix} -\chi^{\mathrm{Vir}(4)}_{1,4} & -\chi^{\mathrm{Vir}(4)}_{1,2} & X^{(3)}_{1,1;1} \\ \chi^{\mathrm{Vir}(4)}_{2,4} & \chi^{\mathrm{Vir}(4)}_{2,2} & X^{(3)}_{1,2;0} \\ -\chi^{\mathrm{Vir}(4)}_{3,4} & -\chi^{\mathrm{Vir}(4)}_{3,2} & X^{(3)}_{1,3;1} \end{pmatrix},$$

$$A^{(-)}=\begin{pmatrix} Y^{(4)}_{1,2;1} & -Y^{(4)}_{2,2;0} & Y^{(4)}_{3,2;1} \\ Y^{(4)}_{1,4;1} & -Y^{(4)}_{2,4;0} & Y^{(4)}_{3,4;1} \\ \chi^{\mathrm{Vir}(3)}_{1,1} & \chi^{\mathrm{Vir}(3)}_{1,2} & \chi^{\mathrm{Vir}(3)}_{1,3} \end{pmatrix},$$

$$[A^{(-)}]^{-1}=\begin{pmatrix} \chi^{\mathrm{Vir}(4)}_{1,2} & \chi^{\mathrm{Vir}(4)}_{1,4} & X^{(3)}_{1,1;1} \\ -\chi^{\mathrm{Vir}(4)}_{2,2} & -\chi^{\mathrm{Vir}(4)}_{2,4} & X^{(3)}_{1,2;0} \\ \chi^{\mathrm{Vir}(4)}_{3,2} & \chi^{\mathrm{Vir}(4)}_{3,4} & X^{(3)}_{1,3;1} \end{pmatrix}. \tag{3.15}$$

Exploiting the fact that the matrices $A^{(+)}$ and $[A^{(+)}]^{-1}$ are inverses of each other, we may write,

$$\chi^{\mathrm{Vir}(3)}_{1,i}(q)=\frac{\varepsilon_{ijk}(-1)^{j+k}\,\chi^{\mathrm{Vir}(4)}_{j,4}(q)\,\chi^{\mathrm{Vir}(4)}_{k,2}(q)}{\det[A^{(+)}]^{-1}}, \tag{3.16}$$

where,

$$\det[A^{(+)}]^{-1}=\varepsilon_{abc}(-1)^{a+b}\,\chi^{\mathrm{Vir}(4)}_{a,4}(q)\,\chi^{\mathrm{Vir}(4)}_{b,2}(q)\;X^{(3)}_{1,c,\sigma}(q)$$

with $\sigma=0$ or 1 for $1-c$ odd or even respectively. By virtue of (2.1),

$$\det[A^{(+)}]^{-1}=1\,.$$

The result (3.14) follows from the observation that $\det A^{(-)}=-\det A^{(+)}$ when $\tilde{k}^{+}=2$.

When $\tilde{k}^{+}=3$, the situation is slightly different since the matrices $A^{(+)}$ and $A^{(-)}$ do not have the same determinant up to a sign. With matrices $A^{(+)}$ and $[A^{(+)}]^{-1}$ taken to be,

$$A^{(+)}=\begin{pmatrix} Y^{(5)}_{1,5;0} & -Y^{(5)}_{2,5;1} & Y^{(5)}_{3,5;0} & -Y^{(5)}_{4,5;1} \\ Y^{(5)}_{1,3;0} & -Y^{(5)}_{2,3;1} & Y^{(5)}_{3,3;0} & -Y^{(5)}_{4,3;1} \\ Y^{(5)}_{1,1;0} & -Y^{(5)}_{2,1;1} & Y^{(5)}_{3,1;0} & -Y^{(5)}_{4,1;1} \\ \chi^{\mathrm{Vir}(4)}_{2,1} & \chi^{\mathrm{Vir}(4)}_{2,2} & \chi^{\mathrm{Vir}(4)}_{2,3} & \chi^{\mathrm{Vir}(4)}_{2,4} \end{pmatrix},$$

