Commun. Math. Phys. 162, 481–498 (1994)

Communications in
Mathematical
Physics

Self Duality of the Gauge Field Equations and the Cosmological Constant

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Received: 22 April 1993/in revised form: 9 July 1993

Abstract: This paper considers the Einstein equations coupled with the nonabelian gauge and Higgs fields. It is shown that, when cosmic string solutions are sought in the Einstein–Georgi–Glashow system and the Einstein–Weinberg–Salam system governing the gravitational-electromagnetic-weak interaction forces, the self duality conditions lead to positive values of the cosmological constant which can be expressed by some fundamental parameters in particle physics.

1. Introduction

In quantum field theory, phase transitions are described by a generalized order parameter, called the Higgs field, which is defined on spacetime and takes values in a range space. The spacetime symmetry, or the external symmetry, gives rise to Einstein's theory of general relativity or the theory of gravitation while the range space symmetry, or the internal symmetry, leads to the Yang–Mills–Higgs gauge theory or the field theory of electromagnetic and nuclear (strong and weak) forces. Therefore the coupling of the Einstein and the Yang–Mills–Higgs theories should naturally lead to a unified theoretical framework to house gravitational, electromagnetic, and nuclear forces. In fact, recent developments in cosmology and particle physics have already witnessed an exciting interaction of these two traditionally different areas and a lot of progress has been made in understanding some important issues. For example, it has been recognized that, due to the spontaneously broken symmetry, the coupled Einstein–Yang–Mills–Higgs equations may provide a class of interesting solutions called topological defects. The symmetry-breaking scales are realized by the gauge groups corresponding to various stages of the phase transitions after the Big Bang. These stable defects may be domain walls, monopoles, or strings but the former two types of solutions are disastrous for cosmological models and only the string solutions can lead to

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interesting cosmological consequences. See Vilenkin [V] for a detailed review on this subject. Strings, or cosmic strings, are cylindrically symmetric solutions of the Einstein–Yang–Mills–Higgs system that are regarded as giving rise to seeds for accretion of matter to form galaxies. Even in the simplest case where the gauge group $G=U(1)$, the system is very difficult to solve and there are only heuristic arguments or numerical simulations in the general setting of the problem in order to make any progress. However, the investigations of Linet [L1, L2] and Comtet–Gibbons [CG] showed that, when the cosmological term is absent, there exists a critical phase so that the full $U(1)$ system allows a reduction into a self-dual system called the Einstein–Bogomol'nyi equations. Using this system, we have constructed a continuous family of finite-energy cosmic string solutions [SY2, CHMcY]. Actually, the existence of such solutions implies the vanishing of the cosmological constant Λ (see Sect. 2) and this simple but important observation motivates the work of the present paper.

The constant Λ, introduced by Einstein himself, has a rich history in theoretical physics. According to modern ideas, Λ should have a significant value in the early universe when the phase transitions were taking place in the beginning stages and maintained at rather high energy levels. At that time, the symmetry-breaking scales were realized by larger gauge groups. Then the result $\Lambda=0$ by assuming the presence of self-dual $U(1)$ strings simply indicates that the strings obtained belong to a later stage of the universe. Thus an interesting question arises:

For what gauge group G, the Einstein–Yang–Mills–Higgs equations permit self-dual cosmic string solutions which yield a positive Λ?

This paper is devoted to an answer of this question. We shall show that for the important Einstein–Georgi–Glashow equations where $G=SO(3)$ or $SU(2)$ and for the Einstein–Weinberg–Salam equations where $G=SU(2)\times U(1)$, the existence of self-dual cosmic string solutions lead to positive values of Λ and these values can be expressed by some fundamental parameters in the Georgi–Glashow or Weinberg–Salam models. This work provides another evidence of the close relationship of cosmology and particle physics. Our approach comes from a combination of the elegant work on self-dual electroweak vortices by Ambjorn–Olesen [AO1, AO2, AO3] and the reduction of the Einstein equations for string solutions in the work of Linet [L1, L2] and Comtet–Gibbons [CG].

Note that the construction of a regular stationary solution of the Einstein type equations has always been an interesting question in mathematical physics. There are the Schwarzschild blackhole solution of the vacuum Einstein equations and the Reissner–Nordström solution of the Einstein–Maxwell equations which are singular somewhere. Recently, Smoller–Wasserman–Yau–McLeod showed the existence of a regular stationary solution of the Einstein–Yang–Mills equations when $G=SU(2)$. In these studies, the matter Higgs field is absent and solutions are spherically symmetric. On the other hand, the self-dual $SU(2)$ and $SU(2)\times U(1)$ equations obtained here (as well as the $U(1)$ equations derived in [CG]) provide us new opportunities to get regular stationary solutions of the full Einstein–Yang–Mills–Higgs systems with cylindrical symmetry.

The rest of the paper is organized as follows. In Sect. 2 we recall the derivation of Comtet–Gibbons [CG] and emphasize that in this $U(1)$ case the existence of self-dual strings will inevitably lead to vanishing cosmological constant Λ. In Sect. 3 we work in a conformally flat surface and derive the Einstein–Bogomol'nyi equations for the Georgi–Glashow theory. It will be seen that Λ now takes a positive value if the self-dual strings are present. In Sect. 4 we obtain self duality for the full Einstein–Weinberg–Salam system and express Λ in terms of electroweak coupling constants and the Weinberg angle. In Sect. 5 we prove the existence of self-dual string solutions in the Einstein–Georgi–Glashow system. Section 6 is a brief summary.

2. Self-Dual $U(1)$ Strings and Vanishing Λ

Let $g_{\mu\nu}$ be the metric tensor of a four-dimensional pseudo-Riemannian manifold with signature $(-+++)$, $R_{\mu\nu}$ the Ricci tensor, and R the scalar curvature. Then the Einstein tensor takes the form

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\ .$$

With a suitable normalization of the universal gravitational constant, the Einstein equations in the presence of the cosmological term are written [W]

$$G_{\mu\nu}-\Lambda g_{\mu\nu}=-T_{\mu\nu}\ , \tag{2.1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the matter-gauge sector to be introduced in the Einstein–Yang–Mills–Higgs coupling.

