Commun. Math. Phys. 162, 371–397 (1994)

Communications in
Mathematical
Physics

Analysis of the Static Spherically Symmetric $SU(n)$-Einstein-Yang-Mills Equations

H. P. Künzle

Department of Mathematics, University of Alberta, Edmonton, Canada T6G 2G1.
E-mail: hkunzle@vega.math.ualberta.ca

Received: 27 April 1993/in revised form: 12 October 1993

Abstract: The singular boundary value problem that arises for the static spherically symmetric $SU(n)$-Einstein-Yang-Mills equations in the so-called magnetic case is analyzed. Among the possible actions of $SU(2)$ on a $SU(n)$-principal bundles over space-time there is one which appears to be the most natural. If one assumes that no electrostatic type component is present in the Yang-Mills fields and the gauge is suitably fixed a set of $n-1$ second order and two first order differential equations is obtained for $n-1$ gauge potentials and two metric components as functions of the radial distance. This system generalizes the one for the case $n=2$ that leads to the discrete series of the Bartnick-Mckinnon and the corresponding black hole solutions. It is highly nonlinear and singular at $r=\infty$ and at $r=0$ or at the black hole horizon but it is known to admit at least one series of discrete solutions which are scaled versions of the $n=2$ case. In this paper local existence and uniqueness of solutions near these singular points is established which turns out to be a nontrivial problem for general n. Moreover, a number of new numerical soliton (i.e. globally regular) numerical solutions of the $SU(3)$-EYM equations are found that are not scaled $n=2$ solutions.

1. Introduction

The coupling of Einstein's general relativity with Yang-Mills gauge theories leads to complicated nonlinear systems of equations even in the static spherically symmetric case. If the gauge group is $SU(2)$ and the "Coulomb" part of the gauge potential is set to zero and asymptotical flatness is imposed the resulting singular boundary value problem admits a sequence of regular solutions parametrized by the number of zeros of a convenient gauge potential component. These solutions were numerically discovered by Bartnik and Mckinnon [3] and their existence was proved analytically by Smoller et al. [18–20] for some range of the initial conditions for a suitable gauge potential at the center or at the black hole horizon. Such discrete sequences of solutions have since also been found numerically for a number of other field theories coupled to gravitation like EYM-Higgs fields [1], dilatons [15, 10, 8, 16] and skyrmions [5]. For the $SU(2)$-EYM theory it has turned out that sequences of black hole solutions exist for arbitrary radii of the horizon and appear to approach regular solutions as the black hole radius tends to zero. Most likely these solutions are not stable against time dependent perturbations [22, 7]. There remain many questions, in particular, about the behavior of the "higher energy" solutions. But it is known that their total mass is always bounded and a (not very good) upper bound has been analytically established [13]. Numerical evidence suggests that this upper bound is in fact equal to one (in suitable units) and is approached asymptotically by the solutions whose gauge potential oscillates more and more often.

In this paper we analyze the equations for an Einstein-Yang-Mills system with gauge group $SU(n)$ in a static space-time obtained if the apparently most natural action of the $SU(2)$ group on the principal bundle leaves the gauge connection invariant. We assume that there is no Coulomb type component, i.e. the timelike components of the gauge potential vanishes. One then arrives at a system of $n-1$ second order and 2 first order ordinary differential equations for $n-1$ surviving gauge potentials and two metric functions with singular boundary conditions both at $r=\infty$ and at either $r=0$ or $r=r_H$, the black hole horizon [12]. It is easy to see that this system admits some special solutions by scaling the radial variable as well as most of the dependent variables and reducing it to the case $n=2$ for which existence has been proved. We demonstrate numerically, that there must also be solutions more general than these special scaled ones. One might, in fact, have conjectured that the solutions would be parametrized now by the number of zeros of each of the $n-1$ gauge potentials. However, numerical solutions exist that prove this conjecture wrong. Even in the $SU(3)$ case it is therefore quite difficult to get an idea of the structure of the set of global solutions.

The bulk of the paper is in fact concerned just with the preliminary problem to establish that the local initial conditions at the end points of the interval in r can always be solved uniquely thus showing that the "shooting to a fitting point" numerical technique can always be applied. It turns out that even this apparently straightforward problem is surprisingly complicated, at least for general n. We establish first what initial conditions can be chosen that determine uniquely a formal power series solution and then show that with these initial data a unique analytic solution of the system exists near $r=0$ and $r=\infty$. The scaling argument mentioned above shows that some of these solutions exist for all $r>0$, but most do not. To prove rigorously that global solutions exist that are not scaled $n=2$ solutions will take much more work. Even numerical exploration of the set of global solutions is very cumbersome, at least with the shooting method, since it is hard to choose an appropriate initial point for Newton's technique in the $(2n-1)$-dimensional parameter space. So far the numerical evidence suggests that the masses of these solutions fall in between the masses of the scaled solutions and are bounded by the same upper limit.

The paper is organized as follows. In Sect. 2 some elementary facts about the radial field equations derived in [12] are recalled and the existence of the scaled "diagonal" solutions for the $SU(n)$ case is proved. Section 3 contains the main part of the paper, namely the proof that a local formal power series solution exists in a neighborhood of the singular boundary points. We also find in the process what parameters can be freely chosen at these endpoints to serve as initial conditions for the local solutions. We generalize, in Sect. 4, the local existence proofs of Smoller et al. [18, 20] to $n>2$ and present some new numerical solutions for the $SU(3)$-EYM theory in Sect. 5.

2. The Radial Field Equation, Scaled Solutions

The Einstein and Yang-Mills equations can be formulated in a fairly coordinate independent form on general spherically symmetric space-times as was done in [12]. If all the fields are static, however, and we only consider regular space-times diffeomorphic to $\mathbb{R}^3$ or only the outside of static black hole space-times then it is only a slight restriction to assume that there exists a global Schwarzschild type coordinate system [1]

$$ds^2 = -Ne^{-2\delta}\, dt^2 + N^{-1}\, dr^2 + r^2(d\theta^2 + \sin\theta d\phi^2)\,,$$

where $N = 1 - 2m/r$ and δ are functions of the radial variable r only. Asymptotical flatness requires that $m(r) = m_\infty + O(1/r)$ as well as $\delta(r) = O(1/r)$ as $r \to \infty$. For regularity at the center it is necessary that $N = 1 + O(r^2)$ and δ finite while at a regular (not extreme) horizon $r = r_H$ we have $N(r_H) = 0$ and $N'(r_H) > 0$. Einstein's equations, $R_{\alpha\beta} = 8\pi\big(T_{\alpha\beta} - \frac{1}{2}T^\lambda_\lambda g_{\alpha\beta}\big)$, then reduce to

$$m' = 4\pi\mu r^2 \quad \text{and} \quad \delta' = -4\pi r N^{-1}(\mu + p_r)\,, \tag{2.1}$$

where $' = d/dr$ and μ is the mass-energy density and p_r the radial pressure.

A static spherically symmetric Yang-Mills field can be given by a potential in the form ([2, 12, 4])

$$A = \tilde{A} + \hat{A}\,,$$

where $\tilde{A} = A_0(r)\,dt + A_r(r)\,dr$ and

$$\hat{A} = \Lambda_1 d\theta + (\Lambda_2 \sin\theta + \Lambda_3 \cos\theta)\, d\phi$$

and $\Lambda_k = \Lambda(\tau_k) = \Lambda(\sigma_k/(2i))$ are the components of an equivariant linear map of $\mathfrak{su}(2)$, into the Lie algebra of the gauge group. We consider here only the case when the gauge group is $SU(n)$, with a "standard" irreducible action of $SU(2)$ on the principal bundle (so that $\Lambda_3 = \operatorname{diag}(n-1, n-3, \ldots, -n+3, -n+1) \in \mathfrak{su}(n)$) and we also assume that the "Coulomb" part of the gauge potential vanishes, i.e. $A_0 = 0$. The gauge can then be chosen (see [12]) such that also $A_r = 0$ and the potential, when written as an anti-Hermitian matrix, becomes

$$A = \hat{A} = \frac{1}{2}\begin{pmatrix} i(1-n)\cos\theta d\varphi & w_1\Theta & 0 & \ldots & 0 \\ -w_1\bar{\Theta} & i(3-n)\cos\theta d\varphi & w_2\Theta & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & 0 & -w_{n-1}\bar{\Theta} & i(n-1)\cos\theta \end{pmatrix}$$

where $\Theta := d\theta - i\sin\theta\, d\phi$, and the w_j are real valued functions of r. The Yang-Mills field is

$$F = \hat{F} = -\frac{i}{2}\operatorname{diag}(f_1, \ldots, f_n)\sin\theta\, d\theta \wedge d\phi \tag{2.2}$$

with

$$f_j := w_j^2 - w_{j-1}^2 + 2j - n - 1 \qquad (j = 1, \ldots, n \text{ with } w_0 = w_n = 0)\,.$$

[1] Nevertheless this *is* a restriction (cf. [14]). It is possible that singularities in the solutions of the differential equations obtained in this coordinate system are due to the function r failing to be monotonically increasing outwards. This possibility was also observed in [16]

The Yang-Mills field equations now take the form

$$r^2 N w_j'' + 2(m - rP)w_j' + \tfrac{1}{2}(f_{j+1} - f_j)w_j = 0\,, \tag{2.3}$$

and the energy density and the pressures are given by

$$4\pi\mu = r^{-2}(NG + P)\,, \qquad 4\pi p_r = r^{-2}(NG - P) \quad \text{and} \quad 4\pi p_\theta = r^{-2}P\,, \tag{2.4}$$

where $G := \sum_{j=1}^{n-1} w_j'^2$ and $P := \frac{1}{4} r^{-2} \sum_{j=1}^{n} f_j^2$.

We thus have a system of $n+1$ ordinary differential equations for the $n+1$ functions $m(r)$, $\delta(r)$ and $w_1(r), \ldots, w_{n-1}(r)$, whereby the equation for δ decouples.

Regularity of the metric at the center requires in particular that μ, p_θ and p_r remain finite as $r \to 0$ so that $G(0) = 0$ whence $w_j'(0) = 0\,\forall j$ and

$$f_j(0) = w_j^2(0) - w_{j-1}^2(0) + 2j - n - 1 = 0 \qquad (j = 1, \ldots, n)\,.$$

These equations are readily solved and give

$$w_j^2(0) = j(n-j)\,. \tag{2.5}$$

In order to derive the boundary conditions for $r \to \infty$ we rewrite (2.3) in terms of the variable $\varrho = 1/r$ and find

$$\varrho^2 N \frac{d^2 w_j}{d\varrho^2} + 2\varrho(1 - 3m\varrho + P)\frac{dw_j}{d\varrho} + \frac{1}{2}(f_{j+1} - f_j)w_j = 0\,. \tag{2.6}$$

Now Eqs. (2.1), (2.3) and (2.6) involve only rational functions of r or ϱ so that a solution at any nonsingular point will be analytic. Moreover, for an asymptotically flat Yang-Mills field we would expect that $F = O(r^{-2})$ as $r \to \infty$ so that by (2.2) the f_j must remain finite as $r \to \infty$. It follows that the w_j have finite limits at infinity. Nothing of physical interest is lost therefore if we now assume that

$$\begin{aligned} m &= m_\infty + m_1 \varrho + O(\varrho^2)\,, \\ w_j &= w_{j,\infty} + w_{j,1}\varrho + O(\varrho^2)\,. \end{aligned} \tag{2.7}$$

Then, however, it follows from (2.6) to lowest order that

$$(f_{j+1}(\infty) - f_j(\infty))w_{j,\infty} = 0 \qquad (j = 1, \ldots, n-1)\,. \tag{2.8}$$

Now suppose that $w_{j,\infty} \neq 0$ for $j \in \{k+1, \ldots, l\}$ and that $w_{\infty,k} = w_{\infty,l} = 0$ for $0 \leq k < l \leq n$. Then one derives as in (2.5) that

$$w_{j,\infty}^2 = (j - k + 1)(l - j + 1) \quad \text{for} \quad k \leq j \leq l\,.$$

However, an analysis of higher powers in (2.6) shows that $w_j(\varrho) \equiv 0$, provided one assumes that the w_j are analytic in ϱ at $\varrho = 0$. The system of Eqs. (2.3) then decouples into several systems of smaller n. Such solutions of an $SU(n)$-EYM theory could be considered reduced to several somewhat trivially superposed $SU(n_j)$-EYM solutions with $\sum n_j = n$. Since we do not consider this case we can assume that also

$$\lim_{r \to \infty} w_j^2(r) = j(n-j) =: \gamma_j\,.$$

Remark. In the globally regular case, this decomposition cannot occur (at least, if we assume analyticity) since $w_j(r) \equiv 0$ is incompatible with the regularity condition (2.5) at $r = 0$.

