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Staggered Polarization of Vertex Models with $U_q(\widehat{sl}(n))$-Symmetry

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Abstract: In this paper we give an explicit formula for level 1 vertex operators related to $U_q(\widehat{sl}(n))$ as operators on the Fock spaces. We derive also their commutation relations. As an application we calculate with the vector representation of $U_q(\widehat{sl}(n))$, thereby extending the recent work on the staggered polarization of the XXZ-model.

1. Introduction

The Hamiltonian of the XXZ-model has $U_q(\widehat{sl}(2))$-symmetry in the thermodynamic limit. Recently, on the basis of this fact, the XXZ-model was formulated in the framework of representation theory of $U_q(\widehat{sl}(2))$. Let us explain the scheme described in [1] briefly.

First we recall XXZ-model as it appears in physics. The space of states of the XXZ-model is the infinite tensor product $\cdots \otimes V \otimes V \otimes V \otimes \cdots$, where $V = \mathbf{C}v_+ \otimes \mathbf{C}v_-$ is the two-dimensional vector space. The XXZ-Hamiltonian is the following operator formally acting on the above space:

$$H_{XXZ} = -\frac{1}{2}\sum_{k\in\mathbf{Z}} (\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \frac{q+q^{-1}}{2}\sigma_k^z\sigma_{k+1}^z)\,,$$

where σ^x, σ^y, σ^z are the Pauli matrices on V, σ_k^α acting on the k$^{\text{th}}$ component of $\cdots \otimes V \otimes V \otimes V \otimes \cdots$. Let $U'_q(\widehat{sl}(2))$ denote the subalgebra of $U_q(\widehat{sl}(2))$ with the grading operator d being dropped. It acts on V as follows:

$$e_1 \cdot v_- = v_+\,, \quad f_1 \cdot v_+ = v_-\,, \quad t_1 \cdot v_\pm = q^{\pm 1} v_\pm\,,$$

$$e_0 \cdot v_+ = v_-\,, \quad f_0 \cdot v_- = v_+\,, \quad t_0 \cdot v_\pm = q^{\mp} v_\pm\,.$$

$U'_q(\widehat{sl}(2))$ acts on $\otimes V \otimes V \otimes V \otimes \cdots$ via the iterated coproduct $\Delta^{(\infty)}$.

$$\Delta^{(\infty)}(t_i) = \cdots \otimes t_i \otimes t_i \otimes t_i \otimes \cdots\,,$$

$$\Delta^{(\infty)}(e_i) = \sum \cdots \otimes t_i \otimes t_i \otimes e_i \otimes 1 \otimes 1 \otimes \cdots\,,$$

$$\Delta^{(\infty)}(f_i) = \sum \cdots \otimes 1 \otimes 1 \otimes f_i \otimes t_i^{-1} \otimes t_i^{-1} \otimes \cdots\,.$$

{Formal manipulation shows that

$$[H_{XXZ}, U'_q(\widehat{sl}(2))]=0 .$$

Furthermore, we can identify d with $(H_{CTM}-S)/2$, where

$$H_{CTM}=-\frac{q}{1-q^2}\sum_{k\in \mathbf{Z}} k(\sigma_k^x\sigma_{k+1}^x+\sigma_k^y\sigma_{k+1}^y+\frac{q+q^{-1}}{2}\sigma_k^z\sigma_{k+1}^z)$$

and $S=\frac{1}{2}\sum_{k\in \mathbf{Z}}\sigma_k^z$ is the total spin operator. Letting T denote the shift operator on $\cdots\otimes V\otimes V\otimes V\otimes\cdots$, we can also check formally

$$\frac{2q}{1-q^2}H_{XXZ}=T^2dT^{-2}-d .$$

The above observation holds only in the infinite lattice case. Of course, H_{XXZ} and the action of $U_q(\widehat{sl}(2))$ are not literally well-defined. Nevertheless, when we consider the model in the anti-ferroelectric regime $-1<q<0$, we can construct a well-defined theory on "the space of physical states", which is the subspace consisting of finite excitations over the ground states in $\ldots\otimes V\otimes V\otimes V\otimes\cdots$. The formulation of [1] is based on the (hypothetical) identification

$$\text{"\textbf{the space of physical states}"}=\bigoplus_{0\leq i,j\leq 1} V(\Lambda_i)\otimes V(\Lambda_j)^* ,$$

where $V(\Lambda_i)$ is the level 1 highest weight irreducible $U_q(\widehat{sl}(2))$-module and $V(\Lambda_j)^*$ is the dual module of $V(\Lambda_j)$. The symbol $\otimes$ is to be understood with an appropriate completion, but we will not go into such details in the sequel. To motivate this hypothesis, consider the intertwiner of $U_q(\widehat{sl}(n))$-modules

$$\tilde{\Phi}_{\Lambda_i}^{\Lambda_{1-i}V}: V(\Lambda_i)\to V(\Lambda_{1-i})\otimes V, \qquad (*)$$

called vertex operators ([2]). In fact, such an operator exists, is unique up to a scalar, and gives an isomorphism if we properly define the completion of the right-hand side. Iterating the vertex operators, we get the following isomorphism:

$$V(\Lambda_i)\otimes V(\Lambda_j)^*\cong V(\Lambda_{1-i})\otimes V\otimes V(\Lambda_j)^*\cong V(\Lambda_{0\,\mathrm{or}\,1})\otimes V\otimes\cdots\otimes V\otimes V(\Lambda_j)^* .$$

It tells us that the local structure $\cdots\otimes V\otimes V\otimes\cdots$ in the naive picture is realized in the space $\bigoplus V(\Lambda_i)\otimes V(\Lambda_j)^*$. By composing $(*)$ with a similar vertex operator

$$V\otimes V(\Lambda_i)^*\to V(\Lambda_{1-i})^* ,$$

we get

$$V(\Lambda_i)\otimes V(\Lambda_j)^*\cong V(\Lambda_{1-i})\otimes V\otimes V(\Lambda_j)^*\cong V(\Lambda_{1-i})\otimes V(\Lambda_{1-j})^* .$$

The resulting isomorphism can be identified with the shift operator T. In this manner we can build a well-defined theory on $\bigoplus V(\Lambda_i)\otimes V(\Lambda_j)^*$ that captures all the essential features expected from the physical definition.

It is straightforward to generalize the above formulation to the models related to any quantum affine algebra. In this paper, we consider a multi spin-analogue of the XXZ-model related to the vector representation of $U_q(\widehat{sl}(n))$. Our main results are twofold. One is the bosonization of the level 1 vertex operators (Theorem 3.3, 3.4). The other is the exact calculation of the one-point functions (Theorem 5.2). The structure of this paper is the following. In sect. 2, we review the construction of the level 1 irreducible highest weight $U_q(\widehat{sl}(n))$-modules. In sect. 3, we construct the vertex operators on the bosonic Fock space explicitly. In sect. 4, we explain the mathematical formulation of models. In sect. 5, first, we derive an integral representation for the one-point function by using the bosonization of the vertex operators. Next, by using the commutation relations of the vertex operators, we derive difference equations for the one-point functions. This equation can be solved easily. As a result, we obtain an explicit formula of the one-point functions extending the previous work on the spontaneous staggered polarization for the XXZ-model ([13]).

2. Vertex Operator Representations of $U_q(\widehat{sl}(n))$

In this section, we review the construction of the level 1 irreducible highest weight modules following [5].

