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Quantum R-Matrix and Intertwiners for the Kashiwara Algebra

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Abstract: We study the algebra $B_q(\mathfrak{g})$ presented by Kashiwara and introduce intertwiners similar to q-vertex operators. We show that a matrix determined by 2-point functions of the intertwiners coincides with a quantum R-matrix (up to a diagonal matrix) and give the commutation relations of the intertwiners. We also introduce an analogue of the universal R-matrix for the Kashiwara algebra.

0. Introduction

In a recent work [FR], Frenkel and Reshetikhin developed the theory of q-vertex operators. They showed that n-point correlation functions associated to q-vertex operators satisfy a q-difference equation called the q-deformed Kniznik–Zamolodchikov equation. In the derivation of this equation, a crucial point is that the quantum affine algebra is a quasi-triangular Hopf algebra. By using several properties of the quasi-triangular Hopf algebra and the representation theory of the quantum affine algebra, the equation is described in terms of quantum R-matrices ([FR, IIJMNT]).

In [K1], Kashiwara introduced the algebra $B_q^\vee(\mathfrak{g})$, which is generated by $2 \times \operatorname{rank} \mathfrak{g}$ symbols with the Serre relations and the q-deformed bosonic relations (see Sect. 1, (1.5)) in order to study the crystal base of U^-, where U^- is a maximal nilpotent subalgebra of the quantum algebra $U_q(\mathfrak{g})$ associated to a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$. (In [K1], $B_q^\vee(\mathfrak{g})$ is denoted by $\mathscr{B}_q(\mathfrak{g})$). We shall call this algebra the *Kashiwara algebra*. He showed that U^- has a $B_q^\vee(\mathfrak{g})$-module structure and it is irreducible. He also showed that $B_q^\vee(\mathfrak{g})$ has a similar structure to the Hopf algebra: there is an algebra homomorphism $B_q^\vee(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes B_q^\vee(\mathfrak{g})$. Thus if M is a $U_q(\mathfrak{g})$-module and N is a $B_q^\vee(\mathfrak{g})$-module, then $M \otimes N$ has a $B_q^\vee(\mathfrak{g})$-module structure via this homomorphism.

The purposes of the present paper are the following: first we clarify the algebraic structure of the Kashiwara algebras similar to the Hopf algebra and develop their representation theory and then applying these to the affine case, we obtain direct connection between the quantum R-matrices and 2-point correlation functions for the affine Kashiwara algebra. From these results we can expect new approaches for analyzing the quantum or other type R-matrices.

The organization of this paper is as follows; in Sect. 1, we shall introduce the algebras $B_q(\mathfrak{g})$, $\bar{B}_q(\mathfrak{g})$, $U_q(\mathfrak{g})$ associated to a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$ and algebra morphisms for such algebras. The algebra B_q is obtained by adding the Cartan part to $B_q^\vee$ and the algebra $\bar{B}_q$ is an algebra anti-isomorphic to B_q, where we also call these the Kashiwara algebras. The algebra U_q is an ordinary quantum algebra. The Kashiwara algebra has no natural Hopf algebra structure, but these algebras admit a certain algebra structure similar to the Hopf algebra. In fact, there are the following algebra homomorphisms, $U_q \to U_q \otimes U_q$, $B_q \to B_q \otimes U_q$, $\bar{B}_q \to U_q \otimes \bar{B}_q$, $U_q \to \bar{B}_q \otimes B_q$, an antipode $S: U_q \to U_q$ and an anti-isomorphism $\varphi: \bar{B}_q \to B_q$. By using these, in the former half of Sect. 2, we can consider tensor products and dual modules of B_q-modules, $\bar{B}_q$-modules and U_q-modules. In the latter half of Sect. 2, we discuss properties of the category of highest weight B_q-modules. In Sect. 3, we recall the Killing form of U_q due to [R, T] and give a certain relationship between the algebra $B_q^\vee$ and the Killing form. We also introduce a bilinear pairing $\langle\,|\,\rangle$ for highest weight B_q-module $H(\lambda)$, which is an analogue of an ordinary vacuum expectation value. In Sect. 4, we restrict ourselves to an affine case and consider the following type of intertwiners similar to q-vertex operators;

$$\mathrm{Hom}_{B_q}(H(\lambda), H(\mu) \hat{\otimes} V_z)\,, \tag{0}$$

and examine the condition for existence of such intertwiners. By using the bilinear pairing above for a composition of these intertwiners, we define 2-point functions. By using the relationship between the algebra $B_q^\vee$ and the Killing form, we can explicitly describe a 2-point function as a matrix element of an image of the universal R-matrix. In other words, 2-point functions give matrix elements of the quantum R-matrix up to scalar factors. This result clarifies the new aspects of quantum R-matrices. Here note that we do not derive any type of equation. This point differs from [FR]. Nevertheless, by pure algebraic method we can describe 2-point functions.

In order to explain precisely, we prepare some notations. Let U_q' be a subalgebra of a quantum affine algebra U_q without a scaling element, let V and W be finite dimensional U_q'-modules, let V_{z_1} and W_{z_2} be their affinizations, where z_1 and z_2 are formal variables, let $R^{VW}(z_1/z_2)$ be the image of the universal R-matrix onto $V_{z_1} \otimes W_{z_2}$ and let u_λ (resp. u_λ^r) be a highest weight vector of an irreducible highest weight left (resp. right) B_q-module $H(\lambda)$ (resp. $H^r(\lambda)$).

Theorem (Theorem 5.3). *For* $\Phi_\lambda^{\mu V}(z_1) \in \mathrm{Hom}_{B_q}(H(\lambda), H(\mu) \hat{\otimes} V_{z_1})$ *and* $\Phi_\mu^{\nu W}(z_2) \in \mathrm{Hom}_{B_q}(H(\mu), H(\nu) \hat{\otimes} W_{z_2})$, *we have*

$$\langle u_\nu^r | \Phi_\mu^{\nu W}(z_2)\, \Phi_\lambda^{\mu V}(z_1) | u_\lambda \rangle = q^{(\lambda-\mu,\mu-\nu)} \sigma R^{VW}(z_1/z_2)(v_0 \otimes w_0)\,,$$

where σ: $a \otimes b \to b \otimes a$, *and* $v_0 \in V$ *and* $w_0 \in W$ *are the leading terms of* $\Phi_\lambda^{\mu V}(z_1)$ *and* $\Phi_\mu^{\nu W}(z_2)$ *respectively (see Definition* 4.1).

From this theorem and the unitarity of a quantum R-matrix, we can derive the commutation relation of intertwiners of type (0).

The contents of Sect. 6 is divided from the ones of the previous sections. In this section, for the algebra B_q we give an element $\tilde{\mathscr{R}}$, which is an analogue of the universal R-matrix $\mathscr{R}$. This satisfies, for example, $\tilde{\mathscr{R}}_{12}\tilde{\mathscr{R}}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\tilde{\mathscr{R}}_{13}\tilde{\mathscr{R}}_{12}$, *etc.* We also introduce a projector Γ associated to $\tilde{\mathscr{R}}$, which acts on $H(\lambda)$ and singles out only the highest weight component. In Appendix A, we list some formulae for algebra homomorphisms related to the algebras introduced in this paper and in Appendix B, we recall the theory of the universal R-matrix of U_q.

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1. Preliminary

We shall define the algebras playing a significant role in this paper. First, let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra over $\mathbf{Q}$ with a Cartan subalgebra $\mathfrak{t}$, $\{\alpha_i \in \mathfrak{t}^*\}_{i\in I}$ the set of simple roots and $\{h_i \in \mathfrak{t}\}_{i\in I}$ the set of coroots, where I is a finite index set. We define an inner product on $\mathfrak{t}^*$ such that $(\alpha_i, \alpha_i) \in \mathbf{Z}_{\geqq 0}$ and $\langle h_i, \lambda\rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for $\lambda \in \mathfrak{t}^*$. Set $Q = \bigoplus_i \mathbf{Z}\alpha_i$, $Q_+ = \bigoplus_i \mathbf{Z}_{\geqq 0}\alpha_i$ and $Q_- = -Q_+$. We call Q a root lattice. Let P a lattice of $\mathfrak{t}^*$, i.e. a free $\mathbf{Z}$-submodule of $\mathfrak{t}^*$ such that $\mathfrak{t}^* \cong \mathbf{Q} \bigoplus_{\mathbf{Z}} P$, and $P^* = \{h \in \mathfrak{t} \mid \langle h, P\rangle \subset \mathbf{Z}\}$. Now, we introduce the symbols $\{e_i, e_i'', f_i, f_i''(i\in I), q^h(h\in P^*)\}$. These symbols satisfy the following relations;

$$q^0 = 1, \quad \text{and} \quad q^h q^{h'} = q^{h+h'}, \tag{1.1}$$

$$q^h e_i q^{-h} = q^{\langle h, \alpha_i\rangle} e_i, \tag{1.2a}$$

$$q^h e_i'' q^{-h} = q^{\langle h, \alpha_i\rangle} e_i'', \tag{1.2b}$$

$$q^h f_i q^{-h} = q^{-\langle h, \alpha_i\rangle} f_i, \tag{1.3a}$$

$$q^h f_i' q^{-h} = q^{-\langle h, \alpha_i\rangle} f_i', \tag{1.3b}$$

$$[e_i, f_j] = \delta_{i,j}(t_i - t_i^{-1})/(q_i - q_i^{-1}), \tag{1.4}$$

$$e_i'' f_j = q_i^{\langle h_i, \alpha_j\rangle} f_j e_i'' + \delta_{i,j}, \tag{1.5}$$

$$f_i' e_j = q_i^{\langle h_i, \alpha_j\rangle} e_j f_i' + \delta_{i,j}, \tag{1.6}$$

$$\sum_{k=1}^{1-\langle h_i, \alpha_j\rangle} (-1)^k X_i^{(k)} X_j X_i^{(1-\langle h_i, \alpha_j\rangle - k)} = 0, \quad (\text{for } X_i = e_i, e_i'', f_i, f_i' \text{ and } i \neq j), \tag{1.7}$$

where q is transcendental over $\mathbf{Q}$ and we set $q_i = q^{(\alpha_i, \alpha_i)/2}$, $t_i = q_i^{h_i}$, $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{k=1}^n [k]_i$ and $X_i^{(n)} = X_i^n/[n]_i!$.