It is easily seen that if $(m(r), w_1(r), \ldots, w_{n-1}(r))$ is a solution of the system (2.1), (2.3) then also $(m(r), \varepsilon_1 w_1(r), \ldots, \varepsilon_n w_{n-1}(r))$ is a solution for any choice of the $\varepsilon_j \in \{1, -1\}$. Moreover, no physical quantities (Yang-Mills field, energy density, pressures) depend on the choice of the ε_j. We may therefore assume without loss of generality that

$$w_j(0) = \sqrt{j(n-j)} \quad (j = 1, \ldots, n-1). \tag{2.9}$$

The observation of Bartnik and Mckinnon [3] in the $SU(2)$ case that $|w(r)| \leq 1$ for a regular solution of the boundary value problem also generalizes to $SU(n)$. We have the

Proposition 1. *If a solution $(m, w_1, \ldots, w_{n-1})$ of Eqs.* (2.1) *and* (2.3) *is smooth on* $[0, \infty)$ *or on* $[r_H, \infty)$ *and m and all w_j have finite nonzero limits as $r \to \infty$ then*

$$w_j^2(r) \leq w_j^2(\infty) \quad \forall r > 0 (or\, r > r_H) \quad and \quad j = 1, \ldots, n-1\,,$$

with equality occurring only for the trivial solution (Schwarzschild metric and zero YM-*curvature).*

Proof. Let $v_j = \sup\limits_r w_j^2(r)$ and suppose that $w_j(r_j) = \pm\sqrt{v_j}$ for some $r_j \in (0, \infty)$ or (r_H, ∞). Then $w_j(r_j)$ is an absolute maximum (minimum) so that $w_j'(r_j) = 0$ and $w_j''(r_j) \leq 0 (\geq 0)$. It then follows from (2.3) that

$$w_j^2(r_j) \leq 1 + \tfrac{1}{2}\left[w_{j-1}^2(r_j) + w_{j+1}^2(r_j)\right], \tag{2.10}$$

and therefore

$$w_j^2(r_j) \leq 1 + \tfrac{1}{2}\left[\sup_r w_{j-1}^2(r) + \sup_r w_{j+1}^2(r)\right],$$

so that

$$v_j \leq 1 + \tfrac{1}{2}\left[v_{j-1} + v_{j+1}\right] \quad \text{for} \quad 1 \leq j \leq n-1\,, \qquad (v_0 = v_n = 0). \tag{2.11}$$

These conditions together with $v_j \geq 0$ are easily seen to define a bounded convex set B in $\mathbb{R}^{n-1}$. (If $v, u \in B$, $\lambda \in [0,1]$, then $\lambda v_j + (1-\lambda)u_j \leq 1 + \frac{1}{2}\{[\lambda v_{j-1} + (1-\lambda)w_{j-1}] + [\lambda v_{j+1} + (1-\lambda)w_{j+1}]\}$ and $\lambda v_j + (1-\lambda)u_j \geq 0$ so that $\lambda v + (1-\lambda)u \in B$.) If the inequalities in (2.11) are replaced by equalities a system of linear equations is obtained for the v_j that is equivalent to the one for the $w_j^2(0)$ above thus showing that the extremal values are attained at $r = 0$ and $r = \infty$. Now suppose that w_j attains its maximal value $\sqrt{j(n-j)}$ at r_*, where $0 < r_* < \infty$ or $r_H < r_* < \infty$. It then follows from (2.3) that $w_j''(r_*) > 0$, a contradiction unless w_{j+1}^2 and w_{j-1}^2 also attain their maximal value at the same r_*. It follows that all w_j^2 attain their maximal value at the same value r_*. But then all first and second derivatives of all w_j vanish at r_* which is a regular point of the system (2.3, 2.1). So we get only the solution with m and all w_j constant and thus all $f_j = 0$. □

In view of this result it is now natural to scale the variables so that their absolute values are bounded by 1. We thus replace the $w_j(r)$ by

$$u_j(r) := w_j(r)/\sqrt{\gamma_j}\,,$$

and therefore have also, in view of (2.9),

$$u_j(0) = 1\,.$$

Equations (2.3) and (2.1) now become

$$r^2 N u_j'' + 2(m - rP)u_j' + \tfrac{1}{2}(q_{j+1} - q_j)u_j = 0\,, \tag{2.12}$$
$$m' = NG + P\,, \tag{2.13}$$
$$\delta' = -2G/r\,, \tag{2.14}$$

where now

$$\begin{gathered} q_j = \gamma_j u_j^2 - \gamma_{j-1}u_{j-1}^2 + 2j - n - 1\,,\\ G = \sum_{j=1}^{n-1} \gamma_j u_j'^2\,, \qquad P = \tfrac{1}{4} r^{-2} \sum_{j=1}^{n} q_j^2\,. \end{gathered} \tag{2.15}$$

Apart from the well known special solution given by $w_j(r) \equiv 0$ for all $j = 1 \dots (n-1)$ which leads to the Reissner-Nordström metric (and therefore to a black hole) there is another rather special case, namely when all the w_j are proportional, i.e. (ignoring an insignificant sign in each u_j)

$$u_1 = u_2 = \dots = u_{n-1} = u(r)\,.$$

Then (2.12) becomes

$$r^2 N u'' + 2(m - rP)u' + (1 - u^2)u = 0\,,$$

which is the same equation as (2.12) for $n = 2$ except that now

$$\begin{aligned} P &= \tfrac{1}{4} r^{-2} \sum_{j=1}^{n} (2j - n - 1)^2 (1 - u^2)^2 = \tfrac{1}{12}(n-1)n(n+1) r^{-2} (1 - u^2)^2\,,\\ G &= \tfrac{1}{6}(n-1)n(n+1)u'^2\,. \end{aligned}$$

The equation reduce exactly to those for $n = 2$ if we scale them as follows,

$$\begin{aligned} r &= \lambda_n x\,,\\ u(r) &= u(\lambda_n x)\,,\\ m(r) &= \lambda_n \tilde{m}(x) = \lambda_n \tilde{m}(r/\lambda_n)\,,\\ \delta(r) &= \tilde{\delta}(x) = \tilde{\delta}(r/\lambda_n) \end{aligned} \tag{2.16}$$

where $\lambda_n = \sqrt{\tfrac{1}{6}(n-1)n(n+1)}$ (so that $\lambda_2 = 1$).

For this reason we will write the general $SU(n)$ equations also in these scaled variables. They then become with $\mathbf{f} = (u_1, \dots, u_{n-1}, m)$,

$$\mathscr{F}_j[\mathbf{f}](x) := x^2 \tilde{N} \frac{d^2 u_j}{dx^2} + 2(\tilde{m} - \kappa_n x \tilde{P}) \frac{du_j}{dx} + \frac{1}{2}(q_{j+1} - q_j)u_j = 0$$
$$(j = 1, \dots, n-1)\,, \tag{2.17}$$
$$\mathscr{F}_n[\mathbf{f}](x) := \frac{d\tilde{m}}{dx} - \kappa_n(\tilde{N}\tilde{G} + \tilde{P}) = 0\,, \tag{2.18}$$
$$\frac{d\tilde{\delta}}{dx} + \frac{2\kappa_n \tilde{G}}{x} = 0\,, \tag{2.19}$$

where $\kappa_n := \lambda_n^{-2}$, q_j is still given by (2.15), and

$$\tilde{G} = \sum_{j=1}^{n-1} \gamma_j \left(\frac{du_j}{dx}\right)^2, \quad \tilde{P} = \frac{1}{4x^2} \sum_{j=1}^{n} q_j^2 \quad \text{and} \quad \tilde{N} = 1 - \frac{2\tilde{m}}{x}. \tag{2.20}$$

(We drop the tildes for $\tilde{N}$, $\tilde{m}$, $\tilde{P}$, $\tilde{G}$ and $\tilde{\delta}$ from now on when referring to Eq. (2.17) to (2.20).) Since Eq. (2.19) decouples we will not consider it any further, and we write $\mathscr{F} = (\mathscr{F}_1, \ldots, \mathscr{F}_n)$ so that the system (2.17), (2.18) becomes

$$\mathscr{F}[\mathbf{f}](x) = 0. \tag{2.21}$$

In view of the results of Smoller et al. [18–20] we therefore have immediately.

Proposition 2. *There exists a countably infinite family of globally regular solutions of the $SU(n)$-Einstein-Yang-Mills equations on a static spherically symmetric space-time that is diffeomorphic to $\mathbb{R}^4$. For any choice $r_H > 0$ there is also an infinite discrete family of static spherically symmetric solutions regular outside and on a black hole horizon of radius r_H.*

Remark. These global existence proofs were obtained by showing that for certain initial values at $x = 0$ a global solution exists with $m(x)$ tending to a finite limit and $u(x) \to \pm 1$ as $x \to \infty$. That the solutions have the asymptotic behavior (2.7) is not proved yet (although very likely in view of the success of the numerical two-point shooting method that uses these expansions).

In the case $n > 2$ it seems reasonable to expect also solutions for which the different u_j are not equal. But their global existence may be even more difficult to establish analytically. We will present some numerical evidence for their existence in Sect. 5. But first we analyze the system of differential equations locally at the critical points $x = 0$, $x = \infty$ and where $N = 0$ (i.e. at the horizon).

3. Formal Power Series Solutions at the Singular Points

In order to find suitable initial conditions for the system (2.21) at the critical points we first derive the formal power series solutions.

