

# Why the General Zakharov–Shabat Equations Form a Hierarchy

**L.A. Dickey**

Dept. Math., University of Oklahoma, Norman, OK 73019, USA. Internet: ldickey@nsfuvax.math.uoknor.edu

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**Abstract.** The totality of all Zakharov–Shabat equations (ZS), i.e., the zero-curvature equations with rational dependence on a spectral parameter, if properly defined, can be considered as a hierarchy. The latter means a collection of commuting vector fields in the same phase space. Further properties of the hierarchy are discussed, such as additional symmetries, an analogue to the string equation, a Grassmannian related to the ZS hierarchy, and a Grassmannian definition of soliton solutions.

## 0. Introduction.

We are accustomed to the fact that integrable systems appear not one at a time but in big families called hierarchies. So, first of all, the KdV ( $n=2$ ) hierarchy was invented (Gardner, Green, Kruskal, and Miura made the first and the most important discovery, the KdV equation in the proper sense; later on all the higher KdV were found by Gardner). Then this was generalized to every  $n$  (Gelfand and Dickey, who used fractional powers of operators). Thus, infinitely many generalized KdV hierarchies were found. They were unified to a single one large KP hierarchy (Kyoto school: Sato et al. [1]). Another line of developments was connected with equations generated by a linear first order differential operator with matrix coefficients linearly dependent on a spectral parameter (Albowitz, Kaup, Newell and Segur for matrices  $2 \times 2$  and Dubrovin in a general case, let us call this hierarchy AKNS-D). Later this was generalized to operators with a polynomial dependence on the spectral parameter. Thus, for every degree of a polynomial,  $m$ , there is a hierarchy, a generalized AKNS-D.<sup>1</sup>

More than that, there is a very general type of equations proposed by Zakharov and Shabat (see [2]): the equation of zero curvature, where matrices depend on some spectral parameter as rational functions. (The above mentioned hierarchies where operators depend on a parameter as polynomials, i.e., have a single pole at infinity, represent special case of these equations). The ZS equations usually have been treated individually, not as a hierarchy.

<sup>1</sup> More detail can be found, e.g., in [3].

Which property permits to consider a hierarchy as a single whole, as an entity? Geometrically speaking, a differential equation is a flow in a phase space. We call a family of differential equations a hierarchy if they act in the same phase space and commute. Then each of equations determines a symmetry for each other. Let us consider this as a characteristic property of a hierarchy.

We see our goal in this paper in the construction of a theory where all the Zakharov–Shabat (ZS) equations can be considered as one hierarchy, in the above sense.

We discuss also problems such as a Grassmannian approach to the ZS hierarchy, the existence of additional symmetries and a definition of an analogue to the string equation.

The study of integrable systems is interesting in several aspects. One is a desire to expand the class of exactly solvable equations or to find new solutions to already known integrable equations. This can be important in applications, though one must realize that there are rather few integrable systems among all equations significant for physics or engineering. The other aspect is that this area of mathematics provides very rich algebraic structures which more and more find their way to modern physical theories. The fact that there are large hierarchies of commuting equations is of great importance just from this point of view.

### 1. Definition of the ZS Hierarchy.

Let  $a_k, k = 1, \dots, m$  be a given set of complex numbers. Let, for every  $k$ ,

$$\hat{w}_k = \sum_0^\infty w_{ki}(z - a_k)^i,$$

be a formal series. The entries of  $n \times n$  matrices  $w_{ki}, w_{ki, \alpha\beta}$  are just letters. We consider the algebra  $\mathcal{A}_w$  of polynomials of all these entries and  $(\det w_{k0})^{-1}$ . The formal series  $\hat{w}_k$  can be inverted within this algebra. Let

$$R_{k\alpha} = \hat{w}_k E_\alpha \hat{w}_k^{-1}; \quad R_{k\alpha l} = R_{k\alpha}(z - a_k)^{-l},$$

where  $E_\alpha$  is a matrix with only one non-vanishing element, equal to 1, on the  $(\alpha, \alpha)$  place.

We have two kinds of objects. Such quantities as  $\hat{w}_k$  and  $R_{k\alpha l}$  are formal series, or jets, at the points  $a_k$ . The algebra of all such jets will be called  $J_k$  and  $J = \bigoplus J_k$ . If  $j_k \in J_k$  is a jet then  $j_k^-$  symbolizes principal part, i.e., a sum of negative powers of  $z - a_k$ , and  $j_k^+$  the rest of the series. Correspondingly, the jet algebras split into two parts,  $J_k = J_k^+ \oplus J_k^-$ . If the principal part contains a finite number of terms (and we tacitly assume this unless the opposite is said or is evident from a context) it can be considered as a global meromorphic function; the algebra of global meromorphic functions is  $G$ . Global functions are objects of the second kind. A global function gives rise to a jet at every  $a_k$ . In particular,  $j_k^-$  can be considered as a jet at a point  $a_{k_1}$ , different from  $a_k$ , more precisely, as an element of  $J_{k_1}^+$ .

**Definitions.** (i) A hierarchy corresponding to a fixed set  $\{a_k\}$  is the totality of equations

$$\partial_{k\alpha l} \hat{w}_{k_1} = \begin{cases} -R_{k\alpha l}^+ \hat{w}_{k_1}, & k = k_1 \\ R_{k\alpha l}^- \hat{w}_{k_1}, & \text{otherwise} \end{cases}, \quad \partial_{k\alpha l} = \partial / \partial t_{k\alpha l}. \tag{1}$$

In the second case  $R_{kzl}^-$  is considered as an element of  $J_{k_1}^+$ , see above;  $t_{kzl}$  are some variables.

(ii) A ZS hierarchy is an inductive limit of hierarchies with fixed sets  $\{a_k\}$ , with respect to a natural embedding of a hierarchy corresponding to a subset into a hierarchy corresponding to a larger set, as a subhierarchy.

Further in this section, for simplicity of writing, we shall unite indices  $\alpha$  and  $l$  into one subscript  $a = (\alpha, l)$  and write  $\partial_{ka}$  and  $R_{ka}$  instead of  $\partial_{k\alpha l}$  and  $R_{k\alpha l}$ .