2.1. Notations. Throughout this paper, we fix a real number $q(-1<q<0)$ and a positive integer n. We denote $(q^k-q^{-k})/(q-q^{-1})$ and $\prod_{k=0}^{\infty}(1-aq^k)$ by $[k]$ and $(a;q)_\infty$ respectively. Most notations concerning Lie algebras follows [14]. Let P be a free $\mathbf{Z}$-module

$$P:=\bigoplus_{i=0}^{n-1}\mathbf{Z}\Lambda_i\oplus\mathbf{Z}\delta\ .$$

We call it the weight lattice. We define P^* as follows:

$$P:=\mathrm{Hom}(P,\mathbf{Z})=\bigoplus_{i=0}^{n-1}\mathbf{Z}h_i\oplus\mathbf{Z}d\ .$$

The pairing is given by $\langle\Lambda_i,h_j\rangle=\delta_{ij}$, $\langle\Lambda_i,d\rangle=0$, $\langle\delta,h_j\rangle=0$, $\langle\delta,d\rangle=1$. The indices are extended cyclically such as $\Lambda_i=\Lambda_{i+n}$, etc. Let $\alpha_0=-\Lambda_{n-1}+2\Lambda_0-\Lambda_1+\delta$, $\alpha_j=-\Lambda_{j-1}+2\Lambda_j-\Lambda_{j+1}$ $(1\leqq j\leqq n-1)$ be the simple roots. The invariant bilinear form on P is given by $(\alpha_i|\alpha_j)=-\delta_{ij-1}+2\delta_{ij}-\delta_{ij+1}$ and $(\delta|\delta)=0$. The projection to the classical weight lattice is given by $\bar{\Lambda}_i=\Lambda_i-\Lambda_0$, $\bar{\delta}=0$. $U_q(\widehat{sl}(n))$ is the $\mathbf{C}$-algebra generated by the symbols $\{t_i^{\pm}(=q^{\pm h_i}),q^d,e_i,f_i,(i=0,\cdots,n-1)\}$ which satisfy the following defining relations:

$$t_it_j=t_jt_i,\ t_ie_jt_i^{-1}=q^{\langle\alpha_j,h_i\rangle}e_j,\ t_if_jt_i^{-1}=q^{-\langle\alpha_j,h_i\rangle}f_j\ ,$$

$$[e_i,f_j]=\delta_{ij}\frac{t_i-t_i^{-1}}{q-q^{-1}}\ ,$$

$$\sum_{k=0}^{b}(-1)^k\begin{bmatrix}b\\k\end{bmatrix}e_i^ke_je_i^{b-k}=0,\ \sum_{k=0}^{b}(-1)^k\begin{bmatrix}b\\k\end{bmatrix}f_i^kf_jf_i^{b-k}=0\ ,$$

where $b=1-\langle\alpha_i,h_j\rangle$, $\begin{bmatrix}b\\k\end{bmatrix}=\dfrac{[b]!}{[k]![b-k]!}$, $[k]!=[1][2]\cdots[k]$. Throughout this paper, we denote $U_q(\widehat{sl}(n))$ by U_q. U_q' is the subalgebra of U_q generated by $\{t_i,e_i,f_i\}$. We denote the irreducible highest weight U_q (or U_q')-module with highest weight λ by $V(\lambda)$. We fix a highest weight vector of $V(\lambda)$ and denote it by $|\lambda\rangle$. The coproduct Δ and antipode S are given as follows:

$$\Delta(q^h)=q^h\otimes q^h,\ \Delta(e_i)=e_i\otimes 1+t_i\otimes e_i,\ \Delta(f_i)=f_i\otimes t_i^{-1}+1\otimes f_i\,,$$
$$S(q^h)=q^{-h},\ S(e_i)=-t_i^{-1}e_i,\ S(f_i)=-f_it_i\,.$$

When W is a U_q (or U'_q)-module, we introduce the left module structure on the dual space W^* by $x\cdot u^*(v)=u^*(S(x)\cdot v)$ for $x\in U_q$ (orU'_q), $u^*\in W^*$ and $v\in W$. If W has a weight decomposition $\bigoplus_\lambda W_\lambda$, we define the completion $\hat{W}=\prod_\lambda W_\lambda$. Normally we omit.

2.2. Drinfeld Generators of $\boldsymbol{U}_q$. We introduce another set of generators of U_q ([4]).

Definition. *$\mathscr{A}$ is the* **C**-*algebra generated by the symbols* $\{\gamma^{\pm\frac{1}{2}}, K_i, a_i(k), x_i^{\pm}(l)$ $(1\leqq i\leqq n-1,\ k\in\mathbf{Z}\backslash\{0\},\ l\in\mathbf{Z})\}$ *which satisfy the following defining relations*:

1) $$\gamma^{\pm\frac{1}{2}}\in \textit{Center of } \mathscr{A},\quad \gamma^{\frac{1}{2}}\gamma^{-\frac{1}{2}}=1\,,$$

$$K_iK_j=K_jK_i,\quad K_iK_i^{-1}=K_i^{-1}K_i=1\,,$$

2) $$[a_i(k),a_j(l)]=\delta_{k+l,0}\frac{[(\alpha_i|\alpha_j)k]}{k}\frac{\gamma^k-\gamma^{-k}}{q-q^{-1}}\,,$$

3) $$[a_i(k),K_j]=0\,,$$

4) $$K_ix_j^{\pm}(k)K_i^{-1}=q^{\pm\langle\alpha_j,h_i\rangle}x_j^{\pm}(k)\,,$$

5) $$[a_i(k),x_j^{\pm}(l)]=\pm\frac{[\langle\alpha_j,h_i\rangle k]}{k}\gamma^{\mp\frac{|k|}{2}}x_j^{\pm}(k+l)\,,$$

6) $$x_i^{\pm}(k+1)x_j^{\pm}(l)-q^{\pm(\alpha_i|\alpha_j)}x_j^{\pm}(l)x_i^{\pm}(k+1)$$
$$=q^{\pm(\alpha_i|\alpha_j)}x_i^{\pm}(k)x_j^{\pm}(l+1)-x_j^{\pm}(l+1)x_i^{\pm}(k)\,,$$

7) $$[x_i^+(k),x_j^-(l)]=\frac{\delta_{ij}}{q-q^{-1}}(\gamma^{(k-l)/2}\psi_i(k+l)-\gamma^{(l-k)/2}\varphi_i(k+l))\,,$$

where

$$\begin{cases}\displaystyle\sum_{k=0}^{\infty}\psi_i(k)z^{-k}=K_i\exp\left((q-q^{-1})\sum_{k=1}^{\infty}a_i(k)z^{-k}\right), \\ \displaystyle\sum_{k=0}^{\infty}\varphi_i(-k)z^{-k}=K_i^{-1}\exp\left(-(q-q^{-1})\sum_{k=1}^{\infty}a_i(-k)z^{-k}\right).\end{cases}$$

8) $$[x_i^{\pm}(k),x_j^{\pm}(l)]=0\quad \textit{for } \langle\alpha_i,h_j\rangle=0\,,$$

9) $$\{x_i^{\pm}(k)x_i^{\pm}(l)x_j^{\pm}(m)-(q+q^{-1})x_i^{\pm}(k)x_j^{\pm}(m)x_i^{\pm}(l)+x_j^{\pm}(m)x_i^{\pm}(k)x_i^{\pm}(l)\}$$
$$+\{x_i^{\pm}(l)x_i^{\pm}(k)x_j^{\pm}(m)-(q+q^{-1})x_i^{\pm}(l)x_j^{\pm}(m)x_i^{\pm}(k)+x_j^{\pm}(m)x_i^{\pm}(l)x_i^{\pm}(k)\}$$
$$=0\quad \textit{for } \langle\alpha_j,h_i\rangle=-1\,.$$ □

We know the following theorem.