$$[A^{(+)}]^{-1}=\begin{pmatrix} \chi^{\mathrm{Vir}(5)}_{1,5} & \chi^{\mathrm{Vir}(5)}_{1,3} & \chi^{\mathrm{Vir}(5)}_{1,1} & X^{(4)}_{2,1;0} \\ -\chi^{\mathrm{Vir}(5)}_{2,5} & -\chi^{\mathrm{Vir}(5)}_{2,3} & -\chi^{\mathrm{Vir}(5)}_{2,1} & X^{(4)}_{2,2;1} \\ \chi^{\mathrm{Vir}(5)}_{3,5} & \chi^{\mathrm{Vir}(5)}_{3,3} & \chi^{\mathrm{Vir}(5)}_{3,1} & X^{(4)}_{2,3;0} \\ -\chi^{\mathrm{Vir}(5)}_{4,5} & -\chi^{\mathrm{Vir}(5)}_{4,3} & -\chi^{\mathrm{Vir}(5)}_{4,1} & X^{(4)}_{2,4;1} \end{pmatrix}. \tag{3.17}$$

it is easy to set that,

$$\chi^{\mathrm{Vir}(4)}_{2,i}(q)=\frac{\varepsilon_{ijkl}(-1)^{j+k+l}\,\chi^{\mathrm{Vir}(5)}_{j,5}(q)\,\chi^{\mathrm{Vir}(5)}_{k,3}(q)\,\chi^{\mathrm{Vir}(5)}_{l,1}(q)}{\det[A^{(+)}]^{-1}}. \tag{3.18}$$

However, because the following bilinear in $m=5$ Virasoro characters is equal to 1 (see Appendix),

$$\varepsilon_{ab}(\chi_{a,1}^{\mathrm{Vir}(5)}(q)+\chi_{a,5}^{\mathrm{Vir}(5)}(q))\chi_{b,3}^{\mathrm{Vir}(5)}(q)=1\,, \tag{3.19}$$

one obtains from (3.18) that,

$$\chi_{2,1}^{\mathrm{Vir}(4)}(q)=\frac{\chi_{2,1}^{\mathrm{Vir}(5)}(q)-\chi_{2,5}^{\mathrm{Vir}(5)}(q)}{\det[A^{(+)}]^{-1}}\,,$$

$$\chi_{2,2}^{\mathrm{Vir}(4)}(q)=\frac{\chi_{1,1}^{\mathrm{Vir}(5)}(q)-\chi_{1,5}^{\mathrm{Vir}(5)}(q)}{\det[A^{(+)}]^{-1}}\,. \tag{3.20}$$

We have used the well known symmetry properties of unitary Virasoro characters,

$$\chi_{r,s}^{\mathrm{Vir}(m)}(q)=\chi_{m-r,m+1-s}^{\mathrm{Vir}(m)}(q)\,.$$

It is now straightforward to conclude from the identities (2.9) and the expression (3.20) that

$$\det[A^{(+)}]^{-1}=\frac{1}{\chi_{2,2}^{\mathrm{Vir}(3)}(q)}\,.$$

The number of unknown functions, $X_{r,s;\sigma}^{(4)}(q)$, in the matrix $[A^{(-)}]^{-1}$ is however too high to derive its determinant, even with the help of the identities (2.9). This precisely shows the limit of our approach to generate more identities of the type described in Sect. 2. Indeed, the higher value $\bar{k}^{+}$ takes, the higher number of unknown branching functions $Y_{r,s;\sigma}^{(\bar{k}^{+}+2)}(q)$ and functions $X_{r,s;\sigma}^{(\bar{k}^{+}+1)}(q)$ one gets. For instance, when $\bar{k}^{+}=4$, the fact that $A^{(+)}$ and $[A^{(+)}]^{-1}$ are inverse matrices implies that the $m=5$ Virasoro characters are obtained as,

