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Symplectic Structures Associated to Lie-Poisson Groups

A. Yu. Alekseev [1,2,*] **A. Z. Malkin** [2,**,***]

[1] Laboratoire de Physique Théorique et Hautes Énergies****, Paris, France*****
[2] Institute of Theoretical Physics, Uppsala University, Box 803 S-75108, Uppsala, Sweden

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Abstract: The Lie-Poisson analogues of the cotangent bundle and coadjoint orbits of a Lie group are considered. For the natural Poisson brackets the symplectic leaves in these manifolds are classified and the corresponding symplectic forms are described. Thus the construction of the Kirillov symplectic form is generalized for Lie-Poisson groups.

Introduction

The method of geometric quantization [9] provides a set of Poisson manifolds associated to each Lie group G. The dual space $\mathcal{G}^*$ of the corresponding Lie algebra $\mathcal{G}$ plays an important role in this theory. The space $\mathcal{G}^*$ carries the Kirillov-Kostant Poisson bracket which mimics the Lie commutator in $\mathcal{G}$. Having chosen a basis $\{\varepsilon^a\}$ in $\mathcal{G}$, we can define structure constants f_c^{ab}:

$$[\varepsilon^a, \varepsilon^b] = \sum_c f_c^{ab} \varepsilon^c , \tag{1}$$

where $[\,,]$ is the Lie commutator in $\mathcal{G}$. On the other hand, we can treat any element ε^a of the basis as a linear function on $\mathcal{G}^*$. The Kirillov-Kostant Poisson bracket is defined so that it resembles formula (1):

$$\{\varepsilon^a, \varepsilon^b\} = \sum_c f_c^{ab} \varepsilon^c . \tag{2}$$

The Kirillov-Kostant bracket has two important properties:

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*** On leave of absence from St Petersburg University
**** Unité Associée au C N.R S., URA 280
***** LPTHE, Paris-VI, Tour 16 – 1er étage, 4 place Jussieu, F-75252 Paris Cedex 05, France

i. the r.h.s. of (2) is linear in ε^c,
ii. the group G acts on $\mathscr{G}^*$ by means of the coadjoint action and preserves the bracket (2).

The Kirillov-Kostant bracket is always degenerate (e.g. at the origin in $\mathscr{G}^*$). According to the general theory of Poisson manifolds [2, 15] the space $\mathscr{G}^*$ splits into the set of symplectic leaves. Usually it is not easy to describe symplectic leaves of a Poisson manifold. Fortunately, an effective description exists in this very case. Symplectic leaves coincide with orbits of the coadjoint action of G in $\mathscr{G}^*$. Kirillov obtained an elegant expression for the symplectic form Ω on the orbit [9]:

$$\Omega_X(u,v) = \langle X, [\varepsilon_u, \varepsilon_v]\rangle \,. \tag{3}$$

Here $\langle\, , \rangle$ is the canonical pairing between $\mathscr{G}$ and $\mathscr{G}^*$. The value of the form is calculated at the point X on the pair of vector fields u and v on the orbit. The elements ε_u, ε_v of the algebra $\mathscr{G}$ are defined as follows:

$$u|_X = \mathrm{ad}^*(\varepsilon_u)X\,, \tag{4}$$

where ad^* is the coadjoint action of $\mathscr{G}$ on $\mathscr{G}^*$. The purpose of this paper is to generalize formula (3) for Lie-Poisson groups.

Lie group G equipped with a Poisson bracket $\{\, , \}$ is called a Lie-Poisson group when the multiplication in G

$$G \times G \to G\,, \tag{5}$$

$$(g, g') \to gg' \tag{6}$$

is a Poisson mapping. In other words, the bracket of any two functions f and h satisfies the following condition:

$$\{f, h\}(gg') = \{f(gg'), h(gg')\}_g + \{f(gg'), h(gg')\}_{g'}\,. \tag{7}$$

Here we treat $f(gg'), h(gg')$ as functions of the argument g only in the first term of the r.h.s., whereas in the second term they are considered as functions of g'.