At the center, $x = 0$, let

$$u_i = \sum_{k=0}^{\infty} u_k^i x^k \quad \text{and} \quad m = \sum_{k=0}^{\infty} m_k x^k,$$

where we know already that $u_0^i = 1$ and $u_1^i = 0$. Then (2.18) gives

$$\begin{aligned} m_0 &= m_1 = m_2 = 0, \\ m_{k+1} &= \frac{\kappa_n}{k+1}\left(P_k + G_k - 2\sum_{l=2}^{k-2} m_{l+1} G_{k-l}\right) \quad (k \geq 2), \end{aligned} \tag{3.1}$$

$$\begin{aligned} -\gamma_{i-1} u_{k+1}^{i-1} &+ [2\gamma_i - k(k+1)] u_{k+1}^i - \gamma_{i+1} u_{k+1}^{i+1} \\ &= \sum_{l=2}^{k-1} \left(\tfrac{1}{2}\gamma_{i+1} u_l^{i+1} u_{k-l+1}^{i+1} - \gamma_j u_l^i u_{k-l+1}^i + \tfrac{1}{2}\gamma_{i-1} u_l^{i-1} u_{k-l+1}^{i-1}\right. \\ &\quad \left. + u_l^i\left(\tfrac{1}{2} q_{k-l+1}^{i+1} - \tfrac{1}{2} q_{k-l+1}^i - 2l(l-2) m_{k-l+2} - 2\kappa_n l P_{k-l+1}\right)\right), \end{aligned} \tag{3.2}$$

where (always for $k \geq 2$)

$$G_k = \sum_{i=1}^{n-1} \gamma_i \sum_{l=1}^{k-1} (l+1)(k-l+1) u_{l+1}^i u_{k-l+1}^i ,$$

$$P_k = \tfrac{1}{4} \sum_{i=1}^{n} \sum_{l=2}^{k} q_l^i q_{k-l+2}^i , \tag{3.3}$$

$$q_k^i = \sum_{l=0}^{k} (\gamma_i u_l^i u_{k-l}^i - \gamma_{i-1} u_l^{i-1} u_{k-l}^{i-1}) . \tag{3.4}$$

Equation (3.1) determines m_{k+1} in terms of $m_3, \ldots, m_{k-1}$ and all $u_2^i, \ldots, u_k^i$, but Eq. (3.2) requires solving a tridiagonal linear system

$$(\mathbf{A} - k(k+1)\mathbf{I})\mathbf{u}_{k+1} = \mathbf{b}_{k+1} \tag{3.5}$$

for each $k \geq 2$, where $\mathbf{u}_k = (u_k^1, \ldots, u_k^{n-1})^T$,

$$A_j^i = (2\delta_j^i - \delta_j^{i+1} - \delta_j^{i-1})\gamma_j , \qquad (\gamma_0 = \gamma_n = 0) \tag{3.6}$$

or

$$\mathbf{A} = \begin{pmatrix} 2\gamma_1 & -\gamma_2 & 0 & \ldots & 0 & 0 \\ -\gamma_1 & 2\gamma_2 & -\gamma_3 & \ldots & 0 & 0 \\ 0 & -\gamma_2 & 2\gamma_3 & \ldots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \ldots & -\gamma_{n-2} & 2\gamma_{n-1} \end{pmatrix},$$

and $\mathbf{b}_{k+1}$ is the $(n-1) \times 1$-matrix representing the right-hand side of Eq. (3.2). It can be written in the form

$$b_{k+1}^i = -\sum_{l=1}^{k-2} \Bigg[u_{l+1}^i M_{k,l} + \sum_{j=1}^{n-1} \Bigg(\tfrac{1}{2} A_j^i u_{l+1}^j u_{k-l}^j + u_{l+1}^i A_j^i u_{k-l}^j + \tfrac{1}{2} u_{l+1}^i \sum_{r=2}^{k-l-2} A_j^i u_r^j u_{k-l-r}^j \Bigg) \Bigg]$$

with

$$M_{k,l} := 2(l+1)\left[(l-1)m_{k-l+1} + \kappa_n P_{k-l}\right] .$$

We need to show that this system can always be solved and that there are the right number of initial data. This is achieved by the

Theorem 1. *The recurrence relations* (3.1), (3.2) *determine uniquely all coefficients* m_k *and* u_k^i *for* $k > n$ *once* $n-1$ *arbitrary parameters have been chosen, one for each equation with* $k = 1$ *to* $k = n-1$.

The proof consists of several steps. We need to show that the $(n-1) \times (n-1)$ coefficient matrix $\mathbf{A} - k(k+1)\mathbf{I}$ has rank $n-2$ for $k = 2, \ldots, n$ and is nonsingular for $k > n$. Moreover, it must be shown that the vector $\mathbf{b}_k$ lies in the left kernel of $\mathbf{A} - k(k+1)\mathbf{I}$ for $k < n$. Finally, it will be convenient to make a systematic choice of the free parameters that will serve as the initial data of the differential equation at $x = 0$.

Lemma 1. *The matrix* $\mathbf{A}$ *has the eigenvalues* $1 \cdot 2, 2 \cdot 3, \ldots, (n-1) \cdot n$.

Proof. Let $\chi_j(\lambda) := \det(\lambda \mathbf{I}_j \mathbf{A}_j)$, where $\mathbf{A}_j$ is the upper left $(j \times j)$-submatrix of $\mathbf{A}$. Then, by the cofactor expansion along the last column of $\lambda \mathbf{I}_j - \mathbf{A}_j$,

$$\chi_j(\lambda) = (\lambda - 2\gamma_j)\chi_{j-1}(\lambda) - \gamma_j \gamma_{j-1} \chi_{j-2}(\lambda). \tag{3.7}$$

This recurrence relation is of the form of the one for the dual Hahn polynomials [11],

$$R_j(\lambda) = R_j(\lambda; \alpha, \beta, N) \qquad (k = 0, 1, \ldots, N-1)$$

defined by

$$-\lambda R_j(\lambda) = D_j R_{j-1}(\lambda) - (B_j + D_j) R_j(\lambda) + B_j R_{j+1}(\lambda) \tag{3.8}$$

with $R_0(\lambda) = 1$ (and $R_{-1}(\lambda) = R_N(\lambda) = 0$), where $B_j = (N-1-j)(\alpha+1+j)$ and $D_j = j(N + \beta - j)$. For $\hat{R}_j(\lambda) = (-1)^j B_0 B_1 \ldots B_{j-1} R_j(\lambda)$ the recurrence relation becomes

$$-\lambda \hat{R}_j(\lambda) = -B_{j-1} D_j \hat{R}_{j-1}(\lambda) - (B_j + D_j)\hat{R}_j(\lambda) - \hat{R}_{j+1}(\lambda)$$

which agrees with (3.7) if $\alpha = 1$, $\beta = -1$ and $N = n$. It follows that

$$\chi_j(\lambda) = (-1)^j (j+1)!(n-1)(n-2)\ldots(n-j) R_j(\lambda),$$

in particular,

$$\chi(\lambda) = \chi_{n-1}(\lambda) = (-1)^{n-1} n!(n-1)! R_{n-1}(\lambda; 1, -1, n).$$

However, by Eq. (1.19) and (1.1) of [11] we have

$$\begin{aligned} R_{n-1}(k(k+1)) &= Q_k(n-1) \\ &= {}_3F_2(-k, -n+1, k+1; 2, -n+1; 1) = {}_2F_1(-k, k+1; 2; 1) \\ &= \frac{\Gamma(2)\Gamma(1)}{\Gamma(2+k)\Gamma(1-k)} = 0 \quad \text{for} \quad k = 1, 2, \ldots, \end{aligned}$$

where from now on $R_i(\lambda) := R_i(\lambda; 1, -1, n)$ and $Q_k(i) : Q_k(i; 1, -1, n)$. It follows that $\chi(\lambda)$ vanishes when $\lambda = k(k+1)$ for $k = 1, 2, \ldots, n-1$. $\square$

So far we have shown that the $\mathbf{u}_k$ are uniquely determined by (3.2) for $k > n$. Moreover, when $k = 2$ or 3 the right-hand side of (3.2) vanishes so that $\mathbf{u}_2$ and $\mathbf{u}_3$ are determined up to one new parameter each. However, if $3 < k \leq n$ it must still be shown that the linear system is consistent. This is best done by introducing a new basis in the vector space of the $\mathbf{u}_k$, also constructed with the help of the Hahn polynomials. With this method it is also easier to pick the free parameters in a systematic way.

Lemma 2. *The right and left eigenvectors*

$$\mathbf{v}_k = (v_k^1, \ldots, v_k^{n-1})^T \quad \textit{and} \quad \mathbf{w}^k = (w_1^k, \ldots, w_{n-1}^k)$$

of the matrix $\mathbf{A}$ *to the eigenvalue* $k(k+1)$ *are given by*

$$v_k^i = \frac{n-1}{n-i} Q_k(i-1) = \frac{n-1}{n-i} {}_3F_2(-k, -i+1, k+1; 2, -n+1; 1) \tag{3.9}$$

and

$$w_i^k = \frac{1}{n-1} \gamma_i v_k^i,$$

respectively. They satisfy the orthogonality relation

$$\mathbf{w}^k\mathbf{v}_l = \langle \mathbf{v}_k, \mathbf{v}_l \rangle = d_k \delta^k_l = \frac{(n+k)!(n-k-1)!}{(n-1)!(n-2)!k(k+1)(2k+1)}\, \delta^k_l$$

with respect to the scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle := \frac{1}{n-1} \sum_{i=1}^{n-1} \gamma_i x^i y^i \,. \tag{3.10}$$

Proof. For the right eigenvectors we solve $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$ again by recursion and find (choosing $v^1(\lambda) = 1$)

$$v^i(\lambda) = \gamma_i^{-1}[(2\gamma_{i-1} - \lambda)v^{i-1} - \gamma_{i-2}v^{i-2}] \qquad (2 \le i \le n-2)\,, \tag{3.11}$$

so that $v_i(\lambda)$ is a polynomial in λ of degree $i-1$. The last equation is then a polynomial of degree $n-1$, proportional to the characteristic polynomial of A that must vanish. Equation (3.11) can be brought into the form (3.8) by putting $v^i = \theta_i \tilde{v}^i$, $\theta_0 = 1$ and adjusting the factors θ_i. We find

$$v^i(\lambda) = \frac{n-1}{n-i} R_{i-1}(\lambda; 1, -1, n) \qquad (i = 1, \ldots, n-1)$$

so that we obtain for the eigenvector $\mathbf{v}_k$ to the eigenvalue $\lambda = k(k+1)$,

$$\begin{aligned} v^i_k &= \frac{n-1}{n-i} R_{i-1}(k(k+1)) = \frac{n-1}{n-i} Q_k(i-1) \\ &= \frac{n-1}{n-i}\, {}_3F_2(-k, -i+1, k+1; 2, -n+1; 1)\,. \end{aligned} \tag{3.12}$$

Note that, in particular, $v^i_1 = 1$ and $v^1_k = 1\, \forall i, k$ so that the eigenvector to $\lambda = 2$ is $\mathbf{v} = (1, \ldots, 1)^T$ (which can also be verified directly).

The left eigenvectors $\mathbf{w}^k$ of $\mathbf{A}$ are derived in a very similar manner and come out proportional to

$$w^k_i = (i+1)\, {}_3F_2(-k, -i+1, k+1; 2, -n+1; 1) = \frac{1}{n-1} \gamma_i v^i_k \,. \tag{3.13}$$

Since $\mathbf{w}^k \mathbf{v}_l = 0$ for $k \neq l$ the right eigenvectors $\mathbf{v}_k$ are orthogonal with respect to the scalar product (3.10) in $\mathbb{R}^{n-1}$. Directly from the recurrence relation for the Hahn polynomials ([11], Eq. (1.2)) one can derive the normalization

$$\begin{aligned} d_k := \mathbf{w}^k \mathbf{v}_k = \langle \mathbf{v}_k, \mathbf{v}_k \rangle &= (n-1) \sum_{i=1}^{n-1} \frac{i}{n-i} Q_k(i-1)^2 \\ &= \frac{(n+k)!(n-k-1)!}{(n-1)!(n-2)!k(k+1)(2k+1)} \,. \quad \square \end{aligned}$$

It is now clear that for $k = 2, \ldots, n$ the coefficient matrix of (3.5) has a 1-dimensional left and right kernel. Since $\mathbf{b}_2 = \mathbf{b}_3 = 0$ by (3.2) it follows that

$$\mathbf{u}_2 = \beta_1 \mathbf{v}_1 \quad \text{and} \quad \mathbf{u}_3 = \beta_2 \mathbf{v}_2 \,, \tag{3.14}$$

while

$$\mathbf{u}_{k+1} = \overset{*}{\mathbf{u}}_{k+1} + \beta_k \mathbf{v}_k \quad \text{for} \quad k = 2, \ldots, n-1 \,,$$

provided

$$\mathbf{w}^k \mathbf{b}_{k+1} = 0 \quad \text{for} \quad k = 2, \ldots, n-1\,. \tag{3.15}$$

Here $\mathbf{u}^*_{k+1}$ is a special solution and $\beta_2, \ldots, \beta_{n-1}$ are arbitrary parameters. However, since $\mathbf{b}_k$ will depend on the $\beta_1, \ldots, \beta_{k-2}$ it is best to fix the choice of $\mathbf{u}^*_k$ by imposing an additional condition on $\mathbf{u}_k$ before attempting to prove (3.15). We choose

$$\mathbf{w}^k \mathbf{u}_{k+1} = d_k \beta_k \quad \text{for} \quad k = 1, \ldots, n-1\,, \tag{3.16}$$

which is compatible with (3.14).

