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Selberg Trace Formula for Bordered Riemann Surfaces: Hyperbolic, Elliptic and Parabolic Conjugacy Classes, and Determinants of Maass–Laplacians

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Abstract: The Selberg trace formula for automorphic forms of weight $m \in \mathbb{Z}$, on bordered Riemann surfaces is developed. The trace formula is formulated for arbitrary Fuchsian groups of the first kind with reflection symmetry which include hyperbolic, elliptic and parabolic conjugacy classes. In the case of compact bordered Riemann surfaces we can explicitly evaluate determinants of Maass–Laplacians for both Dirichlet and Neumann boundary-conditions, respectively. Some implications for the open bosonic string theory are mentioned.

I. Introduction

Spectral theory of automorphic forms has recently seen some activity in the physical literature, due to its importance in string theory. This theory originates from the work of A. Selberg [50], where the famous so-called "Selberg trace formula" was first presented. Other classical contributions are due to Hejhal [32, 33] and Venkov [56, 57]. String theory gave new interest in this work, first to refine the Selberg trace formula further in order to calculate determinants of Laplacians on Riemann surfaces, and second to develop other versions of the Selberg trace formula. Here the generalization to the fermionic- (super-) string theory was most important, leading to a formulation of a trace formula on super Riemann surfaces, the Selberg super trace formula [8, 25–27].

Because the original contribution of Selberg was founded in the field of number theory, physicists only lately acknowledged its value in periodic orbit theory as founded by Gutzwiller [30, 52] (see also Albeverio et al. for a thorough mathematical treatment of a particular system [1]), who rediscovered the Selberg trace formula years later within his formalism [31], and in quantum field theory on Riemann surfaces, i.e. the Polyakov approach [17–19, 47] to (bosonic-, fermionic- and super-) string theory. In the perturbative expansion of the Polyakov path

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integral one is left with a summation over all topologies of world sheets a string can sweep out, and an integral over the moduli space of Riemann surfaces. This picture is true for bosonic strings as well as for fermionic strings. The partition function for open as well as closed bosonic strings corresponding to a topology without conformal Killing vectors (D'Hoker and Phong [17, 19]) turns out to be

$$Z_0=\sum_g \int d\mu_{WP}[\det(P_1^\dagger P_1)]^{1/2}(\det{}'\Delta_0)^{-D/2}\;, \tag{1.1}$$

where P_1 and Δ_0 are the symmetrized traceless covariant derivative and scalar Laplacian with Dirichlet boundary-conditions, respectively, and $d\mu_{\mathrm{WP}}$ is the Weil–Petersson measure. D denotes the critical dimension which equals 26 for the bosonic string. For the fermionic-, respectively the super-string all quantities have to be replaced by their appropriate super versions, and the critical dimension is $D=10$.

The calculation of determinants of Laplacians on Riemann surfaces is due to several authors. Let us mention the evaluation of these determinants in terms of Selberg zeta-functions by e.g. Bolte and Steiner [11], D'Hoker and Phong [18, 19], Efrat [21], Gilbert [23], Namazie and Rajeev [44], Sarnak [49], Steiner [53], and Voros [58], and in terms of the period matrix and theta-functions, e.g. Alvarez-Gaumé et al. [3] and Manin [40]. Let us note that the Selberg zeta-function approach enabled Gross and Periwal [28] to show that the bosonic string perturbation theory is not (Borel-) summable and hence not finite.

In the perturbative expansion of the bosonic string [17, 19, 47] the classical Selberg trace formula could be applied [23, 44, 53] in a straightforward way, whereas the perturbation theory for the fermionic string required the introduction of the Selberg super-trace formula. Here Baranov, Manin et al. [8] originally started this activity, and it was further developed by Aoki [4] and Grosche [25–27].

It was mainly the closed bosonic string that was dealt with and for which the perturbation theory for scattering amplitudes was developed quite comprehensively, whereas the open bosonic string took somewhat longer to be developed, starting with the pioneering work of Alvarez [2], and until now seems not as well developed as the former one.

The Polyakov path integral approach does not cause too much difficulties, see e.g. Blau et al. [9, 10], Bolte and Steiner [12], Burgess and Morris [14], Carlip [15], Dunbar [20], Jaskólski [35], Luckock [38], Martín-Delgado and Mittelbrunn [41], Ohndorf [45], Rodrigues et al. [16, 48], and Wu [59].

There are (at least) two possibilities to express the scalar determinant: either by the period-matrix and theta-functions, or by appropriately chosen Selberg zeta-functions for the corresponding Dirichlet or Neumann boundary-value problems, respectively. The former approach was discussed by Bolte and Steiner [13], Burgess and Morris [14], Dunbar [20], Losev [37], Luckock [38], Martín-Delgado and Mittelbrunn [41], and Mozorov and Rosly [43] for multiloop expressions. The latter case was treated by Blau et al. [9, 10], and Bolte and Steiner [12, 13].

Of course, while dealing with open strings one has to distinguish Dirichlet and Neumann boundary-conditions, respectively. In particular, relations between the determinants $\det\Delta_\Sigma^{(D)}$ and $\det{}'\Delta_\Sigma^{(N)}$ corresponding to Dirichlet and Neumann boundary-conditions on the bordered surface Σ, and the determinant of the scalar

Laplacian $\det'\Delta_{\hat{\Sigma}}$ for the doubled (closed) surface $\hat{\Sigma}$ could be derived, i.e.

$$\det \Delta_{\Sigma}^{(D)} \cdot \det'\Delta_{\Sigma}^{(N)} = \det'\Delta_{\hat{\Sigma}} . \tag{1.2}$$

For automorphic functions (i.e. automorphic forms of weight $m=0$) the Selberg trace formula for bordered Riemann surfaces already exists and almost the entire theory is due to Venkov [56, 57], see also [55] and the particular case for PSL (2, $\mathbb{Z}$) [54], and Guillope [29]. Independently, later on a more simple version of the Selberg trace formula for bordered Riemann surfaces was developed by Blau and Clements [9] and Bolte and Steiner [13].

In this article we want to develop the Selberg trace formula on bordered Riemann surfaces for automorphic forms of weight m, $m \in \mathbb{Z}$. In doing so we bring together results of Hejhal, who has developed the Selberg trace formula for automorphic forms of weight m on closed Riemann surfaces, with those of Venkov, who has set up the trace formula for automorphic functions on bordered Riemann surfaces. This merge seems still to be missing and appears to be interesting in its own in the theory of automorphic forms and number theory; and it turns out to be useful in string theory, because we are able to calculate determinants of Maass–Laplacians on bordered Riemann surfaces which are needed in the string path integral.

Our paper is organized as follows. In the second section we shortly describe how to construct a bordered Riemann surface and its double, respectively, and we refer to some results of Venkov concerning the relevant conjugacy classes of the Fuchsian groups on bordered Riemann surfaces. Also the Selberg operator on the involuted surface is explicitly constructed.

In the third section the Selberg trace formula for automorphic forms of weight m, $m \in \mathbb{Z}$, is explicitly calculated, first for hyperbolic conjugacy classes only, and then incorporating elliptic and parabolic conjugacy classes.

In the fourth section we calculate determinants of Maass–Laplacians on bordered Riemann surfaces. They will be expressed in terms of the Selberg zeta-function on bordered Riemann surfaces. Since this Selberg zeta-function was thoroughly studied by Venkov [56], we rely on his results and rather do not explicitly recall them.

Dirichlet as well as Neumann boundary-conditions will be considered in the discussion of determinants of Maass–Laplacians on bordered Riemann surfaces.

All principal results will be stated as theorems.

The last section is devoted to a summary and a discussion of the results. These concern the determinants $[\det(P_1^{\dagger}P_1)]^{1/2} = \det(\frac{1}{2}\Delta_1^{(+)})$ and $\det(\frac{1}{2}\Delta_0)$ in terms of the Selberg zeta-function and their growing behaviour for increasing genus, from which follows that the bosonic string theory for closed as well as open strings diverges; furthermore we shortly mention the form of Weyl's and Huber's law for bordered Riemann surfaces as they can be derived from the trace formula.

In Appendix A we provide some results concerning the analytic properties of the Selberg zeta-function on bordered Riemann surfaces, where we generalize some results of Venkov, and in Appendix B an important integral is evaluated.

II. Bordered Riemann Surfaces and Conjugacy Classes

1. Automorphic forms on bordered Riemann surfaces. Let $\tilde{\Sigma}$ be a closed Riemann surface of genus g, and $d_1, \ldots, d_m$ conformal, non-overlapping discs on $\tilde{\Sigma}$. Then

$\Sigma := \tilde{\Sigma} \setminus \{d_1, \ldots, d_m\}$ is a bordered Riemann surface with signature (g, n). $c_i = \partial d_i (i = 1, \ldots, n)$ are the n components of $\partial \Sigma$. Now one takes a copy $\mathscr{I}\Sigma$ of Σ, a mirror image, and glues both surfaces together along $\partial\Sigma$ and $\partial\mathscr{I}\Sigma$, giving the doubled surface $\hat{\Sigma} := \Sigma \cup \mathscr{I}\Sigma$. Furthermore $\Sigma = \hat{\Sigma}/\mathscr{I}$, and $\hat{\Sigma}$ is a closed Riemann surface of genus $\hat{g} = 2g + n - 1$. The uniformization theorem for Riemann surfaces states that $\hat{\Sigma}$ is conformally equivalent to $\Gamma\setminus U$ with the universal covering $U = \hat{\mathbb{C}}, \mathbb{C}, \mathscr{H}$, and Γ a discrete, fixed-point free subgroup of the conformal automorphisms of U. For either $g \geqq 2$, or bordered Riemann surfaces with $\hat{g} \geqq 2$, the relevant universal covering is the Poincaré upper half-plane $\mathscr{H} = \{z = x + iy \mid y > 0, x \in \mathbb{R}\}$ endowed with the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$. Hence, $\hat{\Sigma}$ may be represented as $\hat{\Sigma} \cong \hat{\Gamma}\setminus\mathscr{H}$, where $\hat{\Gamma}$ is a Fuchsian group, i.e. a discrete subgroup of PSL(2, $\mathbb{R}$). Σ and $\hat{\Sigma}$ may be represented as fundamental polygons $\mathscr{F}$ and $\hat{\mathscr{F}}$, respectively, tesselating the entire Poincaré upper half-plane by means of the group action.

Until now we have only considered strictly hyperbolic Fuchsian groups $\hat{\mathscr{F}}$. In this paper we also want to consider groups $\hat{\Gamma}$ which contain elliptic and parabolic elements, but nevertheless possess the above described symmetry.

Let us denote by s the number of inequivalent elliptic fixed points and by κ the number of inequivalent cusps. v_j denotes the order of the generators of the elliptic subgroups $R_j \subset \hat{\Gamma}$ $(1 \leqq j \leqq s)$; this means that $R_j^{v_j} = 1$ for $(1 \leqq j \leqq s,\ 1 < v_j < \infty)$.

If $\hat{\Gamma}$ contains no elliptic elements, i.e. no $\gamma \in \hat{\Gamma}$ has a fixed point on $\mathscr{H}$, $\hat{\Gamma}\setminus\mathscr{H}$ is a regular surface. If, however, elliptic elements are present in $\hat{\Gamma}$, $\hat{\Gamma}\setminus\mathscr{H}$ will only have a manifold structure outside the respective fixed points. Including these turns $\hat{\Gamma}\setminus\mathscr{H}$ into an orbifold. Despite this slight complication any $\hat{\Gamma}\setminus\mathscr{H}$, irrespective of a possible existence of orbifold-points, will in the following be called a hyperbolic surface.

On a regular hyperbolic surface the conjugacy classes $\{\gamma\}_{\hat{\Gamma}}$ of hyperbolic $\gamma \in \hat{\Gamma}$ are in one-to-one correspondence with the closed geodesics on $\hat{\Sigma}$ and we denote by l_γ the *length* of the closed geodesic on $\hat{\Sigma}$ related to $\{\gamma\}_{\hat{\Gamma}}$. The *norm* N_γ of a hyperbolic element $\gamma \in \Gamma$ and the length l_γ are related by $N_\gamma = e^{l_\gamma}$. This one-to-one correspondence is no longer true if elliptic elements are present. However, the norms N_γ of conjugacy classes in $\hat{\Gamma}$ are still properly defined, and we use sometimes the notion "lengths of closed orbits" and "norms of conjugacy classes" irrespective of a possible existence of such orbifold points, keeping in mind that "norms of conjugacy classes" is the more correct one.