Now, we define the algebras $B_q(\mathfrak{g})$, $\bar{B}_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$. In the rest of this paper, we denote the base field $\mathbf{Q}(q)$ by $\mathbf{F}$. The algebra $B_q(\mathfrak{g})$ (resp. $\bar{B}_q(\mathfrak{g})$) is an associative algebra generated by the symbols $\{e_i'', f_i\}_{i\in I}$ (resp. $\{e_i, f_i'\}_{i\in I}$) and q^h $(h\in P^*)$ with the defining relations (1.1), (1.2b), (1.3a), (1.5) and (1.7) (resp. (1.1), (1.2a), (1.3b), (1.6) and (1.7)) over $\mathbf{F}$. The algebra $U_q(\mathfrak{g})$ is an associative algebra generated by the symbols $\{e_i, f_i\}_{i\in I}$ and q^h $(h\in P^*)$ with the defining relations (1.1), (1.2a), (1.3a), (1.4) and (1.7) over $\mathbf{F}$. We shall call algebras $B_q(\mathfrak{g})$ and $\bar{B}_q(\mathfrak{g})$ the *Kashiwara algebras*. ([K1]). Furthermore, we define subalgebras

$$T = \langle q^h \mid h\in P^*\rangle = B_q(\mathfrak{g}) \cap \bar{B}_q(\mathfrak{g}) \cap U_q(\mathfrak{g}),$$

$$B_q^\vee(\mathfrak{g}) \text{ (resp. } \bar{B}_q^\vee(\mathfrak{g})) = \langle e_i'', f_i \text{ (resp. } e_i, f_i') \mid i\in I\rangle \subset B_q(\mathfrak{g}) \text{ (resp. } \bar{B}_q(\mathfrak{g})),$$

$$U_q^+(\mathfrak{g}) \text{ (resp. } U_q^-(\mathfrak{g})) = \langle e_i \text{(resp. } f_i) \mid i\in I\rangle = \bar{B}_q^\vee(\mathfrak{g}) \cap U_q(\mathfrak{g}) \text{ (resp. } B_q^\vee(\mathfrak{g}) \cap U_q(\mathfrak{g})),$$

$$U_q^{\geqq}(\mathfrak{g}) \text{ (resp. } U_q^{\leqq}(\mathfrak{g})) = \langle e_i(\text{resp. } f_i),\ q^h \mid i \in I,\ h \in P^* \rangle$$
$$= \bar{B}_q(\mathfrak{g}) \cap U_q(\mathfrak{g}) \text{ (resp. } B_q(\mathfrak{g}) \cap U_q(\mathfrak{g})) ,$$
$$B_q^+(\mathfrak{g}) \text{ (resp. } \bar{B}_q^-(\mathfrak{g})) = \langle e_i'' \text{ (resp. } f_i') \mid i \in I \rangle \subset B_q^\vee(\mathfrak{g}) (\text{resp } \bar{B}_q^\vee(\mathfrak{g})) ,$$
$$B_q^{\geqq}(\mathfrak{g}) \text{ (resp. } \bar{B}_q^{\leqq}(\mathfrak{g})) = \langle e_i''(\text{resp. } f_i'),\ q^h \mid i \in I,\ h \in P^* \rangle$$
$$\subset B_q(\mathfrak{g}) \text{ (resp. } \bar{B}_q(\mathfrak{g})) .$$

We shall use the abbreviated notations U, B, $\bar{B}$, $B^\vee, \ldots$ for $U_q(\mathfrak{g})$, $B_q(\mathfrak{g})$, $\bar{B}_q(\mathfrak{g})$, $B_q^\vee(\mathfrak{g}), \ldots$ if there is no confusion.

For $\beta = \sum m_i \alpha_i \in Q_+$ we set $|\beta| = \sum m_i$ and

$$U^\pm_{\pm\beta} = \{u \in U^\pm \mid q^h u q^{-h} = q^{\pm\langle h, \beta\rangle} u (h \in P^*)\} ,$$

and call $|\beta|$ a height of β and U^+_β (resp. $U^-_{-\beta}$) a weight space of U^+ (resp. U^-) with a weight β (resp. $-\beta$). We also define B^+_β and $\bar{B}^-_{-\beta}$ by the similar manner.

We shall define weight completions of $L^{(1)} \otimes \cdots \otimes L^{(m)}$, where $L^{(i)} = B$ or U (see [T]).

$$\hat{L}^{(1)} \hat{\otimes} \cdots \hat{\otimes} \hat{L}^{(m)} = \varprojlim_l L^{(1)} \otimes \cdots \otimes L^{(m)} / (L^{(1)} \otimes \cdots \otimes L^{(m)}) L^{+,l} ,$$

where $L^{+,l} = \bigoplus_{|\beta_1| + \cdots + |\beta_m| \geqq l} L^{(1)+}_{\beta_1} \otimes \cdots \otimes L^{(m)+}_{\beta_m}$. (Note that $U \cong U^- \otimes T \otimes U^+$ and $B \cong U^- \otimes T \otimes B^+$.) The linear maps as below Δ, $\Delta^{(r)}$, S, φ, multiplication, etc. are naturally extend for such completions.

Remark 1.1. The algebra $B^\vee$ is introduced in [K1] for studying the crystal base of U^- and called the reduced q-analogue. Note that in [K1] the algebra defined by the relation $e_i' f_j = q_i^{-\langle h_i, \alpha_j \rangle} f_j e_i' + \delta_{ij}$ is mainly studied, but there is no essential difference since both are equivalently related to each other by $q \leftrightarrow q^{-1}$.

We shall introduce the algebra homomorphisms related to the algebras defined above.

Proposition 1.2. (1) *If we define linear maps* Δ: $U \to U \otimes U$, $\Delta^{(r)}$: $B \to B \otimes U$, $\Delta^{(l)}$: $\bar{B} \to U \otimes \bar{B}$ and $\Delta^{(b)}$: $U \to \bar{B} \otimes B$ *by*

$$\Delta(q^h) = \Delta^{(r)}(q^h) = \Delta^{(l)}(q^h) = \Delta^{(b)}(q^h) = q^h \otimes q^h , \tag{1.8}$$

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i , \tag{1.9}$$

$$\Delta^{(r)}(e_i'') = (q_i - q_i^{-1}) \otimes t_i^{-1} e_i + e_i'' \otimes t_i^{-1}, \quad \Delta^{(r)}(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i , \tag{1.10}$$

$$\Delta^{(l)}(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta^{(l)}(f_i') = (q_i - q_i^{-1}) t_i f_i \otimes 1 + t_i \otimes f_i' , \tag{1.11}$$

$$\Delta^{(b)}(e_i) = t_i \otimes \frac{t_i e_i''}{q_i - q_i^{-1}} + e_i \otimes 1, \quad \Delta^{(b)}(f_i) = 1 \otimes f_i + \frac{t_i^{-1} f_i'}{q_i - q_i^{-1}} \otimes t_i^{-1} , \tag{1.12}$$

and extending these to the whole algebras by the rule: $\Delta(xy) = \Delta(x)\Delta(y)$ *and* $\Delta^{(i)}(xy) = \Delta^{(i)}(x)\Delta^{(i)}(y)$ $(i = r, l, b)$, *then they give well-defined algebra homomorphisms.*

(2) *If we define linear maps* S: $U\to U$ *and* φ: $\bar{B}\to B$ *by*

$$S(e_i)=-t_i^{-1}e_i,\quad S(f_i)=-f_it_i,\quad S(q^h)=q^{-h} \tag{1.13}$$

$$\varphi(e_i)=-\frac{1}{q_i-q_i^{-1}}e_i'',\quad \varphi(f_i')=-(q_i-q_i^{-1})f_i,\quad \varphi(q^h)=q^{-h}\,, \tag{1.14}$$

and extending these to the whole algebras by the rule: $S(xy)=S(y)S(x)$ *and* $\varphi(xy)=\varphi(y)\varphi(x)$, *then these maps give well-defined anti-isomorphisms.*

Note that in [K1] a homomorphism similar to $\Delta^{(r)}$ is introduced.

Proof. By direct calculations, we can check all the commutation relations. But it is too complicated to check the Serre relations directly. Since the map Δ is an ordinary comultiplication, we may assume that Δ is well-defined. The formulae (A10), (A11), (A12) and (A13) in Appendix A are useful for checking the Serre relations. For example, from (A10) and the fact: $\Delta^{(l)}_{|U^{\geqq}}=\Delta_{|U^{\geqq}}$, we have

$$\begin{aligned}
&\Delta^{(r)}\left(\sum_{k=1}^{1-\langle h_i,\alpha_j\rangle}(-1)^k e_i''^{(k)}e_j''e_i''^{(1-\langle h_i,\alpha_j\rangle-k)}\right)\\
&=\sigma(S\otimes\varphi)\Delta^{(l)}\varphi^{-1}\left(\sum_{k=1}^{1-\langle h_i,\alpha_j\rangle}(-1)^k e_i''^{(k)}e_j''e_i''^{(1-\langle h_i,\alpha_j\rangle-k)}\right)\\
&=(q_j^{-1}-q_j)(q_i^{-1}-q_i)^{1-\langle h_i,\alpha_j\rangle}\sigma(S\otimes\varphi)\Delta\left(\sum_{k=1}^{1-\langle h_i,\alpha_j\rangle}(-1)^k e_i^{(k)}e_je_i^{(1-\langle h_i,\alpha_j\rangle-k)}\right)=0\,.
\end{aligned}$$

Q.E.D.

Remark 1.3.

(1) If we define an algebra homomorphism ε: $U\to\mathbf{F}$ by $\varepsilon(e_i)=\varepsilon(f_i)=0$ and $\varepsilon(q^h)=1$, then (Δ, S, ε) gives a Hopf algebra structure on U.
(2) The following diagrams are commutative:

$$\begin{array}{ccc}
B & \xrightarrow{\Delta^{(r)}} & B\otimes U\\
\Delta^{(r)}\downarrow & & 1\otimes\Delta\downarrow\\
B\otimes U & \xrightarrow{\Delta^{(r)}\otimes 1} & B\otimes U\otimes U
\end{array}\qquad
\begin{array}{ccc}
\bar{B} & \xrightarrow{\Delta^{(l)}} & U\otimes\bar{B}\\
\Delta^{(l)}\downarrow & & \Delta\otimes 1\downarrow\\
U\otimes\bar{B} & \xrightarrow{1\otimes\Delta^{(l)}} & U\otimes U\otimes\bar{B}
\end{array}$$

Thus for a B (resp. $\bar{B}$)-module L, and U-modules M and N, there is an isomorphism of B (resp. $\bar{B}$)-module;

$$(L\otimes M)\otimes N\cong L\otimes(M\otimes N)\ \ (\text{resp. } (M\otimes N)\otimes L\cong M\otimes(N\otimes L))\,.$$

Hence we write these $L\otimes M\otimes N$ (resp. $M\otimes N\otimes L$). More generally, if M is a B (resp. $\bar{B}$)-module and $N_1,\ldots,N_k$ are U-modules, then $M\otimes N_1\otimes\cdots\otimes N_k$ (resp. $N_1\otimes\cdots\otimes N_k\otimes M$) is a well-defined B (resp. $\bar{B}$)-module.
(3) If M is a $\bar{B}$-module and N is a B-module, then $M\otimes N$ has a U-module structure via $\Delta^{(b)}$.
(4) From (A8) (resp. (A9)) and the coassociative laws of $\Delta^{(r)}$ (resp. $\Delta^{(l)}$) and Δ as in (2), we know that B (resp. $\bar{B}$) has a right (resp. left) U-comodule structure. (see [A].)
(5) The algebra $B^{\geqq}$ (resp. $\bar{B}^{\leqq}$) is isomorphic to $U^{\geqq}$ (resp. $U^{\leqq}$) as an associative algebra, but $B^{\geqq}$ (resp. $\bar{B}^{\leqq}$) has no natural Hopf algebra structure, thus we do not identify them.