Lemma 3. *Whenever* (3.15) *is satisfied then the linear system*

$$\left\{ \begin{array}{c} (\mathbf{A} - k(k+1)\mathbf{I})\mathbf{u}_{k+1} = \mathbf{b}_{k+1} \\ \mathbf{w}^k \mathbf{u}_{k+1} = d_k \beta_k \end{array} \right\} \quad \textit{for} \quad k = 1, \ldots, n-1$$

has a unique solution for given β_k.

Proof. Uniqueness will follow if we show that the coefficient matrix $\tilde{\mathbf{A}}_k$ of the combined system has rank $n-1$. Now $\mathbf{A}_k = \mathbf{A} - k(k+1)\mathbf{I}$ has rank $n-2$. Suppose that the last row $\mathbf{w}^k$ of $\tilde{\mathbf{A}}_k$ is a linear combination of the other rows, i.e. $\mathbf{w}^k = \lambda \mathbf{A}_k$. Then $0 = \mathbf{w}^k \mathbf{A}_k = \lambda \mathbf{A}_k \mathbf{A}_k$. But $\mathbf{A}$ has $n-1$ distinct eigenvalues and is thus diagonalizable so that it follows that $\lambda = \mu \mathbf{w}^k$, whence $\mathbf{w}^k = \mu \mathbf{w}^k \mathbf{A}_k = 0$, a contradiction. To prove existence suppose that $\tilde{\mathbf{x}}^T \tilde{\mathbf{A}}_k = 0$, i.e. $\mathbf{x}^T \mathbf{A}_k + x_n \mathbf{w}^k = 0$, then it follows similarly that $x_n = 0$ and $\mathbf{x}^T = \nu \mathbf{w}^k$, whence $\tilde{\mathbf{x}}^T \tilde{\mathbf{b}}_{k+1} = \nu \mathbf{w}^k \mathbf{b}_{k+1} = 0$. □

To prove that (3.15) is satisfied for $k = 3, \ldots, n-1$ is more easily done in terms of the bases $\{\mathbf{v}_k\}$ and $\{\mathbf{w}^k\}$. We have

$$A^i_j = \sum_{k=1}^{n-1} d_k^{-1} k(k+1) v^i_k w^k_j\,,$$

and let

$$\mathbf{u}_{k+1} = \sum_{l=1}^{n-1} U^l_k \mathbf{v}_l \quad (k = 1, 2, \ldots)\,. \tag{3.17}$$

Equation (3.2) then becomes

$$\begin{aligned} &(a-k)(a+k+1)U^a_k \\ &= -\sum_{j=1}^{k-2} \left\{ U^a_j M_{k,j} + \sum_{r,s=1}^{n-1} \left[\left(\tfrac{1}{2}\, a(a+1) + s(s+1)\right) d^a_{rs} U^r_j U^s_{k-j-1} \right] \right. \\ &\quad \left. + \tfrac{1}{2} \sum_{l=2}^{k-j-2} \sum_{p,r,s,t=1}^{n-1} [p(p+1) d^a_{rp} d^p_{st} U^r_j U^s_{l-1} U^t_{k-j-l-1}] \right\}, \end{aligned} \tag{3.18}$$

where

$$d^a_{rs} := d_a^{-1} \sum_{i=1}^{n-1} w^a_i v^i_r v^i_s\,, \tag{3.19}$$

and (3.16) gives

$$U^k_k = \beta_k\,. \tag{3.20}$$

The consistency condition (3.15) for (3.18) is now obtained by putting $a = k$ in (3.18).

It is clear from (3.18) that

$$U_1^a = 0 \quad \text{for} \quad a > 1 \quad \text{and} \quad U_2^a = 0 \quad \text{for} \quad a > 2\,.$$

We will now show by induction on k that

$$U_k^a = 0 \quad \text{for} \quad a > k \quad (1 \le k < n)\,, \tag{3.21}$$

and that the right-hand side of (3.18) vanishes if $a = k$, thus proving consistency of the linear system. This can be done since strategic components of d^a_{rs} are zero.

Lemma 4.

$$c_{ijk} := \sum_{r=1}^{n-1} \frac{r}{(n-r)^2}\, Q_i(r-1)Q_j(r-1)Q_k(r-1) = 0 \quad \textit{if} \quad i + j \le k \tag{3.22}$$

for all $i, j, k \in \{1, \ldots, n-1\}$, *where* $Q_j(r)$ *is as in* (3.12).

Proof. Clearly $c_{ijk} = c_{(ijk)}$, i.e. c_{ijk} is totally symmetric in its indices. From the orthogonality relation given in Lemma 2 we have, since $Q_1(r-1) = (n-r)/(n-1)$,

$$\begin{aligned} c_{1jk} &= \sum_{r=1}^{n-1} \frac{r}{(n-r)^2}\,\frac{n-r}{n-1}\, Q_j(r-1)Q_k(r-1) \\ &= \frac{1}{n-1} \sum_{r=1}^{n-1} \frac{r}{n-r}\, Q_j(r-1)Q_k(r-1) \\ &= \frac{1}{(n-1)^2}\, d_j \delta_{jk} = 0 \quad \text{if} \quad j \ne k\,, \end{aligned}$$

in particular, if $1+j \le k$. Assume now that (3.22) is true for all $i = 1, 2, \ldots, I < n-1$. To prove the induction step we use the recurrence relation for the Hahn polynomials (Eq. (1.2) of [11]),

$$-rQ_i(r) = d_i Q_{i-1}(r) - (b_i + d_i)Q_i(r) + b_i Q_{i+1}(r)\,, \tag{3.23}$$

where

$$b_i = \frac{(i+2)(n-i-1)}{2(2i+1)} \quad \text{and} \quad d_i = \frac{i(i-1)(n+i)}{2i(2i+1)}$$

Applying (3.23) to $Q_i(r-1)$ and then to $Q_k(r-1)$ in (3.22) gives

$$c_{ijk} = [-d_{i-1}c_{i-2,jk} + d_k c_{i-1,k-1} + (b_{i-1} + d_{i-1} - b_k - d_k)c_{i-1,jk} + b_k c_{i-1,jk+1}]/b_{i-1}\,.$$

Choose now $i = I + 1$ in this equation and assume that $(I+1) + j \le k$. Then on the right-hand side, by the induction hypothesis, $c_{I-1,jk} = 0$, since $I - 1 + j < k$, $c_{Ij,k-1} = 0$, since $I + k \le k - 1$, $c_{Ij,k-1} = 0$, since $I + j \le k - 1$, and $c_{Ij,k+1} = 0$, since $I + j < k + 1$. Thus all terms of the right-hand side vanish completing the induction step. □

Now we have, in view of (3.19), (3.12) and (3.13),

$$d^a_{rs} = (n-1)^2 d_a^{-1} c_{ars} = 0 \quad \text{if} \quad r + s \le a\,.$$

Making now the induction hypothesis (3.21) for $k = 1, \ldots, K$ we have for (3.18) with $k = K + 1$ for the first term on the right-hand side

$$-\sum_{j=1}^{K-1} U_j^a M_{K+1,j} = 0 \quad \text{if} \quad a > K + 1\,,$$

since $j < K - 1$ in the sum. Using (3.21) in the second term gives

$$-\sum_{j=1}^{K-1} \sum_{r=1}^{j} \sum_{s=1}^{K-j} \left[\tfrac{1}{2}\, a(a+1) + s(s+1)\right] d^a_{rs} U_j^r U_{K-j}^s\,,$$

but now $r + s \leq j + K - j = K < a$ in this sum so that $d^a_{rs} = 0$ in every term. Similarly, for the last term,

$$-\tfrac{1}{2} \sum_{j=1}^{K-1} \sum_{l=2}^{K-j-1} \sum_{p=1}^{n-1} \sum_{r=1}^{j} \sum_{s=1}^{l-1} \sum_{t=1}^{K-l-j} [p(p+1) d^a_{rp} d^p_{st} U_j^r U_{l-1}^s U_{K-l-j}^t]\,,$$

so that $s + t \leq K - j - 1$, and therefore $d^p_{st} = 0$ unless $p < K - j - 1$. But in the latter case $r + p \leq K - 2 < a$ which means that the factor $d^a_{rp} = 0$. So all terms in the sum individually vanish. This proves the induction step and completes the proof of Theorem 1.

Fortunately, the asymptotic expansion at $x = \infty$ is obtained in a very similar manner. Substituting $x = 1/z$ in (2.21) gives

$$z^2 N \frac{d^2 u_i}{dz^2} + 2z(1 - 3mz + \kappa_n P) \frac{du_i}{dz} + \frac{1}{2} (q_{i+1} - q_i) u_i = 0\,,$$

$$z^2 \frac{dm}{dz} + \kappa_n (NG + P) = 0\,,$$

with

$$G = z^4 \sum_{i=1}^{n-1} \gamma_i \left(\frac{du_i}{dz}\right)^2, \qquad P = \frac{1}{4} z^2 \sum_{i=1}^{n} q_i^2 \quad \text{and} \quad N = 1 - 2mz\,.$$

A power series ansatz,

$$u_i = \sum_{k=0}^{\infty} u_k^i z^k \quad \text{and} \quad m = \sum_{k=0}^{\infty} m_k z^k\,,$$

where we also write m_∞ for m_0 and we can assume $u_0^i = 1$ (since an overall sign can be put in later), leads to the recurrence relations (for $k \geq 0$)

$$m_{k+1} = \frac{-\kappa_n}{k+1} \left(P_k + G_k - 2m_\infty G_{k-1} - 2 \sum_{l=0}^{k-4} m_{l+1} G_{k-l-2} \right) \tag{3.24}$$

$$\begin{aligned} - \gamma_{i-1} u_k^{i-1} &+ [2\gamma_i - k(k+1)] u_k^i - \gamma_{i+1} u_k^{i+1} \\ &= \sum_{l=1}^{k-1} \big(\tfrac{1}{2}\, \gamma_{i+1} u_l^{i+1} u_{k-l}^{i+1} - \gamma_i u_l^i u_{k-l}^i + \tfrac{1}{2}\, \gamma_{i-1} u_l^{i-1} u_{k-l}^{i-1} \\ &\quad + u_l^i \big(\tfrac{1}{2}\, q_{k-l}^{i+1} - \tfrac{1}{2}\, q_{k-l}^i - 2l(l+2) m_{k-l-1} - 2\kappa_n l P_{k-l-2} \big) \big)\,, \end{aligned} \tag{3.25}$$

where q_k^i is given by (3.4),

$$G_k = \sum_{i=1}^{n-1} \gamma_i \sum_{l=0}^{k-2} (l+1)(k-l-1) u_{l+1}^i u_{k-l-1}^i \quad \text{and} \quad P_k = \tfrac{1}{4} \sum_{i=1}^{n} \sum_{l=1}^{k-1} q_l^i q_{k-l}^i \,.$$

