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Coherent States of the q-Canonical Commutation Relations

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Abstract: For the q-deformed canonical commutation relations $a(f)a^\dagger(g) = (1-q)\langle f, g\rangle \mathbb{1} + qa^\dagger(g)a(f)$ for f, g in some Hilbert space $\mathscr{H}$ we consider representations generated from a vector Ω satisfying $a(f)\Omega = \langle f, \varphi\rangle\Omega$, where $\varphi \in \mathscr{H}$. We show that such a representation exists if and only if $\|\varphi\| \leqq 1$. Moreover, for $\|\varphi\| < 1$ these representations are unitarily equivalent to the Fock representation (obtained for $\varphi = 0$). On the other hand representations obtained for different unit vectors φ are disjoint. We show that the universal C*-algebra for the relations has a largest proper, closed, two-sided ideal. The quotient by this ideal is a natural q-analogue of the Cuntz algebra (obtained for $q=0$). We discuss the conjecture that, for $d<\infty$, this analogue should, in fact, be equal to the Cuntz algebra itself. In the limiting cases $q = \pm 1$ we determine all irreducible representations of the relations, and characterize those which can be obtained via coherent states.

1. Introduction

In this paper we study some new aspects of a set of commutation relations, depending on a parameter $q \in (-1, 1)$ studied by various authors on quite different motivations. Greenberg [15] introduced these relations as an interpolation between Bose ($q=1$) and Fermi ($q=-1$) statistics. He was particularly interested in the observable consequences of a hypothetical small deviation from the Pauli principle. However, due to problems with field theoretical localizability [16] and thermodynamic stability [34], a naive particle interpretation of systems satisfying these relations is problematic. Speicher [33] introduced these relations as a new kind of quantum "noise," which could be used as a driving force in a quantum stochastic differential equation [23]. From the point of view of C*-algebra theory the relations became interesting as an example of a C*-algebra defined in terms of generators and relations. In this context it was observed that the relations reduce for $q=0$ to those studied by Cuntz [9].

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The special case of a single generator, the so-called q-oscillator, was introduced by Biedenharn [4] and Macfarlane [27] as a means of constructing representations of quantum groups. In fact, the q-oscillator also appears as a subalgebra of the quantum group $S_\nu U(2)$ [35]. The q-oscillator can be studied in full detail by representing the generator as a weighted unilateral shift (in mathematical terminology) or as a Bose creation operator multiplied with a suitable function of the number operator (in physical terminology). This has been noted in a large number of papers. We will use this representation in the present paper to obtain information about the non-trivial case of several generators.

In this case most early work [15, 6, 13] focussed on showing that the scalar product in the q-analogue of Fock space is positive definite. On the other hand, from the C*-algebraic point of view the most immediate and natural problem arising from the relations was to characterize the norm-closed operator algebra generated by any realization of the relations by bounded operators on a Hilbert space. Here the case $q=0$ served as a model: for $q=0$ this algebra must be either isomorphic to the one obtained in the Fock representation, called the Cuntz–Toeplitz algebra, or a quotient of the Cuntz–Toeplitz algebra by its unique two-sided ideal (isomorphic to the compact operators), known as the Cuntz algebra. For $q \neq 0$ the first important step was made in [18], where we showed that for $|q| < \sqrt{2}-1 \approx .41$ the same results hold. In particular, the C*-algebras generated with q in this range are exactly the same as for $q=0$. The condition $|q| < \sqrt{2}-1$ is certainly not optimal, and all the results known to us are compatible with the conjecture (which we will refer to as "*Conjecture C*," see Sect. 4) that the results of [18] hold for all $|q|<1$. However, no decisive progress towards proving this conjecture has been made since [18]. Based on an improved understanding of the Fock representation [36], Dykema and Nica [12] managed to extend the interval for q slightly, but only for the algebra generated in the Fock representation. More importantly, they established, for the Fock representation only, the existence of the homomorphism between the algebras for $q=0$ and for general $-1<q<1$, which according to Conjecture C should be an isomorphism. We will briefly describe and apply their results in Sect. 4.

The main aim of this paper is to study the q-analogue of a structure which is well-known in the limiting cases $q=0, \pm 1$, namely the generalization of the Fock state to the so-called coherent states. In the case of a single relation such states appear in [26], although, due to a different choice of generators, their work makes sense only in the Fock representation, and gives states different from ours. We will determine all coherent states, and discuss under what circumstances they generate the same representation, or are mutually singular. Using coherent states, we show that the universal C*-algebra generated by the relations has a unique largest closed two-sided ideal. (If Conjecture C holds this ideal is also the only proper ideal, and isomorphic to the compact operators). The quotient of the algebra by this ideal is then simple, and the natural analogue of the Cuntz algebra for $q \neq 0$. Finally, we consider the limiting case $q=-1$, and compute all irreducible representations of the relations with Clifford algebra methods. It turns out that in this degenerate case the coherent states exhaust only a small subclass of irreducible representations.

We emphasize that when we talk about representations in the sequel we always mean *-representations of some involutive algebra by *bounded* operators on a Hilbert space. Thus even if the relations may have interesting unbounded realizations we do not consider them.

2. q-Relations and Coherent States

The following Proposition introduces the "q-relations" which are the object of our study.

Proposition. *Let $\mathscr{H}$ be a Hilbert space, and let $q\in\mathbb{R}$, $|q|<1$. Then there is a C*-algebra $\mathscr{E}_{\mathscr{H}}(q)$ generated by elements $a^\dagger(f)$ for $f\in\mathscr{H}$, such that $f\mapsto a^\dagger(f)$ is linear, and*

$$a(f)a^\dagger(g)=(1-q)\langle f,g\rangle\mathbb{1}+qa^\dagger(g)a(f)\,, \tag{1}$$

where $a(f):=a^\dagger(f)^*$. For $q=1$, and orthogonal unit vectors $e_1,\ldots,e_n\in\mathscr{H}$ the bound

$$\sum_{i=1}^{n} a^\dagger(e_i)a(e_i)\leqq\mathbb{1} \tag{2}$$

holds.

Moreover, $\mathscr{E}_{\mathscr{H}}(q)$ is uniquely determined by the following universal property: whenever $\tilde{\mathscr{E}}$ is a C-algebra containing elements $\tilde{a}^\dagger(f)$ satisfying the above conditions, there is a unique unital homomorphism φ: $\mathscr{E}_{\mathscr{H}}(q)\to\tilde{\mathscr{E}}$ such that $\varphi(a^\dagger(f))=\tilde{a}^\dagger(f)$.*

The proof of this result is given in [18]. Note that in comparison with [18, 6] we have changed the normalization of the operators $a^\dagger(f)$. This modification makes no essential difference for $|q|<1$. However, it removes the singularity of the relations for $q\to 1$ and simplifies all algebraic expressions. Moreover, it was shown in [28] that with this normalization the algebras $\mathscr{E}_{\mathbb{C}}(q)$ form a continuous field of C*-algebras [10]. We may consider the relations (1) for all $q\in\mathbb{R}\cup\{\infty\}$, where for $q=\infty$ we set $a^\dagger(g)a(f)=\langle f,g\rangle\mathbb{1}$. The study of the case $|q|\geqq 1$ is then reduced to the case $|q|\leqq 1$ by the symmetry

$$\begin{aligned} q&\mapsto q^{-1}\\ a^\dagger(f)&\mapsto a(\bar{f}) \end{aligned} \tag{3}$$

for some antiunitary operator $f\mapsto\bar{f}$.

The crucial feature of the relations (1) is that they allow us to order any polynomial in the generators in such a way that in every monomial all operators $a(f)$ are to the right of every $a^\dagger(f)$. This normal ordered, or "Wick ordered" form of a polynomial is unique [2, 3, 20], hence we can define a linear functional ω on the polynomial algebra over the relations by choosing an arbitrary multilinear expression for $\omega(a^\dagger(f_1)\cdots a^\dagger(f_n)a(g_1)\cdots a(g_m))$. Since such monomials generate $\mathscr{E}_{\mathscr{H}}(q)$ this is also a way to parametrize all states on the C*-algebra $\mathscr{E}_{\mathscr{H}}(q)$. The following theorem introduces the coherent states on $\mathscr{E}_{\mathscr{H}}(q)$ using such a parametrization.

