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Bounds on the Growth of the Support of a Vortex Patch

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Abstract: We study the time evolution of the support of a vortex patch evolving in $\mathbb{R}^2$ according to the Euler Equation for an incompressible fluid and we bound its growth. Furthermore we discuss the same problem in the framework of a simplified model. Finally we consider a similar problem for the Navier-Stokes flow.

1. Introduction

In this paper we study the behavior of a non-viscous incompressible fluid in $\mathbb{R}^2$. In particular we consider the so-called vortex patch, that is a system in which the vorticity $\omega(x, 0)$ is proportional to the characteristic function χ of a region Λ_0:

$$\omega(x, 0) = a\chi(\Lambda_0) \qquad a \in \mathbb{R}\,. \tag{1.1}$$

We suppose that initially Λ_0 has a bounded diameter $2R_0$. Then we evolve the vorticity by means of the Euler Equations: $\omega(x, 0) \to \omega(x, t)$. As it is well known $\omega(x, t)$ has the form:

$$\omega(x, t) = a\chi(\Lambda_t)\,. \tag{1.2}$$

Denote by $2R_t$ the diameter of Λ_t. We want to control its growth in time. This problem is interesting both for theoretical reasons, to understand deeply the dynamical behavior of the Euler equation and for applied ones (see for instance pollution problems).

In the present paper we want to find α, b for which we have:

$$R_t \le (R_0^{1/\alpha} + bt)^\alpha \quad \text{for } t \ge 0\,. \tag{1.3}$$

It is trivial to observe that the boundedness of the velocity of the flow particles assures that Eq. (1.3) holds with $\alpha = 1$; however this bound is very bad and not so interesting. We want to improve this estimate. We observe that the main part of the

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vorticity remains in a bounded region (as consequence of stability results for circular vortex patch, see [WaP85, MaP85, Dri88]), but thin filaments have a complicate motion and may be pushed away. On a particle of the extreme part of the filament mainly act two fields: the first is due to the vorticity near the center, the second is due to the vorticity contained in a region close to the particle. The first one is easily controlled and gives a velocity field less than a constant times R_t^{-2}. The second is more complicated as we shall see later. In any case, also if we neglect the last term, we cannot obtain a bound better than

$$\frac{d}{dt} R_t \leq \frac{\text{constant}}{R_t^2}, \tag{1.4}$$

which implies Eq. (1.3) with $\alpha = \frac{1}{3}$. This seems the optimal estimate that we can find using the general argument only. In the present paper we prove exactly this bound.

This result is valid for *any* initial configuration. For particular initial data the bound on R_t might be better. Actually different parts of the vortex patch could turn around with different speeds so that a sort of homogenization happens asymptotically in time. As a consequence, the radial velocity decreases in time faster as the circular symmetry is reached. Moreover, when a particle of a filament is far from the center of vorticity, it moves slowly, so that the freedom degrees of particles near the center are "fast variables" for it. Their action can be averaged on a large interval of time. This is a further effect of homogenization (temporal homogenization). An example of this effect arises in the so-called Kirchhoff ellipse (see the appendix). In conclusion, if the homogenization happens, it is natural to suppose that $\alpha \ll \frac{1}{3}$. We believe that the main part of the initial condition gives rise to some homogenization, but it is very difficult to prove it and to exclude that some "resonant" situation could happen for which bound (1.3), with $\alpha = \frac{1}{3}$, is the best possible. In the next section we obtain bounds valid for *any* initial data and we do not take care of this interesting but difficult phenomenon. In Sect. 3 we discuss a simplified model more easy to investigate that exhibits this homogenization effect. For numerical pictures see [Dri89] and references quoted therein.

Finally in Sect. 4 we study the problem for a Navier-Stokes flow. In this case the viscosity produces an immediate diffusion and the vorticity has an unbounded support. However we can study the growth of a region out of which the vorticity mass is exponentially negligible. We find a bound analogous to that obtained in the pure diffusive case.

2. Euler Flow

Consider an incompressible non-viscous fluid of unitary density moving in $\mathbb{R}^2$. The Euler Equation in term of the vorticity reads:

$$\partial_t \omega(x,t) + (u \cdot \nabla)\omega(x,t) = 0\,, \tag{2.1}$$

$$\nabla \cdot u(x,t) = 0\,, \tag{2.2}$$

$$\omega \equiv \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1\,, \qquad \omega(x,t) = \omega_0\,, \qquad x \equiv (x_1, x_2) \in \mathbb{R}^2\,. \tag{2.3}$$

Here $u = (u_1, u_2)$ denotes the velocity field.

If u decays at infinity, as we suppose in this paper, we can reconstruct the velocity field by means of ω as

$$u(x,t) = \int K(x-y)\omega(x,t)\,dy\,, \tag{2.4}$$

$$K = \nabla^{\perp} G(x)\,, \tag{2.5}$$

where

$$\nabla^{\perp} \equiv (\partial_2, -\partial_1) \tag{2.6}$$

and

$$G(x) = -\frac{1}{2\pi} \ln|x|\,. \tag{2.7}$$

As well known, Eq. (2.1) means that the vorticity is constant along the particle paths, which are the characteristic of the Euler Equation. Therefore

$$\omega(x,t) = \omega(x_0(x,-t),0)\,, \tag{2.8}$$

where the trajectory $x(x_0,t)$ of the fluid particle initially in x_0 satisfies:

$$\frac{d}{dt} x(x_0,t) = u(x(x_0,t),t)\,, \qquad x(x_0,0) = x_0\,, \tag{2.9}$$

$$u(x,t) = \int K(x-y)\omega(x,t)\,dy\,. \tag{2.10}$$

We want to study the Euler Equation when the initial data have the form

$$\omega_0 = a\chi(\Lambda_0)\,, \qquad a \in \mathbb{R} \tag{2.11}$$

and $\chi(\Lambda_0)$ denotes the characteristic function of $\Lambda_0 \subset \mathbb{R}^2$. Then we need a weak formulation of Eqs. (2.1), (2.2), (2.3) which is given by Eqs. (2.8), (2.9), (2.10) or by the equivalent form

$$\frac{d}{dt} \omega_t[f] = \omega_t[u \cdot \nabla f]\,, \tag{2.12}$$

where f is a smooth function and

$$\omega_t[f] \equiv \int f(x)\omega(x,t)\,dx\,. \tag{2.13}$$

In conclusion

$$\omega(x,t) = a\chi(\Lambda_t)\,, \tag{2.14}$$

where Λ_t is the time evolution of Λ_0 by Eqs. (2.9), (2.10), i.e.

$$x(x_0,t) \in \Lambda_t \quad \text{iff} \quad x_0 \in \Lambda_0\,. \tag{2.15}$$

As well known the Lebesgue measure of Λ_t is equal to the measure of Λ_0. (For the standard results on the Euler Equation see, for instance, [MaP94]).

We prove the following result:

Theorem 2.1. *Suppose that*

$$\Lambda_0 \subset \Sigma(R_0)\,, \tag{2.16}$$

where $\Sigma(R)$ in the circle with center in the origin and radius R. Then for any $\alpha \geq \frac{1}{3}$ there is a constant $b > 0$ such that

$$\Lambda_t \subset \Sigma(R_t) \tag{2.17}$$

with

$$R_t \leq (R_0^{1/\alpha} + bt)^\alpha \quad \textit{for} \quad t \geq 0\,. \tag{2.18}$$

(Of course the sharpest bound is obtained when $\alpha = \frac{1}{3}$.)

The technique of the proof is inspired to the papers [Mar88, MaP92], developed however in a different context.

The strategy of the proof is the following: we find an upper bound for the radial component of the velocity field u_r computed in a point x such that $|x| = R_t$. In particular we prove that

$$|u_r| \leq R_t^{-\beta}\,. \tag{2.19}$$

(from now on C denotes a positive constant). Then

$$\frac{d}{dt} R_t \leq C R_t^{-\beta}\,. \tag{2.20}$$

This differential inequality implies Eq. (2.18) with

$$\alpha = \frac{1}{1+\beta}\,. \tag{2.21}$$

We remark that the bound (2.18) becomes more sharp as α decreases (i.e. β increases). So α must be as small as possible. We shall see that for $\alpha = \frac{1}{2}$ (i.e. $\beta = 1$) the proof is quite simple, while the general case is more inolved.

Proof for $\alpha = \frac{1}{2}$.

From now on for simplicity we suppose $a = 1$ and meas $\Lambda_0 = 1$.

We write

$$u_r = \frac{x}{|x|} \cdot u(x,t) = \frac{x}{|x|} \cdot \int K(x-y)\omega(x,t)\,dy = \frac{x}{|x|} \cdot \int\limits_{\Lambda_t} K(x-y)\,dy\,. \tag{2.22}$$

Hence

$$|u_r| \leq \left| \int\limits_{\Lambda_t} K(x-y)\,dy \right|. \tag{2.23}$$

We divide the integration domain Λ_t into two sets:

$$A_t = \Sigma\left(\frac{R_t}{2}\right) \cap \Lambda_t \tag{2.24}$$

and

$$A_2 = \Sigma^c\left(\frac{R_t}{2}\right) \cap \Lambda_t \tag{2.25}$$

(where Σ^c is the complement of Σ). Then

$$\int\limits_{\Lambda_t} K(x-y)\,dy = \int\limits_{A_1} + \int\limits_{A_2}\,. \tag{2.26}$$

The first integral is easily evaluated by using the obvious inequality

$$|K(x)| \le C|x|^{-1}. \tag{2.27}$$

Hence

$$\left| \int_{A_1} K(x-y)\,dy \right| \le \frac{C}{R_t}. \tag{2.28}$$

To evaluate the second integral we observe that a bound on it is obtained considering the integral performed on a circular domain centered on x and with area equal to meas A_2:

$$\left| \int_{A_2} K(x-y)\,dy \right| < \left| \int_{\eta} K(-y)\,dy \right| = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{\eta} \frac{1}{|y|}\,dy = \eta, \tag{2.29}$$

where

$$\pi\eta^2 = \text{meas}\, A_2 \Rightarrow \eta = \sqrt{\frac{\text{meas}\, A_2}{\pi}}. \tag{2.30}$$

We denote by $m_t(r)$ the vorticity out of $\Sigma(r)$ at time t. Then we use the fact that the quantity

$$I = \int \omega(x,t)x^2\,dx \quad \text{(moment of inertea)} \tag{2.31}$$

is conserved during the motion (as can be verified by direct computation).

We have trivially

$$I \ge r^2 m_t(r) \tag{2.32}$$

which implies

$$m_t(r) \le Cr^{-2}. \tag{2.33}$$

Observing that $m_t\left(\frac{R}{2}\right) = \text{meas}\, A_2$, inserting Eq. (2.33) in Eq. (2.29), from Eq. (2.23) we have

$$|u_r| \le C\frac{1}{R_t} \tag{2.34}$$

that is Eq. (2.19) with $\beta = 1$ $\left(\text{i.e. } \alpha = \frac{1}{2}\right)$. □

Proof of Theorem 2.1. Define

$$r_t = (R_0^3 + b_1 t)^{1/3}, \tag{2.35}$$

where b_1 shall be fixed later.

