

## Effective Action in Spherical Domains

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**Abstract:** The effective action on an orbifolded sphere is computed for minimally coupled scalar fields. The results are presented in terms of derivatives of Barnes  $\zeta$ -functions and it is shown how these may be evaluated. Numerical values are shown. An analytical, heat-kernel derivation of the Cesàro-Fedorov formula for the number of symmetry planes of a regular solid is also presented.

### 1. Introduction

In an earlier work [1] we have shown that the  $\zeta$ -function,  $\zeta_\Gamma(s)$ , on orbifold-factored spheres,  $S^d/\Gamma$ , for a conformally coupled scalar field, is given by a Barnes  $\zeta$ -function, [2],  $\zeta_d(s, a|\mathbf{d})$ , where the  $d_i$  are the *degrees* associated with the tiling group  $\Gamma$ . The *free-field Casimir energy on the space-time*  $\mathbb{R} \times S^d/\Gamma$  was given as the value of the  $\zeta$ -function at a negative integer which evaluated to a generalised Bernoulli function. In the present work we wish to consider the effective action on orbifolds  $S^d/\Gamma$  which this time are to be looked upon as Euclidean space-times. In particular we will discuss  $d = 2$  and  $d = 3$ , concentrating on the former.

The simplifying assumption in our previous work was that of conformal coupling on  $\mathbb{R} \times S^d/\Gamma$ . This made the relevant eigenvalues *perfect squares and allowed us to use known generating functions to incorporate the degeneracies*. From the point of view of field theories on the space-times  $S^d/\Gamma$ , retaining this assumption would be rather artificial. A more appropriate choice would be minimal coupling, or possibly conformal coupling, on  $S^d/\Gamma$ . (These coincide for  $d = 2$ .)

The quantities in which we are interested are  $\zeta'_\Gamma(0)$  and  $\zeta_\Gamma(0)$ . The latter determines the divergence in the effective action and the former is, up to a factor and a finite addition, the renormalised effective action (i.e. half the logarithm of the functional determinant).

## 2. Eigenvalues, Degeneracies and Zeta Functions

For the aforementioned conformal coupling, the eigenvalues of the second order operator  $-\Delta_2 + \xi R$  ( $\xi = (d - 1)/4d$ ) are

$$\lambda_n = \frac{1}{4} (n + d - 2)^2 \tag{1}$$

with degeneracies that we shall leave unspecified here.

In our previous work [1] we showed that the corresponding Neumann and Dirichlet  $\zeta$ -functions on  $S^d/\Gamma$  were,

$$\zeta_N^{(C)}(s) = \zeta_d(2s, (d - 1)/2 | \mathbf{d} ), \tag{2}$$

$$\zeta_D^{(C)}(s) = \zeta_d(2s, \sum d_i - (d - 1)/2 | \mathbf{d} ), \tag{3}$$

where the general definition of the Barnes  $\zeta$ -function is

$$\begin{aligned} \zeta_d(s, a | \mathbf{d} ) &= \frac{i\Gamma(1 - s)}{2\pi} \int_L d\tau \frac{\exp(-a\tau) (-\tau)^{s-1}}{\prod_{i=1}^d (1 - \exp(-d_i\tau))} \\ &= \sum_{\mathbf{m}=0}^{\infty} \frac{1}{(a + \mathbf{m} \cdot \mathbf{d})^s}, \quad \text{Re } s > d. \end{aligned} \tag{4}$$

This shows that the eigenvalues are given specifically by

$$\lambda_n = (a + \mathbf{m} \cdot \mathbf{d})^2, \tag{5}$$

the degeneracies coming from coincidences. The parameter  $a$  is  $(d - 1)/2$  in the Neumann case and comparison with the previous form shows that the integer  $n = 2\mathbf{m} \cdot \mathbf{d} + 1$ ,  $\mathbf{m} = \mathbf{0}$  upwards. For Dirichlet conditions,  $a = \sum d_i - (d - 1)/2$ , and then  $n = 2\mathbf{m} \cdot \mathbf{d} - 1$  with  $\mathbf{m} = (1, 1)$  upwards. The interpretation in two dimensions is that the angular momentum is  $L = \mathbf{m} \cdot \mathbf{d}$  for Neumann and  $\mathbf{m} \cdot \mathbf{d} - 1$  for Dirichlet conditions.

Turning to minimal coupling, ( $\xi = 0$ ), the eigenvalues of the Laplacian are

$$\lambda_n = (a + \mathbf{m} \cdot \mathbf{d})^2 - \frac{(d - 1)^2}{4}, \tag{6}$$

and the corresponding  $\zeta$ -function is

$$\zeta(s) = \sum_{\mathbf{m}} \frac{1}{((a + \mathbf{m} \cdot \mathbf{d})^2 - (d - 1)^2/4)^s}. \tag{7}$$

The origin  $\mathbf{m} = \mathbf{0}$  is to be omitted for Neumann conditions, when the  $\zeta$ -function is denoted by  $\bar{\zeta}(s)$ .

Consider a sum of the form

$$\zeta(s) = \sum_{\mathbf{m}} \frac{1}{((a + \mathbf{m} \cdot \mathbf{d})^2 - \alpha^2)^s}, \tag{8}$$

so that

$$\bar{\zeta}(s) = \zeta(s) - (a^2 - \alpha^2)^{-s}. \tag{9}$$

For minimal coupling,  $\alpha = (d - 1)/2$ , while for conformal coupling in  $d$ -dimensions,  $\alpha = 1/2$ . We concentrate on minimal coupling.

