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Quantum Group $SU(1,1) \rtimes \mathbf{Z}_2$ and "Super-Tensor" Products

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Abstract: A quantum analogue of the group $SU(1,1) \rtimes \mathbf{Z}_2$ – the normalizer of $SU(1,1)$ in $SL_2(\mathbf{C})$ – is introduced and studied. Although there is *no* correctly defined tensor product in the category of $*$-representations of the quantum algebra $\mathbf{C}[SU(1,1)]_q$ of regular functions, some categories of $*$-representations of $\mathbf{C}[SU(1,1) \rtimes \mathbf{Z}_2]_q$ turn out to be endowed with a certain $\mathbf{Z}_2$*-graded* structure which can be considered as a "super"-generalization of the monoidal category structure. This "quantum effect" may be considered as a step to understanding the concept of quantum topological locally compact group.

In fact, there seems to be a *family* of quantum groups $SU(1,1) \rtimes \mathbf{Z}_2$ parameterized by unitary characters $\beta \in \mathbf{T}^1$ of the fundamental group of the two-dimensional symplectic leaf of $SU(1,1)/T$, where T is the subgroup of diagonal matrices.

It is shown that the *quasi-classical* analogues of the results of the paper are connected with the decomposition of Schubert cells of the flag manifold $SL_2(\mathbf{C})_{\mathbf{R}}/B$ (where B is the Borel subgroup of upper-triangular matrices) into symplectic leaves.

1. Introduction

The theory of quantum groups arose out from the quantum inverse scattering method (QISM) developed by L.D. Faddeev's school. The fundamental concepts of the theory were developed by V.G. Drinfeld (cf. [1]). Other approaches to quantum groups can be found in [3, 4, 15, 23].

Let us take the Hopf algebra of regular functions on a quantum algebraic complex group. A Hopf $*$-algebra structure on it is referred to as a *real* (when $q \in \mathbf{R} \setminus \{0\}$) or an *imaginary* (when $|q|=1$) form of the quantum algebraic complex group[2].

The notion of quantum *topological* real group is generally believed to involve some topological Hopf $*$-algebras which could become the analogues of algebras of

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[2] A more general approach to the notion of real and imaginary forms can be found in [14].

continuous functions. Thus, S.L. Woronowicz found in [23] a successful axiomatics for quantum *compact* groups in the language of Hopf C^*-algebras.

As to the quantum *non-compact* groups, no axiomatics does still exist, although some positive results have been obtained, those which do not require necessarily any topological algebra of functions. One can mention, for instance, harmonic analysis on the quantum group $SU(1, 1)$ developed in [9, 10, 22].

Nevertheless, an attempt undertaken by S.L. Woronowicz in [24] to construct the quantum group $SU(1, 1)$ on the C^*-algebra level has led him to a result of inexistence. Actually, this negative result is based on the fact that there is *no* correctly defined tensor product in the category of $*$-representations of the quantum algebra $\mathbf{C}[SU(1, 1)]_q$ of regular functions.

Without giving any complete axiomatics in the present paper, I have tried to look at the picture from a bit different point of view. The result is a certain "quantum effect" observed in the paper which shows that, although a quantum group $SU(1, 1)$ does not exist on the C^*-algebra level, one can possibly consider the quantum group $SU(1, 1) \rtimes \mathbf{Z}_2$ in a certain sense. Let me explain the general idea.

An intuitive reason why we would have been glad to have something like a *tensor product* in a category of $*$-representations of the quantum algebra of regular functions is a desire to have a quantum analogue of the group multiplication (as far as representations of the quantum algebra of functions correspond to symplectic leaves).

As to quantum *compact* groups, the tensor product is obviously correctly defined, since the quantum algebra of regular functions is represented always by *bounded* operators. In the case of the quantum group $M(2)$ of the motions of the Euclidean plane (cf. [8, 24]), the tensor product also can be correctly defined (because of another reason, though), and this very fact made it possible to construct in [24] the quantum group on the C^*-algebra level.

When we go onto quantum *non-compact* groups, the problem becomes quite untrivial because the quantum algebra of regular functions is represented usually by *unbounded* operators.

While in the case of the quantum group $M(2)$ this difficulty can be overcome, it is impossible to define correctly tensor product of two infinite-dimensional irreducible $*$-representations of $C[SU(1, 1)]_q$, since a certain symmetric operator *cannot be extended* to a self-adjoint one *in accordance* with the other operators of the representations, as was shown in [24].

However, there is a certain extension of the point of view. It involves two principal steps. *First*, one should go onto the quantum group $SU(1, 1) \rtimes \mathbf{Z}_2$, the normalizer of $SU(1, 1)$ in $SL_2(\mathbf{C})$. *Second*, one should abandon, at least for a while, the C^*-algebra level and look for some already existing structure on the category of $*$-representations of $\mathbf{C}[SU(1, 1) \rtimes \mathbf{Z}_2]_q$.

Is there no natural monoidal category structure? Well, but there is another, a bit surprising, structure which is worth considering. One should just notice it.

The Poisson Lie group $N_{SL_2(\mathbf{C})}(SU(1, 1)) \simeq SU(1, 1) \rtimes \mathbf{Z}_2$, the normalizer of $SU(1, 1)$ in $SL_2(\mathbf{C})$, is the union of two connected components $SU(1, 1) \cup SU(1, 1) \cdot w$ of $SL_2(\mathbf{C})$ where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. But first a couple of notation remarks.

Throughout the paper we use the following shortened notation: $\mathfrak{R}_+ = \mathbf{C}[SU(1, 1)]_q$, where $\mathfrak{R}_+ = (\mathfrak{R}, *)$, $\mathfrak{R} = \mathbf{C}[SL_2(\mathbf{C})]_q$. The quantum analogue of the algebra of regular functions on the left coset $SU(1,1)\cdot w$ is the $*$-algebra $\mathfrak{R}_- = (\mathfrak{R}, \sharp)$ introduced in Sect. 2. Then, $\mathfrak{S} = \mathfrak{R}_+ \oplus \mathfrak{R}_-$ can be equipped with a Hopf $*$-algebra structure, and may be thought of as the quantum algebra of regular functions on $SU(1,1)\rtimes \mathbf{Z}_2$ (denoted also $\mathfrak{S} = \mathbf{C}[SU(1,1)\rtimes \mathbf{Z}_2]_q$).

The main observation of the paper is as follows. For each given $\beta \in \mathbf{R}/\mathbf{Z}$, one can consider a certain subcategory $\mathscr{C}_\beta$ of the category of $*$-representations of $\mathfrak{S}$. Given infinite-dimensional irreducible $*$-representations $\pi_1^\pm, \pi_2^\mp$ of $\mathfrak{R}_\pm$ from $\mathscr{C}_\beta$, one can define a $*$-representation

$$\pi_1^+ \otimes \pi_2^+ \oplus \pi_1^- \otimes \pi_2^- \tag{1.1}$$

of $\mathfrak{R}_+ \subset \mathfrak{S}$ from $\mathscr{C}_\beta$, and a $*$-representation

$$\pi_1^+ \otimes \pi_1^- \oplus \pi_2^- \otimes \pi_2^+ \tag{1.2}$$

of $\mathfrak{R}_- \subset \mathfrak{S}$ from $\mathscr{C}_\beta$, although each direct summand in (1.1), (1.2) *cannot* be correctly defined.

The arising structure might be treated, for instance, as follows. Given a pair σ_1, σ_2 of $*$-representations of $\mathfrak{S}$ from $\mathscr{C}_\beta$, one can consider a set $P(\sigma_1, \sigma_2)$ which parameterizes "*different*" (in the sense of a certain equivalence relation) tensor products of σ_1 and σ_2:

$$\sigma_1 \otimes_p \sigma_2, \quad p \in P(\sigma_1, \sigma_2).$$

For instance, $P(\pi_1^+ \oplus \pi_1^-, \pi_2^+ \oplus \pi_2^-)$, as is easy to see, consists of just one element. If $P(\sigma_1, \sigma_2) = \emptyset$, this means that there is *no* tensor product of σ_1 and σ_2. In general, the "different" tensor products are parameterized by different "means" of assignment a term of the form "$\pi^+ \otimes \pi^\pm$" to each term of the form "$\pi^- \otimes \pi^\mp$."

Remark. The categories $\mathscr{C}_\beta$ are parameterized by unitary characters $\beta \in \mathbf{T}^1$ of the fundamental group of a non-degenerate symplectic leaf of the flag manifold $G_{\mathbf{R}}/B$, where B is the Poisson subgroup of upper-triangular matrices (see the present paper). In fact, β seems to parameterize a *family* of quantum groups $SU(1,1)\rtimes \mathbf{Z}_2$.

Now I would like to say a few words about the principal tool of the paper, namely, the quantum adjoint actions of quantum algebra of functions. Thus, the construction of the $*$-representations (1.1), (1.2) is given through some *geometric realization* where the quantum adjoint actions of $\mathfrak{R}_+$ (left and right) play a principal role.

Note that the quantum adjoint actions are defined for arbitrary Hopf algebra. However, they are quite used to be considered only in the case of quantum (or classical) universal enveloping algebras. That is why I would like to stress that throughout the paper we consider the quantum adjoint actions only of a quantum algebra of functions.

The $*$-algebra $\mathfrak{R}_-$ mentioned above is an example of what I call a shadow of $\mathfrak{R}_+$-bimodule $*$-algebra in the present paper, a term which I use mainly for convenience in order not to repeat too many words each time. This term means the following. The left and right quantum adjoint actions ad_q and ad'_q make $\mathfrak{R}_+$ into an $\mathfrak{R}_+$-bimodule $*$-algebra[3]. Roughly speaking, that $\mathfrak{R}_- = (\mathfrak{R}, \sharp)$ is a shadow of

[3] That is, both a $*$-algebra and an $\mathfrak{R}_+$-bimodule with compatible structures, see [7, 10, 11] and Sect. 3 of the present paper.

$\mathfrak{R}_+ = (\mathfrak{R}, *)$ means that ad_q and ad'_q make $\mathfrak{R}_-$ into an $\mathfrak{R}_+$-bimodule $*$-algebra as well (although $\mathfrak{R}_-$ is not a Hopf $*$-algebra).

As to the general case of a real form G_0 of a quantum simple complex Lie group G, the shadows of $\mathbf{C}[G_0]_q$ are shown to be parameterized by the finite abelian group $\bar{W} = N_G(G_0)/G_0$, where $N_G(G_0) \simeq G_0 \rtimes \bar{W}$ is the normalizer of G_0 in G. They are "bricks" in the construction of the Hopf $*$-algebra $\mathbf{C}[G_0 \rtimes \bar{W}]_q$.

We will observe that the shadows are nothing but quantum analogues of algebras of regular functions on the connected components of $G_0 \rtimes \bar{W}$.

The *quasi-classical* analogue of the left (or right) quantum adjoint action of the quantum algebra of functions is the local right (or left) *dressing action* of the Drinfeld's dual Poisson Lie group G_0^* on G_0 (or on $G_0 \rtimes \bar{W}$) respectively. The geometric realization of the $*$-representations (1.1), (1.2) constructed in the paper implies the following quasi-classical picture.

Namely, the quasi-classical analogue of some important special case of the $*$-representation (1.1) is the *global* right dressing action of G_0^* on a Schubert cell (namely, $\mathbf{C} \subset \mathbf{C}P^1$) of the flag manifold $G_{\mathbf{R}}/B$ (namely, $\mathbf{C}P^1$) (or the global left dressing action of $B \backslash G_{\mathbf{R}}$), while that of each direct summand of (1.1) is the *local* right (left) dressing action on the corresponding non-degenerate symplectic leaf of the Schubert cell (namely, the inner and the outer part of the unit disc). The fact that those direct summands cannot be correctly defined corresponds to the fact that the above-mentioned *local* action *cannot* be extended to a *global* one (recall that $SU(1,1)^*$ acts on $\mathbf{C}$ by translations and dilations).

Remark. Note that, if G_0 is compact, Schubert cells of the flag manifold are always symplectic manifolds. But if G_0 is non-compact, this is not the case.

The *structure* of the paper is as follows. First of all, note that throughout the paper $q \in \mathbf{R} \backslash \{0\}$, $|q| < 1$.

In Sect. 2 the main objects $\mathfrak{R}_\pm$ and $\mathfrak{S}$ are introduced, and their irreducible $*$-representations are described.

In Sect. 3 the notion of module $*$-algebra and an auxiliary notion of its shadows are introduced and studied. The results obtained here are applied in Sect. 4 to the case of the quantum algebra of regular functions on a real form G_0 of a quantum simple complex Lie group G. The shadows of $\mathbf{C}[G_0]_q$ are described and the Hopf $*$-algebra $\mathbf{C}[G_0 \rtimes \bar{W}]_q$ is constructed.

Throughout Sects. 5 and 6 we suppose $G_0 = SU(1,1)$. Here the geometric realization of the "tensor products" (1.1), (1.2) is obtained. In Sect. 7 we consider the quasi-classical picture. Appendix is devoted to the proof of Theorem 6.4 (about the decomposition of the $*$-representations (1.1), (1.2) into irreducible ones).

Remark. A similar effect takes place in the category of $*$-representations of the quantum universal enveloping algebra $U_q\mathfrak{su}(1,1)$ that certain sums of tensor products can be correctly defined, while each of the summands cannot (announced in [11]).

2. Algebras $\mathfrak{R}_\pm$, $\mathfrak{S}$ and their $*$-Representations

Throughout the paper $q \in \mathbf{R} \backslash \{0\}$, $|q| < 1$.

Recall that the Hopf $*$-algebra $\mathfrak{R}_+ = \mathbf{C}[SU(1,1)]_q$ is the pair $(\mathfrak{R}, *)$, where $\mathfrak{R} = \mathbf{C}[SL_2(\mathbf{C})]_q$ (cf. [1]) is the Hopf algebra generated by all the matrix elements of finite-dimensional representations of the quantum enveloping algebra $U_q\mathfrak{sl}_2(\mathbf{C})$ (cf. [1, 4]), $*$ is the antilinear involutive algebra antiautomorphism and coalgebra automorphism of $\mathfrak{R}$ given below.

