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Deformations of Enveloping Algebra of Lie Superalgebra $sl(m, n)$

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Abstract: The q-deformations of the universal enveloping algebra of $sl(m, n)$ are considered, a Poincaré–Birkhoff–Witt type theorem is proved for these deformations, and the extra relations which are needed to define $sl(m, n)$ as a contragredient algebra in addition to the Serre-type relations are identified with proof.

1. Introduction

The q-deformations of the enveloping algebras of some classical Lie superalgebras have been discussed by several authors (see [1, 3, 4, 9] and the references therein). It has been realized that in general the Serre-type relations are not sufficient to define $G = sl(m, n)$ as a contragredient Lie superalgebra (see [4, 9]), and extra conditions must be added. In order to define the q-deformation of the enveloping algebra $U(G)$, it is necessary to deform these extra relations. Though some extra relations were introduced in [4, 9], it has not been proved that the Serre-type relations together with these extra relations are the defining relations of G as a contragredient Lie superalgebra with the standard Cartan matrix of G (we will see that this is a tensor problem in Sect. 4 below). Also, a Poincaré–Birkhoff–Witt type theorem for the q-deformation of $U(G)$ (defined in [4] or [9]) is still lacking. As indicated in [9], an adequate Poincaré–Birkhoff–Witt theorem is important in showing that a q-deformation of $U(G)$ is a decent deformation.

In this paper, we consider a somewhat different approach to the problem of deforming the enveloping algebra of a classical Lie superalgebra. We start with the following characterization of a Lie superalgebra [5]: every Lie superalgebra can be specified by three objects: the Lie algebra G_0, the G_0-module G_1, and the homomorphism of G_0-modules φ: $S^2G_1 \to G_0$, with the sole condition

$$\varphi(a, b)c + \varphi(b, c)a + \varphi(c, a)b = 0 \quad \text{for } a, b, c \in G_1 . \tag{1.1}$$

Our observation is that, since for a classical Lie superalgebra $G = G_0 + G_1$, the even part G_0 of G is a reductive Lie algebra, the q-deformation $U_q(G_0)$ of $U(G_0)$ is well understood, and for the finite dimensional G_0-module G_1, there is the corresponding

$U_q(G_0)$-module, which is the q-deformation of G_1, thus, in order to deform $U(G)$, we only need to deform (1.1). In the present paper, we only consider the case $sl(m, n)$, the other classical Lie superalgebras will be treated in another paper.

In Sect. 2, we describe the q-deformations of $U(G)$ (the deformation is not unique, see the definitions in Sect. 2 and the discussion at the end of Sect. 2). These algebras tend to $U(G)$ as q tends to 1. In Sect. 3, we prove a q-analog of the Poincaré–Birkhoff–Witt theorem for our deformations of $U(G)$. In Sect. 4, by considering the decomposition of the G_0-module S^2G_1, we will prove that the q-deformation of $U(G)$ defined by [4, 9] is isomorphic to one of our deformations, and thus prove that it is a reasonable q-deformation of $U(G)$ and a Poincaré–Birkhoff–Witt type theorem holds for this algebra.

2. Deforming $U(sl(m, n))$

Recall that ([5]) $G=sl(m,n)$ can be viewed as the set of all $(m+n)^2$ matrices $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ over the complex number field $\mathbf{C}$, where α is an $m\times m$ matrix, β is an $m\times n$ matrix, γ is an $n\times m$ matrix, δ is an $n\times n$ matrix, and $\mathrm{tr}\alpha=\mathrm{tr}\delta$.The even part G_0 of G consists of matrices of the form $\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}$, the odd part G_1 of G consists of matrices of the form $\begin{pmatrix}0 & \beta\\ \gamma & 0\end{pmatrix}$, and $G_0\cong sl(m)\oplus sl(n)\oplus\mathbf{C}$. Let

$$H=\{\mathrm{diag}(a_1, a_2, \dots, a_{m+n}):\ \sum_{i=1}^{m} a_i=\sum_{j=1}^{n} a_{m+j},\ a_t\in\mathbf{C},\ 1\leqq t\leqq m+n\}\ .$$

Then H is a Cartan subalgebra of G. The corresponding root system will be denoted by R. The roots can be expressed in terms of linear functions $\varepsilon_1, \dots, \varepsilon_m$; $\delta_1=\varepsilon_{m+1}, \dots, \delta_n=\varepsilon_{m+n}$. Let R_0 be the set of even roots, let R_1 be the set of odd roots, then

$$R_0=\{\varepsilon_i-\varepsilon_j;\ \delta_i-\delta_j:\ i\neq j\},\quad R_1=\{\pm(\varepsilon_i-\delta_j)\}\ .$$

Let

$$R_0^+=\{\varepsilon_i-\varepsilon_j;\ \delta_i-\delta_j:\ i<j\},\quad R_1^+=\{\varepsilon_i-\delta_j\}\ ,$$

and let $R_0^-,=-R_0^+$, $R_1^-=-R_1^+$. We choose

$$\{\varepsilon_1-\varepsilon_2,\ \varepsilon_2-\varepsilon_3, \dots, \varepsilon_m-\delta_1,\ \delta_1-\delta_2, \dots, \delta_{n-1}-\delta_n\} \tag{2.1}$$

as a simple root system.