We list several formulae for these operations in Appendix A.

2. Representation Theory of the Kashiwara Algebra

We shall discuss the representation theory of the algebra $B_q(\mathfrak{g})$. In the rest of this paper, we assume that all representations below have a weight space decomposition and each weight space is finite dimensional, where for a vector space M with a T-module structure, a weight space M_λ with weight $\lambda \in \mathfrak{t}^*$ is defined by $\{u \in M \mid q^h u = q^{\langle h, \lambda \rangle} u (h \in P)\}$.

2.1. Dual modules. Let M be a left B-module and h: $\bar{B} \to B$ an anti-isomorphism (e.g. φ in Sect. 1). Then the dual space $M^* = \mathrm{Hom}_{\mathbf{F}}(M, \mathbf{F})$ has a left $\bar{B}$-module structure by

$$(xu, v) = (u, h(x)v), \quad \text{for } x \in \bar{B}, \quad u \in M^*, v \in M . \tag{2.1}$$

We denote it by M^{*h}. Similarly, for a $\bar{B}$-module N and an anti-isomorphism g: $B \to \bar{B}$, the dual space N^* has a left B-module structure and we denote it by N^{*g}.

Let M be a $\bar{B}$-module, N be a U-module and g be as above. Then we can give a left B-module structure on $\mathrm{Hom}_{\mathbf{F}}(M, N)$ by

$$(xf)(u) = \sum x_{(2)} f(g(x_{(1)})u), \quad \text{for } x \in B, \ f \in \mathrm{Hom}_{\mathbf{F}}(M, N), u \in M , \tag{2.2}$$

where we denote $\Delta^{(r)}(x) = \sum x_{(1)} \otimes x_{(2)} \in B \otimes U$. Note that there is an isomorphism as a B-module,

$$\mathrm{Hom}_{\mathbf{F}}(M, N) \cong M^{*g} \otimes N . \tag{2.3}$$

Similarly, for B-modules M and N, we give a U-module structure on $\mathrm{Hom}_{\mathbf{F}}(M, N)$ by,

$$(yf)(u) = \sum y_{(2)} f(h(y_{(1)})u), \quad \text{for } y \in U, \ f \in \mathrm{Hom}_{\mathbf{F}}(M, N), u \in M ,$$

where $\Delta^{(b)}(y) = \sum y_{(1)} \otimes y_{(2)} \in \bar{B} \otimes B$.

Proposition 2.1. *Let L be a $\bar{B}$-module, M be a B-module, N be a U-module and φ: $\bar{B} \to B$ be as in Sect.* 1. *Then we obtain an isomorphism of vector spaces;*

$$\mathrm{Hom}_U(L \otimes M, N) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_{\mathbf{F}}(L, N)) . \tag{2.4}$$

Remark that $L \otimes M$ has a U-module structure via $\Delta^{(b)}$ and $\mathrm{Hom}_{\mathbf{F}}(L, N)$ has a B-module structure via $\Delta^{(r)}$ according to (2.2).

Proof. We define a map Φ: $\mathrm{Hom}_U(L \otimes M, N) \to \mathrm{Hom}_B(M, \mathrm{Hom}_{\mathbf{F}}(L, N))$ as follows: for $f \in \mathrm{Hom}_U(L \otimes M, N)$, $\Phi(f)$ is given by

$$\begin{array}{rl} \Phi(f)(y)\colon & L \to N \\ & x \mapsto f(x \otimes y) \end{array}, \quad \text{for } y \in M .$$

First we check the well-definedness of Φ i.e. B-linearity of $\Phi(f)$. For $P \in B$, $x \in L$ and $y \in M$ by the definition of Φ, we get $(\Phi(f)(Py))(x) = f(x \otimes Py)$. From (2.2) we can act P on $\Phi(f)(y)$ as follows:

$$\begin{aligned}(P\Phi(f)(y))(x) &= \sum P_{(2)} \Phi(f)(y)(\varphi^{-1} P_{(1)} x) \\ &= \sum P_{(2)} f(\varphi^{-1} P_{(1)} x \otimes y) \\ &= \sum f(P_{(2)} \varphi^{-1} P_{(1)} x \otimes P_{(3)} y) \\ &= \sum f(\varphi^{-1}(P_{(1)} \varphi P_{(2)}) x \otimes P_{(3)} y) ,\end{aligned} \tag{2.5}$$

where $(1 \otimes \Delta^{(b)})\Delta^{(r)}(P)=\sum P_{(1)} \otimes P_{(2)} \otimes P_{(3)}$. From (A2) in Appendix A, the last formula in (2.5) is equal to $f(x \otimes Py)$. Hence $\Phi(f)$ is B-linear. The injectivity of Φ is trivial. For $k \in \mathrm{Hom}_B(M, \mathrm{Hom}_{\mathbf{F}}(L, N))$, we define $\Psi(k) \in \mathrm{Hom}_U(L \otimes M, N)$ by $\Psi(k)(x \otimes y)=(k(y))(x)$ $(x \in L, y \in M)$. We can easily check the well-definedness of Ψ and $\Phi \circ \Psi(k)=k$. Q.E.D.

From Proposition 2.1 and (2.3), for a B-modules L, M and a U-module N, there is an isomorphism;

$$\mathrm{Hom}_U({}^rL^{*\varphi} \otimes M, N) \cong \mathrm{Hom}_B(M, \hat{L} \otimes N)\,, \tag{2.6}$$

where ${}^rL^{*\varphi}$ is a restricted dual module of L defined by ${}^rL^{*\varphi}=\bigoplus_\lambda L_\lambda^*$, $\hat{L}$ is a weight completion of L defined by $\hat{L}=\prod_\lambda L_\lambda$ and note that as a B-module: $({}^rL^{*\varphi})^{*\varphi^{-1}} \cong \hat{L}$. Similarly, we obtain

Corollary 2.2. *For $B^{\gtrless}$-modules L, M and $U^{\gtrless}$-module N, there is an isomorphism,*

$$\mathrm{Hom}_{U^{\gtrless}}({}^rL^{*\varphi} \otimes M, N) \cong \mathrm{Hom}_{B^{\gtrless}}(M, \hat{L} \otimes N)\,.$$

Note that in the rest of this paper the expression $L \hat{\otimes} N$ implies $\hat{L} \otimes N$.

2.2. *Highest weight B-modules. We shall discuss highest weight B-modules.*

Proposition 2.3. *For $\lambda \in \mathfrak{t}^*$, we set*

$$H(\lambda)=B/\sum_i Be_i''+\sum_{h \in P^*} B(q^h-q^{\langle h,\lambda\rangle})\,,$$

$$H^r(\lambda)=B/\sum_i f_iB+\sum_{h \in P^*} (q^h-q^{\langle h,\lambda\rangle})B\,. \tag{2.7}$$

Then for an arbitrary λ, $H(\lambda)$ (resp. $H^r(\lambda)$) *is an irreducible highest weight left* (resp. *right*) *B-module and is a free and rank one U^-* (resp. B^+)*-module.*

We denote the highest weight vector $1 \bmod \sum_i Be_i''+\sum_{h \in P^*} B(q^h-q^{\langle h,\lambda\rangle})$ by u_λ and $1 \bmod \sum_i f_iB+\sum_{h \in P^*} (q^h-q^{\langle h,\lambda\rangle})B$ by u_λ^r.

Proof. We show only for $H(\lambda)$. In [K1], it is shown that the subalgebra $U^- \subset U$ has a $B^\vee$-module structure and it is isomorphic to an irreducible $B^\vee$-module $B^\vee/\sum_i B^\vee e_i''$. Since $B^\vee$ is a subalgebra of B, $H(\lambda)$ is regarded as a $B^\vee$-module. We can easily obtain the following isomorphism of $B^\vee$-modules and then of the U^--module,

$$B^\vee/\sum_i B^\vee e_i'' \cong U^- \xrightarrow{\sim} H(\lambda)\,,$$

$$X \mapsto Xu_\lambda\,, \tag{2.8}$$

Hence $H(\lambda)$ is irreducible as a $B^\vee$-module and then irreducible as a B-module. Q.E.D.

Let $\mathcal{O}(B)$ (resp. $\mathcal{O}^r(B)$) be the category of left (resp. right) B-modules M such that M has a weight space decomposition and for any element $u \in M$ there exists $l>0$ such that $e_{i_1}''e_{i_2}'' \cdots e_{i_l}''u=0$ (resp. $uf_{i_1}f_{i_2} \cdots f_{i_l}=0$) for any $i_1, i_2, \ldots, i_l \in I$ (see [K1]).

Proposition 2.4. (*See Remark* 3.4.10 [K1].) *The category $\mathcal{O}(B)$* (resp. $\mathcal{O}^r(B)$) *is semi-simple,* (i.e. *any object is a direct sum of simple objects*) *and for any simple object M there exists $\lambda \in \mathfrak{t}^*$ such that $M \cong H(\lambda)$* (resp. $M \cong H^r(\lambda)$) *as a B-module.*

Proof. We shall show only for $\mathcal{O}(B)$. Let M be a simple object of $\mathcal{O}(B)$ and v_λ be a highest weight vector of M with a highest weight λ, where a highest weight vector implies a weight vector annihilated by any e_i'' $(i \in I)$. Here we set u_λ a highest weight vector of $H(\lambda)$. We can easily know that a map

$$\begin{array}{rl} \pi\colon H(\lambda) & \to M \\ Pu_\lambda & \mapsto Pv_\lambda \end{array} \quad (P \in B)$$

is B-linear and surjective. The kernel of π is a B-submodule of $H(\lambda)$, and by Proposition 2.3, the kernel of π is 0. Hence π is injective. Next we show the semi-simplicity of $\mathcal{O}(B)$. First note that if $N \subset M$ are objects in $\mathcal{O}(B)$, then M/N is also an object in $\mathcal{O}(B)$. Let M be a non-simple object of $\mathcal{O}(B)$. Without a loss of generality, we may assume that M has two highest weight vectors u and v. By the argument in this proof, Bu and Bv are simple. We have $M = Bu + Bv$ and then B-module M/Bu has only one highest weight vector $\bar{v}$ and $M/Bu \cong B\bar{v}$. By the argument in this proof, we have $B\bar{v} \cong Bv$, since $wt(v) = wt(\bar{v})$. Thus the following exact sequence splits:

$$0 \to Bu \to M \to B\bar{v} \to 0 .$$

Therefore, we obtain the desired result. Q.E.D.

Note that lowest weight $\bar{B}$-modules, e.g. $H(\lambda)^*$ have similar properties.

3. Bilinear Forms

In this section, after recalling the Killing form of U, we give an interpretation of the Killing form of U by the algebra $B^\vee$. We also introduce a bilinear pairing similar to a vacuum expectation value.