$$\chi_{1,i}^{\mathrm{Vir}(5)}(q)=\frac{1}{\det[A^{(+)}]^{-1}}\,\varepsilon_{ijklm}(-1)^{j+k+l}\chi_{j,6}^{\mathrm{Vir}(6)}(q)\chi_{k,4}^{\mathrm{Vir}(6)}(q)\chi_{l,2}^{\mathrm{Vir}(6)}(q)X_{3,m;\sigma}^{(5)}(q)\,,$$

$$\chi_{3,i}^{\mathrm{Vir}(5)}(q)=\frac{1}{\det[A^{(+)}]^{-1}}\,\varepsilon_{ijklm}(-1)^{j+k+l}\chi_{j,6}^{\mathrm{Vir}(6)}(q)\chi_{k,4}^{\mathrm{Vir}(6)}(q)\chi_{l,2}^{\mathrm{Vir}(6)}(q)X_{1,m;\sigma}^{(5)}(q)\,, \tag{3.21}$$

so that potential identities between unitary Virasoro characters depend on the unknown functions $X_{r,s;\sigma}^{(5)}(q)$.

4. Conclusions

We have presented and proven two sets of identities between unitary minimal Virasoro characters at levels $m=3$, 4, 5. The first identity (2.1) suggests a strong connection between the Ising and the tricritical Ising models since the $m=3$ Virasoro characters are obtained as bilinears of $m=4$ Virasoro characters. It may also imply an as yet unknown mechanism which produces a conformal field theory with central charge $c=\frac{1}{2}$ when considering two copies of a $c=\frac{7}{10}$ theory in a much less trivial way than their tensor product. Such a mechanism would be a very interesting alternative to twisting the energy momentum tensor of a given theory in order to alter its central charge.

The second identity (2.9) is more involved since it gives the tricritical Ising model characters as bilinears in the Ising model characters and the six combinations of $m=5$ Virasoro characters which do not appear in the spectrum of the three state Potts model. A field theoretic interpretation of these identities would certainly shed new light on the underlying structure of minimal Virasoro theories.

It would also be important to investigate the generalisation of these identities to higher level Virasoro characters. Our approach, which involves the study of $N=4$ superconformal characters and their branching functions into $SU(2)\times SU(2)$ characters is quite limited at present due to the lack of information on the analytic structure of the branching functions. A more direct approach based on a deeper field theoretic understanding would undoubtedly reveal more relations between Virasoro characters of different levels.

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A. Appendix: Proof of the Identities

We give here a complete proof of the vectorial identity (2.1) between $m=3$ and $m=4$ unitary Virasoro characters, as well as a proof of one of the identities (2.9) between $m=3$, 4, 5 Virasoro characters. The other identities in this second set can be proven by similar methods. We use the infinite product representation of $m=3$ and $m=4$ characters, given for instance in [20],

$$\chi_{1,2}^{\mathrm{Vir}(3)}(q)=q^{\frac{1}{24}}\prod_{n=0}^{\infty}(1+q^{n+1}),$$

$$\chi_{1,1}^{\mathrm{Vir}(3)}(q)\pm\chi_{1,3}^{\mathrm{Vir}(3)}(q)=q^{-\frac{1}{48}}\prod_{n=0}^{\infty}(1\pm q^{n+\frac{1}{2}}), \tag{A.1}$$

and,

$$\chi_{2,1}^{\mathrm{Vir}(4)}(q)=q^{-\frac{7}{240}+\frac{7}{16}}\prod_{n=0}^{\infty}\frac{(1+q^{5n+1})(1+q^{5n+4})(1+q^{5n+5})}{(1-q^{5n+2})(1-q^{5n+3})},$$