In the framework of the Poisson theory the natural action of a group on a manifold is the Poisson action [4, 13]. It means that the mapping

$$G \times M \to M \tag{8}$$

is a Poisson one. In Poisson theory this property replaces property (ii) of the Kirillov-Kostant bracket. There exist direct analogues of the coadjoint orbits for Lie-Poisson groups. Our goal in this paper is to obtain an analogue of formula (3). However, it is better to begin with the Lie-Poisson analogue of the cotangent bundle T^*G described in Sect. 2. The symplectic form for this case is obtained in Sect. 3 and then in Sect. 4 the analogue of the Kirillov form appears as a result of reduction. Section 1 is devoted to an exposition of the Kirillov theory. In Sect. 5 some examples are considered.

When speaking about Lie-Poisson theory, the works of Drinfeld [5], Semenov-Tian-Shansky [13], Weinstein and Lu [10] must be mentioned. We follow these papers when representing the known results.

The theory of Lie-Poisson groups is a quasiclassical version of the theory of quantum groups. So we often use the attribute "deformed" instead of "Lie-Poisson." Similarly we call the case when the Poisson bracket on the group is equal to zero the "classical" one.

1. Symplectic Structures Associated to Lie Groups

For the purpose of selfconsistency we shall collect in this section some well-known results concerning Poisson and symplectic geometry associated to Lie groups. The most important part of our brief survey is a theory of coadjoint orbits. Our goal is to rewrite the Kirillov symplectic form so that a generalization can be made straightforward.

Let us fix notations. The main object of our interest is a Lie group G. We denote the corresponding Lie algebra by $\mathcal{G}$. The linear space $\mathcal{G}$ is supplied with Lie commutator $[\,,\,]$. If $\{\varepsilon^a\}$ is a basis in $\mathcal{G}$ we can define structure constants f_c^{ab} in the following way:

$$[\varepsilon^a, \varepsilon^b] = \sum_c f_c^{ab} \varepsilon^c . \tag{9}$$

The Lie group G has a representation which acts in $\mathcal{G}$. It is called an adjoint representation:

$$\varepsilon^g \equiv \mathrm{Ad}(g)\varepsilon . \tag{10}$$

The corresponding representation of the algebra $\mathcal{G}$ is realized by the commutator:

$$\mathrm{ad}(\varepsilon)\eta = [\varepsilon, \eta] . \tag{11}$$

We denote elements of the algebra $\mathcal{G}$ by small Greek letters.

Let us introduce a space $\mathcal{G}^*$ dual to the Lie algebra $\mathcal{G}$. There is a canonical pairing $\langle\,,\,\rangle$ between $\mathcal{G}^*$ and $\mathcal{G}$ and we may construct a basis $\{l_a\}$ in $\mathcal{G}^*$ dual to the basis $\{\varepsilon^a\}$ so that

$$\langle l_a, \varepsilon^b \rangle = \delta_a^b . \tag{12}$$

We use small Latin letters for elements of $\mathcal{G}^*$. Each vector ε from $\mathcal{G}$ defines a linear function on $\mathcal{G}^*$:

$$H_\varepsilon(l) = \langle l, \varepsilon \rangle . \tag{13}$$

In particular, a linear function H^a corresponds to an element ε^a of the basis in $\mathcal{G}$.

By duality the group G and its Lie algebra $\mathcal{G}$ act in the space $\mathcal{G}^*$ via the coadjoint representation:

$$\langle \mathrm{Ad}^*(g)l, \varepsilon \rangle = \langle l, \mathrm{Ad}(g^{-1})\varepsilon \rangle , \tag{14}$$

$$\langle \mathrm{ad}^*(\varepsilon)l, \eta \rangle = -\langle l, [\varepsilon, \eta] \rangle . \tag{15}$$

The space $\mathcal{G}$ can be considered as a space of left-invariant or right-invariant vector fields on the group G. Let us define the universal right-invariant one-form θ_g on G which takes values in $\mathcal{G}$:

$$\theta_g(\varepsilon) = -\varepsilon . \tag{16}$$

We treat ε in the l.h.s. of formula (16) as a right-invariant vector field whereas in the r.h.s. as an element of $\mathcal{G}$. The minus in the r.h.s. of (16) reflects the fact that the isomorphism of the algebra of right-invariant vector fields on the Lie group and the corresponding Lie algebra is nontrivial and may be represented by $-id$ at the group unit. Since the one-form θ_g and the vector field ε are right-invariant the result does not depend on the point g of the group. θ_g is known as the Maurer-Cartan form.