A standard way of obtaining information about an expression such as (8) is to perform a binomial expansion to produce a sum of known  $\zeta$ -functions, in the present case a sum of Barnes  $\zeta$ -functions,

$$\zeta(s) = \sum_{r=0}^{\infty} \alpha^{2r} \frac{s(s+1)\dots(s+r-1)}{r!} \zeta_d(2s+2r, a | \mathbf{d}). \tag{10}$$

From this, the value of  $\zeta(s)$  at a nonpositive integer is easily found. For example the important value  $\zeta(0)$  is given by

$$\zeta(0) = \zeta_d(0, a | \mathbf{d}) + \frac{1}{2} \sum_{r=1}^u \frac{\alpha^{2r}}{r} N_{2r}(d) \tag{11}$$

and, more generally, we have

$$\begin{aligned} \zeta(-n) &= \sum_{r=0}^n (-\alpha^2)^r \binom{n}{r} \zeta_d(2r-2n, a | \mathbf{d}) \\ &+ \frac{(-1)^n}{2} \sum_{r=1}^u \frac{n!(r-1)!}{(r+n)!} \alpha^{2n+2r} N_{2r}, \end{aligned} \tag{12}$$

where  $u = d/2$  if  $d$  is even and  $u = (d-1)/2$  if  $d$  is odd.

$N_r(d)$  is the residue defined by

$$\zeta_d(s+r, a | \mathbf{d}) \rightarrow \frac{N_r(d)}{s} + R_r(d) \quad \text{as } s \rightarrow 0, \tag{13}$$

where  $1 \leq r \leq d$ . Expressions for the residue and remainder involve generalised Bernoulli functions and can be found in Barnes [2]. For shortness, their dependence on the parameter  $a$  is not indicated.

The form of the residues given by Barnes [2] is

$$N_r(d) = \frac{(-1)^{r+d}}{(r-1)!} {}_dS_1^{(r+1)}(a),$$

where  ${}_dS_1^{(r+1)}(a)$  is the  $(r+1)$ <sup>th</sup> derivative of Barnes' generalised Bernoulli polynomial  ${}_dS_1(a)$ . The general relation with the more usual polynomials, [5], will not be given here. Specific forms are

$$\begin{aligned} {}_dS_1^{(d+1)}(a) &= \frac{1}{\prod d_i}, & {}_dS_1^{(d)}(a) &= \frac{2a - \sum d_i}{2 \prod d_i}, \\ {}_dS_1^{(d-1)}(a) &= \frac{1}{12 \prod d_i} \left( 6a^2 - 6a \sum d_i + \sum d_i^2 + 3 \sum_{i<j} d_i d_j \right). \end{aligned} \tag{14}$$

Barnes also gives the values

$$\zeta_d(-n, a | \mathbf{d}) = \frac{(-1)^d}{n+1} {}_dS_{1+n}^{(1)}(a) = \frac{(-1)^d}{\prod d_i (d+n)!} B_{d+n}^{(d)}(a | \mathbf{d}). \tag{15}$$

From (9), (11), (14) and (15) we find, for two dimensions,

$$\begin{aligned} \zeta_D(0) &= \frac{1}{12d_1d_2} (3 - 3(d_1 + d_2) + (d_1 + d_2)^2 + d_1d_2), \\ \zeta_N(0) &= \zeta_D(0) - 1. \end{aligned} \tag{16}$$

This corrects our previous expression [1] obtained by an incorrect manipulation of the heat-kernel.

### 3. The Derivative of the Zeta Function

The derivative at  $s = 0$  is a little more difficult to find. From (10) a first step is

$$\begin{aligned} \zeta'(0) &= 2\zeta'_d(0, a | \mathbf{d}) + \sum_{r=1}^u \frac{\alpha^{2r}}{r} \left( R_{2r} + \frac{1}{2} N_{2r} \sum_1^{r-1} \frac{1}{k} \right) \\ &+ \sum_{r=u+1}^{\infty} \frac{\alpha^{2r}}{r} \zeta_d(2r, a | \mathbf{d}). \end{aligned} \tag{17}$$

The integral representation of the Barnes  $\zeta$ -function allows the final sum in (17) to be written as

$$\begin{aligned} &\sum_{r=u+1}^{\infty} \frac{\alpha^{2r}}{r\Gamma(2r)} \int_0^{\infty} d\tau \frac{\tau^{2r-1} \exp(-a\tau)}{\prod_i (1 - \exp(-d_i \tau))} \\ &= 2 \int_0^{\infty} \exp(-a\tau) \left( \cosh \alpha\tau - \sum_{r=0}^u \frac{(\alpha\tau)^{2r}}{(2r)!} \right) \frac{d\tau}{\tau \prod_i (1 - \exp(-d_i \tau))}. \end{aligned} \tag{18}$$

In the Neumann case  $a = \alpha$  and there is an infrared, logarithmic divergence at infinity caused by the zero mode which will be taken care of by the transition to  $\bar{\zeta}$ , (9).

Although the integral converges nicely at  $\tau = 0$ , the individual terms of the integrand do not. It is enough to introduce another ultraviolet analytic regularisation and define the intermediate quantity,

$$2 \int_0^{\infty} \exp(-a\tau) \left( \cosh \alpha\tau - \sum_{r=0}^u \frac{(\alpha\tau)^{2r}}{(2r)!} \right) \frac{\tau^{s-1} d\tau}{\prod_i (1 - \exp(-d_i \tau))} \tag{19}$$

whose  $s = 0$  limit gives (18).

After continuation, (19) integrates to

$$\Gamma(s) (\zeta_d(s, a - \alpha | \mathbf{d}) + \zeta_d(s, a + \alpha | \mathbf{d})) - 2 \sum_{r=0}^u \frac{\alpha^{2r}}{(2r)!} \Gamma(s + 2r) \zeta_d(s + 2r, a | \mathbf{d}). \tag{20}$$

As  $s$  tends to zero, each term in (20) yields a pole and a finite remainder. The poles must cancel and so

$$\zeta_d(0, a - \alpha | \mathbf{d}) + \zeta_d(0, a + \alpha | \mathbf{d}) - 2\zeta_d(0, a | \mathbf{d}) = \sum_{r=1}^u \frac{\alpha^{2r}}{r} N_{2r}(d). \tag{21}$$

This condition is an identity between generalised Bernoulli functions. Combined with (11) it produces the symmetrical expression

$$\zeta(0) = \frac{1}{2} (\zeta_d(0, a - \alpha | \mathbf{d}) + \zeta_d(0, a + \alpha | \mathbf{d})). \tag{22}$$

