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Renormalization of Random Jacobi Operators

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Abstract: We construct a Cantor set $\mathcal{J}$ of limit-periodic Jacobi operators having the spectrum on the Julia set J of the quadratic map $z \mapsto z^2 + E$ for large negative real numbers E. The density of states of each of these operators is equal to the unique equilibrium measure μ on J. The Jacobi operators in $\mathcal{J}$ are defined over the von Neumann-Kakutani system, a group translation on the compact topological group of dyadic integers. The Cantor set $\mathcal{J}$ is an attractor of the iterated function system built up by the two renormalisation maps $\Phi_\pm : L = \psi(D_\pm^2 + E) \mapsto D_\pm$. To prove the contraction property, we use an explicit interpolation of the Bäcklund transformations by Toda flows. We show that the attractor $\mathcal{J}$ is identical to the hull of the fixed point L_+ of Φ_+.

1. Introduction

Random Jacobi operators are discrete one-dimensional Laplacians and are discrete approximations of one-dimensional random Schrödinger operators. The literature about such operators is huge and a part is by now covered by text books like [CFKS, CL, C, PF].

Dynamical systems obtained by *iteration of rational maps* have a rich structure. Among these systems, the quadratic map $z \mapsto z^2 + E$ is studied best. For reviews in the large literature we refer to [Bla, CG, Ere, M].

Toda differential equations are *integrable Hamiltonian systems* and are discretisations of the Korteweg de Vries systems. According to the chosen boundary condition, the investigation of the Toda systems touches different areas in mathematics. We refer to [FT, Tod, Per, K1].

The subject of this article is located in the intersection of the above three fields. We study random Jacobi operators having the symmetry of being invariant under a

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scale transformation given by a doubling of the lattice spacing, a squaring of the operator and an adaptation of the energy:

$$\mathscr{T} : D \mapsto \psi(D^2 + E) ,$$

where ψ is the restriction of the Laplacian to the coarser lattice. More precisely, assuming that D is an off-diagonal disorder Jacobi matrix $(Du)_n = d_n u_{n+1} + d_{n-1} u_{n-1}$, the matrix $L = D^2 + E$ is given by $Lu_n = a_n u_{n+2} + a_{n-2} u_{n-2} + b_n u_n$, with $a_n = d_n d_{n+1}$ and $b_n = E + d_n^2 + d_{n-1}^2$. This operator is the direct sum of two Jacobi matrices. One of them is $L_{nm} = [\psi(D^2 + E)]_{nm} = [D^2 + E]_{2n,2m}$. Because L has also diagonal entries, the map $\mathscr{T}$ can not be iterated. However, we have shown in [K2] that the inverse of $\mathscr{T}$ can always be computed in the class of random Jacobi operators: given L, there exist two new Jacobi operators $D_\pm$ defined over a new renormalized dynamical system such that $\mathscr{T}(D_\pm) = L$. The entries of $D_\pm$ are constructed from the Titchmarsh-Weyl functions of L. The aim of this work is to study the iteration of the maps

$$\Phi_\pm : L = \psi(D_\pm^2 + E) \mapsto D_\pm .$$

These maps on operators correspond on the spectral level to the two inverses $\phi_\pm$ of the quadratic map $z \mapsto z^2 + E$. Here is the link between random Jacobi operators and the iteration of the quadratic map.

Before we outline the content of our results, we mention the earlier works of Baker, Barnsley, Bellissard, Bessis, Geronimo, Harrington, Mehta and Moussa, who constructed semi-infinite Jacobi operators $\tilde{L} \in \mathscr{B}(l^2(\mathbb{N}))$ with spectra on Julia sets J (see [Bak, BBM, BGM, BMM, BGH2, BGH3]). Such operators have also the equilibrium measure on the Julia set as the density of states and satisfy the fixed point equation $\psi(\tilde{L}^2 + E) = \tilde{L}$. The side diagonal $d_n = [\tilde{L}]_{n,n+1}$ of $\tilde{L}$ begins with

$$d_0 = [\tilde{L}]_{01} = 0, d_1 = [\tilde{L}]_{12} = \sqrt{E}, d_2 = [\tilde{L}]_{23} = 1, \dots ,$$

where the entries d_k are obtained recursively using $d_{2n+1}^2 = -d_{2n}^2 - E$, $d_n^2 = d_{2n}^2 d_{2n-1}^2$. Our set up is different in that we construct Jacobi matrices by renormalisation maps $\Phi_\pm$ which can only be defined in $\mathscr{B}(l^2(\mathbb{Z}))$ and not in $\mathscr{B}(l^2(\mathbb{N}))$, whereas the fixed point equation of the renormalisation equation has a solution also in $\mathscr{B}(l^2(\mathbb{N}))$. For the construction of the attractor, we have to work in an algebra of random operators rather then in $\mathscr{B}(l^2(\mathbb{Z}))$.

The set of random Jacobi operators forms a fiber bundle over the topological group $\mathscr{U}$ of dynamical systems: over each dynamical system is defined the Banach space of random Jacobi operators which is a subspace of the crossed product of $L^\infty(X)$ with the dynamical system. The factorization result in [K2] can be restated in saying that the $2:1$ integral extension map Φ on $\mathscr{U}$ can be lifted to two renormalization maps $\Phi_\pm$ defined on an open set of the bundle. A pair (T, L), where L is a Jacobi operator over the dynamical system (X, T, m) is mapped into a pair $(S, D_\pm)$, where $\psi(D_\pm^2 + E) = L$ and S is an integral extension of T satisfying $S^2 = T$. We will show that for large enough real $-E$, both renormalization maps $\Phi_\pm$ are contractions on an open set of the bundle forming a so called *hyperbolic iterated function system* having an attractor $\mathscr{J}$ which is a Cantor set in the fiber $\mathscr{L}$ over the von Neumann Kakutani system.

The spectrum of $L = \psi(D^2 + E)$ and the spectrum of D are related by $\sigma(D)^2 + E = \sigma(L)$ and the spectrum of each operator $L \in \mathscr{J}$ is the *Julia set*

J of the quadratic map $z \mapsto z^2 + E$. Moreover, we will show that the *density of states* of L is the unique *equilibrium measure* on J. The *Lyapunov exponent* turns out to be the potential theoretical *Green function* of the Julia set and the *determinant* $z \mapsto \det(L - z)$ of an operator $L \in \mathcal{J}$ is the *Böttcher function* which conjugates the map $z \mapsto z^2 + E$ to $z \mapsto z^2$ in a neighborhood of ∞.

The factorization $L = \psi(D^2 + E)$ is the key for isospectral *Bäcklund transformations,* translations by one unit on the finer lattice. A main tool to prove our results is the following interpolation of Bäcklund transformations $BT_\pm$ by a Toda flow with a time-dependent Hamiltonian $H_E(L) = \pm\mathrm{tr}(h_E(L)) = \mp\frac{1}{2}w(E)$, where $w(E)$ is the *Floquet exponent* of L. The interpolating Toda flow is

$$\frac{d}{dE}BT_\pm(L) = \mp\frac{1}{2}\left[\left(\frac{1}{L-E}\right)^+ - \left(\frac{1}{L-E}\right)^-, L\right] .$$

It follows from this Toda interpolation that $BT_\pm(L)$ is unitarily equivalent to L. It is here, where the theory of integrable systems enters. We mention, that for the KdV equation, there exists a similar interpolation of the Bäcklund transformation given by a time-dependent KdV flow

$$\frac{d}{dE}BT(L) = -2DG_{xx}(E, L) ,$$

where $G_{xx}(E, L) = (L - E)^{-1}_{xx}$ is the Green function of the Schrödinger operator $L = -D^2 + q$. (See [McK] p. 31.)

The hull $\mathcal{O}$ of the fixed point L_+ of Φ_+ is the set of all translates of $L_+(T_x)$, where T_x is the translation belonging to any element x in the group X of dyadic integers. Every $L \in \mathcal{J}$ belongs to some $\omega \in \Omega = \{-1, 1\}^{\mathbb{N}}$ by

$$L = \Phi(\omega) = \lim_{n\to\infty} \Phi_{\omega_1}\Phi_{\omega_2}\dots\Phi_{\omega_n}K .$$

A change of alphabet $1 \mapsto 0, -1, \mapsto 1$ identifies an element $\omega \in \Omega = \{-1, 1\}^{\mathbb{N}}$ with an element $x(\omega) \in X = \{0, 1\}^{\mathbb{N}}$. We will prove that $\Phi(\omega) = L_+(T_{x(\omega)})$, which implies $\mathcal{O} = \mathcal{J}$ and means that each element of the attractor of the iterated function system is obtained by an explicitly known translation of the fixed point L_+.