Theorem 2.2 ([4]). *The following correspondence gives an isomorphism* $U_q' \cong \mathscr{A}$:

$$t_j \mapsto K_j, \quad e_j \mapsto x_j^{\pm}(0), \quad f_j \mapsto x_j^-(0) \quad (1 \leqq j \leqq n-1),$$

$$t_0 \mapsto \gamma K_1^{-1} \cdots K_{n-1}^{-1},$$

$$e_0 \mapsto [x_{n-1}^-(0), [x_{n-2}^-(0), \cdots [x_2^-(0), x_1^-(1)]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} K_1^{-1} \cdots K_{n-1}^{-1},$$

$$f_0 \mapsto K_1 \cdots K_{n-1} [[\cdots [x_1^+(-1), x_2^+(0)]_q \cdots, x_{n-2}^+(0)]_q, x_{n-1}^+(0)]_q .$$

Here we have set $[A, B]_q = AB - qBA$. □

2.3. Group Algebra $C[\bar{P}]$. For the constuction of representations, it is enough to consider only $\mathbf{C}[\bar{Q}]$, where $\bar{Q} = \bigoplus_{j=1}^{j=n-1} \alpha_j$ is the classical root lattice. But, for the construction of the vertex operators it is convenient to define $\mathbf{C}[\bar{P}]$ ($\bar{P} = \bigoplus_{j=2}^{j=n-1} \alpha_j \oplus \bar{\Lambda}_{n-1}$: the classical weight lattice). In fact, we use a central-extension of the group algebra of $\bar{P}$.

Definition. $\mathbf{C}[\bar{P}]$ *is the* $\mathbf{C}$*-algebra generated by the symbols* $\{e^{\alpha_2}, \cdots, e^{\alpha_{n-1}}, e^{\bar{\Lambda}_{n-1}}\}$ *which satisfy the following defining relations*:

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i | \alpha_j)} e^{\alpha_j} e^{\alpha_i} \qquad (2 \leqq i \leqq n-1),$$

$$e^{\alpha_i} e^{\bar{\Lambda}_{n-1}} = (-1)^{\delta_{in-1}} e^{\bar{\Lambda}_{n-1}} e^{\alpha_i} \qquad (2 \leqq i \leqq n-1).$$ □

For $\alpha = m_2\alpha_2 + \cdots + m_{n-1}\alpha_{n-1} + m_n\bar{\Lambda}_{n-1}$ ($\in \bar{P}$), we denote $e^{m_2\alpha_2} \cdots e^{m_{n-1}\alpha_{n-1}} e^{\bar{\Lambda}_{n-1}}$ by e^{α}. For example, $e^{\alpha_1} = e^{-2\alpha_2} e^{-3\alpha_3} \cdots e^{-(n-1)\alpha_{n-1}} e^{n\bar{\Lambda}_{n-1}}$, $e^{\bar{\Lambda}_i} = e^{-\alpha_{i+1}} e^{-2\alpha_{i+2}} \cdots e^{-(n-i-1)\alpha_{n-1}} e^{n\bar{\Lambda}_{n-1}}$. A simple calculation shows the following.

Proposition 2.3.

1) $$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i | \alpha_j)} e^{\alpha_j} e^{\alpha_i} \quad (1 \leqq i, j \leqq n-1),$$

2) $$e^{\alpha_1} e^{\bar{\Lambda}_{n-1}} = (-1)^{n-1} e^{\bar{\Lambda}_{n-1}} e^{\alpha_1},$$

3) $$e^{\alpha_i} e^{\bar{\Lambda}_1} = (-1)^{n\delta_{i1}} e^{\bar{\Lambda}_1} e^{\alpha_i} \quad (1 \leqq i \leqq n-1),$$

4) $$e^{\bar{\Lambda}_1} e^{\bar{\Lambda}_{n-1}} = (-1)^n e^{\bar{\Lambda}_{n-1}} e^{\bar{\Lambda}_1}.$$

□

2.4. Construction of Representations. Let

$$W_i := \mathbf{C}[a_j(-k)(1 \leqq j \leqq n-1, k \in \mathbf{Z}_{>0})] \otimes \mathbf{C}[\bar{Q}] e^{\bar{\Lambda}_i} \quad (0 \leqq i \leqq n-1).$$

We define the operators $a_j(k)$ $(1 \leqq j \leqq n-1)$, ∂_α, e^{α} $(\alpha \in \bar{Q})$, d on W_i as follows: for $f \otimes e^{\beta} = a_{i_1}(-n_1) \ldots (-n_k) \otimes e^{\beta} \in W_i$,

$$a_j(k) \cdot f \otimes e^{\beta} = \begin{cases} a_j(k) f \otimes e^{\beta} & (k<0) \\ [a_j(k), f] \otimes e^{\beta} & (k>0) \end{cases},$$

$$\partial_\alpha \cdot f \otimes e^{\beta} = (\alpha | \beta) f \otimes e^{\beta},$$

$$e^{\alpha} \cdot f \otimes e^{\beta} = f \otimes e^{\alpha} e^{\beta},$$

$$d \cdot f \otimes e^{\beta} = \left(-\sum_{l=1}^{k} n_l - \frac{(\beta | \beta)}{2} + \frac{(\bar{\Lambda}_i | \bar{\Lambda}_i)}{2} \right) f \otimes e^{\beta}.$$

Let

$$X_j^{\pm}(z) := \sum_{k\in Z} x_j^{\pm}(k) z^{-k-1} \quad (1 \leqq j \leqq n-1).$$

We define the action of U_q,

$$\gamma \mapsto q, \ K_j \mapsto q^{\partial_{\alpha_j}} \quad (1 \leqq j \leqq n-1),$$

$$X_j^{\pm}(z) \mapsto \exp\left(\pm \sum_{k=1}^{\infty} \frac{a_j(-k)}{[k]} q^{\mp\frac{k}{2}} z^k \right) \exp\left(\mp \sum_{K=1}^{\infty} \frac{a_j(k)}{[k]} q^{\mp\frac{k}{2}} z^{-k} \right) e^{\pm\alpha_j} z^{\pm\partial_{\alpha_j}}.$$

We know the following theorem.

Theorem 2.4([5]). *By the above action, W_i becomes the irreducible highest weight module with highest weight Λ_i, and $1 \otimes e^{\bar{\Lambda}_i}$ is a highest weight vector of W_i.* □

From now on, we identify W_i and $1 \otimes e^{\bar{\Lambda}_i}$ with $V(\Lambda_i)$ and $|\Lambda_i\rangle$ respectively.

3. Construction of Vertex Operators

In this section, we construct the vertex operators on W_i explicitly.