In this section, we discuss the standard $U(1)$ Higgs theory in the Bogomol'nyi critical phase. The Lagrangian reads

$$\mathscr{L}=\frac{1}{4}g^{\mu\mu'}g^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'}+\frac{1}{2}g^{\mu\nu}[D_\mu\phi][D_\nu\phi]^\dagger+\frac{1}{8}[|\phi|^2-1]^2\ , \tag{2.2}$$

where ϕ is a complex scalar matter field, $D_\mu\phi=\partial_\mu\phi-\mathrm{i}A_\mu\phi$ is the gauge-covariant derivative, A_μ is a real valued gauge vector field, and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the electromagnetic field. Thus a variation with respect to the Riemannian metric in the action $\int\mathscr{L}\sqrt{-g}\,\mathrm{d}x$ leads to the following expression of the energy-momentum tensor of the matter-gauge sector

$$T_{\mu\nu}=g^{\mu'\nu'}F_{\mu\mu'}F_{\nu\nu'}+\frac{1}{2}[D_\mu\phi(D_\nu\phi)^\dagger+(D_\mu\phi)^\dagger D_\nu\phi]-g_{\mu\nu}\mathscr{L}\ .$$

We now assume the string ansatz for the metric tensor so that

$$\begin{aligned}\mathrm{d}s^2&=g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\\&=-\mathrm{d}t^2+\mathrm{d}z^2+g_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k,\quad j,k=1,2\ ,\end{aligned} \tag{2.3}$$

where $\{g_{jk}\}$ is the metric of a two-dimensional Riemannian manifold M which can always be assumed to be locally conformally flat (the existence of isothermal coordinate chart on 2-surfaces), $x^0=t$, $x^3=z$, and for the gauge and matter fields A_μ, ϕ so that A_μ, ϕ depend only on the coordinates on M and $A_\mu=(0,0,A_1,A_2)$. Thus $T_{\mu\nu}$ verifies

$$T_{00}=\mathscr{L},\quad T_{33}=-\mathscr{L},\quad T_{03}=T_{0j}=T_{3j}=0\ ,$$

$$T_{jk}=g^{j'k'}F_{jj'}F_{kk'}+\frac{1}{2}[D_j\phi(D_k\phi)^\dagger+(D_j\phi)^\dagger D_k\phi]-g_{jk}\mathscr{L}\ ,$$

where

$$\mathscr{L}=\frac{1}{4}g^{jj'}g^{kk'}F_{jk}F_{j'k'}+\frac{1}{2}g^{jk}[D_j\phi][D_k\phi]^\dagger+\frac{1}{8}[|\phi|^2-1]^2$$

is the energy density of the matter-gauge sector.

On the other hand, due to the string assumption (2.3), the Einstein tensor is simplified to

$$-G_{00}=G_{33}=\frac{1}{2}R\,,$$

$$G_{\mu\nu}=0 \text{ for other values of } \mu, \nu\,,$$

where, and in the sequel, R is the scalar curvature of $(M, \{g_{jk}\})$. Consequently, the Einstein equations (2.1) become

$$\begin{aligned}&\frac{1}{2}R=\Lambda+\mathscr{L}\,,\\ &&x\in M\,.\\ &\Lambda g_{jk}=T_{jk},\quad j,k=1,2\,,\end{aligned}\tag{2.4}$$

To proceed further, we recall that the equations of motion of the matter-gauge sector defined by the Lagrangian (2.2),

$$\frac{1}{\sqrt{-g}}D_\mu[g^{\mu\nu}\sqrt{-g}(D_\nu\phi)]=\frac{1}{2}[|\phi|^2-1]\phi\,,$$

$$\frac{1}{\sqrt{-g}}\partial_\mu[g^{\mu'\nu}g^{\mu\nu'}\sqrt{-g}F_{\mu'\nu'}]=\frac{\mathrm{i}}{2}g^{\mu\nu}[\phi(D_\mu\phi)^\dagger-\phi^\dagger(D_\mu\phi)]\,,$$

are satisfied by the solutions of the self-dual system

$$\begin{aligned}&D_j\phi\pm\mathrm{i}\varepsilon_j^kD_k\phi=0\,,\\ &&x\in M\,,\\ &F_{jk}\pm\frac{1}{2}\varepsilon_{jk}[|\phi|^2-1]=0\,,\end{aligned}\tag{2.5}$$

where ε_{jk} is the skew-symmetric Levi–Civita tensor with $\varepsilon_{12}=\sqrt{-g}$, $\varepsilon_j^k=g^{kl}\varepsilon_{jl}$, and a solution $(\{g_{jk}\}, A_j, \phi)$ of (2.4)–(2.5) is called a self-dual cosmic string solution of the Einstein-matter-gauge system under consideration.

A lengthy calculation gives us

$$4g^{jk}T_{jk}=[\varepsilon^{jk}F_{jk}+(|\phi|^2-1)][\varepsilon^{j'k'}F_{j'k'}-(|\phi|^2-1)] \quad \text{in } M\,.$$

Therefore a solution of (2.5) must fulfill the condition $g^{jk}T_{jk}\equiv 0$ in M. Inserting this fact into the second equation in (2.4), we obtain $\Lambda=0$ as desired.

Thus the assumption of the presence of self-dual $U(1)$ strings (whose existence is proved in [SY2, CHMcY]) immediately leads to the vanishing of the cosmological constant.

In the next two sections, we study the relationship of self duality and the values of Λ in the Georgi–Glashow and Weinberg–Salam models. We shall work for simplicity in a framework in which the 2-surface M where the strings reside is conformally $\mathbb{R}^2$. The main reason for taking this non-intrinsic approach is that we will follow a standard procedure to obtain the complex scalar W-boson field from appropriate components of the nonabelian gauge vector field. Such a reduction requires a global coordinate system.

3. The Einstein–Georgi–Glashow Coupling

In this section we consider the Einstein theory coupled with the $SO(3)$ Georgi–Glashow model ignoring the Higgs field. There is a massive W-boson. The vortex condensation of this model (without gravity) was studied by Ambjorn–Olesen [AO1] and the existence of vortices was proved in Yang [Y]. See also Spruck–Yang [SY2].

Let $\{t_a\}_{a=1,2,3}$ be a set of generators of $SO(3)$ satisfying the commutation relation

$$[t_a, t_b] = \mathrm{i}\varepsilon_{abc}t_c, \quad a, b, c = 1, 2, 3 .$$

Then the $SO(3)$ gauge potential A_μ can be expressed in the matrix form

$$A_\mu = A_\mu^a t_a .$$

As in [AO1], introduce the complex W-vector boson by setting

$$W_\mu = \frac{1}{\sqrt{2}}[A_\mu^1 + \mathrm{i}A_\mu^2] .$$

Then the Lagrangian of the $SO(3)$ matter-gauge sector in the presence of a gravitational metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ under consideration is

$$\mathscr{L} = \frac{1}{4} g^{\mu\mu'} g^{\nu\nu'} F_{\mu\nu}^a F_{\mu'\nu'}^a + m_W^2 g^{\mu\nu} W_\mu^\dagger W_\nu ,$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}e[A_\mu, A_\nu] ,$$

$m_W > 0$ is the mass of the W-particle, and $-e$ is the electron charge.