The linear system (3.25) is again of the form

$$(\mathbf{A} - k(k+1)\mathbf{I})\mathbf{u}_k = \mathbf{b}_k \,,$$

where $\mathbf{A}$ is as in (3.6) and $\mathbf{b}_k$ is now given by

$$b_k^i = -\sum_{l=1}^{k-1} \left[u_l^i M_{k,l} + \sum_{j=1}^{n-1} \left(\tfrac{1}{2} A_j^i u_l^j u_{k-l}^j + \tfrac{1}{2} u_l^i \sum_{r=0}^{k-l} A_j^i u_r^j u_{k-l-r}^j \right) \right]$$

with

$$M_{k,l} := -2l(l+2)m_{k-l-1} + 2\kappa_n l P_{k-l-2} \,.$$

Again, the coefficient matrix is singular for $k = 1, \dots, n-1$, and the system can be supplemented by the conditions

$$\mathbf{w}^k \mathbf{u}_k = d_k \alpha_k \quad \text{for} \quad k = 1, \dots, n-1 \,. \tag{3.26}$$

Then we have the

Theorem 2. *The recurrence relations* (3.24), (3.25), *together with* (3.26) *determine uniquely all coefficients m_k and u_k^i in terms of the parameters m_∞ and $\alpha_1, \dots, \alpha_{n-1}$.*

Proof. The method is completely analogous to the one in the proof of Theorem 1 once we put

$$\mathbf{u}_k = \sum_{l=1}^{n-1} U_k^l \mathbf{v}_l \quad (k = 1, \dots, n-1) \,. \quad \square$$

Finally, we consider solutions with a *regular black hole horizon* at $x = x_H > 0$, so that $N(x_H) = 0$ and $\nu = dN/dx(x_H) > 0$. While this is also a singular initial value problem finding a formal power series solution $(u_1(x), \dots, u_{n-1}(x), m(x))$ at $x = x_H$ is completely straightforward. If with $t = x - x_H$ we let

$$u_i = \sum_{k=0}^{\infty} u_{i,k} t^k \quad \text{and} \quad m = \sum_{k=0}^{\infty} m_k t^k \,,$$

then $m_0 = \frac{1}{2} x_H$ and the $n-1$ values $u_{i,0} = u_i(x_H)$ can be assigned freely subject to the condition that

$$\nu = \frac{1}{x_H} \left(1 - \tfrac{1}{2} \kappa_n \sum_{i=1}^{n} q_{i,0}^2 \right) > 0 \,,$$

where

$$q_{i,0} = q_i(x_H) = \gamma_i u_{i,0}^2 - \gamma_{i-1} u_{i-1,0}^2 + 2i - n - 1 \,.$$

In terms of these data the coefficients u_{k+1}^i and m_{k+1} become polynomials of the coefficients of lower order divided by ν. The formal power series of $u_i(x)$ and $m(x)$ are thus completely determined in terms of the $n-1$ values $u_i(x_H)$. Note, incidentally, that

$$\nu x_H < 1 \,. \tag{3.27}$$

At this point it is worth observing that our choice of the initial data β_i at $x = 0$ and α_i at $x = \infty$ is quite convenient.

From the results of Sects. 1 and 2 it is clear that choosing all β_i and all α_i except the first to be zero leads to a solution with all functions $u_i(x)$ being the same. We have also already observed in Sect. 2 that changing the sign of any $u_i(x)$ leads to another local solution of the system (2.21) so that in the regular case we could normalize $u_i(0)$ to be 1. But the signs of the $u_i(\infty)$ and α_i are still arbitrary.

It turns out that there is at least one other symmetry of this system that can be exploited to reduce the number of parameter values that must be investigated to find numerical solutions of the boundary value problem.

Proposition 3. *Under*

$$\pi : (u_1(x), \dots, u_{n-1}(x), m(x)) \mapsto (u_{n-1}(x), \dots, u_1(x), m(x)), \tag{3.28}$$

the set of local solutions of (2.21) *is mapped into itself.*

Proof. Since $\gamma_i = \gamma_{n-i}$ we have (in the notation of (2.21)) $(q_i \circ \pi)[\mathbf{f}](x) = \gamma_i u_{n-i}^2 - \gamma_{i-1} u_{n-i+1}^2 + 2i - n - 1 = -\gamma_{n-i+1} u_{n-i+1}^2 + \gamma_{n-i} u_{n-i}^2 - 2(n-i+1) + n + 1 = -q_{n-i+1}[\mathbf{f}]$ so that $Q_i \circ \pi) = Q_{n-i}$, and therefore $P \circ \pi = P$ and similarly $G \circ \pi = G$. It follows immediately that $(\mathscr{F} \circ \pi)[\mathbf{f}] = \pi \mathscr{F}[\mathbf{f}]$. $\square$

Equations (2.2), (2.4), and (3.28) then imply that for such solutions the Yang-Mills field (up to a relabeling of the basis of the Lie algebra) and the stress-energy tensor as well as the total gravitational mass m_∞ are the same. These solutions are thus physically equivalent.

Since on the black hole horizon $x = x_H$ the initial data are simply the values of $u_i(x_H)$ it is clear how to generate the other solution when one is known. It is not quite so obvious which initial data at $x = 0$ and at $x = \infty$ generate solutions related by π, but due to our particular choice of these data we still have a simple rule.

Proposition 4. *For initial data β_i and $\tilde{\beta}_i$ to generate solutions* $\mathbf{f}$ *and* $\pi(\mathbf{f})$ *of* (2.21), *respectively, it is necessary and sufficient that*

$$\tilde{\beta}_i = (-1)^{i+1} \beta_i .$$

Similarly, initial data $\{\alpha_i, m_\infty\}$ and $\{\tilde{\alpha}_i, \tilde{m}\}$ at $x = \infty$ generate solutions related by π if and only if

$$\tilde{\alpha}_i = (-1)^{i+1} \alpha_i \quad \textit{and} \quad \tilde{m}_\infty = m_\infty .$$

Proof. Both at $x = 0$ and $x = \infty$ the result follows immediately from (3.17) and (3.20) since

$$v_k^i = (-1)^{k+1} v_k^{n-i}, \qquad (k = 1, \dots, n-1)$$

which, in view of (3.9) is a consequence of the following lemma. $\square$

Lemma 5. *The Hahn polynomials $Q_k(i) = Q_k(i; 1, -1, n)$ satisfy the relation*

$$i Q_k(i-1) = (-1)^{k+1}(n-i) Q_k(n-i-1), \qquad (1 \le i, k < n). \tag{3.29}$$

Proof. Since by the definition

$$Q_k(i) = {}_3F_2(-k, -i, k+1; 2, -n+1, 1) = \sum_{l=0}^{k} \frac{(-k)_l (-i)_l (k+1)_l}{l!(l+1)!(-n+1)_l},$$

where $(a)_n = a(a+1)\ldots(a+n-1)$, Eq. (3.29) follows from

$$G(k,y,z) := \sum_{j=0}^{k} \frac{(-k)_j(k+1)}{j!(j+1)!(z)_j}[(-y)_{j+1} + (-1)^k(y+z-1)_{j+1}] = 0\,,$$
$$k = 1,2,\ldots; y,z \in \mathbb{C}$$

for $y = i$ and $z = 1-n$. The expression G being a polynomial of degree $k+1$ in y vanishes identically if it, as well as the $k+1$ first forward differences with respect to y, vanish for $y = 0$. But if $\Delta G(k,y,z) := G(k,y+1,z) - G(k,y,z)$, then for $m = 0,\ldots,k+1$,

$$\Delta^m G(k,y,z) = \sum_{j=0}^{k} \frac{(-k)_j(k+1)_j}{j!(z)_j(j-m+1)!} \times [(-1)^m(-y)_{j-m+1} + (-1)^k(y+z+m-1)_{j-m+1}]$$

so that, for $y = 0$,

$$\Delta^m G(k,0,z) = \frac{1}{(z)_{m-1}}\left[(-1)^m \frac{(-k)_{m-1}(k+1)_{m-1}}{(m-1)!} + (-1)\sum_{j=0}^{k-m+1} \frac{(-k)_j(k+1)_{j+m-1}}{j!(j+1)!}\right].$$

Since $(k+1)_{j+m-1} = (k+1)_{m-1}(k+m)_j$ the term with the sum in the last expression becomes

$$\begin{aligned}
&(-1)^k \frac{(-k)_{m-1}(k+1)_{m-1}}{(m-1)!} \sum_{j=0}^{k-m+1} \frac{(-k+m-1)_j(k+m)_j}{j!(m)_j} \\
&= (-1)^k \frac{(-k)_{m-1}(k+1)_{m-1}}{(m-1)!}\, {}_2F_1(-k+m-1, k+m; m; 1) \\
&= (-1)^k \frac{(-k)_{m-1}(k+1)_{m-1}}{(m-1)!} \frac{(-k)_{k-m+1}}{(m)_{k-m+1}} \\
&= \frac{(-k)_{m-1}(k+1)_{m-1}}{(m-1)!}(-1)^{m+2k+1}\,,
\end{aligned}$$

so that it cancels the first term, showing that $\Delta^m G(k,0,z) = 0$. □

4. Local Existence and Uniqueness Proofs

The standard local existence theorems for systems of ordinary differential equations do not apply at the singular points $x = 0$ or $z = 0$ or where $N = 0$. Thus to prove that the power series constructed in Sect. 3 define unique regular solutions for a particular choice of parameters one must either prove that they converge (e.g. by a variation of Cauchy's majorant method) or adapt the fixed point method to this singular case (see, e.g. [9]). It turns out that the method of [18] can be generalized to $n > 2$ in a fairly straightforward way. One could prove existence and uniqueness of a C^{n+a}-solution with the appropriate initial conditions but, for simplicity and since we really expect our solution to be analytic, we will treat only the analytic case, i.e. we show that a unique analytic solution exists which will then be given locally by the power series of Sect. 3 and will therefore also depend analytically on the parameters. Since limits of sequences of real analytic functions need not be analytic we must work with complex analytic functions of a complex variable x. But this causes no problem since all the constructions in Sect. 3 go through for complex x (and even complex initial data).

Near $x = 0$ we write

$$u_i(x) = U_i(x) + v_i(x) \quad \text{and} \quad m(x) = M(x) + \mu(x)\,,$$

where

$$U_i(x) := \sum_{k=0}^{n} u_k^i x^k \quad \text{and} \quad M(x) := \sum_{k=0}^{n-1} m_k x^k\,,$$

and the u_k^i and the m_k are the coefficients of the power series for u_i and m, respectively, obtained in Sect. 3. They are thus polynomials in x and in the parameters $\beta_1, \ldots, \beta_{n-1}$. On the other hand v_i and μ represent the remainder terms and vanish to order n and $n-1$, respectively, at $x = 0$. More precisely, if $B_R := \{x \in \mathbb{C} \,|\, |x| < R\}$ for given $R > 0$, and k a nonnegative integer, let

$$\begin{aligned}
D_R^0 &: = \{f : B_R \to \mathbb{C} \,|\, f \text{ analytic}\}\,,\\
D_R^{k+1} &: = \{f \in D_R^0 \,|\, f(0) = f'(0) = \ldots = f^{(k)}(0) = 0\}\,,\\
D_R^{-k} &: = \{f : B_R \backslash \{0\} \to \mathbb{C} \,|\, f \text{ analytic with a pole of order } k \text{ at } x = 0\}\,,
\end{aligned}$$

and define

$$\|f\|_0 \equiv \|f\|_\infty := \sup_{x \in B_R} |f(x)|\,, \qquad f \in D_R^0\,,$$

as well as

$$\|f\|_{k+1} := \sup_{x_1, x_2} \left| \frac{f^{(k)}(x_2) - f^{(k)}(x_1)}{x_2 - x_1} \right| \quad \text{for} \quad f \in D_R^{k+1}\,.$$

Here $\sup\limits_{x_1,x_2}$ stands for $\sup\limits_{x_1,x_2 \in B_R}$ with $x_1 \neq x_2$. Clearly, $D_R^k \subset D_R^l$ is continuously imbedded for $k > l \geq 0$. We define no norm on D_R^k for $k < 0$.