3.1. Vertex Operators. We review the definition and some properties of the vertex operators. ([2, 3]) Let V be a finite dimensional representation of U'_q. The affinization of V is the following U_q-module V_z,

$$V_z = V \otimes \mathbf{C}[z, z^{-1}].$$

We define the U_q-module structure on V_z as follows:

$$e_i \cdot (v \otimes z^m) = e_i \cdot v \otimes z^{m+\delta_{i0}}, \quad f_i \cdot (v \otimes z^m) = f_i \cdot v \otimes z^{m-\delta_{i0}},$$

$$t_i \cdot (v \otimes z^m) = t_i \cdot v \otimes z^m, \quad q^d \cdot (v \otimes z^m) = mv \otimes z^m.$$

Definition. *The vertex operator is a U_q-homomorphism of the following form:*
Type I:

$$\tilde{\Phi}_{\lambda}^{\mu V}(z): V(\lambda) \to V(\mu) \hat{\otimes} V_z,$$

Type II:

$$\tilde{\Phi}_{\lambda}^{V\mu}(z): V(\lambda) \to V_z \hat{\otimes} V(\mu).$$

□

The symbol $\hat{\otimes}$ means $W_1 \widehat{\otimes} W_2$. From now on, we omit it. We know the following theorem about the existence of the vertex operators.

Theorem 3.1([3]).

$$\mathrm{Hom}_{U_q}(V(\lambda), V(\mu) \otimes V_z) \cong \{ v \in V \mid \textit{the weight of } v = \lambda - \mu \bmod \delta$$
$$\textit{and } e_i^{\langle \mu, h_i \rangle + 1} \cdot v = 0 \text{ for } i = 0, \cdots, n-1 \},$$

where $\Phi \in \mathrm{Hom}_{U_q}(V(\lambda), V(\mu) \otimes V_z)$ corresponds to v via the relation $\Phi|\lambda\rangle = |\mu\rangle \otimes v +$ (terms of positive powers in z). □

We define the components of the vertex operators as follows.

$$\tilde{\Phi}_{\lambda}^{\mu V}(z)|u\rangle=\sum_{j=0}^{n-1}\tilde{\Phi}_{\lambda}^{\mu V}j(z)|u\rangle\otimes v_j \quad \text{for } |u\rangle\in V(\lambda)\,,$$

where $\{v_j\}$ is a set of basis of V. For the type II, the components are also defined similarly. Using the components, we define similar vertex operators

$$\tilde{\Phi}_{\lambda V}^{\mu}(z)\colon V(\lambda)\otimes V_z\to V(\mu)\otimes \mathbf{C}[z,z^{-1}]$$

by

$$\tilde{\Phi}_{\lambda V}^{\mu}(z)(|v\rangle\otimes v_i)=\tilde{\Phi}_{\lambda}^{\mu V^*}j(z)|v\rangle \quad \text{for } |v\rangle\in V(\lambda)\,.$$

Here $x\in U_q$ acts on $V(\mu)\otimes\mathbf{C}[z,z^{-1}]$ as $x\otimes 1$.
Now, we specialize V to the vector representation,

$$V=\mathbf{C}\,v_0\oplus\cdots\oplus\mathbf{C}v_{n-1}\,.$$

The U_q'-module structure on V is the following:

$$e_i\cdot v_j=\delta_{ij}v_{i-1},\quad f_i\cdot v_j=\delta_{i-1j}v_i,\quad t_i\cdot v_j=q^{\delta_{ij+1}-\delta_{ij}}v_i\,.$$

$(V^*)_z$ is denoted by V_z^*. The action of U_q on V_z^* is the following:

$$e_i\cdot(v_j^*\otimes z^m)=-q^{-1}\delta_{i-1j}v_i^*\otimes z^{m+\delta_{i0}},\quad f_i\cdot(v_j^*\otimes z^m)=-q\delta_{ij}v_{i-1}^*\otimes z^{m-\delta_{i0}}$$

$$t_i\cdot v_j\otimes z^m=q^{-\delta_{ij+1}-\delta_{ij}}v_j\otimes z^m,\quad q^d\cdot v_j\otimes z^m=mv_j\otimes z^m\,.$$

In our case, by the above theorem, only

$$\tilde{\Phi}_{\Lambda_{i+1}}^{\Lambda_i V}(z);\quad \tilde{\Phi}_{\Lambda_i}^{\Lambda_{i+1}V^*}(z);\quad \tilde{\Phi}_{\Lambda_{i+1}}^{V\Lambda_i}(z);\quad \tilde{\Phi}_{\Lambda_i}^{V^*\Lambda_{i+1}}(z)$$

are non-trivial. Furthermore, each of them is unique up to a scalar. Here, we take the following normalization:

$$\tilde{\Phi}_{\Lambda_{i+1}}^{\Lambda_i V}(z)|\Lambda_{i+1}\rangle=|\Lambda_i\rangle\otimes v_i+(\text{terms of positive powers in } z)\,,$$

$$\tilde{\Phi}_{\Lambda_i}^{\Lambda_{i+1}V^*}(z)|\Lambda_i\rangle=|\Lambda_{i+1}\rangle\otimes v_i^*+(\text{terms of positive powers in } z)\,.$$

For the type II, we take a similar normalization.

3.2. Coproduct of $a_i(k)$, $x_i^{\pm}(l)$ and action of $a_i(k)$, $x_i^{\pm}(l)$ on V_z. The coproduct of Drinfeld generators is not known in full. But the "main terms" are calculated in [7] for $U_q(\widehat{sl}(2))$. The case of $U_q(\widehat{sl}(n))$ is quite similar.

Proposition 3.2.A. *For $k\geqq 0$, $l>0$,*

$$\Delta(x_i^+(k))=x_i^+(k)\otimes\gamma^k+\gamma^{2k}K_i\otimes x_i^+(k)$$
$$+\sum_{j=0}^{k-1}\gamma^{(k-j)/2}\psi_i(-k+j)\otimes\gamma^{k-j}x_i^+(j)\quad \operatorname{mod} UN_-\otimes UN_+^2\,,$$

$$\Delta(x_i^+(-l))=x_i^+(-l)\otimes\gamma^{-l}+K_i^{-1}\otimes x_i^+(-l)$$
$$+\sum_{j=1}^{l-1}\gamma^{(l-j)/2}\varphi_i(-l+j)\otimes\gamma^{-l+j}x_i^+(-j)\quad \operatorname{mod} UN_-\otimes UN_+^2\,,$$

$$\Delta(x_i^-(l))=x_i^-(l)\otimes K_i+\gamma^k\otimes x_i^-(l)$$
$$+\sum_{j=1}^{k-1}\gamma^{k-j}x_i^-(j)\otimes\gamma^{(j-k)/2}\psi_i(k-j)\mod UN_-^2\otimes UN_+ ,$$
$$\Delta(x_i^-(-k))=x_i^-(-k)\otimes\gamma^{-2k}K_i^{-1}+\gamma^{-k}\otimes x_i^-(-k)$$
$$+\sum_{j=0}^{k-1}\gamma^{j-k}x_i^-(-j)\otimes\gamma^{-(k+3j)/2}\varphi_i(j-k)\mod UN_-^2\otimes UN_+ ,$$
$$\Delta(a_i(l))=a_i(l)\otimes\gamma^{\frac{l}{2}}+\gamma^{\frac{3l}{2}}\otimes a_i(l)\mod UN_-\otimes UN_+ ,$$
$$\Delta(a_i(-l))=a_i(-l)\otimes\gamma^{-\frac{3l}{2}}+\gamma^{-\frac{l}{2}}\otimes a_i(-l)\mod UN_-\otimes UN_+$$

where, $UN_\pm$, $UN_\pm^2$ *are the left ideals generated by* $\{x_i^\pm(k)\}$, $\{x_i^\pm(k)x_j^\pm(l)\}$. □