(It would be possible to take its own spectral family  $E_{k\alpha}$ ,  $\alpha = 1, \dots, n$ , for each point  $a_k$ , those spectral projectors not necessarily commuting for different  $k$ ; however, it is easy to see that this is not a real generalization since it can be reduced to the same definition by a substitution  $\hat{w}_k \mapsto \hat{w}_k c_k^{-1}$ , where  $c_k$  are matrices reducing  $E_{k\alpha}$  to the diagonal form,  $E_{k\alpha} = c_k E_{\alpha} c_k^{-1}$ .)

**Lemma. Equalities**

$$\partial_{k\alpha l} R_{k_1 \alpha_1 l_1} = \begin{cases} -[R_{k\alpha l}^+, R_{k_1 \alpha_1 l_1}], & k = k_1 \\ [R_{k\alpha l}^-, R_{k_1 \alpha_1 l_1}], & \text{otherwise} \end{cases}$$

hold.

*Proof.* It easily can be obtained from the definition of  $R_{k\alpha l}$ .  $\square$

**Theorem.** All operators  $\partial_{k\alpha l}$  commute.

*Proof.* One has to prove  $[\partial_{k_1 a_1}, \partial_{k_2 a_2}] \hat{w}_{k_3} = 0$  in 3 cases: i) all of  $k_i$  coincide, ii) only two of them coincide, iii) all are distinct.

i) We have

$$\begin{aligned} (\partial_{ka_1} \partial_{ka_2} - \partial_{ka_2} \partial_{ka_1}) \hat{w}_k &= -\partial_{ka_1} R_{ka_2}^+ \hat{w}_k - (1 \Leftrightarrow 2) \\ &= [R_{ka_1}^+, R_{ka_2}^-]_k^+ \hat{w}_k + R_{ka_2}^+ R_{ka_1}^+ \hat{w}_k - (1 \Leftrightarrow 2) \\ &= R_{ka_1}^+ R_{ka_2}^+ \hat{w}_k + [R_{ka_1}, R_{ka_2}^-]_k^+ \hat{w}_k \\ &\quad - [R_{ka_2}^+, R_{ka_1}]_k^+ \hat{w}_k - R_{ka_1}^+ R_{ka_2}^+ \hat{w}_k \\ &= [R_{ka_1}, R_{ka_2}]_k^+ \hat{w}_k = 0, \end{aligned}$$

since  $R_{ka_1}$  and  $R_{ka_2}$  commute. A notation  $A_k^-$  means the principal part of an expansion of  $A$  in powers of  $z - a_k$ . Similarly,  $A_k^+$ .

ii) First we consider

$$\begin{aligned} (\partial_{ka_1} \partial_{ka_2} - \partial_{ka_2} \partial_{ka_1}) \hat{w}_{k_1} &= \partial_{ka_1} R_{ka_2}^- \hat{w}_{k_1} - \partial_{ka_2} R_{ka_1}^- \hat{w}_{k_1} \\ &= -[R_{ka_1}^+, R_{ka_2}^-]_k^- \hat{w}_{k_2} + R_{ka_2}^- R_{ka_1}^- \hat{w}_{k_1} - (1 \Leftrightarrow 2) \\ &= -(R_{ka_1}^+ R_{ka_2}^-)_k^- \hat{w}_{k_1} + (R_{ka_2}^- R_{ka_1}^-)_k^- \hat{w}_{k_1} - (1 \Leftrightarrow 2) \\ &= -(R_{ka_1} R_{ka_2})_k^- \hat{w}_{k_1} \end{aligned}$$

which is zero since  $R_{ka_1}$  and  $R_{ka_2}$  with the same  $k$  commute.

Then we take

$$\begin{aligned} (\partial_{ka} \partial_{k_1 a_1} - \partial_{k_1 a_1} \partial_{ka}) \hat{w}_k &= \partial_{ka} R_{k_1 a_1}^- \hat{w}_k + \partial_{k_1 a_1} R_{ka}^+ \hat{w}_k \\ &= [R_{ka}^-, R_{k_1 a_1}]_{k_1}^- \hat{w}_k - R_{k_1 a_1}^- R_{ka}^+ \hat{w}_k \\ &\quad + [R_{k_1 a_1}, R_{ka}]_k^+ \hat{w}_k + R_{ka}^+ R_{k_1 a_1}^- \hat{w}_k \end{aligned}$$

$$\begin{aligned}
&= [R_{ka}^-, R_{k_1 a_1}]_{k_1}^- \hat{w}_k - [R_{k_1 a_1}^-, R_{ka}^+] \hat{w}_k \\
&\quad + [R_{k_1 a_1}^-, R_{ka}^+]_k^+ \hat{w}_k + [R_{k_1 a_1}^-, R_{ka}^-]_k^+ \hat{w}_k \\
&= ([R_{ka}^-, R_{k_1 a_1}]_{k_1}^- - [R_{ka}^-, R_{k_1 a_1}] + [R_{ka}^-, R_{k_1 a_1}]_k^-) \hat{w}_k .
\end{aligned}$$

In the parentheses there is a function  $[R_{ka}^-, R_{k_1 a_1}]$  minus its principal parts in both the poles,  $a_k$  and  $a_{k_1}$ . Thus, this is a constant. This expression approaches zero when  $z \rightarrow \infty$  which implies that the constant is zero.

iii)

$$\begin{aligned}
[\partial_{k_1 a_1}, \partial_{k_2 a_2}] \hat{w}_{k_3} &= \partial_{k_1 a_1} R_{k_2 a_2}^- \hat{w}_{k_3} - \partial_{k_2 a_2} R_{k_1 a_1}^- \hat{w}_{k_3} \\
&= [R_{k_1 a_1}^-, R_{k_2 a_2}^-]_{k_2}^- \hat{w}_{k_3} + R_{k_2 a_2}^- R_{k_1 a_1}^- \hat{w}_{k_3} - (1 \Leftrightarrow 2) \\
&= ([R_{k_1 a_1}^-, R_{k_2 a_2}^-]_{k_1}^- + [R_{k_1 a_1}^-, R_{k_2 a_2}^-]_{k_2}^- - [R_{k_1 a_1}^-, R_{k_2 a_2}^-]) \hat{w}_{k_3} .
\end{aligned}$$

The expression in the parentheses vanish by the same reason as in the previous case.  $\square$

