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Roots of Unity: Representations of Quantum Groups

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Abstract: Representations of Quantum Groups $U_\varepsilon(g_n)$, g_n any semi-simple Lie algebra of rank n, are constructed from arbitrary representations of rank $n-1$ quantum groups for ε a root of unity. Representations which have the maximal dimension and number of free parameters for irreducible representations arise as special cases.

1. Introduction

Deformations of semi-simple Lie algebras [18, 19] appear as a common algebraic structure in the field of low dimensional integrable systems. In many cases the deformation parameter is an N-th root of unity, where N can correspond, e.g. to the number of states per site or to the lattice size in a two dimensional model. We will denote the deformation parameter by ε, if the parameter is an N-th root of unity (N the smallest integer such that $\varepsilon^N = 1$) and by q in the general case.

The theories of chiral Potts [4, 5] type models, which saw dramatic developments in recent years [6–8, 12, 24], are closely tied to the representation theory of the quantum group $U_\varepsilon(sl(n, \mathbf{C}))$ in the case of ε being an N-th root of unity. The progress in the theories of chiral Potts models was partly stimulated by the better understanding of its deep connection to the representation theory of quantum groups.

The representation theory in the case of ε an N-th root of unity is much richer than for generic q, and several deep results by De Concini, Kac, Procesi [15, 17, 16] and Lusztig [20–23] exist, laying the foundations of the general representation theory in the roots of unity case. Also considerable progress has been made in directly constructing representations of quantum groups $U_\varepsilon(g_n)$. Accelerated by the development of chiral Potts type models, much interest was devoted to find non-highest weight representations of $U_\varepsilon(g_n)$ [25, 1–3, 9–11]. Finite dimensional

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non-highest weight representations, which do not exist in the representations theory of $U_q(g_n)$, are a new interesting feature of the representation theory in the roots of unity case. Non-highest representations of minimal dimension play a role similar to that played by fundamental representations of Lie algebras [10]. The free parameters which are characteristic of non-highest weight representations appear in the form of spectral parameters in chiral Potts type models.

In this article we will show, that starting from an arbitrary representation of $U_\varepsilon(g_{n-1})$ one can construct a representation of $U_\varepsilon(g_n)$. The Lie algebras g_n and g_{n-1} will usually, but not always lie in the same series of Lie algebras.

De Concini, Kac [15] showed that the maximal dimension and number of parameters for representations of $U_\varepsilon(g_n)$ for odd N is given by $N^{\Delta^+(g_n)}$ and $\dim g_n$, respectively. In this expression $\Delta^+(g_n)$ $(=1/2(\dim g_n - n))$ is the number of positive roots of the Lie algebra g_n. We use here and in the following the term dimension of a representation to denote the dimension of its representation space. The representations which will be constructed in this article, are of dimension greater than or equal to

$$N^{\Delta^+(g_n)-\Delta^+(g_{n-1})} .$$

For ε-deformations in the case of the A_n, B_n, C_n and D_n series such representations will be given in Sect. 3. The exceptional E_6, E_7, E_8, F_4 and G_2 cases of quantum groups are discussed in Sect. 4. The number of free parameters in all constructed representations is greater than or equal to $2(\Delta^+(g_n)-\Delta^+(g_{n-1}))$. Representations of maximal dimension and number of free parameters arise as special cases for odd N and are discussed in Sect. 5. These representations coincide with the maximal cyclic representations of Date et al. [13] in the A_n case, and with the representations of [26] in the B_n, C_n and D_n cases. Conclusions are given in Sect. 6, and in the appendix two relations which are important for the construction of representations in the case of $U_q(E_8)$ and $U_q(F_4)$ are written down explicitly.

2. Definition of $U_q(g_n)$

In this article we will use the definition of quantum groups given by the relations among its Chevalley generators. The quantized universal enveloping algebra $U_q(g_n)$ of a semi-simple Lie algebra g_n of rank n is generated by $4n$ Chevalley generators $\{e_i, f_i, t_i^{\pm 1}\}$ which satisfy the commutation relations

$$\left.\begin{aligned} &t_i t_j = t_j t_i, \quad t_i t_i^{-1} = t_i^{-1} t_i = 1 \\ &t_i e_j t_i^{-1} = q^{d_i a_{ij}} e_j, \quad t_i f_j t_i^{-1} = q^{-d_i a_{ij}} f_j \\ &[e_i, f_i] = \delta_{ij} \{t_i\}_{q^{d_i}} \end{aligned}\right\} \tag{1}$$

and the Serre relations

$$\left.\begin{aligned} &\sum_{v=0}^{1-a_{ij}} (-1)^v \begin{bmatrix} 1-a_{ij} \\ v \end{bmatrix}_{q^{d_i}} e_i^{1-a_{ij}-v} e_j e_i^v = 0 \\ &\sum_{v=0}^{1-a_{ij}} (-1)^v \begin{bmatrix} 1-a_{ij} \\ v \end{bmatrix}_{q^{d_i}} f_i^{1-a_{ij}-v} f_j f_i^v = 0 \end{aligned}\right\} (i \neq j) . \tag{2}$$

The matrix a_{ij} is the Cartan matrix of the Lie algebra g_n, d_i non-zero integers satisfying $d_i a_{ij} = d_j a_{ji}$. The q-Gaussian is given as

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q!\,[n]_q}, \quad [m]_q! = \prod_{j=1}^{m} \frac{q^j - q^{-j}}{q - q^{-1}}$$

and the curly bracket is defined by $\{x\}_q = (x - x^{-1})/(q - q^{-1})$. Further, we extend the algebra by adding the elements $t_i^{1/k}$, $k \in \mathbf{Z}$. In case we specialize q to be a primitive N-th root of unity the letter ε is used instead ($\varepsilon^N = 1$). We will not make use of the Hopf algebra structure of $U_q(g_n)$.

Some of the results in this article are most conveniently given in terms of Weyl algebra generators. The Weyl algebra W is defined by the commutation relation among its two generators x, z,

$$xz = qzx \,. \tag{3}$$

We denote by $\tilde{W}$ a copy of this algebra with generators $\tilde{x}$, $\tilde{z}$.

In the roots of unity case one can define a N dimensional representation σ_{gh}: $W \to \mathrm{End}\,(\mathbf{C}^N)$ of W, depending on the two parameters g, h, by

$$\sigma_{gh}(x) = g \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & 0 & 1 \\ 1 & & \cdots & & 0 \end{pmatrix} \qquad \sigma_{gh}(z) = h \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon^2 & \cdots & 0 \\ & & & \ddots & \\ 0 & & \cdots & & \varepsilon^{N-1} \end{pmatrix} .$$

We will retain the notation σ_{gh} also for representations of tensor products of Weyl algebras W. In this cases the letters g, h denote the set of parameters $g := \{g_1, \ldots, g_l\}$ and $h := \{h_1, \ldots, h_l\}$, with l the number of W algebras in $\bigotimes_{1 \leq i \leq l} W_i$.