2. The von Neumann Kakutani System

The set $\mathcal{U}$ of *automorphisms* of a Lebesgue space (X, m) is a complete topological group when equipped with the uniform topology given by the metric $d(T, S) = m\{x \in X \mid T(x) \neq S(x)\}$. For $T \in \mathcal{U}$, we call (X, T, m) a *dynamical system.*

Given a function $f \in L^1(X, \mathbb{N} \setminus \{0\})$, a new dynamical system (X^f, T^f, m^f) called the *integral extension* is defined as follows (see [CFS]). Define $X^f := \{(x, i) \mid x \in X \text{ and } 1 \leq i \leq f(x)\}$ and a probability measure m^f on X^f by $m^f((Y, i)) = m(Y)/\int f\, dm$. This measure is preserved by the transformation

$$T^f(x, i) := \begin{cases} (x, i+1) , & \text{if } i + 1 < f(x) , \\ (T(x), 1) , & \text{if } i + 1 = f(x) . \end{cases}$$

This construction gives also a map $\Phi^f : \mathcal{U} \to \mathcal{U}$ because the integral transformation is again an automorphism of (X, m) after an identification of the Lebesgue spaces (X, m) and (X^f, m^f).

Proposition 2.1. *For $\int f\, dm \neq 1$, the map Φ^f is a contraction on $\mathscr{U}$ and has a unique fixed point T^f.*

Proof. The contraction property follows because for all $T_1, T_2 \in \mathscr{U}$

$$m(\{T_1(x) \neq T_2(x)\}) = \left(\int f\, dm\right)^{-1} \cdot m(\{T_1(x) \neq T_2(x)\}) \,.$$

Apply Banach's fixed point theorem in the complete metric space $\mathscr{U}$. □

We consider now the special case of an integral extension with $f \equiv 2$, where the identification of $X = [0, 1]$ with X^f is given by

$$X = [0,1] \to (X, \{1\}) \cup (X, \{2\}) = X^f = [0, 1/2] \cup [1/2, 1] \,.$$

In order to fix the ideas we can write a dynamical system as a measurable map $T : [0,1] \to [0,1]$ leaving invariant the Lebesgue measure on $[0, 1]$ and define Φ^f by

$$\Phi^f(T)(x) = \begin{cases} x + 1/2 \,, & \text{if } x \in X_1 = [0, 1/2) \,, \\ T(2x-1)/2 \,, & \text{if } x \in X_2 = [1/2, 1] \,. \end{cases}$$

The unique fixed point T of $\Phi = \Phi^f$, with $f(x) = 2$ is called the *von Neumann-Kakutani system*. It is by construction a piecewise translation of intervals

$$T(x) = x + 1 - C_{n+1} \,, \quad \text{for } C_n \leq x < C_{n+1} \,,$$

where $C_0 = 0$ and $C_n = \sum_{i=1}^{n} 2^{-i}$, $n > 0$. The system (X, T, m) is ergodic and has a discrete spectrum

$$\hat{G} = \{e^{2\pi i k 2^{-n}} \mid k \in \mathbb{Z}, n > 0\} \subset \mathbb{T}$$

(see [P, F]). T is conjugated to a group translation on the compact abelian group G of dyadic integers, the dual group of $\hat{G} \subset \mathbb{T}^1$. The group G is the space of sequences $\omega = \{\omega_1, \omega_2, \ldots\}$ in $\{0,1\}^{\mathbb{N}}$ with group operation

$$(\omega + \eta)_n = \omega_n + \eta_n + \rho_{n-1} \pmod 2 \,,$$

where $\rho_0 = 0$ and $\rho_n \in \{0, 1\}$ is equal to 1 if and only if $\omega_n + \eta_n + \rho_{n-1} \geq 2$. The group translation $T_1 : \omega \mapsto \omega + (1, 0, 0, \ldots)$ is conjugated by $\omega \mapsto \sum_{n=1}^{\infty} \omega_n 2^{-n} \in [0, 1]$ to the map T on the interval I.

3. Random Jacobi Operators

The *crossed product* $\mathscr{K}$ of $L^\infty(X, \mathbb{C})$ with the dynamical system (X, T, m) is a C^* algebra consisting of operators $K = \sum_{n \in \mathbb{Z}} K_n \tau^n$ with convolution multiplication

$$KM = \sum_n (KM)_n \tau^n = \sum_{k+m=n} (K_k M_m(T^k))\tau^n$$

and norm $|||K||| = |\, ||K(x)||\, |_\infty$, where $K(x)$ is the infinite matrix

$$[K(x)]_{mn} = K_{n-m}(T^m x) \,.$$

The adjoint of an operator is defined by requiring $\tau^* = \tau^{-1}$. A trace on $\mathscr{X}$ is given by $\mathrm{tr}(K) = \int_X K_0 \, dm$. A *random Jacobi operator* L is an element in $\mathscr{X}$ of the form

$$L = a\tau + a(T^{-1})\tau^* + b$$

with $a, b \in L^\infty(X, \mathbb{C})$. We denote by $\mathscr{L} \subset \mathscr{X}$ the complex Banach space of random Jacobi operators. We call $M(L) := \exp\left(\int \log |a| \, dm\right)$ the *mass* of a Jacobi operator. If $\log |a| \geq \delta > 0$ for some $\delta > 0$, we say, the operator has a *positive definite mass.* Notice that random Jacobi operators are only normal, if a, b are real. Denote by $\Phi(\mathscr{X})$ the C^*-algebra corresponding to the renormalized system $(X, \Phi(T), m)$. As long as we consider only one renormalization step, we denote the renormalized dynamical system with (Y, S, n) and the von Neumann algebra with $\mathscr{Y}$ and elements in $\mathscr{Y}$ by $B = \sum_n B_n \sigma^n$, where σ is the symbol in $\mathscr{Y}$ corresponding to τ in $\mathscr{X}$. Call ψ the map $\mathscr{Y} = \Phi(\mathscr{X}) \to \mathscr{X}$

$$\psi : K = \sum_n K_n \sigma^n \mapsto \sum_n \tilde{K}_n \tau^n ,$$

where $\tilde{K}_n(x) = K_{2n}(x)$ for $x \in X_1 = X$. The mapping ψ gives for almost all $x \in X_1$

$$[\psi(K)(x)]_{nm} = [K(x)]_{2n,2m} \, .$$

Let $L \in \mathscr{L}$ be a random Jacobi operator having positive mass. For E outside a ball containing the spectrum of L, the *Titchmarsh-Weyl functions* are given by

$$m_\pm(x) = a(x)\frac{u_\pm(Tx)}{u_\pm(x)} , \qquad n_\pm(x) = a(T^{-1}x)\frac{u_\pm(T^{-1}x)}{u_\pm(x)} ,$$

where $u_\pm(x) \in \mathbb{R}^\mathbb{Z}$ are solutions of $L(x) = Eu(x)$ with $\sum_{\pm n > 0} |(u_\pm)_n(x)|^2 < \infty$. These functions are measurable according to the multiplicative ergodic theorem of Oseledec and are bounded. Using $Lu_\pm = au_\pm(T) + a(T^{-1})u_\pm(T^{-1}) + bu_\pm = Eu_\pm$ and the definition of $m_\pm, n_\pm$, we get

$$\begin{aligned} m_\pm + n_\pm &= E - b \, , \\ m_\pm \cdot n_\pm(T) &= a^2 \, . \end{aligned}$$

New random Jacobi operators

$$D_\pm = \sqrt{c_\pm}\sigma + \sqrt{c_\pm(S^{-1})}\sigma^* \in \mathscr{Y}$$

are obtained with functions $c_\pm \in L^\infty(Y, \mathbb{C})$ defined by requiring that for $x \in X = X_1$,

$$c_\pm(x) = -m_\pm(x), \; c_\pm(S^{-1}x) = -n_\pm(x) \, .$$

The sign of $D_\pm$ is specified if we take the principal branch of the square root for $\sqrt{-m_\pm}$ and the branch $\sqrt{-n_\pm}$ such that $a = \sqrt{m_\pm n_\pm(T)}$. We get then

$$\begin{aligned} c_\pm(x) + c_\pm(S^{-1}x) &= -E + b(x) \, , \\ c_\pm(x) \cdot c_\pm(Sx) &= a^2(x) \, . \end{aligned}$$

As c is defined on Y, these formulas extend the functions $a, b \in L^\infty(X, \mathbb{C})$ to functions in $L^\infty(Y, \mathbb{C})$.