The finite remainder in (20) is

$$\begin{aligned} &\zeta'_d(0, a - \alpha | \mathbf{d}) + \zeta'_d(0, a + \alpha | \mathbf{d}) - 2\zeta'_d(0, a | \mathbf{d}) - \sum_{r=1}^u \frac{\alpha^{2r}}{r} R_{2r}(d) \\ &- \gamma(\zeta_d(0, a - \alpha | \mathbf{d}) + \zeta_d(0, a + \alpha | \mathbf{d}) - 2\zeta_d(0, a | \mathbf{d})) \\ &- \sum_{r=1}^u \frac{\alpha^{2r}}{r} \psi(2r)N_{2r}(d) \end{aligned} \tag{23}$$

which, in view of (21), can be written

$$\begin{aligned} &\zeta'_d(0, a - \alpha | \mathbf{d}) + \zeta'_d(0, a + \alpha | \mathbf{d}) - 2\zeta'_d(0, a | \mathbf{d}) \\ &- \sum_{r=1}^u \frac{\alpha^{2r}}{r} R_{2r}(d) - \sum_{r=1}^u \frac{\alpha^{2r}}{r} (\psi(2r) + \gamma)N_{2r}(d). \end{aligned} \tag{24}$$

Combining this with (17) we have finally

$$\zeta'(0) = \zeta'_d(0, a - \alpha | \mathbf{d}) + \zeta'_d(0, a + \alpha | \mathbf{d}) - \sum_{r=1}^u \frac{\alpha^{2r}}{2r} (2\psi(2r) - \psi(r) + \gamma)N_{2r}(d). \tag{25}$$

The fact that the remainders have cancelled, suggests that there is a more rapid route to this result.

Apart from the final term, (25) is the expression that would have been obtained by a naive application of the ‘‘surrogate’’  $\zeta$ -function method which is based on the product nature of the eigenvalues,  $(a - \alpha + \mathbf{m} \cdot \mathbf{d})(a + \alpha + \mathbf{m} \cdot \mathbf{d})$ , in (8) followed by an application of the rule  $\ln \det(AB) = \ln \det A + \ln \det B$ . This method is suspect, as discussed by Allen [3] and by Chodos and Myers [4]. Allen [3] derives a particular ‘‘correction’’ term as in (25). He also points out that (22) could be expected on the basis of the eigenvalue factorisation, being the average of the regularised dimensions of the operator factors.

The final term in (25) can be rewritten

$$\sum_{r=1}^u \frac{\alpha^{2r}}{2r} (2\psi(2r) - \psi(r) + \gamma)N_{2r}(d) = \sum_{r=1}^u \frac{\alpha^{2r}}{r} N_{2r}(d) \sum_0^{r-1} \frac{1}{2k + 1}.$$

In order to evaluate the effective action we must substitute the appropriate values of  $a$  and  $\alpha$  for Neumann and Dirichlet conditions into (25). In the former case it is also necessary to remove the zero mode, i.e. to use  $\bar{\zeta}$ . The relevant quantity then is the  $\Gamma$ -modular form  $\varrho$ , defined by, [2],

$$\lim_{\varepsilon \rightarrow 0} \zeta'_r(0, \varepsilon | \mathbf{d}) = -\ln \varepsilon - \ln \varrho_r(\mathbf{d}). \tag{26}$$

We find the following basic expressions

$$\zeta'_N(0) = -\ln \varrho_d(\mathbf{d}) + \zeta'_d(0, d - 1 | \mathbf{d}) + \ln(d - 1) - \sum_{r=1}^u \frac{\alpha^{2r}}{r} N_{2r}(d) \sum_0^{r-1} \frac{1}{2k + 1} \tag{27}$$

and

$$\zeta'_D(0) = \zeta'_d(0, d_0 | \mathbf{d}) + \zeta'_d(0, d_0 + d - 1 | \mathbf{d}) - \sum_{r=1}^u \frac{\alpha^{2r}}{r} N_{2r}(d) \sum_0^{r-1} \frac{1}{2k + 1} \tag{28}$$

where  $d_0 = \sum_i d_i - d + 1$  is the number of reflecting planes in  $\Gamma$ . We recall that  $\alpha = (d - 1)/2$  for minimal coupling and that in (27),  $N$  is evaluated at  $a = (d - 1)/2$  and in (28) at  $a = d_0 + (d - 1)/2$ .

In the case of the two-sphere, (27) and (28) become

$$\zeta'_N(0) = -\ln \varrho_2(\mathbf{d}) + \zeta'_2(0, 1 | \mathbf{d}) - \frac{1}{4g} \tag{29}$$

and

$$\zeta'_D(0) = \zeta'_2(0, d_0 | \mathbf{d}) + \zeta'_2(0, d_0 + 1 | \mathbf{d}) - \frac{1}{4g}, \tag{30}$$

where we have set  $g = d_1 d_2$ , the order of the rotational part of  $\Gamma$ .

For the three-sphere

$$\zeta'_N(0) = -\ln \varrho_3(\mathbf{d}) + \zeta'_3(0, 2 | \mathbf{d}) + \ln 2 + \frac{d_0}{2g}, \tag{31}$$

and

$$\zeta'_D(0) = \zeta'_3(0, d_0 | \mathbf{d}) + \zeta'_3(0, d_0 + 2 | \mathbf{d}) - \frac{d_0}{2g}, \tag{32}$$

where  $g = d_1 d_2 d_3$ . We note the change of sign in the last term.

Equations (29) to (32) are the calculational formulae we shall use in the rest of this paper. It is also possible to evaluate the derivative of the  $\zeta$ -function at negative integers,  $\zeta'(-n)$ . This would be relevant if we were interested in the effective action on product spaces like  $\mathbb{R} \times \mathbb{R}^k \times S^d/\Gamma$ . A few details are presented in the appendix.

Although our main interest is in minimal coupling, it should be mentioned that the result (25) can be used immediately for massive fields, assuming that the appropriate value of  $\alpha$  is real. This means that the mass  $\kappa$  is restricted to the region  $0 \leq \kappa \leq (d - 1)/2$ . For larger masses a slightly different continuation is needed.