3. The A_n, B_n, C_n and D_n Series

In this section we want to demonstrate how one can construct, starting from an arbitrary representation of a quantum group $U_q(g_{n-1})$ a representation of $U_q(g_n)$, in the cases of the A_n, B_n, C_n and D_n series of quantum groups. The next section will be devoted to the more complicated case of exceptional quantum groups. As a first step we investigate the algebra $\bar{U}_q(g_{n-1})$ which is defined as the algebra given by $U_q(g_n)$ generators $\mathscr{F}_i$, $(1 \leq i \leq n-1)$ and $\mathscr{E}_i$, $\mathscr{T}_i^{\pm 1}$, $(1 \leq i \leq n)$ which satisfy the defining relations (1,2). Representations of algebras $\bar{U}_q(g_{n-1})$ arise immediately from representations of $U_q(g_{n-1})$, and in turn representations of $U_q(g_n)$ itself will be defined in terms of $\bar{U}_q(g_{n-1})$ representations. This will be discussed in the following.

The Cartan matrices in the B_n and C_n correspond to Dynkin diagrams with the first root being the shortest or longest root, respectively. For the D_n case we fix the notation in a way such that the Dynkin diagram nodes 1 and 2 are both connected with node 3. The integers $d_i = 1$ for $2 \leq i \leq n$ in all cases and $d_1 = 1, 1/2, 2, 1$ in the A_n, B_n, C_n and D_n series, respectively.

Lemma 3.1. *Let* $\{f_i, e_i, t_i, t_i^{-1}\}$ $(i = 1, \ldots, n-1)$ *be the generators of* $U_q(g_{n-1})$ *and* $\{\mathscr{E}_i, \mathscr{T}_i, \mathscr{T}_i^{-1}\}$ $(i = 1, \ldots, n)$, $\{\mathscr{F}_i)$ $(i = 1, \ldots, n-1)$ *the generators of* $\bar{U}_q(g_{n-1})$.

Then one obtains an algebra homomorphism $\bar{\rho}$ from $\bar{U}_q(g_{n-1})$ to $U_q(g_{n-1})$ by taking $\bar{\rho}(\mathscr{F}_i)=f_i$, $\bar{\rho}(\mathscr{E}_i)=e_i$, $\bar{\rho}(\mathscr{T}_i^{\pm 1})=t_i^{\pm 1}$ $(i=1,\ldots,n-1)$ and $\bar{\rho}(\mathscr{E}_n)=0$. The action of the algebra homomorphism $\bar{\rho}$ on the generators $\mathscr{T}_n^{\pm 1}$ depends on the complex parameters λ_n and is defined by

$$\bar{\rho}(\mathscr{T}_n)=\begin{cases} q^{\lambda_n}\prod_{i=1}^{n-1} t_i^{-\frac{i}{n}} & \text{for } \bar{U}_q(A_{n-1}) \\ q^{\lambda_n}\prod_{i=1}^{n-1} t_i^{-1} & \text{for } \bar{U}_q(B_{n-1}) \\ q^{\lambda_n} t_1^{-\frac{1}{2}}\prod_{i=2}^{n-1} t_i^{-1} & \text{for } \bar{U}_q(C_{n-1}) \\ q^{\lambda_n} t_1^{-\frac{1}{2}} t_2^{-\frac{1}{2}}\prod_{i=3}^{n-1} t_i^{-1} & \text{for } \bar{U}_q(D_{n-1}) \end{cases}$$

The above algebra homomorphism $\bar{\rho}$ can be extended to define representations of the quantum group $U_q(g_n)$. The above lemma, together with the theorem below shows how any representation of $U_q(g_{n-1})$ gives rise to a representation of $U_q(g_n)$ in the A_n, B_n, C_n and D_n series.

Theorem 3.2. *The following formulas define algebra homomorphisms ρ from the quantum groups $U_q(g_n)$ to $(\bigotimes_i W)\otimes\bar{U}_q(g_{n-1})$, for g_n the A_n, B_n, C_n and D_n series of semi simple Lie algebras. The composition $\pi:=(\sigma_{gh}\otimes \mathrm{id})\cdot\rho$ defines an algebra homomorphism in the roots of unity case.*

a) $\rho\colon U_q(A_n)\rightarrow(\bigotimes_{i=1}^{n} W_i)\otimes\bar{U}_q(A_{n-1})$

$$\rho(f_i)=\{z_{i-1}z_i^{-1}\}x_i+x_{i-1}^{-1}x_i\mathscr{F}_{i-1},\quad \rho(e_i)=\{z_i z_{i+1}^{-1}\mathscr{T}_i\}x_i^{-1}+\mathscr{E}_i$$

$$\rho(t_i)=z_{i-1}^{-1}z_i^2 z_{i+1}^{-1}\mathscr{T}_i$$

b) $\rho\colon U_q(B_n)\rightarrow(\bigotimes_{i=1}^{n} W_i)\otimes(\bigotimes_{i=1}^{n-1}\tilde{W}_i)\otimes\bar{U}_q(B_{n-1})$

$$\rho(f_i)=\{z_{i+1}z_i^{-1}\}x_i+\{\tilde{z}_i\tilde{z}_{i+1}^{-1}\}x_{i+1}^{-1}x_i\tilde{x}_{i+1}+x_{i+1}^{-1}x_i\tilde{x}_{i+1}\tilde{x}_i^{-1}\mathscr{F}_i,\quad (i>1)$$

$$\rho(f_1)=\{z_2 z_1^{-1/2}\}_{q^{1/2}}x_1+\{\tilde{z}_2^{-1}z_1^{1/2}\}_{q^{1/2}}x_2^{-1}x_1\tilde{x}_2+x_2^{-1}\tilde{x}_2\mathscr{F}_1$$

$$\rho(e_i)=\{z_i z_{i-1}^{-1}\tilde{z}_{i-1}^{-1}\tilde{z}_i^2\tilde{z}_{i+1}^{-1}\mathscr{T}_i\}x_i^{-1}+\{\tilde{z}_i\tilde{z}_{i+1}^{-1}\mathscr{T}_i)\tilde{x}_i^{-1}+\mathscr{E}_i,\quad (i>1)$$

$$\rho(e_1)=\{z_1^{1/2}\tilde{z}_2^{-1}\mathscr{T}_1\}_{q^{1/2}}x_1^{-1}+\mathscr{E}_1$$

$$\rho(t_i)=z_{i+1}^{-1}z_i^2 z_{i-1}^{-1}\tilde{z}_{i-1}^{-1}\tilde{z}_i^2\tilde{z}_{i+1}^{-1}\mathscr{T}_i,\quad (i>1),\quad \rho(t_1)=z_2^{-1}z_1\tilde{z}_2^{-1}\mathscr{T}_1$$

c) $\rho\colon U_q(C_n)\rightarrow(\bigotimes_{i=1}^{n} W_i)\otimes(\bigotimes_{i=1}^{n-1}\tilde{W}_i)\otimes\bar{U}_q(C_{n-1})$