In order to construct a convenient fundamental domain and representation of the involution $\mathscr{I}$ on it, one takes, according to Sibner [51] and Venkov [56], $\hat{\Sigma}$ as a symmetric Riemann surface with reflection symmetry $\mathscr{I}$. Then $\hat{\mathscr{F}}$ may be chosen as the interior of a fundamental polygon in $\mathscr{H}$ with $4\hat{g} + 2n - 2 + 2(s + \kappa)$ edges, and area [33, p. 2]

$$\mathscr{A}(\hat{\mathscr{F}}) = 2\pi \left[2(\hat{g} - 1) + \kappa + \sum_{j=1}^{s} \left(1 - \frac{1}{v_j}\right) \right]. \tag{2.1}$$

The fundamental polygon $\hat{\mathscr{F}}$ is chosen to be symmetric with respect to the imaginary axis. That is, we can translate the polygon $\hat{\mathscr{F}}$ in such a way that the side across which $\mathscr{I}$ is a reflection, say the boundary curve c_n, runs along the y-axis in the Poincaré upper half-plane. Here $\mathscr{I}$ takes on the form

$$\mathscr{I} : z \to -\bar{z} . \tag{2.2}$$

The other sides are among the edges of the fundamental polygon. This choice of $\hat{\mathscr{F}}$ is adopted for convenience and in no way reduces the generality of our considerations.

Now let $\hat{\Gamma}$ be a Fuchsian group for the doubled surface $\hat{\Sigma}$, and $\bar{\Gamma} \subset \mathrm{SL}(2, \mathbb{R})$ such that $\hat{\Gamma} = \bar{\Gamma}/\{\pm \mathbb{1}\}$. Let $\chi : \bar{\Gamma} \to U(1)$ be any multiplier system with $\chi(-\mathbb{1}) = e^{-i\pi m}$ [32, p. 357]. $\chi(\gamma)$ will also be denoted by χ_γ. We define ($m \in \mathbb{Z}$)

$$j(\gamma, z) = \left(\frac{cz+d}{c\bar{z}+d}\right)^{m/2} = \left(\frac{cz+d}{|cz+d|}\right)^m, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}, z \in \mathscr{H} . \tag{2.3}$$

Obviously we have

$$\begin{aligned} &|j(\gamma, z)| = 1 , \\ &j(\gamma\sigma, z) = j(\gamma, \sigma z)\, j(\sigma, z), \quad \forall \gamma, \sigma \in \bar{\Gamma} . \end{aligned} \tag{2.4}$$

We define automorphic forms $f(z)$ of weight m to be $\mathbb{C}$-valued functions on $\mathscr{H}$ having the property

$$f(\gamma z) = \chi_\gamma^m j(\gamma, z) f(z), \quad \forall \gamma \in \bar{\Gamma} . \tag{2.5}$$

The set of such differentiable automorphic forms will be denoted by $C^\infty(\chi, m)$, and $\mathscr{L}^2(\chi, m)$ then is the space of square integrable automorphic forms, i.e.

$$\int_{\hat{\Gamma} \backslash \mathscr{H}} dV(z) |f(z)|^2 < \infty, \quad dV(z) = \frac{dxdy}{y^2} . \tag{2.6}$$

We consider the operator $D_m = -y^2(\partial_x^2 + \partial_y^2) + imy\partial_x$ acting on $C^\infty(\chi, m)$. There exists a (unique) self-adjoint extension of this operator on $\mathscr{L}^2(\chi, m)$ which we also denote by D_m and which is known as the Maass–Laplacian. For a general Fuchsian group of the first kind D_m has a discrete and a continuous spectrum. If $\hat{\Gamma} \backslash \mathscr{H}$ is compact, however, there is only a discrete spectrum. In case of a non-cocompact Fuchsian group it is not known in general whether Maass–Laplacians have infinitely many eigenvalues. Only for arithmetic Fuchsian groups the discrete spectra are known to be infinite, see e.g. [56].

We now introduce the point pair invariant

$$\begin{aligned} k(z, w) &= \left(\frac{w-\bar{z}}{z-\bar{w}}\right)^{m/2} \Phi\left(\frac{|z-w|^2}{\Im(z)\Im(w)}\right) \\ &= i^m \left(\frac{w-\bar{z}}{|w-\bar{z}|}\right)^m \Phi\left(\frac{|z-w|^2}{\Im(z)\Im(w)}\right) \end{aligned} \tag{2.7}$$

for some $\Phi \in C_c^2(\mathbb{R})$ for $z, w \in \mathscr{H}$. $k(z, w)$ has the properties

$$\begin{aligned} &k(z, w) = \overline{k(w, z)} , \\ &k(\gamma z, \gamma w) = j(\gamma, z) k(z, w) j^{-1}(\gamma, w) \end{aligned} \tag{2.8}$$

for all $\gamma \in \hat{\Gamma}$. We introduce the automorphic kernel by [32, p. 360]

$$K(z, w) = \frac{1}{2} \sum_{\gamma \in \hat{\Gamma}} \chi_\gamma^m j(\gamma, w) k(z, \gamma w) , \tag{2.9}$$

and the factor "1/2" is included because $\gamma\in\bar{\Gamma}$ runs through γ and $-\gamma$, respectively. The automorphic kernel has the properties

$$K(z,w)=\overline{K(w,z)}\ ,$$
$$K(\gamma z,\sigma w)=\chi_\gamma^m j(\gamma,z)K(z,w)j^{-1}(\sigma,w)\chi_\sigma^{-m} \tag{2.10}$$

for all $\gamma,\sigma\in\hat{\Gamma}$. This construction of the automorphic kernel is valid for arbitrary Fuchsian groups.

In the theory of symmetric spaces it is convenient to consider the following isomorphic model of $\mathscr{H}$. One defines the positive definite symmetric matrices

$$z(x;y)=\begin{pmatrix} y+x^2/y & x/y \\ x/y & 1/y \end{pmatrix},\quad (x\in\mathrm{I\!R},\ y>0)\ . \tag{2.11}$$

If $g\in\mathrm{SL}(2,\mathrm{I\!R})$, the group action has the form

$$gz(x;y)=g[z(x;y)]g^t\ , \tag{2.12}$$

where g^t is the transpose of g. In this model it is easy to implement the involution $\mathscr{I}$ in terms of the matrix

$$\mathscr{I}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.13}$$

where $\mathscr{I}$ is viewed as an element in $\mathrm{GL}(2,\mathrm{I\!R})/\{\pm\mathbb{1}\}$.

We define even and odd automorphic forms, respectively, by having the property $[f\in\mathscr{L}^2(\chi,m)]$,

$$f(\mathscr{I}z)=\chi(\mathscr{I})f(z)\ , \tag{2.14}$$

where we have extended the multiplier system χ from $\bar{\Gamma}$ to $\bar{\Gamma}\cup\bar{\Gamma}\mathscr{I}$ by setting: $\chi(\gamma\mathscr{I})=\chi(\gamma)\chi(\mathscr{I})$ for $\gamma\in\bar{\Gamma}$. We have $\chi(\mathscr{I})=+1$ for Neumann, and $\chi(\mathscr{I})=-1$ for Dirichlet boundary-conditions.

It is well-known [32] that any eigen-function $\phi\in C^2(\mathscr{H})$ of D_m, $D_m\phi=\lambda\phi$, is also an eigen-function of the integral operator L,

$$(L\phi)(z)=\int_{\mathscr{H}} dV(w)k(z,w)\phi(w)=\Lambda(\lambda)\phi(z)\ . \tag{2.15}$$

$\Lambda(\lambda)$ only depends on Φ, m and λ. On the doubled Riemann surface $\hat{\Sigma}$, i.e. concerning the Fuchsian group $\hat{\Gamma}$, we are lead to a natural definition of the Selberg integral operator $\hat{L}_\pm$ acting on even and odd $f\in\mathscr{L}^2(\chi,m)$, respectively, as follows

$$\begin{aligned}(\hat{L}_\pm f)(z)&=\int_{\hat{\mathscr{F}}} dV(w)\hat{K}_\pm(z,w)f(w)\\ &=\frac{1}{2}\int_{\hat{\mathscr{F}}} dV(w)K(z,w)f(w)\pm\frac{1}{2}\int_{\hat{\mathscr{F}}} dV(w)K(z,-\bar{w})f(w)\ ,\end{aligned} \tag{2.16}$$

with the $\pm$-sign for Dirichlet and Neumann boundary-conditions on $\partial\Sigma$, respectively. Therefore we obtain for the automorphic kernel the expression

$$\hat{K}_\pm(z,w)=\frac{1}{4}\sum_{\gamma\in\bar{\Gamma}}\chi_\gamma^m j(\gamma,w)k(z,\gamma w)\pm\frac{1}{4}\sum_{\gamma\in\bar{\Gamma}}\chi_\gamma^m j(\gamma,-\bar{w})k[z,\gamma(-\bar{w})]\ . \tag{2.17}$$

In case $\hat{\Gamma}$ is strictly hyperbolic, i.e. besides the identity it contains only hyperbolic elements, it is known [32] that the trace of the Selberg operator L is given by

$$\operatorname{tr}(L)=\int_{\hat{\mathscr{F}}} dV(z)K(z,z)=\sum_{n=0}^{\infty} h(p_n)\,, \tag{2.18}$$

where $\Lambda(\lambda_n)=\Lambda(p_n^2+\frac{1}{4})=h(p_n)$. One now obtains the following

Theorem 2.1 [32, p. 448, 50, 56, p. 76]. *The Selberg trace formula for automorphic forms of weight $m\in\mathbb{Z}$ on compact Riemann surfaces has the form*

$$\sum_{n=0}^{\infty} h(p_n)=-\frac{\mathscr{A}(\hat{\mathscr{F}})}{8\pi^2}\int_0^{\infty}\frac{\cosh\frac{m}{2}u}{\sinh\frac{u}{2}}g'(u)\,du+\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{\chi_\gamma^{mk}l_\gamma g(kl_\gamma)}{2\sinh\frac{kl_\gamma}{2}}\,. \tag{2.19}$$

Here on the left-hand side n labels all eigenvalues $\lambda_n=\frac{1}{4}+p_n^2$ of D_m, where only one root p_n is counted. $h(p)$ is an even function in p and has the properties

i) *$h(p)$ is holomorphic in the strip $|\Im(p)|\leqq\frac{1}{2}+\varepsilon$, $\varepsilon>0$.*
ii) *$h(p)$ has to decrease faster than $|p|^{-2}$ for $p\to\pm\infty$.*
iii) *$g(u)=\pi^{-1}\int_0^{\infty}h(p)\cos(\pi p)\,dp$.*

Let us note that the first term on the right-hand side in the trace formula of Theorem 2.1 can also be written as

$$-\frac{\mathscr{A}(\hat{\mathscr{F}})}{8\pi^2}\int_0^{\infty}\frac{\cosh\frac{m}{2}u}{\sinh\frac{u}{2}}g'(u)\,du=\frac{\mathscr{A}(\hat{\mathscr{F}})}{4\pi}\left\{\int_{-\infty}^{\infty}dp\,ph(p)\frac{\sinh(2\pi p)}{\cosh(2\pi p)+\cos(\pi m)}\right.$$
$$\left.+\sum_{n=0}^{[(m-1)/2]}(m-2n-1)h\left[\frac{\mathrm{i}}{2}(m-2n-1)\right]\right\}, \tag{2.20}$$

where $[m]$: integer part of m.