Recall that $\mathfrak{R}$ is generated by t_{ij} $(i,j=1,2)$ and the relations

$$\begin{gathered}
t_{11}t_{12}=qt_{12}t_{11},\quad t_{11}t_{21}=qt_{21}t_{11}\,,\\
t_{12}t_{22}=qt_{22}t_{12},\quad t_{21}t_{22}=qt_{22}t_{21}\,,\\
t_{12}t_{21}=t_{21}t_{12},\quad t_{11}t_{22}-t_{22}t_{11}=(q-q^{-1})t_{12}t_{21}\,,\\
t_{11}t_{22}-qt_{12}t_{21}=1\,,\\
\Delta(t_{ij})=\sum_{k=1,2}t_{ik}\otimes t_{kj},\quad \varepsilon(t_{ij})=\delta_{ij}\,,\\
S\colon\begin{pmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{pmatrix}\mapsto\begin{pmatrix}t_{22} & -q^{-1}t_{12}\\ -qt_{21} & t_{11}\end{pmatrix},
\end{gathered}\tag{2.1}$$

where Δ is the comultiplication, S the antipode and ε the counit (cf. [1]).

The antilinear involutive algebra antiautomorphism $*$ given by

$$t_{11}^*=t_{22},\quad t_{12}^*=qt_{21}\tag{2.2}$$

equips $\mathfrak{R}_+$ with a Hopf $*$-algebra structure, i.e. the following conditions are satisfied (what is referred to as $*$'s being coalgebra automorphism):

$$\begin{gathered}
(ab)^*=b^*a^*,\quad \Delta(a^*)=\Delta(a)^*\,,\\
\bar{\omega}^2(a)=a,\quad \varepsilon(a^*)=\overline{\varepsilon(a)}\,,
\end{gathered}$$

for each $a, b\in\mathfrak{R}$ where

$$\bar{\omega}(a)=(S(a))^*$$

(cf. [3, 9, 10, 16]).

The quasi-classical analogue of $\mathfrak{R}_+$ is the algebra of regular functions on the real Poisson Lie group $SU(1,1)$ considered as the pair $(SL_2(\mathbf{C}),\omega)$, where ω is an involutive antiholomorphic Poisson Lie group automorphism of $SL_2(\mathbf{C})$ such that $SU(1,1)=\{g\in SL_2(\mathbf{C})\,|\,\omega(g)=g\}$.

Consider the $*$-algebra $\mathfrak{R}_-=(\mathfrak{R},\sharp)$ where $\sharp$ is the antilinear involutive algebra antiautomorphism of $\mathfrak{R}$ given by

$$t_{11}^\sharp=-t_{11}^*=-t_{22},\quad t_{12}^\sharp=-t_{12}^*=-qt_{21}\tag{2.3}$$

(note that $\sharp$ is not a coalgebra automorphism, so that $\mathfrak{R}_-$ is not a Hopf $*$-algebra).

It is easy to see that the quasi-classical analogue of $\mathfrak{R}_-$ is the algebra of regular functions on the left Poisson coset $SU(1,1)\cdot w\in SL_2(\mathbf{C})$ where $w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

We will see that $*$-representations of $\mathfrak{R}_\pm$ are usually given by *unbounded* operators. That is why one should carefully introduce the notion of $*$-representation of $\mathfrak{R}_\pm$.

2.1. Definition. *Suppose that $\mathfrak{A}$ is a $*$-algebra, $\mathfrak{H}$ a Hilbert space. A left (right) $\mathfrak{A}$-module V dense in $\mathfrak{H}$ is called unitarizable left (right) $\mathfrak{A}$-module if*[4]

$$(a \cdot v_1, v_2) = (v_1, a^* \cdot v_2) \tag{2.4}$$

for each $a \in \mathfrak{A}$, $v_1, v_2 \in V$.

Throughout the paper we put $x = t_{12} t_{21}$. This element generates the subalgebra of spherical functions (see [9, 10, 22]).

2.2 Definition. *A unitarizable left (right) $\mathfrak{R}_\pm$-module V is called self-adjoint if the operator defined on V by the action of a admits closure (denoted $\sigma_V(a)$) for any $a \in \mathfrak{R}_\pm$, and the operator $\sigma_V(x)$ is self-adjoint.*

2.3 Definition. (i) *Two self-adjoint unitarizable left (right) $\mathfrak{R}_\pm$-modules V and V' are said to be closure equivalent if $\sigma_V(a) = \sigma_{V'}(a)$ for any $a \in \mathfrak{R}_\pm$.*

(ii) *A $*$-representation of $\mathfrak{R}_\pm$ is a closure equivalence class of self-adjoint unitarizable left $\mathfrak{R}_\pm$-modules (right ones give rise to $*$-antirepresentations).*

(iii) *We say that a $*$-representation σ is irreducible if any representative of σ in the corresponding closure equivalence class is irreducible, and that σ is unitarily equivalent to σ' if there exist such representatives V and V' of σ and σ' respectively that V is unitarily equivalent to V'.*

The category of unitarizable left $\mathfrak{R}_\pm$-modules admits tensor product. Given two unitarizable left $\mathfrak{R}_{\iota_i}$-modules V_i ($\iota_i = \pm$, $i = 1, 2$), one can define the unitarizable left $\mathfrak{R}_{\iota_1 \iota_2}$-module $V_1 \otimes V_2$ as follows:

$$a\colon v_1 \otimes v_2 \mapsto \Delta(a) \cdot (v_1 \otimes v_2)$$

for each $a \in \mathfrak{R}_{\iota_1 \iota_2}$, $v_i \in V_i$ ($i = 1, 2$).

But they are of no use if we want to construct a quantum *topological* group because of an abnormally immense set of unitary equivalence classes of irreducible unitarizable left $\mathfrak{R}_\pm$-modules.

However, $*$-representations in their turn have a property which is able to drive away from further investigations. As is shown in Sect. 6, given two $*$-representations π_1, π_2, the unitarizable module $V_{\pi_1} \otimes V_{\pi_2}$ *is not* self-adjoint in general. Moreover, there does not exist a self-adjoint unitarizable left module V which extends this tensor product within the same Hilbert space, i.e. such that $V_{\pi_1} \otimes V_{\pi_2} \subset V \subset \mathfrak{H} = \overline{V_{\pi_1} \otimes V_{\pi_2}}$.

The idea how to overcome this obstacle is outlined in the introduction and realized below in this paper.

2.4. Proposition. (i) *Each irreducible $*$-representation of $\mathfrak{R}_+$ is unitarily equivalent to one of the following:*

- *one-dimensional $*$-representations ζ_φ ($\varphi \in [0, 2\pi)$), given by*

$$\zeta_\varphi\colon \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mapsto \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$$

- *infinite-dimensional $*$-representations $\pi_\varphi^{+,(\beta)}$ ($\varphi \in [0, 2\pi)$, $\beta \in (-\frac{1}{2}, \frac{1}{2})$) given by*

$$\pi_\varphi^{+,(\beta)}(t_{11})\colon e_k^+ \mapsto (1 + q^{2k+2\beta-1})^{\frac{1}{2}} e_{k-1}^+ ,$$

[4] The dot is the notation for the action on the module.

$$\pi_\varphi^{+,(\beta)}(t_{12})\colon e_k^+\mapsto e^{i\varphi}q^{k+\beta+\frac{1}{2}}e_k^+ ,$$

$$\pi_\varphi^{+,(\beta)}(t_{21})\colon e_k^+\mapsto e^{-i\varphi}q^{k+\beta-\frac{1}{2}}e_k^+ ,$$

$$\pi_\varphi^{+,(\beta)}(t_{22})\colon e_k^+\mapsto(1+q^{2k+2\beta+1})^{\frac{1}{2}}e_{k+1}^+ , \tag{2.5}$$

where $\{e_k^+\}_{k\in\mathbf{Z}}$ *is an orthonormal basis of the space of representation (called canonical).*

(ii) *Each irreducible* $*$*-representation of* $\mathfrak{R}_-$ *is unitarily equivalent to one of the infinite-dimensional* $*$*-representations* π_φ^- $(\varphi\in[0,2\pi))$ *given by*

$$\pi_\varphi^-(t_{11})\colon e_k^-\mapsto i(q^{-2(k+1)}-1)^{\frac{1}{2}}e_{k+1}^- ,$$

$$\pi_\varphi^-(t_{12})\colon e_k^-\mapsto e^{i\varphi}q^{-k}e_k^- ,$$

$$\pi_\varphi^-(t_{21})\colon e_k^-\mapsto -e^{-i\varphi}q^{-k-1}e_k^- ,$$

$$\pi_\varphi^-(t_{22})\colon e_k^-\mapsto i(q^{-2k}-1)^{\frac{1}{2}}e_{k-1}^- , \tag{2.6}$$

where $\{e_k^-\}_{k\in\mathbf{Z}_+}$ *is an orthonormal basis of the space of representation (called canonical).*

The proof is quite standard and based on the idea that, given an irreducible $*$-representation π, the spectrum of $\pi(x)$ is the closure of a segment of a *geometric progression* with multiplier q^2 (because of $t_{11}x=q^2xt_{11}$, $t_{22}x=q^{-2}xt_{22}$).

Thus, we see that

$$\operatorname{spec}\pi_\varphi^{+,(\beta)}(x)=\mathfrak{M}_+^{(\beta)}\cup\{0\},\quad \operatorname{spec}\pi_\varphi^-(x)=(\mathfrak{M}_-)^{-1} ,$$

$$\operatorname{spec}\zeta_\varphi(x)=\{0\} ,$$

where $\mathfrak{M}_+^{(\beta)}=\{q^{2(k+\beta)}\}_{k\in\mathbf{Z}}$, $\mathfrak{M}_-=\{-q^{2k+1}\}_{k\in\mathbf{Z}_+}$.

Remark. Denote $V_\varphi^{+,(\beta)}$ and V_φ^- the self-adjoint unitarizable left modules generated by $\{e_k^+\}_{k\in\mathbf{Z}}$ and $\{e_k^-\}_{k\in\mathbf{Z}_+}$ respectively.

Consider the $*$-algebra $\mathfrak{S}=\mathfrak{R}_+\oplus\mathfrak{R}_-$. It is easy to check that it can be equipped with a Hopf $*$-algebra structure as follows:

$$\Delta(a,0)=\sum_k(a_k',0)\otimes(a_k'',0)+\sum_k(0,a_k')\otimes(0,a_k'') ,$$

$$\Delta(0,a)=\sum_k(a_k',0)\otimes(0,a_k'')+\sum_k(0,a_k')\otimes(a_k'',0) ,$$

$$S(a,b)=(S(a),S(b)),\quad \varepsilon(a,b)=\varepsilon(a)$$

$(a,b\in\mathfrak{R})$ whenever $\Delta(a)=\sum_k a_k'\otimes a_k''$.

The Hopf $*$-algebra $\mathfrak{S}$ is a quantum analogue of the algebra of regular functions on the Poisson Lie group $SU(1,1)\cup SU(1,1)\cdot w\simeq SU(1,1)\rtimes\mathbf{Z}_2$, the action of $\mathbf{Z}_2$ on $SU(1,1)$ given by conjugation by w, which is nothing but the *normalizer* of $SU(1,1)$ in $SL_2(\mathbf{C})$.

The general construction of $\mathfrak{R}_-$ and $\mathfrak{S}$ will be given in the subsequent two sections.

We call a unitarizable left $\mathfrak{S}$-module $*$-representation, if its restrictions to both $\mathfrak{R}_+$ and $\mathfrak{R}_-$ give rise to $*$-representations of $\mathfrak{R}_\pm$. It is easy to see that each $*$-representation of $\mathfrak{S}$ is the direct sum of its restrictions to $\mathfrak{R}_+$ and $\mathfrak{R}_-$, and the set of unitary equivalence classes of irreducible $*$-representations of $\mathfrak{S}$ is just the union of those of $\mathfrak{R}_+$ and $\mathfrak{R}_-$.

Note that irreducible $*$-representations of $\mathfrak{R}_+$ and $\mathfrak{R}_-$ are pairs (Σ, β_Σ), where Σ is a symplectic leaf of $SU(1,1)$ or $SU(1,1)\cdot w$ respectively, β_Σ is a character of the fundamental group $\pi_1(\Sigma)$ of the leaf. Their quasi-classical analogues are realized in the sections of the corresponding linear bundles.

In particular, ζ_φ corresponds to the one-dimensional leaf $\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$, while $\pi_\varphi^{+,(\beta)}$ corresponds to the pair $(\Sigma_\varphi^+, \beta\in\mathbf{R}/\mathbf{Z})$, where Σ_φ^+ is the two-dimensional symplectic leaf of $SU(1,1)$ given by $t_{12}\neq 0$, $\arg t_{12}=\varphi$ (note that Σ_φ^+ is equivalent to the outer part $\mathscr{D}_+=\{z\in\mathbf{C}\,|\,|z|>1\}$ of the unit disc). At last, π_φ^- corresponds to the two-dimensional symplectic leaf Σ_φ^- of $SU(1,1)$ given by $t_{12}\neq 0$, $\arg t_{12}=\varphi$ (note that Σ_φ^- is equivalent to the inner part $\mathscr{D}_-=\{z\in\mathbf{C}\,|\,|z|<1\}$ of the unit disc).

It is a little bit more convenient to parameterize irreducible $*$-representations of $\mathfrak{S}$ by quadruples $(\varphi, C, \Sigma, \beta_\Sigma)$, where φ corresponds to a point of the maximal torus $T_0\subset SU(1,1)$ of diagonal matrices, C is a Schubert cell of the flag manifold $SL_2(\mathbf{C})_\mathbf{R}/B\simeq\mathbf{C}P^1$ (where B is the Borel subgroup of the upper-diagonal matrices), namely, $\{\infty\}$ or $\mathbf{C}$, Σ is a *non-degenerate* symplectic leaf of C (that is, such that $\dim\Sigma=\dim C$), namely, $\{\infty\}$ or $\mathscr{D}_\pm$ respectively, β_Σ is a character of $\pi_1(\Sigma)$. The non-degenerate symplectic leaves of the flag manifolds are the images of the corresponding symplectic leaves of $SU(1,1)\rtimes\mathbf{Z}_2\subset SL_2(\mathbf{C})$ via the canonical projection.

Remark. From now on we *fix* β and consider the subcategory $\mathscr{C}_\beta$ of such $*$-representations of $\mathfrak{S}$ that the spectrum of $\sigma(x)$ is a subset of $\mathfrak{M}_+^{(\beta)}\cup\mathfrak{M}_-^{-1}\cup\{0\}$ for each $*$-representation σ from $\mathscr{C}_\beta$. Sometimes β will be omitted for convenience in such expressions as $\pi_\varphi^{+,(\beta)}$, $V_\varphi^{+,(\beta)}$ and the like.