Let $t^* > 0$ a time such that for any t, $0 \le t \le t^*$, the inequality $R_t \le r_t$ holds. (t^* is surely larger than zero for b_1 large because of the boundness of the velocity field). Then we prove in $|x| = r_t$ a bound like (2.19) with $\beta = 2$ with $\beta = 2$ and so a bound like (2.18) with $\alpha = \frac{1}{3}$ with the same b_1. In the proof b_1 is obtained independent of t^*. Hence $t^* \to \infty$ and the theorem is proved for $b = b_1$.

To evaluate the radial velocity field in x, $|x| = r_t$, we divide the circle $\Sigma(r_t)$ into many different annulii:

$$\Sigma(r_t) = \sum_{k=1}^{k^*} [\Sigma(a_k) - \Sigma(a_{k-1})] \cup [\Sigma(r_t) - \Sigma(a_{k^*})], \tag{2.36}$$

where

$$\begin{gathered} a_0 = 0, \qquad a_1 = R_0, \qquad a_k = 2a_{k-1}, \\ k^* \text{ is such that } a_{k^*+1} \le r_t \text{ and } a_{k^*+2} > r_t. \end{gathered} \tag{2.37}$$

The radial velocity in x in given by Eq. (2.22). It can be expressed by the sum of the contributions obtained when the vorticity is contained in each annulus:

$$\begin{aligned} &\frac{x}{|x|} \cdot \int_{[\Sigma(a_k) - \Sigma(a_{k-1})] \cap \Lambda_t} K(x-y)\, dy \\ &= \frac{x}{|x|} \cdot \int_{[\Sigma(a_k) - \Sigma(a_{k-1})] \cap \Lambda_t} K(x)\, dy \\ &\quad + \frac{x}{|x|} \cdot \int_{[\Sigma(a_k) - \Sigma(a_{k-1})] \cap \Lambda_t} [K(x-y) - K(x)]\, dy. \end{aligned} \tag{2.38}$$

The first term in the right-hand side of Eq. (2.38) vanishes because of $x \cdot K(x) = 0$. Moreover, by the explicit form of $K(x)$, we have

$$|K(x-y) - K(x)| < C \frac{\gamma}{r_t(r_t - \gamma)} \quad \text{if} \quad |y| < \gamma. \tag{2.39}$$

Hence

$$\begin{aligned} &\left| \frac{x}{|x|} \cdot \int_{[\Sigma(a_k) - \Sigma(a_{k-1})] \cap \Lambda_t} [K(x-y) - K(x)]\, dy \right| \\ &\le C \frac{a_k}{r_t(r_t - a_k)} \left| \int_{[\Sigma(a_k) - \Sigma(a_{k-1})] \cap \Lambda_t} dy \right. \\ &\quad \text{(by estimate (2.33))} \\ &\le \frac{C}{a_k r_t(r_t - a_k)} = \frac{C}{2^{k-1} r_t(r_t - 2^{k-1} R_0)} \end{aligned} \tag{2.40}$$

We put Eq. (2.40) into Eq. (2.38) and we have

$$\sum_{k=1}^{k^*} \left| \frac{x}{|x|} \cdot \int_{[\Sigma(a_k) - \Sigma(a_{k-1})] \cap \Lambda_t} K(x-y)\, dy \right| \le \frac{C}{r_t^2} \sum_{k=1}^{k^*} 2^{-k} \le \frac{C}{r_t^2}. \tag{2.41}$$

It remains to evaluate the integral (2.22) in the domain $[\Sigma(r_t) - \Sigma(a_{k^*})] \cap \Lambda_t$. It is easy to do in the region in which the integrand is bounded. In fact its contribution is smaller than Cr_t^{-2}, as it follows from estimate (2.33) which reads

$$m_t(a_{k^*}^{-2}) < C a_{k^*}^{-2}, \tag{2.42}$$

and the obvious observation that $a_{k^*} > \frac{r_t}{4}$. When the integrand becomes unbounded we need a more accurate estimate on $m_t(a_{k^*})$. To evaluate $m_t(a_{k^*})$ at time t, we prove that it depends on $m_t(a_{k^*} - R_0)$ at a time before and so on, starting an iterative procedure. More precisely we introduce a function $W_R \in C^\infty(\mathbb{R}^2)$, $r \to W_R(r)$ depending only on $|r|$, not increasing in $|r|$,

$$W_R(r) = \begin{matrix} 1 & \text{if} & |r| \leq R \\ 0 & \text{if} & |r| > R + R_0 \end{matrix}, \tag{2.43}$$

and such that for some $C_1 > 0$,

$$|\nabla W_R(r)| \leq C_1 , \tag{2.44}$$

$$|\nabla W_R(r) - \nabla W_R(r')| \leq C_1 |r - r'| . \tag{2.45}$$

Then we define a smooth version of $m_t(R)$:

$$\mu_t(R) = 1 - \int W_R(x)\omega(x,t)\, dx . \tag{2.46}$$

Hence, using the Euler equation (2.12), we have:

$$\begin{aligned} \frac{d}{dt}\mu_t(R) &= -\int_{\Lambda_t} dx\, u(x,t)\cdot \nabla W_R(x) \\ &= -\int_{\Lambda_1} dx \int_{\Lambda_t} dy\, \nabla W_R(x)\cdot K(x-y) \text{ (by the antisymmetry of } K) \\ &= -\frac{1}{2}\int_{\Lambda_t} dx \int_{\Lambda_t} dy\, [\nabla W_R(x) - \nabla W_R(y)]\cdot K(x-y) . \end{aligned} \tag{2.47}$$