Proposition 3.1. ([R, T]) (1) *There exists a unique bilinear form*

$$(\ ,\ )\colon U^{\geqq} \times U^{\leqq} \to \mathbf{F} , \tag{3.1}$$

satisfying the following properties;

$$(x, y_1 y_2) = (\Delta(x), y_1 \otimes y_2), \quad (x \in U^{\geqq}, y_1, y_2 \in U^{\leqq}) , \tag{3.2}$$

$$(x_1 x_2, y) = (x_2 \otimes x_1, \Delta(y)), \quad (x_1, x_2 \in U^{\geqq}, y \in U^{\leqq}) , \tag{3.3}$$

$$(q^h, q^{h'}) = q^{-(h|h')} (h, h' \in P^*) , \tag{3.4}$$

$$(T, f_i) = (e_i, T) = 0 , \tag{3.5}$$

$$(e_i, f_j) = \delta_{ij}/(q_i^{-1} - q_i) , \tag{3.6}$$

where $(\,|\,)$ *is an invariant bilinear form on* $\mathfrak{t}$ ([Kac]).
(2) *The bilinear form* $(\ ,\ )$ *enjoys the following properties*:

$$(xq^h, yq^{h'}) = q^{-(h|h')}(x, y), \quad \textit{for } x \in U^{\geqq}, y \in U^{\leqq}, h, h' \in P^*. \tag{3.7}$$

For any $\beta \in Q_+$, $(\ ,\ )_{|U_\beta^+ \times U_{-\beta}^-}$ *is non-degenerate and* $(U_\gamma^+, U_{-\delta}^-) = 0$, *if* $\gamma \neq \delta$. (3.8)

We call this bilinear form the Killing form of U.

By using the relation (1.5), it is easy to see that the algebra $B^\vee$ has the following decomposition:

$$B^\vee = \mathbf{F} \oplus \left(\sum_i f_i B^\vee + \sum_i B^\vee e_i'' \right). \tag{3.9}$$

Hence for any $x \in B^\vee$ there is a unique constant c such that $x \equiv c \bmod \sum_i f_i B^\vee + \sum_i B^\vee e_i''$. We denote this c by $\imath(x)$.

There is the following connection between $\imath$ and the Killing form of U.

Proposition 3.2. *Let $\imath$ be as above and $(\ ,\ )$ the Killing form of U. For any $u \in U^+$ and $v \in U^-$,*

$$\imath(\varphi(u)v) = (u, v) . \tag{3.10}$$

Note that since $u \in U^+ = \bar{B}^\vee \cap U$, $\varphi(u) \in B^\vee$ and then $\varphi(u)v \in B^\vee$.

Proof. We may assume u and v are weight vectors. If $\mathrm{wt}(u) + \mathrm{wt}(v) \neq 0$, trivially $\imath(\varphi(u)v) = (u, v) = 0$. For $u \in U_\beta^+$ and $v \in U_{-\beta}^-$ $(\beta \in Q_+)$, it is enough to show

$$\varphi(u)v \equiv (u, v) \bmod \sum_i B^\vee e_i'' . \tag{3.11}$$

We shall show by the induction on $|\beta| =$ height of β. Set $l = |\beta|$. Without a loss of generality, we can set $u = e_{i_1} e_{i_2} \cdots e_{i_l}$ and $v = f_{j_1} f_{j_2} \cdots f_{j_l}$, where $\alpha_{i_1} + \cdots + \alpha_{i_l} = \alpha_{j_1} + \cdots + \alpha_{j_l} = \beta$,

$$\left\{ \prod_{k=1}^{l} (q_{i_k}^{-1} - q_{i_k}) \right\} \varphi(u)v = e_{i_l}'' \cdots e_{i_2}'' e_{i_1}'' f_{j_1} f_{j_2} \cdots f_{j_l}$$

$$= q_{i_1}^{\langle h_{i_1}, \alpha_{j_1} + \quad + \alpha_{j_l} \rangle} e_{i_l}'' \cdots e_{i_2}'' f_{j_1} \cdots f_{j_l} e_{i_1}''$$

$$+ \sum_{m=1}^{l} q_{i_1}^{\langle h_{i_1}, \alpha_{j_1} + \cdots + \alpha_{j_{m-1}} \rangle} \delta_{i_1, j_m} e_{i_l}'' \cdots e_{i_2}'' f_{j_1} \cdots f_{j_{m-1}} f_{j_{m+1}} \cdots j_{j_l} .$$

Thus, by the hypothesis of the induction,

$$(q_{i_1}^{-1} - q_{i_1}) \varphi(u)v$$

$$\equiv \sum_{m=1}^{l} q_{i_1}^{\langle h_{i_1}, \alpha_{j_1} + \cdot \quad + \alpha_{j_{m-1}} \rangle} \delta_{i_1, j_m} \varphi(e_{i_2} \cdot\cdot e_{i_l}) f_{j_1} \cdot\cdot f_{j_{m-1}} f_{j_{m+1}} \cdot\cdot f_{j_l} \bmod \sum_i B^\vee e_i''$$

$$\equiv \sum_{m=1}^{l} q_{i_1}^{\langle h_{i_1}, \alpha_{j_1} + \cdot\cdot + \alpha_{j_{m-1}} \rangle} \delta_{i_1, j_m} (e_{i_2} \cdot\cdot e_{i_l}, f_{j_1} \cdot\cdot f_{j_{m-1}} f_{j_{m+1}} \cdot\cdot f_{j_l}) \bmod \sum_i B^\vee e_i'' . \tag{3.12}$$

On the other hand, from the formulae (3.2)–(3.8) and the explicit form of $\Delta(f_i)$,

$$(e_{i_1} \cdots e_{i_l}, f_{j_1} \cdots f_{j_l}) = (e_{i_2} \cdots e_{i_l} \otimes e_{i_1}, \Delta(f_{j_1} \cdots f_{j_l}))$$

$$= \sum_{m=1}^{l} (e_{i_2} \cdots e_{i_l} \otimes e_{i_1}, f_{j_1} \cdots f_{j_{m-1}} f_{j_{m+1}} \cdots f_{j_l} \otimes t_{j_1}^{-1} \cdots t_{j_{m-1}}^{-1} f_{j_m} t_{j_{m+1}}^{-1} \cdots t_{j_l}^{-1})$$

$$= \sum_{m=1}^{l} (e_{i_2} \cdots e_{i_l}, f_{j_1} \cdots f_{j_{m-1}} f_{j_{m+1}} \cdots f_{j_l})(e_{i_1}, t_{j_1}^{-1} \cdots t_{j_{m-1}}^{-1} f_{j_m} t_{j_{m+1}}^{-1} \cdots t_{j_l}^{-1})$$

$$= \sum_{m=1}^{l} q_{j_m}^{\langle h_{j_1}+\cdots+h_{j_{m-1}}, \alpha_{j_m}\rangle}(e_{i_1}, f_{j_m})(e_{i_2}\cdots e_{i_l}, f_{j_1}\cdots f_{j_{m-1}} f_{j_{m+1}}\cdots f_{j_l})$$

$$= \sum_{m=1}^{l} \frac{q_{j_1}^{\langle h_{j_1}, \alpha_{j_1}+\cdots+\alpha_{j_{m-1}}\rangle}\delta_{i_1, j_m}}{(q_{i_1}^{-1}-q_{i_1})}(e_{i_2}\cdots e_{i_l}, f_{j_1}\cdots f_{j_{m-1}} f_{j_{m+1}}\cdots f_{j_l}) . \tag{3.13}$$

From the equality of (3.12) and (3.13), we get the desired result. Q.E.D.

We shall define a bilinear pairing similar to vacuum expectation values. For $\lambda \in \mathfrak{t}^*$ we define a bilinear pairing $\langle\ |\ \rangle$: $H^r(\lambda)\times H(\lambda)\to \mathbf{F}$ as follows: similar to (3.9) the algebra B has a decomposition,

$$B = T \oplus \left(\sum_i f_i B + \sum_i B e_i''\right) . \tag{3.14}$$

Let Ω: $B\to T$ be a canonical projection. Here we can define a T-valued pairing E: $B\times B\to T$ by $E(x, y)=\Omega(xy)$ for $x, y\in B$. By the definition of $E(\ ,\ )$ and the associativity of B, we have

$$E(xy, z) = E(x, yz) \quad \text{for } x, y, z \in B . \tag{3.15}$$

We define π_λ: $T\to \mathbf{F}$ by $tu_\lambda = \pi_\lambda(t)u_\lambda$ for $t\in T$. A bilinear pairing $\langle\ |\ \rangle$: $H^r(\lambda)\times H(\lambda)\to\mathbf{F}$ is given by $\langle u|v\rangle = \pi_\lambda(E(P, Q))$, where $u=u_\lambda^r P$ and $v=Qu_\lambda$ $(P, Q\in B)$. It is clear that this is well-defined, i.e. it does not depend on a choice of P and Q.

Proposition 3.3. *There is a unique and non-degenerate bilinear pairing* $\langle\ |\ \rangle$: $H^r(\lambda)\times H(\lambda)\to\mathbf{F}$ *such that*

$$\langle ux|v\rangle = \langle u|xv\rangle, (x\in B) \quad \textit{and} \quad \langle u_\lambda^r|u_\lambda\rangle = 1 . \tag{3.16}$$

Proof. If we assume the existence, then the uniqueness immediately follows from (3.16). The existence follows from the construction above and (3.15). We shall show non-degeneracy. Let $\{P_i\}\subset U^+$ and $\{Q_i\}\subset U^-$ be bases dual to each other with respect to the Killing form such that each basis element is a weight vector. By Proposition 3.2, we get

$$\varphi(P_i)Q_j \equiv \delta_{i,j} \bmod \sum_i f_i B^\vee + \sum_i B^\vee e_i'' . \tag{3.17}$$

Hence

$$\langle u_\lambda^r \varphi(P_i)|Q_j u_\lambda\rangle = \delta_{i,j} .$$

Moreover, by Proposition 2.3, $\{u_\lambda^r\varphi(P_i)\}$ and $\{Q_i u_\lambda\}$ are bases of $H^r(\lambda)$ and $H(\lambda)$ respectively. Thus we have completed the proof of Proposition 3.3. Q.E.D.

From the property (3.16), we shall use the expression $\langle u|x|v\rangle$ for $\langle ux|v\rangle = \langle u|xv\rangle$ $(u\in H^r(\lambda),\ v\in H(\lambda)$ and $x\in B)$.

4. Intertwiners

In this section and the next section, we restrict $\mathfrak{g}$ to be an affine Lie algebra. We shall study the following type of intertwiners, which is an analogue of so-called

"q-vertex operators" ([FR, DJO]):

$$\mathrm{Hom}_{B_q(\mathfrak{g})}(H(\lambda), H(\mu)\hat{\otimes} V_z)\,, \tag{4.1}$$

where V_z is a representation of $U=U_q(\mathfrak{g})$ (see below).