2 Theorem. *Let $|q|\leqq 1$, and $\varphi\in\mathscr{H}$ with $\|\varphi\|\leqq 1$. Then there is a unique state ω_φ on $\mathscr{E}_{\mathscr{H}}(q)$ such that*

$$\omega_\varphi(a^\dagger(f)X)=\langle\varphi,f\rangle\omega_\varphi(X) \tag{4}$$

for all $f\in\mathscr{H}$, and all $X\in\mathscr{E}_{\mathscr{H}}(q)$. The state ω_φ is pure. For $\|\varphi\|>1$, there is no state satisfying (4).

We will call ω_φ the *coherent state* associated with φ. This terminology originated in quantum optics, where these states are used to describe states of the electromagnetic radiation field [22, 17]. The special state ω_0 is called the *Fock state*. If $\|\varphi\|=1$, we will call ω_φ a *peripheral coherent* state. For any φ we will denote by π_φ the GNS-representation associated with ω_φ, and call it the *coherent representation* associated with φ. For the special case $q=0$, coherent states in this sense have been studied in [7].

The proof of Theorem 2 is based on an analysis of the case of a single relation. We summarize the relevant results in the following lemma. The assumption that a is bounded is essential for this result, i.e. there are also unbounded operators satisfying the relation, and the conclusion fails for these.

3 Lemma. *Let* $|q|<1$, *and let* $a\equiv a(e_1)$ with $\|e_1\|=1$ *be a bounded operator on a Hilbert space* $\mathscr{R}$ *satisfying the relation* $aa^*=(1-q)\mathbb{1}+qa^*a$.
(1) *Then a is reduced by a unique decomposition* $\mathscr{R}\cong(\mathscr{R}_0\otimes\mathscr{R}')\oplus\mathscr{R}_1$, *such that*

(a) *if* $\mathscr{R}_1\neq\{0\}$, $a\upharpoonright\mathscr{R}_1$ *is unitary.*
(b) *if* $\mathscr{R}'\neq\{0\}$, $a\upharpoonright\mathscr{R}_0\otimes\mathscr{R}'$ *acts as* $a=a_0\otimes\mathbb{1}$, *where* a_0 *is given explicitly as the weighted shift*

$$a_0^*|n\rangle=(1-q^{n+1})^{1/2}|n+1\rangle\,, \tag{5}$$

where $|n\rangle$ *for* $n=0,1,\ldots$ *is an orthonormal basis of* $\mathscr{R}_0$.

(2)

$$\|a\|=\begin{cases}1 & \text{for } q>0, \quad \text{or } a \text{ unitary}\\ \sqrt{1-q} & \text{for } q<0, \quad \text{and } a \text{ not unitary}\,.\end{cases} \tag{6}$$

(3) *There are functions* $\beta_+(q)<\infty$ *and* $\beta_-(q)>0$ *such that*

$$\beta_-(q)\mathbb{1}\leqq a^n(a^*)^n\leqq\beta_+(q)\mathbb{1}\,, \tag{7}$$

uniformly for $n\in\mathbb{N}$. *In particular, the spectral radius of a is equal to* 1.

(4) *Let* $a^*\xi=\lambda\xi$ *for* $\xi\neq0$. *Then* $\xi\in\mathscr{R}_1$, $|\lambda|=1$, *and* $a\xi=\bar\lambda\xi$.

Proof. For (1) see [18]; for (2) see [6, 18].
(3) For the unitary part $a\upharpoonright\mathscr{R}_1$ we only need $\beta_-(q)\leqq1\leqq\beta_+(q)$, which will be true for the $\beta_\pm$ constructed below. Hence it suffices to take $a=a_0$. Then

$$a_0^n(a_0^*)^n|k\rangle=\lambda_{k+1}\cdots\lambda_{k+n}|k\rangle\,,$$

where $\lambda_k=(1-q^k)$. We will take $\beta_\pm$ as the supremum (resp. infimum) over all products $\prod_{k\in M}\lambda_k$ for $M\subset\mathbb{N}$. Explicitly,

$$\beta_+(q)=\begin{cases}1 & q\geqq0\\ \prod_{k=1}^{\infty}(1-q^{2k+1}) & q\leqq0\end{cases},$$

$$\beta_-(q)=\begin{cases}\sum_{k=1}^{\infty}(1-q^k) & q\geqq0\\ \prod_{k=1}^{\infty}(1-q^{2k}) & q\leqq0\end{cases}. \tag{8}$$

Since these products (related to Theta functions, and to "q-factorials" [1]) are absolutely convergent, $\beta_\pm(q)$ is finite and non-zero for all q, $|q|<1$. For computing the spectral radius we let $n \to \infty$ in the inequality

$$\beta_-(q)^{1/2n} \leqq \|a^n\|^{1/n} \leqq \beta_+(q)^{1/2n} .$$

(4) Given the decomposition it suffices to show that $a_0^* \xi = \lambda \xi$ implies $\xi = 0$. This follows immediately from the weighted shift structure (5) of a_0^*, by solving the recursion for the coefficients ξ_n in $\xi = \sum_n \xi_n |n\rangle$. ∎

Consider the GNS-representation π_φ associated with the coherent state ω_φ. This has a cyclic vector Ω_φ, which is a joint eigenvector of the generators, i.e.

$$a(f)\Omega_\varphi = \langle f, \varphi \rangle \Omega_\varphi . \tag{9}$$

Conversely, any unit vector satisfying (9) will give the coherent state via $\omega_\varphi(X) = \langle \Omega_\varphi, X\Omega_\varphi \rangle$. Therefore, in order to show that ω_φ is positive, it is sufficient to exhibit such a vector in a representation which is known to be positive. Now the Fock representation π_0 has been proven to be positive [6, 13, 36, 20]. Hence it suffices to find such vectors in the Fock representation. The basic construction for such vectors can be carried out in the case of a single generator. For Boson commutation relations the operator transforming the vacuum into a coherent state is well-known to be $\exp(za^*)$. For the q-relations a similar rôle is played by the "q-exponential" function Exp_q, defined by the functional equation [21]

$$\mathbf{D}_q \mathrm{Exp}_q(z) \equiv \frac{\mathrm{Exp}_q(z) - \mathrm{Exp}_q(qz)}{z - qz} = \mathrm{Exp}_q(z) . \tag{10}$$

The q-exponential satisfies no simple addition formula, and therefore the operator connecting different coherent states can only be expressed as a quotient of two such exponentials. Rather than defining first the q-exponential, and then studying its invertibility, we define, in the following lemma, all these quotients at the same time. The connection with the q-exponential is $V_{\alpha 0}(z) = \mathrm{Exp}_q(\alpha z/(q-1))$.

4 Lemma.

(1) *Let* $|q| < 1$, *and* $\alpha, \beta \in \mathbb{R}$. *Then the functional equation*

$$V_{\alpha\beta}(qz) = \frac{1-\alpha z}{1-\beta z} V_{\alpha\beta}(z); \quad V_{\alpha\beta}(0) = 1 \tag{11}$$

has a unique analytic solution near $z = 0$, *which is analytic for* $|\alpha z| < 1$. *For* $|\alpha z| < 1$, *and* $|\beta z| < 1$, *and* $\gamma \in \mathbb{R}$ *we have* $V_{\alpha\beta} V_{\beta\gamma} = V_{\alpha\gamma}$.