$\Phi(x)$ is the kernel function of the operator valued function $h(\sqrt{D_m-\frac{1}{4}})$, where D_m denotes the Maass–Laplacian. $\Phi(x)$ and $g(u)$ are connected through

$$g(u)=(-1)^{m/2}\int_{-\infty}^{\infty}d\zeta\left(\frac{\zeta+2\mathrm{i}\cosh\frac{u}{2}}{\zeta-2\mathrm{i}\cosh\frac{u}{2}}\right)^{m/2}\Phi(\zeta^2+4\sinh^2\tfrac{u}{2})\,, \tag{2.21}$$

$$\Phi(x)=-\frac{1}{\pi}\int_{-\infty}^{\infty}Q'(x+t^2)\left(\frac{\sqrt{x+4+t^2}-t}{\sqrt{x+4+t^2}+t}\right)^{m/2}dt\,, \tag{2.22}$$

with $Q(w)=g(u)$, where $w=4\sinh^2\frac{u}{2}$. $Q(w)$ can also be expanded according to [32, p. 384] $(w\geqq 0)$

$$Q(w)=\int_w^{\infty}\frac{du\Phi(u)}{\sqrt{u-w}}(u+4)^{-m/2}\left[\sum_{l=0}^{m/2}\binom{m}{2l}(u-w)^l\mathrm{i}^{2l}(w+4)^{-l+m/2}\right]. \tag{2.23}$$

Let us denote by κ the number of inequivalent cusps of $\hat{\Gamma}$ (i.e. the number of zero interior angles of the fundamental polygon $\hat{\mathscr{F}}$). To each cusp there is associated an *Eisenstein series*

$$e(z,s,\alpha)=\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}y^s(\gamma z) \tag{2.24}$$

$z\in\mathscr{H}$, $\Re(s)>1$, $\alpha=1,\ldots,\kappa$, with Γ_α being the stabilizer of the cusp α. In the spectral decomposition of D_m on $\mathscr{L}^2(\chi, m)$ these Eisenstein series span the continuous spectrum.

Let us now restrict to Dirichlet boundary-conditions. In this case only the odd automorphic forms survive in the spectral expansion of the automorphic kernel. A glance on the continuous spectrum shows that the Eisenstein series $e(z, s, \alpha)$ drop out, according to a result of Venkov [56, p. 121]. In the case of Dirichlet boundary-conditions we are thus left with the spectral expansion of the automorphic kernel in *odd discrete* eigen-functions Ψ_n on $\mathscr{H}$,

$$\hat{K}_D(z, w)=\sum_n h(p_n)\Psi_n(z)\Psi_n(w) . \tag{2.25}$$

In the case of Neumann boundary-conditions both the discrete and continuous spectrum contribute to the spectral expansion of the automorphic kernel. Using even eigen-functions Φ_n and Eisenstein series $e(z, s, \alpha)$, respectively, we get

$$\hat{K}_N(z, w)=\sum_n h(p_n)\Phi_n(z)\Phi_n(w) + \frac{1}{4\pi}\int_{-\infty}^{\infty} dp\, h(p) \sum_{\alpha=1}^{\kappa} e\left(z, \frac{1}{2}+ip, \alpha\right)\overline{e\left(w, \frac{1}{2}+ip, \alpha\right)} . \tag{2.26}$$

Let us denote the composition of a $\gamma\in\bar{\Gamma}$ and $\mathscr{I}$ by $\rho=\gamma\mathscr{I}$. In order to investigate the various conjugacy classes for the formulation of the Selberg trace-formula for bordered Riemann surfaces, we have to distinguish the original conjugacy classes which appear already for closed Riemann surfaces from the additional conjugacy classes of the $\gamma\mathscr{I}$. The new conjugacy classes can be characterized by their traces. We consider first compact Riemann surfaces, i.e. compact polygons as fundamental domains. The case of closed Riemann surfaces gives us hyperbolic and elliptic conjugacy classes which correspond to $|\mathrm{tr}(\gamma)|>2$ and $|\mathrm{tr}(\gamma)|<2$, respectively. Let us denote by $g\in\mathrm{PSL}(2, \mathbb{R})$ some arbitrary element. As it turns out we have to consider two cases for conjugacy classes of the ρ's. The first is for $\mathrm{tr}(\rho)\neq 0$. The relative centralizer Γ_ρ of ρ then is of the form

$$\begin{pmatrix} b & 0 \\ 0 & -b^{-1} \end{pmatrix}, \quad (\mathrm{mod}\pm 1) , \tag{2.27}$$

where we define $\Gamma_\rho:=\{\gamma\in\hat{\Gamma}\,|\,\gamma^{-1}\rho\gamma=\rho\}$, and the relative conjugacy classes by $\{\rho\}_{\hat{\Gamma}}:=\{\rho'\in\hat{\Gamma}\mathscr{I}\,|\,\rho'=\gamma^{-1}\rho\gamma, \gamma\in\hat{\Gamma}\}$. Γ_ρ consists of hyperbolic elements and the identity, and since $\hat{\Gamma}$ is discrete it is generated by a single hyperbolic element. The second case is given by $\mathrm{tr}(\rho)=0$. Then the relative centralizers consist of elements of the form

$$\rho_1=\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad \rho_2=\begin{pmatrix} 0 & d \\ -d^{-1} & 0 \end{pmatrix}, \quad (\mathrm{mod}\pm 1) . \tag{2.28}$$

ρ_2 is an elliptic element of order two. Thus Γ_ρ consists of hyperbolic, elliptic and the identity element. However, due to the construction $\rho_1^n\rho_2 (n\in\mathbb{Z})$ we see that we can generate infinitely many elliptic conjugacy classes, which is impossible, since $\hat{\Gamma}$ is discrete. Therefore the relative centralizer of ρ with $\mathrm{tr}(\rho)=0$ consists either of hyperbolic elements and the identity or by a single elliptic generator of order two. The explicit computation reveals that for cocompact groups only the former case is possible, the latter leading to a divergency.

The conjugacy classes of $\rho\in\hat{\Gamma}\mathscr{I}$ can therefore be distinguished in two ways [9, 13] according to their squares $\rho^2\in\hat{\Gamma}$. Let $\rho\in\hat{\Gamma}$ be primitive, that is not a positive power of any other element of $\hat{\Gamma}\mathscr{I}$. Then

i) $\rho=\rho_i$, $\rho_i^2\in\{C_i\}_{\hat{\Gamma}}$, $i=1,\ldots,n$. The $\{C_i\}_{\hat{\Gamma}}$ are the conjugacy classes of the C_i in $\hat{\Gamma}$ which correspond to the closed geodesics c_i on $\hat{\Sigma}$.

ii) $\rho=\rho_p$, ρ_p^2 being a primitive element in $\hat{\Gamma}$ and $\rho_p^2\neq\{C_i\}_{\hat{\Gamma}}$.

Thus it follows that the sum over conjugacy classes for $\rho\in\hat{\Gamma}\mathscr{I}$ is divided into first the conjugacy classes of the C_i in $\hat{\Gamma}$, which correspond to the closed geodesics c_i on $\tilde{\Sigma}$, and second into conjugacy classes such that for all $\rho\in\hat{\Gamma}\mathscr{I}$ there is a unique description $\gamma=k^{-1}\rho^{2n-1}k\,(n\in\mathbb{N})$, for $\rho\in\hat{\Gamma}\mathscr{I}$ inconjugate and primitive, and $k\in\Gamma_{\rho^2}\backslash\hat{\Gamma}$. In the notation of Venkov [56] the relative conjugacy classes with $\mathrm{tr}(\rho)=0$ correspond to the case i), and the relative classes with $\mathrm{tr}(\rho)\neq 0$ correspond to ii). In this case $P(\rho)=\rho^2$ generates the relative centralizer Γ_ρ, whereas this is generated by $P(\rho)=P(\gamma\mathscr{I})=\gamma$ in the former case. Also, for any γ under consideration with $\mathrm{tr}(\gamma\mathscr{I})\neq 0$, $(\gamma\mathscr{I}\gamma\mathscr{I})$ is hyperbolic.

In addition, we call a relative conjugacy class $\{\rho\}_{\hat{\Gamma}}$ primitive if it is not an odd power of any other relative class $\{\rho'\}_{\hat{\Gamma}}$.

Let us continue by considering a non-compact fundamental polygon with corresponding non-cocompact Fuchsian group $\hat{\Gamma}$. Besides the already known relative conjugacy classes of $\rho\in\hat{\Gamma}\mathscr{I}$ there appear additional classes with $\mathrm{tr}(\rho)=0$ for which the relative centralizers Γ_ρ are generated by single elliptic elements of order two. These have been excluded in the cocompact case. For each such $\rho=\gamma\mathscr{I}$ there then exists an element $g\in\mathrm{PSL}(2,\mathbb{R})$ having the properties

$$g\gamma\mathscr{I}g^{-1}=\mathscr{I}\,, \tag{2.29}$$

$$g\Gamma_{\gamma\mathscr{I}}g^{-1}=\left\{\mathbb{1}_2,\begin{pmatrix}0 & a\\ -1/a & 0\end{pmatrix}(\mathrm{mod}\pm 1)\right\}, \tag{2.30}$$

where $a\geqq 1$. These classes play the rôle of the parabolic classes in the classical Selberg trace formula. Evaluating all contributions, we can write down the following

Theorem 2.2 [56, p. 137]. *The Selberg trace formula for automorphic functions* ($m=0$) *obeying Dirichlet boundary-conditions on arbitrary bordered Riemann surfaces* (*therefore including hyperbolic, elliptic and parabolic conjugacy classes*) *is given by*:

$$\sum_{n=0}^{\infty}h(p_n)=\frac{\mathscr{A}(\hat{\mathscr{F}})}{8\pi}\int_{-\infty}^{\infty}p\tanh\pi p\,h(p)\,dp+\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma g(kl_\gamma)}{4\sinh\dfrac{kl_\gamma}{2}}$$

$$+\sum_{\{R\}_p}\sum_{k=1}^{\nu-1}\frac{1}{8\nu\sin(k\pi/\nu)}\int_{-\infty}^{\infty}h(p)\frac{\cosh[\pi(1-2k/\nu)p]}{\cosh\pi p}dp$$

$$-\sum_{\substack{\{\rho\}_p\\ \mathrm{tr}(\rho)\neq 0}}\sum_{k=1}^{\infty}\frac{l_{\rho^2}g[(k-\frac{1}{2})l_{\rho^2}]}{4\cosh\frac{1}{2}(k-\frac{1}{2})l_{\rho^2}}-\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{l_{c_i}g(kl_{c_i})}{2\cosh\dfrac{kl_{c_i}}{2}}$$

$$-\frac{\kappa}{4\pi}\int_{-\infty}^{\infty} h(p)\,\Psi(\tfrac{1}{2}+\mathrm{i}p)\,dp$$

$$+\frac{g(0)}{2}\left[\sum_{\substack{\{\rho\}_p;\hat{\Gamma}_{\rho,\mathrm{ell}}\\ \mathrm{tr}(\rho)=0}} \ln\left(\frac{a(\rho)}{\nu(\rho)}\right)-\kappa\ln 2-\frac{1}{2}\sum_{i=1}^{n} l_{C_i}\right], \tag{2.31}$$

where $\lambda_n=\frac{1}{4}+p_n^2$ on the left runs through the set of all eigenvalues of the Dirichlet Laplacian ($m=0$), and the summation on the right is taken over all primitive conjugacy classes $\{R\}_p$ with $\mathrm{tr}(R)<2$, $\{\gamma\}_p$ with $|\mathrm{tr}(\gamma)|>2$, and $\{\rho\}_p$, $\mathrm{tr}(\rho)\neq 0$. The lengths l_{C_i} are twofold degenerate, since C_i and C_i^{-1} both have to be included into the sum. $a(\rho)$ and $\nu(\rho)$ denote specific quantities appearing for the conjugacy class of $\rho\in\hat{\Gamma};\hat{\Gamma}_{\rho,\mathrm{ell}}$, $\mathrm{tr}(\rho)=0$, and will be explained later on in the derivation of Theorem 3.2. $\Psi(z)=\Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the Γ-function.