3. Module $*$-Algebras and Shadows

The construction of $\mathfrak{R}_-$ is based on the notion of *module $*$-algebras* and an auxiliary notion of their *shadows*. Namely, as is shown below, $\mathfrak{R}_+$ is equipped with a natural $\mathfrak{R}_+$-bimodule $*$-algebra structure given by the *left* and *right adjoint actions* ad_q and ad'_q. We will see that $\mathfrak{R}_-$ is nothing but the only non-trivial shadow of this $\mathfrak{R}_+$-bimodule $*$-algebra. We will show in the subsequent section how this construction might be generalized to the case of arbitrary real form of a quantum simple complex Lie group.

Recall first the notion of $\mathfrak{A}_0$-module $*$-algebra (which can be found also in [7, 10, 11]). Throughout the paper we suppose that all the algebras are "good enough" (that is, satisfy some very natural conditions).

3.1. Definition. (i) *Suppose that $\mathscr{F}$ is both a $*$-algebra and a left (right) $\mathfrak{A}_0$-module. Then, $\mathscr{F}$ is called left (right) $\mathfrak{A}_0$-module $*$-algebra if*

- *the multiplication map $m\colon \mathscr{F}\otimes\mathscr{F}\to\mathscr{F}$ is an $\mathfrak{A}_0$-module morphism,*
- *The $*$-structures in $\mathfrak{A}_0$ and $\mathscr{F}$ are compatible in the following sense:*

$$(a\cdot f)^*=\bar{\omega}(a)\cdot f^* \tag{3.1}$$

for each $a\in\mathfrak{A}_0, f\in\mathscr{F}$, where $\bar{\omega}(a)=(S(a))^*$.

(ii) *Suppose that* $\mathscr{F}_1$ *and* $\mathscr{F}_2$ *are left (right)* $\mathfrak{A}_0$*-module algebras. A map* $A: \mathscr{F}_1 \to \mathscr{F}_2$ *is called left (right)* $\mathfrak{A}_0$*-module* $*$*-algebra morphism if* A *is both an algebra* $*$*-homomorphism and a left (right)* $\mathfrak{A}_0$*-module morphism.*

Remark. Suppose that $\mathfrak{A}$ is a Hopf algebra, $\mathscr{F}$ is both an algebra and a left (right) $\mathfrak{A}$-module. The definition of left (right) $\mathfrak{A}$*-module algebra* can be obtained by removing all the symbols '$*$' and the condition (3.1) from the above definition.

3.2. Definition. (i) *Let* $\mathscr{F}$ *be a left (right)* $\mathfrak{A}$*-module algebra,* $(\mathscr{F}, *)$ *a left (right)* $\mathfrak{A}_0$*-module* $*$*-algebra. If a* $*$*-algebra* $(\mathscr{F}, \sharp)$ *equipped with the action of* $\mathfrak{A}$ *on* $\mathscr{F}$ *becomes a left (right)* $\mathfrak{A}_0$*-module* $*$*-algebra, we will call it left (right) pre-semishadow of* $(\mathscr{F}, *)$.

Two left (right) pre-semishadows are called equivalent if they are equivalent as left (right) $\mathfrak{A}_0$*-module* $*$*-algebras.*

(ii) *A left (right) pre-semishadow* $(\mathscr{F}, \sharp)$ *of* $(\mathscr{F}, *)$ *is called left (right) semishadow if there exists at least one unitarizable left (right)* $(\mathscr{F}, \sharp)$*-module.*

3.2.′ Definition. *Let* $(\mathscr{F}, *)$ *be a* $\mathfrak{A}_0$*-bimodule* $*$*-algebra. If a* $*$*-algebra* $(\mathscr{F}, \sharp)$ *equipped with the left and right actions of* $\mathfrak{A}_0$ *becomes both a left and a right semishadow of* $(\mathscr{F}, *)$*, we will call it shadow of* $(\mathscr{F}, *)$.

One of the examples of $\mathfrak{A}_0$-module $*$-algebra is the algebra of functions on (*quantum*) *real* G_0*-spaces* where G_0 is a real Lie group.

Recall that in the reasonable examples one can take $\mathfrak{B}_0 \subset \mathfrak{A}_0^*$ so that the Hopf $*$-algebra structure on $\mathfrak{A}_0$ induces a Hopf $*$-algebra structure on $\mathfrak{B}_0$ as follows:

$$\langle \xi\eta, a \rangle = \langle \xi \otimes \eta, \Delta(a) \rangle, \quad \langle \Delta(\xi), a \otimes b \rangle = \langle \xi, ab \rangle ,$$

$$\langle S(\xi), a \rangle = \langle \xi, S(a) \rangle, \quad \varepsilon(\xi) = \langle \xi, 1 \rangle ,$$

$$\langle \xi^*, a \rangle = \overline{\langle \xi, \bar{\omega}(a) \rangle}, \quad \langle \bar{\omega}(\xi), a \rangle = \overline{\langle \xi, a^* \rangle} , \tag{3.2}$$

where $a, b \in \mathfrak{A}_0$, $\xi, \eta \in \mathfrak{B}_0$, $\langle \cdot , \cdot \rangle$: $\mathfrak{B}_0 \otimes \mathfrak{A}_0 \subset \mathfrak{A}_0^* \otimes \mathfrak{A}_0 \to \mathbf{C}$ is the natural pairing.

Recall the left and right $\mathfrak{B}_0$-module $*$-algebra structures on $\mathfrak{A}_0$ given by the well known *right regular representation* and *the left regular antirepresentation* respectively:

$$\mathscr{R}(\xi)a = \langle \mathrm{id} \otimes \xi, \Delta(a) \rangle, \quad \mathscr{L}(\xi)a = \langle \xi \otimes \mathrm{id}, \Delta(a) \rangle ,$$

where $\xi \in \mathfrak{B}_0$, $a \in \mathfrak{A}_0$.

When $\mathfrak{A}_0 = \mathbf{C}[G_0]_q$ is the quantum algebra of regular functions, we can put $\mathfrak{B}_0 = U_q\mathfrak{g}_0 = (U_q\mathfrak{g}, *)$. In this case $\mathscr{R}$ and $\mathscr{L}$ are the quantum analogues of the right and left regular representations of G_0 in functions on G_0 respectively.

Remark. Note that the left or right semishadows of $\mathbf{C}[G_0]_q$ with respect to $\mathscr{R}$ or $\mathscr{L}$ are easily seen to be the quantum analogues of the algebras of regular functions on *Poisson right cosets* $gG_0 \subset G_{\mathbf{R}}$ or *left* ones $G_0 g$ (where $g \in G_{\mathbf{R}}$) respectively.

Therefore, if we consider $\mathbf{C}[G_0]_q$ as the corresponding $U_q\mathfrak{g}_0$-bimodule $*$-algebra, its shadows correspond to the connected components of the Poisson Lie group $N_G(G_0)$, the normalizer of G_0 in $G_{\mathbf{R}}$.

The following example of $\mathfrak{A}_0$-module $*$-algebra is of a particular interest for us. The proposition given below is a direct consequence of the definition of Hopf $*$-algebra.

3.3. Proposition. *Let $\mathfrak{A}_0$ be a Hopf $*$-algebra. The left and right quantum adjoint actions* ad_q *and* ad_q' *of* $\mathfrak{A}_0$ *on itself given by*

$$\mathrm{ad}_q(a)b=\sum_k a_k' b S(a_k''), \quad \mathrm{ad}_q'(a)b=\sum_k S(a_k')ba_k''$$

whenever $\Delta(a)=\sum_k a_k' \otimes a_k''(a, b\in\mathfrak{A}_0)$ *equip* $\mathfrak{A}_0$ *with an* $\mathfrak{A}_0$*-bimodule* $*$*-algebra structure.*

This case is considered in the present paper for quantum algebras of regular functions on real quantum groups. Some simple examples follow.

Example 1. It is easy to show that $\mathfrak{R}_+=(\mathfrak{R}, *)$ and $\mathfrak{R}_-=(\mathfrak{R}, \sharp)$ are the only pre-shadows (and shadows) of $\mathfrak{R}_+$, and that they are not equivalent.

Example 2. Consider $\mathbf{C}[SU(2)]_q=(\mathfrak{R}, \natural)$, where $\natural$ is given by

$$t_{11}^{\natural}=t_{22}, \quad t_{12}^{\natural}=-qt_{21} .$$

It is easy to see that the only pre-shadows of $(\mathfrak{R}, \natural)$ are $(\mathfrak{R}, \natural)$ and $(\mathfrak{R}, \flat)$, where $\flat$ is given by

$$t_{11}^{\flat}=-t_{11}^{\natural}=-t_{22}, \quad t_{12}^{\flat}=-t_{12}^{\natural}=qt_{21} .$$

However, $(\mathfrak{R}, \flat)$ is not a shadow because of

$$t_{11}^{\flat}t_{11}+t_{21}^{\flat}t_{21}=-1 . \tag{3.3}$$

We will see later that, when G_0 is compact, $\mathbf{C}[G_0]_q$ has only *one* shadow, namely, itself (the situation is trivial).

Denote $\mathfrak{T}$ the subgroup of non-zero group-like elements of $\mathfrak{A}^*$, that is, $\mathfrak{T}=\{t\in \mathfrak{A}^* \mid t\neq 0, \Delta(t)=t\otimes t\}$. Denote $\mathfrak{T}_0$ the subgroup of non-zero Hermitian group-like elements of $\mathfrak{A}_0^*$, that is, $\mathfrak{T}_0=\{t\in\mathfrak{T} \mid t^*=t\}$.

3.4. Theorem. (i) $(\mathfrak{A}, \sharp)$ *is a left or right pre-semishadow of* $\mathfrak{A}_0=(\mathfrak{A}, *)$ *if and only if there exists* $t_0\in\mathfrak{T}_0$ *such that*

$$\sharp=*\circ\mathscr{R}(t_0) \quad or\ \sharp=*\circ\mathscr{L}(t_0)$$

respectively.

(ii) *Left or right pre-semishadows* $(\mathfrak{A}, \sharp)$ *and* $(\mathfrak{A}, \natural)$ *are equivalent if and only if there exists* $t\in\mathfrak{T}$ *such that*

$$\natural=\mathscr{R}(t)\circ\sharp\circ\mathscr{R}(t^{-1}) \quad or\ \natural=\mathscr{L}(t)\circ\sharp\circ\mathscr{L}(t^{-1})$$

respectively.

(iii) $(\mathfrak{A}, \sharp)$ *is a left* (*or right*) *pre-semishadow of* $\mathfrak{A}_0$ *with respect to* ad_q (*or* ad_q') *if and only if it is a right pre-semishadow with respect to* $\mathscr{L}$ (*or a left one with respect to* $\mathscr{R}$).

The theorem immediately follows from the following easy lemma.

3.5. Lemma. *Let* $\mathfrak{A}$ *be a Hopf algebra, and suppose that* $\mathfrak{A}$ *is equipped with the left or right* $\mathfrak{A}$*-module algebra structure given by* ad_q *or* ad_q' *respectively. A linear map* $\gamma\colon \mathfrak{A}\to\mathfrak{A}$ *is a left or right* $\mathfrak{A}$*-module algebra automorphism if and only if there exists* $t\in\mathfrak{T}$ *such that* $\gamma=\mathscr{R}(t)$ *or* $\gamma=\mathscr{L}(t)$ *respectively.*

Remark. From now on we call left (right) (pre)-semishadows and shadows of $\mathfrak{A}_0$ with respect to the left (right) quantum adjoint action simply left (right) (pre)-semishadows and shadows without a special reference to the action.

3.6. Theorem. *The left pre-semishadow* $(\mathfrak{A}, \sharp_t)$ *is a left semishadow if and only if there exists* $u \in \mathfrak{A}_0^*$ *such that* $t = \bar{\omega}(u^{-1})u$, $\varepsilon(u) = 1$ [5].

3.7. Corollary. $(\mathfrak{A}, \sharp)$ *is a shadow if and only if there exists* $u \in \mathfrak{A}_0^*$ *such that* $\bar{\omega}(u^{-1})u \in \mathfrak{Z}_0$ *where* $\mathfrak{Z}_0$ *is the intersection of* $\mathfrak{T}_0$ *and the center of* $\mathfrak{A}_0^*$, *and*

$$\begin{aligned}\sharp &= \mathscr{R}(u^{-1}) \circ * \circ \mathscr{R}(u) \\ &= \mathscr{L}(u^{-1}) \circ * \circ \mathscr{L}(u)\,.\end{aligned}$$

4. The Normalizer of a Quantum Real Form

Let G_0 be a real form of a simple complex Lie group G such that there exists a compact Cartan subgroup of G_0 [6], $\mathfrak{g}_0$ and $\mathfrak{g}$ the Lie algebras of G_0 and G respectively. Recall that $\mathfrak{g}_0 = \{\xi \in \mathfrak{g} \mid \omega_0(\xi) = -\xi^* = \xi\}$ and $G_0 = \{g \in G \mid \omega(g) = g\}$, where ω_0 is an antilinear involutive automorphism of $\mathfrak{g}$, and ω is the corresponding antiholomorphic involutive automorphism of G.

Choose a compact Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}_0$, and let $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$ be the corresponding Cartan subalgebra in $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ (which depends, of course, on the choice of positive roots).

Throughout the paper we consider the *standard* Poisson Lie group structure on G_0 given by the Manin triple $(\mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_0, \mathfrak{g}_0^*)$ where $\mathfrak{g}_0^* = \mathfrak{n}_+ \oplus i\mathfrak{t}$. Both $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$ are isotropic with respect to the non-degenerate symmetric bilinear scalar product on $g_{\mathbf{R}}$ given by imaginary part of the Killing form (the definition of Manin triple can be found in [1]).

The quantization of this Poisson Lie group structure is the quantum algebra $\mathfrak{A}_0 = \mathbf{C}[G_0]_q$ of regular functions. Recall that $\mathfrak{A}_0 = (\mathfrak{A}, *)$, where $\mathfrak{A} = \mathbf{C}[G]_q$ is the Hopf algebra generated by matrix elements of finite-dimensional representations of the quantum universal enveloping algebra $U_q\mathfrak{g}$ (cf. [1]), $* = S \circ \bar{\omega}$, where $\bar{\omega}$ is the quantization of ω_0. Various concrete examples of quantum real forms (and imaginary ones, that is when $|q| = 1$) can be found in [3].