#### 4. The Derivative of the Barnes Zeta Function

We turn now to the evaluation of the derivatives needed in (29) and (30). A preliminary step is to remove any common factors of the degrees  $d_1$  and  $d_2$  by setting  $d_i = ce_i$  with  $e_1$  and  $e_2$  coprime so that the denominator function in (4) equals  $c(b + \mathbf{m} \cdot \mathbf{d})$ , where  $b = a/c$ .

The summation in (4) is rewritten by introducing the residue classes with respect to  $\mathbf{e}$ . On setting

$$m_1 = n_1 e_2 + p_2, \quad m_2 = n_2 e_1 + p_1,$$

where  $0 \leq p_i \leq e_i - 1$ , the denominator function in (4) equals  $c(b + e_1 e_2 (n_1 + n_2) + p_2 e_1 + p_1 e_2)$  and the sum over  $\mathbf{m}$  becomes

$$\begin{aligned} \zeta_2(s, a | \mathbf{d}) &= c^{-s} \sum_{\mathbf{P}} \sum_{n=0}^{\infty} \frac{1+n}{(b + e_1 e_2 n + p_2 e_1 + p_1 e_2)^s} \\ &= \frac{c^{2-s}}{g} \sum_{\mathbf{P}, n} \frac{1}{(b + N)^{s-1}} + \left(\frac{c}{g}\right)^s \sum_{\mathbf{P}} (1 - w_b) \sum_{n=0}^{\infty} \frac{1}{(n + w_b)^s}, \end{aligned} \tag{33}$$

where

$$N = e_1 e_2 n + p_2 e_1 + p_1 e_2, \quad \text{and} \quad w_b = \frac{b}{e_1 e_2} + \frac{p_1}{e_1} + \frac{p_2}{e_2}.$$

Consider the integer  $N = e_1 e_2 n + p_2 e_1 + p_1 e_2$ . As  $n$  ranges over 0 to  $\infty$ , and the  $p_i$  over their domains,  $N$  will likewise run over this infinite range with the exception of some integers  $< e_1 e_2$  at the beginning, specifically those integers that equal  $p_2 e_1 + p_1 e_2 \pmod{e_1 e_2}$  for  $p_2 e_1 + p_1 e_2 > e_1 e_2$ . We denote these missing integers by  $\nu_i$ . Apart from these terms, the first sum in the second line of (33) will immediately give a single Hurwitz  $\zeta$ -function,

$$\begin{aligned} \zeta_2(s, a | \mathbf{d}) &= \frac{c^{2-s}}{g} \left( \zeta_R(s-1, b) - \sum_i \frac{1}{(b + \nu_i)^{s-1}} \right) \\ &+ \left( \frac{c}{g} \right)^s \sum_{\mathbf{P}} (1 - w_b) \zeta_R(s, w_b). \end{aligned} \tag{34}$$

Equation (34) is a convenient form for numerical evaluation. It provides an explicit analytical continuation of this *integral* Barnes  $\zeta$ -function.

For the derivative at  $s = 0$  we find, after inserting the known values of the Hurwitz  $\zeta$ -function and its derivative,

$$\begin{aligned} \zeta_2'(0, a | \mathbf{d}) &= \frac{c^2}{g} \left( \zeta_R'(-1, b) + \sum_i (\nu_i + b) \ln(\nu_i + b) \right) \\ &- \frac{c^2}{g} \left( \zeta_R(-1, b) - \sum_i (b + \nu_i) \right) \ln c \\ &+ \sum_{\mathbf{P}} (1 - w_b) \left( \ln(\Gamma(w_b)/\sqrt{(2\pi)}) - (1/2 - w_b) \ln(g/c) \right). \end{aligned} \tag{35}$$

Letting  $a$  tend to zero in (35) and comparing with the definition of  $\ln \varrho$ , (26), one finds that

$$\begin{aligned} \ln \varrho_2(\mathbf{d}) &= -\frac{c^2}{g} \left( \zeta_R'(-1) + \sum_i \nu_i \ln \nu_i \right) - \frac{c^2}{g} \left( \frac{1}{12} + \sum_i \nu_i \right) \ln c \\ &- \sum_{\mathbf{P}}' (1 - w_0) \left( \ln(\Gamma(w_0)/\sqrt{(2\pi)}) \right. \\ &\left. - \left( \frac{1}{2} - w_0 \right) \ln(g/c) \right) - \frac{1}{2} \ln(g/2\pi c), \end{aligned} \tag{36}$$

where  $w_0 = p_1/e_1 + p_2/e_2$  and the dash means that the term  $p_1 = p_2 = 0$  is to be omitted from the sum.

Barnes gives a formula in terms of the multiple  $\Gamma$ -function,

$$\zeta_r'(0, a | \mathbf{d}) = \ln \left( \frac{\Gamma_r(a)}{\varrho_r(\mathbf{d})} \right). \tag{37}$$

Formal expressions for the functional determinants are thus

$$e^{-\zeta_N'(0)} = e^{1/4g^2} \frac{\varrho_2^2(\mathbf{d})}{\Gamma_2(1)} \tag{38}$$

and

$$e^{-\zeta'_D(0)} = e^{1/4g^2} \frac{\varrho_2^2(\mathbf{d})}{\Gamma_2(d_0)\Gamma_2(d_0 + 1)}. \tag{39}$$

Our results, (34) and (35), (36), can be thought of as computational formulae for these functions in terms of simpler ones.

It is not necessary to rearrange the summation as in (33). We have done so in order to extract the term  $\zeta'_R(s - 1, b)$ . If the summation is left as in the first line of (33), it can immediately be turned into a sum of Hurwitz  $\zeta$ -functions,

$$\zeta_2(s, a|\mathbf{d}) = \left(\frac{c}{g}\right)^s \sum_{\mathbf{P}} (\zeta'_R(s - 1, w_b) + (1 - w_b)\zeta'_R(s, w_b)). \tag{40}$$

Then we have the alternative form

$$\begin{aligned} \zeta'_2(0, a|\mathbf{d}) &= \ln(c/g) \sum_{\mathbf{P}} (\zeta'_R(-1, w_b) + (1 - w_b)\zeta'_R(0, w_b)) \\ &\quad + \sum_{\mathbf{P}} (\zeta'_R(-1, w_b) + (1 - w_b)\zeta'_R(0, w_b)) \\ &= \frac{1}{12} (6a^2 - 6a(d_0 + 1) + (d_0 + 1)^2 + g) \ln(c/g) \\ &\quad + \sum_{\mathbf{P}} (\zeta'_R(-1, w_b) + (1 - w_b) \ln(\Gamma(w_b)/\sqrt{2\pi})). \end{aligned} \tag{41}$$

In this way we do not need to find the missing integers (nor even the common factor  $c$ ) but the price is the multiple evaluation of  $\zeta'_R(-1, w_b)$  by a numerical procedure. There is no difficulty in this but (34) is faster and more accurate. Equation (41) constitutes a useful check.