The test function h must satisfy the following properties:

i) *$h(p)$ is an even function in p,*
ii) *$h(p)$ is analytic in the strip $|\Im(p)|<\frac{1}{2}+\varepsilon$ for some $\varepsilon>0$,*
iii) *$h(p)$ has to decrease faster than $|p|^{-2}$ for $p\to\pm\infty$.*

2. The Selberg operator on the doubled Riemann surface. We return with our discussion to the Selberg operator (2.16) with the automorphic kernel (2.17). We consider the second term with an odd automorphic function ϕ

$$\begin{aligned}
&\frac{1}{2}\int_{\hat{\mathscr{F}}} dV(w)\,K(z,\mathscr{I}w)\,\phi(w)\\
&=\frac{1}{4}\sum_{\gamma\in\bar{\Gamma}}\chi_\gamma^m\int_{\hat{\mathscr{F}}} dV(w)\,j(\gamma,-\bar{w})\,k[z,\gamma(-\bar{w})]\,\phi(w)\\
&=\frac{1}{4}\sum_{\gamma\in\bar{\Gamma}}\chi_\gamma^m\int_{\hat{\mathscr{F}}} dV(-\bar{w})\,j(\gamma,w)\,k(z,\gamma w)\,\phi(-\bar{w})\\
&=-\frac{1}{4}\sum_{\gamma\in\bar{\Gamma}}\chi_\gamma^m\int_{\hat{\mathscr{F}}} dV(-\bar{w})\,j(\gamma,w)\,k(z,w)\,\chi_\gamma^{-m}j^{-1}(\gamma,w)\,\phi(w)\\
&=-\frac{1}{4}\sum_{\gamma\in\bar{\Gamma}}\int_{\gamma\hat{\mathscr{F}}} dV(-\bar{w})\,k(z,w)\,\phi(w)\\
&=\frac{1}{4}\int_{\mathscr{H}} dV(w)\,k[z,(-\bar{w})]\,\phi(w)\\
&=\frac{1}{4}\int_{\mathscr{H}} dV(w)\,\overline{k(-\bar{z},w)\,\overline{\phi(w)}}=\frac{1}{2}\overline{(L\bar{\phi})(-\bar{z})}
\end{aligned}\tag{2.32}$$

due to the property $k(z,\mathscr{I}w)=\overline{k(\mathscr{I}z,w)}$. Therefore we have obtained for an odd automorphic function the following identity:

$$(\hat{L}_-\phi)(z)=\frac{1}{2}(L\phi)(z)-\frac{1}{2}\overline{(L\bar{\phi})(-\bar{z})}\,. \tag{2.33}$$

Now let ϕ be an eigen-function of the operator D_m

$$D_m\phi=\lambda\phi\ , \tag{2.34}$$

then we have ($\lambda\in\mathbb{R}$)

$$\lambda\bar{\phi}=\overline{D_m\phi}=D_{-m}\bar{\phi}\ , \tag{2.35}$$

since $\overline{D_m}=D_{-m}$, and it follows that if ϕ is an eigen-function of D_m with eigenvalue λ, then $\bar{\phi}$ will be an eigen-function of D_{-m} with the same eigenvalue. Therefore also

$$\begin{aligned}(\hat{L}_-\phi)(z)&=\frac{1}{2}(L\phi)(z)-\frac{1}{2}\overline{(L\bar{\phi})(-\bar{z})}\\&=\frac{1}{2}[\Lambda(\lambda)+\Lambda'(\lambda)]\,\phi(z)=:\widehat{\Lambda(\lambda)}\,\phi(z)\ .\end{aligned} \tag{2.36}$$

D_m has a purely discrete spectrum with $\lambda_1\leqq\lambda_2\leqq,\ldots,\lambda_n\to\infty$ $(n\to\infty)$ and only finitely many eigenvalues are negative. We expand the automorphic kernel according to

$$\hat{K}(z,w)=\sum_{n=1}^{\infty}\widehat{\Lambda(\lambda_n)}\,\phi_n(z)\overline{\phi_n(w)}\ . \tag{2.37}$$

Hence we get for the trace

$$\operatorname{tr}(\hat{L}_-)=\frac{1}{2}\sum_{n=1}^{\infty}[\Lambda(\lambda_n)+\Lambda'(\lambda_n)]\ , \tag{2.38}$$

and there remains the question of the relation between Λ and Λ'.

We consider the Selberg operators $L_{(\pm)}$ corresponding to the Maass–Laplacians $D_{\pm m}$. Then we are given the eigenvalue problems $L_{(\pm)}\phi_\pm=\Lambda_\pm(\lambda_\pm)\phi_\pm$, and seek for the relation between Λ_+ and Λ_-. As is well-known [32, p. 364–9] we have

$$\Lambda_+(\lambda_+)=\int_{\mathscr{H}}dV(z)k_+(\mathrm{i},z)g_+(z)\ , \tag{2.39}$$

where g_+ is an eigen-function of D_m on the entire $\mathscr{H}$ (for the explicit evaluation take e.g. $g(z)=[\Im(z)]^s$). The function Λ depends on k, respectively Φ, and on m. $L_{(-)}$ is defined in the same way as $L_{(+)}$ by replacing $+\to-$ in (2.39) and λ is an eigenvalue of D_m and D_{-m}, see (2.33, 2.34). The functions $g_\pm$ are solutions of the differential equations

$$D_{\pm m}g_\pm(r)=\lambda g_\pm(r) \tag{2.40}$$

and depend only on $r=|w|$, where $w=(z-\mathrm{i})/(z+\mathrm{i})$, $z\in\mathscr{H}$ [32, pp. 366–8; 22, pp. 304–5]. Here

$$D_{\pm m}=-\frac{1}{4}(1-r)^2\left(\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\phi^2\right)\pm\mathrm{i}\frac{m}{2}(1-r^2)\partial_\phi-\frac{1}{4}m^2(1-r^2)\ . \tag{2.41}$$

Since $D_{\pm m}g_\pm(r)$ depends only on m^2 but not on m, we obtain $g_+(r)=g_-(r)$ and thus

$$\Lambda_+(\lambda)=\overline{\Lambda_+(\lambda)}=\int_{\mathscr{H}}dV(z)\overline{k_+(\mathrm{i},z)}g_+(z)=\int_{\mathscr{H}}dV(z)k_-(\mathrm{i},z)g_-(z)=\Lambda_-(\lambda)\ . \tag{2.42}$$

Therefore the spectrum of D_m depends only on $|m|$ and we conclude $\Lambda_-(\lambda)=\Lambda_+(\lambda)$; hence $\Lambda(\lambda)=\Lambda'(\lambda)$, and we have found that an odd eigen-function of D_m is also an eigen-function of $\hat{L}$. We get

$$\operatorname{tr}(\hat{L}_-)=\sum_{n=1}^{\infty}\Lambda(\lambda_n)\,. \tag{2.43}$$

This now yields with the usual identification $\Lambda(\lambda_n)=\Lambda(\frac{1}{4}+p_n^2)=h(p_n)$,

$$\begin{aligned}\operatorname{tr}(\hat{L}_-)&=\sum_{n=1}^{\infty}\Lambda(\lambda_n)=\sum_{n=1}^{\infty}h(p_n)=\int_{\hat{F}}\hat{K}(z,z)\,dV(z)\\&=\frac{1}{4}\sum_{\gamma\in\bar{\Gamma}}\chi_\gamma^m\int_{\hat{\mathscr{F}}}dV(z)j(\gamma,z)k(z,\gamma z)\\&-\frac{1}{4}\sum_{\gamma\in\bar{\Gamma}}\chi_\gamma^m\int_{\hat{\mathscr{F}}}dV(z)j(\gamma,-\bar{z})k[z,\gamma(-\bar{z})]\,.\end{aligned} \tag{2.44}$$

The first term is already know from the usual Selberg trace formula for closed Riemann surfaces [cf. (2.19), Theorem 2.1]. In the next chapter we are going to calculate the second one.

III. The Selberg Trace Formula for Bordered Riemann Surfaces

1. *The fundamental domain $\hat{\mathscr{F}}$ is compact.* For convenience we set $\rho=\gamma\mathscr{I}$ and use the classification of the inverse-hyperbolic transformations according to $\rho\in\bar{\Gamma}\mathscr{I}$, respectively, $\rho^2\in\bar{\Gamma}$. We obtain

$$\begin{aligned}&\sum_{\gamma\in\bar{\Gamma}}\chi_\gamma^m\int_{\hat{\mathscr{F}}}dV(z)j(\gamma,-\bar{z})k[z,\gamma(-\bar{z})]\\&\quad=\sum_{\rho\in\bar{\Gamma}\mathscr{I}}\chi_{\rho\mathscr{I}}^m\int_{\hat{\mathscr{F}}}dV(z)j(\rho,z)k(z,\rho z)=:\sum_{\rho\in\bar{\Gamma}\mathscr{I}}A(\rho)\\&\quad=\sum_{\rho_p}\sum_{k=0}^{\infty}A(\rho_p^{2k+1})+\sum_{i=1}^{n}\sum_{\rho_\iota}\sum_{k=0}^{\infty}A(\rho_i^{2k+1})\,.\end{aligned} \tag{3.1}$$

We now get

$$\begin{aligned}&\sum_{\rho}\sum_{k=0}^{\infty}A(\rho^{2k+1})\\&=\sum_{\{\rho\}}\sum_{\sigma\in\{\rho\}}\sum_{k=0}^{\infty}\chi^m(\sigma^{2k+1}\mathscr{I})\int_{\hat{\mathscr{F}}}dV(z)j(\sigma^{2k+1},z)k(z,\sigma^{2k+1}z)\\&=\sum_{\{\rho\}}\sum_{\gamma\in\Gamma_{\rho^2}\backslash\bar{\Gamma}}\sum_{k=0}^{\infty}\chi^m(\gamma^{-1}\rho^{2k+1}\gamma\mathscr{I})\int_{\hat{\mathscr{F}}}dV(z)j(\gamma^{-1}\rho^{2k+1}\gamma,z)k(z,\gamma^{-1}\rho^{2k+1}\gamma z)\\&=\sum_{\{\rho\}}\sum_{\gamma\in\Gamma_{\rho^2}\backslash\bar{\Gamma}}\sum_{k=0}^{\infty}\chi_\rho^{m(2k+1)}\chi_{\mathscr{I}}^m\int_{\hat{\mathscr{F}}_\gamma}dV(z)j(\rho^{2k+1},\gamma z)k(\gamma z,\rho^{2k+1}\gamma z)\end{aligned}$$

$$=\sum_{\{\rho\}}\sum_{k=0}^{\infty}\chi_\rho^{m(2k+1)}\chi_{\mathscr{I}}^m\sum_{\gamma\in\Gamma_{\rho^2}\backslash\bar{\Gamma}}\int_{\gamma\hat{\mathscr{F}}}dV(z)j(\rho^{2k+1},z)k(z,\rho^{2k+1}z)$$

$$=\sum_{\{\rho\}}\sum_{k=0}^{\infty}\chi_\rho^{m(2k+1)}\chi_{\mathscr{I}}^m\int_{\Gamma_{\rho^2}\backslash\mathscr{H}}dV(z)j(\rho^{2k+1},z)k(z,\rho^{2k+1}z)$$

$$=\sum_{\{\rho\}}\sum_{k=0}^{\infty}\chi_\rho^{m(2k+1)}\chi_{\mathscr{I}}^m B_k(\rho)\,. \tag{3.2}$$

By an overall conjugation we can arrange that $\rho^2z=Nz$, therefore $\rho z=\sqrt{N}(-\bar{z})$. This yields

$$\begin{aligned}B_k(\rho)&=\int_{\Gamma_{\rho^2}\backslash\mathscr{H}}dV(z)j(\rho^{2k+1},z)k(z,\rho^{2k+1}z)\\&=\int_1^N\frac{dy}{y^2}\int_{-\infty}^{\infty}dx\,k[z,N^{k+\frac{1}{2}}(-\bar{z})]\\&=(-1)^{m/2}\int_1^N\frac{dy}{y^2}\int_{-\infty}^{\infty}dx\left(\frac{\bar{z}}{z}\right)^{m/2}\Phi\left(\frac{|z+N^{k+\frac{1}{2}}\bar{z}|^2}{N^{k+\frac{1}{2}}y^2}\right)\\&=\frac{(-1)^{m/2}\ln N}{2\cosh\frac{u}{2}}\int_{-\infty}^{\infty}d\zeta\left(\frac{\zeta+2i\cosh\frac{u}{2}}{\zeta-2i\cosh\frac{u}{2}}\right)^{m/2}\Phi(\zeta^2+4\sinh^2\tfrac{u}{2})\\&=\frac{l_0g(u)}{2\cosh\frac{u}{2}}\,,\end{aligned} \tag{3.3}$$

with the abbreviation $u=(2k+1)\ln\sqrt{N}=(k+\frac{1}{2})l_{\rho^2}$ and $g(u)$ as in (2.21). In order to finish the computation for compact domains we have to consider elliptic elements in the trace formula. They can be parameterized by matrices of the form

$$R=\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix},\quad (0<\phi<\pi)\,, \tag{3.4}$$

such that we obtain

$$R\colon z\to\frac{z\cos\phi-\sin\phi}{z\sin\phi+\cos\phi}\,. \tag{3.5}$$

The relevant contribution in the trace formula has the form

$$\begin{aligned}\int_{\mathscr{F}(R)}dV(z)K(z,z)&=\sum_{\{R\}}\chi_R^m\int_{\Gamma_R\backslash\mathscr{H}}dV(z)k(z,Rz)j(R,z)\\&=\sum_{\{R\}}\frac{\chi_R^m}{\nu}\int_{\mathscr{H}}dV(z)k(z,Rz)j(R,z)=:\sum_{\{R\}}A(R)\end{aligned} \tag{3.6}$$

with ν the order of the element $R\in\Gamma$. The evaluation of the relevant terms can be found in Hejhal [33] and the result is

$$A(R)=\frac{i\chi_R^m e^{i(m-1)\phi}}{2\nu\sin\phi}\int_{-\infty}^{\infty}du\,g(u)\frac{e^{(m-1)u/2}(e^u-e^{2i\phi})}{\cosh u+\cos(\pi-2\phi)}\,. \tag{3.7}$$