We want to estimate this term for $R = nR_0$, $n > 1$. We divide the integration domain into the following sets:

$$T_h = \{(x,y) \in \Lambda_t \times \Lambda_t \,|\, x \notin \Sigma(nR_0), y \in [\Sigma(a_h) - \Sigma(a_{h-1})]\} \quad \text{if} \quad h < k , \tag{2.48}$$

$$T_h = \{(x,y) \in \Lambda_t \times \Lambda_t \,|\, x \notin \Sigma(nR_0), y \notin \Sigma(a_{k-1})\} \quad \text{if} \quad h = k \tag{2.49}$$

$$S_h = \{(x,y) \in \Lambda_t \times \Lambda_t \,|\, y \notin \Sigma(nR_0), x \in [\Sigma(a_h) - \Sigma(a_{h-1})]\} \quad \text{if} \quad h < k , \tag{2.50}$$

$$S_h = \{(x,y) \in \Lambda_t \times \Lambda_t \,|\, y \notin \Sigma(nR_0), x \notin \Sigma(a_{k-1})\} \quad \text{if} \quad h = k , \tag{2.51}$$

where a_h are defined in Eq. (2.37).

We choose k such that $a_{k+1} \leq nR_0$ and $a_{k+2} > nR_0$.

Notice that the integrand in Eq. (2.47) vanishes in the complement of $\bigcup_{h=1}^{k} (T_h \cup S_h)$.

Thanks to the identity $\nabla W_{nR_0}(x) \cdot K(x) = 0$ and the fact that $\nabla W_{nR_0}(y) = 0$ if $y \in [\Sigma(a_h) - \Sigma(a_{h-1})]$, $h < k$, the contribution on the integral (2.47) due to T_h,

$h < k$ is bounded by

$$\frac{1}{2}\left|\int_{\Lambda_t} dx \int_{\Lambda_t \cap [\Sigma(a_h) - \Sigma(a_{h-1})]} dy\, \nabla W_{nR_0}(x) \cdot [K(x-y) - K(y)]\right|$$
$$\leq \text{(by (2.44), the fact that } \nabla W_{nR_0}(x) = 0 \text{ if } |x| < nR_0 \text{ and inequality (2.39))}$$
$$\leq C\,\frac{2^h}{(nR_0)^2}\,[m_t(\Sigma(a_h)) - m_t(\Sigma(a_{h-1}))]m_t(nR_0) \leq \text{(by (2.33))}$$
$$\leq C\,\frac{m_t(nR_0)}{(nR_0)^2 2^h}\,. \tag{2.52}$$

To estimate the contribution due to T_k, we use the obvious inequality $|K(x)| \leq C|x|^{-1}$, Eq. (2.45) and the bound

$$|\{\nabla W_R(x) - \nabla W_R(y)\} \cdot K(x-y)| \leq C\,, \tag{2.53}$$

and we obtain that it is smaller than

$$C\,\frac{m_t(nR_0)}{(nR_0)^2}\,. \tag{2.54}$$

In conclusion, let $A_3 = \bigcup_{h=1}^{k} T_h$, then

$$\left|\int_{A_3} dx\, dy\, [\nabla W_{nR_0}(x) - \nabla W_{nR_0}(y)] \cdot K(x-y)\right|$$
$$\leq C\,\frac{m_t(nR_0)}{(nR_0)^2}\sum_{h=1}^{k} 2^{-h} < C\,\frac{m_t(nR_0)}{(nR_0)^2}\,. \tag{2.55}$$

The terms due to $\bigcup_{h=1}^{k} S_h$ can be handed exactly in the same way. Hence

$$\left|\frac{d}{dt}\,\mu_t(nR_0)\right| \leq C\,\frac{m_t(nR_0)}{(nR_0)^2}\,. \tag{2.56}$$

We observe now that trivially

$$m_t(nR_0) \leq \mu_t((n-1)R_0)\,. \tag{2.57}$$

Hence the identity

$$\mu_t(nR_0) = \mu_0(nR_0) + \int_0^t d\tau\,\frac{d}{d\tau}\,\mu_\tau(nR_0) \tag{2.58}$$

gives, by Eqs. (2.56), (2.57),

$$\mu_t(nR_0) \leq C\int_0^t d\tau\,\frac{\mu_t((n-1)R_0)}{(nR_0)^2} \tag{2.59}$$

because $\mu_0(nR_0) = 0$ for $n \geq 1$.

We are now able to apply an iterative procedure.

We start from $n = 2^{k^*-1} - 1$ and we arrive at 1. We apply many times Eq. (2.59) and then Eq. (2.57). We have

$$m_t(a_{k^*}) \le \frac{C^n t^n}{(n!)^3}. \tag{2.60}$$

Since

$$t < r_t^3 b_1^{-3} \tag{2.61}$$

and

$$r_t < Cn, \tag{2.62}$$

using the Stirling formula

$$\ln n! > n(\ln n - 1) \tag{2.63}$$

we have

$$m_t(a_{k^*}) < \frac{C^n n^{3n}}{b_1^n (n!)^3} < C^n b_1^{-n}. \tag{2.64}$$

We choose b_1 large enough (independently of n) to obtain

$$m_t(a_{k^*}) < C^n b_1^{-n} < C_2 n^{-4}. \tag{2.65}$$

We have so obtained an accurate bound on $m_t(a_{k^*})$. To evaluate the radial field in x, $|x| = r_t$, due to the vorticity out of $\Sigma(a_{k^*})$ we use the same technique of Eq. (2.29) and we prove that it is smaller than

$$C_3(a_{k^*})^{-2} < C_4 r_t^{-2}. \tag{2.66}$$

In conclusion adding estimate (2.41) to estimate (2.66), we obtain that the radial velocity field in x, $|x| = r_t$ is smaller than

$$\frac{C + C_4}{r_t^2} \tag{2.67}$$

with C independent of b_1 and C_4 decreasing as b_1 increases. Equation (2.67) gives a bound like (2.18) with $\alpha = \frac{1}{3}$ and $b = 3(C + C_4)$. So we can choose b_1 large such that $b_1 > 3(C + C_4)$, and then $R_t \le r_t$ for any time. □

In Theorem 2.1 b depends, of course, on the initial conditions. It is natural to prove that $b \to 0$ as the initial vortex patch becomes circular. More precisely:

Theorem 2.3. *Let b defined by Eq.* (2.18). *Then*

$$b \to 0 \quad as \quad I \to I_0, \tag{2.68}$$

where I is the moment of inertia defined by Eq. (2.18) *and $I_0 = (2\pi)^{-1}$ is the moment of inertia of a circular vortex patch of unitary mass.*

Proof. The proof is based on an improvement of the bound (2.33).