Put $A_\mu^3 = f_\mu$ and

$$f_{\mu\nu} = \partial_\mu f_\nu - \partial_\nu f_\mu .$$

Thus $\mathscr{L}$ is reduced after a calculation to

$$\begin{aligned}\mathscr{L} = {} & \frac{1}{4} g^{\mu\mu'} g^{\nu\nu'} f_{\mu\nu} f_{\mu'\nu'} + \frac{1}{2} g^{\mu\mu'} g^{\nu\nu'} (D_\mu W_\nu - D_\nu W_\mu)(D_{\mu'} W_{\nu'} - D_{\nu'} W_{\mu'})^\dagger \\ & + m_W^2 g^{\mu\nu} W_\mu^\dagger W_\nu + \mathrm{i}e g^{\mu\mu'} g^{\nu\nu'} f_{\mu\nu} W_{\mu'}^\dagger W_{\nu'} \\ & - \frac{e^2}{2}[(g^{\mu\mu'} W_\mu^\dagger W_{\mu'}^\dagger) g^{\nu\nu'} W_\nu W_{\nu'} - (g^{\mu\mu'} W_\mu^\dagger W_{\mu'})^2] ,\end{aligned}$$

where $D_\mu = \partial_\mu - \mathrm{i}e f_\mu$. Varying the metric $\{g_{\mu\nu}\}$ in $\mathscr{L}$ leads to the following expression of the energy-momentum tensor:

$$\begin{aligned}T_{\mu\nu} = {} & g^{\mu'\nu'} f_{\mu\mu'} f_{\nu\nu'} + 2\mathrm{Re}\{g^{\mu'\nu'}(D_\mu W_{\mu'} - D_{\mu'} W_\mu)(D_\nu W_{\nu'} - D_{\nu'} W_\nu)^\dagger\} \\ & + m_W^2 [W_\mu W_\nu^\dagger + W_\nu W_\mu^\dagger] \\ & + \mathrm{i}e g^{\mu'\nu'} [f_{\mu'\mu}(W_{\nu'}^\dagger W_\nu - W_{\nu'} W_\nu^\dagger) + f_{\mu'\nu}(W_{\nu'}^\dagger W_\mu - W_{\nu'} W_\mu^\dagger)] \\ & + e^2[(g^{\mu'\nu'} W_{\mu'}^\dagger W_{\nu'})(W_\mu^\dagger W_\nu + W_\nu^\dagger W_\mu) \\ & - (W_\mu^\dagger W_\nu^\dagger)(g^{\mu'\nu'} W_{\mu'} W_{\nu'}) - (g^{\mu'\nu'} W_{\mu'}^\dagger W_{\nu'}^\dagger)(W_\mu W_\nu)] - g_{\mu\nu} L .\end{aligned} \tag{3.1}$$

Besides, the equations of motion of the matter-gauge Lagrangian $\mathscr{L}$ are

$$\frac{1}{\sqrt{-g}} D_\mu[g^{\mu\mu'} g^{\nu\nu'} \sqrt{-g}(D_{\mu'} W_{\nu'} - D_{\nu'} W_{\mu'})]$$

$$= -\mathrm{i}e g^{\mu\mu'} g^{\nu\nu'} f_{\mu'\nu'} W_\mu + m_W^2 g^{\mu\nu} W_\mu$$

$$-e^2[g^{\mu\nu} W_\mu^\dagger (g^{\mu'\nu'} W_{\mu'} W_{\nu'}) - (g^{\mu'\nu'} W_{\mu'} W_{\nu'}^\dagger) g^{\mu\nu} W_\mu] ,$$

$$\frac{1}{\sqrt{-g}} \partial_\mu [g^{\mu'\nu} g^{\mu\nu'} \sqrt{-g} f_{\mu'\nu'}] = \mathrm{i}e g^{\mu\nu} g^{\mu'\nu'} [D_\mu W_{\nu'} - D_{\nu'} W_\mu]^\dagger W_{\mu'}$$

$$-\mathrm{i}e g^{\mu\nu} g^{\mu'\nu'} [D_\mu W_{\nu'} - D_{\nu'} W_\mu] W_{\mu'}^\dagger$$

$$-\frac{\mathrm{i}e}{\sqrt{-g}} \partial_\mu [g^{\mu'\nu} g^{\mu\nu'} \sqrt{-g}(W_{\mu'}^\dagger W_{\nu'} - W_{\nu'}^\dagger W_{\mu'})] . \tag{3.2}$$

We are now at a position to derive the self-dual conditions. Assume that the string metric is defined on a conformally flat surface. Then (2.3) becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{e}^\eta [(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2] . \tag{3.3}$$

A symmetry consideration shows that it may be consistent to assume that

$$W_0 = W_3 = 0, \; f_0 = f_3 = 0 ,$$

W_j, f_j $(j=1, 2)$ depend only on x^k $(k=1, 2)$, and there is a complex scalar field W so that (see Ambjorn–Olesen [AO1])

$$W_1 = W, \quad W_2 = \mathrm{i}W .$$

Thus, in view of the expression (3.1), we have

$$T_{\mu\nu} = 0, \quad \mu \neq \nu .$$

Moreover,

$$T_{00} = -T_{33} = \mathscr{L} ,$$

where $\mathscr{L}$ can be written

$$\mathscr{L} = \frac{1}{2}\mathrm{e}^{-2\eta} f_{12}^2 + \mathrm{e}^{-2\eta} |D_1 W + \mathrm{i}D_2 W|^2 + 2m_W^2 \mathrm{e}^{-\eta} |W|^2$$

$$-2e\mathrm{e}^{-2\eta} f_{12} |W|^2 + 2e^2 \mathrm{e}^{-2\eta} |W|^4$$

$$= \mathrm{e}^{-2\eta} |D_1 W + \mathrm{i}D_2 W|^2 + \frac{1}{2}\mathrm{e}^{-2\eta}\left[f_{12} - \left(\frac{m_W^2}{e}\mathrm{e}^\eta + 2e|W|^2\right)\right]^2$$

$$-\frac{1}{2}\frac{m_W^4}{e^2} + \frac{m_W^2}{e}\mathrm{e}^{-\eta} f_{12} . \tag{3.4}$$

Besides, the other two nonvanishing components of $T_{\mu\nu}$ are

$$\begin{aligned}T_{11}&=T_{22}\\&=e^{-\eta}f_{12}^2+2e^{-\eta}|D_1W+iD_2W|^2+2m_W^2|W|^2\\&\quad-4ee^{-\eta}f_{12}|W|+4e^2e^{-\eta}|W|^4-e^{\eta}\mathscr{L}\\&=e^{-\eta}|D_1W+iD_2W|^2+\frac{1}{2}e^{-\eta}\left[f_{12}-\left(\frac{m_W^2}{e}e^{\eta}+2e|W|^2\right)\right]^2\\&\quad+\frac{m_W^2}{e}\left[f_{12}-\left(\frac{m_W^2}{e}e^{\eta}+2e|W|^2\right)\right]+\frac{1}{2}\frac{m_W^4}{e^2}e^{\eta}\,.\end{aligned}$$

The form of $\mathscr{L}$ suggests the following curved-space version of the self-dual Bogomol'nyi equations (see [AO1] in the flat case $\eta=0$)

$$\begin{aligned}D_1W+iD_2W&=0\,,\\f_{12}&=\frac{m_W^2}{e}e^{\eta}+2e|W|^2\,.\end{aligned}\tag{3.5}$$

It can be verified directly that (3.5) implies the equation of motion (3.2). Furthermore, since the full Einstein equations (2.1) are again reduced to (2.4), we see that the consistency in (2.4) requires that the cosmological constant take the unique value

$$\Lambda=\frac{m_W^4}{2e^2}\,.\tag{3.6}$$

Inserting (3.6) into (2.4) and using (3.5) in (3.4), we see that the Einstein system (2.3) is simplified into the single equation

$$\frac{1}{2}R=\frac{m_W^2}{e}e^{-\eta}f_{12}\,.\tag{3.7}$$

On the other hand, it is well-known that [A] the scalar curvature R has the expression

$$R=-e^{-\eta}\Delta\eta\,.$$

Therefore (3.7) becomes

$$\Delta\eta+\frac{2m_W^2}{e}f_{12}=0\,.\tag{3.8}$$

Thus we have derived the Einstein–Bogomol'nyi system composed of Eqs. (3.5) and (3.8). Any solution of (3.5) and (3.8) also satisfies the original Einstein–Georgi–Glashow system (2.1) (or (2.4)) and (3.2). A solution of (3.5) and (3.8) is called a self-dual cosmic string solution. We have shown that the presence of such strings requires the fulfillment of (3.6).