### 5. The Point Groups

A limited test of our formulae is provided by the dihedral case,  $\Gamma = [q]$  in Coxeter's notation [7, 8]. (Schönflies would write  $C_{qv}$  and it is  $\mathbf{C}_q[\mathbf{D}_q]$  in Polya and Meyer [9, 10]. Table 2 in [11] has a complete list of equivalents.) The degrees are  $d_1 = q$ ,  $d_2 = 1$ , so  $c = 1$ ,  $g = q$  and  $d_0 = q$ . There are no missing integers  $\nu_i$  and, furthermore,  $p_2 = 0$ . The fundamental domain is the lune, or digon,  $(qq1)$ .

For  $q = 1$  there is a single, equatorial reflection plane, the fundamental domain being a hemisphere, (111) (a spherical triangle with every angle equal to  $\pi$ ). An alternative notation for this domain is  $\mathbf{A}_1$ , [7]. In this extreme case,  $p_1$  is also zero and the expressions rapidly collapse to

$$\ln \varrho_2(1, 1) = -\zeta'_R(-1) + \ln \sqrt{2\pi}, \quad \zeta'_2(0, 1 | 1, 1) = \zeta'_R(-1)$$

and

$$\zeta'_2(0, 2 | 1, 1) = \zeta'_R(-1) + \ln \sqrt{2\pi}.$$

Thus, on the hemisphere, from (29) and (30),

$$\zeta'_N(0) = 2\zeta'_R(-1) - \ln \sqrt{2\pi} - \frac{1}{4}, \quad \zeta'_D(0) = 2\zeta'_R(-1) + \ln \sqrt{2\pi} - \frac{1}{4}, \tag{42}$$

which agree with the results exhibited by Weisberger [12].



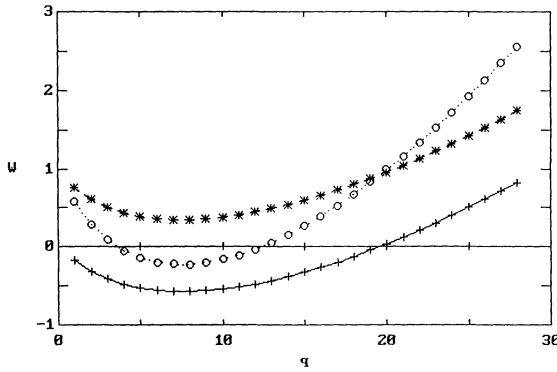


Fig. 1. Effective action,  $W$ , for  $[q]$ . + Dirichlet; \* Neumann; o Rotation

The sum of the Neumann and Dirichlet expressions should reduce to the full-sphere result derived by, e.g. Hortaçsu, Rothe and Schroer [13] and later by Weisberger [14]. We find

$$\zeta'_{S^2}(0) = 4\zeta'_R(-1) - \frac{1}{2} \approx -1.161684575,$$

agreeing with these earlier calculations. There are many discussions on spheres bounded equatorially by spheres.

We give the explicit formulae for the next value of  $q$ ,  $q = 2$ , corresponding to a quartersphere,

$$\zeta'_N(0) = \zeta'_R(-1) - \ln \sqrt{2\pi} - \frac{1}{8}, \quad \zeta'_D(0) = \zeta'_R(-1) + \ln \sqrt{2\pi} - \frac{1}{8}. \quad (43)$$

Adding these expressions gives half the full-sphere value.

The results for higher values of  $q$  are shown in Fig. 1, where we plot the effective action  $W = -\zeta'(0)/2$ . It is shown in the appendix that

$$\zeta'_N(0) - \zeta'_D(0) = -\ln(2\pi) \quad (44)$$

for all  $[q]$ , as born out by the numbers.

For completeness we record the value of  $\zeta_D(0)$  obtained from (16), relevant for the conformal anomaly,

$$\zeta_D(0) = \frac{1}{12q} (1 + q^2). \quad (45)$$

We turn now to the extended dihedral group,  $[q, 2]$ , or order  $4q$ , obtained from  $[q]$  by adding a perpendicular reflection. It is the complete symmetry group of the dihedron. (In [9, 10] this group is  $\mathbf{D}_{q^2}$  ( $q$  even) and  $\mathbf{D}_1[\mathbf{D}_{2q}]$  ( $q$  odd). The Schönflies equivalent is  $D_{q^2}$ .)

If  $q$  is odd,  $c = 1$ ,  $d_1 = q$ ,  $d_2 = 2$ ,  $d_0 = q + 1$  and  $g = 2q$ , while, if  $q$  is even,  $c = 2$ ,  $e_1 = q/2$  and  $e_2 = 1$ . For odd  $q$ , the missing integers are  $1, 3, \dots, q - 2$ . There are no missing integers if  $q$  is even.

The fundamental domain is the spherical triangle  $(22q)$ . When  $q = 1$  this domain is the quartersphere lune and the results coincide with (43). The group isomorphisms are  $[1, 2] \cong [2] \cong [1] \times [1]$  (or  $\mathbf{C}_2[\mathbf{D}_2] \cong \mathbf{D}_1[\mathbf{D}_2]$  since  $\mathbf{C}_2 \cong \mathbf{D}_1$ ).

Generally one has  $[q, 2] \cong [q] \times [1]$ , in particular,  $[2, 2] \cong [1] \times [1] \times [1]$  which corresponds to three perpendicular reflections with the eightsphere,  $(222)$ , as fundamental domain.