Similarly, the universal left-invariant one-form μ_g can be introduced:

$$\mu_g(\varepsilon) = \varepsilon , \qquad \mu_g = \mathrm{Ad}(g^{-1})\theta_g , \tag{17}$$

where ε is a left-invariant vector field, Ad acts on values of θ_g.

In the case of matrix group G the invariant forms θ_g and μ_g look as follows:

$$\theta_g = dg\, g^{-1}, \tag{18}$$

$$\mu_g = g^{-1} dg\,. \tag{19}$$

For any group G there exist two covariant differential operators ∇_{L} and ∇_{R} taking values in the space $\mathscr{G}^*$. These are left and right derivatives:

$$\langle \nabla_{\mathrm{L}} f, \varepsilon\rangle (g) = -\frac{d}{dt} f(\exp(t\varepsilon)g)\,, \tag{20}$$

$$\langle \nabla_{\mathrm{R}} f, \varepsilon\rangle (g) = \frac{d}{dt} f(g\exp(\varepsilon))\,, \tag{21}$$

where exp is the exponential map from a Lie algebra to a Lie group. The simple relation for left and right derivatives of the same function f holds:

$$\nabla_{\mathrm{R}} f = -\,\mathrm{Ad}^*(g^{-1})\nabla_{\mathrm{L}} f\,. \tag{22}$$

From the very beginning the linear space $\mathscr{G}^*$ is not supplied with a natural commutator. Nevertheless, we define the commutator $[\,,\,]^*$ and put is equal to zero:

$$[l,m]^* = 0\,. \tag{23}$$

The main technical difference of the deformed theory from the classical one is that the commutator in $\mathscr{G}^*$ is nontrivial. As a consequence, the corresponding group G^* becomes nonabelian. This fact plays a crucial role in the consideration of Lie-Poisson theory. In the classical case the Lie algebra $\mathscr{G}^*$ is just abelian and the group G^* coincides with $\mathscr{G}^*$.

The space $\mathscr{G}^*$ carries a natural Poisson structure invariant with respect to the coadjoint action of G on $\mathscr{G}^*$. Let us remark that the differential of any function on $\mathscr{G}^*$ is an element of the dual space, i.e. of the Lie algebra $\mathscr{G}$. It gives us a possibility to define the following Kirillov-Kostant Poisson bracket:

$$\{f,h\}(l) = \langle l, [df(l), dh(l)]\rangle\,. \tag{24}$$

In particular, for linear functions H_ε the r.h.s. of (24) simplifies:

$$\{H_\varepsilon, H_\eta\} = H_{[\varepsilon,\eta]}\,, \tag{25}$$

$$\{H^a, H^b\} = \sum_c f_c^{ab} H^c\,. \tag{26}$$

The last formula simulates the commutation relations (1).

In the general situation the space $\mathscr{G}^*$ supplied with Poisson bracket (24) is not a symplectic manifold. The Kirillov-Kostant bracket is degenerate. For example, in the simplest case of $\mathscr{G} = su(2)$ the space $\mathscr{G}^*$ is 3-dimensional. The matrix of the Poisson bracket is antisymmetric and degenerates as any antisymmetric matrix in an odd-dimensional space.

The relation between symplectic and Poisson theories is the following. Any Poisson manifold with degenerate Poisson bracket splits into a set of symplectic leaves. A symplectic leaf is defined so that its tangent space at any point consists of the values of all hamiltonian vector fields at this point:

$$v_h(f) = \{h,f\}\,. \tag{27}$$

Each symplectic leaf inherits the Poisson bracket from the manifold. However, being restricted onto the symplectic leaf the Poisson bracket becomes nondegenerate, and we can define the symplectic two-form Ω so that:

$$\Omega(v_f, v_h) = \{f, h\}. \tag{28}$$

The relation (28) defines Ω completely because any tangent vector to the symplectic leaf can be represented as a value of some hamiltonian vector field.