Then $v_i \in D_R^{n+1}$, $\mu \in D_R^n$ and, if we let $\sigma = (v_1, \ldots, v_{n-1}, w_1, \ldots, w_{n-1}, \mu)$ then a local solution of the system (2.21) can be regarded as a fixed point of a map $T : \sigma \mapsto \tilde{\sigma}$ given by

$$\begin{aligned}
\tilde{v}(x) &= \int_0^x w_i(s)\, ds\,,\\
\tilde{w}_i(x) &= \int_0^x N(s)^{-1} (F_i \circ \xi)(s)\, ds\,, \qquad (4.1)\\
\tilde{\mu}(x) &= \int_0^x (\mathscr{M} \circ \xi)(s)\, ds\,,
\end{aligned}$$

where the path from 0 to x can be chosen arbitrarily in B_R.

Here $\xi : t \mapsto (t, v_i(t), w_i(t), \mu(t))$ and

$$F_i := -2x^{-2}(m - \kappa_n xP)u_i' - \tfrac{1}{2}x^{-2}(q_{i+1} - q_i), \tag{4.2}$$

$$\mathscr{M} := \kappa_n(NG + P) \tag{4.3}$$

are functions on a subset of $\mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \times \mathbb{C}$ which are polynomials in v_j, w_j and μ with coefficients that are analytic in x except at $x = 0$ where some have poles.

More specifically, if we denote somewhat symbolically, for example, a homogeneous polynomial of degree d in $v_1, \ldots, v_{n-1}$ with a coefficient that is a function of x in D_R^k ($k \in \mathbb{Z}$) by $f_k v^d$, then

$$\begin{aligned} F_i &= f_{n-1} + f_0\mu + f_{-2}v + f_{-2}v^2 + f_{-2}v^3 + f_{-2}v^4 \\ &\quad + (f_1 + f_{-2}\mu + f_{-1}v + f_{-3}v^2 + f_{-3}v^3 + f_{-3}v^4)w \end{aligned} \tag{4.4}$$

and

$$\mathscr{M} = f_n + f_1\mu + f_0 v + f_{-2}v^2 + f_{-2}v^3 + f_{-2}v^4 + f_1 w + f_0 w^2 + f_0\mu w + f_{-1}\mu w^2,$$

as follows from (4.2), (4.3) if the form of the first terms in the power series for u_i and m is used. The coefficient functions are complicated expressions depending on the polynomials $U_i(x)$ and $M(x)$ but can be considered fixed.

Let now

$$\|\sigma\| := \max\left(a \max_i(\|v_i\|_{n+1}), b\max_i(\|w_i\|_n), c\|\mu\|_n\right), \tag{4.5}$$

where a, b, c are positive constants to be chosen later and define for some $\varrho > 0$,

$$X_R := \{\sigma \in D_R^{n+1} \times \ldots \times D_R^{n+1} \times D_R^n \times \ldots \times D_R^n \times D_R^n \mid \|\sigma\| \le \varrho\}. \tag{4.6}$$

We need to show that for a given (small enough) ϱ (i.e. a small enough neighborhood of 0 in the space of parameters β_i) there exists a $R > 0$ such that (i) X_R is a complete metric space, (ii) T maps X_R into itself and (iii) T is a contraction. This will follow from repeated use of

Lemma 6.
(a) $f \in D_R^k, g \in D_R^l \Rightarrow fg \in D_R^{k+l}\ \forall k, l \in \mathbb{Z}$,
(b) $\|f^{(k)}\|_0 = \|f\|_k$ ($f \in D_R^0$, $k \ge 0$),
(c) $\|f\|_k \le R^{l-k}\|f\|_l$ ($f \in D_R^l$, $l \ge k \ge 0$),
(d) $\|fg\|_k \le \sum_{i=0}^{k} \binom{k}{i} \|f\|_i\|g\|_{k-i}$, ($f, g \in D_R^0$, $k \ge 0$),
(e) $\|fg\|_j \le 2^j R^{k+l-j}\|f\|_k\|g\|_l$ ($0 \le j \le \min(k,l)$, $f \in D_R^k$, $g \in D_R^l$),
(f) $\|x^k f\|_l \le 2^l k! R^k \|f\|_l$ ($k, l \ge 0$, $f \in D_R^l$),
(g) $\|x^{-k} f\|_l \le \frac{1}{(l+1)\ldots(l+k)} \|f\|_{k+l}$, ($k, l \ge 0$, $f \in D_R^k$).

Proof. Part (a) is obvious. For (b) we have

$$\begin{aligned} \|f^{(k)}\|_0 &= \sup_{x_1 \in B_R} \left| \lim_{x_2 \to x_1} \frac{f^{(k-1)}(x_2) - f^{(k-1)}(x_1)}{x_2 - x_1} \right| \\ &\le \sup_{x_1, x_2} \left| \frac{f^{(k-1)}(x_2) - f^{(k-1)}(x_1)}{x_2 - x_1} \right| = \|f\|_k \end{aligned}$$

and

$$\|f\|_k \leq \sup_{x_1,x_2} \frac{1}{|x_2 - x_1|} \int_{x_1}^{x_2} |f^{(k)}(s)| ds \leq \|f^{(k)}\|_0 ,$$

and for (c)

$$\begin{aligned} \|f\|_k = \|f^{(k)}\|_0 &= \sup_{x_2 \in B_R} |f^{(k)}(x_2) - f^{(k)}(0)| \\ &= \sup_{x_1,x_2} |x_2 - x_1| \left| \frac{f^{(k)}(x_2) - f^{(k)}(x_1)}{x_2 - x_1} \right| \leq R\|f\|_{k+1} , \end{aligned} \tag{4.7}$$

if $f \in D_R^k$ so that the result follows by induction.

Part (d) follows from $\|fg\|_1 \leq \|f\|_0 \|g\|_1 + \|f\|_1 \|g\|_0$ and Leibniz's rule, part (e) from (d) and (c). Part (f) is derived easily from (d) with the help of

$$\|x^k\|_l = \begin{cases} (k-l+1)\dots kR^{k-l} & \text{if } l \leq k , \\ 0 & \text{if } l > k . \end{cases}$$

To prove (g) note first that $f \in D_R^k \Rightarrow x^{-k} f \in D_k^0$ and

$$x^{-k} f(x) = \int_0^1 dt_k t_k^{k-1} \int_0^1 \dots \int_0^1 dt_1 f^{(k)}(t_1 \dots t_k x) ,$$

which follows by induction on k from $x^{-1} f(x) = \int_0^1 f'(tx)\, dt$. Then, if $f \in D_R^k$ and $l > 0$,

$$\begin{aligned} \|x^{-k} f\|_l &\leq \sup_{x_1,x_2} |x_2 - x_1|^{-1} \left| \int_0^1 dt_k t_k^{k+l-1} \dots \int_0^1 dt_1 t_1^l f^{(k+l-2)}(t_1 \dots t_k x)\Big|_{x_1}^{x_2} \right| \\ &\leq \int_0^1 dt_k t_k^{k+l-1} \dots \int_0^1 dt_1 t_1^l \|f\|_{k+l} = \frac{1}{(l+1)\dots(l+k)} \|f\|_{k+l} . \end{aligned} \tag{4.8}$$

The proof for $l = 0$ uses (b) and is very similar. □

To show that X_R is a complete metric space assume that $\{\sigma_k\}$ is a Cauchy sequence in X_R. Then the components of σ_k are Cauchy sequences in D_R^{n+1} or D_R^n and, by (b) of Lemma 6 their $(n+1)$st derivatives are Cauchy sequences in D_R^0. But D_R^0 is a complete metric space since these analytic functions converge uniformly on compact subsets of B_R to analytic functions $\hat{v}_i(x), \hat{w}_i(x), \hat{\mu}$, respectively (cf. [9]). Define $v_i(x)$ by integrating $\hat{v}_i(x) n + 1$ times from 0 to x and similarly for $w_i(x)$ and $\mu(x)$. This defines $\sigma(x)$ as an analytic function and $\sigma_k \to \sigma$ and $\|\sigma\| \leq \varrho$ follow.

To show that $T: X_R \to X_R$ observe that from $\|\sigma\| \leq \varrho$ it follows that $\|v_i\|_{n+1} < \varrho$, $\|w_i\|_n \leq \varrho$, $\|\mu\|_n \leq \varrho$. Now $\|\tilde{v}_i\|_{n+1} = \|w_i\|_n \leq \varrho$ and

$$\|\tilde{\mu}\|_n = \sup_{x_1,x_2} |y - x|^{-1} |(\mathscr{M} \circ \xi)^{(n-2)}|_x^y| \leq \|f_n\|_{n-1} + \dots + \|f_1 \mu w^2\|_{n-1} .$$

Also $\|f_n\|_{n-1} \le R\|f_n\|_n = \text{const}\, R$, $\|f_1\mu\|_{n-1} \le \text{const}\, R\|\mu\|_n$ by parts (e) and (c) of Lemma 6, where the constants are positive and depend on the functions f, i.e. on the initial conditions. For the other terms we have, for example, $\|f_{-2}v^2\|_{n-1} = \|x^{-2}f_0v^2\|_{n-1} \le \text{const}\,\|(x^{-1}v)(x^{-1}v)\|_{n-1} \le \text{const}\, R^{n+1}\|x^{-1}v\|_n^2 \le \text{const}\, R^{n+1}\|v\|_{n+1}^2 \le \text{const}\, R^{n+1}\varrho^2$. All other terms are handled similarly so that it follows that $\|\tilde{\mu}\|_{n-1} \le R(\text{const} + \text{const}\,\varrho + O(\varrho^2))$. In the same way one finds that $\|\tilde{w}_i\|_{n-1} \le \text{const} + \text{const}\,\|\mu\|_n + \text{const}\,\|v\|_{n+1} \le \text{const} + \text{const}\,\varrho + \text{const}\, O(\varrho^2)$. Thus $\|T(\sigma)\| \le \varrho$ provided both ϱ and R are small enough.