Let us finish this section by writing the coupled equations (3.5) and (3.8) as a second order elliptic partial differential equation.

It is well-known that [JT] the first equation in (3.5) says that the zero set of W is discrete and these zeros all have integral multiplicities. Let the zeros of W be denoted by $p_1, \ldots, p_N$ (a zero of multiplicity m is counted as m zeros). Then the substitution $u = \ln |W|^2$ reduces (3.5) into the form

$$\Delta u = -2m_W^2 e^{\eta} - 4e^2 e^u + 4\pi \sum_{n=1}^{N} \delta_{p_n} \quad \text{in} \quad \mathbb{R}^2 . \tag{3.9}$$

Furthermore, using (3.5) in (3.8), we obtain

$$\Delta \eta = -\frac{2m_W^4}{e^2} e^{\eta} - 4m_W^2 e^u . \tag{3.10}$$

Hence we have seen that the Einstein–Bogomol'nyi system (3.5) and (3.8) is equivalent to the coupled elliptic equations (3.9)–(3.10). A solution of (3.9)–(3.10) gives rise to an N-string solution of the Einstein–Georgi–Glashow system (2.3) and (3.2) with strings located at $p_1, \ldots, p_N$.

The special form of (3.10) allows a further simplification of the system. In fact, inserting (3.10) into (3.9), we get

$$\Delta u = \frac{e^2}{m_W^2} \Delta \eta + 4\pi \sum_{n=1}^{N} \delta_{p_n} .$$

Namely,

$$w = u - \frac{e^2}{m_W^2} \eta - 2 \sum_{n=1}^{N} \ln |x - p_n|$$

is a harmonic function in $\mathbb{R}^2$. For simplicity we assume $w \equiv 0$. As a consequence, the system (3.9)–(3.10) is reduced to the single equation

$$\Delta \eta = -\frac{2m_W^4}{e^2} e^{\eta} - 4m_W^2 \prod_{n=1}^{N} |x - p_n|^2 e^{\frac{e^2}{m_W^2}\eta} . \tag{3.11}$$

The existence of solutions of this interesting nonlinear equation is not difficult to establish when $p_1, \ldots, p_n$ coincide. See Sect. 5.

Remark 3.1. So far, it seems that most known stationary solutions of the Einstein type equations are radially symmetric. Equation (3.11) gives us a highly tractable form to obtain non-radially symmetric solutions. We intend to pursue this direction of study later. Note that the structure of (3.11) belongs to a class of 2-dimensional elliptic partial differential equations which are not yet well understood.

4. The Einstein–Weinberg–Salam System

Let t_a ($a = 1, 2, 3$) be the generators of $SU(2)$ introduced in Sect. 3 (note that $SU(2)$ and $SO(3)$ have the same Lie algebra) and set

$$t_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Then the gauge group $SU(2) \times U(1)$ in the Weinberg–Salam electroweak theory transforms a complex doublet ϕ according to the rules

$$\phi \mapsto \exp(-\mathrm{i}\omega_a t_a)\phi, \quad \omega_a \in \mathbb{R}, \quad a = 1, 2, 3,$$

$$\phi \mapsto \exp(-\mathrm{i}\xi t_0)\phi, \quad \xi \in \mathbb{R} .$$

The $SU(2)$ and $U(1)$ gauge fields are denoted by $A_\mu = A_\mu^a t_a$ (or $A_\mu = (A_\mu^a)$) and B_μ respectively, where both A_μ^a and B_μ are real 4-vectors. Besides, the field strength tensors and the $SU(2)\times U(1)$ gauge-covariant derivative are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_1[A_\mu, A_\nu] ,$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu ,$$

$$D_\mu \phi = \partial_\mu \phi + ig_1 A_\mu^a t_a \phi + ig_2 B_\mu t_0 \phi ,$$

where $g_1, g_2 > 0$ are coupling constants.

In the presence of the gravitational metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, the Lagrangian density of the bosonic sector of the Weinberg–Salam theory is

$$\mathscr{L} = \frac{1}{4} g^{\mu\mu'} g^{\nu\nu'} F^a_{\mu\nu} F^a_{\mu'\nu'} + \frac{1}{4} g^{\mu\mu'} g^{\nu\nu'} G_{\mu\nu} G_{\mu'\nu'} + g^{\mu\nu} (D_\mu \phi)(D_\nu \phi)^\dagger + \lambda(\varphi_0^2 - \phi^\dagger \phi)^2 ,$$

where $\lambda > 0$ is a constant and $\varphi_0 > 0$ is the vacuum expectation value of the Higgs field ϕ.

We now go to the standard unitary gauge. We introduce the new vector fields P_μ and Z_μ as a rotation of the pair A_μ^3 and B_μ:

$$P_\mu = B_\mu \cos\theta + A_\mu^3 \sin\theta ,$$

$$Z_\mu = -B_\mu \sin\theta + A_\mu^3 \cos\theta .$$

Thus D_μ becomes

$$D_\mu = \partial_\mu + ig_1(A_\mu^1 t_1 + A_\mu^2 t_2) + iP_\mu(g_1 \sin\theta t_3 + g_2 \cos\theta t_0)$$
$$+ iZ_\mu(g_1 \cos\theta t_3 - g_2 \sin\theta t_0) .$$

As usual, if the coupling constants $g_1, g_2 > 0$ are so chosen that the electron charge satisfies

$$e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} ,$$

then there is an angle θ: $0 < \theta < \pi/2$ (the Weinberg angle) so that

$$e = g_1 \sin\theta = g_2 \cos\theta .$$

In this situation, the operator D_μ has the expression

$$D_\mu = \partial_\mu + ig_1(A_\mu^1 t_1 + A_\mu^2 t_2) + iP_\mu eQ + iZ_\mu eQ' .$$

Here $eQ = e(t_3 + t_0)$ and $Q' = \cot\theta t_3 - \tan\theta t_0$ are charge and neutral charge operators, respectively.