Therefore we have derived the following

Theorem 3.1. *The Selberg trace formula on compact bordered Riemann surfaces for automorphic forms of weight m obeying Dirichlet boundary-conditions is given by*

$$\sum_{n=1}^{\infty} h(\rho_n) = -\frac{\mathscr{A}(\hat{\mathscr{F}})}{16\pi^2} \int_0^{\infty} \frac{\cosh \frac{mu}{2}}{\sinh \frac{u}{2}} g'(u)\,du + \frac{1}{4} \sum_{\{\gamma\}_p} \sum_{k=1}^{\infty} \frac{\chi_\gamma^{mk} l_\gamma g(kl_\gamma)}{\sinh \frac{kl_\gamma}{2}}$$

$$+\frac{i}{4} \sum_{\{R\}_p} \sum_{k=1}^{\nu-1} \frac{\chi_R^{mk} e^{i(m-1)k\pi/\nu}}{\nu \sin(k\pi/\nu)} \int_{-\infty}^{\infty} du\, g(u) \frac{e^{(m-1)u/2}(e^u - e^{2ik\pi/\nu})}{\cosh u + \cos[\pi - 2(k\pi/\nu)]}$$

$$-\frac{1}{4} \sum_{\{\rho\}_p} \sum_{k=0}^{\infty} \frac{\chi_\rho^{m(2k+1)} \chi_{\mathscr{I}}^m l_{\rho^2} g[(k+\frac{1}{2}) l_{\rho^2}]}{\cosh \frac{1}{2}[(k+\frac{1}{2}) l_{\rho^2}]}$$

$$-\frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{\chi_{C_i}^{mk} l_{C_i} g(kl_{C_i})}{\cosh \frac{kl_{C_i}}{2}} - \frac{L}{4} g(0)\,. \tag{3.8}$$

Here we have abbreviated $\sum_{i=1}^{n} l_{C_i} = L$. $h(p)$ *is an even function in* p *with the corresponding* $g(u)$ *given in* (2.21) *and has the properties*

i) $h(p)$ *is holomorphic in the strip* $|\Im(p)| \leqq \frac{1}{2} + \varepsilon$, $\varepsilon > 0$.
ii) $h(p)$ *has to decrease faster than* $|p|^{-2}$ *for* $p \to \pm\infty$.

Note that for Neumann boundary-conditions the last three terms change their signs. □

2. *The fundamental domain* $\hat{\mathscr{F}}$ *is non-compact.* To evaluate the trace formula in the case when also parabolic conjugacy classes are present we recall the enumeration before Theorem 2.2. In order that the regularization of the terms which corresponds to the parabolic conjugacy classes is actually possible we require the following property of the multiplier system:

$$\tilde{\kappa} := \sum_{\{S\}} \chi_S^m = \sum_{\substack{\{\rho\};\hat{\Gamma}_{\rho,\mathrm{ell}} \\ \mathrm{tr}(\rho)=0}} \chi_\rho^m\,. \tag{3.9}$$

We include all relevant conjugacy classes. There are the hyperbolic ones $\{\gamma\}_{\hat{\Gamma}}$, the inverse hyperbolic ones $\{\gamma\mathscr{I}\}_{\hat{\Gamma}}$, $\mathrm{tr}(\gamma\mathscr{I}) \neq 0$, the elliptic ones $\{R\}_{\hat{\Gamma}}$, the parabolic ones $\{S\}_{\hat{\Gamma}}$, $\mathrm{tr}(S) = 2$, and the inverse elliptic ones $\{\gamma\mathscr{I}\}_{\hat{\Gamma}}$, $\mathrm{tr}(\gamma\mathscr{I}) = 0$. Following Venkov [56, p. 132] we hence have to consider

$$\mathrm{tr}(L) = \frac{1}{2} \int_{\hat{\mathscr{F}}} \sum_{\{\gamma\}} [k(z, \gamma z) - k(z, \rho z)]\, dV(z)$$

$$= \frac{1}{2} \mathscr{A}(\hat{\mathscr{F}}) \Phi(0) + \frac{1}{2} \sum_{\{\gamma\}} \chi_\gamma^m \int_{\hat{\mathscr{F}}(\gamma)} dV(z) k(z, \gamma z)$$

$$+ \frac{1}{2} \sum_{\{R\}} \chi_R^m \int_{\hat{\mathscr{F}}(R)} dV(z) k(z, Rz)$$

$$-\frac{1}{2}\sum_{\substack{\{\rho\}\\ \mathrm{tr}(\rho)\neq 0}}\chi_\rho^m\int_{\hat{\mathscr{F}}(\rho)}dV(z)k(z,\rho z)-\frac{1}{2}\sum_{\substack{\{\rho\};\hat{\Gamma}_{\rho,\mathrm{hyp}}\\ \mathrm{tr}(\rho)=0}}\chi_\rho^m\int_{\hat{\mathscr{F}}(\rho)}k(z,\rho z)dV(z)$$

$$+\frac{1}{2}\lim_{Y\to\infty}\int_{\hat{\mathscr{F}}_Y}dV(z)$$

$$\times\left\{\sum_{\{S\}}\chi_S^m\sum_{\gamma'\in\hat{\Gamma}_S\backslash\Gamma}k(z,\gamma'^{-1}S\gamma'z)-\sum_{\substack{\{\rho\};\hat{\Gamma}_{\rho,\mathrm{ell}}\\ \mathrm{tr}(\rho)=0}}\chi_\rho^m\sum_{\gamma'\in\hat{\Gamma}_\rho\backslash\Gamma}k(z,\gamma'^{-1}\rho\gamma'z)\right\}, \tag{3.10}$$

with some properly defined compact domain $\hat{\mathscr{F}}_Y$ depending on a large parameter Y, and where the sum is taken over all hyperbolic conjugacy classes $\{\gamma\}$, elliptic conjugacy classes $\{R\}$ and parabolic conjugacy classes $\{S\}$ in $\bar{\Gamma}$, over all relative non-degenerate classes $\{\gamma\mathscr{I}\}$ with $\mathrm{tr}(\gamma\mathscr{I})\neq 0$, and all relative conjugacy classes $\{\gamma\mathscr{I}\}$ with $\mathrm{tr}(\gamma\mathscr{I})=0$. Parabolic transformations have the form

$$S=\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix},\quad (n\in\mathbb{Z}\backslash\{0\})\,. \tag{3.11}$$

For the sake of clarity, let us recapitulate some calculations of Hejhal [33, pp. 406]. We consider

$$\begin{aligned}A_Y(S)&:=\int_0^Y\frac{dy}{y^2}\int_0^1 dx\sum_{n\neq 0}k(z,z+n)\\ &=2\Re\sum_{n=1}^{\infty}\frac{1}{n}\int_{n/Y}^{\infty}\left(\frac{2-\mathrm{i}u}{2+\mathrm{i}u}\right)^{m/2}\Phi(u^2)du\\ &=2(C+\ln Y)\Re\int_0^{\infty}\left(\frac{2-\mathrm{i}u}{2+\mathrm{i}u}\right)^{m/2}\Phi(u^2)du\\ &\quad+2\Re\int_0^{\infty}\left(\frac{2-\mathrm{i}u}{2+\mathrm{i}u}\right)^{m/2}\Phi(u^2)\ln u\,du+O(Y^{-1/2})\,.\end{aligned} \tag{3.12}$$

Here $C=0.577215665\ldots$ denotes Euler's constant. By means of the relations (2.21, 2.22), $A_Y(S)$ can be reformulated in the following form:

$$\begin{aligned}A_Y(S)&=(C+\ln Y+\ln 2)\,g(0)\\ &\quad-\frac{1}{\pi}\Re\int_0^{\infty}dw\,Q'(w)\int_{-\pi/2}^{\pi/2}\left(\frac{\sinh\omega+\mathrm{i}\cos\phi}{\cosh\omega+\sin\phi}\right)^m\ln\frac{\cos\phi}{\sinh\omega}d\phi+O(Y^{-1/2})\,,\end{aligned} \tag{3.13}$$

where we have abbreviated $\sinh\omega=2/\sqrt{w}$, $\cosh\omega=\sqrt{(w+4)/w}$. The ϕ-integral can be evaluated by a contour integration [33, pp. 408] and one finally obtains by taking into account all inequivalent parabolic conjugacy classes,

$$\begin{aligned}&\frac{1}{2}\int_{\hat{\mathscr{F}}}dV(z)\sum_{\substack{\{S\}\\ |\mathrm{tr}(S)|=2}}\sum_{\gamma\in\bar{\Gamma}(S)\backslash\bar{\Gamma}}k(z,\gamma^{-1}S\gamma z)\\ &=\frac{\tilde{\kappa}}{2}(\ln Y-\ln 2)g(0)+\frac{\tilde{\kappa}}{8}h(0)-\frac{\tilde{\kappa}}{4\pi}\int_{-\infty}^{\infty}h(p)\,\Psi(1+\mathrm{i}p)\,dp\\ &\quad+\frac{\tilde{\kappa}}{4}\int_0^{\infty}\frac{g(u)}{\sinh\frac{u}{2}}\left(1-\cosh\frac{mu}{2}\right)du+O(Y^{-1/2})\,.\end{aligned} \tag{3.14}$$

As we know from the discussion in Sect. II, there may be some $\rho\in\hat{\Gamma}\mathscr{I}$, $\operatorname{tr}\rho=0$, such that the relative centralizer Γ_ρ is generated by an elliptic element of order two. We now want to deal with these ρ. First we assume that in (2.30) $g=\mathbb{1}_2$, that is

$$\gamma_a=\begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}, \quad (\operatorname{mod}\pm 1)\,, \tag{3.15}$$

with some $a\geqq 1$, is the elliptic generator of order two of Γ_ρ. We therefore have to consider

$$\int\limits_{\bigcup_{\gamma'\hat{\mathscr{F}}_Y,\,\gamma'\in\Gamma}} k(z,\rho z)=|\Gamma_\rho| \int\limits_{\bigcup_{\gamma'\hat{\mathscr{F}}_Y,\,\gamma'\in\Gamma\mathscr{I}\setminus\hat{\Gamma}}} k(z,\rho z)\,, \tag{3.16}$$

where $|\Gamma_\rho|=\text{order}\,[\Gamma_\rho]=2$, which yields an additional factor 1/2.