We now need to prove that T is a contraction, i.e. that $\|T\sigma_2 - T\sigma_1\| < \|\sigma_2 - \sigma_1\|$ for small enough R and ϱ, or, since T is differentiable, $\|d_\sigma T\| \le k < 1$, where

$$\|d_\sigma T\| = \sup_{\|\Sigma\|=1} \|(d_\sigma T)(\Sigma)\|\,.$$

Let

$$\Sigma = (V_1, \dots, V_{n-1}, W_1, \dots, W_{n-1}, L) \in X_R$$

and

$$(d_\sigma T)(\Sigma) = \tilde{\Sigma} = (\tilde{V}_1, \dots, \tilde{V}_{n-1}\tilde{W}_1, \dots, \tilde{W}_{n-1}, \tilde{L})$$

then

$$\tilde{V}_i = \sum_{j=1}^{n-1}\left(\frac{\partial \tilde{v}_i}{\partial v_j} V_j + \frac{\partial \tilde{v}_i}{\partial w_j} W_j\right) + \frac{\partial \tilde{v}_i}{\partial \mu} L\,,$$

and similarly for $\tilde{W}_j$ and $\tilde{L}$. It follows from (4.1) that

$$\tilde{V}_i(x) = \int_0^x W_i(s)\,ds\,,$$

so that

$$\|\tilde{V}_i\|_{n+1} = \sup_{x,y} |y-x|^{-1}|W_i^{(n-1)}(y) - W_j^{(n-1)}(x)| = \|W_i\|_n\,.$$

Similarly we find from (4.3)

$$\tilde{L}(x) = 2\kappa_n \int_0^x ds\left(- s^{-1}GL + N\sum_{k=1}^{n-1} \gamma_k u_k' W_k - s^{-2}\sum \gamma_k u_k Q_k V_k\right),$$

so that

$$\begin{aligned}\|\tilde{L}\|_n &\le 2\kappa_n \| - x^{-1}GL + N\sum_{k=1}^{n-1}\gamma_k u_k' W_k - x^{-2}\sum_{k=1}^{n-1}\gamma_k u_k Q_k V_k\|_{n-1} \\ &\le R^2(C_1\|L\|_n + C_2 \max_i \|W_i\|_n + C_3 \max_i \|V_i\|_{n+1})\,, \qquad (4.9)\end{aligned}$$

where C_k denote constants depending on σ, but not on R and where parts (c), (f) and (g) of Lemma 6 have been used together with the observation that $G, u_k' \in D_R^1$ and $Q_k := \frac{1}{2}(q_{k+1} - q_k) \in D_R^0$.

Differentiating $\tilde{w}_i$ in (4.1) gives

$$\tilde{W}_i(x) = \int_0^x \left(l_i L + \omega W_i + \sum_{k=1}^{n-1} \mathscr{V}_{ik} V_k \right),$$

where

$$\begin{aligned} l_i &= 2(x^{-1}N^{-2}F_i - x^{-2}N^{-1}u_i'), \\ \omega &= -2x^{-2}N^{-1}(m - \kappa_n xP), \\ \mathscr{V}_{ik} &= x^{-2}N^{-1}[A_{ik}u_i u_k - Q_k(\delta_{ik} + 4\kappa_n \gamma_k u_k u_i')], \end{aligned}$$

and therefore

$$\|\tilde{W}_i\|_n \le \|l_i L\|_{n-1} + \|\omega W_i\|_{n-1} + \sum_{k=1}^{n-1} \|\mathscr{V}_{ik} V_k\|_{n-1}.$$

From (4.4) we have $F_i \in D_R^{n-1}$ and since from the power series expansion (and the remark after (3.12)) $U_i'(x) = 2\beta_1 x \,\forall i$ it follows that

$$l_i = -4\beta_1 x^{-1}[1 + O(x)] \in D_R^{-1}.$$

Similarly it is seen that

$$\mathscr{V}_{ik} = A_{ik} x^{-2}[1 + O(x)],$$

while $\omega \in D_R^1$. Using repeatedly Lemma 6 we therefore have the following estimates

$$\|\omega W_i\|_{n-1} \le O(R^2)\, \|W_i\|_n, \tag{4.10}$$

$$\|l_i L\|_{n-1} = \left(\frac{4}{n} |\beta_1| + O(R) \right) \|L\|_n, \tag{4.11}$$

$$\begin{aligned} \left\| \sum_{k=1}^{n-1} \mathscr{V}_{ik} V_k \right\|_{n-1} &\le \sum_{k=1}^{n-1} \|(x^{-2}V_k)\,(A_{ik} + O(R))\|_{n-1} \\ &\le \sum_{k=1}^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} \|x^{-2}V_k\|_l \|A_{ik} + O(R)\|_{n-1-l} \\ &\le \sum_{k=1}^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{1}{(l+1)(l+2)} \|V_k\|_{l+2} \|A_{ik} + O(R)\|_{n-1-l} \\ &\le \sum_{k=1}^{n-1} \left(\frac{1}{n(n+1)} |A_{ik}| + O(R) \right) \\ &\le \left(\frac{n}{n+1} + O(R) \right) \max_k \|V_k\|_{n+1}, \end{aligned} \tag{4.12}$$

where we used the fact that there are only three terms in the sum the matrix A being tridiagonal.

From (4.5) we now have $a\|V_i\|_{n+1}$, $b\|W_i\|_n$, $c\|L\|_n \le \|\Sigma\|$ so that

$$\|\tilde{\Sigma}\| \le \max\left\{ \frac{a}{b}, \frac{n}{n+1} \quad \frac{b}{a} + \frac{4|\beta_1|}{n} \frac{b}{c} + O(R), O(R) \right\} \|\Sigma\|.$$

Thus the constants a, b, c can be chosen such that $\|\tilde{\Sigma}\| < \|\Sigma\|$ provided R is small enough, for example,

$$a = \frac{2n+1}{2n+2}, \qquad b = 1, \qquad c > 8|\beta_1|\left(1 + \frac{1}{2n}\right).$$

This proves that $T: X_R \to X_R$ is a contraction for small enough R. In summary we have the

Theorem 3. *For any (small enough) choice of the parameters $\beta_1, \ldots, \beta_{n-1}$ there exists an $R > 0$ such that the formal power series for $u_i(x)$ and $m(x)$ constructed in Sect. 3 converges for $|x| < R$. Hence the system of Eq. (2.21) admits a unique local solution near $x = 0$ which moreover depends analytically on the parameters β_i.*

The proof of the existence and uniqueness of solutions with the chosen asymptotic properties at $x = \infty$ is almost identical and will be omitted.

The corresponding result at a *regular black hole horizon* is proved similarly and only an outline will be given. Let again the black hole horizon be given by x_H and put $x = x_H + t$. We use the same terminology as for the initial value problem at $x = 0$ and define the dependent variables $v_i(t)$, $w_i(t)$ by what remains after the initial conditions have been imposed, i.e.

$$u_i = u_{i,0} + u_{i,1}t + v_i(t) \quad \text{and} \quad m = \tfrac{1}{2}x_H + \mu(t).$$

The initial value problem is then again equivalent to finding a fixed point of the map $T: X_\varepsilon \to X_\varepsilon$ given by (4.1) and (4.2)/(4.3) except that now $X_\varepsilon := D^2_\varepsilon \times \ldots \times D^2_\varepsilon \times D^1_\varepsilon \times \ldots \times D^1_\varepsilon \times D^1_\varepsilon$ again with a norm of the form (4.5). Observing that all quantities (4.1) and (4.2), (4.3) are now finite at $t = 0$ except $N = (t - 2\mu)/x$ we find that

$$\|\tilde{v}_i\|_2 = \|w_i\|_1, \qquad \|\tilde{\mu}\|_1 \le \kappa_n \|NG + P\|_0 \quad \text{and} \quad \|\tilde{w}_i\|_1 \le \|N^{-1}F_i\|_0.$$

Since also $F_i \in D^1_\varepsilon$ it follows that $T: X_\varepsilon \to X_e$ for small enough ϱ.

To show that T is contracting one calculates again the differential dT with the same notation getting instead of (4.9)

$$\|\tilde{L}\|_1 \le C_1\varepsilon\|L\|_1 + C_2\varepsilon^2 \max_i \|W_i\|_1 + C_3\varepsilon^2 \max_i \|V_i\|_2.$$

However, the quantities in (4.10) have now different limits at $x = x_H$, namely

$$\begin{aligned}
l_i &= -\frac{2}{x_H}(u_{i,1} + 2x_H u_{i,2})\frac{1}{t} + O(1) = \frac{l_{-1,i}}{t} + O(1),\\
\omega &= -\frac{\nu x_H}{t} + O(1),\\
\mathscr{V}_{ik} &= \frac{1}{x_H t}(A_{ik}u_{i,0}u_{k,0} - Q_{i,0}\delta_{ik} - \kappa_n\gamma_k u_{k,0}u_{i,1}Q_{k,0}) + O(1),
\end{aligned}$$

so that

$$\|\tilde{W}_i\|_1 \le (|l_{i,-1}| + O(\varepsilon))\,\|L\|_1 + \left[\frac{\nu}{x_H} + O(\xi)\right]\|W_i\|_1 + O(\varepsilon)\max_k \|V_k\|_2,$$

and

$$\|\tilde{\Sigma}\| \le \max\left\{\frac{a}{b}, \nu x_H + \frac{b}{C}\max_i |l_{i,-1}| + O(\varepsilon), O(\varepsilon)\right\}\|\Sigma\|.$$

Recalling (3.27) we see that T becomes contracting for small enough ε if we choose, for example,

$$0 < a < 1\,, \qquad b = 1 \quad \text{and} \quad c > \frac{\max_i |l_{i,-1}|}{|1 - \nu x_H|}\,.$$

5. Some Numerical Results

At this stage it is unfortunately not very clear what rigorous technique could be used to analyze the set of all global solutions of this boundary value problem. Even finding solutions numerically is quite difficult and very time consuming. As in [13] we use the "shooting to a fitting point" method ([17], Sect. 17.2) which incorporates a fourth order adaptive step size Runge-Kutta integration and choose as initial data $(\alpha_1, \ldots, \alpha_{n-1}, m_\infty)$ at $x = \infty$ and either the $(\beta_1, \ldots, \beta_{n-1})$ at $x = 0$ or the values of $(u_1(x_H), \ldots, u_{n-1}(x_H))$. Just to determine the possible value of the total mass m_∞ one can simply integrate upwards from $x_0 = \varepsilon$ or $x_0 = x_H + \varepsilon$ (where ε, for example, 0.1 or 0.01) and choose the initial parameters such that $|u_i(x_H)| < 1$ and $N > 0$ for as large values of x as possible. An algorithm was used that finds initial conditions at $x = 0$ or $x = x_H$ which maximize the value $x_{\max}$ of x at which the solution will finally violate one of the necessary conditions for an acceptable smooth global solution, namely $|u_i(x)| \leq 1$, $N(x) > 0$. While this technique does not give reliable values for m_∞ it determines approximate values of β_i or $u_i(x_H)$. Similarly, approximate values for m_∞ and α_i can be found by integrating downwards from a large value x_∞ of x, say $x_\infty = 10^4$, (checking also that $m(x) > 0$). The parameter sets thus obtained (for which the upward and the downward solutions are defined on overlapping intervals) are then used in the "shooting to a fitting point" method with the hope that Newton's method will converge. This happens very rarely, but when it does there is a very high probability that a genuine solutions is being approximated.

The numerical integration is quite delicate already in the $SU(2)$ case since it appears that N always has one minimum value (near $x = 1$) that seems to approach 0 for those solutions with highly oscillating functions u_i. One approaches here an extremal black hole ($N = 0$, $N' = 0$) which is a very singular limit. Therefore no solutions where u_i has more than about seven zeros have been found, but Lavrelashvili and Maison [15] were able to approximate analytically near this point a function analogous to the u_i in a somewhat simpler dilaton model.

In view of this difficulty it is best to integrate with the highest possible precision and to start off the integrations after summing the power series at x_0 and x_∞ with as many terms as necessary to achieve a precision of about 10^{-10}. This is done easily (for arbitrary n) directly by means of the recursion relations (3.1, 3.2) and (3.24, 3.25). These recurrence relations themselves were derived with the help of Maple and Mathematica and verified to solve the differential equations. The validity of the corresponding code in C-language was tested with some numerical parameters against the values produced in the Mathematica program and also against the Runge-Kutta numerical integration by evaluating the series at a few small x-values and integrating the differential equation between the two points.

Once some promising initial values have been found at both ends they must be paired and fed into the shooting algorithm. We consider a solution found if the sum of all jumps in $|u_i|$, $|u_i'|$ and m is less than about 10^{-10}. Since Newton's method very rarely converges even for such a carefully selected starting point in a $2n - 1$-dimensional parameter space (for $n > 2$), whenever it does, there can be little doubt that the solution is genuine. It is harder to judge the error in the parameter values (and the zeros of the $u_i s$). Some information is obtained from the way the parameters change from one iteration to the next in Newton's method, and also by varying the initial and the fitting points and the precision of numerical integration. The values of m_∞ and of the β_i are probably accurate to about five or six decimal places, but the α values are less precise.