$$\chi_{2,2}^{\mathrm{Vir}(4)}(q)=q^{-\frac{7}{240}+\frac{3}{80}}\prod_{n=0}^{\infty}\frac{(1+q^{5n+2})(1+q^{5n+3})(1+q^{5n+5})}{(1-q^{5n+1})(1-q^{5n+4})},$$

$$\chi_{1,1}^{\mathrm{Vir}(4)}(q)\pm\chi_{3,1}^{\mathrm{Vir}(4)}(q)=q^{-\frac{7}{240}}\prod_{n=0}^{\infty}\frac{(1\pm q^{5n+\frac{3}{2}})(1\pm q^{5n+\frac{5}{2}})(1\pm q^{5n+\frac{7}{2}})}{(1-q^{5n+2})(1-q^{5n+3})},$$

$$\chi_{1,2}^{\mathrm{Vir}(4)}(q)\pm\chi_{3,2}^{\mathrm{Vir}(4)}(q)=q^{-\frac{7}{240}+\frac{1}{10}}\prod_{n=0}^{\infty}\frac{(1\pm q^{5n+\frac{1}{2}})(1\pm q^{5n+\frac{5}{2}})(1\pm q^{5n+\frac{9}{2}})}{(1-q^{5n+1})(1-q^{5n+4})}, \tag{A.2}$$

The vectorial identity (2.1) is equivalent to the set of three scalar identities,

$$\chi_{1,2}^{\mathrm{Vir}(3)}(q)=\frac{1}{2}\Big([\chi_{1,1}^{\mathrm{Vir}(4)}(q)+\chi_{3,1}^{\mathrm{Vir}(4)}(q)]\,[\chi_{1,2}^{\mathrm{Vir}(4)}(q)-\chi_{3,2}^{\mathrm{Vir}(4)}(q)]$$

$$+[\chi_{1,1}^{\mathrm{Vir}(4)}(q)-\chi_{3,1}^{\mathrm{Vir}(4)}(q)]\,[\chi_{1,2}^{\mathrm{Vir}(4)}(q)+\chi_{3,2}^{\mathrm{Vir}(4)}(q)]\Big), \tag{A.3}$$

$$\chi_{1,1}^{\mathrm{Vir}(3)}(q)\pm\chi_{1,3}^{\mathrm{Vir}(3)}(q)=\chi_{2,2}^{\mathrm{Vir}(4)}(q)\left[\chi_{1,1}^{\mathrm{Vir}(4)}(q)\mp\chi_{3,1}^{\mathrm{Vir}(4)}(q)\right] \pm\chi_{2,1}^{\mathrm{Vir}(4)}(q)\left[\chi_{1,2}^{\mathrm{Vir}(4)}(q)\mp\chi_{3,2}^{\mathrm{Vir}(4)}(q)\right]. \tag{A.4}$$

When using the product representation (A.1) and (A.2), Eqs. (A.3) and (A.4) may be rewritten as,

$$2\prod_{n=0}^{\infty}\frac{(1+q^{n+1})(1-q^{n+1})}{(1-q^{10n+5})(1-q^{5n+5})}=\prod_{n=0}^{\infty}(1+q^{5n+\frac{3}{2}})(1+q^{5n+\frac{7}{2}})(1-q^{5n+\frac{1}{2}})(1-q^{5n+\frac{9}{2}}) +\prod_{n=0}^{\infty}(1-q^{5n+\frac{3}{2}})(1-q^{5n+\frac{7}{2}})(1+q^{5n+\frac{1}{2}})(1+q^{5n+\frac{9}{2}}), \tag{A.5}$$

and,

$$\prod_{n=0}^{\infty}\frac{(1\pm q^{n+\frac{1}{2}})(1-q^{n+1})}{(1-q^{5n+5})(1+q^{5n+5})(1\mp q^{5n+\frac{5}{2}})} =\prod_{n=0}^{\infty}(1+q^{5n+2})(1+q^{5n+3})(1\mp q^{5n+\frac{3}{2}})(1\mp q^{5n+\frac{7}{2}}) \pm q^{\frac{1}{2}}\prod_{n=0}^{\infty}(1+q^{5n+1})(1+q^{5n+4})(1\mp q^{5n+\frac{1}{2}})(1\mp q^{5n+\frac{9}{2}})\,. \tag{A.6}$$