For a properly defined asymptotic expansion of the corresponding integral we remove from $\mathscr{H}$ two regions, denoted by $B_1(Y)=\{z\in\mathscr{H}\,|\,x\geqq Y\}$ and $B_2(Y)=\gamma_a B_1(Y)$, respectively, i.e. we consider [56, p. 134]

$$B(Y)=\mathscr{H}-B_1(Y)-B_2(Y)\,. \tag{3.17}$$

Since we have the entire domain $\hat{\mathscr{F}}$ taken into account, we must in the sequel consider the domain $B(Y)\cup\mathscr{I}B(Y)$. First let us insert a $n=0$ "parabolic term" into the automorphic kernel to study the $k(z,-\bar{z})$-behaviour in the integral; this gives the integral

$$\begin{aligned}\frac{1}{2}\int\limits_{B(Y)\cup\mathscr{I}B(Y)} k(z,-\bar{z})\,dV(z)&=\frac{(-1)^{m/2}}{8}\int\limits_{B(Y)\cup\mathscr{I}B(Y)}\frac{dt\,dy}{y\sqrt{t}}\left(\frac{2+i\sqrt{t}}{2-i\sqrt{t}}\right)^{m/2}\Phi(t)\\
&=\underbrace{\frac{1}{4}\Re\int\limits_{a^2/Y}^{Y}\frac{dy}{y}\int\limits_0^\infty\frac{dt}{\sqrt{t}}\left(\frac{2+i\sqrt{t}}{2-i\sqrt{t}}\right)^{m/2}\Phi(t)}_{=:A_1(Y)}\\
&\quad+\underbrace{\frac{1}{4}\Re\int\limits_0^{a^2/Y}\frac{dy}{y}\int\limits_{4(a^2/yY-1)}^\infty\frac{dt}{\sqrt{t}}\left(\frac{\sqrt{t}-2i}{\sqrt{t}+2i}\right)^{m/2}\Phi(t)}_{=:\,A_2(Y)}\,.\end{aligned} \tag{3.18}$$

By a similar argument as for the previous integral one obtains by making use of the relations (2.21, 2.22) that [33, pp. 399–400]

$$A_1(Y)=\frac{1}{2}(\ln Y-\ln a)g(0)\,. \tag{3.19}$$

Using elementary partial integration it follows further that

$$A_2(Y)=A_2=-\frac{1}{2}g(0)\ln 2+A_3\,, \tag{3.20}$$

with the quantity A_3 given by

$$
\begin{aligned}
A_3 &= \frac{1}{4}\Re \int_0^\infty \frac{dt}{\sqrt{t}} \ln(t+4) \left(\frac{2+i\sqrt{t}}{2-i\sqrt{t}}\right)^{m/2} \Phi(t) \\
&= -\frac{1}{4}\Re \int_0^\infty \frac{dt}{\sqrt{t}} \ln(t+4) \left(\frac{2+i\sqrt{t}}{2-i\sqrt{t}}\right)^{m/2} \\
&\quad \times \int_{-\infty}^{\infty} dy\, Q'(t+y^2) \left(\frac{\sqrt{t+4+y^2}-y}{\sqrt{t+4+y^2}+y}\right)^{m/2} \\
&= -\frac{1}{4\pi}\Re \int_{-\infty}^{\infty} d\xi \int_{\xi^2}^{\infty} dw\, Q'(w) \frac{\ln(w-\xi^2+4)}{\sqrt{w-\xi^2}} \left(\frac{2+i\sqrt{w-\xi^2}}{\sqrt{w+4}+\xi}\right)^m \\
&= -\frac{1}{4\pi}\Re \int_0^\infty dw\, Q'(w) \\
&\quad \times \int_{-\pi/2}^{\pi/2} \left(\frac{\sinh\omega + i\cos\phi}{\cosh\omega+\sin\phi}\right)^m \left[2\ln 2 + \ln\left(1+\frac{\cos^2\phi}{\sinh^2\omega}\right)\right] d\phi \\
&= \frac{1}{2} g(0)\ln 2 \\
&\quad -\frac{1}{4\pi}\Re \int_0^\infty dw\, Q'(w) \int_{-\pi/2}^{\pi/2} \left(\frac{\sinh\omega + i\cos\phi}{\cosh\omega+\sin\phi}\right)^m \ln\left(1+\frac{\cos^2\phi}{\sinh^2\omega}\right) d\phi\,, \qquad (3.21)
\end{aligned}
$$

where ω is the same quantity as defined in (3.13). The last integral is evaluated in Appendix B. We then obtain for the sum A_1+A_2,

$$
\begin{aligned}
A_1(Y)+A_2(Y) &= \frac{1}{2}(\ln Y - \ln a) g(0) + \frac{1}{8} h(0) \\
&\quad + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(p)\left[\Psi(\tfrac{1}{2}+ip) - \Psi(1+ip)\right] dp\,. \qquad (3.22)
\end{aligned}
$$

To finish the discussion we have to consider

$$
\int_{\bigcup_{\{\gamma\}} g\gamma\mathscr{F}_Y} k(z,\rho z)\, dV(z)\,, \qquad (3.23)
$$

for some appropriate $g \in \mathrm{SL}(2,\mathbb{R})$ (see (2.30)). According to Venkov [56, pp. 136], this has the consequence that the asymptotic behaviour in the limit $Y\to\infty$ is changed such that the integral (3.18) is calculated with respect to the variable vY instead of Y. The fixed number v is denoted by $v(\rho)$. Similarly, a is denoted by $a(\rho)$. Therefore we must multiply the result of (3.19) by $q(\hat{\mathscr{F}})$ which denotes the number of classes $\{\rho\}_{\tilde{\Gamma}}$ having the property $\mathrm{tr}(\gamma\mathscr{I})=0$ and Γ_ρ being generated by an elliptic element of order two. Because we know that all terms in the trace formula must be finite we deduce from a comparison of (3.14) and (3.18) and $q(\hat{\mathscr{F}})=4\tilde{\kappa}$. Thus we have proven the following

Theorem 3.2. *The Selberg trace formula on arbitrary bordered Riemann surfaces for automorphic forms of weight m obeying Dirichlet boundary-conditions, $m\in\mathbb{Z}$, is given by*

$$\sum_{n=1}^{\infty} h(p_n) = -\frac{\mathscr{A}(\hat{\mathscr{F}})}{4\pi}\int_0^{\infty}\frac{\cosh\frac{mu}{2}}{\sinh\frac{u}{2}}g'(u)\,du + \frac{1}{4}\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{\chi_\gamma^{mk}\, l_\gamma g(kl_\gamma)}{\sinh\frac{kl_\gamma}{2}}$$
$$+\frac{i}{4}\sum_{\{R\}_p}\sum_{k=1}^{\nu-1}\chi_R^{mk}\frac{e^{i(m-1)k\pi/\nu}}{\nu\sin(k\pi/\nu)}\int_{-\infty}^{\infty}du\, g(u)\frac{e^{(m-1)u/2}(e^u-e^{2ik\pi/\nu})}{\cosh u+\cos[\pi-2(k\pi/\nu)]}$$
$$-\frac{1}{4}\sum_{\{\rho^2\}_p}\sum_{k=0}^{\infty}\frac{\chi_\rho^{m(2k+1)}\chi_{\mathscr{I}}^m l_{\rho^2} g[(k+\frac{1}{2})l_{\rho^2}]}{\cosh[\frac{1}{2}(k+\frac{1}{2})l_{\rho^2}]} - \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{\chi_{C_i}^{mk} l_{C_i} g(kl_{C_i})}{\cosh\frac{kl_{C_i}}{2}}$$
$$+\frac{g(0)}{2}\left[\frac{1}{4}\sum_{\substack{\{\rho\};\hat{\Gamma}_{\rho,\mathrm{ell}}\\ \mathrm{tr}(\rho)=0}}\chi_\rho^m\ln\left(\frac{a(\rho)}{\nu(\rho)}\right)-\tilde{\kappa}\ln 2-\frac{L}{2}\right]+\frac{\tilde{\kappa}}{8}h(0)$$
$$-\frac{\tilde{\kappa}}{4\pi}\int_{-\infty}^{\infty}h(p)\Psi(\tfrac{1}{2}+ip)\,dp+\frac{\tilde{\kappa}}{4}\int_0^{\infty}\frac{g(u)}{\sinh\frac{u}{2}}\left(1-\cosh\frac{um}{2}\right)du\,. \qquad (3.24)$$

$h(p)$ denotes an even function in p with the corresponding $g(u)$ given in (2.21) *and has the properties*

i) *$h(p)$ is holomorphic in the strip $|\Im(p)|\leqq\frac{1}{2}+\varepsilon$, $\varepsilon>0$.*
ii) *$h(p)$ has to decrease faster than $|p|^{-2}$ for $p\to\pm\infty$.*

Note that for Neumann boundary-conditions the inverse-hyperbolic terms change their signs. In this case, however, the parabolic terms are quite different, due to the additional presence of the continuous spectrum represented by Eisenstein-series, see e.g. *Ref.* [32]. □

IV. Determinants of Maass–Laplacians

In this section we are going to calculate determinants of Maass–Laplacians on bordered Riemann surfaces. Some examples of calculations of the scalar determinant are due to Bolte and Steiner [13] and Blau et al. [9, 10]. In particular, we calculate the determinant of the operator $\Delta_m^{(\pm)}=D_m+m(m\pm 1)$, because $\Delta_m^{(+)}$ is the relevant operator in string theory. First we only consider the case where the Fuchsian group $\hat{\Gamma}$ is strictly hyperbolic, since then it is known that the discrete spectrum of D_m (and of $\Delta_m^{(\pm)}$) is infinite and no continuous spectrum appears. We denote the omission of zero-modes by primes and define the determinants by the zeta-function regularization, i.e. we set

$$\det{}'(\Delta_m^{(\pm)}):=\exp\left(-\frac{d}{ds}\zeta_m^{(\pm)}(s)\right)\Bigg|_{s=0} \qquad (4.1)$$

$$\zeta_m^{(\pm)}(s):=\mathrm{tr}'(\Delta_m^{(\pm)})^{-s}=\frac{1}{\Gamma(s)}\int_0^{\infty}dt\, t^{s-1}[e^{-tm(m\pm 1)}\mathrm{tr}(e^{-tD_m})-N_m^{(\pm)}]\,. \qquad (4.2)$$

Here $N_m^{(\pm)}$ denote the numbers of respective zero-modes. In the case of Dirichlet boundary-conditions no zero-modes are present, and we discuss this case first. With the heat-kernel function $h(p)=\mathrm{e}^{-t(p^2+\frac{1}{4})}$ we determine $g(u)=\mathrm{e}^{-u^2/4t-t/4}/\sqrt{4\pi t}$ and consider the trace formula (3.8) for this test function. We split up $\zeta_m(s)$ into a term $\zeta_I(s)$ corresponding to the first summand on the r.h.s. of (3.8), a term $\zeta_\Gamma(s)$ corresponding to the second, fourth and fifth summand, and a term $\zeta_{g(0)}(s)$ corresponding to the last summand, respectively. We then find by means of (3.8)

$$\zeta_\Gamma'(0)=-\frac{1}{2}\ln Z(m+1)\,. \tag{4.3}$$

Here we have introduced the Selberg zeta-function for bordered Riemann surfaces and Dirichlet boundary-conditions according to [56, p. 139] as

$$\begin{aligned}Z(s)=&\prod_{\{\gamma\}_p}\prod_{k=0}^{\infty}[1-\chi_\gamma^m \mathrm{e}^{-l_\gamma(s+k)}]\\ &\times\prod_{\{\rho\}_p}\prod_{k=0}^{\infty}\left(\frac{1+\chi_\rho^m \mathrm{e}^{-l_\rho(s+k)}}{1-\chi_\rho^m \mathrm{e}^{-l_\rho(s+k)}}\right)^{(-1)^k\chi_{\mathscr{I}}^m}\\ &\times\prod_{i=1}^{n}\prod_{k=0}^{\infty}\left(\frac{1}{1-\chi_{C_i}^m \mathrm{e}^{-l_{C_i}(s+k)}}\right)^{2(-1)^k},\end{aligned} \tag{4.4}$$