For $\lambda\in H^*$, let $V(\lambda)$ be the irreducible highest weight G_0-module with the highest weight λ. As a G_0-module, $G_1\cong V(\lambda_1)\oplus V(\lambda_2)$, where $\lambda_1=\varepsilon_1-\delta_n$, $\lambda_2=-(\varepsilon_m-\delta_1)$, and the corresponding highest weight vectors are $e_{1,m+n}$ and $e_{m+1,m}$ respectively, where e_{ij} denotes the $(m+n)^2$ matrix with 1 at the ij-entry and 0 elsewhere. Denote the representation of G_0 on G_1 by ϕ and identify e_{ij} with its image in $V(\lambda_1)\oplus V(\lambda_2)$. The map φ: $S^2G_1\to G_0$ is given by

$$\varphi(e_{ij}, e_{ts})=e_{ij}e_{ts}+e_{ts}e_{ij}=\delta_{jt}e_{is}+\delta_{si}e_{tj}\ , \tag{2.2}$$

where either $m+1\leqq i\leqq m+n$ and $1\leqq j\leqq m$, or $1\leqq i\leqq m$ and $m+1\leqq j\leqq m+n$, and similar conditions hold for t and s. Formula (2.2) can also be given by using the following basis of G_0: e_{ij}, $i \neq j$, with $1 \leqq i \leqq m$ and $1 \leqq j \leqq m$, or $m+1 \leqq i \leqq m+n$, $m+1 \leqq j \leqq m+n$; $h_i = e_{ii} - e_{i+1,i+1}$, $1 \leqq i \leqq m-1$, or $m+1 \leqq i \leqq r$, where $r = m+n-1$, and $h_m = e_{mm} - e_{m+1,m+1}$. With this basis, (2.2) becomes

$$\varphi(e_{ij}, e_{ts}) = \begin{cases} h_i + \cdots + h_m - h_{m+1} - \cdots - h_{j-1}, & i=s>j=t\,, \\ h_j + \cdots + h_m - h_{m+1} - \cdots - h_{i-1}, & i=s<j=t\,, \\ \delta_{jt} e_{is} + \delta_{si} e_{tj}, & \text{otherwise}\,. \end{cases} \tag{2.3}$$

We rewrite it as

$$\varphi(e_{ij}, e_{ts}) = \sum c_{ab}^{ijts} e_{ab} + \sum c_f^{ijts} h_f\,. \tag{2.4}$$

Then $c_{ab}^{ijts} = 0, 1$, and $c_f^{ijts} = 0, \pm 1$.

Let $U_\varphi(G)$ be the associative algebra with 1 generated by the vector space $G_0 \oplus V(\lambda_1) \oplus V(\lambda_2)$ subject to the following relations:

(1) The usual defining relations of G_0 hold for the elements of G_0.

(2) For $x \in G_0$, $v \in V(\lambda_1) \oplus V(\lambda_2)$,

$$xv - vx = \phi(x)v\,. \tag{2.5}$$

(3) For v_1 and $v_2 \in V(\lambda_1) \oplus V(\lambda_2)$,

$$v_1 v_2 + v_2 v_1 = \varphi(v_1, v_2)\,. \tag{2.6}$$

Then as an associative algebra, $U_\varphi(G)$ is isomorphic to $U(G)$, the enveloping algebra of G.

Let q be an indeterminate over $\mathbf{C}$, let $\mathscr{A} = \mathbf{C}[q, q^{-1}]$ and let $\mathbf{F}$ be the quotient field of $\mathscr{A}$. Let $U_q(G_0)$ be the associative algebra over $\mathbf{F}$ with 1 generated by E_i, F_i, $i \in \{1, 2, \ldots, r\} \backslash \{m\}$, and $K_i^{\pm 1}$, $i \in \{1, 2, \ldots, r\}$, with relations:

$$\begin{gathered} K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i\,, \\ K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j\,, \\ E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}\,, \\ E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i, \quad a_{ij} = 0\,, \\ E_i^2 E_j - (q_i + q_i^{-1}) E_i E_j E_i + E_j E_i^2 = 0, \quad |i-j| = 1\,, \\ F_i^2 F_j - (q_i + q_i^{-1}) F_i F_j F_i + F_j F_i^2 = 0, \quad |i-j| = 1\,, \end{gathered} \tag{2.7}$$

where a_{ij} is the (ij)-entry of the Cartan matrix (a_{ij}) of G corresponds to the simple root system chosen in (2.1) and

$$q_i = \begin{cases} q, & \text{if } 1 \leqq i \leqq m\,, \\ q^{-1}, & \text{if } m+1 \leqq i \leqq r\,. \end{cases} \tag{2.8}$$

The comultiplication Δ, the antipole S and the counit ε of $U_q(G_0)$ are defined by

$$\Delta(K_i) = K_i \otimes K_i,\ \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,\ \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i\,, \tag{2.9}$$

$$S(K_i) = K_i^{-1},\ S(E_i) = -K_i^{-1} E_i,\ S(F_i) = -F_i K_i\,, \tag{2.10}$$

$$\varepsilon(E_i) = 0,\ \varepsilon(F_i) = 0,\ \varepsilon(K_i) = 1\,. \tag{2.11}$$

By [6], the G_0-modules $V(\lambda_1)$ and $V(\lambda_2)$ admit q-deformations $V_q(\omega_1)$ and $V_q(\omega_2)$, which are simple highest weight $U_q(G_0)$-modules with highest weights

$$\omega_1=(q_1, 1, \ldots, 1, q_{m+n}^{-1})=(q, 1, \ldots, 1, q),$$

and

$$\omega_2=(1, \ldots, q_m, q_{m+1}^{-1}, \ldots, 1)=(1, \ldots, q, q, \ldots, 1),$$

respectively. Let $V_q=V_q(\omega_1)\oplus V_q(\omega_2)$, denote the $U_q(G_0)$-action on V_q by ϕ_q. Fix a highest weight vector of $V_q(\omega_1)$ and denote it by $E_{1,m+n}$, fix a highest weight vector of $V_q(\omega_2)$ and denote it by $E_{m+1,m}$.

We use the action of $U_q(G_0)$ on V_q to construct a basis of $V_q(\omega_1)$,

$$\{E_{ij}\colon 1\leqq i\leqq m,\ m+1\leqq j\leqq m+n\},$$

and a basis of $V_q(\omega_2)$

$$\{E_{ij}\colon m+1\leqq i\leqq m+n,\ 1\leqq j\leqq m\}$$

as follows. For $1\leqq i\leqq m$ and $m+1\leqq j\leqq m+n$, set

$$E_{ij}=(-1)^{m+n-j}\phi_q((F_j\cdots F_{m+n-1})(F_{i-1}\cdots F_1))E_{1,m+n}, \tag{2.12}$$

where we have adopted the following convenience: if $i=1$, $F_{i-1}\cdots F_1=1$, and if $j=m+n$, $F_j\cdots F_{m+n-1}=1$. For $m+1\leqq i\leqq m+n$ and $1\leqq j\leqq m$, set

$$E_{ij}=(-1)^{m-j}\phi_q((F_j\cdots F_{m-1})(F_i\cdots F_{m+1}))E_{m+1,m}, \tag{2.13}$$

where if $j=m$, $F_j\cdots F_{m-1}=1$ and if $i=m+1$, $F_i\cdots F_{m+1}=1$.