Proposition 3.1. *The random Jacobi operators* $D_\pm = \sqrt{c_\pm}\sigma + \sqrt{c_\pm(S^{-1})}\sigma^* \in \mathscr{Y}$ *are bounded for* $|E| > |||L|||$ *and satisfy*

$$\psi(D_\pm^2 + E) = L\ .$$

The operator $BT_\pm(L) := \psi(D_\pm^2(S) + E)$ *is a Bäcklund transform of* L *and is isospectral to* L*. The operators* $D_\pm$ *are selfadjoint if* L *is selfadjoint and* E *is real below the spectrum of* L*.*

Since the proof that $BT_\pm$ is isospectral given in [K2] is quite rough, we add here a detailed proof. We will see in the next paragraph that L and $BT_\pm(L)$ are even unitarily equivalent. We call

$$\Phi_\pm : \mathscr{L} \to \Phi(\mathscr{L}),\ L \mapsto D_\pm$$

renormalisation maps. They are parameterized by an energy $E \in \mathbb{C}$ and are defined on an open (possibly empty) set $\mathscr{V}$ of $\mathscr{L}$.

Proof. The relation $\psi((D_\pm)^2) = L - E$ follows from the definition of the Titchmarsh-Weyl functions:

$$\begin{aligned}\psi((D_\pm)^2) &= \psi\left(\sqrt{c_\pm \cdot c_\pm(S)}\sigma^2 + (c_\pm + c_\pm(S^{-1})) + \sqrt{c_\pm(S^{-2}) \cdot c_\pm(S^{-1})}\sigma^{-2}\right)\\ &= a\tau + b - E + a(T^{-1})\tau^* = L - E\ .\end{aligned}$$

If E is real and below the spectrum of L, the functions $c^\pm$ are positive and $D_\pm$ are real and selfadjoint.

In order to prove that $BT_\pm(L)$ is isospectral to L we take first the periodic ergodic case, where $N = |X|$ is finite and where we can build for each periodic $N \times N$ Jacobi matrix L of positive mass a periodic $2N \times 2N$ Jacobi matrix D such that $D^2 + E$ is the direct sum of two $N \times N$ matrices L and $BT_\pm(L)$. The spectrum of periodic Jacobi operators is generically simple and the multiplicity of their eigenvalues is ≤ 2.

(i) Assume first that L has N simple nonzero eigenvalues. The Jacobi matrix D has a spectrum $\pm\lambda_1, \dots, \pm\lambda_N$ symmetric with respect to the imaginary axis because if λ is an eigenvalue with the eigenvector $(u_1, u_2, \dots, u_{2N-1}, u_{2N})$ then $-\lambda$ is an eigenvalue with the eigenvector $(u_1, -u_2, \dots, u_{2N-1}, -u_{2N})$. The matrix $D^2 + E$ is the direct sum of the two Jacobi matrices $L, BT_\pm(L)$ and has the eigenvalues $\lambda_i^2 + E$, each with multiplicity exactly 2 coming from the two signs of λ_i. As L has by assumption simple spectrum and each eigenvalue $\lambda_i^2 + E$ of $D^2 + E$ has multiplicity 2, we obtain by the pigeon-hole-principle that both L and $BT_\pm(L)$ have the spectrum $\{\lambda_i^2 + E \mid i = 1, \dots, N\}$.

(ii) In the case, when L is periodic with not necessarily simple spectrum, the claim follows because the renormalisation maps are continuous and the spectrum depends continuously on the matrix and invertible matrices having a simple spectrum are dense in the finite-dimensional vector space of N periodic Jacobi matrices.

(iii) In the general infinite-dimensional case, we can approximate a Jacobi matrix $L(x)$ in the weak operator topology by periodic Jacobi matrices $L^{(N)}(x)$ and the spectra of these approximations converge by the theorem of Avron-Simon for $N \to \infty$ to the spectrum of $L(x)$. The Bäcklund transformed matrices $BT_\pm(L^{(N)}(x))$ converge for $N \to \infty$ in the weak operator topology to $BT_\pm(L(x))$ because the Titchmarsh-Weyl functions depend continuously on the matrices. The spectrum of $BT_\pm(L(x))$ is the same as the spectrum of $L(x)$. $\square$

Random Jacobi operators form a fiber bundle over the space $\mathscr{U}$ of dynamical systems. Over each dynamical system T is defined the fiber $\mathscr{L}$ of Jacobi operators over this system. Given $E \in \mathbb{C}$, there is an open (possibly empty) subset of this fiber bundle, where the renormalisation maps $\Phi_\pm$ make sense. A pair (T, L), where L is a Jacobi operator over the dynamical system (X, T, m) is mapped into a pair $(S, D_\pm)$, where $\psi(D_\pm^2 + E) = L$ and $\Phi(T) = S$ is the $2:1$ integral extension of T.

4. Bäcklund Transformations

For $|E| > |||L|||$, the Bäcklund transformations $BT_\pm$ are given by

$$L = a\tau + a(T^{-1})\tau^* + b \mapsto BT_\pm(L) = a_\pm\tau + a_\pm(T^{-1})\tau^* + b_\pm \ ,$$

where by construction

$$b_\pm = b + n_\pm - n_\pm(T), \ a_\pm^2 = a^2\frac{m_\pm(T)}{m_\pm} \ .$$

We have shown in [K2] Proposition 4.4 that

$$\lim_{E\to-\infty} BT_+(L) = L(T) \ , \ \lim_{E\to-\infty} BT_-(L) = L$$

and that in the periodic case the transformations can be interpolated by time-dependent Toda flows in $\mathscr{L}$ (see Theorem 4.5 in [K2]). Because we want to estimate the Fréchet derivative of the Bäcklund transformations near $-\infty$, we have to refine the analysis of Bäcklund transformations and to determine explicitly the Hamiltonian Toda flow which does the interpolation. The projections

$$K = \sum_{n=-\infty}^{\infty} K_n\tau^n \mapsto K^\pm = \sum_{\pm n>0} K_n\tau^n$$

yield the decomposition $K = K^- + K_0 + K^+$. Define $K \mapsto K^\Delta$ to be the projection from $\mathscr{X}$ to $\mathscr{L}$. For a *Hamiltonian* $H(L) = \mathrm{tr}(h(L))$ with analytic h, the differential equation

$$\dot{L} = [h'(L)^+ - h'(L)^-, L] = [L^+ - L^-, h'(L)^\Delta] \ ,$$

is *a random Toda system* [K1]. In order to get local existence of the flow, the domain of analyticity of h must be sufficiently large. We will consider also complex time t as well as time-dependent Hamiltonians H.

The *Floquet exponent* $w(E)$ of $L = a\tau + a(T^{-1})\tau^* + b$ is defined as

$$w(E) = -\mathrm{tr}(\log(L - E)), \ \ \mathrm{Im}(E) > 0 \ .$$

By the *Thouless formula* it is also defined on $\mathbb{R}$. The Lyapunov exponent $\lambda(E) = -\mathrm{Re}(w(E)) - \int \log|a(x)| \, dm(x)$ makes sense for all $E \in \mathbb{C}$ and the derivative $w'(L)$ is bounded for all E in the resolvent set of L.