($l_\rho=\frac{1}{2}l_{\rho^2}$) and use has been made of the identity

$$\begin{aligned}\ln&\prod_{\{\rho\}_p}\prod_{k=0}^{\infty}\left(\frac{1+\chi_\rho^m \mathrm{e}^{-l_\rho(s+k)}}{1-\chi_\rho^m \mathrm{e}^{-l_\rho(s+k)}}\right)^{(-1)^k\chi_{\mathscr{I}}^m}\\ &=\sum_{\{\rho\}}\sum_{k=0}^{\infty}\chi_{\mathscr{I}}^m 2(-1)^k\sum_{l=1}^{\infty}\frac{1}{2l-1}[\chi_\rho^m \mathrm{e}^{-(s+k)l_\rho}]^{2l-1}\\ &=\sum_{\{\rho^2\}}\sum_{k=0}^{\infty}\frac{\chi_\rho^{m(2k+1)}\chi_{\mathscr{I}}^m \mathrm{e}^{-sl_{\rho^2}(k+\frac{1}{2})}}{(k+\frac{1}{2})(1+\mathrm{e}^{-l_{\rho^2}(k+\frac{1}{2})})},\end{aligned} \tag{4.5}$$

together with the expansion

$$\ln\frac{1+x}{1-x}=2\sum_{k=1}^{\infty}\frac{x^{2k-1}}{2k-1}\,. \tag{4.6}$$

We have adopted the notation of Venkov [56] for the Selberg zeta-function on bordered Riemann surfaces with some small modifications, e.g., the definition of Venkov differs by a power of two. The analytical properties of this zeta-function are shortly discussed in Appendix A. The term in the trace formula proportional to $g(0)$ gives the contribution

$$\zeta_{g(0)}'(0)=\frac{L}{8}(m+1)\,. \tag{4.7}$$

The term $\zeta_I(s)$ corresponding to the zero-length contribution has been calculated in [11]. One has only to replace $(g-1)=\mathscr{A}/4\pi\rightarrow\mathscr{A}(\hat{\mathscr{F}})/8\pi$ in the relevant formulae. Therefore we obtain for the determinant of the operator Δ_m^+ for $m \in \mathbb{Z}$,

$$\det(\Delta_m^+) = \sqrt{Z\left(1+\frac{m}{2}\right)}$$

$$\times \exp\left[-\frac{L}{8}(m+1) + \frac{\mathscr{A}(\hat{\mathscr{F}})}{8\pi} \left\{ (m+1)\ln(2\pi) + 4\zeta'(-1) - \frac{1}{2}(m+1)^2 \right.\right.$$

$$-4 \sum_{3 \leqq k \leqq m/2+1} \ln \Gamma(k) + 4\left(\frac{m}{2} - \frac{1}{2} - \left[\frac{m}{2}\right]\right) \ln \Gamma\left(\frac{m}{2}+1\right)$$

$$\left.\left. + \sum_{0 \leqq n \leqq (m-1)/2} (m-2n-1)\ln(mn+m-n^2-n) \right\} \right] . \qquad (4.8)$$

Here empty sums are understood as to be ignored and $[x]$ is the integer part of x. $\zeta'(-1)$ denotes the derivative of the Riemann-zeta-function $\zeta(s)$ at $s=-1$. The cases $m=0, 2$, respectively, reduce to

$$\det(\Delta_0^+) = \sqrt{Z(1)} \exp\left\{ -\frac{L}{8} + \frac{\mathscr{A}(\hat{\mathscr{F}})}{8\pi} \left[-\frac{1}{2} + \ln(2\pi) + 4\zeta'(-1) \right] \right\} , \qquad (4.9)$$

$$\det(\Delta_2^+) = \sqrt{Z(2)}$$

$$\times \exp\left\{ -\frac{3L}{8} + \frac{\mathscr{A}(\hat{\mathscr{F}})}{8\pi} \left[-\frac{9}{2} + 3\ln(2\pi) + 4\zeta'(-1) - 2\ln\Gamma(2) \right] \right\} . \qquad (4.10)$$

Note that the construction of the Selberg zeta-function on bordered Riemann surfaces guarantees that in the case of Neumann boundary-conditions the power of the terms involving the inverse-hyperbolic conjugacy classes change their signs, such that the product of the determinants of Dirichlet and Neumann boundary-conditions, denoted by superscripts D and N, respectively, just give in a natural way the determinant of the corresponding operator on the Riemann surface $\hat{\Sigma}$, denoted by the superscript $\hat{\Sigma}$, i.e.

$$\det(\Delta_m^{(D,+)}) \times \det{}'(\Delta_m^{(N,+)}) = \det{}'(\Delta_m^{(\hat{\Sigma},+)}) \propto \begin{cases} Z'_{\hat{\Sigma}}(1), & \text{for } m=0 , \\ Z_{\hat{\Sigma}}(\frac{m}{2}+1), & \text{for } m \neq 0 , \end{cases} \qquad (4.11)$$

where $Z_{\hat{\Sigma}}$ is the Selberg zeta-function on the entire Riemann surface $\hat{\Sigma}$.

Note that for $m=0$ in (4.9–4.11) one must take $Z(1)$ for Dirichlet boundary-conditions, and $Z'(1)$ for Neumann boundary-conditions. The difference is caused by the existence or absence of a zero for $Z(s)$ at $s=1$ in the Neumann- and Dirichlet-case, respectively. In the Neumann-case the zero at $s=1$ corresponds to the zero-mode of the Laplacian, which must be subtracted in the definition of the determinant. In the Dirichlet-case neither a zero of $Z(s)$ nor a zero-mode of the Laplacian exists.

As already mentioned in Sect. II it is not clear for a general non-cocompact Fuchsian group $\hat{\Gamma}$ that the operator D_m on $\mathscr{L}^2(\chi, m)$ has infinitely many eigenvalues. In fact, for non-arithmetic groups a conjecture by Phillips and Sarnak [46] says that this will not be the case. One therefore is confronted with the problem of defining a suitable zeta-regularized determinant of D_m (or of $\Delta_m^{(\pm)}$). If a continuous spectrum is present one can apply quantum mechanical scattering theory, and one finds that the relevant S-matrix (see e.g. [21, 56]) has poles in the complex s-plane for $0<\Re(s)<1/2$ corresponding to scattering resonances (for some explicit examples see e.g. Koyama [36]). By the Selberg trace formula one can conclude [56] that the total number of eigenvalues and resonances with $(\Im(s))^2 \leqq \lambda$ grows asymptotically like $\mathscr{A}(\hat{\mathscr{F}})\lambda/4\pi$ for $\lambda \to \infty$, and thus is infinite. Conventionally [21], one therefore defines the spectral zeta-function $\zeta_m(s)$ by an inclusion of the resonances in addition to the eigenvalues. It then turns out that the relation (4.11) remains valid. There do indeed occur complicated constants (depending on the topology of $\hat{\Sigma}$) multiplying the Selberg zeta-function, but the relevant contribution, which is not constant on the respective Teichmüller space of quasi-conformal deformations of $\hat{\Sigma}$, is determined by the Selberg zeta-function; and this is also the information needed for the application of determinantal formulae in string theory.

V. Discussion and Summary

In this paper we have studied the Selberg trace formula on bordered Riemann surfaces for automorphic forms of weight m, $m \in \mathbb{Z}$. Our paper is thus a merge of the work of Hejhal and Venkov, and a generalization of Bolte and Steiner, who have studied the Selberg trace formula for automorphic forms of weight m on closed Riemann surfaces and for automorphic functions (i.e. for weight zero) on bordered surfaces, respectively. Our fundamental results are the trace formulae (3.8, 3.24) as formulated in Theorems 3.1 and 3.2, where we first included only hyperbolic and inverse-hyperbolic conjugacy classes, and then all possible conjugacy classes for bordered Riemann surfaces, i.e. elliptic as well as parabolic ones.

By means of this trace formula we could derive determinants of Maass–Laplacians on bordered Riemann surfaces in terms of the Selberg zeta-functions on these surfaces. This could be done for Dirichlet (4.8) as well as Neumann boundary-conditions, and a fundamental relation connecting both was derived (4.11). These expressions have previously not been available, due to the lack of the relevant trace formula, cf., Refs. [9, 13]. Of course, the reasoning of Gross and Periwal [28] on the divergence of the perturbation expansion concerns only closed bosonic strings, but it is reasonable to expect that their arguments are not altered considerably in our case.

The growing behaviour of the string integrand, which depends via the scalar- and vector-Laplacian determinants on the Selberg zeta-functions, in our case $\sqrt{Z(1)}$ and $\sqrt{Z(2)}$, respectively, can be kept under control in such a way, that at most an exponential behaviour appears. The blowing up of the bosonic perturbative string theory, both for closed and open string, is eventually due to the factorial growths of the volume of the moduli space for increasing genus. Let us further note that the study of trace formulae in general can also be useful in the theory of quantum chaos [5–7, 42]. We have not discussed in detail the analytic properties of the Selberg zeta-function on bordered Riemann surfaces (cf. [56]), but it is well-known that these functions have a so-called "trivial" and "non-trivial" structure. The nontrivial structure stems from the eigenvalues of the Maass-Laplacians on the corresponding Riemann surfaces. In general one finds these zeros on the critical line $\Re(s)=1/2$. The trivial structure is due to the zero-length term and the elliptic and parabolic conjugacy classes. The zero-length term usually generates zeros of a particular multiplicity $\propto \mathscr{A}(\mathscr{F})$, which is only altered (reduced) by the presence of elliptic conjugacy classes. The parabolic conjugacy classes, however, generally introduce an additionally analytical structure, e.g. *poles* might appear. This is the case both for closed and bordered Riemann surfaces. Let us finally note that from the explicit form of the trace formula (3.24) Weyl's law for the Dirichlet boundary-value problem can be derived. It reads [56, p. 143]

$$N\left(p^2+\frac{1}{4}\right)\simeq\frac{\mathscr{A}(\mathscr{F})}{4\pi}p^2-\frac{\kappa}{2\pi}p\ln p+\frac{c^{(\mathscr{F})}+\kappa}{2\pi}p+O\left(\frac{p}{\ln p}\right), \tag{5.1}$$

with the quantity $c^{(\mathscr{F})}$ defined in Appendix A. $N(\lambda)$ denotes the number of eigenvalues of D_m not exceeding λ. Notice that D_m is defined by using the Poincaré (hyperbolic) metric on the doubled surface. Analogously, Huber's [34] law can be derived, giving for the number $N_\gamma(L)$ of closed geodesics on the bordered surface with lengths up to a given value L the estimate

$$N_\gamma(L)\sim \mathrm{Ei}\,(L)\sim\frac{e^L}{L}\,. \tag{5.2}$$

These results can e.g. be achieved by means of the test-function $h(p)=\cos(pL)\,\mathrm{e}^{-t(p^2+\frac{1}{4})}$ in the relevant trace formula.

Summarizing our results, we have contributed to the theory of automorphic forms as well as to the theory of open strings by explicitly evaluating and discussing the relevant determinants in the Polyakov path integral in terms of Selberg zeta-functions.

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Appendix A

We cite some results from the study of the analytic properties of the Selberg zeta-function. The analytic properties are studied by choosing a particular test-function in the trace formula. It is also possible to derive these properties by introducing the determinant [53] which for $m=0$ is given by

$$\mathscr{D}_i(z)=\det(-\Delta_i+z)\,, \tag{A.1}$$

where $i=D, N, \Delta$ denotes Dirichlet, Neumann boundary-conditions, or the determinant on the entire domain $\hat{\Sigma}$, and formulating the Selberg zeta-function in terms of this determinant, i.e. [53, 58]

$$Z(s)=s(s-1)\mathscr{D}_\Delta[s(s-1)]\left[(2\pi)^{1-s}\,\mathrm{e}^{\tilde{C}+s(s-1)}G(s)\,G(s+1)\right]^{\mathscr{A}(\mathscr{F})/2\pi} \tag{A.2}$$

with $\tilde{C}=\frac{1}{4}+\frac{1}{2}\ln 2\pi-2\zeta'(-1)$. The obvious relation $\mathscr{D}_D(z)\mathscr{D}_N(z)=\mathscr{D}_\Delta(z)$ is, of course, fulfilled for all z and can be obtained via the regularization

$$\mathscr{D}_i(z)=\mathscr{D}_i(0)\,\mathrm{e}^{\gamma_i z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{E_n}\right)\mathrm{e}^{-z/E_n}\right], \tag{A.3}$$

(compare also [53, 58]), and where γ_i is a generalized Euler constant for Riemann surfaces [13, 53]. This shows again (4.11).