Remark. Note that, if we consider the standard Poisson Lie group structures described above with respect to different choices of positive roots of $\mathfrak{h}$, we get, in general, non-isomorphic Poisson Lie groups parameterized by W/W_1, where W is the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$, W_1 is a subgroup of W described below in the section.

For instance, when $G = SL_{m+n}(\mathbf{C})$, $G_0 = SU(m, n)$, the non-equivalent standard Poisson Lie group structures on $SU(m, n)$ give rise to non-isomorphic quantum algebras of regular functions introduced in [3] and denoted there by

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5 For right semishadows the condition looks as follows: $t = u\bar{\omega}(u^{-1})$.

6 This condition is necessary to consider a quantum *real* form (that is, when $q \in \mathbf{R} \setminus \{0\}$). For instance, in the case $G_0 = SL_n(\mathbf{R})$ (cf. [3]) we would have a quantum *imaginary* form instead (that is, when $|q| = 1$).

$\mathbf{C}[SU(\pm 1, \pm 1, \ldots, \pm 1)]_q$, where the number of pluses and the number of minuses are m and n respectively.

We consider the following *real* Poisson Lie group structure on the real group $G_{\mathbf{R}}$. Namely, we see from the above Manin triple that $\mathfrak{g}_{\mathbf{R}}$ is the double Lie algebra with respect to $\mathfrak{g}_0$. The induced real Poisson Lie group structure on $G_{\mathbf{R}}$ is given by the Manin triple $(\mathfrak{g}_{\mathbf{R}} \oplus \mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_0 \oplus \mathfrak{g}_0^*)$, where $\mathfrak{g}_{\mathbf{R}}$ is embedded into $\mathfrak{g}_{\mathbf{R}} \oplus \mathfrak{g}_{\mathbf{R}}$ as the diagonal.

Recall that the subgroup of zero-dimensional symplectic leaves of G_0 is *the* maximal compact torus $T_0 = \exp \mathfrak{t}$, and that of $G_{\mathbf{R}}$ is *the* maximal torus $T = \exp \mathfrak{h}$. The center Z_0 of G_0 (which coincides with the center of G in the case when there exists a compact Cartan subgroup of G_0) is a finite subgroup of T_0.

As is well known, there exists a *natural isomorphism* $\mathfrak{i}\colon \mathfrak{T} \to T$ such that $\mathfrak{i}\mathfrak{T}_0 = T_0$ and $\mathfrak{i}\mathfrak{Z}_0 = Z_0$, where $\mathfrak{Z}_0$ is the intersection of $\mathfrak{T}_0$ with the center of $\mathfrak{A}^*$.

Given a left (or right) pre-semishadow $(\mathbf{C}[G]_q, \sharp_t)$, $(t \in \mathfrak{T}_0$, consider its quasi-classical analogue $(\mathbf{C}[G], \sharp_{t'})$, where $t' = \mathfrak{i}(t), f^{\sharp_{t'}}(g) = f(\omega_{t'}(g)^{-1})$, $\omega_{t'}(g) = \omega(gt')$ (or $\omega_{t'}(g) = \omega(t'g)$ respectively). As follows from Theorem 3.4.(iii), the set of equivalence classes of irreducible $*$-representations of this Poisson algebra, i.e. the set $X_{t'} = \{g \in G_{\mathbf{R}} \mid \omega_{t'}(g) = g\}$, must be either empty or coincide with a Poisson left coset $G_0 u \subset G_{\mathbf{R}}$ (or a Poisson right coset uG_0 respectively).

It is clear that the left (right) pre-semishadow $(\mathbf{C}[G]_q, \sharp_t)$ is a left (right) semishadow if and only if $X_{t'}$ is not empty. Hence, shadows correspond to such Poisson cosets that $G_0 u = uG_0$ which means that u belongs to the normalizer of G_0.

We see that a left coset $G_0 u$ is the set of fixed points with respect to the involution $g \mapsto \omega(g \cdot u^{-1}) \cdot u = \omega(g) \cdot \omega(u^{-1})u$. Therefore, it is a Poisson submanifold of $G_{\mathbf{R}}$ if and only if $\omega(u^{-1})u$ is a zero-dimensional symplectic leaf of G, that is, $\omega(u^{-1})u \in T$ (in fact, it belongs to T_0), which is the direct analogue of Theorem 3.6.

It is easy to show that it is equivalent to $u \in G_0 N(T)$, where $N(T)$ is the normalizer of T. Therefore, equivalence classes of left semishadows of $\mathbf{C}[G_0]_q$ are parameterized by $G_0 \backslash G_0 N(T)/T \simeq W_0 \backslash W$, where W is the Weyl group of $\mathfrak{g}$, W_0 is the subgroup of W (not normal in general) generated by simple reflections with respect to *compact* roots. Analogously one can show that right semishadows are naturally parameterized by W/W_0.

As to the shadows, they correspond to such $u \in G_{\mathbf{R}}$ that $\omega(u^{-1})u$ belongs not simply to T_0 but to Z_0. This means that $ugu^{-1} = \omega(u) g \omega(u)^{-1}$ for each $g \in G$. It follows that the conjugation by u commutes with ω, therefore, $u \in N_G(G_0)$, where $N_G(G_0)$ is the normalizer of G_0 in G.

Thus, we see that shadows of $\mathbf{C}[G_0]_q$ are parameterized by the finite group $\bar{W} = N_G(G_0)/G_0$. Note that $\bar{W}$ is in fact Abelian, since the homomorphism $\bar{\tau}\colon \bar{W} \to Z_0$ induced by $\tau\colon N_G(G_0) \to Z$, $g \mapsto \omega(g^{-1})g$ is easily seen to be injective.

Let us return to the quantum picture. One can show that the involution on $\mathbf{C}[G]_q$ which defines the left semishadow corresponding to $W_0 w$ $(w \in W)$ can be given by

$$\sharp = \mathscr{R}(\tilde{w}^{-1}) \circ * \circ \mathscr{R}(\tilde{w}) , \tag{4.1}$$

where $\tilde{w}$ is the element of the *quantum Weyl group* corresponding to $w \in W$. Analogously, for the right semishadow corresponding to wW_0 one has

$$\sharp = \mathscr{L}(\tilde{w}^{-1}) \circ * \circ \mathscr{L}(\tilde{w}) . \tag{4.2}$$

The quantum Weyl group was introduced in [20] and studied, for instance, in [6, 12, 18–20]. Recall that the quantum analogue of $w\in W$ is a certain Gelfand–Naimark–Segal state $\tilde{w}\in \mathbf{C}[G]_q^*$ with respect to certain irreducible representations of $\mathbf{C}[G]_q$ and certain vectors in the spaces of the representations.

As far as the shadows are concerned, note that $\bar{W}\simeq W_1/W_0$, where W_1 is the subgroup of W which consists of the elements whose action on $\mathfrak{h}$ commutes with the Cartan involution $\theta=\omega_0\circ\omega_{0,\mathrm{comp}}$ and preserves roots of $\mathfrak{t}$ in $\mathfrak{h}$. Therefore, the involution on $\mathbf{C}[G]_q$ which defines the shadow corresponding to $\bar{w}\in\bar{W}$ can be given by either (4.1) or (4.2) where $w\in W_1$ represents $\bar{w}$.

Recall that the homomorphism $\bar{\tau}\colon \bar{W}\to Z_0$ is injective. Therefore, the shadows corresponding to distinct elements of $\bar{W}$ are not equivalent. Note also that, if G_0 is compact, $\bar{W}$ is trivial.

Let us summarize the obtained results in the following statements.

4.1. Theorem. (i) *The equivalence classes of left semishadows of* $\mathbf{C}[G_0]_q$ *are parameterized by* $W_0\backslash W$, *with the involution on* $\mathbf{C}[G]_q$ *given by* (4.1), *where w represents the corresponding coset from* $W_0\backslash W$.

(ii) *The equivalence classes of right semishadows of* $\mathbf{C}[G_0]_q$ *are parameterized by* W/W_0, *with the involution on* $\mathbf{C}[G]_q$ *given by* (4.2), *where w represents the corresponding coset from* W/W_0.

(iii) *The equivalence classes of shadows of* $\mathbf{C}[G_0]_q$ *are parameterized by the finite Abelian group* $\bar{W}\simeq W_1/W_0\simeq N_G(G_0)/G_0$, *with the involution on* $\mathbf{C}[G]_q$ *given by either* (4.1) *or* (4.2), *where* $w\in W_1$ *represents the corresponding element of* W_1/W_0.

Remark. (i) The quasi-classical analogues of the left (or right) semishadows of $\mathbf{C}[G_0]_q$ are the algebras of regular functions on the Poisson left (or right) cosets of the form G_0w (or wG_0 respectively), where $w\in N(T)$.

(ii) The quasi-classical analogues of the shadows of $\mathbf{C}[G_0]_q$ are the algebras of regular functions on the connected components of $N_G(G_0)\simeq G_0\rtimes\bar{W}$, the normalizer of G_0 in G.

Now we are going to construct the quantum algebra $\mathbf{C}[G_0\rtimes\bar{W}]_q$ of regular functions on the quantum *disconnected* group $G_0\rtimes\bar{W}$. It is given by the following proposition.

4.2. Theorem. *Define a $*$-algebra* $\mathbf{C}[G_0\rtimes\bar{W}]_q$ *as the coproduct of all its shadows in the category of $*$-algebras*:

$$\mathbf{C}[G_0\rtimes\bar{W}]_q=\bigoplus_{\bar{w}\in\bar{W}}(\mathbf{C}[G]_q,\sharp_{\bar{w}}), \tag{4.3}$$

where $\sharp_{\bar{w}}$ *defines the shadow corresponding to* $\bar{w}\in\bar{W}$. *This means that all the summands in* (4.3) *are $*$-subalgebras of the coproduct, and the product of any elements from distinct summands is zero.*

The following formulae define a Hopf $*$-algebra structure on $\mathbf{C}[G_0\rtimes\bar{W}]_q$:

$$\Delta(j_{\bar{w}}(a))=\sum_{\bar{w}_1\bar{w}_2=\bar{w}}(j_{\bar{w}_1}\otimes j_{\bar{w}_2})\Delta(a)\,,$$

$$S(j_{\bar{w}}(a))=j_{\bar{w}^{-1}}(S(a)),\quad \varepsilon(j_{\bar{w}}(a))=\delta_{\bar{w},1}\varepsilon(a)\,, \tag{4.4}$$

where $\bar{w}\in\bar{W}$, $a\in\mathbf{C}[G]_q$, $j_{\bar{w}}\colon \mathbf{C}[G]_q\to\mathbf{C}[G_0\rtimes\bar{W}]_q$ *is the embedding of the* $\bar{w}$*-th summand in* (4.3).

Remark. When $G_0 = SU(1,1)$, the Hopf $*$-algebra $\mathbf{C}[G_0 \rtimes \bar{W}]_q$ is nothing but the Hopf $*$-algebra $\mathfrak{S} = \mathbf{C}[SU(1,1) \rtimes \mathbf{Z}_2]_q$ introduced in Sect. 2.

Remark. Note that the group $\bar{W}$ is usually too small for the above construction to be sufficient to obtain a result in the general case similar to the result obtained in the following sections in the case $G_0 = SU(1,1)$. For instance, when $G_0 = SU(m,n)$, $\bar{W} \simeq \mathbf{Z}_2$ if $m = n$ and is trivial otherwise. So, while there are embeddings of $SU(2)$'s and $SU(1,1)$'s into G_0 corresponding to the embeddings of $\mathfrak{sl}(2)$-triples into $\mathfrak{g}$, they cannot be lifted to embeddings of the normalizers.

Recall, however, that we have several standard Poisson Lie group structures on G_0. They all induce the same Poisson Lie group structure on their double group $G_{\mathbf{R}}$, so they can be considered as different Poisson Lie subgroups of $G_{\mathbf{R}}$. It seems to be likely that a generalization of the result obtained in the present paper might involve in some sense all those Poisson Lie subgroups and their normalizers.

5. Geometric Realization of Tensor Products of Unitarizable Modules

Throughout Sect. 5 and Sect. 6 we consider the case of $\mathfrak{S} = \mathbf{C}[SU(1,1) \rtimes \mathbf{Z}_2]_q$. In this section we make use of the left quantum adjoint action in order to obtain some geometric realization of tensor products of *irreducible self-adjoint unitarizable left $\mathfrak{S}$-modules.* In the subsequent section this realization enables us to construct the $*$*-representations* (1.1), (1.2) of $\mathfrak{S}$.

We call sometimes an $\mathfrak{S}$-module $\mathfrak{R}_\pm$-module if the action of $\mathfrak{R}_\mp \subset \mathfrak{S}$ on it is trivial.

Let $\mathrm{Fun}_q^\pm$ be the $*$-algebra generated by t_{ij} $(i,j = 1,2)$, functions of a real variable x, relations (2.1) and

$$t_{11} f(x) = f(xq^2) t_{11}, \quad t_{12} f(x) = f(x) t_{12},$$

$$t_{22} f(x) = f(xq^{-2}) t_{22}, \quad t_{21} f(x) = f(x) t_{21},$$

$$t_{12} t_{21} = x$$

with the involution given by (2.2) on Fun_q^+ and by (2.3) on Fun_q^-.

The $\mathfrak{R}_+$-module $*$-algebra structure can be obviously extended from $\mathfrak{R}_\pm \subset \mathrm{Fun}_q^\pm$ to the whole $\mathrm{Fun}_q^\pm$.

Consider the $\mathfrak{R}_+$-module $*$-algebra $\mathrm{Fun}(\mathscr{H})_q^\pm$ of $\mathrm{Fun}_q^\pm$ generated by $y = t_{11} t_{12}$, $y^* = q^2 t_{22} t_{21}$ and functions of x. Note that the involutions $*$ and $\sharp$ coincide on $\mathrm{Fun}(\mathscr{H})_q^\pm$, therefore, $\mathrm{Fun}(\mathscr{H})_q^+ = \mathrm{Fun}(\mathscr{H})_q^-$ which we denote simply $\mathrm{Fun}(\mathscr{H})_q$ from now on.