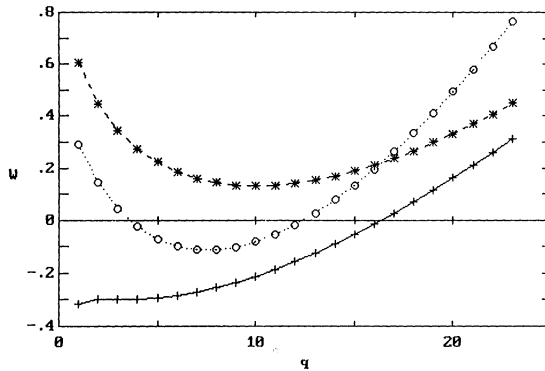


Fig. 2. Effective action,  $W$ , for  $[q, 2]$ . + Dirichlet; \* Neumann; o Rotation

The hemisphere, quartersphere and eightsphere are the intersections of  $S^2$  with  $(\mathbb{R}^+ \times \mathbb{R}^2)$ ,  $(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R})$  and  $(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$ , respectively. The positive real axis,  $\mathbb{R}^+$ , is the positive root space of the  $SU(2)$  algebra,  $A_1$  (cf. [6]). Figure 2 displays values of  $W$  for bigger orders.

The rotation part of  $[q, 2]$  is the complete symmetry group of the regular  $q$ -gon,  $\{q\}$ , and is the dihedral group in its guise as a group of rotations. Coxeter denotes it by  $[q, 2]^2$  and Polyá and Meyer by  $D_q$ . As stated, its structure is  $[q, 2]^2 \cong \mathcal{D}_q$ . When  $q$  is odd there is the curious isomorphism  $[2q] \cong [2, q]$ .

It is only a matter of substitution to work out the values of (29) and (30) for the other reflection groups which are the complete symmetry groups of the spherical tessellations  $\{3, 3\}$ ,  $\{3, 4\}$ , and  $\{3, 5\}$ . We find  $-\zeta'(0)/2$  for (Dirichlet, Neumann)-conditions to be  $(0.45603, -0.34216)$  for  $T_d = [3, 3]$ ,  $(0.2508, 0.001915)$  for  $O_h = [3, 4]$  and  $(-0.10538, 0.45014)$  for  $I_h = [3, 5]$ .

The fundamental domain of  $[p, q]$  is the spherical triangle  $(pqr)$ . The rotational part of  $[p, q]$ , i.e.  $[p, q]^+$ , is often denoted by  $(p, q, r)$ .

### 7. The Cesàro-Fedorov Formula

It is interesting to check the formula (16) by remembering that  $\zeta(0)$  is a local object related to the constant term in the short-time expansion of the heat-kernel. The general formula for a two-dimensional domain,  $\mathcal{M}$ , with boundary  $\partial\mathcal{M} = \bigcup \partial\mathcal{M}_i$  is

$$\zeta(0) = \frac{1}{24\pi} \int_{\mathcal{M}} R dA + \frac{1}{12} \sum_i \int_{\partial\mathcal{M}_i} \kappa(l) dl + \frac{1}{24\pi} \sum_{\alpha} \frac{\pi^2 - \alpha^2}{\alpha}, \tag{46}$$

where the  $\alpha$  sum runs over all inward facing angles at the corners of  $\partial\mathcal{M}$ .

In the present case  $R = 2$  and the extrinsic curvature,  $\kappa$ , vanishes since the boundaries of the fundamental domains are geodesic. Therefore

$$\zeta(0) = \frac{1}{24} \left( \frac{2}{g} + p + q + r - 1 \right), \tag{47}$$

where we have used the standard formula for the area of a spherical triangle. This agrees with (16) if the formula

$$2d_0(d_0 - 1) = g(p + q + r - 3) \tag{48}$$

is taken into account. In fact our derivation can be thought of as an analytical proof of this relation which is a slight generalisation of Eq. 4.51 in [7]. (Coxeter has  $r = 2$  and  $g = 2N_1$ .)

Coxeter indicates a purely geometric proof and points out that (48) is equivalent to a formula discovered numerologically by Cesàro [15] and is a special case of an earlier result of Fedorov [see 16]. Equation (48) is virtually identical to the equation on p. 177 of [15] with the correspondances  $X = d_0$ ,  $n = r$ ,  $p = p$ , and  $q = q$ . An extension to higher dimensions is possible using the generalisation of (46) that includes the results of Fedosov on polyhedral domains, [17].

### 7. Scaling and Limits

The results given so far are for a unit sphere. For radius  $R$ , simple scaling gives the relation

$$\zeta'(0; R) = \zeta'(0) + 2 \ln R \zeta(0), \tag{49}$$

where  $\zeta'(0) = \zeta'(0; 1)$  and  $\zeta(0) = \zeta(0; 1) = \zeta(0; R)$ .

The effective action should incorporate an arbitrary scaling length,  $L$ , by

$$W_L = -\frac{1}{2} \zeta'(0; R) + \ln L \zeta(0) = -\frac{1}{2} \zeta'(0) + \ln(L/R) \zeta(0).$$

The figures show just the first term.

Consider the dihedral case [6] and let  $q$  and  $R$  tend to infinity in such a way that the equatorial width of the fundamental domain ( $qq1$ ) remains fixed at  $\beta \equiv \pi R/q$ . From (49) and (45), whence  $\zeta(0) \rightarrow q/12$ , we have

$$\zeta'(0; R) \rightarrow \lim_{q \rightarrow \infty} \zeta'(0) + \frac{q}{6} \ln \left( \frac{\beta q}{\pi} \right). \tag{50}$$

The area of ( $qq1$ ) is  $A_q = 2\beta^2 q/\pi$  and requiring the density,  $\zeta'(0; R)/A_q$ , to remain finite as  $q \rightarrow \infty$  entails the leading behaviour

$$\zeta'(0) \rightarrow -\frac{q}{6} \ln q + O(q). \tag{51}$$

Numerically we find

$$\zeta'(0) \rightarrow -\frac{q}{6} \ln q + 0.497509q \approx -\frac{q}{6} \ln(q/19.79) \tag{52}$$

so that the density becomes

$$\frac{\zeta'(0; R)}{A_q} \rightarrow \frac{\pi}{12\beta^2} \ln(6.299\beta). \tag{53}$$