With fixed $m_t(r)$, we observe that the configuration which gives a minimal I is the following: Λ_0 is the sum of two regions, the first consisting of an annulus of interior radius r and area $m_t(r)$, the second one consisting in a circle of area $1 - m_t(r)$. Then

$$I \ge 2\pi \int_r^{r_1} r^3\, dr + 2\pi \int_0^{r_2} r^3\, dr, \tag{2.69}$$

where r_1 and r_2 are defined by the relations

$$2\pi \int_{r}^{r_1} r\,dr = m_t(r)\,, \tag{2.70}$$

$$2\pi \int_{0}^{r_2} r\,dr = 1 - m_t(r)\,. \tag{2.71}$$

Hence

$$\begin{aligned} I &\geq \frac{\pi}{2}\left\{\left[\frac{m_t(r)}{\pi} + r^2\right]^2 - r^4 + \left[\frac{1 - m_t(r)}{\pi}\right]^2\right\} \\ &= \frac{m_t(r)^2}{\pi} + m_t(r)\left(r^2 - \frac{1}{\pi}\right) + \frac{1}{2\pi}\,, \end{aligned} \tag{2.72}$$

and so

$$m_t(r) \leq \frac{I - (2\pi)^{-1}}{r^2 - \pi^{-1}}\,. \tag{2.73}$$

Obviously $r \geq R_0 > (\pi)^{-1/2}$ and so that bound (2.33) holds with a constant vanishing as $I \to I_0$. Since this bound appears as a multiplicative factor in all the later steps, b also vanishes as $I \to I_0$. □

Remark. Until now we have considered ω_0 of the form of a characteristic function of a set but all the previous considerations apply without any change if ω_0 is a generic bounded function of definite sign with a compact support.

3. A Simplified Model

In the previous section we have neglected completely the effects of the homogenization. The nonlinear nature of the model makes it very difficult to take them into account. To simplify the problem we can introduce some new, more tractable, models, that are schematic approximations of the vortex patch dynamics and which exhibit the homogenization effects. We discuss briefly one of them:

$$\omega_0 = \chi(\Lambda_0)\,, \tag{3.1}$$

$$\Lambda_0 = \Sigma\left(\frac{1}{\sqrt{\pi}}\right) \cup \Omega_0 \cup \Xi_0\,, \tag{3.2}$$

where Σ is the circle of radius $\dfrac{1}{\sqrt{\pi}}$ and

$$\Omega_0 = \left\{x \in \mathbb{R}^2 \,\middle|\, \frac{1}{\sqrt{\pi}} \leq |x| \leq f(\theta)\right\} \qquad (\varrho, \theta) = \text{polar coordinates}\,, \tag{3.3}$$

$f(\theta)$ being a smooth periodic function such that $f(\theta) \geq \dfrac{1}{\sqrt{\pi}}$. Finally Ξ_0 is a compact set.

We assume that the initial vorticity ω_0 evolves in ω_t,

$$\omega_t = \chi(\Lambda_t) \tag{3.5}$$

and Λ_t is obtained from Λ_0 by the following rules: a point of $\Sigma\left(\frac{1}{\sqrt{\pi}}\right) \cup \Omega_0$ moves according the velocity field produced by $\Sigma\left(\frac{1}{\sqrt{\pi}}\right)$ alone, while Ξ_0 evolves in the velocity field produced by $\Sigma\left(\frac{1}{\sqrt{\pi}}\right) \cup \Omega_0$ but not by itself.

Of course this model is a first order approximation for Ω_0 and Ξ_0 small of the stationary solution $\Lambda_0 = \Sigma\left(\frac{1}{\sqrt{\pi}}\right)$.

We can easily prove that this model exhibits the property of spatial homogenization, as we show briefly. In particular we find a bound on the growth of R_t:

Theorem 3.1. *Suppose*

$$\Lambda_0 \subset \Sigma(R_0)\,, \tag{3.6}$$

then Λ_t

$$\Lambda_t \subset \Sigma(R_t)\,, \tag{3.7}$$

where

$$R_t \le (R_0^3 + b\ln(1+t))^{1/3} \qquad t \ge 0\,. \tag{3.8}$$

Proof. We only sketch the proof. We evaluate the radial velocity field in the point x, $|x| = R_t$. We prove that it is smaller than Ct^{-1}. Then Eq. (3.8) follows from the integration of the differential inequality (2.20).

The main observation for the proof is the following: points of Ω_0 with different ϱ move with different angular velocity. In fact the angular velocity γ of a point of radial coordinate ϱ is

$$\gamma = \frac{1}{2\varrho^2}\,. \tag{3.9}$$

Hence during the motion points with different ϱ have different phases and this fact makes it rise to a spatial homogenization.