4.1. Notations. We shall prepare notations. (See [KMN², Kac, DJO].)

Set $I=\{0,1,\ldots,n\}$ and $(\langle h_i,\alpha_j\rangle)_{0\leqq i,j\leqq n}$ coincides with an affine Cartan matrix in [Kac] except for the type $A_{2n}^{(2)}$. For this type we reverse the ordering of vertices since we need that $\delta-\alpha_0\in\sum_{i=1}^n \mathbf{Z}\alpha_i$ for a generator of null roots δ. Let c be a canonical center of $\mathfrak{g}$, $\{\Lambda_i\}_{i\in I}$ a set of fundamental weights and $d\in\mathfrak{t}$ a scaling element. Now, since $\mathfrak{g}$ is affine, $\dim\mathfrak{t}=\#I+1$. Thus we can write $\mathfrak{t}=\bigoplus_i\mathbf{Q}h_i\oplus\mathbf{Q}d$, $\mathfrak{t}^*=\bigoplus_i\mathbf{Q}\Lambda_i\oplus\mathbf{Q}\delta$, $P=\bigoplus_i\mathbf{Z}\Lambda_i\oplus\mathbf{Z}\delta$ and $P^*=\bigoplus_i\mathbf{Z}h_i\oplus\mathbf{Z}d$. We set $\mathfrak{t}^*_{cl}=\mathfrak{t}^*/\mathbf{Q}\delta$ and $(P_{cl})^*=\bigoplus_{i=0}^n\mathbf{Z}h_i$. Let $cl\colon P\to P_{cl}$ be a canonical projection and set $P_{cl}=cl(P)$. We fix a map $af\colon P_{cl}\to P$ by $af\circ cl(\alpha_i)=\alpha_i$ $(i\neq 0)$ and $af\circ cl(\Lambda_0)=\Lambda_0$ so that $cl\circ af=\mathrm{id}$ and $af\circ cl(\alpha_0)=\alpha_0-\delta$. For a fixed $k\in\mathbf{Q}$, we set $(\mathfrak{t}^*)_k=\{\lambda\in af(\mathfrak{t}^*_{cl})\,|\,\langle c,\lambda\rangle=k\}$ and we say that $\lambda\in(\mathfrak{t}^*)_k$ has a *level* k. The subalgebra of U (resp. B) generated by $\{e_i(\text{resp. } e_i''), f_i\,|\,i\in I\}$ and $q^h(h\in(P_{cl})^*)$ is denoted by U' (resp. B').

For a finite dimensional U'-module V and a formal variable z, we define an affinization $V_z=\mathbf{F}[z,z^{-1}]\otimes V$ with a U-module structure as follows:

$$\begin{aligned} &e_i(z^n\otimes u)=z^{n+\delta_{i0}}\otimes e_iu,\quad f_i(z^n\otimes u)=z^{n-\delta_{i0}}\otimes f_iu\,,\\ &\mathrm{wt}(z^n\otimes u)=n\delta+af(\mathrm{wt}\,u)\,. \end{aligned} \tag{4.2}$$

4.2. Condition for existence. We shall examine the condition for existence of the intertwiners of B-modules of type (4.1) by the similar way of [DJO].

Definition 4.1. *For λ, $\mu\in(\mathfrak{t}^*)_k$ and $\Phi\in\mathrm{Hom}_B(H(\lambda), H(\mu)\hat{\otimes} V_z)$ and the highest weight vector u_λ and u_μ, write the image of u_λ by Φ*

$$\Phi u_\lambda=u_\mu\otimes v_{lt}+\cdots,$$

where $\cdots$ implies terms of the form $u\otimes v$ with $u\in\bigoplus_{\xi\neq\mu}H(\mu)_\xi$. We call v_{lt} the leading term of Φ.

Proposition 4.2. *The map sending Φ to its leading term gives an isomorphism;*

$$\mathrm{Hom}_B(H(\lambda), H(\mu)\hat{\otimes} V_z)\xrightarrow{\sim}(V_z)_{\lambda-\mu}\,.$$

Proof. Let $\mathbf{F}u_\lambda$ be one dimensional $B^{\geqq}$-module with defining relations: $e_i''u_\lambda=0$ and $q^hu_\lambda=q^{\langle h,\lambda\rangle}u_\lambda$. We prepare the following lemma.

Lemma 4.3. *We have the following isomorphism;*

$$\begin{array}{ccc} \mathrm{Hom}_B(H(\lambda), H(\mu)\hat{\otimes} V_z) & \xrightarrow{\sim} & \mathrm{Hom}_{B^{\geqq}}(\mathbf{F}u_\lambda, H(\mu)\hat{\otimes} V_z)\\ \Phi & \mapsto & \Phi_{|\mathbf{F}u_\lambda} \end{array} \tag{4.3}$$

Proof of Lemma 4.3. By B-linearity of Φ, one gets $B^{\geqq}$-linearity of $\Phi_{|\mathbf{F}u_\lambda}$ and if $\Phi u_\lambda=0$, then $\Phi=0$. Hence the map (4.3) is well-defined and injective. To show the surjectivity, take a vector $v\in H(\mu)\hat{\otimes} V_z$ such that $\mathrm{wt}(v)=\lambda$ and $e_i''v=0$ for all $i\in I$. By the property of the category $\mathcal{O}(B)$ (Proposition 2.4), the B-module Bv is isomorphic to $H(\lambda)$ as a B-module. Hence we obtain the surjectivity. Q.E.D.

From Corollary 2.2, we have the following isomorphism:

$$\text{R.H.S. of (4.3)} \cong \operatorname{Hom}_{U^{\geqq}}({}^rH(\mu)^{*\varphi} \otimes \mathbf{F}u_\lambda, V_z)\,. \tag{4.4}$$

Here note that $\Delta^{(b)}(U^{\geqq}) \subset U^{\geqq} \otimes B^{\geqq}$, and as a $U^{\geqq}(=\bar{B} \cap U)$-module ${}^rH(\mu)^{*\varphi}$ is isomorphic to

$$U^{\geqq} / \sum_{h \in P^*} U^{\geqq}(q^h - q^{-\langle h, \mu \rangle})\,.$$

It is easy to see that R.H.S. of (4.4) is isomorphic to $(V_z)_{\lambda-\mu}$. Q.E.D.

5. 2-Point Functions and Commutation Relations of Intertwiners

In this section we show that a matrix determined by "2-point functions" coincides with a quantum R-matrix up to a diagonal matrix and give commutation relations for intertwiners.

5.1. 2-point functions. First we shall define "2-point functions" for the intertwiners of B-modules introduced in Sect. 4. We fix $k \in \mathbf{Q}$. For $\Phi_\lambda^{\mu V}(z_1) \in \operatorname{Hom}_B(H(\lambda), H(\mu) \hat{\otimes} V_{z_1})$ and $\Phi_\mu^{\nu W}(z_2) \in \operatorname{Hom}_B(H(\mu), H(\nu) \hat{\otimes} W_{z_2})$ $(\lambda, \mu, \nu \in (\mathfrak{t}^*)_k)$, we use an abbreviated notation $\Phi_\mu^{\nu W}(z_2)\Phi_\lambda^{\mu V}(z_1)$ for $(\Phi_\mu^{\nu W}(z_2) \otimes \mathrm{id}_{V_{z_1}})\Phi_\lambda^{\mu V}(z_1)$. With this notation, the following is called a *2-point function*:

$$\langle u_\nu^r | \Phi_\mu^{\nu W}(z_2) \Phi_\lambda^{\mu V}(z_1) | u_\lambda \rangle \in \mathbf{F}\left[\left[\frac{z_1}{z_2}\right]\right] \otimes W \otimes V\,.$$

We shall give an explicit description of 2-point functions. For a B-module $H(\lambda)$, ${}^rH(\lambda)^{*\varphi}$ means the restricted dual module $\bigoplus_\xi (H(\lambda)^*)_\xi$ as in Sect. 2. Here ${}^rH(\lambda)^{*\varphi}$ is an irreducible lowest weight left $\bar{B}$-module with a lowest weight vector denoted by u_λ^* such that $f_i' u_\lambda^* = 0$ for any $i \in I$, $q^h = q^{-\langle h, \lambda \rangle} u_\lambda^*$ for any $h \in P^*$, $(u_\lambda^*, u_\lambda) = 1$ and $(u_\lambda^*, v) = 0$ for $v \in \bigoplus_{\mu \neq \lambda} H(\lambda)_\mu$. From Proposition 2.1 and the formula (2.6), there is an isomorphism for $\lambda, \mu \in (\mathfrak{t}^*)_k$;

$$\Psi\colon \operatorname{Hom}_U({}^rH(\mu)^{*\varphi} \otimes H(\lambda), V_z) \xrightarrow{\sim} \operatorname{Hom}_B(H(\lambda), H(\mu) \hat{\otimes} V_z)\,. \tag{5.1}$$

We translate this in terms of dual bases as follows. Let $\{u_i\} \subset H(\mu)$ and $\{u_i^*\} \subset {}^rH(\mu)^{*\varphi}$ be bases dual to each other such that $u_\mu \in \{u_i\}$. Then for $x \in H(\lambda)$ and $\phi \in \operatorname{Hom}_U({}^rH(\mu)^{*\varphi} \otimes H(\lambda), V_z)$, Ψ is given by

$$\Psi(\phi)(x) = \sum_i u_i \otimes \phi(u_i^* \otimes x)\,. \tag{5.2}$$

The following lemma is immediate from (5.2) and the definition of the leading term.

Lemma 5.1. *Let Ψ and ϕ be as above. Then $\phi(u_\mu^* \otimes u_\lambda)$ is a leading term of $\Psi(\phi)$.*

Lemma 5.2. *Let $\{P_i\} \subset U^+$ and $\{Q_i\} \subset U^-$ be bases dual to each other with respect to the Killing form such that each basis element is a weight vector and $1 \in \{P_i\}$ (and then $1 \in \{Q_i\}$). Then for any $\lambda \in \mathfrak{t}^*$, $\{P_i u_\lambda^*\} \subset {}^rH(\lambda)^{*\varphi}$ and $\{Q_i u_\lambda\} \subset H(\lambda)$ are bases dual to each other.*

Proof. First note that for $u\in {}^rH(\lambda)^{*\varphi}$, $v\in H(\lambda)$, $x\in\bar{B}$ and $y\in B$,

$$(xu, yv)=(u, \varphi(x)yv)=(\varphi^{-1}(y)xu, v)\,. \tag{5.3}$$

From Proposition 3.2 and (5.3), we get $(P_iu_\lambda^*, Q_ju_\lambda)=\delta_{i,j}$ and from Proposition 2.3 (and a similar one for lowest $\bar{B}$-modules), we know that $\{P_iu_\lambda^*\}$ and $\{Q_iu_\lambda\}$ are bases. Q.E.D.

Let $\mathscr{R}$ be a universal R-matrix and $\mathscr{R}'(z)$ a modified universal R-matrix as in (B10) (see Appendix B.). Let V and W be finite dimensional U'-modules and V_{z_1} and W_{z_2} their affinizations. We denote the image of the universal R-matrix onto a U-module $V_{z_1}\otimes W_{z_2}$ by $R^{VW}(z)=\pi_{V\otimes W}(\mathscr{R}'(z))$, where $z=z_1/z_2$. This coincides with a quantum R-matrix on $V\otimes W$ up to a scalar factor.