The Jacobi triple product identity,

$$\sum_{n=-\infty}^{+\infty} z^n q^{n^2}=\prod_{n=0}^{\infty}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1})\,, \tag{A.7}$$

with $q\to q^{\frac{5}{2}}$, $z\to\pm q^{\frac{5}{2}-i}$ for $i=\frac{1}{2},\frac{3}{2},1$ and 3, allows us to rewrite (A.5) and (A.6) as,

$$2\prod_{n=0}^{\infty}\frac{(1+q^{n+1})(1-q^{n+1})(1-q^{5n+5})}{(1-q^{10n+5})} =\left[\sum_{n=-\infty}^{+\infty} q^{\frac{5}{2}n^2+n}\right]\left[\sum_{m=-\infty}^{+\infty}(-1)^m q^{\frac{5}{2}m^2+2m}\right] +\left[\sum_{n=-\infty}^{+\infty} q^{\frac{5}{2}n^2+2n}\right]\left[\sum_{m=-\infty}^{+\infty}(-1)^m q^{\frac{5}{2}m^2+m}\right] =2q^{-\frac{1}{2}}\left[\theta_{2,10}(q)\theta_{4,10}(q)-\theta_{6,10}(q)\theta_{8,10}(q)\right]\,, \tag{A.8}$$

and,

$$\prod_{n=0}^{\infty}\frac{(1\pm q^{n+\frac{1}{2}})(1-q^{n+1})(1-q^{5n+5})}{(1+q^{5n+5})(1\mp q^{5n+\frac{5}{2}})} =\left[\sum_{n=-\infty}^{+\infty} q^{\frac{5n^2}{2}+\frac{n}{2}}\right]\left[\sum_{m=-\infty}^{+\infty}(\mp 1)^m q^{\frac{5}{2}m^2+m}\right] \pm q^{\frac{1}{2}}\left[\sum_{n=-\infty}^{+\infty} q^{\frac{5}{2}n^2+\frac{3n}{2}}\right]\left[\sum_{m=-\infty}^{+\infty}(\mp 1)^m q^{\frac{5}{2}m^2+2m}\right]$$

$$=q^{-\frac{5}{40}}([\theta_{1,10}(q)+\theta_{9,10}(q)]\,[\theta_{2,10}(q)\mp\theta_{8,10}(q)]$$
$$\pm[\theta_{3,10}(q)+\theta_{7,10}(q)]\,[\theta_{4,10}(q)\mp\theta_{6,10}(q)])\,, \tag{A.9}$$

respectively.

The product of two theta functions at levels k and k' is

$$\theta_{m,k}(q)\,\theta_{m',k'}(q)=\sum_{\ell=1}^{k+k'}\theta_{mk'-m'k+2\ell kk',kk'(k+k')}(q)\,\theta_{m+m'+2\ell k,k+k'}(q)\,. \tag{A.10}$$

In particular, we have,

$$\theta_{2,10}(q)\,\theta_{4,10}(q)-\theta_{6,10}(q)\,\theta_{8,10}(q)=[\theta_{2,20}(q)-\theta_{18,20}(q)]\,[\theta_{6,20}(q)-\theta_{14,20}(q)]$$
$$=\eta^2(q)\chi^{\mathrm{Vir}(4)}_{2,1}(q)\,\chi^{\mathrm{Vir}(4)}_{2,2}(q)\,, \tag{A.11}$$