According to [7], there exists a braid group action on $U_0(G_0)$. By using this braid group action, we can construct root vectors of $U_q(G_0)$. We denote the root vector corresponds to $\varepsilon_i-\varepsilon_j$ by E_{ij}, where $i\neq j$, $1\leqq i\leqq m$ and $1\leqq j\leqq m$, or $m+1\leqq i\leqq m+n$ and $m+1\leqq j\leqq m+n$. Note that according to our notations, $E_{i,i+1}=E_i$, $E_{i+1,i}=F_i$, $i\neq m$, $1\leqq i\leqq r$.

Le $U_{q,\mathscr{A}}(G_0)$ be the $\mathscr{A}$-algebra of $U_q(G_0)$ generated by E_i, F_i, $K_i^{\pm}$, and

$$[K_i;0]=\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}}. \tag{2.14}$$

Let I_1 be the ideal of $U_{q,\mathscr{A}}(G_0)$ generated by $q-1$ and K_i-1, $1\leqq i\leqq r$, then $U_{q,\mathscr{A}}(G_0)/I_1\cong U(G_0)$. And by [2, Prop. 1.5], under this isomorphism, $E_{ij}\to e_{ij}$, $[K_i; 0]\to h_i$, where $h_i=e_{ii}-e_{i+1,i+1}$, $i\neq m$, and $h_m=e_{mm}+e_{m+1,m+1}$. Let $H_i=[K_i;0]$, $1\leqq i\leqq r$.

Let $V_{q,\mathscr{A}}$ be the $\mathscr{A}$-submodule of V_q spanned by the E_{ij}'s. Observe that one can define $\mathscr{A}$-module homomorphisms

$$\varphi_{c(q)}\colon S(V_{q,\mathscr{A}})\to U_{q,\mathscr{A}}(G_0),$$

such that as $q\to 1$, $\varphi_{c(q)}\to\varphi$, that is one can deform φ. For example, one can choose elements $c_*^{ijts}(q)\in\mathscr{A}$ suitably (e.g. $c_*^{ijts}(q)=c_*^{ijts}$), such that

(1) $c_*^{ijts}(q)=0$ if both e_{ij} and $e_{ts}\in V(\lambda_i)$, $i=1, 2$; and

(2) $c_*^{ijts}\to c_*^{ijts}$ under isomorphism $\mathbf{C}\cong\mathscr{A}/(q-1)$,

where $c_*^{ijts} \in \mathbf{C}$ are defined by (2.4), and then define an $\mathscr{A}$-module homomorphism

$$\varphi_{c(q)}: S^2(V_{q,\mathscr{A}}) \to U_{q,\mathscr{A}}(G_0),$$

by

$$\varphi_{c(q)}(E_{ij}, E_{st}) = \sum c_{ab}^{ijts}(q) E_{ab} + \sum c_f^{ijts}(q) H_f . \tag{2.15}$$

Since the E_{ij}'s form an $\mathbf{F}$-basis of V_q, $\varphi_{c(q)}$ can be extended to an $\mathbf{F}$-linear map from $S^2(V_q)$ to $U_q(G_0)$.

Remark. The condition (1) on $c_*^{ijts}(q)$ is to ensure that

$$\varphi_{c(q)}(x, y) = 0, \quad \text{for all } x, y \in V_q(\omega_i),\ i = 1, 2.$$

We shall assume this condition for our choice of $\varphi_{c(q)}$.

Now we are ready to define the q-deformation of $U(G)$ corresponding to a fixed $\varphi_{c(q)}$. Let $U_{\varphi_{c(q)}}(G)$ be the associative $\mathbf{F}$-algebra with 1 generated by $U_q(G_0)$ and V_q with the following conditions:

(1) the multiplication restricted to $U_q(G_0)$ is the same as the multiplication in $U_q(G_0)$,

(2) for $x \in U_q(G_0)$ and for the generators E_i, F_i, K_i of $U_q(G_0)$,

$$K_i v K_i^{-1} = \phi_q(K_i) v,\ E_i v - v E_i = \phi_q(E_i) v,\ F_i v - v F_i = \phi_q(F_i) v , \tag{2.16}$$

(3) for $v_1, v_2 \in V_q$,

$$v_1 v_2 + v_2 v_1 = \varphi_{c(q)}(v_1, v_2) . \tag{2.17}$$

Let $E_m = E_{m,m+1} (\in V_q(\omega_1))$, and let $F_m = E_{m+1,m} (\in V_q(\omega_2))$. We have

Lemma 2.1. *As an associative* $\mathbf{F}$*-algebra* $U_{\varphi_{c(q)}}(G)$ *is generated by* E_i, F_i, $K_i^{\pm 1}$, $1 \leqq i \leqq r$, *and* 1. *Furthermore, these generators satisfy relations* (2.7) *with the following modifications: when* $i = m$,

$$E_m F_m + F_m E_m = c_m(q) H_m , \tag{2.18}$$

where $c_m(q) \in \mathscr{A}$ *is chosen in the definition of* $\varphi_{c(q)}$, *and*

$$E_m^2 = F_m^2 = 0 . \tag{2.19}$$

Proof. By definition, $U_q(G_0)$ is generated by E_i, F_i, $1 \leqq i \leqq r$, $i \neq m$, and $K_i^{\pm 1}$, $1 \leqq i \leqq r$. Also, an an irreducible $U_q(G_0)$-module, $V_q(\omega_1)$ is generated by any nonzero element $v \in V_q(\omega_1)$, in particular, it is generated by E_m. Similarly, $V_q(\omega_2)$ is generated by F_m as a $U_q(G_0)$-module. Thus the first statement follows. The second statement follows directly from the definition of $\varphi_{c(q)}$. ■

We will discuss the generating relations of $U_{\varphi_{c(q)}}(G)$ in Sect. 4.