Proposition 3.2.B. *The action of* $a_i(l)$, $x_i^\pm(k)$ *on* V_z *is the following*:

$$x_i^+(k)\mapsto(q^iz)^kE_{ii-1} ,$$
$$x_i^-(k)\mapsto(q^iz)^kE_{i-1i} ,$$
$$a_i(l)\mapsto\frac{[l]}{l}(q^iz)^l(q^{-l}E_{i-1i-1}-q^lE_{ii}) ,$$

where E_{ij} *is the matrix unit of* EndV *such that* $E_{ij}v_l=\delta_{jl}v_i$. □

3.3. Vertex Operators of Type I. First, we consider the vertex operator $\tilde{\Phi}_{\Lambda_{i+1}}^{\Lambda_iV}(z):V(\Lambda_{i+1})\to V(\Lambda_i)\otimes V_z$. We can determine the $(n-1)^{\text{th}}$ component as follows. By Prop. 3.2, we get the following commutation relations:

$$[\tilde{\Phi}_{n-1}^V(z),X_j^+(w)]=0 ,$$
$$t_j\tilde{\Phi}_{n-1}^Vt_j^{-1}=q^{\delta_{j,n-1}}\tilde{\Phi}_{n-1}^V(z) ,$$
$$[a_j(k),\tilde{\Phi}_{n-1}^V(z)]=\delta_{j,n-1}q^{\frac{2n+3}{2}k}\frac{[k]}{k}z^k\tilde{\Phi}_{n-1}^V(z) ,$$
$$[a_j(-k),\tilde{\Phi}_{n-1}^V(z)]=\delta_{j,n-1}q^{-\frac{2n+1}{2}k}\frac{[k]}{k}z^{-k}\tilde{\Phi}_{n-1}^V(z) ,$$
$$\text{for } 1\leqq j\leqq n-1 .$$

The above conditions determine the form of $\tilde{\Phi}_{\Lambda_{i+1}n-1}^{\Lambda_iV}(z)$ completely under the normalization conditions in Sect. 3.1. The other components are determined by one of the interwining conditions

$$\tilde{\Phi}_{\Lambda_{i+1}j-1}^{\Lambda_iV}(z)=[\tilde{\Phi}_{\Lambda_{i+1}j}^{\Lambda_iV}(z),f_j]_q .$$

Hence, the other components are represented by the integral of the currents.

For the vertex operators $\tilde{\Phi}_{\Lambda i}^{\Lambda_{i+1}V^*}(z)$: $V(\Lambda_i)\to V(\Lambda_{i+1})\otimes V_z^*$, we have the similar commutation relations this time for the 0^{th} component. We summarize the results.

Theorem 3.3.

1) $$\tilde{\Phi}^{\Lambda_i V}_{\Lambda_{i+1} n-1}(z)=\exp\left(\sum_{k=1}^{\infty} a^*_{n-1}(-k)\,q^{\frac{2n+3}{2}}z^k\right)$$

$$\times\exp\left(\sum_{k=1}^{\infty} a^*_{n-1}(k)\,q^{-\frac{2n+1}{2}k}z^{-k}\right)$$

$$\times e^{\bar{\Lambda}_{n-1}}(q^{n+1}z)^{\partial_{\bar{\Lambda}_{n-1}}+\frac{n-i-1}{n}}$$

$$\times(-1)^{\left(\partial_{\bar{\Lambda}_1}-\frac{n-i-1}{n}\right)(n-1)}(-1)^{\frac{1}{2}(n-i)(n-i-1)}\quad(i=0,\cdots,n-1),$$

$$\tilde{\Phi}^{\Lambda_i V}_{\Lambda_{i+1} j-1}(z)=[\tilde{\Phi}^{\Lambda_i V}_{\Lambda_{i+1} j}(z),f_j]_q\quad(j=1,\cdots,n-1).$$

2) $$\tilde{\Phi}^{\Lambda_{i+1}}_{\Lambda_i V_0}(z)=\exp\left(\sum_{k=1}^{\infty} a^*_1(-k)\,q^{\frac{3}{2}k}z^k\right)$$

$$\times\exp\left(\sum_{k=1}^{\infty} a^*_1(k)\,q^{-\frac{1}{2}k}z^{-k}\right)$$

$$\times e^{\bar{\Lambda}_1}((-1)^{n-1}z)^{\partial_{\bar{\Lambda}_1}+\frac{i}{n}}q^i(-1)^{in+\frac{1}{2}i(i+1)}$$

$$(i=0,\cdots,n-1),$$

$$\tilde{\Phi}^{\Lambda_{i+1}V^*}_{\Lambda_i j}(z)=[f_j,\tilde{\Phi}^{\Lambda_{i+1}V^*}_{\Lambda_i j-1}(z)]_{q^{-1}}\quad(j=1,\cdots,n-1),$$

where $a^*_{n-1}(k)=\sum_{l=1}^{n-1}\frac{-[lk]}{[k][nk]}a_l(k)$, $a^*_1(k)=\sum_{l=1}^{n-1}\frac{-[(n-l)k]}{[k][nk]}a_l(k)$.

The coefficients of $a^*_{n-1}(k)$ *and* $a^*_1(k)$ *are determined by the conditions*

$$[a_i(k),a^*_{n-1}(-k)]=\delta_{i,n-1}\frac{[k]}{k},\ [a_i(k),a^*_1(-k)]=\delta_{i1}\frac{[k]}{k}.$$

□

3.4. Vertex Operators of Type II. We can also apply the same method for the vertex operators of type II.

Theorem 3.4.

1) $$\tilde{\Phi}^{V\Lambda_i}_{\Lambda_{i+1}0}(z)=\exp\left(-\sum_{k=1}^{\infty} a^*_1(-k)\,q^{\frac{1}{2}k}z^k\right)\exp\left(-\sum_{k=1}^{\infty} a^*_1(k)\,q^{-\frac{3}{2}k}z^{-k}\right)$$

$$\times e^{-\bar{\Lambda}_1}((-1)^{n-1}qz)^{-\partial_{\bar{\Lambda}_1}+\frac{n-i-1}{n}}q^{-i}(-1)^{in+\frac{1}{2}i(i+1)}$$

$$(i=0,\cdots,n-1),$$

$$\tilde{\Phi}^{V\Lambda_i}_{\Lambda_{i+1}j}(z)=[\tilde{\Phi}^{V\Lambda_i}_{\Lambda_{i+1}j-1}(z),e_j]_q\quad(j=1,\cdots,n-1).$$

2) $\tilde{\Phi}^{V^* \Lambda_{i+1}}_{\Lambda_i n-1}(z)=\exp\left(-\sum_{k=1}^{\infty} a^*_{n-1}(-k)q^{\frac{2n+1}{2}k}z^k\right)\exp\left(-\sum_{k=1}^{\infty} a^*_{n-1}(k)q^{-\frac{2n+3}{2}k}z^{-k}\right)$

$\times e^{-\bar{\Lambda}_{n-1}}(q^{n+1}z)^{-\partial_{\bar{\Lambda}_{i-1}}+\frac{i}{n}}$

$\times(-1)^{\left(\partial_{\bar{\Lambda}_1}-\frac{n-i}{n}\right)(n-1)}(-1)^{\frac{1}{2}(n-i)(n-i-1)}$

$(i=0,\cdots,n-1)$,

$\tilde{\Phi}^{V^* \Lambda_{i+1}}_{\Lambda_{i+1} j-1}(z)=[e_j, \tilde{\Phi}^{V^* \Lambda_i}_{\Lambda_{i+1} j}(z)]_{q^{-1}} \quad (j=1,\cdots,n-1)$.