Geometrically, it might be imagined that in the limit  $R = \infty$ , since the sphere becomes flat, the rescaled lune,  $(\infty\infty 1)$ , would be an infinite strip of width  $\beta$ . Defining the strip coordinates  $x = R\phi$  and  $y = R(\pi/2 - \theta)$ , the spherical Laplacian does become the usual Cartesian one as  $R \rightarrow \infty$ . However, the influence of the infinitely sharp corners at the poles persists, even though they are infinitely distant, producing an anomaly density of  $\pi/12\beta^2$ . On the rectangular strip, infinite or not, the integrated anomaly equals 1/2 and so the density vanishes in the infinite case.

### 8. The Three-Sphere

The expressions for the three-sphere are (31) and (32). Then we require,

$$\zeta_3(s, a|\mathbf{d}) = \sum_{\mathbf{m}} \frac{1}{(a + m_1d_1 + m_2d_2 + m_3d_3)^s}.$$

We will not attempt to extract a single  $\zeta$ -function as we did previously but will just reduce the sum to a finite one over Hurwitz  $\zeta$ -functions in a not very symmetrical nor economic fashion.

The residue classes

$$m_2 = d_1n_2 + p_1, \quad m_1 = d_2n_1 + p_2$$

are introduced so that the denominator function reads  $(a + d_1d_2(n_1 + n_2) + p_2d_1 + p_1d_2 + m_3d_3)$ . The sums over  $n_1$  and  $n_2$  can be transformed by defining  $n = n_1 + n_2$  and doing the sum over  $n_1 - n_2$  to yield the intermediate form

$$\zeta_3(s, a|\mathbf{d}) = \sum_{p_1, p_2} \sum_{n, m_3=0}^{\infty} \frac{1 + n}{(a + d_1d_2n + p_2d_1 + p_1d_2 + m_3d_3)^s}.$$

The further residue classes

$$m_3 = d_1d_2n_3 + p_3, \quad n = d_3n_4 + p_4$$

are introduced and the sum and difference defined by

$$n_+ = n_4 + n_3, \quad n_- = n_4 - n_3.$$

The denominator is independent of  $n_-$  while the numerator equals  $1 + d_3(n_+ + n_-)2 + p_4$ . Since the range of  $n_-$  is symmetrical about zero (from  $-n_+$  to  $n_+$  in steps of 2) the  $n_-$  term gives nothing and there is a factor of  $(1 + n_+)$  multiplying the rest. The sum may therefore be written

$$\zeta_3(s, a|\mathbf{d}) = \sum_{\mathbf{p}, n} \frac{(1 + n)(1 + d_3n/2 + p_4)}{(f + gn)^s},$$

where  $\mathbf{p} = (p_1, p_2, p_3, p_4)$ ,  $f = a + d_1d_2p_4 + p_2d_1 + p_1d_2 + p_3d_3$ ,  $g = d_1d_2d_3$  and we have set  $n = n_+$  for notational simplicity.

The numerator is reorganised to

$$(1 + n)(1 + d_3n/2 + p_4) = \frac{d_3}{2g^2} (F + G(f + gn) + (f + gn)^2),$$

where

$$F = (A - g + d_1d_2p_4)(A - d_1d_2p_4 - 2d_1d_2), \quad G = g + 2d_1d_2 - 2A$$

with  $A$  being the combination  $A = a + d_1p_2 + d_2p_1 + d_3p_3$ .

Thus, finally, we arrive at a finite sum of Hurwitz  $\zeta$ -functions,

$$\begin{aligned} \zeta_3(s, a|\mathbf{d}) = \frac{d_3}{2g^2} \sum_{\mathbf{p}} \left[ \frac{F}{g^s} \zeta_R\left(s, \frac{f}{g}\right) + \frac{G}{g^{s-1}} \zeta_R\left(s - 1, \frac{f}{g}\right) \right. \\ \left. + \frac{1}{g^{s-2}} \zeta_R\left(s - 2, \frac{f}{g}\right) \right], \end{aligned} \tag{54}$$

which constitutes a possible, but inefficient, continuation of the Barnes  $\zeta$ -function.

### 9. The Honeycomb Groups

The three-dimensional analogues of the polyhedral tessellations,  $\{p, q\}$ , of the two-sphere are the spherical honeycombs  $\{p, q, r\}$ , [7, 8, 18]. The reflection groups  $[p, q, r]$  are their complete symmetry groups, the fundamental domains being subspaces of the honeycomb cells. A numerical calculation using (54) and (32) produces the following typical results for the Dirichlet effective actions. For  $[3, 3, 3]$ ,  $W \approx 44.4$  and for  $[3, 3, 4]$ ,  $W \approx -427.25$ .

### 10. Conclusion

The results of this paper are strictly technical. We have achieved our aim of presenting calculable formulae for the functional determinants of minimally coupled scalar fields on the fundamental domains of finite reflection groups. The problem has devolved upon an evaluation of the derivative of the Barnes  $\zeta$ -function.

We could also extend our previous results on the vacuum energies [1] to minimal coupling using the expressions for  $\zeta(-n)$ , (12), and  $\zeta'(-n)$ . This straightforward exercise will not be done here.

The Cesàro-Fedorov formula for the number of symmetry planes of a (regular) solid proved in Sect. 6 is one of a number of similar relations in higher dimensions derivable from expressions for the coefficients in the short-time expansion of the heat-kernel. The details will be presented elsewhere.

The conformal transformations taking a fundamental domain into the upper half-plane are known, and so the results here described should also be obtainable using standard conformal techniques. This will be recounted at another time.

### Appendix

In this appendix we first work out an expression for the derivative of the  $\zeta$ -function (8) at negative integers,  $\zeta'(-n)$ . For brevity we do not display the dependence of the Barnes  $\zeta$ -function on the  $\mathbf{d}$ .