Theorem 5.3. *For intertwiners $\Phi_\lambda^{\mu V}(z_1)\in \mathrm{Hom}_B(H(\lambda), H(\mu)\hat{\otimes} V_{z_1})$ and $\Phi_\mu^{\nu W}(z_2)\in \mathrm{Hom}_B(H(\mu), H(\nu)\hat{\otimes} W_{z_2})$, we set $v_0\in V_{z_1}$ and $w_0\in W_{z_2}$ be leading terms of $\Phi_\lambda^{\mu V}(z_1)$ and $\Phi_\mu^{\nu W}(z_2)$ respectively. Then the 2-point function is given by*

$$\langle u_\nu^r|\Phi_\mu^{\nu W}(z_2)\Phi_\lambda^{\mu V}(z_1)|u_\lambda\rangle=q^{(\lambda-\mu,\mu-\nu)}\sigma\circ R^{VW}(z_1/z_2)(v_0\otimes w_0)\,,$$

where σ: $a\otimes b\to b\otimes a$.

Proof. Let Ψ be as in (5.1). We set $\phi_1=\Psi^{-1}(\Phi_\lambda^{\mu V}(z_1))$ and $\phi_2=\Psi^{-1}(\Phi_\mu^{\nu W}(z_2))$. Let $\{P_i\}$ and $\{Q_i\}$ be as in Lemma 5.2. From (5.2) and Lemma 5.2, for $x\in H(\lambda)$ we have

$$\Phi_\mu^{\nu W}(z_2)\Phi_\lambda^{\mu V}(z_1)(x)=\sum_{i,j} Q_ju_\nu\otimes\phi_2(P_ju_\nu^*\otimes Q_iu_\mu)\otimes\phi_1(P_iu_\mu^*\otimes x)\,,$$

and then 2-point function can be written by

$$\begin{aligned}\langle u_\nu^r|\Phi_\mu^{\nu W}(z_2)\Phi_\lambda^{\mu V}(z_1)|u_\lambda\rangle&=\sum_i \phi_2(u_\nu^*\otimes Q_iu_\mu)\otimes\phi_1(P_iu_\mu^*\otimes u_\lambda)\\ &\in\mathbf{F}\left[\left[\frac{z_1}{z_2}\right]\right]\otimes W\otimes V\,.\end{aligned} \tag{5.4}$$

By the intertwining property of ϕ_i $(i=1,2)$ and the fact that $e_i''u_\lambda=0$ and $f_i'u_\mu^*=0$ for any $i\in I$, we have

$$P_i\phi_1(u_\mu^*\otimes u_\lambda)=\phi_1(\Delta^{(b)}(P_i)(u_\mu^*\otimes u_\lambda))=\phi_1(P_iu_\mu^*\otimes u_\lambda)\,,$$

$$Q_i\phi_2(u_\nu^*\otimes u_\mu)=\phi_2(\Delta^{(b)}(Q_i)(u_\nu^*\otimes u_\mu))=\phi_2(u_\nu^*\otimes Q_iu_\mu)\,.$$

Hence (5.4) can be rewritten by

$$\langle u_\nu^r|\Phi_\mu^{\nu W}(z_2)\Phi_\lambda^{\mu V}(z_1)|u_\lambda\rangle=\sigma\left(\sum_i P_i\otimes Q_i\right)\cdot\{\phi_1(u_\mu^*\otimes u_\lambda)\otimes\phi_2(u_\nu^*\otimes u_\mu)\}\,. \tag{5.5}$$

From (B9) in Appendix B, on a vector $u\otimes v$ ($\mathrm{wt}(u)=\xi$ and $\mathrm{wt}(v)=\eta$) we have

$$\mathscr{R}=q^{-(\xi,\eta)}\sum_i P_i\otimes Q_i\,. \tag{5.6}$$

From Lemma 5.1, $\phi_1(u_\mu^*\otimes u_\lambda)\otimes\phi_2(u_\nu^*\otimes u_\mu)=v_0\otimes w_0$. Therefore by the formulae (5.5) and (5.6), we obtain the desired result. Q.E.D.

Fix bases C and C' of V and W respectively such that each basis element is a weight vector. For a pair $(v_i, w_j)\in C\times C'$ let $\Phi^{\mu_i V}_{\lambda,(i)}(z_1)\in \mathrm{Hom}_B(H(\lambda), H(\mu_i)\hat{\otimes} V_{z_1})$ and $\Phi^{v_j W}_{\mu_i,(j)}(z_2)\in \mathrm{Hom}_B(H(\mu_i), H(v_j)\hat{\otimes} W_{z_2})$ be intertwiners with leading terms v_i and w_j respectively. Let $\Xi(z_1, z_2)$, $D\in \mathrm{End}(V\otimes W)$ be matrices defined by

$$\Xi(z_1, z_2)\colon v_i\otimes w_j\mapsto \sigma\langle u^r_{v_j}|\Phi^{v_j W}_{\mu_i,(j)}(z_2)\Phi^{\mu_i V}_{\lambda,(i)}(z_1)|u_\lambda\rangle\,,$$

$$D\colon v_i\otimes w_j\mapsto q^{(\mathrm{wt}(v_i),\,\mathrm{wt}(w_j))}v_i\otimes w_j\,.$$

From Theorem 5.3, we obtain the following;

Corollary 5.4. *With the notations as above, we have*

$$\Xi(z_1, z_2)=DR^{VW}(z_1/z_2)\,.$$

5.2. Commutation relations. Let V and W be finite dimensional U'-modules. We assume that $V_{z_1}\otimes W_{z_2}$ is an irreducible U-module. Let C and C' be bases of V and W as in 5.1. Now, we fix $v_0\in C$, $w_0\in C'$, $\lambda, v\in(\mathfrak{t}^*)_k$ such that $\lambda-v=af(\mathrm{wt}(v_0)+\mathrm{wt}(w_0))$ and let $\Phi^{vV}_{\mu}(z)$ and $\Phi^{\mu W}_{\lambda}(z)$ be intertwiners such that their leading terms are $v_0\in C$ and $w_0\in C'$ respectively. Here note that we identify $v\in V$ and $w\in W$ with $1\otimes v\in V_z$ and $1\otimes w\in W_z$ respectively. We set

$$E=\{(v, w)\in C\times C'\,|\,af(\mathrm{wt}(v))+af(\mathrm{wt}(w))=af(\mathrm{wt}(v_0))+af(\mathrm{wt}(w_0))\}\,.$$

For a pair $(v_i, w_i)\in E$, we set $\Phi^{\mu_i V}_{\lambda,(i)}(z)$ and $\Phi^{vW}_{\mu_i,(i)}(z)$ be intertwiners such that their leading terms are v_i and w_i respectively.

For a U'-modules $V\otimes W$, from the uniqueness and the unitarity of quantum R-matrices, there exists some function $f(x)$ such that

$$R^{VW}(z_1/z_2)\sigma R^{WV}(z_2/z_1)\sigma=f(z_1/z_2)\,\mathrm{id}_{V\otimes W}\,. \tag{5.7}$$

We define $W_i(z_1/z_2)$ by,

$$R^{VW}(z_1/z_2)^{-1}(v_0\otimes w_0)=\sum_i q^{(\mathrm{wt}(v_i),\,\mathrm{wt}(w_i))}(v_i\otimes w_i)\,W_i(z_1/z_2)\,. \tag{5.8}$$

Proposition 5.5. *With the notations as above, we have the following commutation relation (in the sense of a matrix element):*

$$\sigma\circ R^{VW}(z_1/z_2)\Phi^{vV}_{\mu}(z_1)\Phi^{\mu W}_{\lambda}(z_2)=q^{(\lambda-\mu,\,\mu-v)}f(z_1/z_2)$$
$$\times\sum_i \Phi^{vW}_{\mu_i,(i)}(z_2)\Phi^{\mu_i V}_{\lambda,(i)}(z_1)\,W_i(z_1/z_2)\,.$$

Proof. From (5.7) and Theorem 5.3, we have

$$\begin{aligned}f(z_1/z_2)(v_0\otimes w_0)&=R^{VW}(z_1/z_2)\sigma R^{WV}(z_2/z_1)\sigma(v_0\otimes w_0)\\&=q^{-(\lambda-\mu,\,\mu-v)}R^{VW}(z_1/z_2)\langle u^r_v|\Phi^{vV}_{\mu}(z_1)\Phi^{\mu W}_{\lambda}(z_2)|u_\lambda\rangle\,.\end{aligned} \tag{5.9}$$

On the other hand, from (5.8) and Theorem 5.3,

$$v_0\otimes w_0=\sigma\sum_i\langle u^r_v|\Phi^{vW}_{\mu_i,(i)}(z_2)\Phi^{\mu_i V}_{\lambda,(i)}(z_1)|u_\lambda\rangle\,W_i(z_1/z_2)\,. \tag{5.10}$$

From (5.9), (5.10), the intertwining property of $\sigma \circ R^{VW}(z)$ and B-linearity of elements in $\mathrm{Hom}_B(H(\lambda), H(\mu)\hat{\otimes} V_z)$, we obtain the desired result. Q.E.D.

Example. Set $\mathfrak{g}=\widehat{\mathfrak{sl}_2}$ and $V=\mathbf{F}u_+ \oplus \mathbf{F}u_-$. A U-module structure of V_z is given by

$$\begin{aligned}&e_0(z^n u_+)=z^{n+1}u_-, e_0(z^n u_-)=0, f_0(z^n u_+)=0, f_0(z^n u_-)=z^{n-1}u_+ ,\\ &e_1(z^n u_+)=0, e_1(z^n u_-)=z^n u_+, f_1(z^n u_+)=z^n u_-, f_1(z^n u_-)=0 ,\\ &\mathrm{wt}(z^n u_\pm = n\delta \pm (\Lambda_1-\Lambda_0) .\end{aligned}$$

Set

$$(z)_\infty=\prod_{i=0}^{\infty}(1-q^{4i}z),\quad \rho(z)=\frac{(q^2z)_\infty^2}{(z)_\infty(q^4z)_\infty},\quad \Theta(z)=(z)_\infty(q^4z^{-1})_\infty(q^4)_\infty .$$

An explicit form of the image of the universal R-matrix onto $V_{z_1}\otimes V_{z_2}$ is described in [DFJMN], therefore 2-point functions are given as follows:

$$\langle u_\nu^r|\Phi_\mu^{\nu V}(z_2)\Phi_\lambda^{\mu V}(z_1)|u_\lambda\rangle=\rho(z_1/z_2)$$

$$\times\begin{cases} u_\pm\otimes u_\pm & \text{if } \lambda-\mu=\mu-\nu=\pm(\Lambda_1-\Lambda_0) ,\\ \dfrac{q^{-1}-q}{1-q^2z_1/z_2}\dfrac{z_1}{z_2}u_+\otimes u_- +\dfrac{1-z_1/z_2}{1-q^2z_1/z_2}u_-\otimes u_+ & \text{if } \lambda-\mu=\nu-\mu=\Lambda_1-\Lambda_0 ,\\ \dfrac{1-z_1/z_2}{1-q^2z_1/z_2}u_+\otimes u_- +\dfrac{q^{-1}-q}{1-q^2z_1/z_2}u_-\otimes u_+ & \text{if } \lambda-\mu=\nu-\mu=\Lambda_0-\Lambda_1 ,\end{cases}$$

where we normalize intertwiners so that their leading term is u_+ or u_-. Note that we take the normalization $(\alpha_i,\alpha_i)=2$, thus we have $(\Lambda_i,\Lambda_j)=\delta_{i1}\delta_{j1}/2$. The function in (5.7) is given by

$$f(z)=q^{-1}\frac{\Theta(q^2z)^2}{\Theta(z)\Theta(q^4z)} .$$

6. An Element $\tilde{\mathscr{R}}$ and a Projector Γ

In this section we do not restrict $\mathfrak{g}$ to be an affine Lie algebra. We introduce an element $\tilde{\mathscr{R}}$, which satisfies the properties similar to those of the universal R-matrix.