## 2. Gauge.

If we let  $\hat{v}_k = c \hat{w}_{k_1}$  where  $c$  is a matrix depending of variables  $\{t_{k\alpha l}\}$  in an arbitrary way, then  $\hat{v}_k$  satisfy equations

$$\partial_{k\alpha l} \hat{v}_{k_1} = A_{k\alpha l} \hat{v}_{k_1} + \begin{cases} -R_{k\alpha l}^+ \hat{v}_{k_1}, & k = k_1 \\ R_{k\alpha l}^- \hat{v}_{k_1}, & \text{otherwise} \end{cases} \quad (2)$$

with new  $R_{k\alpha l} = \hat{v}_k E_\alpha (z - a_k)^{-1} \hat{v}_k^{-1}$ , and  $A_{k\alpha l} = \partial_{k\alpha l} c \cdot c^{-1}$  which implies

$$\partial_{k_1 \alpha_1 l_1} A_{k\alpha l} - \partial_{k\alpha l} A_{k_1 \alpha_1 l_1} + [A_{k\alpha l}, A_{k_1 \alpha_1 l_1}] = 0 . \quad (3)$$

Equations (2) are slightly more general than (1). We say that  $\hat{w}_k$  and  $\hat{v}_k$  are gauge-equivalent. Conversely, if some  $\hat{v}_k$  satisfy (2) with the property (3) for  $A$ , then, integrating equations  $\partial_{k\alpha l} c = A_{k\alpha l} c$  (which are compatible by virtue of this property), one can find a gauge-equivalent  $\hat{w}_k$  satisfying (1). We can, e.g., normalize a solution by a condition  $\sum_i \hat{w}_i(a_i) = I$ .

The following lemma can be useful:

**Lemma.** *Given solutions of (2) with some  $A_{k\alpha l}$  not depending on  $z$  for any  $(k\alpha l)$ , being  $\sum_i \hat{v}_i(a_i) = I$ . Then there is some  $c$  such that  $A_{k\alpha l} = \partial_{k\alpha l} c \cdot c^{-1}$  and, therefore,  $\hat{v} = \{\hat{v}_i\}$  is gauge equivalent to a solution of (1).*

*Proof.* First of all from the assumption we get

$$A_{k\alpha l} = R_{k\alpha l}^+(a_k) \hat{v}_k(a_k) - \sum_{k_1 \neq k} R_{k\alpha l}^-(a_{k_1}) \hat{v}_{k_1}(a_{k_1}) ,$$

thus,  $A_{k\alpha l}$  is a differential polynomial in elements of  $\hat{v}_i$ 's. A new differentiation can be defined in  $\mathcal{A}_v$ :  $\partial_{k\alpha l}^* \hat{v}_i = (\partial_{k\alpha l} - A_{k\alpha l}) \hat{v}_i$ . The quantities  $\hat{v}_k$  satisfy the system (1) with respect to new variables  $\hat{v}_{k\alpha l}^*$  and, therefore,  $\partial_{k\alpha l}^*$  commute, i.e.,  $\partial_{k\alpha l} - A_{k\alpha l}$  commute which is equivalent to Eq. (3). The rest is clear.  $\square$

Functions  $\hat{w}_k$  admit also the following transformations: multiplication on the right by series in  $(z - a_k)^{-1}$  with constant diagonal coefficients. This does not affect the equations (1) (or (2)) at all.

### 3. Differential Operators.

The following proposition readily can be proven by a simple straightforward computation:

**Proposition 1.** *A dressing formula*

$$\hat{w}_{k_1}(\partial_{kz_l} - E_\alpha(z - a_k)^{-l} \delta_{kk_1}) \hat{w}_{k_1}^{-1} = \partial_{kz_l} - B_{kz_l}, \quad B_{kz_l} = R_{kz_l}^- \tag{4}$$

is equivalent to Eq. (1).

The operator  $\partial_{kz_l} - B_{kz_l}$  is assumed to act in  $J_{k_1}$ . However, it does not depend on  $k_1$  at all and can be considered as a global function of  $z$  with the only pole of the  $l^{\text{th}}$  order at  $a_k$ . Let

$$w_k = \hat{w}_k \exp \xi_k, \text{ where } \xi_k = \sum_{l=0}^{\infty} \sum_{\alpha=1}^n t_{kz_l} E_\alpha(z - a_k)^{-l}.$$

**Definition.** *The collection  $w = \{w_k\}$  is the formal Baker function of the hierarchy.*

Equation (4) can be written in terms of the Baker function as

$$w_{k_1} \partial_{kz_l} w_{k_1}^{-1} = \partial_{kz_l} - B_{kz_l}. \tag{5}$$

**Proposition 2.** *All the operators  $\partial_{kz_l} - B_{kz_l}$  commute.*

*Proof.* This is a corollary of the theorem in Sect. 1. and Eq. (5).  $\square$

One can consider arbitrary linear combinations of the above constructed operators,

$$L = \sum_{k, \alpha, l} \lambda_{kz_l} (\partial_{kz_l} - B_{kz_l}) = \partial + U, \tag{6}$$

where  $\partial = \sum_{k, \alpha, l} \lambda_{kz_l} \partial_{kz_l}$  and  $U = -\sum_{k, \alpha, l} \lambda_{kz_l} B_{kz_l}$ . Two such operators commute which yields equations of the Zakharov–Shabat type

$$\partial U_1 - \partial_1 U = [U_1, U].$$

Functions  $U$  and  $U_1$  are rational functions of the parameter  $z$ .

Operators  $\partial$  and  $\partial_1$  can be interpreted as derivatives with respect to new variables  $x$  and  $t$ , i.e., we have a change of variables  $t_{kz_l} = \lambda_{kz_l} x + \mu_{kz_l} t + \dots$ , where dots stand for the rest of the new variables. It is also possible to prove that if for some  $k$  coefficients  $\lambda_{kz_l}$  with the biggest  $l$  are non-vanishing and distinct for all  $\alpha$  then all elements of all  $R_{k\beta m}$  can be expressed as polynomials in elements of  $U$  and their derivatives with respect to  $x$ . A similar statement is true for  $U_1$ .

*Remark 1.* It is possible to give a group theory interpretation to these equations considering a Lie algebra of jets  $J$  and, as the dual space, global meromorphic functions with poles at  $\{a_k\}$ . (See also [8]).