$\mathrm{Fun}(\mathscr{H})_q$ first appeared in [10] as an algebra of functions on a quantum two-sheet hyperboloid $\mathscr{H}$. One of its sheets can be realized as $SU(1,1)/T_0$ and the other one as $SU(1,1) \cdot w/T_0$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

What we consider is a quantum analogue of the transformation of the hyperboloid $\mathscr{H} = \{x \in \mathbf{R},\ y \in \mathbf{C} \mid |y|^2 = x(x+1)\}$ such that its symplectic leaf $\{x > 0\}$ turns into $\mathscr{D}_+ = \{|z| > 1\}$, the leaf $\{x \leqq -1\}$ into $\mathscr{D}_- = \{|z| < 1\}$, and the zero-dimensional leaf $\{x = 0,\ y = 0\}$ goes away into infinity $\infty \in \mathbf{C}P^1 \simeq SL_2(\mathbf{C})_{\mathbf{R}}/B \supset (SU(1,1) \rtimes \mathbf{Z}_2)/T_0$, where B is the Borel subgroup of upper-triangular matrices.

Namely, consider the $\mathfrak{R}_+$-module $*$-algebra $\mathrm{Fun}(\mathbf{C})_q$ generated by $z=t_{11}t_{21}^{-1}$, $z^*=t_{22}t_{12}^{-1}$ and functions of $r=x^{-1}$. In the new generators the relations look as follows:

$$\begin{gathered} zf(r)=f(rq^{-2})z, \quad zz^*=1+q^{-1}r\,, \\ z^*f(r)=f(rq^2)z^*, \quad z^*z=1+qr\,, \end{gathered} \tag{5.1}$$

$$t_{11}\colon z^kf(r)\mapsto q^{-k+1}z^k\frac{(1+q^{-1}r)f(rq^{-2})-f(r)}{r}\,,$$

$$t_{12}\colon z^kf(r)\mapsto -q^{-k-1}z^{k+1}\frac{f(r)-f(rq^2)}{r}\,,$$

$$t_{21}\colon (z^*)^kf(r)\mapsto q^{k+1}(z^*)^{k+1}\frac{f(rq^{-2})-f(r)}{r}\,,$$

$$t_{22}\colon (z^*)^kf(r)\mapsto -q^{k-1}(z^*)^k\frac{f(r)-(1+qr)f(rq^2)}{r}\,, \tag{5.2}$$

where $k\in\mathbf{Z}$ (as follows from (5.1), $z^{-1}=z^*(1+q^{-1}r)^{-1}$).

There are the following unitary equivalence classes of infinite-dimensional irreducible $*$-representations of $\mathrm{Fun}(\mathbf{C})_q$: π^+ (the restriction of π_φ^+) corresponding to $\mathscr{D}_+$ and π^- (the restriction of π_φ^-) corresponding to $\mathscr{D}_-$. There are also the one-dimensional $*$-representations $z\mapsto e^{i\varphi}$, $z^*\mapsto e^{-i\varphi}$, $f(r)\mapsto f(0)$ corresponding to the points of $S^1=\{|z|=1\}$.

Let $C_c^\infty(\mathscr{D}_+)_q^{(\beta)}$ and $C_c^\infty(\mathscr{D}_-)_q$ be the ideals in $\mathrm{Fun}(\mathbf{C})_q$ generated by those functions of r whose supports are *finite* subsets of the geometric progressions $\mathfrak{M}_+^{(-\beta)}=\{q^{2(k-\beta)}\}_{k\in\mathbf{Z}}$ and $\mathfrak{M}_-=\{-q^{2k+1}\}_{k\in\mathbf{Z}_+}$ respectively (which are the sets of eigen-values of r in the corresponding $*$-representations $\pi^{+,(\beta)}$ and π^-).

It is easy to see that $C_c^\infty(\mathscr{D}_\pm)_q$ is an $\mathfrak{R}_+$-module $*$-algebra. By the construction, it can be thought of as a quantum analogue of the algebra of smooth functions in $\mathscr{D}_\pm$ with compact supports.

5.1. Definition. *Suppose that $\mathfrak{A}_0$ is a Hopf $*$-algebra, $\mathscr{F}$ is an $\mathfrak{A}_0$-module $*$-algebra. A linear functional $\nu\colon\mathscr{F}\to\mathbf{C}$ is called quasi-invariant integral on $\mathscr{F}$ if there exists a positive group-like element $\chi\in\mathfrak{A}_0^*$ such that*

$$\begin{gathered} \nu(a\cdot f)=\langle\chi,a\rangle\nu(f)\,, \\ \nu(f^*)=\overline{\nu(f)}, \quad \nu(f^*f)>0\ (f\neq 0) \end{gathered} \tag{5.3}$$

for each $a\in\mathfrak{A}_0$, $f\in\mathscr{F}$.

If $\chi=1$, ν is called invariant integral on $\mathscr{F}$.

5.2. Proposition. *The linear functional $\nu_\pm\colon C_c^\infty(\mathscr{D}_\pm)_q\to\mathbf{C}$ given by*

$$\nu_\pm(f)=(q^{-1}-q)\cdot\mathrm{tr}\,\pi^\pm(fr)$$

is a quasi-invariant integral on $C_c^\infty(\mathscr{D}_\pm)_q$, the associated element $\chi\in\mathfrak{R}_+^*$ given by $\chi=\chi_0^2=\chi_0^*\chi_0$, where the group-like element χ_0 is given by

$$\left\langle\chi_0,\begin{pmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{pmatrix}\right\rangle=\begin{pmatrix}q^{\frac{1}{2}} & 0\\ 0 & q^{-\frac{1}{2}}\end{pmatrix}.$$

Proof. It is easy to see that

$$\nu_{\pm}(z^k f(r)) = \delta_{k,0} \cdot (q^{-1} - q) \cdot \sum_{r \in \mathfrak{M}_{\pm}} f(r) r \,. \tag{5.4}$$

The remainder of the proof is just a straightforward computation with use of (5.2) and (5.4). One can show that the quasi-invariant integral is unique up to multiplying by a positive constant.

It is easy to see that the *unique* up to unitary equivalence irreducible $*$-representation $\pi^{\pm}$ of $C_c^{\infty}(\mathscr{D}_{\pm})_q$ is *faithful*. Therefore, $C_c^{\infty}(\mathscr{D}_{\pm})_q$ may be identified with the image

$$\mathscr{F}(V_{\varphi}^{\pm}) = C_c^{\infty}(\mathscr{D}_{\pm})_q|_{V_{\varphi}^{\pm}} \,,$$

which is the algebra of such linear operators in $V_{\varphi}^{\pm}$ that their matrices with respect to the canonical basis $\{e_k^{\pm}\}$ contain finitely many non-zero elements.

Let us equip $\mathscr{F}(V_{\varphi}^{\pm})$ with an $\mathfrak{R}_+$-module structure by twisting that on $C_c^{\infty}(\mathscr{D}_{\pm})_q$:

$$a : f|_{V_{\varphi}^{\pm}} \mapsto (\gamma(a) \cdot f)|_{V_{\varphi}^{\pm}} \,, \tag{5.5}$$

where $a \in \mathfrak{R}_+$, $f \in C_c^{\infty}(\mathscr{D}_{\pm})_q$ and $\gamma = \mathscr{R}(\chi_0^{-1})$ is an automorphism of $\mathfrak{R}$.

Consider the scalar product on $\mathscr{F}(V_{\varphi}^{\pm})$ given by

$$(f_1, f_2) = \nu_{\pm}(f_2^* f_1) \,. \tag{5.6}$$

5.3. Theorem. *The $\mathfrak{R}_+$-module $\mathscr{F}(V_{\varphi}^{\pm})$ is unitarizable and unitary equivalent to $V_{\varphi}^{\pm} \otimes V_{\pi+\varphi}^{\pm}$, a unitary intertwiner given by*

$$e_m^{\pm} \otimes e_n^{\pm} \mapsto (q^{-1} - q)^{-\frac{1}{2}} e^{\pm i(m-n)\varphi} |r_n^{(\pm)}|^{-\frac{1}{2}} \times (-q r_n^{(\pm)})_{\mp(m-n)}^{-\frac{1}{2}} z^{\mp(m-n)} \delta_n^{(\pm)}(r) \,, \tag{5.7}$$

where $r_n^{(+)} = q^{-2(n+\beta)} \in \mathfrak{M}_+^{(-\beta)}$, $r_n^{(-)} = -q^{2n+1} \in \mathfrak{M}_-$, and $\delta_n^{(\pm)}(r)$ is the "δ-function" on $\mathfrak{M}_{\pm}$ which takes the value 1 at the point $r_n^{(\pm)}$ and 0 at all other points, $(a)_{\alpha} = (a)_{\infty}/(q^{2\alpha}a)_{\infty}$, $(a)_{\infty} = \prod_{k=0}^{\infty}(1 - aq^{2k})$.

Proof. That $\mathscr{F}(V_{\varphi}^{\pm})$ is unitarizable is provided by (5.3), (5.5) and (5.6).

Let us prove that $\mathscr{F}(V_{\varphi}^{\pm})$ is unitarily equivalent to $V_{\varphi}^{\pm} \otimes V_{\pi+\varphi}^{\pm}$. Consider the conjugate $\mathfrak{R}_+$-module $(V_{\varphi}^{\pm})^*$ defined as follows:

$$\langle a \cdot \psi, v \rangle = \langle \psi, S(a) \cdot v \rangle$$

for each $a \in \mathfrak{R}_+$, $\psi \in (V_{\varphi}^{\pm})^*$, $v \in V_{\varphi}^{\pm}$. Note that $(V_{\varphi}^{\pm})^*$ is not unitarizable.

Since the action of $\mathfrak{R}_+$ on $C_c^{\infty}(\mathscr{D}_{\pm})_q$ comes from the quantum adjoint action, we see that the $\mathfrak{R}_+$-modules $C_c^{\infty}(\mathscr{D}_{\pm})_q$ and $V_{\varphi}^{\pm} \otimes (V_{\varphi}^{\pm})^*$ are equivalent, the intertwiner being given by the obvious identifications of these both to a space of finite-dimensional linear operators in $V_{\varphi}^{\pm}$. As soon as we twist the action on $C_c^{\infty}(\mathscr{D}_{\pm})_q$ so as to get the action on $\mathscr{F}(V_{\varphi}^{\pm})$, this intertwiner becomes a unitary operator.

Recall now that $\Delta(\gamma(a)) = (\mathrm{id} \otimes \gamma)\Delta(a)$ for each $a \in \mathfrak{R}_+$. Therefore, $\mathscr{F}(V_{\varphi}^{\pm})$ is unitarily equivalent to $V_{\varphi}^{\pm} \otimes (V_{\varphi}^{\pm})_{\gamma}^*$, where $(V_{\varphi}^{\pm})_{\gamma}^*$ is the unitarizable module obtained from $(V_{\varphi}^{\pm})^*$ by twisting the action of $\mathfrak{R}_+$ by γ. It is an easy computation to see that $(V_{\varphi}^{\pm})_{\gamma}^*$ is unitary equivalent to $V_{\pi+\varphi}^{\pm}$, the unitary intertwiner given by

$$J : e_n^{\pm} \mapsto (q^{-1} - q)^{-\frac{1}{2}} \cdot |r_n^{(\pm)}|^{-\frac{1}{2}} \psi_n^{\pm} \,,$$

where $\{\psi_n^{\pm}\}$ is the basis of $(V_{\varphi}^{\pm})^*$ dual to the basis $\{e_n^{\pm}\}$ of $V_{\varphi}^{\pm}$ (that is, $\langle \psi_m^{\pm}, e_n^{\pm} \rangle = \delta_{mn}$). This proves our assertion.

The explicit expression (5.7) is obtained as follows. The right-hand side is equal to $(q^{-1}-q)^{-\frac{1}{2}} \cdot |r_n^{(\pm)}|^{-\frac{1}{2}} E_{mn}$, where E_{mn} is the operator in $V_{\varphi}^{\pm}$ given by $E_{mn}: e_k^{\pm} \mapsto \delta_{kn} e_m^{\pm}$. Now we use the formulae (2.5), (2.6).

Let $\mathscr{F}(V_{\varphi_2}^{\pm}, V_{\varphi_1}^{\pm})$ and $\mathscr{F}(V_{\varphi_2}^{\pm}, V_{\varphi_1}^{\mp})$ be the vector spaces of such linear operators from $V_{\varphi_2}^{\pm}$ to $V_{\varphi_1}^{\pm}$ and to $V_{\varphi_1}^{\mp}$ respectively that their matrices with respect to the canonical bases $\{e_k^{\pm}\}$ contain finitely many non-zero elements. Consider the $\mathfrak{R}$-module structure on $\mathscr{F}(V_{\varphi_2}^{\pm}, V_{\varphi_1}^{\pm})$ and $\mathscr{F}(V_{\varphi_2}^{\pm}, V_{\varphi_1}^{\mp})$ given by

$$a: f \mapsto \sum_k \pi_{\varphi_1}^{\pm}(a_k') f \pi_{\varphi_2}^{\pm}(S(\gamma(a_k''))) \quad \text{and}$$

$$a: f \mapsto \sum_k \pi_{\varphi_1}^{\mp}(a_k') f \pi_{\varphi_2}^{\pm}(S(\gamma(a_k''))) \quad \text{respectively}$$

whenever $\Delta(a) = \sum_k a_k' \otimes a_k''$ (and, therefore, $\Delta(\gamma(a)) = \sum_k a_k' \otimes \gamma(a_k'')$).

Consider the scalar products on $\mathscr{F}(V_{\varphi_2}^{\pm}, V_{\varphi_1}^{\pm})$ and $\mathscr{F}(V_{\varphi_2}^{\pm}, V_{\varphi_1}^{\mp})$ given by

$$(f_1, f_2) = \nu_{\pm}((f_1, f_2)_F), \quad \text{where} \quad (f_1, f_2)_F = f_2^* f_1$$

is an $\mathscr{F}(V_{\varphi_2}^{\pm})$-valued scalar product which is $\mathscr{F}(V_{\varphi_2}^{\pm})$-linear with respect to the right action of $\mathscr{F}(V_{\varphi_2}^{\pm})$.

The idea of the proof of the following theorem does not differ significantly from that of the proof of Theorem 5.3.

5.4. Theorem. (i) *The $\mathfrak{R}_+$-module $\mathscr{F}(V_{\varphi_2}^{\pm}, V_{\varphi_1}^{\pm})$ is unitarizable and unitarily equivalent to $V_{\varphi_1}^{\pm} \otimes V_{\pi+\varphi_2}^{\pm}$.*

(ii) *The $\mathfrak{R}_-$-module $\mathscr{F}(V_{\varphi_2}^{\pm}, V_{\varphi_1}^{\mp})$ is unitarizable and unitarily equivalent to $V_{\varphi_1}^{\mp} \otimes V_{\pi+\varphi_2}^{\pm}$.*

6. "Super-Tensor" Products of Irreducible $*$-Representations

Now we make use of the geometric realization of tensor products of irreducible self-adjoint unitarizable $\mathfrak{S}$-modules obtained in the previous section in order to construct the "super-tensor" products (1.1) and (1.2). But first we explain why we cannot correctly define tensor products "without the quotes."