□

3.5. Commutation Relations of the Vertex Operators. In [12], by solving the q-KZ equations the authors get the commutation relations of the vertex operators of type I related to $U_q(\widehat{sl}(n))$. These formulas can be derived directly by using our explicit formulas for vertex operators. First, we write down the matrixs coefficient of $\bar{R}_{V^*V}(z_1/z_2) \in \mathrm{End}_{\mathbf{C}} V^*_{z_1} \otimes V_{z_2}$,

$$\bar{R}_{V^*V}(z)(v^*_i \otimes v_j)=v^*_i \otimes v_j \ (i \neq j), \quad \bar{R}_{V^*V}(z)(v^*_i \otimes v_i)=\sum_{j=0}^{n-1} a_{ij} v^*_j \otimes v_j ,$$

$$\text{where } a_{ij}=\begin{cases} \dfrac{(q-q^{-1})z}{1-z} & (i>j) \\ \dfrac{q-q^{-1}z}{1-z} & (i=j) \\ \dfrac{q-q^{-1}}{1-z} & (i<j) . \end{cases}$$

Proposition 3.5.

1) $$\tilde{\Phi}^{\Lambda_{i+1}}_{\Lambda_i V}(z)\tilde{\Phi}^{\Lambda_i V}_{\Lambda_{i+1}}(z)=\frac{(q^{2n};q^{2n})_\infty}{(q^2;q^{2n})_\infty}\mathrm{id}_{V(\Lambda_{i+1})},$$

2) $$\tilde{\Phi}^{\Lambda_i V}_{\Lambda_{i+1}}(z)\tilde{\Phi}^{\Lambda_{i+1}}_{\Lambda_i V}(z)=\frac{(q^{2n};q^{2n})_\infty}{(q^2;q^{2n})_\infty}\mathrm{id}_{V(\Lambda_i)\otimes V},$$

3) $$\tilde{\Phi}^{\Lambda_i V}_{\Lambda_{i+1}}(z_2)\tilde{\Phi}^{\Lambda_{i+1}V^*}_{\Lambda_i}(z_1) = -q\left(\frac{z_1}{z_2}\right)^{-\delta_{i0}} r\left(\frac{z_1}{z_2}\right)P\bar{R}_{V^*V}\left(\frac{z_1}{z_2}\right)\tilde{\Phi}^{\Lambda_i V^*}_{\Lambda_{i-1}}(z_1)\tilde{\Phi}^{\Lambda_{i-1}V}_{\Lambda_i}(z_2),$$

where $r(z)=\dfrac{(z;q^{2n})_\infty(q^{2n+2}z^{-1};q^{2n})_\infty}{(q^2z;q^{2n})_\infty(q^{2n}z^{-1};q^{2n})_\infty}, \quad Pv^*_i \otimes v^*_i = v_j \otimes v^*_i$.

Proof. Formulas 1), 2) follow from simple calculations. We know the uniqueness of the vertex operator $V(\Lambda_i) \to V(\Lambda_i) \otimes V^*_{z_1} \otimes V_{z_2}$. (For the details see [2, 3].) So, the left- and right-hand sides of 3) coincide up to a scalar factor. By comparing the $v_n \otimes v^*_1$ component of both sides, we get the above equation. □

4. Vertex Model

In this section, we give a mathematical definition of the model treated in this paper ([1, 10, 12]).

4.1. Space of States. We know the integrable generalization of the XXZ-model related to any quantum affine algebra $U_q(\hat{\mathfrak{g}})$. Let V_z be a finite dimensional representation of $U'_q(\hat{\mathfrak{g}})$ with a spectral parameter z and $R(z_1/z_2)\in\mathrm{End}(V_{z_1}\otimes V_{z_2})$ be the R-matrix for $U'_q(\hat{\mathfrak{g}})$. We define the model on the infinite lattice $\cdots\otimes V\otimes V\otimes V\otimes\cdots$. Let h be the operator on $V\otimes V$ such that

$$PR(z_1/z_2)=(1+uh+\cdots)\times\text{const.}\ (u\to 0)\,,$$

$$P\colon \text{the transposition, } e^u=z_1/z_2\,.$$

We define the Hamiltonian $\mathscr{H}$ as follows:

$$\mathscr{H}=\sum_{k\in\mathbf{Z}} h_{l+1\,l}\,,$$

where $h_{l+1\,l}$ is $\cdots\otimes 1\otimes 1\otimes h\otimes 1\otimes 1\otimes\cdots$ acting the l^{th} component and $l+1^{\text{th}}$ component. We can check immediately

$$[U'_q(\hat{\mathfrak{g}}),\mathscr{H}]=0\,.$$

When $\mathfrak{g}=sl(2)$ and V_z is two-dimensional $U'_q(\widehat{sl}(2))$-module, $\mathscr{H}$ becomes H_{XXZ}.

From now on, we specialize $\mathfrak{g}$ to $sl(n)$ and V_z to the vector representation of U'_q in Sect. 3.1. Later, when we solve the difference equations for the one-point functions, we find it convenient to pass to an equivalent representation V_ζ^{pr} defined by

$$V=\mathbf{C}u_1\oplus\cdots\oplus\mathbf{C}u_{n-1}\,,$$

$$e_i\cdot u_j=\delta_{ij}u_{i-1}\zeta,\ f_i\cdot u_j=\delta_{i-1j}u_i\zeta^{-1},\ t_i\cdot u_j=q^{\delta_{i-1j}-\delta_{ij}}u_j\,.$$

The equivalence is given by

$$V_z\to V_\zeta^{pr},\ v_i\mapsto u_i\zeta^{-i},\ z=\zeta^n\,.$$

We will refer to V_z and V_ζ^{pr} as the homogeneous picture and the principal picture, respectively. As explained in the introduction, we take

$$\mathrm{End}_{\mathbf{C}}\left(\bigoplus_{i=0}^{n-1}V(\Lambda_i)\right)\cong\bigoplus_{i,j}V(\lambda_i)\otimes V(\lambda_j)^*$$

as the space of states $\mathscr{F}$. $\mathscr{F}$ is understood naively as the subspace of the infinite tensor product $\cdots\otimes V\otimes V\otimes V\otimes\cdots$. We give the left and right action of U on $\mathscr{F}$ as follows:

$$x\cdot f=\sum x_{(1)}\circ f\circ S(x_{(2)})\,,$$

$$f\cdot x=\sum S^{-1}(x_{(2)})\circ f\circ x_{(1)}\,,$$

$$\text{where } f\in\mathscr{F},\ x\in U,\ \Delta(x)=\sum x_{(1)}\otimes x_{(2)}\,.$$

The space $\mathscr{F}$ regarded as the right module is denoted by $\mathscr{F}^r$. Let

$$\mathscr{F}_{ij}=\mathrm{Hom}(V(\lambda_j),V(\lambda_i))\cong V(\lambda_i)\otimes V(\lambda_j)^*\,.$$