Fix a $\varphi_{c(q)}$, let $U_{\varphi_{c(q)},\mathscr{A}}(G)$ be the $\mathscr{A}$-subalgebra of $U_{\varphi_{c(q)}}(G)$ generated by $U_{q,\mathscr{A}}(G_0)$ together with E_m and F_m. Then $U_{\varphi_{c(q)},\mathscr{A}}(G) \supset V_{q,\mathscr{A}}$, and we have

Proposition 2.2. *As* $\mathbf{C}$*-algebras,* $U_{\varphi_{c(q)},\mathscr{A}}(G)/I_1 \cong (U(G)$, *where* I_1 *is the ideal of* $U_{\varphi_{c(q)},\mathscr{A}}(G)$ *generated by* $K_i - 1$, $1 \leqq i \leqq r$, *and* $q - 1$.

Proof. We know that $U_{q,\mathscr{A}}(G_0)/I_1 \cong U(G_0)$ and the image of $V_{q,\mathscr{A}}$ is isomorphic to $G_1 = V(\lambda_1) \oplus V(\lambda_2)$ as a $U(G_0)$-module (see [6]), so we can identify them. With these identifications, $U_{\varphi_{c(q)},\mathscr{A}}(G)/I_1$ is generated by $U(G_0) \oplus V(\lambda_1) \oplus V(\lambda_2)$ with the same generating relations as $U_\varphi(G)$, since conditions (2.15)–(2.17) induce conditions (2.4)–(2.6). Hence the proposition follows. ■

The algebra $U_{\varphi_{c(q)}}(G)$ is a $\mathbf{Z}_2$-graded algebra with the grading given by

$$\deg(E_i)=\deg(F_i)=0,\ i\neq m;\ \deg(K_i^{\pm 1})=0,\ 1\leqq i\leqq r\ ,$$

and

$$\deg(E_m)=\deg(F_m)=1\ .$$

The $\mathbf{F}$-algebra $U_{\varphi_{c(q)}}(G)$ is a Hopf algebra with comultiplication Δ, antipode S and counit ε defined as in (2.9)–(2.11) without the restriction $i\neq m$ for E_i and F_i. The adjoint action of $U_{\varphi_{c(q)}}(G)$ on itself is denoted by ad_q. Thus for $x, y\in U_{\varphi_{c(q)}}(G)$, if $\Delta x=\sum a_i\otimes b_i$, then

$$\mathrm{ad}_q(x)y=\sum(-1)^{\deg b_i \deg y}a_i y S(b_i)\ .$$

Since $U_{\varphi_{c(q)}}(G)$ depends on the definition of $\varphi_{c(q)}$, $U_{\varphi_{c(q)}}(G)$ is not unique. For example, in Lemma 2.1, the choice of $c_m(q)$ makes it clear that one may choose $c_m(q)$ up a factor $f(q)\in\mathscr{A}$ and a term $g(q)\in\mathscr{A}$ such that $f(q)\to 0$ as $q\to 1$. Different choices of the $c_*^{ijts}(q)$ lead to nonisomorphic deformations of $U(G)$.

3. A q-Analog of the Poincaré–Birkhoff–Witt Theorem

Choose a deformation $\varphi_{c(q)}$ of φ, and define $U_{\varphi_{c(q)}}(G)$ as in Sect. 2. Let $\mathscr{U}=U_{\varphi_{c(q)}}(G)$, $\mathscr{U}_{\mathscr{A}}=U_{\varphi_{c(q)},\mathscr{A}}(G)$, $\mathscr{U}_0=U_q(G_0)$, $\mathscr{U}_{0,\mathscr{A}}=U_{q,\mathscr{A}}(G_0)$. Let $\mathscr{U}^+$, $\mathscr{U}^-$, $\mathscr{U}^0$ be the subalgebras (with 1) of $\mathscr{U}$ generated by the E_i, the F_i and the $K_i^{\pm}$ respectively. Note that since E_m is a lowest weight vector of $V_q(\omega_1)$, we have $\mathscr{U}^+\supset V_q(\omega_1)$, and by our definition of $\varphi_{c(q)}$, for any v_1 and $v_2\in V_q(\omega_1)$, $\varphi_{c(q)}(v_1, v_2)=0$ (see the remark in Sect. 2), hence F_i and $K_i\notin\mathscr{U}^+$. Similarly, $\mathscr{U}^-\supset V_q(\omega_2)$, E_i and $K_i\notin\mathscr{U}^-$.

We order the root vectors E_{ij} (of $\mathscr{U}_0$) corresponding to the positive roots of G_0 (i.e. $i<j$) as follows:

$$E_{ij}<E_{st}\ \text{iff}\ i<s\ \text{or}\ i=s\ \text{but}\ j<t\ . \tag{3.1}$$

Let $N=[n(n-1)+m(m-1)]/2$. Denote these positive root vectors according to the ordering defined by (3.1) as

$$E_{\beta_1}, E_{\beta_2}, \ldots, E_{\beta_N}\ , \tag{3.2}$$

and the corresponding negative root vectors of $\mathscr{U}_0$ as

$$F_{\beta_1}, F_{\beta_2}, \ldots, F_{\beta_N}\ . \tag{3.3}$$

We also order the elements E_{ij}, $1\leqq i\leqq m, m+1\leqq j\leqq m+n$, of $V_q(\omega_1)$ according to relation (3.1) and denote them by

$$E_{\gamma_1}, E_{\gamma_2}, \ldots, E_{\gamma_{mn}}\ . \tag{3.4}$$

The elements E_{ji} of $V_q(\omega_2)$ are ordered correspondingly as

$$F_{\gamma_1}, F_{\gamma_2}, \ldots, F_{\gamma_{mn}}\ . \tag{3.5}$$

The proofs of Lemma 3.1 and Lemma 3.2 below are similar to the proofs given by [8, II].