The following result (actually, its $\mathfrak{R}_+$-part) was first obtained in [24]. We give another proof based on the geometric realization what provides us also with a nice quasi-classical analogues of the results.

6.1. Theorem. (i) *There does not exist a self-adjoint unitarizable $\mathfrak{R}_+$-module V dense in the Hilbert space $\mathfrak{H} = \overline{V_{\varphi}^{\pm} \otimes V_{\pi+\varphi}^{\pm}}$ and such that*

$$V_{\varphi}^{\pm} \otimes V_{\pi+\varphi}^{\pm} \subset V\,.$$

(ii) *There does not exist self-adjoint unitarizable $\mathfrak{R}_{\pm}$-modules $V_{\pm}$ dense in the Hilbert spaces $\mathfrak{H}_+ = \overline{V_{\varphi_1}^{\pm} \otimes V_{\pi+\varphi_2}^{\pm}}$ and $\mathfrak{H}_- = \overline{V_{\varphi_1}^{\mp} \otimes V_{\pi+\varphi_2}^{\pm}}$ respectively and such that*

$$V_{\varphi_1}^{\pm} \otimes V_{\pi+\varphi_2}^{\pm} \subset V_+, \quad V_{\varphi_1}^{\mp} \otimes V_{\pi+\varphi_2}^{\pm} \subset V_-\,.$$

Remark. In other words, there are no correctly defined tensor products $V_{\varphi_1}^{\pm} \otimes V_{\pi+\varphi_2}^{\pm}$ and $V_{\varphi_1}^{\mp} \otimes V_{\pi+\varphi_2}^{\pm}$.

Proof of Theorem 6.1. (i) Consider the action of $x = t_{12}t_{21}$ in the subspace of $\mathfrak{H}$ generated by the vectors of the form $e_k^{\pm} \otimes e_k^{\pm}$. By (5.7), the closure of this operator is unitary equivalent to the closure of the operator given by the action of $\gamma(x) = x$ in the subspace of $C_c^{\infty}(\mathscr{D}_{\pm})_q$ generated by functions $f(r)$.

The L^2-closure of this subspace is $L^2(\mathfrak{M}_{\pm}, dv_{\pm})$, where the measure $dv_{\pm}$ is given by $\int f(r)dv_{\pm} = v_{\pm}(f(r))$ and $\mathfrak{M}_+$ is supposed to be $\mathfrak{M}_+^{(-\beta)}$ (recall that $r = x^{-1}$). By (5.2), our operator is the minimal closed operator in this space given by the second order q-difference expression

$$x : f(r) \mapsto -(q^{-1}-q)^2 \cdot D(1+r)Df(r)\,, \tag{6.1}$$

where $(Df)(r) = \dfrac{f(rq^{-1}) - f(rq)}{rq^{-1} - rq}$.

This operator is symmetric but not self-adjoint. Therefore, $V_{\varphi}^{\pm} \otimes V_{\pi+\varphi}^{\pm}$ does not give rise to a $*$-representation of $\mathfrak{R}_+$. Assume that there exists a self-adjoint extension V of $V_{\varphi}^{\pm} \otimes V_{\pi+\varphi}^{\pm}$, and come to a contradiction.

By the assumption, the closure $\sigma(x)$ of the operator given by the action of x in V restricted to $L^2(\mathfrak{M}_{\pm}, dv_{\pm})$ is a self-adjoint extension of the minimal closed operator given by (6.8).

As is well known, any self-adjoint extension is given by a boundary condition of the form

$$\cos\alpha \cdot f(\pm 0) + \sin\alpha \cdot (Df)(\pm 0) = 0\,. \tag{6.2}$$

$L^2(\mathfrak{M}_{\pm}, dv_{\pm})$ is invariant also with respect to $\sigma(t_{11})$, as follows from (5.2). However, as (5.2) shows, $\sigma(t_{11})$ does not respect the initial domain of $\sigma(x)$ as it is given by (6.2). Thus, we have come to a contradiction.

(ii) This statement follows from the previous one in the following way. For instance, assume that there exists a self-adjoint extension $\tilde{\mathscr{F}}(V_{\varphi_2}^{\pm}, V_{\varphi_1}^{\pm})$ of $\mathscr{F}(V_{\varphi_2}^{\pm}, V_{\varphi_1}^{\pm})$.

Let $\tilde{\mathscr{F}}(V_{\varphi_2}^{\pm})$ be the maximal algebra of operators in $V_{\varphi_2}^{\pm}$ such that $vf \in \tilde{\mathscr{F}}(V_{\varphi_2}^{\pm}, V_{\varphi_1}^{\pm})$ for each $v \in \tilde{\mathscr{F}}(V_{\varphi_2}^{\pm}, V_{\varphi_1}^{\pm})$, $f \in \tilde{\mathscr{F}}(V_{\varphi_2}^{\pm})$. One can show that $\tilde{\mathscr{F}}(V_{\varphi_2}^{\pm})$ can be equipped with an $\mathfrak{R}_+$-module structure which extends that of $\mathscr{F}(V_{\varphi_2}^{\pm})$ so that it gives rise to a $*$-representation of $\mathfrak{R}_+$, what contradicts with the assumption.

Remark. We will see in Sect. 7 that the quasi-classical analogue of the $\mathfrak{R}_+$-module $*$-algebra structure on $C_c^{\infty}(\mathscr{D}_{\pm})_q$ is the local action of the dual Poisson Lie group $SU(1,1)^*$ (isomorphic to the group of the matrices of the form $\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix}$) in $\mathscr{D}_{\pm}$ by translations and dilations.

Thus, the quasi-classical analogue of the above negative result is such an obvious fact that this *local* action cannot be extended to a *global* one. The obstacle at $r = \pm 0$ which prevents to construct a self-adjoint extension of the tensor product in the proof of Theorem 6.1 corresponds to the obstacle at $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$.

However, although the local action of $SU(1,1)^*$ cannot be extended to a global one on $\mathscr{D}_+$ and $\mathscr{D}_-$ separately, it *can* be extended to a global action on $\mathbf{C} = \overline{\mathscr{D}_+ \cup \mathscr{D}_-}$.

This observation prompts us what to do in the quantum case. Namely, one should consider the problem of self-adjoint extensions of $V_{\varphi_1}^{+,(\beta)} \otimes V_{\pi+\varphi_1}^{+,(\beta)} \oplus V_{\varphi_2}^{-} \otimes V_{\pi+\varphi_1}^{-}$ using its realization as

$$\mathscr{F}(V_{\varphi_1}^{+,(\beta)}) \oplus \mathscr{F}(V_{\varphi_2}^{-}) = (C_c^{\infty}(\mathscr{D}_+)_q \oplus C_c^{\infty}(\mathscr{D}_-)_q)|_{\pi_{\varphi_1}^{+,(\beta)} \oplus \pi_{\varphi_2}^{-}}\,.$$

Let $C_c^\infty(\mathbf{C})_q$ be the extension of $C_c^\infty(\mathscr{D}_+)_q \oplus C_c^\infty(\mathscr{D}_-)_q$ which is the ideal in $\mathrm{Fun}(\mathbf{C})_q$ generated by those functions of r whose supports are compact subsets of $\mathfrak{M}^{(-\beta)} = \mathfrak{M}_+^{(-\beta)} \cup \mathfrak{M}_- \cup \{0\}$ and which are *smooth at zero*. Consider the extension

$$\tilde{\mathscr{F}}^{(\beta)}_{\varphi_1, \varphi_2} = C_c^\infty(\mathbf{C})_q|_{\pi^{+,(\beta)}_{\varphi_1} \oplus \pi^-_{\varphi_2}}$$

of $\mathscr{F}(V^{+,(\beta)}_{\varphi_1}) \oplus \mathscr{F}(V^-_{\varphi_2})$, and equip it with the $\mathfrak{R}_+$-module structure given by (5.5) and with the scalar product given by (5.6).

6.2. Theorem. *The unitarizable $\mathfrak{R}_+$-module $\tilde{\mathscr{F}}^{(\beta)}_{\varphi_1, \varphi_2}$ is self-adjoint. We denote the corresponding $*$-representation of $\mathfrak{R}_+$ by $\pi^{+,(\beta)}_{\varphi_1} \otimes \pi^{+,(\beta)}_{\varphi_1+\pi} \oplus \pi^-_{\varphi_2} \otimes \pi^-_{\varphi_2+\pi}$.*

Proof. Indeed, the minimal closed operator in $L^2(\mathfrak{M}^{(-\beta)}, dv)$, where $dv = dv_+ + dv_-$, given by the second order q-difference expression (6.8) and the boundary condition

$$f(+0) = f(-0), \quad (Df)(+0) = (Df)(-0) \tag{6.3}$$

is easily seen to be self-adjoint. The operators given by the action of x in other x-invariant subspaces generated by elements of the form $z^k f(r)$ for each fixed $k \in \mathbf{Z}$ can be considered analogously (another way is to notice that all these parts of the operator given by the action of x are intertwined by the action of powers of t_{12} or t_{21}).

Finally, it is easy to see that all operators given by the actions of t_{ij} $(i,j=1,2)$ respect the smoothness condition. This proves our assertion.

Remark. $C_c^\infty(\mathbf{C})_q$ can be thought of as a quantum analogue of the algebra of smooth functions on $\mathbf{C}$ with compact supports.

Consider now the problem of self-adjoint extensions of the unitarizable $\mathfrak{S}$-module $V^+_{\varphi_1} \otimes V^\pm_{\varphi_2} \oplus V^-_{\varphi_3} \otimes V^\mp_{\varphi_4}$ denoted for convenience by $V_\pm$.

Note that $V_\pm$ is endowed with an $\mathfrak{S}$-equivariant Hilbert $(\tilde{\mathscr{F}}_{\varphi_1, \varphi_3}, \tilde{\mathscr{F}}_{\varphi_2, \varphi_4})$-bimodule structure. This means that there are the left $\tilde{\mathscr{F}}_{\varphi_1, \varphi_3}$-module and the right $\tilde{\mathscr{F}}_{\varphi_2, \varphi_4}$-module structures $m_l: \tilde{\mathscr{F}}_{\varphi_1, \varphi_3} \otimes V_\pm \to V_\pm$ and $m_r: V_\pm \otimes \tilde{\mathscr{F}}_{\varphi_2, \varphi_4} \to V_\pm$ respectively given by

$$(f_1, f_3) \otimes (v', v'') \mapsto (f_1 v', f_3 v''),$$

$$(v', v'') \otimes (f_2, f_4) \mapsto (v' f_2, v'' f_4),$$

where $(v', v'') \in V_\pm$, $(f_i, f_j) \in \tilde{\mathscr{F}}_{\varphi_i, \varphi_j}$ as well as the $\tilde{\mathscr{F}}_{\varphi_1, \varphi_2}$-linear scalar product $(\cdot,\cdot)_l: V_\pm \otimes V_\pm \to \tilde{\mathscr{F}}_{\varphi_1, \varphi_3}$ and the $\tilde{\mathscr{F}}_{\varphi_2, \varphi_4}$-linear scalar product $(\cdot,\cdot)_r: V_\pm \otimes V_\pm \to \tilde{\mathscr{F}}_{\varphi_2, \varphi_4}$ given by

$$((v_1, v_2), (v_3, v_4))_l = (v_1 v_3^*, v_2 v_4^*),$$

$$((v_1, v_2), (v_3, v_4))_r = (v_3^* v_1, v_4^* v_2),$$

such that $m_l, m_r, (\cdot,\cdot)_l$ and $(\cdot,\cdot)_r$ are $\mathfrak{S}$-module morphisms.

Let $\tilde{\mathscr{F}}_\pm$ be the maximal $\mathfrak{S}$-equivariant Hilbert $(\tilde{\mathscr{F}}_{\varphi_1, \varphi_3}, \tilde{\mathscr{F}}_{\varphi_2, \varphi_4})$-bimodule which extends $V_\pm$. The following theorem can be deduced from Theorem 6.2.

6.3. Theorem. *The unitarizable $\mathfrak{R}_\pm$-module $\tilde{\mathscr{F}}_\pm$ is self-adjoint. We denote the corresponding $*$-representation of $\mathfrak{R}_\pm$ by $\pi^+_{\varphi_1} \otimes \pi^\pm_{\varphi_2} \oplus \pi^-_{\varphi_3} \otimes \pi^\mp_{\varphi_4}$.*

Another way to do it is to construct explicitly these "tensor products" in a way similar to the construction of the "tensor product" of Theorem 6.2. The key step here is to find an appropriate self-adjoint extension of the operator given by the action of x restricted to a certain x-invariant subspace.

For instance, the self-adjoint operator $(\pi^{+,(\beta)}_{\varphi_1^{(+)}} \otimes \pi^{+,(\beta)}_{\varphi_2^{(+)}} \oplus \pi^{-}_{\varphi_1^{(-)}} \otimes \pi^{-}_{\varphi_2^{(-)}})(x)$ restricted to the subspace generated by vectors of the form $e_k^{\pm} \otimes e_k^{\pm}$ is unitarily equivalent to the operator in $L^2(\mathfrak{M}^{(-\beta)}\,dv)$ given by (6.8) and (6.3), the intertwiner given by (5.7), where $m=n=k$, the right-hand side multiplied by $(-1)^k e^{ik(\varphi_2^{(\pm)}-\varphi_1^{(\pm)})}$.