Assume now

$$\phi = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} ,$$

where φ is a real scalar field. Then

$$D_\mu \phi = \begin{pmatrix} \frac{ig_1}{2}[A_\mu^1 - iA_\mu^2]\varphi \\ \partial_\mu \varphi - \frac{ig_1}{2\cos\theta} Z_\mu \varphi \end{pmatrix} .$$

As in Sect. 3 (see Ambjorn–Olesen [AO3]), define the complex vector field

$$W_\mu=\frac{1}{\sqrt{2}}(A_\mu^1+\mathrm{i}A_\mu^2)$$

and set $\mathscr{D}=\partial_\mu-\mathrm{i}g_1A_\mu^3$. With the notation $P_{\mu\nu}=\partial_\mu P_\nu-\partial_\nu P_\mu$ and $Z_{\mu\nu}=\partial_\mu Z_\nu-\partial_\nu Z_\mu$, the Lagrangian density takes the form

$$\begin{aligned}\mathscr{L}=&\frac{1}{4}g^{\mu\mu'}g^{\nu\nu'}P_{\mu\nu}P_{\mu'\nu'}+\frac{1}{4}g^{\mu\mu'}g^{\nu\nu'}Z_{\mu\nu}Z_{\mu'\nu'}\\&+\frac{1}{2}g^{\mu\mu'}g^{\nu\nu'}(\mathscr{D}_\mu W_\nu-\mathscr{D}_\nu W_\mu)(\mathscr{D}_{\mu'}W_{\nu'}-\mathscr{D}_{\nu'}W_{\mu'})^\dagger\\&+g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi+\frac{g_1^2}{4\cos^2\theta}\varphi^2g^{\mu\nu}Z_\mu Z_\nu+\frac{g_1^2}{2}\varphi^2g^{\mu\nu}W_\mu^\dagger W_\nu\\&+\frac{1}{2}g_1^2[(g^{\mu\nu}W_\mu^\dagger W_\nu)^2-(g^{\mu\mu'}W_\mu W_{\mu'})(g^{\nu\nu'}W_\nu W_{\nu'})^\dagger]\\&+\mathrm{i}g_1g^{\mu\mu'}g^{\nu\nu'}(Z_{\mu'\nu'}\cos\theta+P_{\mu'\nu'}\sin\theta)W_\mu^\dagger W_\nu+\lambda(\varphi^2-\varphi_0^2)^2\,.\end{aligned}\tag{4.1}$$

Moreover, the equations of motion of the Lagrangian (4.1) are

$$\begin{aligned}&\frac{1}{\sqrt{-g}}\mathscr{D}_\mu[g^{\mu\mu'}g^{\nu\nu'}\sqrt{-g}\\&(\mathscr{D}_{\mu'}W_{\nu'}-\mathscr{D}_{\nu'}W_{\mu'})]=-\mathrm{i}g_1g^{\mu\mu'}g^{\nu\nu'}[Z_{\mu'\nu'}\cos\theta+P_{\mu'\nu'}\sin\theta]W_\mu\\&\qquad+\frac{1}{2}g_1^2\varphi^2g^{\mu\nu}W_\mu\\&\qquad+g_1^2[(g^{\mu'\nu'}W_{\mu'}^\dagger W_{\nu'})g^{\mu\nu}W_\mu-(g^{\mu'\nu'}W_{\mu'}W_{\nu'})g_{\mu\nu}W_\mu^\dagger]\,,\end{aligned}$$

$$\begin{aligned}\frac{1}{\sqrt{-g}}\partial_\mu[g^{\mu'\nu}g^{\mu\nu'}\sqrt{-g}P_{\mu'\nu'}]=&\,\mathrm{i}g_1\sin\theta g^{\mu'\nu}g^{\mu\nu'}[\mathscr{D}_{\mu'}W_{\nu'}-\mathscr{D}_{\nu'}W_{\mu'}]^\dagger W_\mu\\&-\mathrm{i}g_1\sin\theta g^{\mu'\nu}g^{\mu\nu'}[\mathscr{D}_{\mu'}W_{\nu'}-\mathscr{D}_{\nu'}W_{\mu'}]W_\mu^\dagger\\&-\frac{\mathrm{i}}{\sqrt{-g}}g_1\sin\theta\\&\times\partial_\mu[g^{\mu'\nu}g^{\mu\nu'}\sqrt{-g}(W_{\mu'}^\dagger W_{\nu'}-W_{\nu'}^\dagger W_{\mu'})]\end{aligned}$$

$$\begin{aligned}\frac{1}{\sqrt{-g}}\partial_\mu[g^{\mu'\nu}g^{\mu\nu'}\sqrt{-g}Z_{\mu'\nu'}]=&\,\mathrm{i}g_1\cos\theta g^{\mu'\nu}g^{\mu\nu'}[\mathscr{D}_{\mu'}W_{\nu'}-\mathscr{D}_{\nu'}W_{\mu'}]^\dagger W_\mu\\&-\mathrm{i}g_1\cos\theta g^{\mu'\nu}g^{\mu\nu'}[\mathscr{D}_{\mu'}W_{\nu'}-\mathscr{D}_{\nu'}W_{\mu'}]W_\mu^\dagger\\&-\frac{\mathrm{i}}{\sqrt{-g}}g_1\cos\theta\end{aligned}$$

$$\times\partial_\mu[g^{\mu'\nu}g^{\mu\nu'}\sqrt{-g}(W^\dagger_{\mu'}W_{\nu'}-W^\dagger_{\nu'}W_{\mu'})]$$

$$-\frac{g_1^2}{2\cos^2\theta}\varphi^2 g^{\mu\nu}Z_\mu\,,$$

$$\frac{1}{\sqrt{-g}}\partial_\mu[g^{\mu\nu}\sqrt{-g}\,\partial_\nu\varphi]=\frac{g_1^2}{4\cos^2\theta}[g^{\mu\nu}Z_\mu Z_\nu]\varphi$$

$$+\frac{g_1^2}{2}[g^{\mu\nu}W^\dagger_\mu W_\nu]\varphi+2\lambda(\varphi^2-\varphi_0^2)\varphi\,. \qquad (4.2)$$

We shall show that, when λ satisfies a specific condition, (4.2) allows a reduction into a first order system which may be called the Bogomol'nyi or the Ambjorn–Olesen equations [AO3] in curved spaces.

Varying the metric $\{g_{\mu\nu}\}$, we obtain from (4.1) the energy-momentum tensor of the electroweak matter-gauge sector:

$$T_{\mu\nu}=g^{\mu'\nu'}P_{\mu\mu'}P_{\nu\nu'}+g^{\mu'\nu'}Z_{\mu\mu'}Z_{\nu\nu'}+2\mathrm{Re}\{g^{\mu'\nu'}(\mathscr{D}_\mu W_{\mu'}-\mathscr{D}_{\mu'}W_\mu)(\mathscr{D}_\nu W_{\nu'}-\mathscr{D}_{\nu'}W_\nu)^\dagger\}$$

$$+2\partial_\mu\varphi\partial_\nu\varphi+\frac{g_1^2}{2\cos^2\theta}\varphi^2 Z_\mu Z_\nu+g_1^2\varphi^2\,\mathrm{Re}\{W_\mu W^\dagger_\nu\}$$

$$+2g_1^2[g^{\mu'\nu'}W^+_{\mu'}W_{\nu'}]\,\mathrm{Re}\{W_\mu W^\dagger_\nu\}$$

$$-2g_1^2\,\mathrm{Re}\{(g^{\mu'\nu'}W_{\mu'}W_{\nu'})W^\dagger_\mu W^\dagger_\nu\}$$

$$+\mathrm{i}g_1\cos\theta g^{\mu'\nu'}[Z_{\mu'\mu}(W^\dagger_{\nu'}W_\nu-W^\dagger_\nu W_{\nu'})+Z_{\mu'\nu}(W^\dagger_{\nu'}W_\mu-W_{\nu'}W^\dagger_\mu)]$$

$$+\mathrm{i}g_1\sin\theta g^{\mu'\nu'}[P_{\mu'\mu}(W^\dagger_{\nu'}W_\nu-W^\dagger_\nu W_{\nu'})+P_{\mu'\nu}(W^\dagger_{\nu'}W_\mu-W_{\nu'}W^\dagger_\mu)]-g_{\mu\nu}\mathscr{L}.$$