and

$$[\theta_{1,10}(q)+\theta_{9,10}(q)]\,[\theta_{2,10}(q)\mp\theta_{8,10}(q)]$$
$$\pm[\theta_{3,10}(q)+\theta_{7,10}(q)]\,[\theta_{4,10}(q)\mp\theta_{6,10}(q)]$$
$$=[\theta_{3,20}(q)\mp\theta_{17,20}(q)\pm\theta_{7,20}(q)-\theta_{13,20}(q)]$$
$$\times[\theta_{1,20}(q)\mp\theta_{19,20}(q)\pm\theta_{11,20}(q)-\theta_{9,20}(q)]$$
$$=\eta^2(q)[\chi^{\mathrm{Vir}(4)}_{1,1}(q)\pm\chi^{\mathrm{Vir}(4)}_{3,1}(q)]\,[\chi^{\mathrm{Vir}(4)}_{1,2}(q)\pm\chi^{\mathrm{Vir}(4)}_{3,2}(q)]\,. \tag{A.12}$$

We repeatedly used the relation,

$$\theta_{m,20}(q)=\sum_{p=-\infty}^{+\infty}\sum_{\ell=0}^{9}q^{20(10p+\ell)^2+(10p+\ell)m+\frac{m^2}{80}}=\sum_{\ell=0}^{9}\theta_{10m+400\ell,2000}(q)\,, \tag{A.13}$$

and the definition of Virasoro characters (2.2). Using the product representation of level $m=4$ Virasoro characters once more, it is very easy to see from (A.11) and (A.12) that the identities (2.1) hold.

We now proceed to prove the identity (2.9),

$$\chi^{\mathrm{Vir}(4)}_{2,2}(q)=\chi^{\mathrm{Vir}(3)}_{2,2}(q)(\chi^{\mathrm{Vir}(5)}_{1,1}(q)-\chi^{\mathrm{Vir}(5)}_{4,1}(q))\,, \tag{A.14}$$

which we rewrite using (A.1) and (A.2) and the definition of $m=5$ Virasoro characters in terms of theta functions as,

$$\prod_{n=0}^{\infty}(1-q^{n+1})=q^{-\frac{1}{120}}\prod_{n=0}^{\infty}(1+q^{5n+1})(1+q^{5n+4})(1-q^{5n+1})(1-q^{5n+4})$$
$$\times[\theta_{1,30}(q)+\theta_{29,30}(q)-\theta_{11,30}(q)-\theta_{19,30}(q)]\,. \tag{A.15}$$

The infinite product on the RHS can be expressed as a product of level $k=30$ theta functions after use of the Jacobi triple identity,

$$q^{-\frac{1}{120}}\prod_{n=0}^{\infty}(1+q^{5n+1})(1+q^{5n+4})(1-q^{5n+1})(1-q^{5n+4})$$
$$=q^{-4+\frac{5}{24}}\prod_{n=0}^{\infty}(1-q^{15n+15})^{-6}\prod_{r=0}^{2}[\theta^2_{27-10r,30}(q)-\theta^2_{3+10r,30}(q)]\,, \tag{A.16}$$

while the LHS is given by,

$$\prod_{n=0}^{\infty}(1-q^{n+1})=q^{-4+\frac{5}{24}}\prod_{n=0}^{\infty}(1-q^{15n+15})^{-6}\prod_{r=0}^{6}[\theta_{2r+1,30}(q)-\theta_{29-2r,30}(q)] . \tag{A.17}$$