Proposition 4.1. *Bäcklund transformations can be interpolated by random Toda flows with the time-dependent Hamiltonians* $H_E^\pm(L) = \pm\frac{1}{2}w(E)$. *This means*

$$\frac{d}{dE}BT_\pm(L) = -\pm\frac{1}{2}\left[\left(\frac{1}{L-E}\right)^+ - \left(\frac{1}{L-E}\right)^-, L\right] . \tag{1}$$

Proof. We prove the proposition first in the finite-dimensional periodic case $|X| < \infty$ and under the condition that E is real below the spectrum of L. We know from [K2] (see Theorem 4.5 in [K2]) that Bäcklund transformations can then be interpolated by Toda flows. In the coordinates $(d, b) = (\log(a), b)$, the Toda flow $\dot{L} = [L^+ - L^-, h'_\pm(L)^\Delta]$ is

$$\frac{d}{dt} d = h'(L)_0(T) - h'(L)_0,$$
$$\frac{d}{dt} b = e^d \cdot h'(L)_1 - e^{d(T^{-1})} \cdot h'(L)_1(T^{-1}),$$

whereas the Bäcklund transformations $BT_\pm$ are

$$d \mapsto d_\pm(E) = d + \frac{1}{2}\log(m_\pm)(T) - \frac{1}{2}\log(m_\pm),$$
$$b \mapsto b_\pm(E) = b + n_\pm - n_\pm(T).$$

Differentiating these equations with respect to E

$$\frac{d}{dE} d = \frac{1}{2}\frac{\frac{d}{dE}m_\pm}{m_\pm}(T) - \frac{1}{2}\frac{\frac{d}{dE}m_\pm}{m_\pm},$$
$$\frac{d}{dE} b = \frac{d}{dE}n_\pm - \frac{d}{dE}n_\pm(T),$$

and requiring $\left(\frac{d}{dE}d, \frac{d}{dE}b\right) = \left(\frac{d}{dt}d, \frac{d}{dt}b\right)$ gives (up to a L-independent constant function which we put to zero)

$$\begin{aligned} h'(L)_0 &= \frac{1}{2}\frac{\frac{d}{dE}m_\pm}{m_\pm} = \frac{1}{2}\frac{d}{dE}\log(m_\pm), \\ h'(L)_1 &= -\frac{\frac{d}{dE}n_\pm(T)}{a}, \end{aligned} \tag{2}$$

and so

$$\begin{aligned} \operatorname{tr}(h'(L)) &= \int_X h'(L(E))_0 \, dm = \frac{d}{dE}\int \frac{1}{2}\log(m_\pm)\, dm = \pm\frac{1}{2}\frac{d}{dE}w(E) \\ &= \mp\frac{1}{2}\operatorname{tr}((L-E)^{-1}). \end{aligned}$$

Therefore

$$h'_E(L) = \mp\frac{1}{2}(L-E)^{-1}, \quad h_E(L) = \mp\frac{1}{2}\log(L-E)$$

which leads to

$$H_E(L) = \operatorname{tr}(h_E(L)) = \pm\frac{1}{2}w(E).$$

We have proven (1) in the finite-dimensional case with E real below the spectrum of L. The formulas (2) are true in general, if they hold in each finite-dimensional case. One can approximate the operators $L(x)$ by periodic matrices $L^{(N)}(x)$ in the weak operator topology. For $N \to \infty$, we obtain $h(L^{(N)})(x) \to h(L)(x)$ in the weak operator topology and $(m_\pm)^{(N)}(x) \to m_\pm(x)$ for almost all $x \in X$. (By analytic continuation, the formula (1) holds also for complex numbers E satisfying $|E| > |||L|||$.) □

We use this interpolation to estimate the Fréchet derivative $\frac{d}{dL}BT_\pm(L)$ of the Bäcklund transformations near ∞.

Corollary 4.2. *For $|E| \to \infty$, we have $\left\|\frac{d}{dL}BT_\pm(L)\right\| \to 1$, uniformly for L in each fixed ball $B(R) \subset \mathscr{L}$.*

Proof. Since there is a uniform bound for the Fréchet derivative of

$$E^2 \cdot \frac{d}{dE}BT_\pm(L) = \mp\frac{1}{2}\sum_{n=1}^{\infty}\left[\left(\frac{L^n}{E^{n-1}}\right)^+ - \left(\frac{L^n}{E^{n-1}}\right)^-, L\right]$$

on $\mathscr{Y}$, we have

$$\frac{d}{dE}\frac{d}{dL}BT_\pm = O(E^{-2})$$

for $|E| \to \infty$. Therefore

$$\left\|\frac{d}{dL}BT_\pm(L) - I^\pm\right\| = O(E^{-1}),\ |E| \to \infty\ ,$$

where $I^- : \mathscr{L} \to \mathscr{L}$ is the identity operator and $I^+(L) = L(T)$. □

5. Iterated Function Systems

We state now a version of Barnsley's result about *hyperbolic iterated function systems* [Bar]. Such a result holds in the general context of complete metric spaces. We formulate and use it in the case when the hyperbolic iterated function system is acting on a Banach space. The proof of the result is given for the convenience of the reader.

Lemma 5.1. *Given a Banach space $(\mathscr{M}, \|\cdot\|)$ and two differentiable maps $\Phi_+, \Phi_- : \mathscr{Y} \subset \mathscr{M} \to \mathscr{M}$ leaving invariant an open connected bounded subset $\mathscr{Y}$ of $\mathscr{M}$. Assume there exists a common inverse $\mathscr{T}$ of both Φ_+, Φ_- on $\Phi_+(\mathscr{Y}) \cup \Phi_-(\mathscr{Y})$. Suppose, there exists $\lambda < 1$ such that for $L \in \mathscr{Y}$*

$$\left\|\frac{d}{dL}\Phi_\pm(L)\right\| \leq \lambda \tag{3}$$

and $\Phi_+(L) \neq \Phi_-(L)$ for all $L \in \mathscr{Y}$. Then there exists a $\Phi_\pm$ invariant Cantor set $\mathscr{J}$ homeomorphic to $\Omega = \{-1, 1\}^{\mathbb{N}}$ which is the image of the injective map

$$\Phi : \omega = (\omega_1, \omega_2, \ldots) \mapsto \lim_{k\to\infty} \Phi_{\omega_1} \circ \Phi_{\omega_2} \circ \cdots \circ \Phi_{\omega_k}(K) = \Phi(\omega)\ ,$$

where $K \in \mathscr{Y}$ is arbitrary. The map $\mathscr{T}$ restricted to $\mathscr{J}$ is topologically conjugated by Φ to the one-sided Bernoulli shift σ on $\{-1, 1\}^{\mathbb{N}}$: $\mathscr{T} \circ \Phi = \Phi \circ \sigma$.

Proof. The contraction property. Connect two given points $K_0, K_1 \in \mathscr{Y}$ by a piecewise differentiable path $t \mapsto K(t) \in \mathscr{Y}$ so that $K_1 - K_0 = \int_0^1 \dot{K}_t \, dt$. The assumption Equation (3) leads to

$$||\Phi_\pm(K_1) - \Phi_\pm(K_0)|| = \left\| \int_0^1 \frac{d}{dt} \Phi_\pm(K(t)) \, dt \right\| \leq \lambda \cdot ||K_1 - K_0|| \, .$$

Existence of the mapping Φ. For $K \in \mathscr{Y}$ and $\omega \in \Omega$, $n \mapsto \Phi_{\omega_1} \circ \Phi_{\omega_2} \circ \cdots \circ \Phi_{\omega_n}(K)$ is a Cauchy sequence in $\mathscr{Y}$ because

$$\operatorname{diam}(\Phi_{\omega_1} \circ \Phi_{\omega_2} \circ \cdots \circ \Phi_{\omega_n}(\mathscr{Y})) \leq \lambda^n \cdot \operatorname{diam}(\mathscr{Y})$$

and the limit $\Phi(\omega)$ of this Cauchy sequence exists, because $\mathscr{M}$ is complete. The limit is independent of $K \in \mathscr{Y}$ and the map is continuous as

$$||\Phi(\omega)K - \Phi(\eta)K|| \leq \operatorname{diam}(\mathscr{Y}) \cdot \lambda^{-\min\{k \in \mathbb{N} \mid \omega_k \neq \eta_k\}} \, .$$

Call $\mathscr{J} = \Phi(\Omega)$.