*Remark 2.* Here we have a special case of the ZS equation: the functions  $U$  and  $U_1$  vanishing at infinity. If we make a gauge transformation  $w_k \mapsto c w_k$  then  $\partial + U \mapsto c(\partial + U)c^{-1} = \partial + c U c^{-1} - (\partial c) c^{-1}$ , the last term does not vanish at infinity. This yields the general case.

### 4. Additional Symmetry and the String Equation.

It is well-known ([5]) that the KP hierarchy has infinitely many symmetries that are not contained in the hierarchy itself; their characteristic feature is an explicit dependence on the variables  $t$ . They are called “additional symmetries.” The so-called “string” equation is nothing but a condition that our operators do not depend on a parameter of an additional symmetry ([6]). We are going to suggest an additional symmetry and the corresponding string equation for the ZS hierarchy.

Let

$$\partial_z - M_i = w_i \partial_z w_i^{-1} = \partial_z - \partial_z w_i \cdot w_i^{-1} = \partial_z - \partial_z \hat{w}_i \cdot \hat{w}_i^{-1} - \hat{w}_i \xi_{iz} \hat{w}_i^{-1},$$

where  $\xi_{iz} = \partial \xi_i / \partial z = -\sum_a \sum_{l=1}^\infty t_{ixl} E_{\alpha l} (z - a_i)^{-l-1}$ . The quantity  $M_i = \partial_z \hat{w}_i \cdot \hat{w}_i^{-1} + \hat{w}_i \xi_{iz} \hat{w}_i^{-1}$  is a jet at the point  $a_i$ .

Dressing an obvious relation  $[\partial_z, \partial_{k\alpha l}] = 0$  with the help of  $w_i$  at the point  $a_i$  we have

$$[\partial_z - M_i, \partial_{k\alpha l} - B_{k\alpha l}] = 0,$$

i.e.,  $\partial_{k\alpha l} M_i = \partial_z B_{k\alpha l} - [M_k, B_{k\alpha l}]$ . Taking negative and positive parts, we get at the point  $a_i$

$$\begin{aligned} (1) \quad i = k \quad & \partial_{k\alpha l} M_k^- = \partial_z B_{k\alpha l} - [M_k, B_{k\alpha l}]_k^-, \\ & \partial_{k\alpha l} M_k^+ = -[M_k, B_{k\alpha l}]_k^+, \\ (2) \quad i \neq k \quad & \partial_{k\alpha l} M_i^- = -[M_i, B_{k\alpha l}]_i^-, \\ & \partial_{k\alpha l} M_i^+ = \partial_z B_{k\alpha l} - [M_i, B_{k\alpha l}]_i^+. \end{aligned} \tag{7}$$

**Definition.** The additional symmetry is given by the system of differential equations

$$\partial^* \hat{w}_j = \left( -M_j^+ + \sum_{i \neq j} M_i^- \right) \hat{w}_j.$$

The same equation can also be written as

$$\partial^* \hat{w}_j = -\partial_z \hat{w}_j - \sum_i \sum_\alpha \sum_l t_{ixl} l \partial_{i\alpha(l+1)} \hat{w}_j.$$

Formally, there are infinitely many terms in this series. It is possible to freeze all  $t_{ixl}$  as zero, except for a finite number of them.

As it is easy to see, the equation of the additional symmetry implies

$$\partial^* R_{k\alpha l} = \left[ -M_k^+ + \sum_{i \neq k} M_i^-, R_{k\alpha l} \right]. \tag{8}$$

This is an equality in  $J_k$ .

**Proposition.** The additional symmetry commutes with operators of the hierarchy,  $[\partial^*, \partial_{k\alpha l}] = 0$ , i.e., it is a symmetry, indeed.

*Proof.* We have to prove that for all  $j$  a relation  $[\partial^*, \partial_{kzl}] \hat{w}_j = 0$  holds. We consider two cases. (1)  $k \neq j$ .

$$\begin{aligned} \partial^* \partial_{kzl} \hat{w}_j - \partial_{kzl} \partial^* \hat{w}_j &= \partial^* B_{kzl} \hat{w}_j - \partial_{kzl} \left( -M_j^+ + \sum_{i \neq j} M_i^- \right) \hat{w}_j \\ &= \left[ -M_k^+ + \sum_{i \neq k} M_i^-, B_{kzl} \right]_k^- \hat{w}_j + \left[ B_{kzl}, -M_j^+ + \sum_{i \neq j} M_i^- \right]_j \hat{w}_j \\ &\quad + (\partial_z B_{kzl}) \hat{w}_j - [M_j, B_{kzl}]_j^+ \hat{w}_j \\ &\quad + \sum_{i \neq k, j} \left[ M_i^-, B_{kzl} \right]_i^- \hat{w}_j - (\partial_z B_{kzl}) \hat{w}_j + [M_k, B_{kzl}]_k^- \hat{w}_j. \end{aligned}$$

First, discuss the terms with  $i \neq k, j$ . These are

$$\sum_{i \neq k, j} \left[ M_i^-, B_{kzl} \right]_k^- - \sum_{i \neq k, j} \left[ M_i^-, B_{kzl} \right]_i^- + \sum_{i \neq k, j} \left[ M_i^-, B_{kzl} \right]_i^-.$$

This is a difference between a global function  $\sum_{i \neq k, j} [M_i^-, B_{kzl}]$  and all its principal parts, at  $a_k$  and  $a_i$ ; it has to be constant. Taking into account that it vanishes at infinity, we can conclude that this is zero. It remains to calculate

$$\begin{aligned} &(-[M_k^+, B_{kzl}]_k^- + [M_j^-, B_{kzl}]_k^- + [M_j^+, B_{kzl}]_j \\ &\quad - [M_k^-, B_{kzl}]_j - [M_j^-, B_{kzl}]_j^+ + [M_k, B_{kzl}]_k^-) \hat{w}_j. \end{aligned}$$

Notice that in the fourth term the subscript  $j$  can be skipped, this is a global term. The first, the fourth and the sixth terms cancel out. The remaining terms are

$$\begin{aligned} &[M_j^-, B_{kzl}]_k^- + [M_j^+, B_{kzl}]_j - [M_j, B_{kzl}]_j^+ \\ &= [M_j^-, B_{kzl}]_k^- + [M_j^+, B_{kzl}]_j^- - [M_j^-, B_{kzl}]_j^+. \end{aligned}$$

The middle term vanishes. Now,

$$[M_j^-, B_{kzl}]_j^+ = [M_j^-, B_{kzl}]_j - [M_j^-, B_{kzl}]_j^- = [M_j^-, B_{kzl}]_k^-$$

by the same reason: a global function vanishing at infinity is a sum of its principal parts. The obtained term cancels with the remaining one.