Also, the self-adjoint operator $(\pi^{+,(\beta)}_{\varphi_1} \otimes \pi^{-}_{\varphi_2} \oplus \pi^{-}_{\varphi_3} \otimes \pi^{+,(\beta)}_{\varphi_4})(x)$ restricted to the subspace generated by vectors of the form $e^{\mp}_{\pm k} \otimes e^{\pm}_{\mp k}$ $(k \in \mathbf{Z}_+)$ is unitarily equivalent to the operator in $L^2(\mathfrak{N}, dv')$, where $\mathfrak{N}=\mathfrak{N}_+ \cup \mathfrak{N}_- \cup \{0\}$, $\mathfrak{N}_\pm=\{\pm q^{2k-\beta}\}_{k\in\mathbf{Z}_+}$, and $\int f(t)\,dv'=(q^{-1}-q)\cdot\sum_{t\in\mathfrak{N}} tf(t)$, given by the second order q-difference expression

$$
\begin{aligned}
x: f(t) \mapsto (q^{-1}-q)^2 \cdot [p(t) \\
&+ D\sqrt{(1-q^{-\beta}t)(1+q^{\beta+1}t)}\,D\kappa_+(t) \\
&+ D\sqrt{(1+q^{-\beta}t)(1-q^{\beta+1}t)}\,D\kappa_-(t)]\,f(t)
\end{aligned}
\tag{6.4}
$$

(where $p(t)$ is a certain function) and the boundary condition (6.3), the intertwiner given by

$$
e^{\mp}_{\pm k} \otimes e^{\pm}_{\mp k} \mapsto (q^{-1}-q)^{-\frac{1}{2}} \cdot i^k e^{ik(\varphi_1^{(\mp)}-\varphi_2^{(\mp)})} |t_k^{(\pm)}|^{-\frac{1}{2}} \delta_k^{(\pm)}(t)\,,
$$

where $\varphi_i^{(+)}=\varphi_i$, $\varphi_i^{(-)}=\varphi_{i+2}$, $t_k^{(\pm)}=\pm q^{2k-\beta}$, $\delta_k^{(\pm)}$ is the "δ-function" on $\mathfrak{N}$ at the point $t_k^{(\pm)}$ and $\kappa_\pm(t)=\sum_{k\in\mathbf{Z}_+} \delta_k^{(\pm)}(t)$ is the characteristic function on $\mathfrak{N}_\pm$.

The following theorem is proven in the Appendix. It confirms our right to fix β and develop and independent theory for each fixed value of β.

Namely, consider the category $\mathscr{C}_\beta$ of $*$-representations of $\mathfrak{S}$ such that the spectrum of the operator which represents $x=(x,x)$ is contained in $\mathfrak{M}_+^{(\beta)} \cup \mathfrak{M}_-^{-1} \cup \{0\}$. Theorem 6.4 shows that $\mathscr{C}_\beta$ is "closed" with respect to "tensor products."

6.4. Theorem.

$$
\begin{gathered}
\zeta_{\varphi_1} \otimes \pi^{+,(\beta)}_{\varphi_2} \simeq \pi^{+,(\beta)}_{\varphi_1+\varphi_2}, \quad \zeta_{\varphi_1} \otimes \pi^{-}_{\varphi_2} \simeq \pi^{-}_{\varphi_1+\varphi_2}\,, \\
\pi^{+,(\beta)}_{\varphi_1} \otimes \zeta_{\varphi_2} \simeq \pi^{+,(\beta)}_{\varphi_1-\varphi_2}, \quad \pi^{-}_{\varphi_1} \otimes \zeta_{\varphi_2} \simeq \pi^{-}_{\varphi_1-\varphi_2}\,, \\
\zeta_{\varphi_1} \otimes \zeta_{\varphi_2} \simeq \zeta_{\varphi_1+\varphi_2}\,,
\end{gathered}
\tag{6.5}
$$

$$
\pi^{+,(\beta)}_{\varphi_1} \otimes \pi^{+,(\beta)}_{\varphi_2} \oplus \pi^{-}_{\varphi_3} \otimes \pi^{-}_{\varphi_4} \simeq \oplus \int_0^{2\pi} \pi^{+,(\beta)}_{\varphi}\,d\varphi\,,
\tag{6.6}
$$

$$
\pi^{+,(\beta)}_{\varphi_1} \otimes \pi^{-}_{\varphi_2} \oplus \pi^{-}_{\varphi_3} \otimes \pi^{+,(\beta)}_{\varphi_4} \simeq \oplus \int_0^{2\pi} \pi^{-}_{\varphi}\,d\varphi\,.
\tag{6.7}
$$

Remark. Note that, although the "super-tensor" products (6.6) (and (6.7)) are unitarily equivalent for different values of β, there is no canonical unitary equivalence, since the corresponding quantum Clebsch–Gordan coefficients *depend* on β (see the Appendix).

7. The "Tensor Products" and the Dressing Action on the Flag Manifold

In this section we consider the quasi-classical analogues of the results obtained above. First of all, we must consider the quasi-classical analogue of the left (right) quantum adjoint action. This can be done on a more general level.

Let G_0 be a real form of a simple complex Lie group G such that there exists a compact Cartan subgroup of G_0. Consider the standard Poisson Lie group structure on G_0. This structure as well as the induced real Poisson Lie group structure on $G_{\mathbf{R}}$ are described in Sect. 4.

Note that one can consider two different quasi-classical analogues of the quantum algebra $\mathbf{C}[G_0]_q$ of regular functions. The first one is the *Poisson Hopf $*$-algebra* $\mathbf{C}[G_0]$ *of regular functions* on the Poisson algebraic group G_0, the Poisson brackets given by

$$\{f_1 \bmod h\hat{\mathfrak{A}}_0, f_2 \bmod h\hat{\mathfrak{A}}_0\} = h^{-1}[f_1, f_2] \bmod h\hat{\mathfrak{A}}_0 ,$$

where $\hat{\mathfrak{A}}_0 = \mathbf{C}[G_0]_q \otimes \mathbf{C}[[h]]$ is the QFSH-algebra obtained as the quantization of $\mathbf{C}[G_0] \simeq \hat{\mathfrak{A}}_0 / h\hat{\mathfrak{A}}_0$, according to [1] (we suppose $q = e^{-\frac{h}{2}} \in \mathbf{C}[[h]]$).

The second analogue is the *universal enveloping algebra* $U\mathfrak{g}_0^*$ *of the dual Lie bialgebra*. More precisely, these two analogues correspond to some QFSH- and QUE-algebras respectively which are not isomorphic over $\mathbf{C}[[h]]$, but become isomorphic over $\mathbf{C}$ once h is fixed.

As is well known from [1], there is a covariant functor which gives an equivalence of the categories of QFSH- and QUE-algebras. For instance, the QUE-algebra corresponding to the QFSH-algebra $\mathbf{C}[SU(1, 1)]_q \otimes \mathbf{C}[[h]]$ is generated over $\mathbf{C}[[h]]$ by $t'_{ij} = h^{-\delta_{ij}} t_{ij}$ $(i, j = 1, 2)$. Its quasi-classical analogue is $U\mathfrak{su}(1, 1)^*$. The precise description of the functor can be found in [1].

To consider the quasi-classical analogue of the left (right) quantum adjoint action of $\mathbf{C}[G_0]_q$ on itself, one should combine the above two quasi-classical analogues so that we get a left (right) action of $U\mathfrak{g}_0^*$ on $\mathbf{C}[G_0]$. Let us show that it is the *local right* (*left*) *dressing action* of G_0^* on G_0 made into a left (right) action in the usual way. It is just what is called for convenience in this paper left (right) dressing action.

Recall first the definition of the right (left) dressing action (cf. [17, 13]). Suppose that $\mathfrak{g}_0^*$ is realized as the Lie algebra of right (left) invariant differential 1-forms on G_0, the Lie brackets given by

$$[\alpha_1, \alpha_2] = -d\langle \pi\alpha_1, \alpha_2 \rangle + L_{\pi\alpha_2}\alpha_1 - L_{\pi\alpha_1}\alpha_2 ,$$

where L stands for the Lie derivative, $\langle \cdot, \cdot \rangle$: $TG_0 \times T^*G_0 \to \mathbf{C}$ is the natural pairing between the tangent and cotangent bundles, π: $T^*G_0 \to TG_0$ is the *bundle map* associated with the Poisson manifold structure on G_0.

For each $\alpha \in \mathfrak{g}_0^*$, let α_r (α_l respectively) be the right (left) invariant differential 1-form on G_0 such that $\alpha_r(1) = \alpha$ ($\alpha_l(1) = \alpha$ respectively). The map $\alpha \mapsto -\pi\alpha_r$ ($-\pi\alpha_l$ respectively) from $\mathfrak{g}_0^*$ into the Lie algebra of smooth vector fields on G_0 is a Lie algebra homomorphism (antihomomorphism).

The vector fields $-\pi\alpha_r$ ($-\pi\alpha_l$ respectively) are called *right* (*left*) *dressing fields*. They give rise to a *local* left (right) action of G_0^* on G_0 called *right* (*left*) *dressing action*.

Consider the quasi-classical analogue of ad_q. When h tends to zero, the action of $\mathbf{C}[G_0]_q$ on itself given by

$$a\colon b \mapsto h^{-1}(\mathrm{ad}_q - \varepsilon(a))b$$

tends to the action of $\mathbf{C}[G_0]$ on itself given by

$$a\colon b(g) \mapsto \{a(gg'), b(g)\}|_{g'=g^{-1}} .$$

It is easy to see that it is nothing but the differentiation along the right dressing field $-\pi((da)(1)_r)$. The quasi-classical limit of ad'_q can be considered analogously.

At last, the quasi-classical analogue of the action of $\mathbf{C}[G_0]_q$ on its left (right) semishadows is easily seen to be the local right (left) dressing action of G_0^* on $G_{\mathbf{R}}$ restricted to the corresponding Poisson left (right) coset, since G_0^* is canonically embedded into $G_{\mathbf{R}}^* \simeq G_0 \times G_0^*$ (see Sect. 4).

As follows from (4.4), the left (right) quantum adjoint action of $\mathbf{C}[G_0 \rtimes \bar{W}]_q$ is, in fact, a $\mathbf{C}[G_0]_q$-action. Its quasi-classical analogue can be considered also as the local right (left) dressing action of $(G_0 \rtimes \bar{W})^* \simeq G_0^*$ on $G_0 \rtimes \bar{W}$.

Now we consider the quasi-classical analogues of the results obtained in Sect. 5 and Sect. 6. Suppose $G_0 = SU(1,1)$, $G = SL_2(\mathbf{C})$. Recall that in this case $\bar{W} \simeq \mathbf{Z}_2$ and $SU(1,1) \rtimes \mathbf{Z}_2$ is embedded into $SL_2(\mathbf{C})$ as $SU(1,1) \cup SU(1,1)\cdot w$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Recall also that $SU(1,1)^*$ is isomorphic to the group of translations and dilations of the plane and is embedded into $SL_2(\mathbf{C})$ as the group of matrices of the form $\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix}$, where $t>0$, $z\in\mathbf{C}$.

It underlines the negative result of Theorem 6.1 that the *local* right (left) dressing action of $SU(1,1)^*$ on either $SU(1,1)$ or $SU(1,1) \rtimes \mathbf{Z}_2$ *cannot* be extended to a *global* one.

Let us compare our situation with the case of the compact real form G_0 of G. In this case Iwasawa's decomposition holds and the global right (for instance) dressing action $g_-\colon g_+ \mapsto g_+^{g_-}$, where $g_+ \in G_0$, $g_- \in G_0^*$ can be given by

$$g_- g_+ = g_+^{g_-} g'_- ,$$

where $g'_- \in G_0^*$ provided by that the multiplication map $G_0 \times G_0^* \to G_{\mathbf{R}}$ is bijective.

In general, $G_0 G_0^*$ is not even dense in $G_{\mathbf{R}}$. However, there is still the following fact.

7.1. Proposition. *The multiplication map* $(SU(1,1) \rtimes \mathbf{Z}_2) \times SU(1,1)^* \to SL_2(\mathbf{C})$ *is injective and its image is dense in* $SL_2(\mathbf{C})$.

In what follows we denote $G_0 = SU(1,1)$, $G = SL_2(\mathbf{C})$, $B \subset SL_2(\mathbf{C})$ is the Borel subgroup of upper-triangular matrices. We consider below only the case of the *left* quantum adjoint action and the *right* dressing action. Another case can be considered analogously and does not contain anything new.

Since B is a *Poisson* Lie subgroup of $G_{\mathbf{R}}$, the *flag manifold* $G_{\mathbf{R}}/B$ is endowed with a Poisson manifold structure, and the local right dressing action of $G_0^* \subset G_{\mathbf{R}}^*$ on $G_{\mathbf{R}}$ induces a local action on $G_{\mathbf{R}}/B$ which we call also right dressing action.

Note that, since $G_{\mathbf{R}}/B$ is compact, this local action *can* be extended to a global one. The G_0^*-orbits of this action are the *Schubert cells* $C_1 = \{\infty\}$ and $C_w = \mathbf{C}P^1 \setminus \{\infty\}$ parameterized, as is well known, by the Weyl group W, G_0^* acting on $C_w \simeq \mathbf{C}$ by translations and dilations.

If G_0 were $SU(2)$, the corresponding G_0^*-orbits would be the same. However, while in this case they are symplectic leaves, if G_0 is $SU(1,1)$, this is not the case.

Indeed, $G_0 G_0^*/B \simeq \mathscr{D}_+ \cup \{\infty\}$ and $G_0 w G_0^*/B \simeq \mathscr{D}_-$ are Poisson submanifolds of $G_{\mathbf{R}}/B$. It is easy to see that the symplectic leaves of $G_{\mathbf{R}}/B \simeq CP^1$ are $\{\infty\}$, $\mathscr{D}_+$, $\mathscr{D}_-$ and each point of $S^1 = \partial\mathscr{D}_\pm$.

Let us call a symplectic leaf of a Schubert cell *non-degenerate* if its dimension is the same as the dimension of the cell. Recall the parameterization of irreducible $*$-representations of $\mathfrak{S}$ by quadruples (t, C, Σ, β), where $t \in T_0$, C is a Schubert cell of the flag manifold $G_{\mathbf{R}}/B$, Σ is a non-degenerate symplectic leaf of C, β is a unitary character of the fundamental group $\pi_1(\Sigma)$ of Σ (cf. Sect. 2). Thus, the one-dimensional $*$-representations ζ_φ correspond to the leaf $\{\infty\}$, the infinite-dimensional ones $\pi_\varphi^\pm$ to the leaf $\mathscr{D}_\pm$.