$\rho(f_i)$, $\rho(e_i)$, $\rho(t_i)$ as in the B_n case $(i>1)$

$$\rho(f_1)=\{z_1^2\tilde{z}_2^{-2}\}_{q^2}x_2^{-2}x_1\tilde{x}_2^2+\{z_2\tilde{z}_2^{-1}\}x_2^{-1}x_1\tilde{x}_2+\{z_2^2 z_1^{-2}\}_{q^2}x_1+x_2^{-2}\tilde{x}_2^2\mathscr{F}_1$$

$$\rho(e_1)=\{z_1^2\tilde{z}_2^{-2}\mathscr{T}_1\}_{q^2}x_1^{-1}+\mathscr{E}_1,\quad \rho(t_1)=z_2^{-2}z_1^4\tilde{z}_2^{-2}\mathscr{T}_1$$

d) $\rho\colon U_q(D_n)\rightarrow(\bigotimes_{i=1}^{n} W_i)\otimes(\bigotimes_{i=1}^{n-2}\tilde{W}_i)\otimes\bar{U}_q(D_{n-1})$

$\rho(f_i)$, $\rho(e_i)$, $\rho(t_i)$ as in the B_n case $(i>2)$

$$\rho(f_1)=\{z_3z_1^{-1}\}x_1+\{z_2\tilde{z}_3^{-1}\}\tilde{x}_3x_3^{-1}x_1+x_3^{-1}x_1x_2^{-1}\tilde{x}_3\mathscr{F}_2$$

$$\rho(f_2)=\{z_3z_2^{-1}\}x_2+\{z_1\tilde{z}_3^{-1}\}\tilde{x}_3x_3^{-1}x_2+x_3^{-1}x_2x_1^{-1}\tilde{x}_3\mathscr{F}_1$$

$$\rho(e_1)=\{z_1\tilde{z}_3^{-1}\mathscr{T}_1\}x_1^{-1}+\mathscr{E}_1,\quad \rho(e_2)=\{z_2\tilde{z}_3^{-1}\mathscr{T}_2\}x_2^{-1}+\mathscr{E}_2$$

$$\rho(t_1)=z_3^{-1}z_1^2\tilde{z}_3^{-1}\mathscr{T}_1,\quad \rho(t_2)=z_3^{-1}z_2^2\tilde{z}_3^{-1}\mathscr{T}_2\ .$$

In these formulas expressions of type $x_i\otimes\mathscr{F}_j$ were abbreviated writing $x_i\mathscr{F}_j$. Further, we set $\mathscr{F}_0=0$ and $x_i^{-1}=\tilde{x}_i^{-1}=0$, $z_i=\tilde{z}_i=1$ if the index i is out of range.

Taking $\bar{U}_\varepsilon(g_{n-1})$ in an arbitrary representation $\bar{\rho}'$, then the algebra homomorphisms π becomes a representation π' of dimension $N^{\Delta^+(g_n)-\Delta^+(g_{n-1})}$ times the dimension of $\bar{\rho}'$. The number of free parameters in π' becomes $2(\Delta^+(g_n)-\Delta^+(g_{n-1}))$ plus the number of free parameters in $\bar{\rho}'$.

This theorem can be proven by directly verifying the defining relations of the algebras. We omit the details of these calculations.

Remark 3.3. Representations with dimensions equal to $N^{\Delta^+(g_n)-\Delta^+(g_{n-1})}$ and $2(\Delta^+(g_n)-\Delta^+(g_{n-1})+\frac{1}{2})$ free parameters are obtained by taking the trivial representation ρ_0' of $U_\varepsilon(g_{n-1})(\rho_0'(e_i)=0,\ \rho_0'(f_i)=0,\ \rho_0'(t_i)=1,\ 1\leqq i\leqq n-1)$ to define $\bar{\rho}$ in Lemma 3.1. In the case of $U_\varepsilon(A_n)$ the resulting representations π' are minimal cyclic representations [1, 2, 9, 10, 12]. Further representations of dimension $N^{\Delta^+(g_n)-\Delta^+(g_{n-1})}$ were discussed in the quantum $SO(5)$ case in [2] and for quantum $SO(8)$ in [11].

Remark 3.4. The algebra homomorphism π defined above for the A_n series is cyclic in the sense that the Chevalley operators in the algebra homomorphism π to the power of N are non-vanishing scalars for generic values of parameters. The explicit expressions are given in the following formulas:

$$\mathrm{n}(f_i)^N=-\frac{h_{i-1}^N h_i^{-N}(-1)^N+h_{i-1}^{-N}h_i^N}{(q-q^{-1})^N}\,g_i^N+g_{i-1}^{-N}g_i^N\mathscr{F}_{i-1}^N\ ,$$

$$\pi(e_i)^N=-\frac{h_i^N h_{i+1}^{-N}\mathscr{T}_i^N(-1)^N+h_i^{-N}h_{i+1}^N\mathscr{T}_i^{-N}}{(q-q^{-1})^N}\,g_{i+1}^{-N}+\mathscr{E}_i^N\ ,$$

$$\pi(t_i)=h_{i-1}^{-N}h_i^{2N}h_{i+1}^{-N}\mathscr{T}_i^N\ .$$

These formulas also show that restricted representations $(\pi'(f_i)^N=0,\ \pi'(e_i)^N=0,\ \pi'(k_i)^N=1)$, as well as semi-cyclic representations (either $\pi'(f_i)^N=0$ or $\pi'(e_i)^N=0$) can be derived from the cyclic representation by specializing some or all of the free parameters. Cyclic and semi-cyclic representations of $U_\varepsilon(A_n)$ in dimensions higher than the minimal dimensions were discussed in [1, 14].

Theorem 3.2 allows to construct a representation of $U_q(g_n)$ from representations of smaller rank quantum groups $U_q(g_{n-1})$ with g_n and g_{n-1} being Lie algebras of the same series. To show that they need not to be necessarily from the same series we present as an example how a representation of $U_q(C_3)$ follows from a representation of $U_q(A_2)$. We define $\bar{U}_q'(A_2)$ to be the algebra given by the generators $\mathscr{E}_i$, $\mathscr{T}_i^{\pm1}$, $(i=1,2,3)$ and $\mathscr{F}_2$, $\mathscr{F}_3$ of a $U_q(C_3)$ algebra. Similarly to 3.1 one can give an algebra homomorphism $\bar{\rho}$ from $\bar{U}_q'(A_2)$ to $U_q(A_2)$ by setting $\bar{\rho}(\mathscr{E}_1)=0$, $\bar{\rho}(\mathscr{T}_1)=q^{\lambda_1}t_3^{-2/3}t_2^{-4/3}$ and the other generators equal to $U_q(A_2)$ generators.