(2) *Let a be a bounded operator on a Hilbert space* $\mathscr{R}$ *with* $aa^* = (1-q)\mathbb{1} + qa^*a$. *Then, for* $\Omega_\beta \in \mathscr{R}$, *and* $|\alpha| < 1$ *we have the implication*

$$(\alpha - \beta)\Omega_\beta = 0 \Rightarrow (a - \alpha) V_{\alpha\beta}(a^*) \Omega_\beta = 0 , \tag{12}$$

where the function $V_{\alpha\beta}$ *is evaluated on* a^* *in the analytic functional calculus.*

Proof. Let $V_{\alpha\beta}(z) = \sum_k c_k z^k$. Then Eq. (12) together with the iterated relation

$$a(a^*)^k = q^k (a^*)^k a + (1-q^k)(a^*)^{k-1} \tag{13}$$

gives a functional equation for the coefficients c_k:

$$c_{k+1} = \frac{\alpha - q^k \beta}{1 - q^{k+1}} c_k; \quad c_0 = 1 . \tag{14}$$

By an elementary computation this is the same recursion which holds for the coefficients of $V_{\alpha\beta}$ defined through the functional equation. By standard theorems on power series its radius of convergence is $|\alpha|^{-1}$. The chain relation $V_{\alpha\beta}V_{\beta\gamma}=V_{\alpha\gamma}$ follows directly from the functional equation. ∎

Proof of Theorem 2. Let ω be a state satisfying Eq. (4). Then we can compute it on any polynomial in the generators by Wick ordering the polynomial, and then applying successively Eq. (4) and its adjoint $\omega(Xa(g))=\langle g,\varphi\rangle\omega(X)$. Since polynomials are dense in $\mathscr{E}_{\mathscr{H}}(q)$, $\omega=\omega_\varphi$ is uniquely determined. It is also clear that ω_φ must be a pure state, since it is the only state on which the positive elements $(a^\dagger(f)-\langle\varphi,f\rangle\mathbb{1})(a(f)-\langle f,\varphi\rangle\mathbb{1})$ have zero expectation for all $f\in\mathscr{H}$.

If there is a state ω_φ we have, for $\|f\|=1$: $|\langle\varphi,f\rangle|^2=\omega_\varphi(a^\dagger(f)^N a(f)^N)^{1/N}\leqq 1$, since the spectral radius of $a(f)$ is 1 by Lemma 3(2). With $f=\varphi/\|\varphi\|$ this implies $\|\varphi\|\leqq 1$.

For $\varphi=0$ we get the Fock functional, which is known to be positive, and leads to a representation π_0 of the relations by bounded operators $\pi_0(a(f))$, as shown by Eq. (6) in Lemma 3. This is the same as saying that the Fock functional extends to a state of the C*-algebra $\mathscr{E}_{\mathscr{H}}(q)$). We will show that the coherent states ω_φ with $\|\varphi\|\leqq 1$ are, in fact, states on the C*-algebra $\pi_0(\mathscr{E}_{\mathscr{H}}(q))$, and hence, a fortiori, states on $\mathscr{E}_{\mathscr{H}}(q)$.

Let $\|\varphi\|<1$. We know from [6] that ω_0, the Fock state, is positive. By Lemma 3(2), $a^\dagger(\varphi)$ has spectral radius <1. Hence we can apply V_{10} from Lemma 4 to $a^\dagger(\varphi)$ in the analytic functional calculus. Let $V\equiv V_{10}(a^\dagger(\varphi))=V_{\|\varphi\|,0}(a^\dagger(\varphi/\|\varphi\|)$. Then since $V_{0,\|\varphi\|}(a^\dagger(\varphi/\|\varphi\|))=V^{-1}$ we have that $\Omega_\varphi=V\Omega_0$ is non-zero. By Lemma 4 we have $a(\varphi)\Omega_\varphi=\langle\varphi,\varphi\rangle\Omega_\varphi$. On the other hand, when $\psi\perp\varphi$, we have $a(\psi)a^\dagger(\varphi)^n\Omega_0=q^n a^\dagger(\varphi)^n a(\psi)\Omega_0=0$. With the series expansion for V we find $a(\psi)\Omega_\varphi=0$. Combining this with the result for $\varphi=\psi$ we get $a(\psi)\Omega_\varphi=\langle\psi,\varphi\rangle\Omega_\varphi$. Hence $\omega_\varphi(X)=\langle\Omega_\varphi,\pi_0(X)\Omega_\varphi\rangle/\|\Omega_\varphi\|^2$ defines a state on $\mathscr{E}_{\mathscr{H}}(q)$ with the required properties.

Finally, for $\|\varphi\|=1$, let ω_* be a weak*-cluster point of a sequence of states ω_{λ_φ} with $|\lambda|<1$, and $\lambda\to 1$. Then, for $X\in\mathscr{E}_{\mathscr{H}}(q)$ and $f\in\mathscr{H}$,

$$\omega_*(a^\dagger(f)X)=\lim_{\lambda\to *}\langle\lambda\varphi,f\rangle\,\omega_{\lambda_\varphi}(X)=\langle\varphi,f\rangle\,\omega_*(X)\,,$$

where the limit is along any subsequence of λ's along which $\omega_{\lambda_\varphi}\to\omega_*$. Hence ω_* satisfies the defining equation (4) of ω_φ, and as uniquely determined by it, as shown above. ∎

The proof gives more information than just the positivity of ω_φ: by composing the operators $V_{10}(a^\dagger(\varphi))$ and $V_{10}(a^\dagger(\psi))^{-1}$ we get the following consequence:

5 Corollary. *For $\|\varphi\|$, $\|\psi\|\leqq 1$, the states ω_φ and ω_ψ are connected by an invertible element $v_{\varphi\psi}\in\mathscr{E}_{\mathscr{H}}(q)$ via $\omega_\varphi(X)=\omega_\psi(v_{\varphi\psi}^* X v_{\varphi\psi})$.*

The operators $v_{\varphi\psi}$ are Araki's Radon–Nikodym derivatives in the sense of [31] (see also [37].) Since they are defined by norm convergent series, they end up in the C*-algebra $\mathscr{E}_{\mathscr{H}}(q)$, and not merely in some bigger von Neumann algebra.

We close this section with a brief discussion of the coherent states for certain variations of the q-relations found in the literature. Most of the literature is concerned with the Fock representation of the relations with a single generator, and the relations are frequently written in a form explicitly involving the number operator N of the Fock representation. This operator is defined by $\exp(itN)a^\dagger(f)\exp(-itN)=a^\dagger(\exp(it)f)$, and $N\Omega_0=0$, where Ω_0 is the Fock vacuum. We will continue to denote by $a^\dagger(f)$ the generators with the conventions fixed in Proposition 1. Then the generators found elsewhere are $b^\dagger(f)$ with

$$b^\dagger(f)=\beta q^{\alpha N}a^\dagger(f)=\beta a^\dagger(f)q^{\alpha(N+1)},$$

$$b(g)b^\dagger(f)=|\beta|^2(1-q)\langle f,g\rangle q^{2\alpha(N+1)}+q^{2\alpha+1}b^\dagger(f)b(g). \tag{15}$$

The normalization used in this paper agrees with [28], implicitly with [35], and with one of the versions introduced by [27] (written with a different parameter $\tilde{q}=q^{-1/2}$). In most of the papers in the bibliography we have the convention $\beta=(1-q)^{-1/2}$, $\alpha=0$. The existence of a vector $\Psi\neq0$ in Fock space with

$$b(f)\Psi=\langle f,\varphi\rangle\Psi \tag{16}$$

is then equivalent to $\|\varphi\|\leqq(1-q)^{-1/2}$, and the joint eigenvectors of the $b(f)$ are precisely those of the $a(f)$.

On the other hand, when $\alpha>0$, the series for Ψ satisfying (16) diverges for all $f\neq0$, and no joint eigenvectors can be found. The interesting cases are for $\alpha<0$. The coefficients of the power series then decrease more rapidly, and the series defines an entire function. Hence no constraint is placed on $\|\varphi\|$, and the notion of peripheral coherent states makes no sense. This is related to the fact that the relations then explicitly involve the operator N, and hence make sense only in the Fock representation. The relations appeared for the first time (for a single generator, a so-called q-oscillator) in [4] with $\alpha=-1/4$, $|\beta|^2=q^{1/2}(1-q)^{-1/2}$, and in [27] with the same constants, but using $\tilde{q}=q^{-1/2}$. Of potential interest is also the case $\alpha=-1/4$, $|\beta|^2=q(1-q)^{-1}$ in which the relations can be expressed by an ordinary commutator, i.e. $[a(f),a^\dagger(g)]=\langle f,g\rangle q^{-N}$.