(2)  $k = j$ .

$$\begin{aligned} \partial^* \partial_{kzl} \hat{w}_k - \partial_{kzl} \partial^* \hat{w}_k &= -\partial^* R_k^+ \hat{w}_k - \partial_{kzl} \left( -M_k^+ + \sum_{i \neq k} M_i^- \right) \hat{w}_k \\ &= \left\{ \left[ M_k^+ - \sum_{i \neq k} M_i^-, R_{kzl} \right]_k^+ + \left[ R_{kzl}, M_k^+ - \sum_{i \neq k} M_i^- \right] \right. \\ &\quad \left. - [M_k, B_{kzl}]_k^+ + \sum_{i \neq k} [M_i, B_k]_i^- \right\} \hat{w}_k. \end{aligned}$$

Three terms cancel:

$$[M_k^+, R_{kzl}]^+ + [R_{kzl}, M_k^+] - [M_k, R_{kzl}]^+ = [M_k^+, R_{kzl}]^+ - [M_k^+, R_{kzl}]^+ = 0.$$

The remaining terms are

$$\begin{aligned}
 & -[M_i^-, R_{k\alpha l}]_k^+ - [R_{k\alpha l}^+, M_i^-]_k + [M_i, R_{k\alpha l}^-]_i^- \\
 & = -[M_i^-, R_{k\alpha l}^-]_k^+ - [R_{k\alpha l}^+, M_i^-]_k^- + [M_i^-, R_{k\alpha l}^-]_i^- .
 \end{aligned}$$

The middle term vanishes and  $[M_i^-, R_{k\alpha l}^-]_k^+ = [M_i^-, R_{k\alpha l}^-]_i^-$  by the same reason as it was in the first part, and the whole expression vanishes.  $\square$

The variable  $t^*$  is independent of all  $t_{k\alpha l}$ . This implies

$$\partial^* w_k = \partial^*(\hat{w}_k \exp \xi_k) = (\partial^* \hat{w}_k) \exp \xi_k = \left( -M_k^+ + \sum_{i \neq k} M_i^- \right) w_k .$$

From the dressing formula  $\partial_{k\alpha l} - B_{k\alpha l} = w_i \partial_{k\alpha l} w_i^{-1}$  we have that

$$\partial^* B_{k\alpha l} = \left[ -M_k^+ + \sum_{i \neq k} M_i^-, -\partial_{k\alpha l} + B_{k\alpha l} \right] .$$

Taking into account (7), this yields

$$\partial^* B_{k\alpha l} = \left[ -\partial_z + \sum_i M_i^-, -\partial_{k\alpha l} + B_{k\alpha l} \right] .$$

Take a linear combination with coefficients  $\lambda_{k\alpha l}$  of these equations. Then (see Eq. (6))

$$\partial^* U = \left[ -\partial_z + \sum_i M_i^-, \partial + U \right] . \tag{9}$$

**Definition.** A string equation is condition that  $U$  does not depend on  $t^*$ . This yields

$$\left[ \sum_i M_i^-, \partial + U \right] = \partial_z U . \tag{10}$$

In terms of  $\hat{w}_j$ , the string equation has the form

$$\partial_z \hat{w}_j = \sum_{\alpha l} t_{j\alpha l} l R_{j\alpha(l+1)}^+ \hat{w}_j - \sum_{i \neq j} \sum_{\alpha l} t_{i\alpha l} l R_{i\alpha(l+1)}^- \hat{w}_j , \tag{11}$$

and in terms of  $w_j$ :

$$\partial_z w_j = - \sum_{i\alpha l} t_{i\alpha l} l R_{i\alpha(l+1)}^- w_j . \tag{12}$$

We have found here an additional symmetry. A problem arises: are there other symmetries, as it is the case with KP, where there is an infinite series of them labeled with two integer indices? We have no answer to this question yet.

### 5. Grassmannian

We use the following notations:  $C_k$  are disjoint circles around fixed points  $a_k, k = 1, \dots, m$ , and  $\Omega$  is the part of the Riemann sphere outside all the circles;  $H_k$  are Hilbert spaces of vector-functions  $\mathbf{f}_k(z) \in C^n$  on the circles, subspaces  $H_k^+$  consist of functions on  $C^k$  which can be expanded in non-negative powers of  $z - a_k$ , and  $H_k^-$  contain expansions in negative powers. Now,  $H = \bigoplus_k H_k$ , and  $H^+$  consists of  $\{\mathbf{f}_k\}$  such that  $\mathbf{f}_k \in H_k^+$  under an additional constraint  $\sum_k \mathbf{f}_k(a_k) = 0$ .

Finally, let  $H^*$  consist of  $\mathbf{f} = \{\mathbf{f}_k\} \in H$  such that  $\mathbf{f}_k$  are boundary values on the circles  $C_k$  of a holomorphic vector-function in the domain  $\Omega$ ; this function will be denoted by the same letter  $\mathbf{f}$ . It is easy to see that  $H = H^* \oplus H^+$ . Indeed, let  $\mathbf{f} = \{\mathbf{f}_k\} \in H$  be an arbitrary element. Then each  $\mathbf{f}_k$  can be decomposed into  $\mathbf{f}_k = \mathbf{f}_k^- + \mathbf{f}_k^+$ . Elements  $\mathbf{f}_k^-$  are holomorphic outside the corresponding circles  $C_k$ , and elements  $\mathbf{f}_k^+$  are holomorphic inside the corresponding circles. Now,

$$\mathbf{f}_k = \left( \sum_i \mathbf{f}_i^- + \mathbf{c} \right) + (\mathbf{g}_k - \mathbf{c}),$$

where  $\mathbf{g}_k = \mathbf{f}_k^+ - \sum_{i \neq k} \mathbf{f}_i^-$  and  $\mathbf{c} = m^{-1} \sum_k \mathbf{g}_k(a_k)$ . We have,  $\{\sum_i \mathbf{f}_i^- + \mathbf{c}\} \in H^*$  and  $\{\mathbf{g}_k - \mathbf{c}\} \in H^+$ . The decomposition is unique. Let  $P^*$  be the projector  $P^*: H \rightarrow H^*$ .