More precisely, we put ourselves in a reference frame turning around the origin with an angular velocity $\frac{1}{2}$. We denote by

$$\varrho_{\max} = \max_{0\le\theta\le 2\pi} f(\theta)\,, \tag{3.10}$$

we divide the interval $(\varrho_{\max}, 0)$ in N disjoint intervals $I_n = (b_n, b_{n-1})$ such that each point in I_n has made $n-1$ turns and not n turns. We study the size of I_n. We compute the angular velocity γ corresponding to extremal points b_n and b_{n-1}. We have

$$\gamma(b_n) = \frac{2\pi}{t}\,n\,, \tag{3.11}$$

$$\gamma(b_{n-1}) = \frac{2\pi}{t}\,(n-1)\,. \tag{3.12}$$

Hence

$$|\gamma(b_n) - \gamma(b_{n-1})| = \frac{2\pi}{t}\,. \tag{3.13}$$

In I_n the angular velocity changes linearly up to a correction of order t^{-2}, so that the size of I_n is of the order of t^{-1}.

Moreover it is easy to see that the particles with $\varrho \in I_n$ are uniformly distributed in the corresponding annulus up to a correction of the order t^{-1}. Of course a circular distribution cannot produce a radial velocity field. Then the radial velocity field produced by the particles with $\varrho \in I_n$ on a particle in x, $|x| = R_t$ can be bounded by:

$$\leq \frac{C}{R_t^2} \frac{\mu(I_n)}{1+t}, \tag{3.14}$$

where $\mu(I_n)$ is the measure of the set of Ω_0 corresponding to $\varrho \in I_n$, which is constant during the motion because the velocity field is divergence-free.

By summing on n, we have

$$\frac{d}{dt} R_t \leq \frac{C}{R_t^2} \frac{1}{1+t}. \tag{3.15}$$

Hence, by integration of this differential inequality, we obtain Eq. (3.8). □

We do not know if bound (3.8) is the best possible in this model. In fact we have not fully used the temporal homogenization, when we have neglected an overall angular motion. To take into account this temporal homogenization effect we need to consider the sign of the radial velocity variable in time and this is a difficult task. However we notice that for practical purposes bound (3.8) is undistinguishable from a constant bound. Moreover this model, perhaps useful for intermediate times, falls surely asymptotically in time, when the nonlinear effect may become relevant.

4. Navier-Stokes Flow

In this section we study the time evolution in $\mathbb{R}^2$ of a vortex patch, when the viscosity is present. In this case the Euler equation (2.12) modifies in the Navier-Stokes equation:

$$\frac{d}{dt} \omega_t[f] = \omega_t([u \cdot \nabla f] + \nu \omega_t[\Delta f] \qquad \nu > 0. \tag{4.1}$$

Obviously, adding a viscosity term the asymptotic in time behavior of the fluid changes drastically. In fact the fluid diffuses immediately in the whole plane and so an equivalent of Theorem 2.1 is meaningless. However we can study the region $\Sigma(r_t)$ in which the main part of the vorticity is concentrated and bounds its growth in time (that is the equivalent of Theorem 2.3). More precisely we want to find for which $\alpha > 0$ we have

$$\begin{aligned} &m_t(r_t) < C \exp[-\gamma r_t^\delta] \quad && \gamma, \delta > 0, \\ &\text{if} \quad r_t > C t^\alpha && t > 0. \end{aligned} \tag{4.2}$$

(Of course the bound is sharper as α is smaller.) For the Stokes equation (when the transport term $\omega_t([u \cdot \nabla f]$ is absent) the problem is easily solved by the explicit solution and we find that the inequality (4.2) holds for any $\alpha > \frac{1}{2}$. In this section we prove that we obtain the same result when we consider the whole Navier-Stokes equation:

Theorem 4.1. *Suppose that*

$$\omega_0 = \chi(\Lambda_0) \tag{4.3}$$

with

$$\Lambda_0 \subset \Sigma(R_0) \qquad R_0 > 0. \tag{4.4}$$

Then for any $\alpha > \frac{1}{2}$, $0 < \delta < 2 - \alpha^{-1}$ *we have*

$$m_t(r_t) < C\exp[-\gamma r_t^\delta] \quad \gamma > 0 \quad \text{if} \quad r_t > Ct^\alpha \,, \tag{4.5}$$

where $m_t(r)$ *denotes the vorticity mass at time* t *out of a circle of radius* r *and the fluid evolves according to the Navier-Stokes Equation* (4.1).

Proof. The proof is based on the following result:

Lemma 4.1. *Define*

$$I_n(t) = \int\limits_{\mathbb{R}^2} dx\, |x|^{2n}\omega(x,t)\,. \tag{4.6}$$

Then

$$\left|\frac{d}{dt}I_n(t)\right| \begin{array}{ll} = 0 & \text{if } n = 0 \\ \le Cn^2 I_{n-1}(t) & \text{if } n \ge 1\,, \end{array} \tag{4.7}$$

and hence

$$I_n(t) \le \sum_{k=1}^{n} C^k \frac{(n!)^2 t^k}{[(n-k)!]^2 k!}\,. \tag{4.8}$$

Proof of the Lemma. The inequality (4.7) can be proved by direct computation. In fact, using Eq. (4.1), we have:

$$\frac{d}{dt}I_n(t) = A + B\,, \tag{4.9}$$

where

$$A = \int dx\, \omega(x,t) u\cdot\nabla|x|^{2n}\,, \tag{4.10}$$

$$B = \nu\int dx\, \omega(x,t)\Delta|x|^{2n}\,, \tag{4.11}$$

and so the proof of Eq. (4.7) for $n = 0$ is trivial. For $n \ge 1$ we study the two terms A, B separately. The second one can be explicitly computed, remembering that in polar coordinates (r,θ) we have: $\Delta f = r^{-1}\partial_r(r\partial_r f) + r^{-2}\partial_\theta^2 f$. We obtain:

$$B = \nu(2n)^2 I_{n-1}(t)\,. \tag{4.12}$$

Then we study term A, using the expression of the velocity field given by Eq. (2.4) and the antisymmetry of K,

$$\begin{aligned} A &= \int dx \int dy\, \omega(x,t)\omega(y,t)K(x-y)\cdot\nabla|x|^{2n} \\ &= \frac{1}{2}\int dx\int dy\, \omega(x,t)\omega(y,t)\left\{\nabla_x|x|^{2n} - \nabla_y|y|^{2n}\right\}\cdot K(x-y) \\ &= n\int dx\int dy\, \omega(x,t)\omega(y,t)\left\{|x|^{2n-2}x - |y|^{2n-2}y\right\}\cdot K(x-y) \\ &= \frac{n}{2}\int dx\int dy\, \omega(x,t)\omega(y,t)\left[\left\{|x|^{2n-2} - |y|^{2n-2}\right\}(x+y)\cdot K(x-y)\right. \\ &\quad \left. + \left\{|x|^{2n-2} + |y|^{2n-2}\right\}(x-y)\cdot K(x-y)\right]. \end{aligned} \tag{4.13}$$

The last term of the sum vanishes, as we can prove using the explicit form of K:

$$K(x,y) = -\frac{1}{2\pi}\frac{(x-y)^{\perp}}{|x-y|^2}\,; \qquad x^{\perp} \equiv (x_2, -x_1)\,.$$

Hence

$$|A| \leq \frac{n}{4\pi}\int dx \int dy\, \omega(x,t)\omega(y,t)\,\big||x|^{2n-2} - |y|^{2n-2}\big|\,\frac{|x+y|}{|x-y|}\,, \tag{4.14}$$

where we have used the well-known fact that $\omega(x,t)$ remains non-negative during the motion. Using the identity

$$|x|^{2n-2} - |y|^{2n-2} = (x-y)\cdot(x+y)\sum_{k=0}^{n-2}|x|^{2n-4-2k}|y|^{2k} \qquad n \geq 2\,, \tag{4.15}$$

we have

$$\begin{aligned} |A| &\leq \frac{n}{2\pi}\int dx \int dy\, \omega(x,t)\omega(y,t)\,(|x|^2 + |y|^2)\sum_{k=0}^{n-2}|x|^{2n-4-2k}|y|^{2k} \\ &\leq \frac{n}{\pi}\int dx \int dy\, \omega(x,t)\omega(y,t)\sum_{k=0}^{n-1}|x|^{2n-2-2k}|y|^{2k}\,. \end{aligned} \tag{4.16}$$

We observe that for any $k \in [0, n-1]$,

$$|x|^{2n-4-2k}|y|^{2k} \leq |x|^{2n-2} + |y|^{2n-2}\,. \tag{4.17}$$

(To prove it, it is enough to put $|y| = \gamma|x|$ and Eq. (4.17) becomes $\gamma^{2k} \leq 1 + \gamma^{2n-2}$ which is verified for any γ and $0 \leq k \leq n-1$.)

Using Eq. (4.17), Eq. (4.16) gives:

$$|A| \leq \frac{n^2}{\pi}\int dx \int dy\, \omega(x,t)\omega(y,t)\,\{|x|^{2n-2} + |y|^{2n-2}\} = \frac{2n^2}{\pi}\, I_0 I_{n-1}(t)\,. \tag{4.18}$$

Since I_0 is constant during the motion, adding Eq. (4.18) to Eq. (4.12), we have proved Eq. (4.7).

To obtain Eq. (4.8) we integrate Eq. (4.7) n times:

$$\begin{aligned} I_n(t) &\leq I_n(0) + Cn^2\int_0^t dt_1\, I_{n-1}(t_1) \\ &\leq I_n(0) + Cn^2 I_{n-1}(0)t + C^2n^2(n-1)^2\int_0^t dt_1\int_0^{t_1} dt_2 I_{n-2}(t_2) \\ &\leq \sum_{k=0}^{n} I_{n-k}(0)C^k\,\frac{(n!)^2}{[(n-k)!]^2}\,\frac{t^k}{k!}\,. \end{aligned} \tag{4.19}$$

The initial conditions assure that

$$I_{n-k} \leq I_0 R_0^{2(n-k)}, \tag{4.20}$$

and so the lemma is proved. $\square$

We use the previous lemma to bound $m_t(r_t)$. In fact by definition (4.6) we have

$$I_n(t) \geq m(r_t) r_t^{2n}, \tag{4.21}$$

and hence by Eq. (4.8),

$$m_t(r_t) \leq \frac{1}{r_t^{2n}} \sum_{k=0}^{n} C^k \frac{(n!)^2}{[(n-k)!]^2} \frac{t^k}{k!}. \tag{4.22}$$

We suppose now that $t \geq 1$ (otherwise the proof of the theorem is trivial). Then

$$m_t(r_t) \leq \frac{C^n (n+1) M_n t^n}{r_t^{2n}}, \tag{4.23}$$

where

$$M_n = \max_{0 \leq k \leq n} \frac{(n!)^2}{[(n-k)!]^2 k!}. \tag{4.24}$$

We choose $t > C r_t^{1/\alpha}$. Hence

$$m_t(r_t) < C^n n M_n r_t^{(1/\alpha - 2)n}. \tag{4.25}$$

We bound M_n. The maximum of

$$\frac{(n!)^2}{[(n-k)!]^2 k!} = \frac{n^2 (n-1)^2 \ldots (n-k+1)^2}{k!} \tag{4.26}$$

is reached for $k = k^*$ when the term added in the numerator is smaller than k, that is the greater k such that

$$(n-k+1)^2 \geq k, \tag{4.27}$$

that is

$$k^* = \text{Integer part of } \left(n + \tfrac{3}{2} - \left[2n + \tfrac{5}{4}\right]^{1/2}\right) > n - C\sqrt{n} + C. \tag{4.28}$$