6.1. An element $\tilde{\mathscr{R}}$. We follow the notations as in Appendix B. We can define $(\hat{B}\hat{\otimes}\hat{U}^{\otimes n})\hat{}$ and extend $\Delta^{(r)}\otimes 1^{\otimes n}$ by the similar manner as in Appendix B.

Let $\mathscr{R}$ be the universal R-matrix of U (see (B8) in Appendix B.). We define

$$\tilde{\mathscr{R}}=q^{-H}\sum_{\beta\in Q_+}q^{(\beta,\beta)}(k_\beta^{-1}\otimes k_\beta)(\varphi S^{-1}\otimes 1)(C_\beta)\in((\hat{B}\otimes\hat{U})\hat{}\ . \tag{6.1}$$

Here note that left components of C_β belong to $U^{\geqq}$, then the map $\varphi S^{-1}\colon U^{\geqq}\xrightarrow{S^{-1}}U^{\geqq}\xrightarrow{\varphi}B^{\geqq}$ is well-defined, and formally we can write $\tilde{\mathscr{R}}=(\varphi S^{-1}\otimes 1)\mathscr{R}$ since φS^{-1} act as an identity for the Cartan part.

Proposition 6.1. *$\tilde{\mathscr{R}}$ enjoys the following properties;*

$\tilde{\mathscr{R}}$ is invertible and

$$\tilde{\mathscr{R}}^{-1}=\sum_{\beta\in Q_+} q^{(\beta,\beta)}(1\otimes k_\beta)(\varphi\otimes 1)(C_\beta)q^H , \tag{6.2}$$

$$(\Delta^{(r)}\otimes 1)\tilde{\mathscr{R}}=\tilde{\mathscr{R}}_{13}\mathscr{R}_{23} , \tag{6.3}$$

$$(1\otimes\Delta)\tilde{\mathscr{R}}=\tilde{\mathscr{R}}_{13}\tilde{\mathscr{R}}_{12} , \tag{6.4}$$

$$\tilde{\mathscr{R}}\cdot\Delta^{(r)}(X)=(X\otimes 1)\cdot\tilde{\mathscr{R}}\ (X\in U^-) , \tag{6.5}$$

$$\tilde{\mathscr{R}}\cdot(\varphi\otimes S)\sigma\Delta^{(r)}(X)=(\varphi S^{-1}\otimes S\varphi^{-1})\sigma\Delta^{(r)}(X)\cdot\tilde{\mathscr{R}},\ (X\in B^+) . \tag{6.6}$$

Corollary 6.2. *We have the following equation in $(\hat{B}\hat{\otimes}\hat{U}\hat{\otimes}\hat{U})\hat{}$:*

$$\mathscr{R}_{23}\tilde{\mathscr{R}}_{13}\tilde{\mathscr{R}}_{12}=\tilde{\mathscr{R}}_{12}\tilde{\mathscr{R}}_{13}\mathscr{R}_{23} .$$

Proof of Corollary 6.2. From the properties (6.4) and (B1),

$$\begin{aligned}\mathscr{R}_{23}\tilde{\mathscr{R}}_{13}\tilde{\mathscr{R}}_{12}&=\mathscr{R}_{23}(1\otimes\Delta)\tilde{\mathscr{R}}\\ &=(1\otimes\sigma\circ\Delta)\tilde{\mathscr{R}}\cdot\mathscr{R}_{23}\\ &=\tilde{\mathscr{R}}_{12}\tilde{\mathscr{R}}_{13}\mathscr{R}_{23} .\end{aligned}$$

Q.E.D.

Proof of Proposition 6.1. We can derive (6.2), (6.3), (6.4) and (6.6) from the property of $\mathscr{R}$. In fact, (6.2), (6.4) and (6.6) are immediate from (B1)–(B3). To show (6.3), we only need the following:

$$\Delta^{(r)}(\varphi S^{-1}(X))=(\varphi S^{-1}\otimes 1)\Delta(X),\quad \text{for any } X\in U^{\geqq} .$$

This is easily obtained by direct calculations. Hence

$$\begin{aligned}(\Delta^{(r)}\otimes 1)\tilde{\mathscr{R}}&=(\Delta^{(r)}\otimes 1)(\varphi S^{-1}\otimes 1)\mathscr{R}\\ &=(\varphi S^{-1}\otimes 1\otimes 1)(\Delta\otimes 1)\mathscr{R}=(\varphi S^{-1}\otimes 1\otimes 1)\mathscr{R}_{13}\mathscr{R}_{23}=\tilde{\mathscr{R}}_{13}\mathscr{R}_{23} .\end{aligned}$$

In order to show (6.5), we shall prepare some lemmas.

Lemma 6.3. *Let C_β be as in Appendix B. Set $\tilde{C}_\beta=(\varphi S^{-1}\otimes 1)C_\beta$. For any $i\in I$ we have,*

$$[f_i\otimes 1, \tilde{C}_{\beta+\alpha_i}]=\tilde{C}_\beta(t_i\otimes f_i) . \tag{6.7}$$

Proof. We show the following lemma.

Lemma 6.4. *For any $i\in I$, $\beta\in Q_+$ and $u\in U^+_{\beta+\alpha_i}$, we have*

$$[f_i, \varphi S^{-1}(u)]=\frac{\varphi S^{-1}(v)t_i}{q_i^{-1}-q_i} ,$$

where $v\in U^+_\beta$ is uniquely determined by $\Delta(u)=u\otimes 1+vt_i\otimes e_i+\cdots$, where $\cdots$ implies terms whose right component is an element of $\bigoplus_{\beta\neq 0,\alpha_i} U^+_\beta$.

Proof. For $\beta=\sum_j m_j\alpha_j$, assuming that u is a monomial $e_{j_1}e_{j_2}\cdots e_{j_l}$, where $l=|\beta|+1$, we can easily show by the induction on m_i. Q.E.D.

We return to the proof of Lemma 6.3. We write $C_\beta=\sum_r x_r^\beta\otimes y_r^{-\beta}$. We shall show the equality of (6.7) by applying $1\otimes(u,\cdot)$ to both sides of (6.7), where $u\in U_\beta^+$ and $(\,,\,)$ is the Killing form,

$$\begin{aligned}
&1\otimes(u,\cdot)[f_i\otimes 1,\tilde{C}_{\beta+\alpha_i}]\\
&=\left(\sum_r f_i\cdot\varphi S^{-1}((u,y_r^{-\beta-\alpha_i})x_r^{\beta+\alpha_i})-\varphi S^{-1}((u,y_r^{-\beta-\alpha_i})x_r^{\beta+\alpha_i})\cdot f_i\right)\otimes 1\\
&=[f_i,\varphi S^{-1}(u)]\,.
\end{aligned}$$

On the other hand, by Lemma 6.4 and the properties of the Killing form,

$$\begin{aligned}
\{1\otimes(u,\cdot)\}\tilde{C}_\beta(t_i\otimes f_i)&=\sum_r\varphi S^{-1}(x_r^\beta)t_i\otimes(u,y_r^{-\beta}f_i)\\
&=\sum_r\varphi S^{-1}(x_r^\beta)t_i\otimes(\Delta(u),y_r^{-\beta}\otimes f_i)\\
&=\sum_r\varphi S^{-1}(x_r^\beta)t_i\otimes(vt_i,y_r^{-\beta})(e_i,f_i)\\
&=\sum_r\varphi S^{-1}((vt_i,y_r^{-\beta})x_r^\beta)t_i/(q_i^{-1}-q_i)\\
&=\varphi S^{-1}(v)t_i/(q_i^{-1}-q_i)\,.
\end{aligned}$$

Q.E.D.

Let us show (6.5). Multiplying $q^{(\beta+\alpha_i,\beta)}(k_{-\beta-\alpha_i}\otimes k_\beta)$ to both sides of (6.7), we obtain

$$\begin{aligned}
&q^{(\beta+\alpha_i,\beta+\alpha_i)}(f_i\otimes t_i^{-1})(k_{-\beta-\alpha_i}\otimes k_{\beta+\alpha_i})\tilde{C}_{\beta+\alpha_i}\\
&=q^{(\beta+\alpha_i,\beta+\alpha_i)}(k_{-\beta-\alpha_i}\otimes k_{\beta+\alpha_i})\tilde{C}_{\beta+\alpha_i}(f_i\otimes t_i^{-1})+q^{(\beta,\beta)}(k_{-\beta}\otimes k_\beta)\tilde{C}_\beta(1\otimes f_i)\,. \qquad (6.8)
\end{aligned}$$

From (6.8), (B6) and the presentation (B4) we obtain (6.5) Q.E.D.

6.2. Projector Γ. We set $\mathscr{C}=\sum_{\beta\in Q_+}q^{(\beta,\beta)}(k_\beta^{-1}\otimes k_\beta)C_\beta\in\hat{U}\hat{\otimes}\hat{U}$ and set $\tilde{\mathscr{C}}=(\varphi S^{-1}\otimes 1)\mathscr{C}$. From the result of [T](Sect. 4), we know that

$$\begin{aligned}
\mathscr{C}^{-1}&=\sum_{\beta\in Q_+}q^{(\beta,\beta)}(1\otimes k_\beta)(S\otimes 1)(C_\beta)\,,\\
\tilde{\mathscr{C}}^{-1}&=(\varphi S^{-1}\otimes 1)\mathscr{C}^{-1}=\sum_{\beta\in Q_+}q^{(\beta,\beta)}(1\otimes k_\beta)(\varphi\otimes 1)(C_\beta)\,.
\end{aligned}$$

We write $\tilde{\mathscr{C}}^{-1}=\sum_k a_k\otimes b_k$, where $a_k\in B^{\geqq}$ and $b_k\in U^{\geqq}$ and set

$$\Gamma=\sum_k S^{-1}(b_k)a_k\in\hat{B}\,.$$

This is well-defined as an endomorphism of objects in $\mathcal{O}(B)$.

Proposition 6.5. *For any $\lambda\in\mathfrak{t}^*$, we have*

$$\Gamma^2=\Gamma,\quad \Gamma\cdot H(\lambda)=\mathbf{F}u_\lambda\,, \qquad (6.9)$$

and in particular, $\Gamma u_\lambda=u_\lambda$.