Differentiation of (10) first of all leads to

$$\begin{aligned} \zeta'(-n) = & \sum_{r=0}^n (-\alpha^2)^r \binom{n}{r} \left[ 2\zeta'_d(2r - 2n, a) - \zeta_d(2r - 2n, a) \sum_{k=n-r+1}^n \frac{1}{k} \right] \\ & + (-1)^n \sum_{r=n+1}^{n+u} \alpha^{2r} \frac{n!(r - n - 1)!}{r!} \left( R_{2r-2n} + \frac{1}{2} N_{2r-2n} \sum_{k=n+1}^{r-n-1} \frac{1}{k} \right) \\ & + (-1)^n \sum_{r=u+1+n}^{\infty} \alpha^{2r} \frac{n!(r - n - 1)!}{r!} \zeta_d(2r - 2n, a). \end{aligned} \tag{55}$$

We substitute the integral form of the Barnes  $\zeta$ -function into the last term and find it as the  $s \rightarrow -2n$  limit of  $2^n(-1)^n n!$  times

$$2 \int_0^{\infty} \exp(-\alpha\tau) \left( \cosh \alpha\tau - \sum_{r=0}^{n+u} \frac{(\alpha\tau)^{2r}}{(2r)!} \right) \frac{\tau^{s-1} d\tau}{\prod_i (1 - \exp(-d_i\tau))} \tag{56}$$

which equals

$$\Gamma(s)(\zeta_d(s, a - \alpha) + \zeta_d(s, a + \alpha)) - 2 \sum_{r=0}^{u+n} \frac{\alpha^{2r}}{(2r)!} \Gamma(s + 2r)\zeta_d(s + 2r, a). \tag{57}$$

The pole cancellation gives the condition

$$\begin{aligned} &\zeta_d(-2n, a - \alpha) + \zeta_d(-2n, a + \alpha) - 2\zeta_d(-2n, a) \\ &= 2 \sum_{r=1}^n \alpha^{2r} \binom{2n}{2r} \zeta_d(2r - 2n) + 2 \sum_{r=n+1}^{n+u} \alpha^{2r} \frac{(2r - 2n - 1)!}{(2r)!} N_{2r-2n}. \end{aligned} \tag{58}$$

Extracting the finite remainder yields, after using (58),

$$\begin{aligned} &\frac{1}{(2n)!} (\zeta'_d(-2n, a - \alpha) + \zeta'_d(-2n, a + \alpha) - 2\zeta'_d(-2n, a)) \\ &+ \frac{2}{(2n)!} \sum_{r=1}^n \alpha^{2r} \binom{2n}{2r} [(\psi(1 + 2n - 2r) - \psi(1 + 2n)) \\ &\times \zeta_d(2r - 2n, a) - \zeta'_d(2r - 2n, a)] \\ &- 2 \sum_{r=n+1}^{n+u} \alpha^{2r} \frac{(2r - 2n - 1)!}{(2r)!} \\ &\times [(\psi(2r - 2n) + \psi(1 + 2n))N_{2r-2n} + R_{2r-2n}]. \end{aligned} \tag{59}$$

Multiplied by  $2^n(-1)^n n!$ , (59) must be substituted into (55) to yield a calculable formula for  $\zeta'(-n)$ . Doing so reveals that the remainder terms  $R_{2r-2n}$  cancel but, apart from this, there are no other simplifications apparent and we leave the analysis at this point.

We next derive the result (44) starting from (38) and (39) whence

$$e^{-\zeta'_N(0)+\zeta'_D(0)} = \frac{\Gamma_2(d_0)\Gamma_2(d_0 + 1)}{\Gamma_2(1)}. \tag{60}$$

It is necessary to use some properties of the multiple  $\Gamma$ -function.

From (26) and (37) it is obvious that

$$\lim_{a \rightarrow 0} \Gamma_r(a) = \frac{1}{a}. \tag{61}$$

The other properties we need follow from the important recursion formula satisfied by the Barnes  $\zeta$ -function,

$$\zeta_r(s, a + d_i | \mathbf{d}) - \zeta_r(s, a | \mathbf{d}) = -\zeta_{r-1}(s, a | \mathbf{d}'), \tag{62}$$

where  $\mathbf{d}'$  stands for the set of degrees  $\mathbf{d}$  with the  $d_i$  element omitted.

If this equation is differentiated, it quickly results that, [2],

$$\frac{\Gamma_r(a)}{\Gamma_r(a + d_i)} = \frac{\Gamma_{r-1}(a)}{\varrho_{r-1}(\mathbf{d}')}. \tag{63}$$

Setting  $a$  equal to zero in (63) and using (61) it follows that

$$\Gamma_r(d_i) = \varrho_{r-1}(\mathbf{d}'). \tag{64}$$

For the group  $[q]$ , we recall that the degrees are  $\mathbf{d} = (q, 1)$ . Then, choosing  $d_i = d_1 = q$  and setting  $a = 1$ , we have from (63),

$$\frac{\Gamma_2(1)}{\Gamma_2(1+q)} = \frac{\Gamma_1(1)}{\varrho_1(1)},$$

which is clearly independent of  $q$  since the quantities on the right-hand side are calculated for the degree  $\mathbf{d}' = (1)$ . Further, from (64), it is likewise clear that  $\Gamma_2(q)$  is independent of  $q$ . Therefore the quantity in (60),

$$\frac{\Gamma_2(q)\Gamma_2(q+1)}{\Gamma_2(1)} = \frac{\varrho_1^2(1)}{\Gamma_1(1)},$$

is independent of  $q$ . The actual value is  $2\pi$ , agreeing with the particular cases (42) and (43).

Incidentally, from the general formulae (2), (3) and (62) it easily follows that the  $[q]$  conformal  $\zeta$ -functions are related by

$$\zeta_N^{(C)}(s) - \zeta_D^{(C)}(s) = \zeta_1(2s, 1/2 | 1) = \zeta_R(2s, 1/2), \tag{65}$$

so that, in particular,

$$\zeta_N^{(C)'}(0) - \zeta_D^{(C)'}(0) = -\ln 2$$

for all  $[q]$ .

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