Injectivity of $\Phi : \Omega \to \mathscr{J}$. Assume $\Phi_\omega = \Phi_\nu$ with $\omega \neq \nu$ and k is the smallest index with $\omega_k \neq \nu_k$. Since Φ_+, Φ_- have a common inverse, we obtain from $\Phi_\omega = \Phi_\nu$ that for all $k \in \mathbb{N}$

$$\Phi_{\sigma^k(\omega)} = \Phi_{\sigma^k(\nu)} \, ,$$

where σ is the shift $\omega = (\omega_1 \omega_2 \ldots) \mapsto (\omega_2, \omega_3, \ldots)$. With $\omega_n \neq \nu_n$ and the assumption $\Phi_+(L) \neq \Phi_-(L)$ for all $L \in \mathscr{M}$, we get

$$\Phi_{\omega_n} \Phi_{\sigma^n(\omega)} \neq \Phi_{\nu_n} \Phi_{\sigma^n(\omega)}$$

in contradiction to the fact that both sides are equal to $\Phi_{\sigma^{n-1}(\omega)} = \Phi_{\sigma^{n-1}(\nu)}$.

Conjugation to a Bernoulli shift. The map $\Phi : \Omega \to \mathscr{J}$ is a continuous bijection. Since Ω is compact, Φ is a homeomorphism. As $\Phi_{\sigma\omega} = \mathscr{T}\Phi_\omega$, the map Φ conjugates the Bernoulli shift $\sigma : \Omega \to \Omega$ to the map $\mathscr{T} : \mathscr{J} \to \mathscr{J}$. □

The maps Φ_+, Φ_- in the Lemma form a *hyperbolic iterated function system.* The invariant Cantor set $\mathscr{J}$ is called the *attractor* of this system.

6. The Quadratic Map

We will need some facts about the dynamical system on the complex plane $\mathbb{C}$ defined by the *quadratic map* $t : z \mapsto z^2 + E$, where $E \in \mathbb{C}$ is a parameter. The inverse of t is a correspondence $\phi_\pm : x \mapsto \pm\sqrt{x - E}$ with the two branches $\phi_\pm$. A map for measures $\mu \mapsto \phi^*(\mu)$ is defined by

$$\phi^*(\mu)(Y) = \frac{1}{2} \cdot \mu(t(Y)) \, .$$

For $E \neq 0$, the map ϕ^* has a unique fixed point μ in the space of probability measure on $\mathbb{C}$. It is an attractor so that $(\phi^*)^n(\nu) \to \mu$ for all probability measures ν on $\mathbb{C}$ [Bro, L]. This measure is called *equilibrium or electrostatic measure* because μ maximizing the metric entropy of t among all invariant probability measures. The support of μ is the *Julia set J* of t, which is defined as the closure of all repelling periodic orbits of t. The measure μ has the property of being *balanced* [BGH1], which means that for each chosen branch $\phi_\pm$, one has

$$(\mu)(\phi_\pm(Y)) = \frac{1}{2}\mu(Y) .$$

For large $|E|$, the two maps $\phi^\pm$ form a hyperbolic iterated function system having the Julia set as the attractor. It follows from Barnsley's Lemma 5.1 that t is then topologically conjugated to a one-sided Bernoulli shift. The Julia set is then called *hyperbolic* and is a completely disconnected Cantor set. (See [Bla, C, EL] for reviews.)

7. Existence of the Attractor

We return now to the *renormalisation maps* $\Phi_\pm$ acting on the Banach space $\mathscr{L}$ of Jacobi operators defined over the von Neumann Kakutani system (X, T, m). An element $L = a\tau + a(T^{-1})\tau^* + b \in \mathscr{L}$ is called *limit-periodic* if for almost all $x \in X$ the sequences $a_n = a(T^n x)$ and $b_n = b(T^n x)$ are limit-periodic in the sense that they can be approximated in $l^\infty(\mathbb{Z})$ by periodic sequences.

Theorem 7.1. *For large enough real $-E$, the maps Φ_+, Φ_- form a hyperbolic iterated function system defined on an open non-empty set $\mathscr{V} \subset \mathscr{L}$. Each $L \in \mathscr{J}$ is limit-periodic and has the spectrum on the Julia set J of the quadratic map $t : z \mapsto z^2 + E$.*

Proof. Fixing a neighborhood of the Julia set. For large $-E$, there exists an open $\phi_\pm$-invariant connected real neighborhood V of J that does not contain E.

Fixing an open set of Jacobi operators. The open connected set

$$\mathscr{V} = \{L \in \mathscr{L} \mid \sigma(L) \in V, \ \text{L has positive definite mass}\ \}$$

is not empty: take any $L \in \mathscr{L}$ with positive definite mass. There exist constants $\alpha > 0, \beta \in \mathbb{R}$, such that $\sigma(\alpha L + \beta) \in V$.

The renormalisation maps have a common inverse. The inverse of $\Phi_\pm$ is given by

$$\mathscr{T}(D) = \psi(D^2 + E) .$$

The two renormalisation maps have no common image. For large enough $|E|$ and $L \in \mathscr{V}$,

$$\Phi_+(L) \neq \Phi_-(L)$$

because $\Phi_+(L) = \Phi_-(L)$ would imply $m_+ = m_-$ and E would be an eigenvalue. This is not possible, since we have assumed E to be outside the open set V which contains the spectrum of L.

Decomposition of the renormalisation maps. In order to estimate the Fréchet derivative of $\Phi_\pm$, we make the decomposition

$$\Phi_\pm = \varphi \circ \eta_\pm \circ \theta ,$$

where $\varphi : L \mapsto \sqrt{L - E}$ is the square-root giving $D_\pm$ and $\theta(L) = L \oplus L \in \mathscr{X}$ is the unique operator which satisfies

$$\psi(\theta(L)) = L, \ \psi(\theta(L(T))) = L, \ \theta(L)_{2n+1} = 0$$

and

$$\eta_\pm(L \oplus K) = L \oplus BT_\pm K .$$

The mapping φ is defined on the manifold $\eta_\pm \circ \theta(\mathscr{L}) \subset \mathscr{K}$.

The derivative of θ. $\theta : \mathscr{L} \mapsto \mathscr{K}$ is linear and $\left\| \frac{d}{dL}\theta \right\| = ||\theta|| \leq 2$.

The derivative of $\eta_\pm$. We know by Corollary 4.2 that for $|E| \to \infty$,

$$\left\| \frac{d}{dL} BT_\pm(L) \right\| \to 1 ,$$

uniformly for $L \in \mathscr{Y}$. We obtain therefore also

$$\lim_{E \to -\infty} \left\| \frac{d}{dL}\eta_\pm(L) \right\| = 1 .$$

The derivative of φ. The derivative of the map $L \mapsto \sqrt{L - E} = D_\pm$ from the manifold $\eta_\pm \circ \theta(\mathscr{L})$ to $\mathscr{K}$ is given by

$$\frac{d}{dL}\varphi(L)U = \frac{1}{2}(L - E)^{-1/2}U = \frac{1}{2}D_\pm^{-1}U .$$

Because $|||(L - E)^{-1/2}||| \to 0$, for $|E| \to \infty$, we get

$$\lim_{-E \to \infty} \left\| \frac{d}{dL}\varphi(L) \right\| = 0 .$$

The derivative of $\Phi_\pm : \mathscr{Y} \to \mathscr{L}$. It follows from the four previous steps that for $|E| \to \infty$

$$\left\| \frac{d}{dL}\Phi_\pm \right\| = \left\| \frac{d}{dL}(\varphi \circ \eta_\pm \circ \theta) \right\| \leq \left\| \frac{d}{dL}(\varphi) \right\| \cdot \left\| \frac{d}{dL}\eta_\pm \right\| \cdot \left\| \frac{d}{dL}\theta \right\| \to 0 .$$

The hyperbolic iterated function system. We have checked the existence of a common inverse, the contraction property and the disjointness of the two maps $\Phi_\pm$. Lemma 5.1 is thus applicable and we have shown that for large enough $-E$, a hyperbolic iterated function system has a unique attractor $\mathscr{J}$ in $\mathscr{L}$.

Limit-periodicity. Start with (T, L), where T is a periodic dynamical system satisfying $T^N(x) = x$. Every Jacobi matrix $L(x)$ is then periodic. Under the iteration of the renormalisation maps, the periodic Jacobi matrices $\Phi_{\omega_1} \circ \ldots \circ \Phi_{\omega_n}(L)$ converge to $\Phi(\omega)$ which is limit-periodic. □

8. The Hull of the Fixed Point of Φ^+

We will assume in this paragraph that $-E$ is so large that the maps Φ_+, Φ_- are defined and form a hyperbolic iterated function system on the bundle of random Jacobi matrices.