**Definition.** An element of the Grassmannian,  $W \in \text{Gr}$ , is a subspace of  $H$  with the following properties: i) the projection  $P^*: H \rightarrow H^*$  restricted to  $W$  is a bijection, and ii)  $(z - a_1)^{-1}W = (z - a_2)^{-1}W = \dots = (z - a_m)^{-1}W \subset W$ .

We think about vectors as vector-rows. A matrix is said to belong to  $W$  if all its rows do also.

One can consider the following transformation of the Grassmannian. If  $\mathbf{f} = \{\mathbf{f}_k\} \in W$ , then  $\mathbf{f} \exp \xi = \{\mathbf{f}_k \exp \xi_k\}$ , where  $\xi_k = \sum_{l=0}^{\infty} \sum_{\alpha=1}^n t_{k\alpha l} E_{\alpha}(z - a_k)^{-l}$ ,  $\xi = \{\xi_k\}$ . The set of all  $\mathbf{f} \exp \xi$  is called  $W \exp \xi$ . For almost all  $t_{k\alpha l}$  the subspace  $W \exp \xi$  is a new element of the Grassmannian.

**Definition.** A Grassmannian pre-Baker function, corresponding to an element of the Grassmannian  $W$ , is a matrix-function  $w \in W$  such that

$$P^* w \exp(-\xi) = c,$$

where  $c$  is a constant in  $z$  (however, it can depend on variables  $t$ ).

An element  $W \in \text{Gr}$  is invariant with respect to multiplication on the left by a matrix constant in  $z$ , since this leads to linear combinations of rows. The projector  $P^*$  commutes with this multiplication. Therefore, if  $w$  is a pre-Baker function then so is a gauge-equivalent function  $ew$ , where  $e$  is any matrix independent of  $z$ . All pre-Baker functions can be expressed in this way in terms of one of them, e.g., corresponding to  $c = I$  (the normalized pre-Baker function).

Let  $w = \{w_k\}$  be a pre-Baker function and  $\hat{w}_k = w_k \exp(-\xi_k)$ . This means that  $\hat{w}_k$  has a form  $c + w_{k,0} + w_{k,1}(z - a_k) + \dots$  and  $\sum_k w_{k,0} = 0$ . For a normalized function  $c = I$ . Thus, the normalized pre-Baker function has expansions

$$w_k = (I + w_{k,0} + w_{k,1}(z - a_k) + \dots) \exp \xi_k \in W, \quad k = 1, \dots, m; \quad \sum_k w_{k,0} = 0.$$

These equalities are equivalent to the definition of the normalized pre-Baker function. Let  $w$  be a pre-Baker function. We have

$$\begin{aligned} \partial_{k\alpha l} w_i - R_{k\alpha l}^- w_i &= (\partial_{k\alpha l} \hat{w}_i + \delta_{ki} \hat{w}_i E_{\alpha}(z - a_k)^{-l} - R_{k\alpha l}^- \hat{w}_i) \exp \xi_i \\ &= \begin{cases} (\partial_{k\alpha l} \hat{w}_i - R_{k\alpha l}^- \hat{w}_i) \exp \xi_i, & i \neq k \\ (\partial_{k\alpha l} \hat{w}_i + R_{k\alpha l}^+ \hat{w}_i) \exp \xi_i, & i = k. \end{cases} \end{aligned} \tag{13}$$

**Definition.** A pre-Baker function is called a Baker function if for every  $(k\alpha l)$  a relation  $\{\partial_{k\alpha l} w_i\} \in (z - a_j)^{-1} W$  holds (in fact, this subspace does not depend on  $j$ , see the definition of the Grassmannian).

**Proposition.** *A Grassmannian Baker function is a formal Baker function of the hierarchy, in the sense of Sect. 3. Thus, a solution of the hierarchy equations is related to any Grassmannian Baker function.*

*Proof.* The left-hand side of (13),  $\{\partial_{kz}w_i - R_{kz}^-w_i\}$ , belongs to  $(z - a_k)^{-1}W$ . Therefore, the expression in parentheses in the right-hand side, let it be  $g_i$ , is in  $(z - a_k)^{-1}W \exp(-\xi)$ . Then  $\{(z - a_j)g_i(z)\} \in W \exp(-\xi)$  for every  $j$ . On the other hand, this is an element of  $H^+$  plus, maybe, a constant. Let  $\gamma_j$  be arbitrary matrices. Then  $\{\sum_j \gamma_j(z - a_j)g_i(z)\} \in W \exp(-\xi)$ . It is easy to see that choosing matrices  $\gamma_j$  one can achieve that  $\sum_i \sum_j \gamma_j(a_i - a_j)g_i(a_i) = 0$ . Then  $\{\sum_j \gamma_j(z - a_j)g_i(z)\}$  is in  $H^+$  and, therefore, in  $W \exp(-\xi) \cap H^+$ . This implies that  $\{\sum_j \gamma_j(z - a_j)g_i(z)\} = 0$  and  $g_i = 0$ . The latter means that  $\hat{w}_i$  satisfy the equations of the hierarchy.  $\square$

### 6. An Example: Soliton-Type Solution.

First the soliton solution were found in the original paper by Zakharov and Shabat [2]. Here a different construction is presented. This approach is known for a long time: Manin [9], Date [10], maybe even earlier, since Manin refers to Drinfeld. We just tried here to do this in the possibly most general form and connect it to the Grassmannian and to a  $\tau$ -function. In frameworks of the Grassmannian theory this construction looks very natural.

One starts with a specification of an element  $W \in \text{Gr}$ . Consider a linear space  $H^*(D)$  of meromorphic vector-functions  $\mathbf{f}$  in the domain  $\Omega$  with a fixed divisor  $D$  of simple poles  $b_j$ , where  $j = 1, \dots, N$ . Collections of boundary values of these functions,  $\{\mathbf{f}_k\}$ , on the circles  $C_k$  will be denoted by the same letter  $\mathbf{f}$ , and the linear space of them by the same symbol  $H^*(D)$ . This will not lead to any ambiguity.