3. Peripheral Coherent States

A remarkable fact about the peripheral coherent states, i.e. the coherent states ω_φ with $\|\varphi\|=1$, is the following: if $\mathscr{H}'\subset\mathscr{H}$, there is a canonical embedding $\mathscr{E}_{\mathscr{H}'}(q)\subsetneq\mathscr{E}_{\mathscr{H}}(q)$. With respect to this embedding a peripheral coherent state on $\mathscr{E}_{\mathscr{H}'}(q)$ has a *unique* extension to $\mathscr{E}_{\mathscr{H}}(q)$, which is also a peripheral cohererent state. This follows readily from the first item of the following proposition.

7 Proposition. *Let $\varphi\in\mathscr{H}$, with $\|\varphi\|=1$. Then*

(1) *ω_φ is the uniquely characterized by the condition $\omega_\varphi((a^\dagger(\varphi)-\mathbb{1})(a(\varphi)-\mathbb{1}))=0$.*
(2) *For $\dim\mathscr{H}>1$ the kernel of the* GNS*-representation π_φ contains every closed two-sided ideal of $\mathscr{E}_{\mathscr{H}}(q)$.*

Proof. We set $a=a(\varphi)$, for short.
(1) Let Ω denote the GNS-vector of a state ω with $\omega((a^*-\mathbb{1})(a-\mathbb{1}))=0$. Then we have $a\Omega=\Omega$, and since on the subspace generated by the $(a^*)^n\Omega$, a is unitary, we also have $a^*\Omega=\Omega$. Then for any vector $\psi\in\mathscr{H}$, we get

$$\begin{aligned}a(\psi)\Omega&=a(\psi)(a^*)^n\Omega\\&=(1-q^n)\langle\psi,\varphi\rangle(a^*)^{n-1}\Omega+q^n(a^*)^n a(\psi)\Omega.\end{aligned}$$

Since $\|(a^*)^n\|$ is uniformly bounded we can take the limit $n \to \infty$ on the right-hand side, and obtain $a(\psi)\Omega = \langle\psi, \varphi\rangle\Omega$. Hence Ω implements ω_φ.
(2) Let $\mathcal{J} \subset \mathcal{E}_{\mathcal{H}}(q)$ be a closed two-sided ideal, and consider the algebra $\tilde{\mathcal{E}} = \mathcal{E}_{\mathcal{H}}(q)/\mathcal{J}$ with quotient mapping η: $\mathcal{E}_{\mathcal{H}}(q) \to \tilde{\mathcal{E}}$. Since $\dim \mathcal{H} > 1$ we know from Proposition 4 in [18] that $\eta(a) \in \tilde{\mathcal{E}}$ cannot be unitary, and consequently that the spectrum of $\eta(a)$ contains the spectrum of a_0, the generator in the Fock representation. This is the unit disk, and hence the spectrum of $\eta(a)$ includes 1. It follows (by compactness of the state space of a C*-algebra) that there is a representation $\tilde{\pi}: \tilde{\mathcal{E}} \to \mathcal{B}(\mathcal{R})$ in which 1 is an eigenvalue of $\tilde{\pi}(\eta(a))$. But then by part (1) we have

$$\langle \xi, \pi(\eta(X))\xi \rangle = \omega_\varphi(X) , \tag{$*$}$$

where ξ is the corresponding normalized eigenvector. The kernel of π_φ is the set of $Y \in \mathcal{E}_{\mathcal{H}}(q)$ such that $\omega_\varphi(X^*YZ) = 0$ for all $X, Z \in \mathcal{E}_{\mathcal{H}}(q)$. By equation $(*)$ it is now plain that $\mathcal{J} = \ker \eta \subset \ker \pi \circ \eta \subset \ker \pi_\varphi$. ■

The second part of this proposition suggests the following terminology:

8 Definition. *Let $\mathcal{H}$ be a Hilbert space with* $\dim \mathcal{H} > 1$. *Then the q-***Cuntz** *algebra $\mathcal{O}_{\mathcal{H}}(q)$ over $\mathcal{H}$ is the quotient of $\mathcal{E}_{\mathcal{H}}(q)$ by its unique largest ideal. Equivalently, $\mathcal{O}_{\mathcal{H}}(q) = \pi_\varphi(\mathcal{E}_{\mathcal{H}}(q))$ for any peripheral coherent representation.*

Of course, for $q = 0$ the q-Cuntz algebra is just the usual Cuntz algebra $\mathcal{O}_{\dim \mathcal{H}}$. For $\dim \mathcal{H} < \infty$ Conjecture C says that $\mathcal{O}_{\mathcal{H}}(q) \cong \mathcal{O}_{\mathcal{H}}(0)$, and this is proven [18] for $|q| < \sqrt{2} - 1$. We will further extend this interval in Sect. 4, using the results of [12]. When $\dim \mathcal{H} = \infty$, one can show that the Fock representation of $\mathcal{E}_{\mathcal{H}}(q)$ is simple [24]. Hence in that case, $\mathcal{O}_{\mathcal{H}}(0)$ is isomorphic to the Fock representation of $\mathcal{E}_{\mathcal{H}}(q)$.

From the Corollary 6 we know that the non-peripheral coherent representations are all equivalent. For the peripheral coherent representations we know that the C*-algebras $\pi_\varphi(\mathcal{E}_{\mathcal{H}}(q))$ are all equal. However, the von Neumann algebras $\pi_\varphi(\mathcal{E}_{\mathcal{H}}(q))''$ are not: in the following proposition we show that all peripheral coherent representations are disjoint.

9 Proposition. *Let $\varphi, \psi \in \mathcal{H}$, with $\|\varphi\| = \|\psi\| = 1$, and let $\pi: \mathcal{E}_{\mathcal{H}}(q) \to \mathcal{B}(\mathcal{R})$ be any representation.*
(1) *The strong operator limit*

$$P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \pi(a^\dagger(\varphi))^k \tag{15}$$

exists, and is a self-adjoint projection.
(2) *For $\chi \in \mathcal{H}$: $\pi(a(\chi))P(\varphi) = \langle \chi, \varphi \rangle P(\varphi)$.*
(3) *Let $P(0)$ denote the orthogonal projection onto the space*

$$\mathcal{N} = \{\Omega \in \mathcal{R} \mid \forall \varphi \in \mathcal{H}: \pi(a(\varphi))\Omega = 0\}$$

of Fock vectors. Then, for φ, ψ unit vectors in $\mathcal{H}$, or zero, and for $X \in \mathcal{E}_{\mathcal{H}}(q)$:

$$P(\varphi)\pi(X)P(\psi) = \begin{cases} \omega_\varphi(X)P(\varphi) & \varphi = \psi \\ 0 & \varphi \neq \psi \,. \end{cases}$$

Proof. In the proof we will suppress the representation of π for notational convenience. The existence of the limit (1) follows from the Mean Ergodic Theorem (e.g.

Corollary VIII, 5.4 in [11], and the fact that the powers $a^{\dagger}(\varphi)^k$ are uniformly norm bounded by Lemma 3.(3). Let $\chi\in\mathscr{H}$. Then

$$a(\chi)\frac{1}{n}\sum_{k=1}^{n} a^{\dagger}(\varphi)^{n}=\frac{1}{n}\sum_{k=1}^{n}\{(1-q^{k})\langle\chi,\varphi\rangle a^{\dagger}(\varphi)^{k-1}+q^{k}a^{\dagger}(\varphi)^{k}a(\chi)\}$$

$$=\langle\chi,\varphi\rangle\frac{1}{n}\sum_{k=1}^{n} a^{\dagger}(\varphi)^{n}+\mathrm{Rest}$$

with the estimate

$$\|\mathrm{Rest}\|\leqq\frac{1}{n}\|\mathbb{1}-a^{\dagger}(\varphi)^{n}\|+\frac{1}{n}\frac{1}{1-|q|}\beta_{+}(q)\,,$$

and $\beta_{+}(q)$ from Eq. (8). Taking the strong limit $n\to\infty$ we find (2). In particular, we have $a(\varphi)P(\varphi)=P(\varphi)$, which implies $P(\varphi)^{*}P(\varphi)=\text{weak}-\lim(1/n)\cdot\sum_{k=1}^{n}a(\varphi)^{k}P(\varphi)=P(\varphi)$, and hence that $P(\varphi)$ is an orthogonal projection.