Lemma 3.1. *The monomials* $K^\tau = \prod_i K_i^\tau$ *with* τ *running through all functions* $\{1, 2, \ldots, r\} \to \mathbf{Z}$ *form a basis of* $\mathscr{U}^0$. ■

Lemma 3.2. *As vector spaces,* $\mathscr{U} \cong \mathscr{U}^- \otimes \mathscr{U}^0 \otimes \mathscr{U}^+$. *Thus* $\mathscr{U} = \mathscr{U}^- \mathscr{U}^0 \mathscr{U}^+$. ■

For $\sigma: \{1, 2, \ldots, N\} \to \mathbf{Z}_+$, let $E_0^\sigma = \prod_i E_{\beta_i}^{\sigma(i)}$ and $F_0^\sigma = \prod_i F_{\beta_i}^{\sigma(i)}$. For $d: \{1, 2, \ldots, mn\} \to \{0, 1\}$, let $E_1^d = \prod_i E_{\gamma_i}^{d(i)}$ and $F_1^d = \prod_i F_{\gamma_i}^{d(i)}$.

Lemma 3.3. *The elements* $E_0^\sigma E_1^d$ (resp. $F_0^\sigma F_1^d$) *form an* $\mathbf{F}$*-basis of* $\mathscr{U}^+$ (resp. $\mathscr{U}^-$) *with* σ *running through all the functions* $\{1, \ldots, N\}) \to \mathbf{Z}_+$ *and* d *running through all the functions* $\{1, \ldots, mn\} \to \{0, 1\}$.

Proof. We prove that $E_0^\sigma E_1^d$ is a basis of $\mathscr{U}^+$, the proof for $F_0^\sigma F_1^d$ is similar. We first prove that the elements $E_0^\sigma E_1^d$ span $\mathscr{U}^+$. Note that by [7], the E_0^σ's form a basis of $\mathscr{U}_0^+$, the subalgebra of $\mathscr{U}^+$ generated by E_i with $i \neq m$. Thus we only need to prove that the elements of the form uE_1^d with $u \in \mathscr{U}_0^+$ span $\mathscr{U}^+$. Let us call these elements standard. The elements of $\mathscr{U}^+$ can be written as linear combinations of monomials of the form $x_1 x_2 \cdots x_k$, where $x_i \in \mathscr{U}_0^+$ or $x_i = E_{\gamma_j}$ (see (3.4)), and if $x_i \in \mathscr{U}_0^+$, then x_{i-1} and $x_{i+1} \notin \mathscr{U}_0^+$ whenever applicable. A monomial $X = x_1 x_2 \cdots x_k$ of this kind is called semistandard.

Let $X = x_1 x_2 \cdots x_k$ be a semistandard monomial. For $1 \leqq i < j \leqq k$, set

$$t_{ij}(X) = \begin{cases} 0, & \text{if either } x_i = E_{\gamma_s},\ x_j = E_{\gamma_t} \text{ with } s \leqq t; \text{ or } x_i \in \mathscr{U}_0^+, \\ 1, & \text{if either } x_i = E_{\gamma_s},\ x_j = E_{\gamma_t} \text{ with } s > t; \text{ or } x_i = E_{\gamma_s} \text{ and } x_j \in \mathscr{U}_0^+. \end{cases}$$

Define the index of X by

$$i(X) = \sum_{i<j} t_{ij}(X).$$

Note that $i(X) = 0$ iff X is standard. Note also that $E_{\gamma_j} E_{\gamma_j} = 0$. We use induction on k and $i(X)$ to prove that a semistandard monomial X is a linear combination of the standard ones. The case $k = 1$ is clear. Assume the statement is true for $< k$ with $k \geqq 2$. Let $X = x_1 x_2 \cdots x_k$ be a semistandard monomial.

If $i(X) = 0$, there is nothing to prove. Assume $i(X) > 0$. Then we can find x_i and x_{i+1}, such that either $x_i = E_{\gamma_s}$ and $x_{i+1} = E_{\gamma_t}$ with $s > t$; or $x_i = E_{\gamma_s}$ and $x_{i+1} \in \mathscr{U}_0^+$. In the first case, let $X' = x_1 \cdots x_{i+1} x_i \cdots x_k$, consider

$$Y = X + X' = x_1 \cdots \varphi_{c(q)}(x_i, x_{i+1}) \cdots x_k.$$

Note that $i(X') < i(X)$ and Y is a linear combination of semistandard monomials which are shorter than X, so by induction hypothesis, $X = Y - X'$ is a linear combination of the standard ones. In the second case, let $x_{i+1} = u$, then we can assume that $u = E_{i_1} \cdots E_{i_t}$, and use induction on t as follows. For $t = 1$, $u = E_j$, define X' as above, then by (2.16)

$$X = X' - x_1 \cdots \phi_q(E_j) x_i \cdots x_k.$$

Thus X can be written as a linear combination semistandard monomials of lower index or shorter length (let us call them lower terms). Assume that for $< t$, X can be written as a linear combination of lower terms. Consider the case $t > 2$. Let $u' = E_{i_2} \cdots E_{i_t}$. Then by (2.16),

$$X = (\cdots E_{i_1} x_i u' \cdots - \cdots \phi_q(E_{i_1}) x_i u' \cdots).$$

Note that the induction hypothesis (on u) is applicable to the second term on the right, thus we have

$$X \equiv (\cdots E_{i_1} x_i u' \cdots) \text{ (modulo lower terms)} .$$

Continue like this t-times, we arrive at

$$X \equiv (x_1 \cdots u x_i \cdots x_k) \text{ (modulo lower terms)} .$$

Since $i(x_1 \cdots u x_i \cdots x_k) < i(X)$, we see that in this case X can also be written as a linear combination of lower terms. Thus by induction, X can be written as a linear combination of the standard monomials. Hence we have proved that the monomials $E_0^\sigma E_1^d$ span $\mathscr{U}^+$. It remains to prove that they are linearly independent.