Example 3.5. The map $\rho: U_q(C_3)\to(\otimes_{i=1}^{6} W_i)\otimes \bar{U}'_q(A_2)$ given below, defines an algebra homomorphism. The composition $\pi := (\sigma_{gh}\otimes \mathrm{id})\cdot\rho$ gives an algebra homomorphism in the roots of unity case. Taking $\bar{U}'_\varepsilon(A_2)$ in an arbitrary representation $\bar{\rho}'$ one obtains a representation π' of $U_\varepsilon(C_3)$ with dimension N^6 times the dimension of $\bar{\rho}'$. Twelve free parameters arise from taking representation σ_{gh} of $(\otimes_{i=1}^{6} W_i)$ and further free parameters can arise in the representation $\bar{\rho}'$,

$$\rho(f_1)=\{z_1^{-2}\}_{q^2}x_1\ ,$$
$$\rho(f_2)=\{z_2z_4^{-2}\}x_1^{-1}x_2x_4+\{z_3z_5^{-1}\}x_1^{-1}x_4x_5+\{z_1^2z_2^{-1}\}x_2+x_1^{-1}x_3^{-1}x_4x_5\mathscr{F}_3\ ,$$
$$\rho(f_3)=\{z_5z_6^{-2}\}x_2^{-1}x_3x_4^{-1}x_5x_6+\{z_4^2z_5^{-1}\}x_2^{-1}x_3x_5+\{z_2z_3^{-1}\}x_3 + x_2^{-1}x_3x_4^{-1}x_6\mathscr{F}_2\ ,$$
$$\rho(e_1)=\{z_1^2z_2^{-2}z_4^4z_5^{-2}z_6^4\mathscr{T}_1\}_{q^2}x_1^{-1}+\{z_4^2z_5^{-2}z_6^4\mathscr{T}_1\}_{q^2}x_4^{-1}+\{z_6^2\mathscr{T}_1\}_{q^2}x_6^{-1}+\mathscr{E}_1\ ,$$
$$\rho(e_2)=\{z_2z_3^{-1}z_4^{-2}z_5^2z_6^{-2}\mathscr{T}_2\}x_2^{-1}+\{z_5z_6^{-2}\mathscr{T}_2\}x_5^{-1}+\mathscr{E}_2\ ,$$
$$\rho(e_3)=\{z_3z_5^{-1}\mathscr{T}_3\}x_3^{-1}+\mathscr{E}_3\ ,$$
$$\rho(t_1)=z_1^4z_2^{-2}z_4^4z_5^{-2}z_6^4\mathscr{T}_1\ ,$$
$$\rho(t_2)=z_1^{-2}z_2^2z_3^{-1}z_4^{-2}z_5^2z_6^{-2}\mathscr{T}_2\ ,$$
$$\rho(t_3)=z_2^{-1}z_3^2z_5^{-1}\mathscr{T}_3\ .$$

4. The Exceptional E_6, E_7, E_8, F_4 and G_2 Quantum Groups

So far we did not describe the actual construction method which leads to the algebra homomorphisms which were given in the previous section. The procedure will be outlined in the following in the case of the exceptional quantum Lie algebras and the A_n case will appear as an example.

The construction method is based on a set of relations in $U_q(g_n)$. The relations needed to derive algebra homomorphisms of quantum Lie algebras in the E_6, E_7, E_8 and F_4 cases are given in the following. We denote divided powers of f_i generators $f_i^j/[j]!$ by $f_i^{(j)}$.

Relations 4.6. In $U_q(g_n)$ we have the commutation relations (compare [13, 26])

(i) $f_if_i^{(j)}=[j+1]_{q^{d_i}}f_i^{(j+1)}$,

(ii) $f_if_k^{(j_1)}f_i^{(j_2)}=f_k^{(j_1)}f_i^{(j_2+1)}[-j_1+j_2+1]_{q^{d_i}}+f_k^{(j_1-1)}f_i^{(j_2+1)}f_k$,
if $a_{ik}=a_{ki}=-1$,

(iii) $f_if_k^{(j_1)}f_i^{(j_2)}f_k^{(j_3)}=f_k^{(j_1-2)}f_i^{(j_2+1)}f_k^{(j_3+2)}[-j_2+j_3+1]_{q^2}$
$+f_k^{(j_1-1)}f_i^{(j_2+1)}f_k^{(j_3+1)}[-j_1+j_2+2]+f_k^{(j_1)}f_i^{(j_2+1)}f_k^{(j_3)}[-j_1+j_2+1]_{q^2}$
$+f_k^{(j_1-2)}f_i^{(j_2)}f_k^{(j_3+2)}f_i$, if $a_{ki}=-2$, $a_{ik}=-1$,

(iv) $e_if_i^{(j)}=f_i^{(j)}e_i+f_i^{(j-1)}\{q^{d_i(1-j)}t_i\}_{q^{d_i}}$,

(v) $t_if_k^{(j)}=q^{-ja_{ik}d_i}f_k^{(j)}t_i$,

(vi) $f_i f_k^{(j)} = f_k^{(j)} f_i$ and $f_i^{(j_1)} f_k^{(j_2)} = f_k^{(j_2)} f_i^{(j_1)}$ if $a_{ik} = 0$,

$e_i f_k^{(j)} = f_k^{(j)} e_i$, if $i \neq k$,

(vii) Relation in Lemma 7.14, Appendix (used only in the E_8 case),

(viii) Relation in Lemma 7.15, Appendix (used only in the F_4 case).

We will use the above relations to commute single generators f_i, e_i and t_i through monomials of l factors of type

$$f_{i_1}^{(j_1)} \cdots f_{i_l}^{(j_l)} \tag{4}$$

$(l = \Delta^+(g_n) - \Delta^+(g_{n-1}))$. We say a generator f_i, e_i or t_i commutes with such a monomial if the multiplication of the generator on the monomial from the left gives a sum over monomials of the same type multiplied by a single or no generator from the right. In this sense we say that f_i commutes with the monomial $f_k^{(j_1)} f_i^{(j_2)}$ according to relation (iii) in 4.6, if $a_{ik} = a_{ki} = -1$. One can find monomials with l factors for all quantum Lie algebras $U_q(g_n)$ which commute with all of its generators. Moreover, the relations 4.6 will be sufficient to commute the generators of $U_q(E_6)$, $U_q(E_7)$, $U_q(E_8)$ and $U_q(F_4)$ with monomials in the corresponding algebras. The $U_q(G_2)$ case will be treated separately.

Commuting a generator $s \in \{f_i, e_i, t_i\}$ with a monomial of type (4) using exclusively the relations 4.6 we denote by $\mathrm{rel}(s f_{i_1}^{(j_1)} \cdots f_{i_l}^{(j_l)})$. To apply relations 4.6 in this way will be the first step in a construction procedure which leads to representations of $U_q(g_n)$ in the exceptional cases.