To prove (3), let X be a polynomial in the generators, which we may assume to be Wick ordered. Then after finitely many applications of (2) we find that $P(\varphi)XP(\psi)$ is equal to some factors times $P(\varphi)P(\psi)$. If $\varphi=\psi$(possibly $\varphi=\psi=0$), the factors add up to $\omega_{\varphi}(X)$, and the result follows because $P(\varphi)$ is a projection.

It remains to show that $P(\varphi)P(\psi)=0$, when $\varphi\neq\psi$, and $\varphi\neq 0$. Since $P(\varphi)$ is also the weak limit of $(1/n)\sum_{k=1}^{n}a(\varphi)^{k}$ this follows from (2) and the observation that $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n}\langle\varphi,\psi\rangle^{k}$ vanishes, unless $\varphi=\psi$. ■

We can use the universal representation for π. Then the projections $P(\varphi)$ are interpreted as projections in the universal enveloping algebra $\mathscr{E}_{\mathscr{H}}(q)^{**}$. By (4) their central supports in $\mathscr{E}_{\mathscr{H}}(q)^{**}$ are mutually disjoint. Hence, the projections $P(\varphi)$ with $\|\varphi\|=1$, and the single projection $P(0)$ (for all the non-peripheral coherent representations) precisely label the quasi-equivalence classes of coherent representations.

From (3) one readily concludes that any representation space $\mathscr{R}$ can be split into a direct sum $\mathscr{R}=\mathscr{R}_{\varphi}\oplus\mathscr{R}_{\varphi}^{\perp}$, where $\mathscr{R}_{\varphi}$ is the cyclic subspace containing $P(\varphi)\mathscr{R}$. Then the representation restricted to the first summand is a direct multiple of π_{φ} with multiplicity $\dim P(\varphi)\mathscr{R}$. The decomposition into a Fock and a non-Fock sector (of which Lemma 3 is a special case) is obtained for $\varphi=0$. It is especially useful because the orthogonal complement $\mathscr{R}_{0}^{\perp}$ has an interesting description [19, 20]: it consists of all vectors with an "infinite iteration history" with respect to the operators $a^{\dagger}(f)$, i.e. it is the intersection over $n\in\mathbb{N}$ of the closed subspaces generated by all vectors of the form $a^{\dagger}(f_1)\cdots a^{\dagger}(f_n)\Phi$ with $f_1,\ldots,f_n\in\mathscr{R}$, and $\Phi\in\mathscr{R}$. This decomposition can be viewed as an analogue of the "Word decomposition" of a contraction operator in Hilbert space [29].

4. Conjecture C and the Fock Representation

We begin by making precise the Conjecture C mentioned in the introduction. We present it here, not because we are completely convinced of its truth, but because we believe that it presents an excellent target for future research.

10 Conjecture C. *Let* $-1<q<1$, *and let* $\mathscr{H}$ *be a Hilbert space with* $\dim\mathscr{H}=d<\infty$. *Let* $\mathscr{E}_{\mathscr{H}}(q)$, *and* $\mathscr{E}_{\mathscr{H}}(0)$ *denote the universal algebras introduced in Proposition* 1, *and* *denote the respective generators by* $a^\dagger(f)\in\mathscr{E}_{\mathscr{H}}(q)$, *and* $v^\dagger(f)\in\mathscr{E}_{\mathscr{H}}(0)$. *Let*

$$\rho=\left(\sum_{i=1}^{d} a^\dagger(e_i)a(e_i)\right)^{1/2}\in\mathscr{E}_{\mathscr{H}}(q)$$

for some (or any) orthogonal basis $e_1,\ldots,e_d\in\mathscr{H}$.
Then there is a C*-*isomorphism* $\eta:\mathscr{E}_{\mathscr{H}}(0)\to\mathscr{E}_{\mathscr{H}}(q)$ *such that*

$$a^\dagger(f)=\rho\eta(v^\dagger(f)) .$$

Moreover, 0 *is an isolated point in the spectrum of* ρ, *and the eigenprojection corresponding to this eigenvalue is* $\eta\ (\mathbb{1}-\sum_{i=1}^{d} v^\dagger(e_i)v(e_i))$.

Note that this conjecture can only be formulated for finite d, since the sum defining ρ cannot converge in norm (even though it converges strongly in every representation). For this reason the universal C*-algebras for the case of infinitely many generators have to be treated separately. For $q=0$, $d=\infty$ it is well known that $\mathscr{E}_{\mathscr{H}}(0)\cong\mathscr{O}_{\mathscr{H}}(0)$ is simple, whereas it has an ideal isomorphic to the compact operators for $d<\infty$. Analogous phenomena occur for $q\neq0$, at least in the Fock representation [24]. There are a number of interesting equivalent reformulations of the conjecture. The following one is of the form in which this conjecture was proven [18] for all finite d, and the restricted range $|q|<\sqrt{2}-1$.

11 Proposition. *Let* $-1<q<1$, *and let* $\mathscr{H}$ *be a Hilbert space with* $\dim\mathscr{H}=d<\infty$, *and let* $e_1,\ldots,e_d\in\mathscr{H}$ *be an orthonormal basis. Then Conjecture C is equivalent to the conjunction of the following two statements*:

(A) *In the* C*-*algebra* $\mathscr{M}_d(\mathscr{E}_{\mathscr{H}}(q))$ *of* $d\times d$-*matrices with entries in* $\mathscr{E}_{\mathscr{H}}(q)$, *the matrix* $X_{ij}=a(e_i)a^\dagger(e_j)$ *is strictly positive.*

(B) *Let* $\mathscr{R}$ *be a Hilbert space, and let* v_i, $i=1,\ldots,d$ *be bounded operators on* $\mathscr{R}$ *satisfying the relations* $v_iv_j^*=\delta_{ij}\mathbb{1}$. *Then there is a unique positive semidefinite bounded operator* ρ *on* $\mathscr{R}$ *such that* $a^\dagger(e_i)=\rho v_i^*$ *satisfies relations* (1), *and such that* $\mathbb{1}-\sum v_i^*v_i$ *projects onto the kernel of* ρ. *Moreover, this unique* ρ *necessarily lies in the* C*-*algebra generated by the operators* v_i.

Proof. Assume (A). Consider in the universal representation $\pi:\mathscr{E}_{\mathscr{H}}(q)\to\mathscr{B}(\mathscr{R})$ the operators

$$A^\dagger:\mathscr{H}\otimes\mathscr{R}\to\mathscr{R} ,$$
$$A^\dagger:f\otimes\psi\mapsto\pi(a^\dagger(f))\psi ,$$
$$A:\mathscr{R}\to\mathscr{H}\otimes\mathscr{R} ,$$
$$A:\psi\mapsto\textstyle\sum_{i=1}^{d} e_i\otimes a(e_i)\psi .$$

Then $\rho^2=A^\dagger A$, and $X=AA^\dagger$. The polar decomposition of $A^\dagger$ takes the form $A^\dagger=\rho V^\dagger=V^\dagger X^{1/2}$. Since $X>0$, the components v_i *of* $V^\dagger:f\otimes\psi\mapsto=\sum_i\langle e_i,f\rangle v_i\psi$ are in the C*-algebra $\pi(\mathscr{E}_{\mathscr{H}}(q))$, and $VV^\dagger=\mathbb{1}_{\mathscr{H}\otimes\mathscr{R}}$. The latter relation translates into $v_iv_j^*=\delta_{ij}\mathbb{1}$. With $v^\dagger(f):=\sum_i\langle e_i,f\rangle v_i^*$, these are the q-relations for v with $q=0$. Hence by the universal property there is a homeomorphism $\eta:\mathscr{E}_{\mathscr{H}}(0)\to\mathscr{E}_{\mathscr{H}}(q)$ with the required property. Since $\mathscr{E}_{\mathscr{H}}(0)$ has only one proper two-sided ideal, and this ideal is clearly not annihilated by η (consider the Fock representation of $\mathscr{E}_{\mathscr{H}}(q)$), η is injective. It remains to be shown that η is onto. This readily follows from condition (B).