Now, let  $W \subset H^*(D)$  be a subset of meromorphic functions in  $H^*(D)$  satisfying  $Nn$  conditions  $\mathbf{v}(\mu_i) \cdot \eta_i = 0$ , where  $i = 1, \dots, Nn$ ,  $\mu_i \in \Omega$  are arbitrary points, and  $\eta_i$  are given vector columns. Collections of their boundary values are symbolized by the same letter  $W$ , and this, generically, is an element of the Grassmannian. The property ii) of the definition of the Grassmannian is self-evident. Notice that  $(z - a_j)^{-1}W$  consists of those elements which are boundary values of meromorphic functions vanishing at infinity.

Let us prove that the property i) is also satisfied. We have to prove that if there is an element  $\mathbf{f} = \{\mathbf{f}_k\} \in H^*$ , then a unique element  $\mathbf{g} = \{\mathbf{g}_k\} \in W$  can be found such that  $\mathbf{h} = \mathbf{f} - \mathbf{g} = \{\mathbf{f}_k - \mathbf{g}_k\} = \{\mathbf{h}_k\}$  is in  $H^+$ , i.e., its analytical prolongation inside every circle  $C_k$  exists, being  $\sum_k \mathbf{h}_k(a_k) = 0$ . Given  $\mathbf{f}_k$  are boundary values of a holomorphic in  $\Omega$  function  $\mathbf{f}$ . Let  $\mathbf{g} = \mathbf{f} + \mathbf{b}_0 + \sum_{j=1}^N \mathbf{b}_j(z - b_j)^{-1}$ , where  $\mathbf{b}_j$  are vectors that have to be found, in all  $(N + 1)n$  unknown components. First of all we require that  $\mathbf{g} \in W$  which is equivalent to  $Nn$  scalar equations  $\mathbf{g}(\mu_i) \cdot \eta_i = 0$ . Then we impose one more constraint  $\sum_{k=1}^m (\mathbf{b}_0 + \sum_{j=1}^N \mathbf{b}_j(a_k - b_j)^{-1}) = 0$  which gives  $n$  more equations for the unknown coefficients, in all  $(N + 1)n$  equations. Generically, this system can be solved uniquely. After that, boundary values of  $\mathbf{b}_0 + \sum_j \mathbf{b}_j(z - b_j)^{-1}$ , call them  $\mathbf{h}_k$ , belong to  $H^+$ , and  $\mathbf{g}_k = \mathbf{f}_k + \mathbf{h}_k$  that implies  $P^*\mathbf{g} = \mathbf{f}$ , as required.

We are looking for a Baker function in the form

$$w = \Phi(z) \exp \sum_k \xi_k = \left( I + \sum_{j=1}^N B_j(z - b_j)^{-1} \right) \exp \sum_k \xi_k .$$

Here  $B_j$  are matrices. Then  $\hat{w}_k = (I + \sum_{j=1}^N B_j(z - b_j)^{-1}) \exp \sum_{i+k} \xi_i$ . We have the

following equations for elements of the matrices  $B_j$ ,

$$\Phi(\mu_i) \exp \sum_k \xi_k(\mu_i) \cdot \eta_i = 0, \quad i = 1, \dots, Nn$$

or, in coordinates,

$$\sum_{\beta=1}^n \Phi_{\alpha\beta}(\mu_i) y_{\beta i} = 0, \quad \alpha = 1, \dots, n, \quad i = 1, \dots, Nn,$$

where

$$y_{\beta i} = \exp \left[ \sum_{k=1}^m \sum_{l=1}^{\infty} t_{k\beta l} (\mu_i - a_k)^{-l} \right] \eta_{i\beta}.$$

Matrices  $B_j$  are determined by this equation uniquely. We have chosen a gauge in which  $w_i$  are boundary values of a function constant at infinity, therefore  $\partial_{k\alpha l} w_i$  are boundary values of a function vanishing at infinity and  $\{\partial_{k\alpha l} w_i\} \in (z - a_j)^{-1} W$ , i.e. this is a Baker function.

**Proposition.** *The solution to the system (14) is given by the formula*

$$\Phi_{\alpha\beta} = \frac{1}{\Delta} \begin{vmatrix} \delta_{\alpha\beta} & | & y_{\alpha 1} & \dots & y_{\alpha, Nn} \\ (z - b_1)^{-1} \delta_{1\beta} & | & (\mu_1 - b_1)^{-1} y_{11} & \dots & (\mu_{Nn} - b_1)^{-1} y_{1, Nn} \\ \vdots & | & \vdots & \dots & \vdots \\ (z - b_1)^{-1} \delta_{n\beta} & | & (\mu_1 - b_1)^{-1} y_{n1} & \dots & (\mu_{Nn} - b_1)^{-1} y_{n, Nn} \\ \vdots & | & \vdots & \dots & \vdots \\ (z - b_N)^{-1} \delta_{1\beta} & | & (\mu_1 - b_N)^{-1} y_{11} & \dots & (\mu_{Nn} - b_N)^{-1} y_{1, Nn} \\ \vdots & | & \vdots & \dots & \vdots \\ (z - b_N)^{-1} \delta_{n\beta} & | & (\mu_1 - b_N)^{-1} y_{n1} & \dots & (\mu_{Nn} - b_N)^{-1} y_{n, Nn} \end{vmatrix}$$

Here  $\Delta$  is the cofactor of the element  $\delta_{\alpha\beta}$ .

(The structure of the determinant is the following. It has  $Nn + 1$  rows and columns. All the rows except the first one can be parted into  $N$  groups,  $n$  rows in each of them. The rows, except the first one, can be labeled by  $j, \gamma$ , where  $j = 1, \dots, N$  and  $\gamma = 1, \dots, n$ . The columns, except the first one are labeled by  $i = 1, \dots, Nn$ . The non-zero entries of the first column are on the  $(j, \beta)$  places, i.e., on the  $\beta$ th place in each group, and also the upper left element if  $\alpha = \beta$ .)

*Proof.* Left-hand side of Eq. (14) is represented by a determinant where the first column coincides with  $i^{\text{th}}$ , hence it vanishes. Taking into account the division by  $\Delta$ , we see that  $\Phi$  has a desired form  $I + \sum_{j=1}^N B_j (z - b_j)^{-1}$   $\square$

### 7. Expression of the Baker Function in Terms of $\tau$ -functions for Solitons.

The next very natural topic in this context would be a  $\tau$ -function. According to the comon definition of that, introduced by Sato et al., see [1], we could expect a relation between the Baker and the  $\tau$  functions something like

$$w_{k, \alpha\beta} = \frac{\tau_{k, \alpha\beta}(t_{k_1\gamma l} - \delta_{kk_1} \delta_{\gamma\beta} l^{-1} (z - a_k)^l)}{\tau(t)} \exp \sum_i \zeta_i(z). \tag{15}$$

We do not give here a general definition of a  $\tau$ -function and restrict ourselves to the soliton-type solutions. We show that in this case the formula (15) can be written for some  $\tau$ 's, indeed.