In the sequel, we impose again the string metric (3.3) and assume that

$$P_0=P_3=Z_0=Z_3=0\,,$$

$$W_0=W_3=0,\quad W_1=W,\quad W_2=\mathrm{i}W\,,$$

and that P_j, Z_j ($j=1,2$), W, and φ depend only on x^k ($k=1,2$). Then (3.4) still holds with

$$\mathscr{L}=\frac{1}{2}\mathrm{e}^{-2\eta}P_{12}^2+\frac{1}{2}\mathrm{e}^{-2\eta}Z_{12}^2+\mathrm{e}^{-2\eta}|\mathscr{D}_1W+\mathrm{i}\mathscr{D}_2W|^2+\mathrm{e}^{-\eta}|\nabla\varphi|^2$$

$$+g_1^2\mathrm{e}^{-\eta}\varphi^2|W|^2+\frac{g_1^2}{4\cos^2\theta}\mathrm{e}^{-\eta}\varphi^2[Z_1^2+Z_2^2]+2g_1^2\mathrm{e}^{-2\eta}|W|^4$$

$$-2g_1\cos\theta\mathrm{e}^{-2\eta}Z_{12}|W|^2-2g_1\sin\theta\mathrm{e}^{-2\eta}P_{12}|W|^2+\lambda(\varphi^2-\varphi_0^2)^2$$

$$=\frac{1}{2}\mathrm{e}^{-2\eta}\left[P_{12}-\frac{g_1}{2\sin\theta}\varphi_0^2\mathrm{e}^{\eta}-2g_1\sin\theta|W|^2\right]^2$$

$$+\frac{1}{2}\mathrm{e}^{-2\eta}\left[Z_{12}-\frac{g_1}{2\cos\theta}(\varphi^2-\varphi_0^2)\mathrm{e}^{\eta}-2g_1\cos\theta|W|^2\right]^2$$

$$+\mathrm{e}^{-2\eta}|\mathscr{D}_1W+\mathrm{i}\mathscr{D}_2W|^2+\mathrm{e}^{-\eta}\left[\frac{g_1}{2\cos\theta}\varphi Z_j+\varepsilon_j^k\partial_k\varphi\right]^2$$

$$+\left[\lambda-\frac{g_1^2}{8\cos^2\theta}\right][\varphi^2-\varphi_0^2]^2-\frac{g_1^2\varphi_0^4}{8\sin^2\theta}+\frac{g_1\varphi_0^2}{2\sin\theta}\mathrm{e}^{-\eta}P_{12}$$

$$-\frac{g_1\varphi_0^2}{2\cos\theta}\mathrm{e}^{-\eta}Z_{12}-\frac{g_1}{2\cos\theta}\mathrm{e}^{-\eta}\partial_k(\varepsilon_j^kZ_j\varphi^2)\ . \tag{4.3}$$

The form of (4.3) suggests that we may impose the critical condition

$$\lambda=\frac{g_1^2}{2\cos^2\theta} \tag{4.4}$$

and the Bogomol'nyi–Ambjorn–Olesen equations

$$\mathscr{D}_1W+\mathrm{i}\mathscr{D}_2W=0\ ,$$

$$P_{12}=\frac{g_1}{2\sin\theta}\varphi_0^2\mathrm{e}^{\eta}+2g_1\sin\theta|W|^2,$$

$$Z_{12}=\frac{g_1}{2\cos\theta}(\varphi^2-\varphi_0^2)\mathrm{e}^{\eta}+2g_1\cos\theta|W|^2\ ,$$

$$Z_j=-\frac{2\cos\theta}{g_1}\varepsilon_j^k\partial_k\ln\varphi\ . \tag{4.5}$$

In fact, we can examine directly that any solution of (4.5) also satisfies the full equations of motion (4.2) when (4.4) is fulfilled.

We now simplify the gravity sector or the Einstein equations (2.1). In view of (4.4) and (4.5), $\mathscr{L}$ may be written in the form

$$\mathscr{L}=-\frac{g_1^2\varphi_0^4}{8\sin^2\theta}+\frac{g_1\varphi_0^2}{2\sin\theta}\mathrm{e}^{-\eta}P_{12}+\frac{g_1}{2\cos\theta}\mathrm{e}^{-\eta}Z_{12}(\varphi^2-\varphi_0^2)+2\mathrm{e}^{-\eta}|\nabla\varphi|^2\ . \tag{4.6}$$

Furthermore, it is straightforward to check using the last equation in (4.5) that

$$T_{12}=T_{21}$$

$$=2\partial_1\varphi\partial_2\varphi+\frac{g_1^2}{2\cos^2\theta}\varphi^2Z_1Z_2=0\ .$$

The other off-diagonal components $T_{\mu\nu}=0$ ($\mu\neq\nu$) are direct consequences of the string ansatz. Besides, we have

$$T_{11}=\frac{1}{2}\mathrm{e}^{-\eta}P_{12}^2+\frac{1}{2}\mathrm{e}^{-\eta}Z_{12}^2+[\partial_1\varphi]^2-[\partial_2\varphi]^2$$

$$+2g_1^2\mathrm{e}^{-\eta}|W|^4+\frac{g_1^2}{4\cos^2\theta}\varphi^2[Z_1^2-Z_2^2]$$

$$-2g_1\sin\theta\mathrm{e}^{-\eta}P_{12}|W|^2-2g_1\cos\theta\mathrm{e}^{-\eta}Z_{12}|W|^2$$

$$
\begin{aligned}
&-\frac{g_1^2}{8\cos^2\theta}e^{\eta}[\varphi^2-\varphi_0^2]^2\\
&=\frac{1}{2}e^{-\eta}\left[(P_{12}-2g_1\sin\theta|W|^2)-\frac{g_1\varphi_0^2}{2\sin\theta}e^{\eta}\right]\\
&\times\left[(P_{12}-2g_1\sin\theta|W|^2)+\frac{g_1\varphi_0^2}{2\sin\theta}e^{\eta}\right]\\
&+\frac{1}{2}e^{-\eta}\left[(Z_{12}-2g_1\cos\theta|W|^2)-\frac{g_1}{2\cos\theta}e^{\eta}(\varphi^2-\varphi_0^2)\right]\\
&\times\left[(Z_{12}-2g_1\cos\theta|W|^2)+\frac{g_1}{2\cos\theta}e^{\eta}(\varphi^2-\varphi_0^2)\right]\\
&+\left[\partial_1\varphi-\frac{g_1}{2\cos\theta}\varphi Z_2\right]\left[\partial_1\varphi+\frac{g_1}{2\cos\theta}\varphi Z_2\right]\\
&+\left[\frac{g_1}{2\cos\theta}\varphi Z_1-\partial_2\varphi\right]\left[\frac{g_1}{2\cos\theta}\varphi Z_1+\partial_2\varphi\right]+\frac{g_1^2\varphi_0^4}{8\sin^2\theta}e^{\eta}\,.
\end{aligned}
$$

Thus in view of (4.5) again, we find $T_{11}=g_1^2\varphi_0^4e^{\eta}/8\sin^2\theta$. Similarly we can show that T_{22} takes the same value as T_{11}.