The second step introduces the Weyl algebra generators. We shall give a rule Ω which applies on expressions of the following type:

$$q^{r_1 j_1 + \cdots + r_l j_l} f_{i_1}^{(j_1+\alpha_1)} \cdots f_{i_l}^{(j_l+\alpha_l)} s\,, \tag{5}$$

wherein the integer quantities $j_1, \ldots, j_l$ are regarded as "free variables" and r_k, $\alpha_k \in \mathbf{N}$, $1 \leqq k \leqq l$. The factor s is either 1 or a single $U_q(g_n)$ generator f_m, e_m, $t_m^{\pm 1}$, $(1 \leqq m \leqq n)$. If a, b are two expressions of type (5) and β a rational function in q which does not depend on the j_k's, then Ω satisfies $\Omega(a+b) = \Omega(a) + \Omega(b)$ and $\Omega(\beta a) = \beta \Omega(a)$. The rule Ω applies on expressions (5) according to the following assignment:

$$\Omega\colon q^{r_1 j_1 + \cdots + r_l j_l} f_{i_1}^{(j_1+\alpha_1)} \cdots f_{i_l}^{(j_l+\alpha_l)} s \mapsto x_1^{\alpha_1} z_1^{-r_1} \cdots x_l^{\alpha_l} z_l^{-r_l} S\,. \tag{6}$$

If $s = 1, f_m, e_m, t_m^{\pm 1}$ then $S = 1, \mathscr{F}_m, \mathscr{E}_m, \mathscr{T}_m^{\pm 1}$, respectively. Capital $\mathscr{E}_i$, $\mathscr{F}_i$ and $\mathscr{T}_i^{\pm 1}$ are generators of a second $U_q(g_n)$ algebra, and will generate in the coming examples the algebras $\bar{U}_q(g_{n-1})$.

To illustrate the working of the above two steps in the construction procedure of an algebra homomorphism we consider the A_n case of Sect. 3.

Example 4.7. The expressions for the algebra homomorphism ρ from $U_q(A_n)$ to $W_1 \otimes \cdots \otimes W_n \otimes \bar{U}_q(A_{n-1})$ in Theorem 3.2 can be constructed in a unique way using relations 4.6 and Ω. If one defines y to be

$$y = f_1^{(j_1)} f_2^{(j_2)} \cdots f_n^{(j_n)}\,,$$

then the formula for $\rho(f_i)$ in Theorem 3.2, a) is given by $\Omega(\mathrm{rel}(f_i y))$, using relations (vi, ii) in 4.6 and (6). Similarly, one obtains $\rho(e_i) = \Omega(\mathrm{rel}(e_i y))$ and $\rho(t_i) = \Omega(\mathrm{rel}(t_i y))$, using the relations (iv, v, vi) in 4.6 and (6). The monomial y which is used to construct the algebra homomorphism ρ in the $U_q(B_n)$ and $U_q(C_n)$ case (Theorem 3.2, b), c)) is

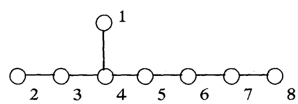
$$y = f_n^{(j_1)} f_{n-1}^{(j_2)} \dots f_2^{(j_{n-1})} f_1^{(j_n)} f_2^{(j_{n+1})} \dots f_n^{(j_{2n-1})} ,$$

and a similar monomial

$$y = f_n^{(j_1)} f_{n-1}^{(j_2)} \dots f_2^{(j_{n-1})} f_1^{(j_n)} f_3^{(j_{n+1})} \dots f_n^{(j_{2n-2})} ,$$

is used to derive the algebra homomorphism in the $U_q(D_n)$ case.

We start the investigation of the exceptional quantum Lie algebras with the $U_q(E_6)$, $U_q(E_7)$, $U_q(E_8)$ cases and define the following numbering of the nodes in the corresponding Dynkin diagrams



The integers d_i are equal to 1 for all i. Let us introduce the algebra $\bar{U}'_q(D_5)$ being generated by $\mathscr{F}_{i \neq 2}$, $\mathscr{E}_i$, $\mathscr{T}_i^{\pm 1} \in U_q(E_6)$. Similarly, one can define $\bar{U}_q(E_6)$ as $\mathscr{F}_{i \neq 7}$, $\mathscr{E}_i$, $\mathscr{T}_i^{\pm 1} \in U_q(E_7)$ and $\bar{U}_q(E_7)$ as given by the generators $\mathscr{F}_{i \neq 8}$, $\mathscr{E}_i$, $\mathscr{T}_i^{\pm 1} \in U_q(E_8)$. Algebra homomorphism $\bar{\rho}$ in case of these algebras arise analogously as in Lemma 3.1 by taking

$$\bar{\rho}(\mathscr{T}_2) = q^{\lambda_2} t_1^{-\frac{3}{4}} t_3^{-\frac{5}{4}} t_4^{-\frac{3}{2}} t_5^{-1} t_6^{-\frac{1}{2}} ,$$

$$\bar{\rho}(\mathscr{T}_7) = q^{\lambda_7} t_1^{-1} t_2^{-\frac{2}{3}} t_3^{-\frac{4}{3}} t_4^{-2} t_5^{-\frac{5}{3}} t_6^{-\frac{4}{3}} ,$$

$$\bar{\rho}(\mathscr{T}_8) = q^{\lambda_8} t_1^{-\frac{3}{2}} t_2^{-1} t_3^{-2} t_4^{-3} t_5^{-\frac{5}{2}} t_6^{-2} t_7^{-\frac{3}{2}} ,$$

respectively.

Let us abbreviate monomials of generators in $U_q(g_n)$ of type

$$f_i^{(j_k)} f_{i \pm 1}^{(j_{k+1})} \dots f_{i \pm l \mp 1}^{(j_{k+l-1})} f_{i \pm l}^{(j_{k+l})}$$

by $f_{ii \pm l}^{(j_k)}$. Using this notation we define the following monomials in $U_q(E_6)$, $U_q(E_7)$ and $U_q(E_8)$,

$$y_6 = f_{63}^{(j_1)} f_1^{(j_5)} f_{46}^{(j_6)} f_{25}^{(j_9)} f_1^{(j_{13})} f_{42}^{(j_{14})} ,$$

$$y_7 = f_{73}^{(j_1)} f_1^{(j_6)} f_{47}^{(j_7)} f_{26}^{(j_{11})} f_1^{(j_{16})} f_{45}^{(j_{17})} f_{34}^{(j_{19})} f_{23}^{(j_{21})} f_1^{(j_{23})} f_{47}^{(j_{24})} ,$$

$$y_8 = f_{83}^{(j_1)} f_1^{(j_7)} f_{48}^{(j_8)} f_{27}^{(j_{13})} f_1^{(j_{19})} f_{46}^{(j_{20})} f_{35}^{(j_{23})} f_{24}^{(j_{26})} f_1^{(j_{29})} f_{42}^{(j_{30})} f_{53}^{(j_{33})} f_{64}^{(j_{36})} f_1^{(j_{39})} ,$$
$$f_{72}^{(j_{40})} f_{84}^{(j_{46})} f_1^{(j_{51})} f_{38}^{(j_{52})} . \tag{7}$$

Similar to Example 4.7 one can obtain the expressions for algebra homomorphisms in the case of $U_\varepsilon(E_6)$, $U_\varepsilon(E_7)$ and $U_\varepsilon(E_8)$ by using relations 4.6 and Ω.