Different notation for X and Ω. The topological space $\Omega = \{1, -1\}^{\mathbb{N}}$ labelling the renormalisation sequence and the dyadic group $X = \{0, 1\}^{\mathbb{N}}$ can be identified by the change of alphabet $1 \mapsto 0, -1 \mapsto 1$. We will use the notation $\omega = \omega(x)$ or $x = x(\omega)$ if x and ω correspond to each other. The addition in Ω is the group operation inherited from the group X. We also use the notation $x_0 = (0,0,0,\ldots)$ for the zero in X and $x_1 = (1,0,0,\ldots)$ for the unit in the ring X of dyadic integers.

The fixed points of $\Phi_\pm$. Call $L_\pm \in \mathcal{J}$ the unique fixed points of $\Phi_\pm$. By definition $L_+ = \Phi(\omega(x_0))$ and $L_- = \Phi(-\omega(x_0))$.

The group structure on the attractor $\mathcal{J}$. The homeomorphism $x \mapsto \omega(x)$ brings the group structure of X to Ω and so to $\mathcal{J}$ by $\Phi(\omega)\Phi(\eta) = \Phi(\omega + \eta)$.

The hull of L_+. Call T_x the group translation on X defined by $T_x(y) = x + y$ and denote by T_ω the analogous group translation on Ω. The group X is acting on $\mathcal{L}$ by $L \mapsto L(T_x)$. The *hull* $\mathcal{O} := \{L_+(T_x) \mid x \in X\}$ of L_+ is a compact set in $\mathcal{L}$ which becomes with the operation $L_+(T_x) \circ L_+(T_y) = L_+(T_{x+y})$ a compact topological group.

Theorem 8.1. *The two sets $\mathcal{J}$ and $\mathcal{O}$ coincide and are as groups isomorphic by the isomorphism $\Phi(\omega) = L_+(T_{x(\omega)})$.*

The proof of this theorem needs some preliminary steps. Denote by ρ the involution $(T, L) \mapsto (T^{-1}, L(T^{-1}))$ on the bundle of random Jacobi operators.

Lemma 8.2. a) $\rho \circ \Phi_+ = \Phi_- \circ \rho$, b) $L_+(T^{k-1}) = L_-(T^k)$, $\forall k \in \mathbb{Z}$.

Proof. a) Given (T, L), we write $m_\pm^{(T,L)}$ for the Titchmarsh-Weyl functions of the operator L over the dynamical system (X, T, m). Using the definitions of these functions, we get

$$n_+^{(T,L)}(x) = m_-^{(T^{-1},L(T^{-1}))}(x) ,$$
$$m_+^{(T,L)}(x) = n_-^{(T^{-1},L(T^{-1}))}(x)$$

which is equivalent to

$$d_+^{(T,L)}(S^{-1}x) = d_-^{(T^{-1},L(T^{-1}))}(x) .$$

Because $\Phi_\pm(L) = d_\pm\sigma + d_\pm(S^{-1})\sigma^*$, this can be rewritten as

$$\rho \circ \Phi_+(L) = \Phi_- \circ \rho(L) .$$

b) Using a), we obtain

$$\Phi_-(\rho L_+) = \Phi_- \circ \rho(L_+) = \rho \circ \Phi_+(L_+) = \rho L_+$$

which shows that ρL_+ is the fixed point of Φ_-. Therefore $L_+(T^{-1}) = L_-$. The claim follows by applying T^k on both sides. □

Define the sets $X_{00} := X = [0,1]$ and $X_{ki} := 2^{-k}[i, i+1] \subset X$ for $0 \le i \le 2^k - 1$. Given $L \in \mathcal{J}$, we define inductively $L_{(0)} := L$, $L_{(k+1)} := L_{(k)}^2 + E \in \mathcal{X}$ and $d_{(k)} := (L_{(k+1)})_{2^k} \in L^\infty(X)$.

Lemma 8.3. $L_+(T^i) \in \mathcal{J}$, $\forall i \in \mathbb{Z}$.

Proof. The spectrum of $L_{(k)}$ is the Julia set J because this is true for $L_{(0)}$ and by the spectral theorem inductively for each $L_{(k)}$. Each $L_{(k)}$ is a random Jacobi operator over the dynamical system (X, T^{2^k}, m) which has the 2^k measurable invariant sets X_{ki}. Each map $T^{(2^k)}$ restricted to such a set X_{ki} is ergodic and the operator $L_{(k)}$ restricted to X_{ki} is an ergodic random Jacobi operator over the dynamical system

$(X_{ki}, T^{(2^k)}, m)$. By definition, $\mathcal{T}^k(L(T^i))$ is defined as $L(T^i)_{(k)}$ restricted to X_{k0} and this is the same operator as $L_{(k)}$ restricted to X_{ki}. The spectrum of $\mathcal{T}^n(L(T^i))$ is therefore also the Julia set J. Since $\mathcal{T}^k(L(T^i)) \in \mathcal{Y}$, we conclude that $L(T^i)$ is in the image of some Φ_ω, where ω is a word of length k. Because this is true for all $k \in \mathbb{N}$, we know that $L(T^i)$ is arbitrarily close to the closed set $\mathcal{J}$ and therefore $L \in \mathcal{J}$. □

Define for $L \in \mathcal{J}$ and $k \in \mathbb{N}, k > 0$

$$\omega_k(L) := -\text{sign} \int_{X_{k0}} \log |d_{(k-1)}| \, dm$$

and call $\omega(L)$ the *code* for L. The next lemma justifies this name.

Lemma 8.4. *For all $L \in \mathcal{J}$, one has $\Phi(\omega(L)) = L$.*

Proof. We know by definition that $\mathcal{T}^k(L)$ is $L_{(k)}$ restricted to X_{k0}. For $x \in X_{k0}$,

$$|d_{(k-1)}(x)| = \sqrt{|m^{\pm}_{(k-1)}(x)|} ,$$

where $m^{\pm}_{(k-1)}$ are the Titchmarsh-Weyl functions of $\mathcal{T}^{k-1}(L)$. The upper-script $\pm$ in $m^{\pm}_{k-1}$ is in correspondence with the fact that $\mathcal{T}^{k-1}(L)$ is in the image of $\Phi_{\pm}$. We see that $\mathcal{T}^{k-1}(L)$ is in the image of $\Phi_{\omega_k(L)}$ for all k. Therefore we get $\Phi(\omega(L)) = L$. □

Lemma 8.5. a) $\omega(\rho(L)) = -\omega(L)$, b) $\omega(L_+(T^n)) = -\omega(L_-(T^{-n}))$.

Proof. a) Lemma 8.2 implies

$$\rho \circ \Phi_\omega \circ \rho = \Phi_{-\omega}$$

for all $\omega \in \Omega$. Let L have the code ω such that $L = \Phi(\omega)$. It follows from

$$\rho(L) = \rho\Phi(\omega) = \rho \circ \Phi_\omega(K) = \Phi_{-\omega}\rho(K) = \Phi(-\omega)$$

that $\rho(L)$ has the code $-\omega$.
b) From $L_+ = \rho(L_-)$ we get $L_+(T^n) = \rho(L_-(T^{-n}))$. Use a) to get

$$\omega(L_+(T^n)) = \omega(\rho(L_-(T^{-n})) = -\omega(L_-(T^{-n})) . \qquad \square$$

Proof of the theorem. We know by Lemma 8.3 that $L_+(T^n) \in \mathcal{J}$. Because $\mathcal{J}$ is closed, it follows that $\mathcal{O} \subset \mathcal{J}$. Our aim is to show that $\Phi(n \cdot \omega(x_1)) = L_+(T^n)$ for all $n \in \mathbb{Z}$.

In order to determine the action of T on the subset of $\Omega = \{-1, 1\}^{\mathbb{N}}$ labelling the points of $\mathcal{O} \subset \mathcal{J}$, we define the matrix

$$M_{ki} := \omega_k(L_+(T^i)), \quad k > 0, \quad i \in \mathbb{Z} .$$

We can read of the code $\omega = \{M_{ki}\}_{k \in \mathbb{N}}$ of $L_+(T^i) = \Phi_+(\omega)$ from the columns of the matrix M.