Inserting the energy-momentum tensor $T_{\mu\nu}$ just obtained into the Einstein equations (2.1) (or (2.4)), we see that there holds the condition

$$\Lambda=\frac{g_1^2\varphi_0^4}{8\sin^2\theta}\,. \tag{4.7}$$

Thus, now, in view of (4.6), (2.1) or (2.4) is equivalent to the single equation

$$\Delta\eta+\frac{g_1\varphi_0^4}{\sin\theta}P_{12}+\frac{g_1}{\cos\theta}[\varphi^2-\varphi_0^2]Z_{12}+4|\nabla\varphi|^2=0\,. \tag{4.8}$$

Therefore we have seen that, under the critical coupling condition (4.4), the Einstein–Weinberg–Salam system (2.1) and (4.2) is reduced to the simpler Einstein–Bogomol'nyi–Ambjorn–Olesen equations (4.5) and (4.8). The presence of such solutions requires that the cosmological constant Λ verify the unique condition (4.7).

We conclude this section by writing (4.5) and (4.8) as a system of second order nonlinear elliptic equations. Assume that the strings are at $p_1,\ldots,p_N$. Let u, v be such that

$$|W|^2=e^u,\quad \varphi=e^v\,.$$

Then it is straightforward to show that (4.5) and (4.8) become

$$
\begin{aligned}
&\Delta u=-g_1^2e^{v+\eta}-4g_1^2e^u+4\pi\sum_{n=1}^{N}\delta(x-p_n)\,,\\
&\Delta v=\frac{g_1^2}{2\cos^2\theta}[e^v-\varphi_0^2]e^{\eta}+2g_1^2e^u\,, \qquad x\in\mathbb{R}^2\,.\\
&\Delta\eta=-\frac{g_1^2}{2}e^{\eta}\left[\frac{(e^v-\varphi_0^2)^2}{\cos^2\theta}+\frac{\varphi_0^4}{\sin^2\theta}\right]-2g_1^2e^{u+v}-|\nabla v|^2e^v\,,
\end{aligned}
\tag{4.9}
$$

The solutions of (4.9) give rise to N-string solutions of the original Einstein–Weinberg–Salam theory.

5. The Existence of Cosmic String Solutions

In this section, we prove the existence of cosmic string solutions of the self-dual system (3.5) and (3.8) derived from the Einstein–Georgi–Glashow model. The existence problem for solutions of the more complicated Einstein–Weinberg–Salam system will be studied elsewhere.

We shall look for an N-string solution so that $p_1 = \cdots = p_N =$ the origin of $\mathbb{R}^2$. Thus, when setting $a = 2m_W^4/e^2$, $b = 4m_W^2$, and $c = e^2/m_W^2$, Eq. (3.11) becomes

$$\Delta\eta = -ae^{\eta} - b|x|^{2N}e^{c\eta}, \quad x \in \mathbb{R}^2 . \tag{5.1}$$

It will be sufficient to find radially symmetric solutions of (5.1).

Let $r = |x|$ be the radial variable. Consider the initial value problem

$$\eta_{rr} + \frac{1}{r}\eta_r = -ae^{\eta} - br^{2N}e^{c\eta}, \quad r > 0 ,$$
$$\eta(0) = \eta_0, \quad \eta_r(0) = 0 , \tag{5.2}$$

where $\eta_0 \in \mathbb{R}$ is arbitrary. It is well known (see [BLP]) that (5.2) has a unique local solution for any η_0 and that such a solution can be extended smoothly to obtain a solution of (5.1) in a neighborhood of the origin.

Lemma 5.1. *The solution of* (5.2) *is globally defined in* $(0, \infty)$.

Proof. Integrating (5.2) in the interval of existence of the solution, we have

$$r\eta_r(r) = -\int_0^r \left[a\rho e^{\eta(\rho)} + b\rho^{2N+1}e^{c\eta(\rho)} \right] d\rho, \quad r > 0 . \tag{5.3}$$

Thus η is decreasing and $|\eta_r|$ cannot blow up in finite $r > 0$. As a consequence, the lemma follows. □

Lemma 5.2. *Let η be the unique global solution of* (5.2). *There exists a constant* β:

$$\beta > 2\max\left\{1, \frac{N+1}{c}\right\} \equiv \beta_1(N) \tag{5.4}$$

so that η is asymptotically $-\beta \ln r$ which is characterized by

$$\lim_{r\to\infty} \frac{\eta(r)}{\ln r} = \lim_{r\to\infty} r\eta_r(r) = -\beta . \tag{5.5}$$

Proof. From (5.3) it is seen that $\eta(r) \to -\infty$ or a finite number as $r \to \infty$. Suppose the latter is true. Then (5.3) says $r\eta_r(r) \to -\infty$ as $r \to \infty$. Therefore there is an $r_0 > 0$ such that

$$r\eta_r(r) < -1, \quad r \geqq r_0 .$$

Hence

$$\eta(r) < -\ln\frac{r}{r_0} + \eta(r_0), \quad r > r_0 ,$$

and $\eta(r) \to -\infty$ as $r \to \infty$, which violates the assumption made earlier.

It is easy to agree that, as $r\to\infty$, $r\eta_r(r)\to$ a finite number or $-\infty$. Let us exclude the latter possibility.

Suppose otherwise that $r\eta_r(r)\to-\infty$ (as $r\to\infty$). Then, for

$$K=\max\left\{\frac{2N+3}{c}, 3\right\},$$

there is an r_0 so that

$$r\eta_r(r)\leqq -K, \quad r\geqq r_0 .$$

This inequality implies

$$\eta(r)\leqq -K\ln\frac{r}{r_0}+\eta(r_0), \quad r\geqq r_0 ,$$

which gives us the immediate consequences

$$\int_0^\infty re^{\eta(r)}\,dr<\infty, \quad \int_0^\infty r^{2N+1}e^{c\eta(r)}\,dr<\infty . \tag{5.6}$$

Thus $r\eta_r(r)\to$ a finite number as $r\to\infty$ and we arrive at a contradiction.

Consequently, (5.5) holds for some constant $\beta>0$. Obviously, $r\eta_r(r)>-\beta$, $r>0$ (see (5.3)). Therefore

$$\eta(r)>-\beta\ln r+\eta(1), \quad r>1 . \tag{5.7}$$

Inserting (5.7) into (5.6), we see that (5.4) follows. □

The estimate (5.4) for the decay exponent β can further be improved.