Suppose that we have a finite sum

$$\sum a_{\sigma,d} E_0^\sigma E_1^d = 0 ,$$

for some $a_{\sigma,d} \in \mathbf{F} \setminus \{0\}$. By clearing denominators of $a_{\sigma,d}$ and factoring out a suitable power of $q-1$, we may assume that all $a_{\sigma,d} \in \mathscr{A}$ and at least one of them does not vanish at $q=1$. By [7], $E_0^\sigma \in \mathscr{U}_{0,\mathscr{A}}$, thus $E_0^\sigma E_1^d \in \mathscr{U}_{\mathscr{A}}^+ = \mathscr{U}^+ \cap \mathscr{U}_{\mathscr{A}}$. By Prop. 2.2, $\mathscr{U}_{\mathscr{A}}/I_1 \cong U(G)$. Denote the image of $E_0^\sigma E_1^d$ under this isomorphism by $e_0^\sigma e_1^d$, then

$$\sum a_{\sigma,d}(1) e_0^\sigma e_1^d = 0 .$$

But by the Poincaré–Birkhoff–Witt theorem of $U(G)$, $e_0^\sigma e_1^d$ are linearly independent over $\mathbf{C}$. Thus we arrived at a contradiction. Hence $E_0^\sigma E_1^d$ are linearly independent. The proof of the lemma is now complete. ∎

The following theorem is an immediate consequence of Lemma 3.1, Lemma 3.2 and Lemma 3.3.

Theorem 3.4. *The monomials $K^\tau F_0^\sigma F_1^d E_0^{\sigma'} E_1^{d'}$ with τ running through all functions $\{1,\ldots,r\} \to \mathbf{Z}$, σ and σ' running through all functions $\{1,\ldots,N\} \to \mathbf{Z}_+$, d and d' running through all functions $\{1,\ldots,mn\} \to \{0,1\}$, form a basis of $\mathscr{U}$.* ∎

The basis of $\mathscr{U}$ described in Thm. 3.4 corresponds to the decomposition $\mathscr{U} = \mathscr{U}^- \mathscr{U}^0 \mathscr{U}^+$. The following theorem corresponds the fact that $\mathscr{U}$ is generated by $\mathscr{U}_0$ and V_q, it gives a q-analog of the Poincaré–Birkhoff–Witt theorem for $\mathscr{U}$ (cf. [5]).

Theorem 3.5. *The monomials $K^\tau E_0^\sigma F_0^{\sigma'} E_1^d F_1^{d'}$ with τ, σ, σ', d, d' as in Thm. 3.4, form a basis of $\mathscr{U}$.*

Proof. We only need to prove that any monomial $u = K^\tau E_0^\sigma E_1^d F_0^{\sigma'} F_1^{d'}$ is a linear combination of the monomials in the theorem, and the monomials in the theorem are linearly independent. We first prove that u is a linear combination of the monomials in the theorem. We only need to work with $E_1^d F_0^{\sigma'}$. Write $E_1^d = x_1 x_2 \cdots x_k$, where $x_i = E_{\gamma_{s(i)}}$ such that $s(1) < s(2) < \cdots < s(k)$. By writing $F_0^{\sigma'}$ as a linear combination of the monomials $F_{i(1)} \cdots F_{i(s)}$, we can assume that $F_0^{\sigma'} = F_{i(1)} \cdots F_{i(s)}$. Now we use induction on k and s to prove that $y = E_0^d F_0^{\sigma'}$ is linear combination of elements of the form $K^a u_0 u_1$, where $K^a \in \mathscr{U}^0$, $u_0 \in \mathscr{U}_0^-$ and u_1 is product of elements of $V_q(\omega_1)$.

For $s=1$, $F_0^{\sigma'} = F_i$, if $k=1$, then by (2.16) we have

$$x_1 F_i = F_i x_1 - \phi_q(F_i) x_1 , \tag{3.6}$$

and the right side is in the desired form. For $k>1$, by using (3.6), we have

$$x_1\cdots x_{k-1}x_kF_i=x_1\cdots x_{k-1}(F_ix_k-\phi_q(F_i)x_k)\,,$$

and by induction of k, we see that the left side can be written as a linear combination of the desired terms for the case $s=1$ and any $k\geqq 1$. *For* $s>1$, by applying the case $s=1$ to $F_{i(1)}$, we have

$$y=x_1\cdots x_kF_{i(1)}F_{i(2)}\cdots F_{i(s)}=\left(\sum \alpha K^a u_0 u_1\right)F_{i(2)}\cdots F_{i(s)}\,,$$

where $\alpha\in\mathbf{F}$, $K^a\in\mathscr{U}^0$, $u_0\in\mathscr{U}_0^-$ and u_1 is a product of elements in $V_q(\omega_1)$. By writing u_1 as linear combination of the E_0^d and use induction on s, we can write

$$u_1F_{i(2)}\cdots F_{i(s)}=\sum \beta K^{a'}u_0'u_1'\,,$$

and we see that y is a linear combination of the desired terms. Hence u is a linear combination of the monomials in the theorem.

It remains to prove that the monomials described by the theorem are linearly independent. The proof of this fact is similar to the proof of Thm. 3.4 by taking into account of the fact that $\mathscr{U}/I_1\cong U(G)$ and the fact that the images of these monomials form a basis of $U(G)$. The proof of Thm. 3.5 is now complete. ■

4. Defining Relations

In this section, we first analyze the generating relations of G as a contragredient Lie superalgebra with the standard Cartan matrix (a_{ij}), where (see [9])

$$a_{ij}=(1+(-1)^{\delta_{i,m}})\delta_{ij}-\delta_{i,\,j+1}-(-1)^{\delta_{i,m}}\delta_{i,\,j-1},\ 1\leqq i,j\leqq r\,,$$

then discuss the relationship between the q-deformation of $U(G)$ given in [4,9] with the deformation of $U(G)$ given in Sect. 2. We assume $m,n\geqq 2$.

Let

$$a'_{ij}=-\delta_{i,\,j+1}-\delta_{i,\,j-1}\,.$$

Then it is easy to see that in addition to the Serre-type relations,

$$\begin{gathered}[h_i,h_j]=0,\quad [e_i,f_j]=\delta_{ij}h_i\,,\\ [h_i,e_j]=a_{ij}e_j,\quad [h_i,f_j]=-a_{ij}f_j\,,\\ (ade_i)^{1-a'_{ij}}e_j=(adf_i)^{1-a'_{ij}}f_j=0,\ \text{if } i\neq j\,,\\ [e_m,e_m]=[f_m,f_m]=0\,,\end{gathered}\tag{4.1}$$

the following relations hold (compare with [4, 9])

$$[e_{m-1},[e_m,[e_{m+1},e_m]]]=[f_{m-1},[f_m,[f_{m+1},f_m]]]=0\,.\tag{4.2}$$

The question is whether (4.1) and (4.2) form a complete set of generating relations of G. We approach this problem by studying the G_0-module S^2G_1.