We build up the matrix M beginning at the top first row and determine inductively one row after the other. The first row is given by

$$M_{1i} = \omega_1(L_+(T^i)) = (-1)^i$$

because $T(X_{01}) = X_{11}$ and $T(X_{11}) = X_{01}$ and

$$\operatorname{sign}\left(\int_{X_{10}} \log |d_{(0)}(x)| \, dm(x)\right) = -\operatorname{sign}\left(\int_{X_{11}} \log |d_{(0)}(x)| \, dm(x)\right) .$$

The $(k+1)$-th row can be constructed from the k-th row using

$$\omega_{k+1}(L_+(T^{2i})) = \omega_k(L_+(T^i)) , \tag{4}$$

$$\omega_k(L_+(T^i)) = -\omega_k(L_+(T^{-(i+1)})) . \tag{5}$$

Proof of formula (4). We know from $\Phi_+(L_+) = L_+$ that $d_{(k-1)}$ restricted to X_{ki} is equal to $d_{(k)}$ restricted to $X_{k+1,2i}$. Therefore

$$\begin{aligned} w_{k+1}(L_+(T^{2i})) &= -\operatorname{sign} \int_{X_{k+1,0}} \log |d_{(k)}(T^{2i}x)| \, dm(x) \\ &= -\operatorname{sign} \int_{X_{k+1,2i}} \log |d_{(k)}(x)| \, dm(x) \\ &= -\operatorname{sign} \int_{X_{k,i}} \log |d_{(k-1)}(x)| \, dm(x) \\ &= -\operatorname{sign} \int_{X_{k,0}} \log |d_{(k-1)}(T^i x)| \, dm(x) = w_k(L_+(T^i)) . \end{aligned}$$

Formula (5) follows from $w_k(L_+(T^i)) = -w_k(L_-(T^i)) = -w_k(L_+(T^{-i+1}))$ which is a consequence of Lemma 8.5 b) and 8.2 b). The constructed matrix

$$M = \begin{pmatrix} \cdot & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdot \\ \cdot & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & \cdot \\ \cdot & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

has the property that if b is the i-th column of M then the $b+\omega(x_1)$ gives the $(i+1)$-th column. To prove this, one checks that if a matrix $\widetilde{M}$ is constructed by this rule for the columns, then $M = \widetilde{M}$.

We know now the action of T on $\mathscr{T}$:

$$T^n \Phi(\omega) = \Phi(\omega + n \cdot \omega(x_1)) .$$

Because the orbit $\{L_+(T^n)\}$ of L_+ is dense in $\mathscr{T}$, it follows that $\mathscr{O} = \mathscr{T}$. Moreover $\Phi(\omega) = L_+(T_{x(\omega)})$ holds for all $\omega \in \Omega$ because the relation holds on a dense set of the group X. $\square$

9. The Density of States

There exists a probability measure dk in $\mathbb{C}$ satisfying for any $f \in C(\mathbb{R})$

$$\mathrm{tr}(f(K)) = \int f(E)\, dk(E)\,,$$

which is called the *density of states* of K. The next lemma gives the relation between the density of states of the renormalized Jacobi operator $\Phi_\pm L$ and the density of states of the operator L.

Lemma 9.1. $dk(\Phi_\pm L) = \phi^* \, dk(L)$.

Proof. Assume first that $|X| = N$ is finite so that L is a N-periodic Jacobi matrix. Denote by $\widetilde{dk}(L)$ the Dirac measure $\frac{1}{N}\sum_{i=1}^{N} \delta(\lambda_i)$, where λ_i are the eigenvalues of L acting on the finite-dimensional vector space of N periodic sequences in $l^2(\mathbb{Z})$. The $2N \times 2N$ periodic Jacobi matrices $D_\pm = \Phi_\pm(L)$ have the eigenvalues $\{\pm\sqrt{\lambda_i - E}\}_{i=1}^{N}$. This implies

$$\widetilde{dk}(\Phi_\pm(L)) = \phi^* \, \widetilde{dk}(L)\,.$$

In the general case, let $L^{(N)}(x)$ be a N-periodic approximation of $L(x)$ such that for $-N/2 \le i,j < N/2$,

$$[L^{(N)}(x)]_{i+N,j+N} = [L^{(N)}(x)]_{ij} = [L(x)]_{ij}\,.$$

By the lemma of Avron-Simon (see [Cyc]), one has the weak convergence $\widetilde{dk}(L^{(N)}(x)) \to dk(L)$ for almost all $x \in X$. The claim follows because $\Phi_\pm(L^{(N)}(x)) \to \Phi_\pm(L(x))$ in the weak operator topology. □

Proposition 9.2. *The density of states of $\Phi(\omega)$ is the unique equilibrium measure μ on J.*

Proof. We know that for $(\phi^*)^n \nu \to \mu$ holds for any probability measure ν on $\mathbb{C}$, where μ is the unique equilibrium measure on the Julia set J. Applying this to $\nu = dk$, the density of states of a Jacobi operator L, we get with Lemma 9.1

$$dk(\Phi_{\omega_1} \circ \ldots \circ \Phi_{\omega_n}(L)) = (\phi^*)^n(\nu) \to \mu\,,$$

for all $\omega \in \Omega$. The density of states of $\Phi(\omega)$ is μ. □

Lemma 9.3. *Every operator* $L = d\tau + d(T^{-1})\tau^* \in \mathcal{J}$ *has the mass* $M(L) = \exp\left(\int \log(d)\, dm = 1\right.$.

Proof. Because every element in $\mathcal{J}$ has the same mass (Theorem 8.1), it is enough to show that $L_+ = d_+\tau + d_+(T^{-1})\tau^*$ has mass 1. From the fixed point equation $\Phi_+(L_+) = L_+$ we get $m_+(x)n_+(Tx) = d_+^2(x)$ for almost all $x \in X_1$. Using this, we calculate

$$\begin{aligned}\int_X \log|d_+|\, dm &= \frac{1}{2}\int_{X_1} \log|m_+|\, dm + \frac{1}{2}\int_{X_1} \log|n_+|\, dm \\ &= \frac{1}{2}\int_{X_1} \log|(d_+)^2|\, dm = \int_{X_1} \log|d_+|\, dm = \frac{1}{2}\int_X \log|d_+|\, dm\end{aligned}$$

which implies $\log(M(L_+)) = \int_X \log|d_+|\,dm = 0$. □

Remark. The above proof shows $\log(M(\phi_\pm(L))) = \frac{1}{2}\log(M(L))$ and the renormalisation maps $\Phi_\pm$ are also contractions in the stronger topology

$$d(L,K) = |||L-K||| + |\log(M(L)) - \log(M(K))|$$

and map an open set of operators with positive mass and spectrum in a connected neighborhood of J into its interior.

The *potential theoretical Green function* g of a compact set $K \subset \mathbb{C}$ is a function $g : \mathbb{C} \to \mathbb{R}$ which is harmonic on $\mathbb{C} \setminus K$, vanishes on K and has the property that $g - \log(z)$ is bounded near $z = \infty$. The Green function exists for the Julia set J of t (see [M]).

Proposition 9.4. *The Lyapunov exponent $E \mapsto \lambda(E)$ of an operator $L \in \mathscr{J}$ is equal to the Green function $g(E)$ of the Julia set J. The Lyapunov exponent λ of $L \in \mathscr{J}$ is vanishing exactly on the spectrum J of L.*

Proof. The density of states of $L \in \mathscr{J}$ is the equilibrium measure on the Julia set J (Proposition 9.2) and gives J the capacity $\gamma(J) = 1$ (see [Bro]). The integral

$$u(z) = -\int \log|z - E'|\,dk(E')$$

is called the *conductor potential*. The relation between the conductor potential, the Green function and the capacity is given by the formula

$$g(z) = -u(z) + \log(\gamma(J)) ,$$

(see [T] Theorem III, 37). Lemma 9.3 together with the Thouless formula $\lambda(z) = \int \log|z - E'|\,dk(E') - \log(M)$ gives $u(z) = -\lambda(z)$. It follows that the Lyapunov exponent λ is equal to the Green function g of the Julia set J, which is by definition vanishing on the Julia set. □

Because ∞ is a super-attractive fixed point of the polynomial map $t(z) = z^2 + E$, there exist new coordinates $\tilde{z} = \zeta(z)$, near ∞ satisfying

$$\zeta \circ t \circ \zeta^{-1}(\tilde{z}) = \tilde{z}^2 .$$

The function ζ is called the *Böttcher function* of the polynomial t. (See for example [M].)