$\mathscr{F}_{ii}$ has the unique canonical element $\mathrm{id}_{V(\Lambda_i)}$. We call it the vacuum and denote it by $|\mathrm{vac}\rangle_i\in\mathscr{F}_{ii}$, ${}_i\langle\mathrm{vac}|\in\mathscr{F}^r_{ii}$. There is a natural inner product between $\mathscr{F}^r_{ij}$ and $\mathscr{F}_{ji}$ as follows:

$$\langle f|g\rangle = \frac{\mathrm{tr}_{V(\Lambda_i)}(q^{-2\rho}fg)}{\mathrm{tr}_{V(\Lambda_i)}(q^{-2\rho})} \quad \text{for } f\in\mathscr{F}^r_{ij},\, g\in\mathscr{F}_{ji}\,,$$

$$\text{where } \rho=\Lambda_0+\Lambda_1+\cdots+\Lambda_{n-1}\,.$$

It is invariant under the action of U_q: $\langle fx|g\rangle=\langle f|xg\rangle$ for $\forall x\in U$.

4.2. Local Structure and Local Operators. We use the vertex operator

$$\tilde{\Phi}^{\Lambda_{i-1}V}_{\Lambda_i}(z):\, V(\Lambda_i)\to V(\Lambda_{i-1})\otimes V_z$$

to incorporate the local structure into $\mathscr{F}$. Setting $z=1$, we obtain the U'_q-homomorphism

$$\tilde{\Phi}^{\Lambda_{i-1}V}_{\Lambda_i}:\, V(\Lambda_i)\to \hat{V}(\Lambda_{i-1})\otimes V\,.$$

Let

$$\tilde{\Phi}^{(m)}_{\Lambda_i}:=\tilde{\Phi}^{\Lambda_{i-m}V}_{\Lambda_{i-m+1}}\cdots\tilde{\Phi}^{\Lambda_{i-2}V}_{\Lambda_{i-1}}\tilde{\Phi}^{\Lambda_{i-1}V}_{\Lambda_i}\,.$$

$\tilde{\Phi}^{(m)}_{\Lambda_i}$ converges and gives the following isomorphism:

$$\mathscr{F}_{ij}=V(\Lambda_i)\otimes V(\Lambda_j)^*\cong \otimes\underbrace{V\otimes\cdots\otimes V}_{\text{m-times}}\otimes V(\Lambda_j)^*\,.$$

By this isomorphism, the local structure is inserted into $\mathscr{F}$. Next, we define the local operators. For $L\in\mathrm{End}\,V^{\otimes m}$, let

$$\mathscr{L}_{(i)}:=(\tilde{\Phi}^{(m)}_{\Lambda_i})^{-1}(\mathrm{id}_{V(\Lambda_{i-m})}\otimes L)(\tilde{\Phi}^{(m)}_{\Lambda_i})\,.$$

By Prop. 3.5, we know

$$(\tilde{\Phi}^{(m)}_{\Lambda_i})^{-1})=\left(\frac{(q^2;q^{2n})_\infty}{(q^{2n};q^{2n})_\infty}\right)^m\tilde{\Phi}^{\Lambda_i}_{\Lambda_{i-1}V}\tilde{\Phi}^{\Lambda_{i-1}}_{\Lambda_{i-2}V}\cdots\tilde{\Phi}^{\Lambda_{i-m+1}}_{\Lambda_{i-m}V}\,,$$

where $\tilde{\Phi}^{\Lambda_i}_{\Lambda_{i-1}V}=\tilde{\Phi}^{\Lambda_i}_{\Lambda_{i-1}V}(1)$. The action of L on $\mathscr{F}_{ij}$ is defined as follows:

$$L\cdot f:=\mathscr{L}_{(i)}\circ f\,.$$

We denote the correlator ${}_i\langle\mathrm{vac}|L|\mathrm{vac}\rangle_i$ by $\langle L\rangle^{(i)}$.

5. Staggered Polarization

The aim of this section is to give $\langle E_{m'm}\rangle^{(i)}$ explicitly.

5.1. Integral Representations. In [9], the authors construct an integral representation of correlators of the XXZ-model by using the trace formula explained in [8] Appendix C. We can apply the same method to $\langle E_{m'm}\rangle^{(i)}$. Put

$$P^m_{m'}(z_1,z_2|x,y|i):=\frac{(q^2;q^{2n})_\infty}{(q^{2n};q^{2n})_\infty}\frac{\mathrm{tr}_{V(\Lambda_i)}(x^{-d}y^{2\bar{\rho}}\tilde{\Phi}^{\Lambda_i}_{\Lambda_{i-1}Vm'}(z_1)\tilde{\Phi}^{\Lambda_{i-1}V}_{\Lambda_i m}(z_2))}{\mathrm{tr}_{V(\Lambda_i)}(x^{-d}y^{2\bar{\rho}})}\,,$$

then $\langle E_{m'm}\rangle^{(i)} = P^{m}_{m'}(z, z\,|\,q^{2n}, q^{-1}\,|\,i)$. Let

$$h(z) = (z; x)_\infty (q^2 z^{-1}; x)_\infty,$$

$$\varphi(z; x) = \prod_{k=1}^{\infty} \frac{(zx^k; q^{2n})_\infty (q^{2n} z^{-1} x^{k-1}; q^{2n})_\infty}{(q^2 z x^k; q^{2n})_\infty (q^{2n+2} z^{-1} x^{k-1}; q^{2n})_\infty},$$

$$\theta_i(z_1, \cdots, z_{n-1}) = y^{(2\bar{\rho}|\bar{\Lambda}_i)} \sum_{\alpha \in \bar{Q}} x^{\frac{(\alpha|\alpha)}{2} + (\alpha|\bar{\Lambda}_i)} z_1^{(\bar{\Lambda}_1|\alpha)} \cdots z_{n-1}^{(\bar{\Lambda}_{n-1}|\alpha)}.$$

We get the following:

$$P^{m}_{m'}(z_1, z_2\,|\,x, y\,|\,i)$$

$$= c \times \frac{\delta_{m'm}(q^2; q^{2n})_\infty (q^2; x)_\infty^{n-1} \varphi(z; x)}{(q^{2n}; q^{2n})_\infty \operatorname{tr}_{V(\Lambda_i)}(x^{-d} y^{2\bar{\rho}})} \left(\prod_{k=1}^{\infty} \frac{(q^2 x^k; q^{2n})_\infty}{(q^{2n} x^k; q^{2n})_\infty} \right)^2$$

$$\times \oint_{q^2 < |w_l| < 1\,(l \neq m)} \frac{d\xi_1 \cdots d\xi_{n-1}}{(2\pi\sqrt{-1})^{n-1} \xi_1 \cdots \xi_{n-1}} \left(\frac{q}{\xi_i} \right)^{1-\delta_{i0}}$$

$$\times \frac{(1 - z w_m) w_{m+1} \cdots w_{n-1}}{h(w_0) \cdots h(w_{m-1}) h(z w_m) h(w_{m+1}) \cdots h(w_{n-1})}$$

$$\times \theta_i(\eta_1, \cdots, \eta_{m-1}, \eta_m z^{-1}, \eta_{m+1} z, \eta_{m+2}, \cdots, \eta_{n-1}),$$

where $w_0 = \frac{q^2}{\xi_1},\ w_1 = \frac{q\xi_1}{\xi_2}, \cdots, w_{n-2} = \frac{q\xi_{n-2}}{\xi_{n-1}},\ w_{n-1} = \frac{\xi_{n-1}}{q^n}$,

$$z = \frac{z_1}{z_2},\quad \eta_j = \frac{w_{j-1}}{w_j} y^2,\quad c = \begin{cases} q^i & (m < i) \\ q^i z^{-1} & (m \geqq i). \end{cases}$$

By this expression, we can verify that

$$\varphi(z; x)^{-1} \operatorname{tr}_{V(\Lambda_i)}(q^{-2\rho} \tilde{\Phi}^{\Lambda_i V^*}_{\Lambda_{i-1}}(z_1) \tilde{\Phi}^{\Lambda_{i-1} V}_{\Lambda_i}(z_2))$$

is a function of $z (= z_1/z_2)$ and regular in $q^{-2n} < |z| < q^{2n}$.