Proof. From (6.8) we obtain $(f_i\otimes t_i^{-1})\tilde{\mathscr{C}}=\tilde{\mathscr{C}}\Delta^{(r)}(f_i)$ for any i, and then $\tilde{\mathscr{C}}^{-1}(f_i\otimes t_i^{-1})=\Delta^{(r)}(f_i)\tilde{\mathscr{C}}^{-1}$. Thus

$$\sum a_kf_i\otimes b_kt_i^{-1}=\sum f_ia_k\otimes t_i^{-1}b_k+a_k\otimes f_ib_k\,. \qquad (6.10)$$

Applying $m \circ \sigma(1 \otimes S^{-1})$ to both sides of (6.10), where σ: $a \otimes b \mapsto b \otimes a$ and m is a multiplication, we have

$$\sum t_i S^{-1}(b_k) a_k f_i = \sum S^{-1}(b_k) t_i f_i a_k - S^{-1}(b_k) t_i f_i a_k = 0 .$$

Thus $\Gamma \cdot f_i = 0$ for any $i \in I$. From this and Proposition 2.3, we get (6.9). Q.E.D.

Example. For $\mathfrak{g} = \mathfrak{sl}_2$, we have

$$\Gamma = \sum_{n \geqq 0} q^{\frac{1}{2}n(n-1)}(-1)^n f^{(n)} e''^n . \tag{6.11}$$

Note that an element similar to (6.11) is introduced in [K1].

Appendix A

We list several formulae for the operations in Sect. 1, which are analogs of the formula for a Hopf algebra:

$$(1 \otimes m)(1 \otimes \varphi \otimes 1)(1 \otimes \Delta^{(b)})\Delta^{(r)}(X) = X \otimes 1 \quad (X \in B) , \tag{A1}$$

$$(m \otimes 1)(1 \otimes \varphi \otimes 1)(1 \otimes \Delta^{(b)})\Delta^{(r)}(X) = 1 \otimes X \quad (X \in B) , \tag{A2}$$

$$(1 \otimes m)(1 \otimes \sigma)(1 \otimes \varphi^{-1} \otimes 1)(\Delta^{(b)} \otimes 1)\Delta^{(l)}(X) = X \otimes 1 \quad (X \in \bar{B}) , \tag{A3}$$

$$(m \otimes 1)(\sigma \otimes 1)(1 \otimes \varphi^{-1} \otimes 1)(\Delta^{(b)} \otimes 1)\Delta^{(l)}(X) = 1 \otimes X \quad (X \in \bar{B}) , \tag{A4}$$

$$(1 \otimes m)(1 \otimes \varphi \otimes 1)(\Delta^{(l)} \otimes 1)\Delta^{(b)}(X) = X \otimes 1 \quad (X \in U) , \tag{A5}$$

$$(1 \otimes m)(\varphi \otimes 1 \otimes 1)(1 \otimes \Delta^{(r)})\Delta^{(b)}(X) = 1 \otimes X \quad (X \in U) , \tag{A6}$$

$$m(\varphi \otimes 1)\Delta^{(b)}(X) = \varepsilon(X) \quad (X \in U) , \tag{A7}$$

$$(1 \otimes \varepsilon)\Delta^{(r)}(X) = X \otimes 1 \quad (X \in B) , \tag{A8}$$

$$(\varepsilon \otimes 1)\Delta^{(l)}(X) = 1 \otimes X \quad (X \otimes \bar{B}) , \tag{A9}$$

$$\Delta^{(l)}\varphi^{-1}(X) = (S^{-1} \otimes \varphi^{-1})\sigma\Delta^{(r)}(X) \quad (X \in B) , \tag{A10}$$

$$\Delta^{(r)}\varphi(X) = (\varphi \otimes S)\sigma\Delta^{(l)}(X) \quad (X \in \bar{B}) , \tag{A11}$$

$$\Delta^{(b)}(X) = (1 \otimes \varphi S^{-1})\Delta(X) \quad (X \in U^+) , \tag{A12}$$

$$\Delta^{(b)}(X) = (\varphi^{-1}S \otimes 1)\Delta(X) \quad (X \in U^-) , \tag{A13}$$

where σ: $a \otimes b \to b \otimes a$ and m is a multiplication m: $a \otimes b \to ab$.

These are obtained by direct calculations. We shall show, for example, (A1). First we show for generators; this is trivial. Next, we assume that x and $y \in B$ satisfy (A1) and write $(1 \otimes \Delta^{(b)})\Delta^{(r)}(u) = \sum u_{(1)} \otimes u_{(2)} \otimes u_{(3)}$. Then we have

$$\begin{aligned}(1 \otimes m)(1 \otimes \varphi \otimes 1)(1 \otimes \Delta^{(b)})\Delta^{(r)}(xy) &= (1 \otimes m)\sum x_{(1)}y_{(1)} \otimes \varphi(y_{(2)})\varphi(x_{(2)}) \otimes x_{(3)}y_{(3)} \\ &= \sum x_{(1)}y_{(1)} \otimes \varphi(y_{(2)})\varphi(x_{(2)})x_{(3)}y_{(3)} \\ &= \sum xy_{(1)} \otimes \varphi(y_{(2)})y_{(3)} = xy \otimes 1 .\end{aligned}$$

Thus we get (A1).

Appendix B

In this appendix, we recall the theory of the universal R-matrix of U (see [D1, T]).

Recall that for the Hopf algebra $(U, \Delta, S, \varepsilon)$ the universal R-matrix $\mathscr{R}$ is an element which enjoys the following properties ([D1, T]):

$$\mathscr{R}\Delta(x)=\Delta'(x)\mathscr{R} \text{ for any } x\in U\,, \tag{B1}$$

$$(\Delta\otimes 1)\mathscr{R}=\mathscr{R}_{13}\mathscr{R}_{23},\quad (1\otimes\Delta)\mathscr{R}=\mathscr{R}_{13}\mathscr{R}_{12}\,, \tag{B2}$$

$$(\varepsilon\otimes \mathrm{id})\mathscr{R}=1\otimes 1=(\mathrm{id}\otimes\varepsilon)\mathscr{R},\quad (S\otimes\mathrm{id})\mathscr{R}=\mathscr{R}^{-1}=(\mathrm{id}\otimes S)\mathscr{R}\,. \tag{B3}$$

We need some preparation to write down the explicit form of $\mathscr{R}$. Let $\hat{U}\hat{\otimes}\hat{U}$ be a weight completion of $U\otimes U$ as in Sect. 1. Let $H\in\mathfrak{t}\otimes\mathfrak{t}$ be a canonical element with respect to the invariant bilinear form on $\mathfrak{t}$. We extend the algebra $\hat{U}\hat{\otimes}\hat{U}$ by adding formal elements $q^{\pm H}$ with the following properties:

$$q^{H}\cdot q^{-H}=q^{-H}\cdot q^{H}=1\otimes 1,\quad q^{\pm H}(q^{h}\otimes q^{h'})=(q^{h}\otimes q^{h'})q^{\pm H}\,, \tag{B4}$$

$$q^{\pm H}(e_i\otimes 1)=(e_i\otimes t_i^{\pm})q^{\pm H},\quad q^{\pm H}(1\otimes e_i)=(t_i^{\pm}\otimes e_i)q^{\pm H}\,, \tag{B5}$$

$$q^{\pm H}(f_i\otimes 1)=(f_i\otimes t_i^{\mp})q^{\pm H},\quad q^{\pm H}(1\otimes f_i)=(t_i^{\mp}\otimes f_i)q^{\pm H}\,, \tag{B6}$$

$$(\Delta\otimes 1)q^{\pm H}=q^{\pm H_{13}}q^{\pm H_{23}},\quad (1\otimes\Delta)q^{\pm H}=q^{\pm H_{13}}q^{\pm H_{12}}\,, \tag{B7}$$

where $q^{\pm H_{ij}}$'s are elements corresponding to $q^{\pm H}$ on the i^{th} and the j^{th} components in tensor products and they commute with each other. Thus, for example, we identify $q^{H_{12}}$ with $q^{H}\otimes 1$. We denote this algebra by $(\hat{U}\hat{\otimes}\hat{U})\hat{}$. From the property (B7), we can also extend $\Delta\otimes 1$ and $1\otimes\Delta$ to the algebra homomorphism $(\hat{U}\hat{\otimes}\hat{U})\hat{}\to(\hat{U}\hat{\otimes}\hat{U}\hat{\otimes}\hat{U})\hat{}$. More generally, we can extend $\hat{U}^{\hat{\otimes} n}$ to $(\hat{U}^{\hat{\otimes} n})\hat{}$ by adding $q^{\pm H_{ij}}$ $(1\leqq i<j\leqq n)$.

By using the Killing form (see Sect. 3) we can carry out Drinfeld's quantum double construction formally and get an explicit presentation of $\mathscr{R}$,

$$\mathscr{R}=q^{-H}\sum_{\beta\in Q_+} q^{(\beta,\beta)}(k_\beta^{-1}\otimes k_\beta)C_\beta\in(\hat{U}\hat{\otimes}\hat{U})\hat{}\,, \tag{B8}$$

where k_β is an element of T given by $k_\beta=\prod_j t_j^{m_j}$ for $\beta=\sum_j m_j\alpha_j$ and C_β is a canonical element of $U_\beta^{+}\otimes U_{-\beta}^{-}$ with respect to the Killing form.

Here, for U-modules V and W, $q^{\pm H}$ can be regarded as an element of $\mathrm{End}(V\otimes W)$ given by $q^{\pm H}(u\otimes v)=q^{\pm(\xi,\eta)}(u\otimes v)$, $(u\in V_\xi$ and $v\in W_\eta)$. (See [Kac] Sect. 2.) In such consideration, $\mathscr{R}$ makes sense as an endomorphism of tensor products of U-modules. For vectors u and v as above we get,

$$\begin{aligned}q^{-H+(\beta,\beta)}(k_\beta^{-1}\otimes k_\beta)C_\beta(u\otimes v)&=q^{-H-(\beta,\beta)}C_\beta(k_\beta^{-1}\otimes k_\beta)(u\otimes v)\\&=q^{-H-(\beta,\beta)+(\beta,\eta-\xi)}C_\beta(u\otimes v)\\&=q^{-(\xi+\beta,\eta-\beta)-(\beta,\beta)+(\beta,\eta-\xi)}C_\beta(u\otimes v)\\&=q^{-(\xi,\eta)}C_\beta(u\otimes v)\,.\end{aligned}$$

Therefore, we obtain

$$\mathscr{R}(u\otimes v)=q^{-(\xi,\eta)}\sum_\beta C_\beta(u\otimes v)\,. \tag{B9}$$

When $\mathfrak{g}$ is an affine Lie algebra, we set

$$\mathscr{R}'(z)=q^{-H+c\otimes d+d\otimes c}\sum_{\beta\in Q_+}q^{(\beta,\beta)}(z^{\langle d,\beta\rangle}k_\beta^{-1}\otimes k_\beta)C_\beta\ , \tag{B10}$$

where c is a canonical central element of $\mathfrak{g}$ and d is a scaling element of $\mathfrak{g}$. This is used to describe the image of the universal R-matrix onto a tensor product of affinization for finite dimensional U'-modules (see [FR, IIJMNT]).

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