Conversely, assume that Conjecture C holds. Then in the universal representation of $\mathcal{E}_{\mathcal{H}}(q)$ the isomorphism η provides a polar decomposition of the operator $A^\dagger$. Since the polar isometry in this case is an isometry, we must have that $AA^\dagger$ has no kernel. If the spectrum of $AA^\dagger$ in $\mathcal{M}_d(\mathcal{E}_{\mathcal{H}}(q))$ had an accumulation point at zero, zero would also have to be an eigenvalue by compactness of the state space, and the universality of the representation. Hence the spectrum must be bounded away from zero (A). Suppose that v_i and ρ are as in (B). Then by the universality of $\mathcal{E}_{\mathcal{H}}(q)$ there is a unique *-representation $\Phi: \mathcal{E}_{\mathcal{H}}(q) \to \mathcal{B}(\mathcal{H})$ such that $\Phi(a^\dagger(e_i)) = \rho v_i^*$. The polar decomposition of $A^\dagger$ in this representation is given by ρ and v_i. On the other hand, by the isomorphism with $\mathcal{E}_{\mathcal{H}}(0)$ we know that $\rho = \Phi(\sum_i a^\dagger(e_i)a(e_i))$ is in the C*-algebra generated by the $v_i^* = \Phi(v^\dagger(e_i))$. ■

Some consequences of Conjecture C would be the following: (1) The Fock representation of $\mathcal{E}_{\mathcal{H}}(q)$ is faithful for all q. (2) $\mathcal{E}_{\mathcal{H}}(q)$ has only one proper ideal, isomorphic to the compact operators. (3) the resulting quotient is isomorphic to the Cuntz algebra $\mathcal{O}_{\mathcal{H}}(0)$. Statement (3) may be extended to a version of Conjecture C on the level of the q-Cuntz algebras $\mathcal{O}_{\mathcal{H}}(q)$. Since $\mathcal{O}_{\mathcal{H}}(q)$ can be obtained as a quotient of *any* other representation of $\mathcal{E}_{\mathcal{H}}(q)$, we can utilise specific information about the Fock representation in approaching this problem.

Dykema and Nica [12], building on results of Zagier [36], were able to verify parts of Conjecture in the Fock representation π_0. For example, they verified statement (A) of Proposition 11 for that representation, by unitary implementation of a homomorphism

$$\eta_0 : \pi_0(\mathcal{E}_{\mathcal{H}}(0)) \to \pi_0(\mathcal{E}_{\mathcal{H}}(q)) \tag{16}$$

satisfying the properties required in Conjecture C, except surjectivity. Their results imply a lower bound

$$\pi_0(X) \geqq \frac{1-q}{1-|q|} \varepsilon(|q|)\mathbb{1} > 0 ,$$

$$\varepsilon(s) = \prod_{k=1}^{\infty} \frac{1-s^k}{1+s^k} = \sum_{k=-\infty}^{\infty} (-1)^k s^{k^2} \tag{17}$$

for $X \in \mathcal{M}_d(\mathcal{E}_{\mathcal{H}}(q))$, $X_{ij} = a(e_i)a^\dagger(e_j)$. (We remind the reader of the difference in normalization between [12], and this paper.) Moreover, they showed surjectivity of η_0 for

$$q^2 < \varepsilon(|q|)$$

$$\text{i.e.} \quad |q| < \approx 0.44 \; . \tag{18}$$

We can immediately translate these results into a partial verification Conjecture C on the Cuntz algebra level:

12 Theorem. *Let* $-1 < q < 1$, *and let* $\mathcal{H}$ *be a Hilbert space with* $\dim \mathcal{H} = d < \infty$. *Let* $\mathcal{O}_{\mathcal{H}}(q)$, *and* $\mathcal{O}_{\mathcal{H}}(0) \equiv \mathcal{O}_d$ *denote the q-Cuntz algebra, and the Cuntz algebra, as in Definition* 8, *and denote by* $\pi_1 : \mathcal{E}_{\mathcal{H}}(q) \to \mathcal{O}_{\mathcal{H}}(q)$ (*or* $\mathcal{E}_{\mathcal{H}}(0) \to \mathcal{O}_{\mathcal{H}}(0)$) *the respective quotient maps. Let* $\rho \in \mathcal{E}_{\mathcal{H}}(q)$ *be as in Conjecture C.*

(1) *Then there is a (not necessarily surjective) C*-homomorphism* $\hat{\eta}: \mathcal{O}_{\mathscr{H}}(0) \to \mathcal{O}_{\mathscr{H}}(q)$ *such that* $\pi_1(a^\dagger(f)) = \pi_1(\rho)\hat{\eta}\pi_1(v^\dagger(f))$.

(2)
$$\pi_1(\rho^2) \geqq \frac{1-q}{1-|q|}\varepsilon(|q|)\mathbb{1} .$$

(3) Let ω_φ *be a peripheral coherent state on* $\mathcal{O}_{\mathscr{H}}(q)$. *Then* $\omega_\varphi \circ \hat{\eta}$ *is the peripheral coherent state on* $\mathcal{O}_{\mathscr{H}}(0)$ *associated with* φ.

(4) *When* $q^2 < \varepsilon(|q|)$, $\hat{\eta}$ *is onto, and hence an isomorphism.*

Proof. The eigenprojection onto the kernel of $\pi_0(\rho)$ is $P_0 \equiv \eta_0(\mathbb{1} - \pi_0(\sum v_i^* v_i)) \in \pi_0(\mathscr{E}_{\mathscr{H}}(q))$. Consider a peripheral coherent representation π_φ of $\pi_0(\mathscr{E}_{\mathscr{H}}(q))$. Since peripheral coherent states are pure, this representation is irreducible. On the other hand, $\pi_\varphi(P_0)$ projects onto the set of Fock vectors in that representation. The invariant subspace generated from a Fock vector is a copy of Fock space, on which the projection $P(\varphi)$, as in Proposition 9, vanishes. On the other hand, $P(\varphi) \neq 0$, so the Fock sector cannot be the whole space, and must be zero by irreducibility. Hence $\pi_\varphi(P_0) = 0$, and $\pi_\varphi(\rho) > 0$. The bound (2) then follows from Eq. (17). Moreover, under the map

$$\mathscr{E}_{\mathscr{H}}(0) \xrightarrow{\pi_0} \pi_0(\mathscr{E}_{\mathscr{H}}(0)) \xrightarrow{\eta_0} \pi_0(\mathscr{E}_{\mathscr{H}}(q)) \xrightarrow{\eta_\varphi} \pi_\varphi(\mathscr{E}_{\mathscr{H}}(q)) \cong \mathcal{O}_{\mathscr{H}}(q)$$

$\mathbb{1} - \sum_i v^\dagger(e_i)v(e_i)$ becomes $\pi_\varphi(P_0) = 0$. Hence it lifts to the quotient as $\hat{\eta}: \mathcal{O}_{\mathscr{H}}(0) \to \mathscr{E}_{\mathscr{H}}(0)$. The properties (1), (4) of this map are readily verified from those proven for the Fock representation.