We put Eq. (4.28) into Eq. (2.24), we neglect the term $[(n-k)!]^2$ and we use the Stirling formula

$$e^{n(\ln n - 1)} < n! \leq n^n.$$

We obtain

$$M_n < \frac{C^n n^{2n}}{(n - C\sqrt{n})^{n - \sqrt{n}}} < C^n n^{n + C\sqrt{n}}. \tag{4.29}$$

Putting Eq. (4.29) in Eq. (4.25), we have

$$m_t(r_t) < \exp\{Cn + n \ln n + C\sqrt{n} \ln n + n(\alpha^{-1} - 2) \ln r_t\}. \tag{4.30}$$

The theorem is proved by choosing

$$n = \text{Integer part of } [Cr_t^\delta] \quad \text{with} \quad \delta < 2 - \alpha^{-1}. \quad \square \tag{4.31}$$

In the previous theorem we have supposed ω_0 of the form of a characteristic function of a set, but the statement and the proof remain unchanged if ω_0 is a generic bounded function of a definite sign with a compact support. On the contrary for non-compact initial data the statement is similar but the proof changes a little.

Theorem 4.2. *Suppose*

$$\omega_0(x) \le C \exp\{-\gamma |x|^\delta\}, \qquad \gamma, \delta > 0. \tag{4.32}$$

Then

$$m_t(r_t) < C \exp[-\gamma' r_t^{\delta'}] \quad \gamma' > 0 \quad \textit{if} \quad r_t > Ct^\alpha \quad \textit{for any} \quad \delta' < \delta < 2 - \frac{1}{\alpha}. \tag{4.33}$$

Proof. The proof follows the lines of the previous one and we sketch it only. We observe that

$$\begin{aligned} I_k(0) &= \int_{\mathbb{R}^2} dx\, |x|^{2k} \omega_0(x) < C \int dx\, |x|^{2k} \exp\{-\gamma |x|^\delta\} \\ &= C\gamma^{-(2k+1)/\delta} \Gamma\left(\frac{2k+1}{\delta} + 1\right) \le C^k k^{2k/\delta}, \end{aligned} \tag{4.34}$$

where $\Gamma(z)$ is the gamma function.

From Eq. (4.22) we have

$$m_t(r_t) < C^n \sum_{k=0}^{n} \frac{(n-k)^{2(n-k)/\delta}}{r_t^{2(n-k)}} \frac{(n!)^2}{[(n-k)!]k!} \frac{t^k}{r_t^{2k}}. \tag{4.35}$$

We choose

$$t < Cr_t^{1/\alpha} \tag{4.36}$$

and

$$n \approx r_t^{\delta'}, \tag{4.37}$$

so that

$$m_t(r_t) < C^n \sum_{k=0}^{n} r_t^{2(\delta'/\delta - 1)(n-k)} \frac{(n!)^2}{[(n-k)!]^2 k!} r_t^{(1/a-2)k}. \tag{4.38}$$

We proceed as in the previous theorem. We look for the maximum of the addendum in the sum which is reached for $k = k^*$, where k^* is the greater k for which:

$$(n-k+1)^2 \ge k r_t^{2\delta'/\delta - 1/\alpha}. \tag{4.39}$$

We put $k^* \approx n^\beta$ and we choose δ' so close to δ (this is the more difficult case) that $n \approx r^{\delta'}$ and $\delta' < \delta < 2 - \alpha^{-1}$ imply $\beta < 1$.

Hence

$$m_t(r_t) < \exp\left\{Cn + 2n\left(\frac{\delta'}{\delta} - 1\right)\ln r_t\right\} \tag{4.40}$$

and, by using Eq. (4.37), the proof is achieved. □

Appendix

For the sake of completeness we prove that a fluid particle moving in the velocity field generated by a Kirchhoff ellipse [Lam32] remains uniformly in time in a bounded region.

It is well known that a vortex patch with an ellipse shape turns around with a constant angular velocity γ without changing its form. We put ourselves in the reference frame in which the ellipse is at rest. In this frame a test particle moves under the action of the velocity field produced by the ellipse plus a tangential velocity of intensity $-\gamma\varrho$:

$$u = \nabla^{\perp}\varphi - \nabla^{\perp}\left(\tfrac{1}{2}\gamma\varrho^2\right), \tag{A.1}$$

where φ is the stream function of the velocity field produced by the ellipse.

It is immediate to verify that the following quantity is conserved during the motion:

$$\varphi - \tfrac{1}{2}\gamma\varrho^2 = \text{const}. \tag{A.2}$$

Since φ diverge at infinity as $\ln\varrho$ it cannot compensate the quadratic term, and so the particle remains in bounded regions.

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Note added in proof. As a corollary, in the present paper we have proved that a fluid particle, initially in x_0 and moving via the Euler equation, goes away from the initial position at most: $\big||x(t)| - |x_0|\big| \leq \text{constant } t^{1/3}$, $t > 1$. A stronger result is obtained for the model of Section 3: $\big||x(t)| - |x_0|\big| \leq \text{constant } [\ln(1+t)^{1/3}]$, $t > 1$. Recently Emanuele Caglioti and Carlotta Maffei in the paper "Asymptotic Behaviour of Vortex Patches: A Case of Confinement", Dipartimento di Matematica, Università "La Sapienza", Roma (Italy) Nota Interna 93/22 (1993), Boll. Unione Matematica Italiana (in press) have studied the model of Section 3 and they have proved the confinement of the system when $f(\theta)$ has a finite number of nondegenerate critical points. A deeper analysis of the model is in progress by the same authors.

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