Lemma 4.1. *As a G_0-module,*

$$\begin{aligned}S^2G_1\cong{}& V(\lambda_1+\lambda_2)\oplus V(2\lambda_1)\oplus V(2\lambda_2)\oplus V(\lambda_3)\\ &\oplus V(\lambda_4)\oplus V(\lambda_5)\oplus V(\lambda_6)\oplus V(0)\,,\end{aligned}$$

where $V(\lambda)$ denotes the highest weight simple G_0-module of highest weight λ, and the λ_i's are given by their numerical marks with respect to the simple root system chosen in in (2.1) (*see also* [5, *p.* 83]) *by the following*:

$$\lambda_1=(1,0,\ldots,0,1), \qquad \lambda_2=(\ldots,0,1;\ 1,0,\ldots),$$
$$\lambda_3=(0,1,0,\ldots,0,1,0), \qquad \lambda_4=(\ldots,0,1,0;\ 0,1,0,\ldots),$$
$$\lambda_5=(1,0,\ldots,0,1;\ 0\ldots), \quad \lambda_6=(\ldots,0;\ 1,0,\ldots,0,1).$$

Proof. We have

$$G_1\otimes G_1\cong V(\lambda_1)\otimes V(\lambda_1)\oplus V(\lambda_2)\otimes V(\lambda_2)$$
$$\oplus V(\lambda_1)\otimes V(\lambda_2)\oplus V(\lambda_2)\otimes V(\lambda_1).$$

Thus,

$$S^2G_1\cong S^2V(\lambda_1)\oplus S^2V(\lambda_2)\oplus V(\lambda_1)\cdot V(\lambda_2),$$

where $V(\lambda_1)\cdot V(\lambda_2)$ denotes the symmetric component of

$$V(\lambda_1)\otimes V(\lambda_2)\oplus V(\lambda_2)\otimes V(\lambda_1).$$

We claim that

$$S^2V(\lambda_1)\cong V(2\lambda_1)\oplus V(\lambda_3), \tag{4.3}$$
$$S^2V(\lambda_2)\cong V(2\lambda_2)\oplus V(\lambda_4), \tag{4.4}$$
$$V(\lambda_1)\cdot V(\lambda_2)\cong V(\lambda_1+\lambda_2)\oplus V(\lambda_5)\oplus V(\lambda_6)\oplus V(0). \tag{4.5}$$

We will prove (4.3); the proofs for (4.4) and (4.5) are similar.

From our notations in Sect. 2, we see that $e_{1,m+n}$ is a highest weight vector of $V(\lambda_1)$, and the vector $e_{1,m+n}\otimes e_{1,m+n}$ generates a copy of $V(2\lambda_1)$ in $S^2V(\lambda_1)$. Computation shows that

$$v=(e_{1,r}\otimes e_{2,m+n}+e_{2,m+n}\otimes e_{1,r})-(e_{1,m+n}\otimes e_{2,r}+e_{2,r}\otimes e_{1,m+n}) \tag{4.6}$$

is a maximal vector (i.e. $G_0^+(v)=0$) of weight λ_3 in $S^2V(\lambda_1)$. Thus there is a copy of $V(\lambda_3)$ in $S^2V(\lambda_1)$. By using the Weyl's formula, we find that $\dim V(2\lambda_1)=mn(m+1)(n+1)/4$ and $\dim V(\lambda_3)=mn(m-1)(n-1)/4$. Hence $\dim V(2\lambda_1)+\dim V(\lambda_3)=\dim S^2V(\lambda_1)$. Thus (4.3) follows. The lemma follows from (4.3)–(4.5). ∎

As a consequence of Lemma 4.1, we conclude that the G_0-module homomorphism

$$\varphi\colon S^2G_1\to G_0\cong sl(m)\oplus sl(n)\oplus \mathbf{C}$$

is given by

$$V(\lambda_1+\lambda_2)\oplus V(2\lambda_1)\oplus V(2\lambda_2)\oplus V(\lambda_3)\oplus V(\lambda_4)\to 0, \tag{4.7}$$
$$V(\lambda_5)\to sl(m),\quad V(\lambda_6)\to sl(n),\quad V(0)\to \mathbf{C}. \tag{4.8}$$

The algebra G_0, the G_0-module $V(\lambda_1)\oplus V(\lambda_2)$, the G_0-module homomorphism φ defined by (4.7) and (4.8) together with (1.1) define G completely. We check that (4.1), (4.2) and (1.1) together imply these conditions as follows.

It is easy to see that (4.1) implies that e_m generates a copy of $V(\lambda_1)$ and f_m generates a copy of $V(\lambda_2)$. We claim that

$$[e_m, f_m]=h_m \Rightarrow \begin{cases} V(\lambda_1)+V(\lambda_2)\to 0\,, \\ V(\lambda_5)\to sl(m)\,, \\ V(\lambda_6)\to sl(n)\,, \\ V(0)\to \mathbf{C}\,. \end{cases} \tag{4.9}$$

$$[e_m, e_m]=[f_m, f_m]=0 \Leftrightarrow V(2\lambda_1)\oplus V(2\lambda_2)\to 0\,. \tag{4.10}$$

$$(4.2)\Leftrightarrow V(\lambda_3)\oplus V(\lambda_4)\to 0\,. \tag{4.11}$$

The equivalence of (4.10) follows from the fact that $e_m\otimes e_m$ and $f_m\otimes f_m$ are generators of $V(2\lambda_1)$ and $V(2\lambda_2)$ respectively. The element

$$\begin{aligned} v_3=&(e_{m,m+1}\otimes e_{m-1,m+2}+e_{m-1,m+2}\otimes e_{m,m+1})\\ &-(e_{m-1,m+1}\otimes e_{m,m+2}+e_{m,m+2}\otimes e_{m-1,m+1}) \end{aligned} \tag{4.12}$$