Corollary 9.5. *The Böttcher function ζ for the polynomial t satisfies $\zeta(z) = \det(L-z)$ for E in a neighborhood of ∞. One has*

$$\det(L - (z^2 + E)) = \det(L - z)^2 .$$

Proof. The Green function g can be expressed as $g(z) = \log|\zeta(z)|$. It follows from

$$|\zeta(z)| = \exp(\lambda(z)) = |\exp(-w(z))| = |\det(L - z)|$$

and the analyticity of ζ and $\det(L-z)$ near ∞ that $\zeta(z) = \det(L-z)$. The known identity $g(z) = g(z^2 + E)/2$ for the Green function g gives then

$$\det(L - (z^2 + E)) = \det(L - z)^2 . \qquad \square$$

It follows from the structure of the Julia set that for the limit-periodic Jacobi operators in $\mathscr{J}$, there is an obvious *gap-labelling*:

Proposition 9.6. *The integrated density of states of $L \in \mathscr{J}$ takes in the gaps exactly the values $l \cdot 2^{-n}$ with $n \in \mathbb{N}$ and $0 \leq l \leq 2^n$.*

Proof. We know from Proposition 9.2 that the density of states is the equilibrium measure on J and so a balanced measure. The inverse of the map t^n has 2^n branches $\phi^{(n,j)}$ labelled by $0 \leq j \leq 2^n$. Each of the sets $\phi^{(n,j)}(J)$ has measure 2^{-n}. If a gap of $L \in \mathscr{J}$ has l sets $\phi^{(n,j)}(J)$ to the left and $2^n - l$ such sets to the right, the integrated density of states of this gap is $l \cdot 2^{-n}$. □

General gap-labelling theorems for limit-periodic operators lead to the same result: it is known that the integrated density of states of an almost periodic Jacobi operator takes the values in the *frequency module* ([DS] Theorem III.1). For limit-periodic operators having as the hull the compact topological group G, the frequency module is the $\mathbb{Z}$ module generated by $\{\alpha \in \mathbb{R} \mid e^{2\pi i\alpha} \in \hat{G}\}$ (see [Bel]). Applying this in our case leads also to Proposition 9.6.

10. Generalisations and Questions

We discuss some generalizations or extensions:

Complex values E. Because an attractor of a hyperbolic iterated function system is structurally stable, the renormalisation is defined on an open subset of $\mathbb{C}$. All the results about the density of states, the Green function, etc. hold also there.

Julia sets of the anti-holomorphic quadratic map. Operators with spectra on the Julia sets of $z \mapsto \bar{t}(z) = \bar{z}^2 + E$ (introduced by Milnor) can be obtained by replacing $\Phi_\pm$ by $\overline{\Phi}_\pm$. The parameter set of $\bar{t}$ analogous to the Mandelbrot set is called the *Mandelbar set* or *tricorn.*

Nonrandom Jacobi matrices. An advantage of doing the renormalisation for random operators over dynamical systems is that one gets immediately the dynamical system over which the Jacobi operators are defined in the limit. We remark that for general Jacobi operators on $l^2(\mathbb{Z})$, there exists a continuum of Bäcklund transformations BT_s [GZ] parametrized by a parameter $s \in [-1, 1]$. For Random Jacobi operators, the random boundary condition selects two of them. It might be possible that the renormalisation for nonrandom Jacobi matrices can be done with an iterated function system Φ_s parametrized by $s \in [-1, 1]$.

Jacobi operators on the strip. The renormalization of Jacobi operators can be generalized to some random Jacobi operators on the strip in the crossed product of $L^\infty(X, M(N, \mathbb{C}))$ with the dynamical system. In the limit of renormalisation, these operators factorize into a direct sum of one-dimensional operators, obtained there.

Higher-dimensional Laplacians. A direct generalization to higher- dimensional random Laplacians is not possible without further modifications. The reason is that for a Laplacian

$$L = D^2 + E = \sum_i a_i\tau_i + a_i(T_i^{-1})\tau_i^* + b ,$$

the cocycles $a_i\tau_i$ satisfy the *zero curvature condition*

$$[a_i\tau_i, a_j\tau_j] = (a_ia_j(T_i) - a_ja_i(T_j))\tau_i\tau_j = 0$$

while the cocycles $d_i\sigma_i$ belonging to the discrete *Dirac operator* $D = \sum_i d_i\sigma_i + d_i(S_i^{-1})\sigma_i^*$ must satisfy the anti-commutation relation

$$\{d_i\sigma_i, d_j\sigma_j\} = (d_i d_j(S_i) + d_j d_i(S_j))\sigma_i\sigma_j = 0,\ i \neq j$$

which prevents a further factorization of D unless one renormalises also the statistics. Dirac operators play a role when doing isospectral deformations of higher-dimensional Laplacians [K3].

Operators with spectra on random Julia sets. The renormalization can be generalized in another way. Instead of taking a constant energy E, we can take a space dependent function $E(x) \leq -R$ for large enough R. We get again the same type of result as before. There exists an attractor above the von Neumann Kakutani system which consists of operators having the spectrum on *random Julia sets* (compare [FS]).

Operators with spectra on Julia sets of higher degree polynomials. By composing the renormalisation map Φ with a linear map of the form $\Phi_{a,b}L = aL + b$ with $a, b \in \mathbb{R}$, one can get operators with spectra on the Julia set of the polynomial $(az + b)^2 + E$ or on the Julia set of finite products of different such polynomials. With $-E_i$ large enough or $|a_i|$ small enough, the renormalisation limits exist.

More general values of E. We don't know how far one can explore the renormalization for general complex parameters E. The results of the literature mentioned in the introduction suggest however that one can find an iterated function system for a large set of E. For complex E, one has to deal with not normal operators. Another problem is that for smaller values of $|E|$, the norm of $\Phi^n(L)$ can blow up under the renormalisation steps even if the spectrum converges to the Julia set. The reason is that E can get closer and closer to the auxiliary spectrum of the matrices. Since the Titchmarsh-Weyl functions are singular at the auxiliary spectrum, the norm can get large or explode. Numerical investigations suggest that for all E in the complement of the Mandelbrot set, there should exist a hyperbolic iterated function system having a Cantor set as an attractor. The scheme can not work for general $E \in \mathbb{C}$ because for $E = 0$, a fixed point of $D \mapsto \Phi^+(D) = \psi(D^2)$ is not almost-periodic. If we do the renormalisation maps numerically for values E approaching 0, there are entries in the Jacobi matrix which begin to blow up. Obviously an invariant set exists but it is no more an attractor.

The case $E = -2$ is interesting because it describes a situation, where the energy E is at the boundary of the Mandelbrot set. The spectrum $[-2, 2]$ is then absolutely continuous with respect to the Lebesgue measure and the corresponding operator is the free Laplacian. Numerically, the renormalisation maps are contractions for all real values $E < -2$ and the attractor $\mathcal{J}$ approaches a single point in the limit $E = -2$.

We mention some open points:

- An obvious problem is to determine the maximal set in $\mathbb{C}$, where the renormalisation mappings $\Phi_\pm$ make sense and to understand what happens at the boundary of this maximal set. Even if a fixed point L_+ of the renormalisation map Φ_+ exists, it is an additional problem to determine its stability or to check if Φ_+ is a contraction. The key problem is to estimate the Fréchet derivative of the Bäcklund transformation BT_+ which we were only able to estimate for large $|E|$.
- The constructed operators have the property that the density of states is the potential theoretical equilibrium measure minimizing the energy on the spectrum. For which random operators is this also true?

• The high symmetry of the constructed operators in $\mathscr{J}$ could allow to determine the isospectral set of such operators. It is tempting to guess that the embedding of the dyadic group G in the infinite-dimensional torus $\mathbb{T}^{\omega}$ corresponds to an embedding of the attractor $\mathscr{J}$ in a not yet explored isospectral set of L_+ which would then be labelled by the infinite-dimensional torus $\mathbb{T}^{\omega}$.

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