Table 1. Total mass, initial values, and number of zeros of u_i for some solutions of the $SU(3)$-EYM theory

							Zeros	
m_∞	$u_1(\infty)$	$u_2(\infty)$	α_1	α_2	β_1	β_2	u_1	u_2
0.82865	−1	−1	−0.8934	0	−0.45372	0	1	1
0.84769	−1	−1	−1.7628	−4.1579	−0.53766	0.26077	1	1
0.93774	−1	1	−4.4302	14.6598	−0.63437	0.33228	1	2
0.97135	1	1	−8.8639	0	−0.65173	0	2	2
0.97354	1	1	−13.4353	−151.01	−0.66743	0.26081	2	2
0.98963	1	−1	−30.3178	512.83	−0.69114	0.34189	2	3
0.99532	−1	−1	−58.9326	0	−0.69704	0	3	3
0.99564	−1	−1	−85.598	−5501.7	−0.69852	0.25790	3	3
0.99830	−1	1	−189.558	18941.5	−0.70139	0.34283	3	4
0.99924	1	1	−366.335	0	−0.70488	0	4	4
0.99929	1	1	−528.489	−205740	−0.70396	0.25735	4	4
0.99987	−1	−1	−2252	0	−0.70617	0	5	5
0.99998	1	1	−13817	0	−0.70638	0	6	6
1.00000	−1	−1	−84757	0	−0.70641	0	7	7

So far we have only searched for numerical (non-scaled) soltions of the $SU(3)$-EYM theory. We need to pick negative values of β_1 and can, in view of Proposition 4, confine ourselves to positive values of β_2. At $x = \infty$, however, all possible values for $u_1(\infty)$, $u_2(\infty)$ and α_2 must be considered while m_∞ must be positive and α_2 negative. Table 1 lists the parameters of the solutions found so far. Recall that all values are normalized by (2.16). For example, the total masses m_∞ of the $SU(3)$ model (in Planck units) are obtained by multiplying those in the table by $\lambda_3 = 2$.

The solutions with $\alpha_2 = \beta_2 = 0$ are the scaled $SU(2)$ solutions. But not only are there solutions in which u_1 and u_2 have different numbers of zeros, there are also solutions in which they have the same number of zeros, but are nevertheless not proportional. Two such solutions are shown in Fig. 1 and Fig. 2. It seems that with the chosen scaling m_∞ still increases with increasing number of zeros of the u_i and approaches an upper limit of 1. Moreover, the scaled $SU(2)$ solutions lead to the smallest mass for a given number of zeros of the (u_1, u_2) pair. The behavior of the matter density and pressure functions and of the Yang-Mills curvature is not qualitatively different from the $n = 2$ case, i.e. from the one for the scaled solutions. Unfortunately it is not yet possible to speculate whether there are more than two, or even infinitely many, solutions for which both u_1 and u_2 have the same number of zeros, although it does seem to be a discrete set.

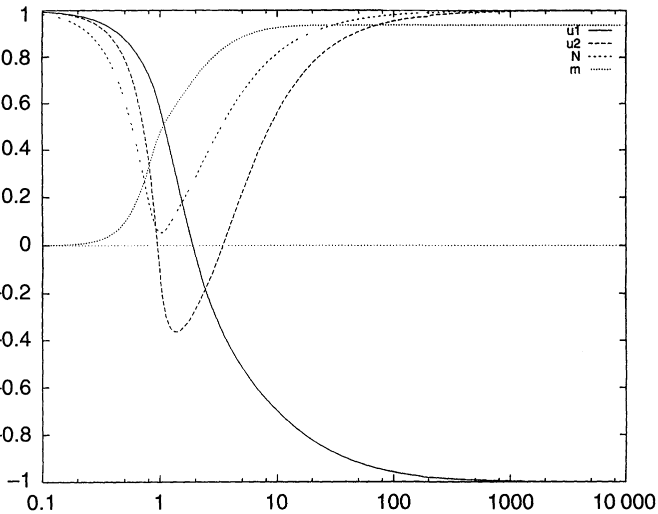

Fig. 1. Regular $SU(3)$-solution with $m_\infty = 0.93774$, $\alpha_1 = -4.4302$, $\alpha_2 = 14.6598$, $\beta_1 = -0.63437$, $\beta_2 = 0.33228$

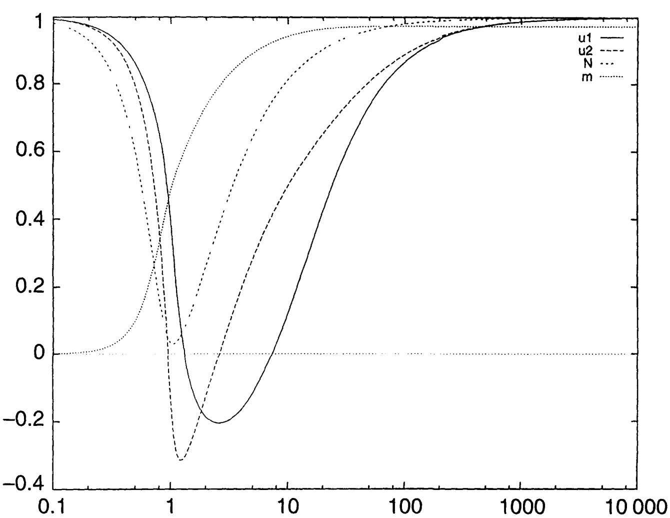

Fig. 2. Regular $SU(3)$-solution with $m_\infty = 0.97354$, $\alpha_1 = -13.4353$, $\alpha_2 = -151.01$, $\beta_1 = -0.66743$, $\beta_2 = 0.26081$

There seems to be no reason why non-scaled black hole $SU(3)$-EYM solutions should not also exist, probably for any radius of the black hole horizon. In spite of a considerable effort, however, no such solution has been yet been found.

Many questions can be asked in view of this fairly strong evidence for the existence of a discrete set of solutions of the $SU(3)$-EYM equations, in particular in the light of the interpretations given in [21]. While in this "purely magnetic" case ($A_0 = 0$) there is no evidence for any continuous family of soliton or black hole solutions, Gal'tsov and Volkov [6] have shown that "superpositions" of the scaled $SU(3)$ solutions with Reissner-Nordström like solutions (parametrized by a continuous charge) may exist. This possibility has not yet been investigated in the general (non-scaled) case. (The corresponding differential equations are considerably more complicated and the singularities at x_H and at ∞ also have a different structure.)

The stability of the presented non-scaled soliton solutions against time dependent perturbations has not yet been investigated. Zhou and Straumann [22] showed that the scaled solutions are unstable, and there is no reason to expect stability for the new non-scaled ones (which are generally "higher resonances").

An interesting conjecture is, however, that all *static*, asymptotically flat "purely magnetic" soliton solutions of the $SU(n)$-EYM equations are spherically symmetric and thus belong to the discrete set of solutions discussed in this paper. This could be expected in view of the black hole uniqueness theorems and similar results for static perfect fluids. But as far as the author knows no work has been done yet towards a proof of this conjecture.

Acknowledgements. Thanks are due for discussions or correspondence to W. A. Al-Salam (in particular for Lemma 5), R. Bartnik, C. Dunkl, M. E. H. Ismail, J. A. Smoller and A. G. Wasserman. Also some corrections and suggestions of the referee are gratefully acknowledged. This work was supported in part by the National Science and Engineering Research Council of Canada

References

1. Aichelburg, P.C., Bizon, P.: Magnetically charged black holes and their stability. Phys. Rev. D (3) **48**, 607–615 (1993)
2. Bartnik, R.: The symmetrically symmetric Einstein Yang-Mills equations. Relatively Today: Proceedings of the third Hungarian relativity workshop (Z. Perjés, ed.), 1989, Tihany, Commack, NY: Nova Science Pub. 1992, pp. 221-240
3. Bartnik, R., Mckinnon, J.: Particlelike solutions of the Einstein Yang-Mills equations. Phys. Rev. Lett. **61**, 141–144 (1988)
4. Brodbeck, O., Straumann, N.: A generalized Birkhoff theorem for the Einstein-Yang-Mills system. J. Math. Phys. **34**, 2412–2423 (1993)
5. Droz, S., Heusler, M., Straumann, N.: New black hole solutions with hair. Phys. Lett. B **268**, 371–476 (1991)
6. Gal'tsov, D.V., Volkov, M.S.: Sphalerons in Einstein-Yang-Mills theory. Phys. Lett. B **273**, 255–259 (1991)
7. Gal'tsov, D.V., Volkov, M.S.: Charged non-abelian $SU(3)$ Einstein-Yang-Mills black holes. Phys. Lett. B **274**, 173–178 (1992)
8. Gregory, R., Harvey, J.: Black holes with a massive dilaton. Phys. Rev. D (3) **47**, 2411–2422 (1993)
9. Hille, E.: Ordinary differential equations in the complex domain. New York: Wiley 1976
10. Horne, J.H., Horowitz, G.T.: Black holes coupled to a massive dilaton. Nucl. Phys. B **399**, 169–196 (1993)
11. Karlin, S., McGregor, J.L.: The Hahn polynomials, formulas and an application. Scripta Mathematica **26**, 33–46 (1961)
12. Künzle, H.P.: $SU(n)$-Einstein-Yang-Mills fields with spherical symmetry. Classical Quantum Gravity **8**, 2283–2297 (1991)
13. Künzle, H.P., Masood-ul-Alam, A.K.M.: Spherically symmetric static $SU(2)$ Einstein-Yang-Mills fields. J. Math. Phys. **21**, 928–935 (1990)
14. Künzle, H.P., Masood-ul-Alam, A.K.M.: Pure Yang-Mills-fields on asymptotically flat curved $\mathbb{R}^3$. Classical Quantum Gravity **10**, 801–804 (1993)
15. Lavrelashvili, G., Maison, D.: Static spherically symmetric solutions of a Yang-Mills field coupled to a dilaton. Phys. Lett. B **295**, 67–72 (1992)
16. Lavrelashvili, G., Maison, D.: Regular and black hole solutions of Einstein-Yang-Mills-dilaton theory. Max-Planck-Institut for Physik München preprint, 1993

17. Press, W.H., Flannery, B.P., Teukolsky, S.A., Vetterling, W.T.: Numerical recipes in C: the art of scientific computing (2nd ed.). Cambridge: Cambridge University Press 1992
18. Smoller, J.A., Wasserman, A.G.: Existence of infinitely many smooth, static, global solutions of the Einstein/Yang-Mills equations. Commun. Math. Phys. **151**, 303–325 (1993)
19. Smoller, J.A., Wasserman, A.G., Yau, S.-T.: Existence of black hole solutions for the Einstein-Yang/Mills equations. Commun. Math. Phys. **154**, 377–401 (1993)
20. Smoller, J.A., Wasserman, A.G., Yau, S.-T., McLeod, J.B.: Smooth static solutions of the Einstein/Yang-Mills equations. Commun. Math. Phys. **143**, 115–147 (1991)
21. Sudarsky, D., Wald, R.M.: Extrema of mass, stationarity and staticity, and solutions to the Einstein-Yang-Mills equations. Phys. Rev. D (3) **46**, 1453–1474 (1992)
22. Zhou, Z., Straumann, N.: Nonlinear perturbations of Einstein-Yang-Mills solitons and non-abelian black holes. Nucl. Phys. B **360**, 180–196 (1991)

Communicated by S.-T. Yau