In order to obtain  $\hat{w}_k$ , one has to multiply  $\Phi$  by  $\exp \sum_{k_1 \neq k} \zeta_{k_1}(z)$  and expand it in powers of  $z - a_k$ . There are two cases i)  $\alpha = \beta$  and ii)  $\alpha \neq \beta$ .

i) Let us transform the determinant adding the first row multiplied by  $-(z - b_j)^{-1}$  to all  $j\beta^{\text{th}}$  rows,  $j = 1, \dots, N$ , i.e., annul all elements of the first column except the first one. Expanding along the first column, we get that  $\Phi_{\beta\beta} = \prod_{j=1}^N (z - b_j)^{-1}$  multiplied by an  $Nn \times Nn$  determinant with the entries:  $(z - \mu_i)(\mu_i - b_j)^{-1} y_{\beta i}$  on the  $(\beta j, i)$  place and  $(\mu_i - b_j)^{-1} y_{\gamma i}$  on the  $(\gamma j, i)$  place for  $\gamma \neq \beta$ . An obvious identity

$$\mu_i - z = (\mu_i - a_k) \exp \left[ - \sum_{l=1}^{\infty} \frac{1}{l} \left( \frac{z - a_k}{\mu_i - a_k} \right)^l \right]$$

implies

$$\hat{w}_{k, \beta\beta} = \prod_{j=1}^N (z - b_j)^{-1} \Delta^{-1} \det(\tilde{T}_{\gamma j, i}^{k\beta\beta}) \exp \sum_{k_1 \neq k} \sum_{l=1}^{\infty} t_{k_1\beta_1} (z - a_{k_1})^{-l},$$

where  $(\tilde{T}_{\gamma j, i}^{k\beta\beta})$  with fixed  $k$  and  $\beta$  is an  $Nn \times Nn$  matrix,  $\gamma j$  is a number of a row and  $i$  that of a column,

$$\tilde{T}_{\gamma j, i}^{k\beta\beta} = \begin{cases} -\frac{\mu_i - a_k}{\mu_i - b_j} \tilde{y}_{\beta i}^{k\beta}, & \text{if } \gamma = \beta \\ 1 & \\ \mu_i - b_j & y_{\gamma i}, \end{cases} \text{ otherwise.}$$

We denoted

$$\tilde{y}_{\gamma i}^{k\beta} = \exp \left[ \sum_{k_1=1}^m \sum_{l=1}^{\infty} (t_{k_1\gamma l} - \delta_{k_1, k} \delta_{\gamma\beta} \frac{1}{l} (z - a_k)^l) (\mu_i - a_{k_1})^{-l} \right] \eta_{i\gamma}.$$

The factor  $\prod_{j=1}^N (z - b_j)^{-1}$  does not play any role in the dressing formula (4), it just cancels out. Thus, except for the factor  $\exp \sum_{k_1} \sum_{l=1}^{\infty} t_{k_1\beta_1} (z - a_{k_1})^{-l}$  the whole dependence on  $z$  is in modified time variables  $t_{k_1\gamma l} \rightarrow t_{k_1\gamma l} - \delta_{k_1, k} \delta_{\gamma\beta} l^{-1} (z - a_k)^l$ . This is just what we need in order to obtain (15). Thus, a  $\tau$  function for a matrix element  $w_{k, \beta\beta}$  is  $\tau_{k, \beta\beta} = \det T^{k\beta\beta}$ , where

$$\tilde{T}_{\gamma j, i}^{k\beta\beta} = \begin{cases} -\frac{\mu_i - a_k}{\mu_i - b_j} \tilde{y}_{\beta i}^{k\beta}, & \text{if } \gamma = \beta \\ 1 & \\ \mu_i - b_j & y_{\gamma i}, \end{cases} \text{ otherwise.}$$

ii) Now the element  $\Phi_{\alpha\beta}$  with  $\alpha \neq \beta$ . The  $(1\beta)$  row multiplied by  $(z - b_1)(z - b_j)^{-1}$  must be subtracted from the  $(j\beta)$  row, for all  $j = 1, \dots, N$ . We have

$$\hat{w}_{k, \beta\beta} = \prod_{j=1}^N (z - b_j)^{-1} \Delta^{-1} \det(\tilde{T}_{\gamma j, i}^{k\alpha\beta}) \exp \sum_{k_1 \neq k} \sum_{l=1}^{\infty} t_{k_1\beta_1} (z - a_{k_1})^{-l},$$

where

$$\tilde{T}_{\gamma j, i}^{k\alpha\beta} = \begin{cases} y_{\alpha i}, & \text{if } j = 1, \gamma = \beta \\ \frac{(\mu_i - a_k)(b_j - b_1)}{(\mu_i - b_j)(\mu_i - b_1)} \tilde{y}_{\beta i}^{k\beta}, & \text{if } j > 1, \gamma = \beta \\ \frac{1}{\mu_i - b_j} y_{\gamma i}, & \text{otherwise.} \end{cases}$$

The determinant  $\det T^{k\alpha\beta}$  is a  $\tau$ -function for  $w_{k, \alpha\beta}$ , i.e.,  $\tau_{k, \alpha\beta}$  where

$$T_{\gamma j, i}^{k\alpha\beta} = \begin{cases} y_{\alpha i}, & \text{if } j = 1, \gamma = \beta \\ \frac{(\mu_i - a_k)(b_j - b_1)}{(\mu_i - b_j)(\mu_i - b_1)} y_{\beta i}, & \text{if } j > 1, \gamma = \beta \\ \frac{1}{\mu_i - b_j} y_{\gamma i}, & \text{otherwise.} \end{cases}$$

*In this particular example we obtained the following fact. There are matrix-functions  $\tau_k(t)$  such that Eq. (15) holds, being  $\tau(t) = \Delta$ .*

It would be interesting to give a general definition of the  $\tau$ -function in terms of the Grassmannian similar to that given in a single-pole case in [4].

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