The study of the analytic properties of the Selberg zeta-function is due to e.g. Bolte and Steiner [13], Hejhal [32], McKean [39], Steiner [53] and Venkov [56], the case of bordered Riemann surfaces was studied in [13, 56]. We treat the general case covered by Theorem 3.2 and use the test-function

$$h(p, s, b) = \frac{1}{(s-\frac{1}{2})^2 + p^2} - \frac{1}{(b-\frac{1}{2})^2 + p^2}, \tag{A.4}$$

which fulfills the requirements of Theorem 3.2 if $\Re(s)$, $\Re(b) > 1$. One finds

$$g(u, s, b) = \frac{1}{2(s-\frac{1}{2})} \left(e^{-(s-\frac{1}{2})|u|} - e^{-(b-\frac{1}{2})|u|}\right). \tag{A.5}$$

We consider the definition of the Selberg zeta-function on bordered Riemann surfaces (4.4), which generalizes the definition of Venkov [56] for $m \neq 0$. Then one derives the following

Theorem A.1 (compare Venkov [56, p. 139] for $m=0$). *The Selberg trace formula for the Selberg zeta-function on bordered Riemann surfaces and Dirichlet boundary-conditions has the form:*

$$\begin{aligned}
\frac{Z'(s)}{Z(s)} &= \left(s - \frac{1}{2}\right) \frac{2\mathscr{A}(\mathscr{F})}{\pi} \left[\Psi\left(s + \frac{m}{2}\right) + \Psi\left(s - \frac{m}{2}\right) - \Psi\left(b + \frac{m}{2}\right) - \Psi\left(b - \frac{m}{2}\right)\right] \\
&\quad - \mathrm{i} \sum_{\{R\}_p} \sum_{k=1}^{\nu-1} \frac{1}{\nu \sin(k\pi/\nu)} \sum_{l=0}^{\infty} \left[\frac{e^{-2\mathrm{i}(k\pi/\nu)(l+1/2-m/2)}}{s+l-m/2} - \frac{e^{2i(k\pi/\nu)(l+1/2+m/2)}}{s+l+m/2}\right] \\
&\quad + 4\left(s - \frac{1}{2}\right) \sum_j \left(\frac{1}{(s-\frac{1}{2})^2 + p_j^2} - \frac{1}{(b-\frac{1}{2})^2 + p_j^2}\right) + \mathrm{const}_1 + \mathrm{const}_2 \left(s - \frac{1}{2}\right) \\
&\quad + 2\tilde{\kappa}\Psi(1-s) - 4\tilde{\kappa}\left(s - \frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{1}{(s-\frac{1}{2})^2 - (k+\frac{1}{2})^2} \\
&\quad + \tilde{\kappa}\left[2\Psi(s) - \Psi\left(s + \frac{m}{2}\right) - \Psi\left(s - \frac{m}{2}\right)\right],
\end{aligned} \tag{A.6}$$

with some constants $\mathrm{const}_{1,2}$. □

In the evaluation of the various integrals some results of Hejhal [33, p. 435] have been used. The zero- and pole-structure can be read off:

Theorem A.2 (Venkov [56, pp. 142]). *$Z(s)$ for bordered Riemann surfaces and Dirichlet boundary-conditions is a meromorphic function of $s \in \mathbb{C}$ of order equal to two.*

The zeros of the function $Z(s)$ are at the following points:

i) *Nontrivial zeros: on the line $\Re(s) = 1/2$, symmetric relative to the point $s = 1/2$, and on the interval $[0, 1]$, symmetric to the point $s = 1/2$. Call these zeros s_j. Each s_j has multiplicity equal twice the multiplicity of the corresponding eigenvalue E_j of the operator D_m of the corresponding Dirichlet boundary-value problem, $E_j = \frac{1}{4} + p_j^2$ and E_j runs through the entire spectrum of D_m.*

ii) *Trivial zeros*:

1) *at the points* $s=-l+m/2$, $l\in\mathbb{N}_0$, *with multiplicity* $\#N_{-l}$ *given by*

$$\#N_{-l}=\frac{2\mathscr{A}(\mathscr{F})}{\pi}\left(l-\frac{m-1}{2}\right)+\tilde{\kappa}+\mathrm{i}\sum_{\{R\}_p}\sum_{k=1}^{\nu-1}\frac{\mathrm{e}^{2\mathrm{i}(k\pi/\nu)(l+1/2+m/2)}}{\nu\sin(k\pi/\nu)}\,. \tag{A.7}$$

2) *at the points* $s=-l-m/2$, $l\in\mathbb{N}_0$, *with multiplicity* $\#N_{-l}$ *given by*

$$\#N_{-l}=\frac{2\mathscr{A}(\mathscr{F})}{\pi}\left(l+\frac{m+1}{2}\right)+\tilde{\kappa}-\mathrm{i}\sum_{\{R\}_p}\sum_{k=1}^{\nu-1}\frac{\mathrm{e}^{2\mathrm{i}(k\pi/\nu)(l+1/2-m/2)}}{\nu\sin(k\pi/\nu)}\,. \tag{A.8}$$

3) *at the points* $s=l\in\mathbb{N}$ *with multiplicity* $\#N_l=2\kappa$.

iii) $Z(s)$ *has poles at the points*

1) $s=l$, $l\in\mathbb{Z}$, *with multiplicity* $\#P_l=2\kappa$.

2) $s=-l$, $l=0,1,2,\ldots$, *with multiplicity* $\#P_l=2\kappa$.

Notice that for $m=0$ *the zeros of* ii1), ii2) *and the poles of* iii2) *combine to zeros with multiplicity*

$$\#N_{-l}=\frac{4\mathscr{A}(\mathscr{F})}{\pi}\left(l+\frac{1}{2}\right)-2\sum_{\{R\}_p}\sum_{k=1}^{\nu-1}\frac{\sin[k\pi(2l+1)/\nu]}{\nu\sin(k\pi/\nu)}\,. \tag{A.9}$$

The full picture of the zero and poles, of course, emerges by combining ii1) *and* iii), i.e. *there remain only poles at* $s=-l$, $l\in\mathbb{N}_0$. *Note, in particular that if no elliptic and parabolic terms are present,* $Z(s)$ *has, of course, no poles, and no zero at* $s=1$; *this is in contrast to the Selberg zeta-function on a closed Riemann surface, where the zero at* $s=1$ *stems from the one-fold zero-mode of the Maass–Laplacian.* □

Theorem A.3 (Venkov [56, p. 143]). *The functional equation for the Selberg zeta function on bordered Riemann surfaces and Dirichlet boundary-conditions has the form*

$$Z(1-s)=Z(s)\psi(s) \tag{A.10}$$

with the function $\Psi(s)$ *given by*

$$\begin{aligned}\psi(s)=\left[\frac{\Gamma(1-s)}{\Gamma(s)}\right]^{2\kappa}\exp\Bigg\{&-4\mathscr{A}(\mathscr{F})\int_0^{s-\frac{1}{2}}t\begin{pmatrix}\tan\pi t\\ \cot\pi t\end{pmatrix}dt+4c^{(\mathscr{F})}\left(s-\frac{1}{2}\right)\\ &+\mathrm{i}\sum_{\{R\}_p}\sum_{k=1}^{\nu-1}\frac{1}{\nu\sin(k\pi/\nu)}\int_0^{s-\frac{1}{2}}\\ &\times\sum_{l=0}^{\infty}\Bigg[\frac{\mathrm{e}^{-2\mathrm{i}(k\pi/\nu)(l+1/2-m/2)}}{s+l-m/2}+\frac{\mathrm{e}^{-2\mathrm{i}(k\pi/\nu)(l+1/2-m/2)}}{s-l+(m-3)/2}\\ &-\frac{\mathrm{e}^{-2\mathrm{i}(k\pi/\nu)(l+1/2+m/2)}}{s+l+(m-1)/2}-\frac{\mathrm{e}^{-2\mathrm{i}(k\pi/\nu)(l+1/2+m/2)}}{s-l-(m-3)/2}\Bigg]dt\Bigg\}\,,\end{aligned} \tag{A.11}$$

where the $\tan\pi t$-, *respectively the* $\cot\pi t$-*term must be taken whether m is even or odd, and the constant* $c^{(\mathscr{F})}$ *is given by*

$$c^{(\mathscr{F})}=\frac{1}{4}\left[\sum_{\substack{\{\rho\};\hat{\Gamma}_{\rho,\mathrm{ell}}\\ \mathrm{tr}(\rho)=0}}\chi_\rho^m\ln\left(\frac{a(\rho)}{\mu(\rho)}\right)-\tilde{\kappa}\ln 2-\frac{L}{2}\right]. \tag{A.12}$$

□

Appendix B

We consider the integral

$$A_4=-\frac{1}{4\pi}\int_0^\infty dw\,Q'(w)I(w)=\frac{1}{4\pi}\int_0^\infty dw\,Q(w)I'(w)\,. \tag{B.1}$$

The integral $I(w)$ is defined by

$$I(w)=\Re\int_{-\pi/2}^{\pi/2}\left(\frac{\sinh\omega+\mathrm{i}\cos\phi}{\cosh\omega+\sin\phi}\right)^m\ln\left(1+\frac{\cos^2\phi}{\sinh^2\omega}\right)d\phi\,. \tag{B.2}$$

Now we perform in $I(w)$ the transformation $t=\tan\phi$ which yields

$$I(w)=\Re\int_{-\infty}^{\infty}\ln\left(\frac{t^2+b^2}{1+t^2}\right)\left(\frac{\sqrt{1+t^2}+\mathrm{i}a}{b\sqrt{1+t^2}+at}\right)^m\frac{dt}{1+t^2}\,, \tag{B.3}$$

with the abbreviations $a=1/\sinh\omega$, $b^2=a^2+1$ and taking positive square roots. Let us consider the contour integral

$$I_C(w)=\oint_C f(z)\,dz\,, \tag{B.4}$$

with $(b_\alpha=1, b)$

$$f(z)=\frac{\ln(z+\mathrm{i}b_\alpha)}{1+z^2}\left(\frac{\sqrt{1+z^2}+\mathrm{i}a}{b\sqrt{1+z^2}+az}\right)^m\,, \tag{B.5}$$

and a contour C running along the real axis from $-R$ to $+R$, and a semi-circle with radius R closing the contour in the upper half-plane. Then

$$2\pi\mathrm{i}\operatorname*{Res}_{z=1}f(z)=\pi\left[\ln(1+b_\alpha)+\mathrm{i}\frac{\pi}{2}\right]\,, \tag{B.6}$$

with no contributions coming from the m-dependent power term. As one can show, the integral over the semi-circle vanishes in the limit $R\to\infty$, the $\int_{-R}^{R}$-integral then gives

$$\int_{-\infty}^{\infty}\frac{\ln(t+\mathrm{i}b_\alpha)}{1+t^2}\left(\frac{\sqrt{1+t^2}+\mathrm{i}a}{b\sqrt{1+t^2}+at}\right)^m dt=\pi\left[\ln(1+b_\alpha)+\mathrm{i}\frac{\pi}{2}\right]\,. \tag{B.7}$$

Finally we take real parts for $b_\alpha=1$ and $b_\alpha=b$, respectively, with the result

$$I(w)=2\pi\left[\ln\left(1+\frac{1}{2}\sqrt{w+4}\right)-\ln 2\right]\,. \tag{B.8}$$

Therefore

$$\begin{aligned}A_4&=\frac{1}{2}\int_0^\infty dw\,Q(w)\frac{d}{dw}\left[\ln\left(1+\frac{1}{2}\sqrt{w+4}\right)-\ln 2\right]\\&=\frac{1}{4}\int_0^\infty\frac{Q(w)\,dw}{w+4+2\sqrt{w+4}}=\frac{1}{4}\int_0^\infty g(u)\tanh\frac{u}{4}\,du\\&=\frac{1}{8}h(0)+\frac{1}{4\pi}\int_{-\infty}^{\infty}h(p)\left[\Psi\left(\frac{1}{2}+\mathrm{i}p\right)-\Psi(1+\mathrm{i}p)\right]dp\,.\end{aligned} \tag{B.9}$$

Here use has been made of the integral [24, p. 356]

$$\int_0^\infty \frac{e^{-\mu x}}{\cosh x}\,dx=\beta\left(\frac{\mu+1}{2}\right), \tag{B.10}$$

and $\beta(x)$ is the β-function defined by [24, p. 945]

$$\beta(x)=\frac{1}{2}\left[\Psi\left(\frac{1+x}{2}\right)-\Psi\left(\frac{x}{2}\right)\right]. \tag{B.11}$$

Insertion into (3.18) yields the result of (3.22).

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