To see (3), recall from Lemma 3 that the eigenvalue equation $\pi_\varphi(a(\varphi) - 1)\omega_\varphi = 0$ can only be satisfied when we also have $\pi_\varphi(a^\dagger(\varphi) - 1)\omega_\varphi = 0$. Hence with a basis $e_1 = \varphi, e_2, \ldots, e_d \in \mathscr{H}$ we get

$$\pi_\varphi(\rho^2)\Omega_\varphi = \sum_i \pi_\varphi(a^\dagger(e_i)a(e_i))\Omega_\varphi = \pi_\varphi(a^\dagger(e_1)a(e_1))\Omega_\varphi = \Omega_\varphi .$$

Since $\pi_\varphi(\rho) > 0$, this entails

$$\pi_\varphi \hat{\eta}(v(f))\Omega_\varphi = \pi_\varphi(\rho)^{-1}\pi_\varphi(a(f))\Omega_\varphi = \langle f, \varphi\rangle \pi_\varphi(\rho)^{-1}\Omega_\varphi = \langle f, \varphi\rangle \Omega_\varphi .$$

Therefore, for $X \in \mathscr{E}_{\mathscr{H}}(0)$, $\omega_\varphi(\hat{\eta}(Xv(f))) = \langle f, \varphi\rangle \omega_\varphi(\hat{\eta}(X))$, which proves (3). ∎

5. The Boundary Points $q = \pm 1$

Apart from Conjecture C and its special cases, an interesting problem concerning the q-relations (1) is to show that they define a continuous field of C*-algebras $\mathscr{E}_{\mathscr{H}}(q)$ in the parameter q in the sense of Dixmier [10]. If Conjecture C holds, i.e. an isomorphism $\eta_q: \mathscr{E}_{\mathscr{H}}(q) \to \mathscr{E}_{\mathscr{H}}(0)$ exists, this problem amounts, for $q \neq \pm 1$, to the question whether the element $\eta_q^{-1}(\rho) \in \mathscr{E}_{\mathscr{H}}(0)$ depends continuously on q. (For $|q| < \sqrt{2} - 1$, this continuity is easily verified from the argument in [18]). The interesting questions arise at the boundaries $q = \pm 1$.

The role of the coherent states is that of a *continuous field of states* in the following sense: for any polynomial X in the variables $a^\dagger(f)$, $a(g)$, and q (with $f, g \in \mathscr{H}$), and for every fixed $\varphi \in \mathscr{H}$, the coherent expectation $\omega_\varphi(X)$ is a continuous function of q. The continuity of the field $q \mapsto \mathscr{E}_{\mathscr{H}}(q)$ is related to the existence of sufficiently many such continuous fields of states.

As a first step towards understanding the continuity at $q=\pm 1$, we compute the algebras $\mathscr{E}_{\mathscr{H}}(\pm 1)$, and their coherent states. Recall that for $q=1$ we have imposed, in Proposition 1, the bound $\sum_i a^\dagger(e_i)a(e_i)\leqq \mathbb{1}$ for any family of orthogonal vectors.

13 Proposition. *Let $\mathscr{H}$ be a Hilbert space. Then $\mathscr{E}_{\mathscr{H}}(+1)$ is isomorphic to the algebra of weakly continuous functions on the unit ball of $\mathscr{H}$. A state on this algebra is coherent if and only if it is pure.*

Proof. We have to show that $\mathscr{E}_{\mathscr{H}}(+1)$ is abelian. Clearly, $[a^\dagger(f), a(g)]=0$ for all f, g. In particular, each $a^\dagger(f)$ is a bounded normal operator. By Fuglede's Theorem [14, 30], $a^\dagger(f)$ and $a^\dagger(g)$ also commute, and $\mathscr{E}_{\mathscr{H}}(1)$ is abelian. A pure state ω must be multiplicative, and is hence determined by its value $\omega(a^\dagger(f))$ on the generators. Since $a^\dagger$ is linear, and since $a^\dagger(f)a(f)\leqq \|f\|^2\mathbb{1}$, this expression must be a bounded linear functional, and hence of the form $\omega(a^\dagger(f))=\langle\varphi, f\rangle$ with $\varphi\in\mathscr{H}$, $\|\varphi\|\leqq 1$. Hence $\omega=\omega_\varphi$ is coherent, and any coherent state is obtained in this way. Note that, for all polynomials X in the generators $a^\dagger(f)$, $a(f)$, the function $\varphi\mapsto\omega_\varphi(X)$ is weakly continuous on the unit ball. On the other hand, the algebra of such polynomials is dense in the algebra of all weakly continuous functions by the Stone–Weierstrass Theorem. ∎

Note that by this proposition the set of coherent states is faithful at $q=+1$. This suggests that they may be useful for proving the continuity at $q=1$, provided one can show that collection of coherent representations is also faithful for $q<1$. In the following proposition we see that faithfulness does not hold at the other limit point $q=-1$, where the relations become

$$a(f)a^\dagger(g)+a^\dagger(g)a(f)=2\langle f, g\rangle\mathbb{1}\ . \tag{21}$$

The proposition is based on well-known results in the theory of Clifford algebras [5, 32, 25], which arise from these relations either by taking f, g to be in a real Hilbert space, and setting $a^\dagger(f)=a(f)$. In even dimensions this is equivalent to taking (21), and adding the relation that the $a^\dagger(f)$ anti-commute with each other. The algebra arising in this way is the Fock representation of (21), and we will refer to it as the CAR-algebra [8]. The point of the following proposition is that no anti-commutation relation is added, but that such a relation automatically holds in every irreducible representation.

14 Proposition. *Let $\mathscr{H}$ be a Hilbert space, and consider the* C*-*algebra $\mathscr{E}_{\mathscr{H}}(-1)$ as defined in Proposition* 1. *Then*

(1) *The elements*
$$a^\dagger(f)a^\dagger(g)+a^\dagger(g)a^\dagger(f)=2\hat{\theta}(f,g)\ ,$$
for $f, g\in\mathscr{H}$ generate the center of $\mathscr{E}_{\mathscr{H}}(q)$.

(2) *The center of $\mathscr{E}_{\mathscr{H}}(-1)$ is isomorphic to $\mathscr{C}(\mathrm{S})$, where* S *is the set of all symmetric bilinear forms $\theta:\mathscr{H}\times\mathscr{H}\to\mathbb{C}$ such that*
$$|\theta(f,g)|\leqq\|f\|\,\|g\|$$
for all f, $g\in\mathscr{H}$, with the coarsest topology making the functions $\theta\mapsto\theta(f,g)$ continuous.

(3) *Let θ be a symmetric bilinear form satisfying the above bound, and let $\mathscr{N}(\theta)$ denote the real subspace of vectors $f\in\mathscr{H}$ such that $\theta(f,f)=\|f\|^2$. Let $r(\theta)$ denote*

the dimension of the complement of $\mathcal{N}(\theta)$ in $\mathcal{H}$, taken as a real Hilbert space. Let $\mathcal{E}_{\mathcal{H}}(-1, \theta)$ denote the quotient of $\mathcal{E}_{\mathcal{H}}(-1)$ by the relations $\hat{\theta}(f, g) = \theta(f, g)\mathbb{1}$. Then

(a) *If $r(\theta)$ is finite and even, $\mathcal{E}_{\mathcal{H}}(-1, \theta)$ is isomorphic to the algebra of $2^{r(\theta)/2}$-dimensional matrices.*
(b) *If $r(\theta)$ is finite and odd, $\mathcal{E}_{\mathcal{H}}(-1, \theta)$ is isomorphic to the direct sum of two copies of the algebra of $2^{(r(\theta)-1)/2}$-dimensional matrices.*
(c) *If $r(\theta)$ is infinite, $\mathcal{E}_{\mathcal{H}}(-1, \theta)$ is isomorphic to the* CAR-*algebra on an infinite dimensional Hilbert space.*

Proof. (1) By an elementary computation one verifies that $\hat{\theta}(f, g)$ commutes with all $a(h)$. Hence $\hat{\theta}(f, g)$ is normal, and by Fuglede's Theorem [30] it must also commute with $a^\dagger(h)$. Hence $\hat{\theta}(f, g)$ is the center for all $f, g \in \mathcal{H}$. Let $\mathcal{C}(\mathrm{S}) \subset \mathcal{E}_{\mathcal{H}}(-1)$ denote the C*-algebra generated by the $\hat{\theta}(f, g)$. Its spectrum space S is the set of those symmetric bilinear forms θ, which may arise in an irreducible representation of $\mathcal{E}_{\mathcal{H}}(-1)$, i.e. those θ for which the relations

$$a(f)a^\dagger(g) + a^\dagger(g)a(f) = 2\langle f, g\rangle\mathbb{1} ,$$
$$a^\dagger(f)a^\dagger(g) + a^\dagger(g)a^\dagger(f) = 2\theta\langle f, g\rangle\mathbb{1} \qquad (*)$$

have a solution by a bounded linear operator $a^\dagger : \mathcal{H} \to \mathcal{B}(\mathcal{R})$ for some Hilbert space $\mathcal{R}$. The rest of the proof depends on the analysis of this set of relations.