Theorem 4.8. *The following expressions define an algebra homomorphism ρ in the case of the quantum groups $U_q(E_6)$, $U_q(E_7)$ and $U_q(E_8)$. The composition $\pi := (\sigma_{gh} \otimes \mathrm{id}) \cdot \rho$ defines an algebra homomorphisms in the roots of unity case. The* *mappings ρ are defined by*

$$\text{e6)}\quad \rho\colon U_q(E_6) \to \left(\bigotimes_{k=1}^{16} W_k \right) \otimes \bar{U}'_q(D_5)\,,$$

$$\text{e7)}\quad \rho\colon U_q(E_7) \to \left(\bigotimes_{k=1}^{27} W_k \right) \otimes \bar{U}_q(E_6)\,,$$

$$\text{e8)}\quad \rho\colon U_q(E_8) \to \left(\bigotimes_{k=1}^{57} W_k \right) \otimes \bar{U}_q(E_7)\,,$$

$$\rho(f_i) = \Omega(\mathrm{rel}(f_i y_n)), \quad \rho(e_i) = \Omega(\mathrm{rel}(e_i y_n))\,,$$

$$\rho(t_i) = \Omega(\mathrm{rel}(t_i y_n)), \quad \forall i, \textit{and } n = 6, 7, 8\,.$$

Taking $\bar{U}'_\varepsilon(D_5)$, $\bar{U}_\varepsilon(E_6)$, $\bar{U}_\varepsilon(E_7)$ in an arbitrary representation $\bar{\rho}'$, one obtains representations π' which dimensions are given respectively by N^{16}, N^{27} and N^{57} times the dimensions of $\bar{\rho}'$. The number of free parameters of the representations π' is 32, 54 *and* 114, *respectively plus the number of free parameters in a representation $\bar{\rho}'$.*

The remaining two cases of exceptional Lie algebras are discussed in the following theorems. We fix the numbering of nodes in the Dynkin diagram for the F_4 case as

$$\underset{1}{\circ} - \underset{2}{\circ} \Rightarrow \underset{3}{\circ} - \underset{4}{\circ}$$

The integers d_i are defined as $d_1 = d_2 = 2$, $d_3 = d_4 = 1$. We write $\bar{U}'_q(B_3)$ for the algebra generated by $\mathscr{F}_{i \neq 4}$, $\mathscr{E}_i$, $\mathscr{T}_i^{\pm 1} \in U_q(F_4)$. Similarly to Lemma 3.1 one can give an algebra homomorphism $\bar{\rho}$ in the case of $\bar{U}'_q(B_3)$ starting from $U_q(B_3)$ and defining $\bar{\rho}(\mathscr{T}_4) = q^{\lambda_4} t_1^{-1/2} t_2^{-1} t_3^{-3/2}$. Using of the relations 4.6 and the operation Ω in the $U_\varepsilon(F_4)$ case, one can again define the algebra homomorphism in a short way. Let y_4 denote the monomial $y_4 = f_{14}^{(j_1)} f_2^{(j_5)} f_{31}^{(j_6)} f_{24}^{(j_9)} f_2^{(j_{12})} f_{31}^{(j_{13})}$.

Theorem 4.9. *The following expressions define an algebra homomorphism ρ from $U_q(F_4)$ to $W_1 \otimes \cdots W_{15} \otimes \bar{U}'_q(B_3)$ in a unique way and by composition with σ_{gh} an algebra homomorphism in the roots of unity case,*

$$\text{f)}\qquad \rho\colon U_q(F_4) \to \left(\bigotimes_{i=1}^{15} W_i \right) \otimes \bar{U}'_q(B_3)\,,$$

$$\rho(f_i) = \Omega(\mathrm{rel}(f_i y_4)), \quad \rho(e_i) = \Omega(\mathrm{rel}(e_i y_4)), \quad \rho(t_i) = \Omega(\mathrm{rel}(t_i y_4)), \quad \forall i\,.$$

Taking $\bar{U}'_q(B_3)$ in an arbitrary representation $\bar{\rho}'$ one obtains a representation π' of dimension N^{15} times the dimension in $\bar{\rho}'$. The number of free parameters in π' is 30 *plus the number of free parameters in $\bar{\rho}'$.*

Let G_2 be defined by the Cartan matrix with $a_{11} = 2$, $a_{12} = -3$, $a_{21} = -1$ and $a_{22} = 2$. The integers d_i are given as $d_1 = 1$, $d_2 = 3$. In the following $\bar{U}'_q(A_1)$ is defined by the $U_q(G_2)$ generators $\mathscr{F}_1$, $\mathscr{E}_1$, $\mathscr{E}_2$, $\mathscr{T}_1^{\pm 1}$, $\mathscr{T}_2^{\pm 1}$. In analogy to Lemma *3.1* one can easily give a corresponding algebra homomorphism $\bar{\rho}$, starting from $U'_q(A_1)$ and taking $\bar{\rho}(\mathscr{T}_2) = q^{\lambda_2} t_1^{-3/2}$.

Theorem 4.10. *The following formulas define an algebra homomorphism ρ from $U_q(G_2)$ to $W_1 \otimes \cdots \otimes W_5 \otimes \bar{U}'_q(A_1)$ and by composition with σ_{gh} an algebra* *homomorphism* $\pi := (\sigma_{gh} \otimes \mathrm{id}) \cdot \rho$ *in the roots of unity case.*

g) $$\rho: U_q(G_2) \to \left(\bigotimes_{i=1}^{5} W_i \right) \otimes \bar{U}'_q(A_1) ,$$

$$\rho(f_1) = \{z_3^3 z_4^{-2}\} x_1^{-1} x_2^{-1} x_3 x_4^2 + \{z_4 z_5^{-3}\} x_1^{-1} x_2^{-1} x_4^2 x_5 + \{x_2^2 x_3^{-3}\} x_1^{-1} x_2 x_3$$
$$+ \{z_2 z_4^{-1}\}\{q^2\} x_1^{-1} x_3 x_4 + \{z_1^3 z_2^{-1}\} x_2 + x_1^{-1} x_2^{-1} x_4 x_5 \mathscr{F}_1 ,$$

$$\rho(f_2) = \{z_1^{-3}\}_{q^3} x_1 ,$$

$$\rho(e_1) = \{z_2 z_3^{-3} z_5^{-3} z_4^2 \mathscr{T}_1\} x_2^{-1} + \{z_4 z_5^{-3} \mathscr{T}_1\} x_4^{-1} + \mathscr{E}_1 ,$$

$$\rho(e_2) = \{z_1^3 z_3^6 z_5^6 z_2^{-3} z_4^{-3} \mathscr{T}_2\}_{q^3} x_1^{-1} + \{z_3^3 z_5^6 z_4^{-3} \mathscr{T}_2\}_{q^3} x_3^{-1} + \{z_5^3 \mathscr{T}_2\}_{q^3} x_5^{-1} + \mathscr{E}_2 ,$$

$$\rho(t_1) = z_1^{-3} z_3^{-3} z_5^{-3} z_2^2 z_4^2 \mathscr{T}_1 ,$$

$$\rho(t_2) = z_1^6 z_3^6 z_5^6 z_2^{-3} z_4^{-3} \mathscr{T}_2 .$$

Taking $\bar{U}'_\varepsilon(A_1)$ *in an arbitrary representation* $\bar{\rho}'$ *one obtains a representation* π' *which dimension (number of free parameters) is* N^5 (10) *times (plus) the dimension (number of free parameters) of* $\bar{\rho}'$*, respectively.*

5. Representations of Maximal Dimensions

The representations constructed in Sects. 3 and 4 depend on the arbitrary representation $\bar{\rho}'$ of the corresponding $\bar{U}_\varepsilon(g_{n-1})$ algebras. In this section we want to show how a natural choice of this representation of $\bar{U}_\varepsilon(g_{n-1})$ leads to representations of $U_\varepsilon(g_n)$, which have maximal dimensions and number of parameters for odd N.