is a generator of $V(\lambda_3)$ (a lowest weight vector), the element

$$\begin{aligned} v_4=&(e_{m+1,m}\otimes e_{m+2,m-1}+e_{m+2,m-1}\otimes e_{m+1,m})\\ &-(e_{m+1,m-1}\otimes e_{m+2,m}+e_{m+2,m}\otimes e_{m+1,m-1}) \end{aligned} \tag{4.13}$$

is a generator of $V(\lambda_4)$ (a highest weight vector), φ maps $V(\lambda_3)$ and $V(\lambda_4)$ to 0 is equivalent to $\varphi(v_3)=\varphi(v_4)=0$, which in turn is equivalent to (4.2). Hence (4.11) holds. Similarly, one can prove (4.9) by using the generators of the simple G_0-modules in (4.9). We list a highest weight vector for each of the simple G_0-modules in (4.9), but omit the proof:

$$V(\lambda_1+\lambda_2):\ e_{1,m+n}\otimes e_{m+1,m}+e_{m+1,m}\otimes e_{1,m+1}\,, \tag{4.14}$$

$$V(\lambda_5):\ \sum_{i=1}^{n}(e_{1,m+i}\otimes e_{m+i,m}+e_{m+i,m}\otimes e_{1,m+i})\,, \tag{4.15}$$

$$V(\lambda_6):\ \sum_{j=1}^{m}(e_{j,m+n}\otimes e_{m+1,j}+e_{m+1,j}\otimes e_{j,m+n})\,, \tag{4.16}$$

$$V(0):\ \sum_{\substack{1\le i\le m\\ m+1\le j\le m+n}}(e_{ij}\otimes e_{ji}+e_{ji}\otimes e_{ij})\,. \tag{4.17}$$

Remark. Condition (4.2) must be added to the Serre-type relations to define G as a contragredient Lie superalgebra. This reflects the fact that φ: $V(\lambda_3)\oplus V(\lambda_4)\to 0$ does not follow from the Serre-type relations. We further note that (similarly, for f_{m-1}, f_m and f_{m+1}):

$$\begin{aligned} [e_{m-1},[e_m,[e_{m+1},e_m]]]=&[[e_{m-1},e_m],[e_{m+1},e_m]]\\ &+[e_m,[e_{m-1},[e_{m+1},e_m]]]\,. \end{aligned}$$

The first term on the right side (call it x_1) was introduced in [4], the second term on the right side (call it x_2) was introduced in [8]. If we denote the left side by x, then it is easy to see that ($e_m^2=0$ and $x=0$) iff ($e_m^2=0$ and $x_1=0$), and iff (($e_m^2=0$ and $x_2=0$). Thus any one of the conditions $x=0$ or $x_1=0$ or $x_2=0$ (and similar conditions for the f_i's) can serve as one of the extra relations that are needed in the definition of G as a contragredient Lie superalgebra.

Lemma 4.2. *One can choose $\varphi_{c(q)}$ such that $\mathscr{U}=U_{\varphi_{c(q)}}(G)$ is generated by E_i, F_i, $K_i^{\pm}$, $1\leqq i\leqq r$, with generating relations* (2.7) *(allow $i=m$ for E_i and F_i) and*

$$E_m^2=F_m^2=0\ , \tag{4.18}$$

$$ad_q(E_{m-1})\,[E_m, ad_q(E_{m+1})E_m]=0\ , \tag{4.19}$$

$$ad_q(F_{m-1})\,[F_m, ad_q(F_{m+1})F_m]=0\ , \tag{4.20}$$

where we have used ad_q and $[x,y]=xy+yx$ to shorten our notations.

Proof. Let us denote the $U_q(G_0)$ action on S^2V_q also by ϕ_q. Note that as a $U_q(G_0)$-module, S^2V_q decomposes as in Lemma 4.1. Thus in order to define $\varphi_{c(q)}$, we only need to specify the images of a set of generators of each component (which are given in (4.12)–(4.17) by using the bases constructed in (2.12) and (2.13)), and require that

$$\phi_q(u)v_0\rightarrow[u, \varphi_{c(q)}(v_0)]\ , \tag{4.21}$$

where $u\in U_{q,\mathscr{A}}(G_0^-)$ (or $\in U_{q,\mathscr{A}}(G_0^+)$), and v_0 is a highest weight vector (or a lowest weight vector) of some simple component of S^2V_q. We let $c_*^{m+1,m,m,m+1}(q)=1$. Then

$$\varphi_{c(q)}(E_m, F_m)=H_m\ .$$

This is sufficient to define the images of the highest weight vectors of the components $V_q(\omega_1\cdot\omega_2)$, $V_q(\omega_5)$, $V_q(\omega_6)$ and $V_q(\omega_0)$, where $V_q(\omega_i)$ is the q-deformation of $V(\lambda_i)$ (note $\omega_0=(1,\ldots,1)$). The images of the highest (lowest) weight vectors of the other components are given by (4.18)–(4.20). ∎

Let $U_q(G)$ be the associative algebra (with 1) over $\mathbf{F}$ generated by E_i, F_i, $K_i^{\pm}$, $1\leqq i\leqq r$, with relations (2.7), (4.18)–(4.20). Let $U_{q,\mathscr{A}}(G)$ be the $\mathscr{A}$-subalgebra of $U_q(G)$ generated by $U_{q,\mathscr{A}}(G_0)$ together with E_m and F_m. Then by Lemma 4.2, we have

Theorem 4.3. (i) *$U_{q,\mathscr{A}}(G)/I_1\cong U(G)$, where I_1 is the ideal of $U_{q,\mathscr{A}}(G)$ generated by $q-1$ and K_i-1, $1\leqq i\leqq r$.* (ii) *A Poincaré–Birkhoff–Witt type theorem (Thm.* 3.5) *holds for $U_q(G)$.* ∎

Note added in proof. After the submission of this paper, the author noted reference [10], in which it is indicated that a Poincaré-Birkhoff–Witt type theorem holds for the q-deformation of $U(sl(m,n))$ defined in [4, 9].

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