5.2. Staggered Polarization. In this subsection we derive the difference equations for one point functions and solve them. These equations can be solved easily up to a pseudo-constant factor and we can determine the factor by the analyticity gained from the integral representations. Following [10], we explain how to derive the difference equations in our context.

Let

$$\tilde{F}^{(i)}\left(\frac{z_1}{z_2}\right) := \operatorname{tr}_{V(\Lambda_i)}(q^{-2\rho} \tilde{\Phi}^{\Lambda_i V^*}_{\Lambda_{i-1}}(z_1) \tilde{\Phi}^{\Lambda_{i-1} V}_{\Lambda_i}(z_2))$$

then, we get the equation

$$\tilde{F}^{(i)}\left(\frac{z_1}{z_2 q^{2n}}\right) = \operatorname{tr}_{V(\Lambda_i)}(q^{-2\rho} \tilde{\Phi}^{\Lambda_i V^*}_{\Lambda_{i-1}}(z_1) \tilde{\Phi}^{\Lambda_{i-1} V^*}_{\Lambda_i}(z_2 q^{2n}))$$

$$= P \operatorname{tr}_{V(\Lambda_{i-1})}(\tilde{\Phi}^{\Lambda_{i-1} V}_{\Lambda_i}(z_2 q^{2n}) q^{-2\rho} \tilde{\Phi}^{\Lambda_i V^*}_{\Lambda_{i-1}}(z_1))$$

$$=(q^{-2\bar{\rho}}\otimes P\mathrm{tr}_{V\Lambda_{i-1}}(q^{-2\rho}\tilde{\Phi}_{\Lambda_i}^{\Lambda_{i-1}V}(z_2)\tilde{\Phi}_{\Lambda_{i-1}}^{\Lambda_i V^*}(z_1))$$

$$=-q\left(\frac{z_1}{z_2}\right)^{-\delta_{i1}}r\left(\frac{z_1}{z_2}\right)(1\otimes q^{-2\bar{\rho}})\bar{R}_{V^*V}\left(\frac{z_1}{z_2}\right)$$

$$\times\mathrm{tr}_{V(\Lambda_{i-1})}(q^{-2\rho}\tilde{\Phi}_{\Lambda_{i-2}}^{\Lambda_{i-1}V^*}(z_1)\tilde{\Phi}_{\Lambda_{i-1}}^{\Lambda_{i-2}V}(z_2))$$

or,

$$\tilde{F}^{(i)}(zq^{-2n})=-qz^{-\delta_{i1}}(1\otimes q^{-2\bar{\rho}})r(z)\bar{R}_{V^*V}(z)\tilde{F}^{(i-1)}(z).$$

We show this equation reduces to scalar equations. Let

$$\bar{F}^{(i)}(z):=\varphi(z;q^{2n})^{-1}\tilde{F}^{(i)}(z),$$

then

$$\bar{F}^{(i)}(zq^{-2n})=-qz^{-\delta_{i1}}(1\otimes q^{-2\bar{\rho}})\bar{R}_{V^*V}(z)\bar{F}^{(i-1)}(z).$$

Let $z=\zeta^n$ and ω be the n^{th} primitive root of 1. We put

$$\sum_{m=0}^{n-1}G_m^{(j)}(\zeta)v_m^*\otimes v_m:=\sum_{i=1}^{n}q^{(n-i)i}\omega^{ij}\zeta^{n-i}\bar{F}^{(i)}(\zeta^n).$$

Let further

$$G^{(j,k)}(\zeta):=\zeta\sum_{m=0}^{n-1}\omega^{km}\zeta^{m-n}G_m^{(j)}(\zeta).$$

Then, we find

$$\frac{G^{(j,k)}(\zeta q^{-2})}{1-\omega^k\zeta q^{-2}}=-q^2\omega^j\zeta^{-1}\frac{G^{(j,k)}(\zeta)}{1-\omega^k\zeta}.$$

Let

$$\Theta_p(z)=(p;p)_\infty(z;p)_\infty(z^{-1}p;p)_\infty.$$

The reduced equation determines $G^{(j,k)}(\zeta)$ as

$$G^{(j,k)}(\zeta)=c_{jk}(\zeta)\frac{1-\omega^k\zeta}{\Theta_{q^2}(\omega^{-j}\zeta)};\ (*)$$

where $c_{jk}(\zeta)$ is a pseudo-constant (i.e. $c_{jk}(\zeta q^{-2})=c_{jk}(\zeta)$). Let us show that $c_{jk}(\zeta)$ is independent of ζ. As $\bar{F}(\zeta^n)$ is regular in $q^{-2}<|\zeta|<q^2$,so is $G^{(j,k)}(\zeta)$. So, when we set $\zeta=e^{2\pi\sqrt{-1}u}$ and $q=e^{\pi\sqrt{-1}\tau}$,

$$c_{jk}(\zeta)=G^{(j,k)}(\zeta)\frac{\Theta_{q^2}(\omega^{-j}\zeta)}{1-\omega^k\zeta}$$

has at most a simple pole in the fundamental region $[0,1]\times[0,\tau]$ in the u-plane. Hence, $c_{jk}(\zeta)$ is an absolute-constant c_{jk}. Moreover, it can be determined by calculating the residue at $\zeta = q^{-2}\omega^j$ of the both sides of the above equation $(*)$. The result is the following:

$$G^{(j,k)}(\zeta) = \begin{cases} n\omega^j C \dfrac{1-\omega^{-j}\zeta}{\Theta_{q^2}(\omega^{-j}\zeta)} & (j+k \equiv 0 \bmod n) \\ 0 & (\text{otherwise}) \end{cases},$$

$$\text{where } C = \frac{(q^2;q^2)_\infty^3 (q^{2n};q^{2n})_\infty}{\varphi(1;q^{2n})(q^2;q^{2n})_\infty} \operatorname{tr}_{V(\Lambda_0)}(q^{-2\rho}).$$

We get the following theorem.

Theorem 5.2. *Let ω be an n^{th} primitive root of 1 and E_{ij} be the matrix unit, then*

$$\sum_{m=0}^{n-1} \omega^{km} \langle E_{mm} \rangle^{(i)} = \frac{\omega^{(i-1)k}(q^2;q^2)_\infty^2}{(q^2\omega^k;q^2)_\infty (q^2\omega^{-k};q^2)_\infty}.$$

□

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