Irreducible representations of quantum groups $\bar{U}_\varepsilon(g_n)$ in the roots of unity case exist only in dimensions smaller than or equal to $N^{\Delta^+(g_n)}$ and the number of free parameters is maximally $\dim(g_n)$, for odd N [15]. The simplest representation of this type is the non-highest weight representation $\pi' := \sigma_{gh} \cdot \rho$ of $U_\varepsilon(A_1)$ which is defined in terms of Weyl algebra generators by

$$\rho(f) = \{z^{-1}\} x, \quad \rho(e) = \{\varepsilon^{\lambda_1} z\} x^{-1}, \quad \rho(t) = \varepsilon^{\lambda_1} z^2 .$$

Representation π' has dimension N and the map σ_{gh} together with the complex parameter λ_1 gives rise to the 3 free parameters of the representation. Starting from this representation one can step by step construct representations of higher rank Lie algebras, using inductively representations $\bar{\rho}$ of of $\bar{U}_\varepsilon(g_{n-1})$ and representations π' of $U_\varepsilon(g_n)$. A representation of $U_\varepsilon(g_n)$ obtained in this way will have $2\Delta^+(g_n)$ number of free parameters induced by the free parameters of the mappings σ_{gh} and in addition n parameters λ_i coming from the definition of $\bar{\rho}$ (see e.g. Lemma 3.1). Also, the dimension of such a representation is induced by the map σ_{gh} and adds up to $N^{\Delta^+(g_n)}$. This gives rise to the following theorem.

Theorem 5.11. *Using inductively the representations of Theorems* 3.2–4.10 *to define representations of* $\bar{U}_\varepsilon(A_1), \ldots, \bar{U}_\varepsilon(g_{n-2}), \bar{U}_\varepsilon(g_{n-1})$ *one obtains representations of* $U_\varepsilon(g_n)$ *which have maximal dimensions and number of free parameters for odd* N*, in all cases of semi-simple Lie algebras* g_n*.*

Remark 5.12. In the case of $U_\varepsilon(A_n)$ one obtains the maximal cyclic representations of [13]. In [13] it is also proven, that this representation is generically irreducible. In the case of quantum $SO(5)$ irreducible representations of maximal dimension were obtained in [3]. For the general B_n, C_n and D_n series the representations of maximal dimension of Theorem 5.11 coincide with those in [26], for which the irreducibility for odd N was established in the simplest examples.

Remark 5.13. The construction of algebra homomorphisms and representations for $U_q(g_n)$ was based on the action of the generators $e_i, f_i, t_i^{\pm 1}$ on monomials of length $\Delta^+(g_n) - \Delta^+(g_{n-1})$. Such monomials were defined e.g. in Example 4.7 and in (7). Adjoining these monomials according to the inductive procedure of Theorem 5.11 gives monomials $f_{i_1}^{(j_1)} \cdots f_{i_l}^{(j_l)}$ of length $l = \Delta^+(g_n)$. In all cases of quantum Lie algebras discussed above the ordering in these monomials corresponds to the ordering of simple reflexions in a longest element in the Weyl group of g_n.

6. Conclusions

Starting from an arbitrary representation of $U_q(A_1)$ one can construct representations for all higher rank semi-simple Lie algebras by "adding" the additional generators which arise with the adding of a new node to the Dynkin diagram. The dependence of the constructed representations of $U_\varepsilon(g_n)$ on the algebra $U_\varepsilon(g_{n-1})$ results in representations of quantum groups in dimensions greater than or equal to $N^{\Delta^+(g_n) - \Delta^+(g_{n-1})}$. Only comparably very few representations in such dimensions were previously known. Even more representations, especially highest weight representations of $U_\varepsilon(g_n)$ can be found by specializing some or all of the free parameters. Although, the constructed representations do not yet lead to representations of quantum groups at ε an N-th root of unity in all possible dimensions for irreducible representations, one could hope that this problem might be settled in the future.

The minimal and maximal cyclic representations of $U_\varepsilon(A_n)$, which are both special cases of the above described representations are basic algebraic structures related to the generalizations of the chiral Potts model in [7, 13, 24]. The next step is to find statistical models related to representations of the other ε-deformed Lie algebras g_n discussed above. In analogy to the $U_q(sl_n)$ case one would expect that starting from the affine extension of an ε-deformation of an arbitrary semi-simple Lie algebra g_n one could find algebraic varieties which determine relations among the spectral parameters to allow the existence and the construction of the intertwining R-matrix of two representations. This article is meant to be a small contribution to the research going in this direction.

But applications of roots of unity representations are not restricted to chiral Potts type models. Many further applications are found to lie in the development of other integrable models, conformal field theory (e.g. fusion rules), and areas in mathematics such as the representation theory of affine Lie algebras or the theory of semi-simple groups over fields of positive characteristic.

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7. Appendix

In the case of applying the construction procedure of Sect. 4 to derive algebra homomorphisms for $U_q(E_6)$, $U_q(E_7)$ quantum Lie algebras only $U_q(A_2)$ type relations in 4.6 were necessary. For the construction of representations in the case of E_8 and F_4 it is necessary to use two further relations in $U_q(E_8)$ and $U_q(F_4)$.

Lemma 7.14. *Let $f_1, \ldots, f_8$ be generators in $U_q(E_8)$. Then the generator f_1 commutes with the monomial $f_4^{(j_1)} f_5^{(j_2)} f_3^{(j_3)} f_4^{(j_4)} f_1^{(j_5)} f_4^{(j_6)} f_5^{(j_7)} f_3^{(j_8)} f_4^{(j_9)}$ as following:*