The unique feature of the relations (1) at $q = -1$ is the symmetry with respect to exchange of a and $a^\dagger$ (up to questions of linearity/antilinearity). Therefore we will consider $\mathcal{H}$ now as a real Hilbert space of dimension $\dim_{\mathbb{R}}(\mathcal{H}) = 2\dim_{\mathbb{C}}(\mathcal{H})$, and introduce the hermitian generators

$$s(f) := \frac{1}{2}(a^\dagger(f) + a(f)) ,$$

which are real linear in $f \in \mathcal{H}$. From $s(f)$ and the complex structure on $\mathcal{H}$ we can recover the original generators by the formula $a^\dagger(f) = s(f) - is(if)$. In terms of the new generators we get the relations

$$s(f)s(g) + s(g)s(f) = \Re\mathrm{e}\{\langle f, g\rangle + \theta(f, g)\} := 2\Theta(f, g)\mathbb{1} . \qquad (**)$$

Clearly, Θ is a symmetric, real-valued form on the real Hilbert space $\mathcal{H}$. Since $\Theta(f,f) = s(f)^2 \geqq 0$, the positivity of Θ is necessary for the existence of a representation, and hence for $\theta \in \mathrm{S}$.

We claim that the positivity of Θ is equivalent to the inequality in item (2) of the proposition. Clearly, $|\theta(f,f)| \leqq \|f\|^2$ is sufficient for $\Theta(f,f) \geqq 0$. Conversely, assume $2\Theta(f,f) = \|f\|^2 + \Re\mathrm{e}\theta(f,f) \geqq 0$. Substituting $f \mapsto if$ in this inequality we get that $|\Re\mathrm{e}\theta(f,f)| = \|f\|^2$. Hence by the Schwarz inequality in the real Hilbert space $\mathcal{H}$ we have $|\Re\mathrm{e}\theta(f, g)| = \|f\| \, \|g\|$, and the result follows by replacing f in this inequality by a complex multiple of f. It is also easy to see that the rank of Θ is equal to $r(\theta)$, as defined in term (3).

In order to prove the characterization of S, the joint spectrum of the central elements $\hat{\theta}(f, g)$, it remains to be shown that for every $\Theta \geqq 0$ there is some representation of $(*)$. We will simultaneously prove (3) by constructing all such representations (assuming $\Theta \geqq 0$), and showing that they have the form given in (3) with $r(\theta) = \operatorname{rank} \Theta$.

We can find an orthonormal basis $\{e_i\} \subset \mathscr{H}$ such that $\Theta(e_i, e_j) = \Theta_i \delta_{ij}$. The generators $s(e_i)$ with $\Theta_i = 0$ have to be zero, and the remaining ones can be multiplied by $\Theta_i^{-1/2}$, so that $(**)$ becomes equivalent to the relations

$$s_i s_j + s_j s_i = 2\delta_{ij} \mathbb{1} , \qquad (***)$$

where the $s_i = s_i^*$, and $i = 1, \ldots$, rank Θ. Hence the isomorphism type of $\mathscr{E}_{\mathscr{H}}(-1, \theta)$ depends only on rank Θ. One readily verifies that rank $\Theta = r(\theta)$, as defined in (3).

For finite $r(\theta)$, (3) follows from the standard results of the representation theory of Clifford algebras (see e.g. Theorems 2 and 3 in Sect. 9, No. 4 of [5]). The CAR-algebras are a special case of these arguments with $\theta = 0$. For infinitely many generators s_i we therefore get an inductive limit which we can take along the simple algebras of even numbers of generators [32], and which is identical with the inductive limit defining the CAR-algebra over an infinite dimensional Hilbert space. We note that in the case (3b) the center is generated by the odd element

$$\hat{s} = s_1 \cdots s_{r(\theta)} ,$$

which is unitary and satisfies $\hat{s}^2 = \pm \mathbb{1}$, depending on $r(\theta)$ modulo 4 [25]. In any case, $\hat{s}$ has two eigenvalues ± 1 or $\pm i$, which label the two irreducible representations with given θ, and are exchanged by the parity automorphism defined by $a^\dagger(f) \mapsto -a^\dagger(f)$.

We have shown that the algebra generated by the elements $\hat{\theta}$ is isomorphic to $\mathscr{C}(\mathrm{S})$ with S as described in item (2). It remains to be shown that this algebra coincides with the center of $\mathscr{E}_{\mathscr{H}}(-1)$. Let the center of $\mathscr{E}_{\mathscr{H}}(-1)$ be $\mathscr{C}(\tilde{\mathrm{S}})$ for some compact space $\tilde{\mathrm{S}}$. Since $\mathscr{C}(\mathrm{S})$ is a subalgebra, we have a canonical continuous surjection $p : \tilde{\mathrm{S}} \to \mathrm{S}$. Whenever $r(\theta)$ is even the relations $(*)$ have only one irreducible representation, which implies that $p^{-1}(\{\theta\})$ is a single point. Otherwise, $p^{-1}(\{\theta\})$ may consist of at most two points, corresponding to the two irreducible representations of $(*)$. The parity automorphism induces a homeomorphism $F : \tilde{\mathrm{S}} \to \tilde{\mathrm{S}}$ which leaves all points with even or infinite $r(\theta)$ fixed. Whenever $r(\theta)$ is odd, and $p^{-1}(\{\theta\})$ consists of two points, these two points are exchanged by F.

Since $\mathscr{H}$ has even or infinite real dimension, $r(\theta)$ is odd or infinite for a dense subset of $\theta \in \mathrm{S}$. Now consider some θ with odd $r(\theta)$, and let $\theta_\alpha \in \mathrm{S}$ be a net with $\theta_\alpha \to \theta$, and $r(\theta_\alpha)$ even for all α. Let $\tilde{\theta}_\alpha$ be the net $\tilde{\mathrm{S}}$ uniquely defined by $p(\tilde{\theta}_\alpha) = \theta_\alpha$. Since F is continuous, and $F\tilde{\theta}_\alpha = \tilde{\theta}_\alpha$ any cluster point $\tilde{\theta}$ of this net must also be fixed under F, and since p is continuous, we must have $p(\tilde{\theta}) = \theta$. But the only way $\tilde{\theta} \in p^{-1}(\{\theta\})$ can be fixed by F is that $p^{-1}(\{\theta\})$ is a single point. It follows that $p : \tilde{\mathrm{S}} \to \mathrm{S}$ is a bijection, and the center of $\mathscr{E}_{\mathscr{H}}(-1)$ coincides with the algebra $\mathscr{C}(\mathrm{S})$ generated by the $\hat{\theta}(f, g)$. ∎

It is clear that the coherent representations of $\mathscr{E}_{\mathscr{H}}(-1)$ are precisely those for which

$$\theta(f, g) = \langle \varphi, f \rangle \langle \varphi, g \rangle \qquad (22)$$

is a rank one operator. The set $\mathscr{N}(\theta)$ of vectors with $\|f\|^2 = \theta(f, f)$ is either null, when $\|\varphi\| < 1$, or is the one-dimensional real subspace spanned by φ when $\|\varphi\| = 1$. Hence for the peripheral coherent states on $\mathscr{E}_{\mathscr{H}}(-1)$ with $\dim \mathscr{H} < \infty$, $r(\theta)$ is odd.

When $\dim \mathscr{H} = 1$, all symmetric bilinear forms on $\mathscr{H}$ are of the form (21). Hence in this case the set of coherent states provides an everywhere faithful family of continuous fields of states. Accordingly, $q \mapsto \mathscr{E}_{\mathscr{H}}(q)$ is a continuous field of C*-algebras [28]. For $\dim \mathscr{H} > 1$ an interesting problem arises here: since the rank one bilinear forms are a low dimensional subset of S it is clear that many irreducible representations of $\mathscr{E}_{\mathscr{H}}(-1)$ are not coherent representations. It is possible to embed states on such non-coherent representations of $\mathscr{E}_{\mathscr{H}}(-1)$ into a continuous field of states for the field $q \mapsto \mathscr{E}_{\mathscr{H}}(q)$?

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