Lemma 5.3. *Let β be the number stated in Lemma 5.2. If $(N+1)/c=1$, we have $\beta=4$. If $(N+1)/c\neq 1$, then β lies in the open range*

$$\beta_2(N)\equiv 4\min\left\{1,\frac{N+1}{c}\right\}<\beta<4\max\left\{1,\frac{N+1}{c}\right\}. \tag{5.8}$$

Proof. From (5.2), we have

$$\frac{d}{dr}(r^2\eta_r^2)=-2ar^2\frac{d}{dr}(e^{\eta})-\frac{2b}{c}r^{2N+2}\frac{d}{dr}(e^{c\eta}) .$$

Integrating the above equation by parts and using (5.4) to drop the boundary terms, we obtain

$$\beta^2=4\int_0^\infty\left\{are^{\eta(r)}+b\left(\frac{N+1}{c}\right)r^{2N+1}e^{c\eta(r)}\right\}dr . \tag{5.9}$$

Thus we are led from Eq. (5.9) to the relation

$$\begin{aligned}4\left(1-\frac{N+1}{c}\right)a\int_0^\infty re^{\eta}\,dr+4\left(\frac{N+1}{c}\right)\beta&=\beta^2\\ =4\beta+4\left(\frac{N+1}{c}-1\right)b\int_0^\infty r^{2N+1}e^{c\eta}\,dr .\end{aligned} \tag{5.10}$$

Thus $\beta=4$ if and only if $(N+1)/c=1$. When $(N+1)/c<1$, we see from (5.10) that $4(N+1)/c<\beta<4$. While, for $(N+1)/c>1$, we have $4<\beta<4(N+1)/c$. Hence the estimate (5.8) follows. □

Using (5.4) and (5.8), we arrive at the inequality

$$\max\{\beta_1(N), \beta_2(N)\}\leqq\beta\leqq 4\max\left\{1, \frac{N+1}{c}\right\}. \tag{5.11}$$

In the case that $(N+1)/c=1$, the lower and upper bounds in (5.11) coincide and we obtain the exact result $\beta=4$. Thus, when $(N+1)/c$ is not too far away from 1, there holds $\beta\approx 4$. However, when $(N+1)/c$ is far away from 1, the range of β stated in (5.11) may be a large *open* interval and in such a situation we are unable to decide the exact values within the interval that β can assume. In particular, it seems to be an interesting question whether each number in the interval (5.11) can be realized as a decay exponent β with a suitable choice of the initial data in (5.2).

Let η be the radially symmetric solution of (5.1) obtained above and

$$u(x)=\frac{e^2}{m_W^2}\eta(x)+2N\ln|x|\,.$$

Since η is asymptotically $-\beta\ln|x|$ as $|x|\to\infty$, where β is a constant satisfying

$$\max\{\beta_1(N), \beta_2(N)\}\leqq\beta\leqq 4\max\left\{1, \frac{(N+1)m_W^2}{e^2}\right\} \tag{5.12}$$

with

$$\beta_1(N)=2\max\left\{1, \frac{(N+1)m_W^2}{e^2}\right\}, \quad \beta_2(N)=4\min\left\{1, \frac{(N+1)m_W^2}{e^2}\right\}$$

(see (5.11)), the function u is asymptotically $-\alpha\ln|x|$ with

$$c\max\{\beta_1(N), \beta_2(N)\}-2N\leqq\alpha\leqq 2\max\left\{\frac{2e^2}{m_W^2}-N, N+2\right\}. \tag{5.13}$$

To obtain a solution of the original system (3.5) and (3.8) from the pair (u, η) constructed above, we put $z=x^1+\mathrm{i}x^2$, $\partial^\dagger=\partial_1+\mathrm{i}\partial_2$, and

$$W(z)=\exp\left[\frac{1}{2}u(z)+\mathrm{i}N\arg z\right],$$

$$f_1=-\mathrm{Re}\left\{\frac{\mathrm{i}}{e}\partial^\dagger\ln W\right\},$$

$$f_2=-\mathrm{Im}\left\{\frac{\mathrm{i}}{e}\partial^\dagger\ln W\right\}.$$

Then (W, f, η) is a solution of (3.5) and (3.8) and

$$|W|^2=\mathrm{O}(r^{-\alpha}),$$

$$\mathrm{e}^\eta=\mathrm{O}(r^{-\beta}),$$

$$f_{12}=\mathrm{O}(r^{-\gamma}), \quad \gamma=\min\{\alpha, \beta\},$$

$$R-\frac{2m_W^4}{e^2}=\mathrm{O}(r^{-(\alpha-\beta)}), \tag{5.14}$$

as $r=|x|\to\infty$, where α, β satisfy (5.12)–(5.13) and R is the scalar curvature of the surface $(\mathbb{R}^2, \delta_{jk}e^{\eta})$. The decay estimates (5.14) say that the cosmic string we have constructed is of finite energy.

Remark 5.1. In fact the property that η is asymptotically $-\beta\ln|x|$, where β lies in the range (5.11) is a consequence of the regularity of any symmetric solution of (5.1) because for such a solution the smoothness at the origin requires

$$\lim_{r\to 0} r\eta_r(r)=0 .$$

This feature ensures already the validity of Lemmas 5.2 and 5.3.

Remark 5.2. In the $U(1)$ case, a necessary and sufficient condition for the existence of an N-string solution has been obtained in [CHMcY] which says for example that if N exceeds an explicit upper bound, there will be an energy-blowup. The result of this section tells us on the other hand that when the gauge group is non-abelian ($G=SU(2)$), in addition to the presence of a positive cosmological constant Λ, finite-energy N-string solutions exist for any number N.

Remark 5.3. A well-known result of Kazdan–Warner [KW] on the prescribed curvature problem for open 2-surfaces says that a function $R\in C^\infty(\mathbb{R}^2)$ is the scalar curvature of a complete Riemannian metric on $\mathbb{R}^2$ if and only if

$$\lim_{r\to\infty}\ \inf_{|x|\geqq r} R(x)\leqq 0 . \tag{5.15}$$

Thus, in view of $R=-e^{-\eta}\Delta\eta$ and (3.10) and (4.9), we see that the obtained gravitational metric cannot be complete because the curvature R is bounded away from zero below by a positive constant, which violates the condition (5.15).

6. Conclusion

We have shown that, when the Einstein theory is coupled with the Georgi–Glashow and the Weinberg–Salam models, the presence of self-dual cosmic strings implies that the cosmological constant Λ must in that situation assume corresponding positive values. In other words, if we accept these two-dimensional self-dual solutions as physical states in suitable phase transition stages of the early universe realized respectively by the coupling models discussed, then, at that time, Λ was positive and might be uniquely and explicitly expressed in terms of some fundamental parameters in particle physics such as the electron charge, the W-particle mass, the Weinberg angle, and the coupling constants.

Acknowledgement. We would like to thank the referee for helpful suggestions.

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Communicated by R. H. Dijkgraaf