$$
\begin{aligned}
&f_1 f_4^{(j_1)} f_5^{(j_2)} f_3^{(j_3)} f_4^{(j_4)} f_1^{(j_5)} f_4^{(j_6)} f_5^{(j_7)} f_3^{(j_8)} f_4^{(j_9)} \\
&\quad = f_4^{(j_1-1)} f_5^{(j_2-1)} f_3^{(j_3-1)} f_4^{(j_4-1)} f_1^{(j_5)} f_4^{(j_6+1)} f_5^{(j_7+1)} f_3^{(j_8+1)} f_4^{(j_9+1)} f_1 \\
&\qquad + f_4^{(j_1-1)} f_5^{(j_2-1)} f_3^{(j_3-1)} f_4^{(j_4-1)} f_1^{(j_5+1)} f_4^{(j_6+1)} f_5^{(j_7+1)} f_3^{(j_8+1)} f_4^{(j_9+1)} [-j_5+j_6+j_9+1] \\
&\qquad + f_4^{(j_1-1)} f_5^{(j_2-1)} f_3^{(j_3-1)} f_4^{(j_4)} f_1^{(j_5+1)} f_4^{(j_6+1)} f_5^{(j_7+1)} f_3^{(j_8+1)} f_4^{(j_9)} [-j_4-j_6+j_8+j_7+1] \\
&\qquad + f_4^{(j_1-1)} f_5^{(j_2-1)} f_3^{(j_3-1)} f_4^{(j_4)} f_1^{(j_5+1)} f_4^{(j_6)} f_5^{(j_7+1)} f_3^{(j_8+1)} f_4^{(j_9+1)} [-j_4+j_9+1] \\
&\qquad + f_4^{(j_1-1)} f_5^{(j_2-1)} f_3^{(j_3)} f_4^{(j_4)} f_1^{(j_5+1)} f_4^{(j_6+1)} f_5^{(j_7+1)} f_3^{(j_8)} f_4^{(j_9)} [-j_3+j_7+1] \\
&\qquad + f_4^{(j_1-1)} f_5^{(j_2)} f_3^{(j_3-1)} f_4^{(j_4)} f_1^{(j_5+1)} f_4^{(j_6+1)} f_5^{(j_7)} f_3^{(j_8+1)} f_4^{(j_9)} [-j_2+j_8+1] \\
&\qquad + f_4^{(j_1-1)} f_5^{(j_2)} f_3^{(j_3)} f_4^{(j_4)} f_1^{(j_5+1)} f_4^{(j_6+1)} f_5^{(j_7)} f_3^{(j_8)} f_4^{(j_9)} [-j_2-j_3+j_4+j_6+1] \\
&\qquad + f_4^{(j_1)} f_5^{(j_2)} f_3^{(j_3)} f_4^{(j_4-1)} f_1^{(j_5+1)} f_4^{(j_6+1)} f_5^{(j_7)} f_3^{(j_8)} f_4^{(j_9)} [-j_1+j_6+1] \\
&\qquad + f_4^{(j_1)} f_5^{(j_2)} f_3^{(j_3)} f_4^{(j_4)} f_1^{(j_5+1)} f_4^{(j_6)} f_5^{(j_7)} f_3^{(j_8)} f_4^{(j_9)} [-j_1-j_4+j_5+1].
\end{aligned}
$$

The proof of this lemma is solely based on the A_2-type Serre relations which define the commutation relations among the f_i generators.

Lemma 7.15. *Let $f_1, \ldots, f_4$ be generators of $U_q(F_4)$. Then the action of f_3 on the monomial $f_4^{(j_1)} f_3^{(j_2)} f_2^{(j_3)} f_1^{(j_4)} f_3^{(j_5)} f_2^{(j_6)} f_3^{(j_7)} f_4^{(j_8)} f_3^{(j_9)} f_2^{(j_{10})} f_3^{(j_{11})}$ is given by*

$$
\begin{aligned}
&f_3 f_4^{(j_1)} f_3^{(j_2)} f_2^{(j_3)} f_1^{(j_4)} f_3^{(j_5)} f_2^{(j_6)} f_3^{(j_7)} f_4^{(j_8)} f_3^{(j_9)} f_2^{(j_{10})} f_3^{(j_{11})} \\
&\quad = f_4^{(j_1-1)} f_3^{(j_2+1)} f_2^{(j_3)} f_1^{(j_4)} f_3^{(j_5-1)} f_2^{(j_6-1)} f_3^{(j_7-1)} f_4^{(j_8)} f_3^{(j_9+1)} f_2^{(j_{10}+1)} f_3^{(j_{11}+1)} f_4 \\
&\qquad + f_4^{(j_1-1)} f_3^{(j_2+1)} f_2^{(j_3)} f_1^{(j_4)} f_3^{(j_5-1)} f_2^{(j_6-1)} f_3^{(j_7-1)} f_4^{(j_8+1)} f_3^{(j_9+1)} f_2^{(j_{10}+1)} f_3^{(j_{11}+1)} \\
&\qquad \times [j_{11}+j_9+1-j_8] \\
&\qquad + f_4^{(j_1-1)} f_3^{(j_2+1)} f_2^{(j_3)} f_1^{(j_4)} f_3^{(j_5-1)} f_2^{(j_6-1)} f_3^{(j_7)} f_4^{(j_8+1)} f_3^{(j_9+1)} f_2^{(j_{10}+1)} f_3^{(j_{11})} \\
&\qquad \times [2j_{10}+1-j_7-j_9] \\
&\qquad + f_4^{(j_1-1)} f_3^{(j_2+1)} f_2^{(j_3)} f_1^{(j_4)} f_3^{(j_5-1)} f_2^{(j_6-1)} f_3^{(j_7)} f_4^{(j_8+1)} f_3^{(j_9)} f_2^{(j_{10}+1)} f_3^{(j_{11}+1)} \\
&\qquad \times [j_{11}+1-j_7] \\
&\qquad + f_4^{(j_1-1)} f_3^{(j_2+1)} f_2^{(j_3)} f_1^{(j_4)} f_3^{(j_5-1)} f_2^{(j_6)} f_3^{(j_7)} f_4^{(j_8+1)} f_3^{(j_9+1)} f_2^{(j_{10})} f_3^{(j_{11})} \\
&\qquad \times [-2j_6+j_7+j_9+1] \\
&\qquad + f_4^{(j_1-1)} f_3^{(j_2+1)} f_2^{(j_3)} f_1^{(j_4)} f_3^{(j_5)} f_2^{(j_6)} f_3^{(j_7-1)} f_4^{(j_8+1)} f_3^{(j_9+1)} f_2^{(j_{10})} f_3^{(j_{11})} \\
&\qquad \times [-j_5+j_9+1]
\end{aligned}
$$

$$+f_4^{(j_1-1)}f_3^{(j_2+1)}f_2^{(j_3)}f_1^{(j_4)}f_3^{(j_5)}f_2^{(j_6)}f_3^{(j_7)}f_4^{(j_8+1)}f_3^{(j_9)}f_2^{(j_{10})}f_3^{(j_{11})}$$
$$\times[-j_5-j_7+j_8+1]$$
$$+f_4^{(j_1)}f_3^{(j_2+1)}f_2^{(j_3)}f_1^{(j_4)}f_3^{(j_5)}f_2^{(j_6)}f_3^{(j_7)}f_4^{(j_8)}f_3^{(j_9)}f_2^{(j_{10})}f_3^{(j_{11})}$$
$$\times[-j_1+j_2+1]\,.$$

The identity of this lemma was obtained using the commutation relations among f_i generators with terms $f_k^{(j)}$ in the cases of $U_q(A_2)$, $U_q(B_2)$ and $U_q(C_2)$.

With the help of the above lemmas it is possible to construct the representation of $U_\varepsilon(E_8)$ and $U_q(F_4)$ in Sect. 4.

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