The identity to prove then reduces to,

$$\prod_{r=0}^{2}[\theta_{9+10r,30}(q)-\theta_{21-10r,30}(q)][\theta_{5,30}(q)-\theta_{25,30}(q)]$$
$$=\prod_{r=0}^{2}[\theta_{7+10r,30}(q)+\theta_{23-10r,30}(q)][\theta_{1,30}(q)+\theta_{29,30}(q)-\theta_{11,30}(q)-\theta_{19,30}(q)], \tag{A.18}$$

which is equivalent to,

$$\begin{aligned}[\theta_{2,15}(q)\theta_{5,15}(q)+\theta_{10,15}(q)\theta_{13,15}(q)][\theta_{1,15}(q)\theta_{12,15}(q)+\theta_{3,15}(q)\theta_{14,15}(q)]\\ =[\theta_{3,15}(q)\theta_{10,15}(q)+\theta_{5,15}(q)\theta_{12,15}(q)]\\ [\theta_{4,15}(q)\theta_{7,15}(q)+\theta_{8,15}(q)\theta_{11,15}(q)] ,\end{aligned} \tag{A.19}$$

as can be seen by using the product of two theta functions at level 30 (A.10) as well as the standard properties,

$$\theta_{m,60}(q)=\sum_{\ell=0}^{29}\theta_{30m+3600\ell,54000}(q) , \tag{A.20}$$

and,

$$\theta_{m,15}(q)=\theta_{2m,60}(q)+\theta_{60-2m,60}(q) . \tag{A.21}$$

This last equality (A.19) can be derived by considering the product,

$$\begin{aligned}&[\theta_{3,30}(q)+\theta_{27,30}(q)][\theta_{7,30}(q)+\theta_{23,30}(q)]\\ &[\theta_{11,30}(q)+\theta_{19,30}(q)][\theta_{13,30}(q)+\theta_{17,30}(q)] ,\end{aligned} \tag{A.22}$$

and using (A.10), (A.20) and (A.21) by pairing the first and second factors together (and the third and fourth factors) and then by pairing the first and third factors together. This completes the proof of (A.14). The other identities of (2.9) can be proved along similar lines.

Finally we wish to show that the bilinear in $m=5$ Virasoro characters (3.19) is equal to 1,

$$\varepsilon_{ab}(\chi_{a,1}^{\mathrm{Vir}(5)}(q)+\chi_{a,5}^{\mathrm{Vir}(5)}(q))\chi_{b,3}^{\mathrm{Vir}(5)}(q)=1 . \tag{A.23}$$

From the definition of Virasoro characters in terms of theta functions (2.2) and using the same techniques as above, we must prove that,

$$\begin{aligned}\eta^2(q)=[\theta_{1,15}(q)-\theta_{11,15}(q)][\theta_{2,15}(q)-\theta_{8,15}(q)]\\ -[\theta_{4,15}(q)-\theta_{14,15}(q)][\theta_{7,15}(q)-\theta_{13,15}(q)] .\end{aligned} \tag{A.24}$$

However, the Euler pentagonal identity [1] gives an expression of the Dedekind function in terms of level 6 theta functions,

$$\eta(q)=q^{\frac{1}{24}}\sum_{m=-\infty}^{+\infty}(-1)^m q^{\frac{3}{2}m^2+\frac{1}{2}m}=\theta_{1,6}(q)-\theta_{5,6}(q) , \tag{A.25}$$

and therefore,

$$\eta^2(q)=[\theta_{1,6}(q)-\theta_{5,6}(q)]^2$$

$$
\begin{aligned}
&= \theta_{0,3}(q)\,\theta_{1,3}(q) - \theta_{2,3}(q)\,\theta_{3,3}(q) \\
&= [\theta_{0,12}(q) + \theta_{12,12}(q)]\,\theta_{1,3}(q) - [\theta_{4,12}(q) + \theta_{8,12}(q)]\,\theta_{3,3}(q) \\
&= [\theta_{1,15}(q) - \theta_{11,15}(q)]\,[\theta_{2,15}(q) - \theta_{8,15}(q)] \\
&\quad - [\theta_{4,15}(q) - \theta_{14,15}(q)]\,[\theta_{7,15}(q) - \theta_{13,15}(q)]\,, \qquad \text{(A.26)}
\end{aligned}
$$

where the product formula (A.10) and (A.21) have been used.

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