As follows from the geometric realization considered in Sect. 5 and Sect. 6, the quasi-classical analogues of the tensor products $V_{\zeta_\varphi} \otimes V_{\zeta_\varphi}$, $V_\varphi^{+,(\beta)} \otimes V_{\pi+\varphi}^{+,(\beta)}$ and $V_\varphi^- \otimes V_{\pi+\varphi}^-$ of *unitarizable* $\mathfrak{R}_+$*-modules* is the *local* right dressing action of $SU(1,1)^*$ on the *symplectic leaves* $\{\infty\}$, $\mathscr{D}_+$ and $\mathscr{D}_-$ respectively, while the quasi-classical analogues of the "tensor products" $\zeta_\varphi \otimes \zeta_\varphi$ and $\pi_{\varphi_1}^{+,(\beta)} \otimes \pi_{\pi+\varphi_1}^{+,(\beta)} \oplus \pi_{\varphi_2}^- \otimes \pi_{\pi+\varphi_2}^-$ is the *global* right dressing action of $SU(1,1)^*$ on the *Schubert cells* $\{\infty\}$ and $\mathbf{C}$ respectively. The negative result of Theorem 6.1 corresponds to the obvious fact that the local action of $SU(1,1)^*$ on $\mathscr{D}_\pm$ by translations and dilations cannot be extended to a global one.

A. Decomposition of the "Tensor Products"

This appendix is devoted to the proof of Theorem 6.4. Of course, (6.5) does not require a special consideration. We prove (6.6) below, (6.7) can be proven analogously and even much simpler.

Denote in short

$$\pi \overset{\text{def}}{=} \pi_{\varphi_1}^{+,(\beta)} \otimes \pi_{\varphi_2}^{+,(\beta)} \otimes \pi_{\varphi_3}^- \otimes \pi_{\varphi_4}^- \tag{A.1}$$

and consider the self-adjoint operator $\pi(x)$. It is clear that what we really need to obtain the decomposition of π is to know the spectrum of $\pi(x)$. The subspace L_m $(m \in \mathbf{Z})$ generated by $e_{k-m}^\pm \otimes e_k^\pm$ $(k \in \mathbf{Z})$ is easily seen to be $\pi(x)$-invariant. Let $\pi(x)_m$ be the restriction of $\pi(x)$ to L_m.

It is easy to show that $\pi(x) \geqq 0$, $\operatorname{Ker} \pi(x) = \{0\}$. The unitary operator $u = q^{\frac{1}{2}} \pi(t_{12}) \pi(x)^{-\frac{1}{2}}$ intertwines $\pi(x)_m$ and $\pi(x)_{m+1}$, therefore, all the self-adjoint operators $\pi(x)_m$ $(m \in \mathbf{Z})$ are unitarily equivalent.

Consider, for instance, $\pi(x)_0$. It is unitarily equivalent to the operator A in $L^2(\mathfrak{M}^{(-\beta)}, dv)$ given by the second order q-difference expression (6.1) and the boundary condition (6.3).

First of all, it is clear that the spectrum of A is *simple*. It is clear also that it is a union of some geometric progressions with ratio q^2 because of $t_{11}x = q^2 x t_{11}$, $t_{22}x = q^{-2} x t_{22}$.

It is easy to show that π can be decomposed into a direct integral of the irreducible representations of the form $\pi_\varphi^{+,(\beta')}$ (no one-dimensional ones, since $\operatorname{Ker} A = \{0\}$). Those geometric progressions which comprise the spectrum of A indicate the possible values of β' for the irreducible components (note that if $\pi_\varphi^{+,(\beta')}$ occurs in the decomposition for some β' and φ, this is the case for the same β' and *all* values of φ).

The following proposition immediately implies (6.6).

A.1. Proposition. *The spectrum of A is simple and equal to* $\mathfrak{M}_+^{(\beta)} \cup \{0\}$ (*where β is the same as fixed in* (B.1)), $\operatorname{Ker} A = \{0\}$.

Proof. The standard way to prove it is to study the asymptotic behavior of eigen-functions of A at infinity. Consider the function

$$f_\lambda(t) = {}_1\Phi_1\left(\begin{matrix} -qt^{-1} \\ q^2 \end{matrix}; q^2, q^2\lambda t\right), \tag{A.2}$$

where $(a)_k = (1-a)(1-aq^2)\dots(1-aq^{2(k-1)})$ and

$${}_1\Phi_1\left(\begin{matrix} a \\ b \end{matrix}; q^2, t\right) = \sum_{k-0}^{\infty} \frac{(-1)^k q^{k(k-1)}(a)_k}{(b)_k(q^2)_k} x^k$$

is a basic hypergeometric function.

The function $f_\lambda(t)$ generates the one-dimensional space of solutions of

$$-(q^{-1}-q)^2 \cdot (D(1+t)Df)(t) = \lambda f(t),$$

$$f(+0) = f(-0), \quad (Df)(+0) = (Df)(-0).$$

Since $A \geqq 0$ and $\operatorname{Ker} A = \{0\}$, we can suppose $\lambda > 0$.

We compare A with some operator A_0 with known spectrum. The operator A_0 is given in $L^2(\mathfrak{M}_+^{(-\beta)}, d\nu_+^{(-\beta)})$ by the second order q-difference expression

$$A_0 = -(q^{-1}-q)^2 \cdot DtD$$

and the boundary condition

$$\lim_{k \to +\infty} (f(q^{2(k-\beta)}) - f(q^{2(k-\beta+1)})) = 0. \tag{A.3}$$

The eigen-functions of A_0 are the so-called zero index Hahn–Exton q-Bessel functions (introduced in [2])

$$f_\lambda^{(0)}(t) = {}_1\Phi_1\left(\begin{matrix} 0 \\ q^2 \end{matrix}; q^2, q^2\lambda t\right)$$

and the spectrum of A_0 is simple and equal to $\mathfrak{M}_+^{(\beta)} \cup \{0\}$, $\operatorname{Ker} A_0 = \{0\}$.

This was announced in [8] where the operator A_0 appeared in harmonic analysis on the quantum group $M(2)$ of the motions of the plane (note that the same operator appears also in the problem of decomposition of tensor products of irreducible $*$-representations of the quantum algebra of regular functions).

The proof which was not included in [8] is based on the fact that, when $\beta = \frac{1}{2}$, A_0 can be approximated by some simpler operators $A_0^{(m)} = q^{2m} T^{-m} A_0^{(0)} T^m$ in the sense that the operators $A_0 - A_0^{(m)}$ are bounded and converge to zero as $m \to +\infty$. The operator T is the shift $(Tf)(t) = f(q^2 t)$ and the operator $A_0^{(0)}$ given in $L^2(\{q^{2k+1}\}_{k \geqq -m}, d\nu_+)$ by the second order q-difference expression

$$A_0^{(0)} = -(q^{-1}-q)^2 \cdot Dt(1-t)D$$

and the boundary condition (A.3) appeared in [21] in harmonic analysis on the quantum group $SU(2)$.

The spectrum of $A_0^{(0)}$ is well-known. It is simple and consists of the values $\lambda_l=\frac{(q^{-l}-q^l)(q^{-l-1}-q^{l+1})}{(q^{-1}-q)^2}$ $(l\in\frac{1}{2}\mathbf{Z})$ of the quadratic Casimir element in the finite-dimensional irreducible $*$-representations of $U_q\mathfrak{su}(2)$. Therefore, we know the spectra of the operators $A_0^{(m)}$, hence the spectrum of A_0. Note that, if we know it for just one value of β, it is easy to obtain it for all values of β.

The following lemma can be proved by some straightforward calculations. The convergence is always provided by terms of the form q^{k^2}.

A.2. Lemma. *The series in the right-hand side of the following identity converges to the left-hand sides absolutely, uniformly in compacta*:

$$ {}_1\Phi_1\left(\begin{matrix}a\\ b\end{matrix};q^2,x\right)=\sum_{k=0}^{\infty}\frac{q^{2k(k-1)}(ax)^k}{(b)_k(q^2)_k}\,{}_1\Phi_1\left(\begin{matrix}0\\ q^{2k}b\end{matrix};q^2,q^{2k}x\right). \tag{A.4}$$

A.3. Corollary. *Put $a=-qt^{-1}$, $b=q^2$, $x=q^2\lambda t$ into (A.3). We get*

$$ {}_1\Phi_1\left(\begin{matrix}-qt^{-1}\\ q^2\end{matrix};q^2,q^2\lambda t\right)=\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(2k+1)}\lambda^k}{(q^2)_k^2} \times {}_1\Phi_1\left(\begin{matrix}0\\ q^{2(k+1)}\end{matrix};q^2,q^{2(k+1)}\lambda t\right).$$

Consider the operator $B: f(t)\mapsto \frac{f(t)-f(q^2t)}{t}$.

A.4. Lemma. *The right-hand side of the following identity converges to the left-hand side absolutely, uniformly in compacta*:

$$ {}_1\Phi_1\left(\begin{matrix}-qt^{-1}\\ q^2\end{matrix};q^2,q^2\lambda t\right)=(-qB)_\infty\,{}_1\Phi_1\left(\begin{matrix}0\\ q^2\end{matrix};q^2,q^2\lambda t\right).$$

Proof. By the recurrence formula (cf. [2])

$$ B:\ {}_1\Phi_1\left(\begin{matrix}0\\ b\end{matrix};q^2,cx\right)\mapsto -\frac{c}{1-b}\,{}_1\Phi_1\left(\begin{matrix}0\\ q^2b\end{matrix};q^2,q^2cx\right),$$

we obtain from (A.4) that

$$ {}_1\Phi_1\left(\begin{matrix}-qt^{-1}\\ q^2\end{matrix};q^2,q^2\lambda t\right)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q^2)_k}B^k\,{}_1\Phi_1\left(\begin{matrix}0\\ q^2\end{matrix};q^2,q^2\lambda t\right) =(-qB)_\infty\,{}_1\Phi_1\left(\begin{matrix}0\\ q^2\end{matrix};q^2,q^2\lambda t\right).$$

Proof of Proposition A.1 (continued). Suppose $\lambda\in\mathfrak{M}_+^{(\beta)}$. By [8, Proposition 11], $f_\lambda^{(0)}(t)$ vanishes faster than t^{-n} for any $n\in\mathbf{N}$ as $t\to+\infty$, $t\in\mathfrak{M}_+^{(-\beta)}$. Note that

Proposition 11 in [8] is a simple consequence of the orthogonality relation for the matrix elements of irreducible $*$-representations of $U_q\mathfrak{m}(2)$ which are expressed in terms of the Hahn–Exton q-Bessel functions and the recurrence formulae for these functions (cf. [2]).

It follows that the functions

$$f_\lambda^{(k)}(t) \stackrel{\text{def}}{=} (-qB)_k f_\lambda^{(0)}(t) \; (k \in \mathbf{Z}_+)$$

also vanish faster than t^{-n} for any $n \in \mathbf{N}$ as $t \to +\infty$, $t \in \mathfrak{M}_+^{(-\beta)}$.

Note that all the above lemmas remain valid when both sides of the identities are multiplied by t^n for any $n \in \mathbf{N}$. Therefore, $t^n f_\lambda^{(k)}(t)$ converge to $t^n f_\lambda(t)$ for any $n \in \mathbf{N}$ absolutely, uniformly in compacta as $k \to +\infty$. Hence, $f_\lambda(t)$ also vanishes faster than t^{-n} for any $n \in \mathbf{N}$ as $t \to +\infty$, $t \in \mathfrak{M}_+^{(-\beta)}$.

This implies that each point of the geometric progression $\mathfrak{M}_+^{(\beta)}$ is an eigenvalue of A. To show that $\mathfrak{M}_+^{(\beta)} \cup \{0\}$ is the whole spectrum of A note that, since for each eigen-value λ of A_0 the corresponding eigen-function $f_\lambda^{(0)}$ *vanishes* faster than t^{-n} for any $n \in \mathbf{N}$ as $t \to +\infty$, $t \in \mathfrak{M}_+^{(-\beta)}$, for any λ' which does not belong to the spectrum of A_0 the corresponding eigen-function $f_{\lambda'}^{(0)}$ *grows* faster than t^n for any $n \in \mathbf{N}$ as $t \to +\infty$, $t \in \mathfrak{M}_+^{(-\beta)}$.

It follows in a similar way that, for each $\lambda' \notin \mathfrak{M}_+^{(-\beta)} \cup \{0\}$, the corresponding eigen-function $f_{\lambda'}(t)$ of A grows faster than any polynomial as $t \to +\infty$, $t \in \mathfrak{M}_+^{(-\beta)}$. This proves Proposition A.1.

In conclusion, I would like to note that, according to (6.6) and (6.7), one can define the *Clebsch–Gordan coefficients for quantum algebra of functions* ("even" and "odd") as follows:

$$e_m^{\pm}(\varphi_1) \otimes e_n^{\pm}(\varphi_2) = \sum_{k \in \mathbf{Z}} \int_0^{2\pi} \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi \\ m & n & k \end{bmatrix}_{q,\beta}^{(\pm\pm)} e_k^{+}(\varphi)\, d\varphi \,,$$

$$e_m^{\pm}(\varphi_1) \otimes e_n^{\mp}(\varphi_2) = \sum_{k \in \mathbf{Z}_+} \int_0^{2\pi} \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi \\ m & n & k \end{bmatrix}_{q,\beta}^{(\pm\mp)} e_k^{-}(\varphi)\, d\varphi \,, \tag{A.5}$$

where $e_k^{\pm}(\varphi)$ stands for the canonical basis of $V_\varphi^{\pm}$ given by (2.5), (2.6).

As follows from (5.7), (A.2) and (5.2), the "even" Clebsch–Gordan coefficients are expressed in terms of the functions

$${}_1\Phi_1\left(\begin{matrix} -qt^{-1} \\ q^{2(k+1)} \end{matrix}; q^2, q^{2(k+1)}\lambda t\right) = \text{const} \cdot (B^k f_\lambda)(t) \tag{A.6}$$

($k \in \mathbf{Z}_+$), where $\lambda \in \mathfrak{M}_+^{(\beta)}$, $t \in \mathfrak{M}_+^{(-\beta)}$. This follows, by the way, that the Clebsch–Gordan coefficients *do depend* on β (see the remark at the end of Sect. 6).

As was shown in [9, 10, 16], the matrix elements $t_{ij}^{l,\varepsilon} \in \mathbf{C}[SU(1,1)]_q^*$ of irreducible $*$-representations of $U_q\mathfrak{su}(1,1)$ are expressed in terms of the q-Jacobi functions. As far as we know precise expression of the Clebsch–Gordan coefficients for quantum algebra of functions, we can apply the technique of [5] to obtain an *addition formula* for the q-Jacobi functions. Namely, one should apply the operator given by the action of

$$\Delta(t_{ij}^{l,\varepsilon}) = \sum_k t_{ik}^{l,\varepsilon} \otimes t_{kj}^{l,\varepsilon}$$

to the right-hand sides and the left-hand sides of (A.5). Therefore, the functions (A.6) appear in that addition formula which involves the "even" Clebsch–Gordan coefficients.

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