If we choose dual bases $\{e_a\}$ and $\{e^a\}$ in tangent and cotangent spaces to the symplectic leaf we can rewrite the bracket and the symplectic form as follows:

$$\{f, h\} = -\sum_{ab} P^{ab} \langle df, e_a\rangle \langle dh, e_b\rangle, \tag{29}$$

$$\Omega = \sum_{ab} \Omega_{ab} e^a \otimes e^b = \frac{1}{2} \sum_{ab} \Omega_{ab} e^a \wedge e^b. \tag{30}$$

Using definition (28) of the form Ω and formulae (29), (30) one can check that the matrix Ω_{ab} is inverse to the matrix P^{ab}:

$$\sum_c \Omega_{ac} P^{cb} = \delta_a^b. \tag{31}$$

For the particular case of the space $\mathscr{G}^*$ with Poisson structure (24), there exists a nice description of the symplectic leaves. They coincide with the orbits of coadjoint action (14) of the group G. Starting from any point l_0 we can construct an orbit

$$O_{l_0} = \{l = \mathrm{Ad}^*(g) l_0, \ g \in G\}. \tag{32}$$

Any point of $\mathscr{G}^*$ belongs to some coadjoint orbit. The orbit O_{l_0} can be regarded as a quotient space of the group G over its subgroup S_{l_0}:

$$O_{l_0} \approx G/S_{l_0}, \tag{33}$$

where S_{l_0} is defined as follows:

$$S_{l_0} = \{g \in G, \ \mathrm{Ad}^*(g) l_0 = l_0\}. \tag{34}$$

In the case of $G = SU(2)$ the coadjoint action is represented by rotations in the 3-dimensional space $\mathscr{G}^*$. The orbits are spheres and there is one exceptional zero radius orbit which is just the origin. The group S_{l_0} is isomorphic to $U(1)$ and corresponds to rotations around the axis parallel to l_0. For the exceptional orbit $S_{l_0} = G$ and the quotient space G/G is a point.

Let us denote by p_{l_0} the projection from G to O_{l_0}:

$$p_{l_0} : g \to l_g = \mathrm{Ad}^*(g) l_0. \tag{35}$$

We may investigate the symplectic form Ω on the orbit directly. However, for technical reasons it is more convenient to consider its pull-back $\Omega_{l_0}^G = p_{l_0}^* \Omega$ defined on the group G itself. We reformulate Kirillov's famous result in the following form. Let O_{l_0} be a coadjoint orbit of the group G and p_{l_0} be the projection (35). The Poisson structure (24) defines a symplectic form Ω on O_{l_0}.

Theorem 1. *The pull-back of Ω along the projection p_{l_0} is the following:*

$$\Omega^G_{l_0} = \tfrac{1}{2}\langle dl_g \overset{\wedge}{,} \theta_g\rangle . \tag{36}$$

Let us remark that dl_g and θ_g are $\mathscr{G}^*$ and $\mathscr{G}$ valued 1-forms. In formula (36) we wedge them as differential forms (the sign $\wedge$) and apply the canonical pairing $\langle , \rangle$ to their values.

We do not prove formula (36), but the proof of its Lie-Poisson counterpart in Sect. 3 will fill this gap. Let us make only a few remarks. First of all, the form $\Omega^G_{l_0}$ actually is a pull-back of some two-form on the orbit O_{l_0}. Then, $\Omega^G_{l_0}$ is a closed form:

$$d\Omega^G_{l_0} = 0 . \tag{37}$$

This is a direct consequence of the Jacobi identity for the Poisson bracket (24). The form $\Omega^G_{l_0}$ is exact, while the original form Ω belongs to a nontrivial cohomology class. The left-invariant one-form

$$\alpha = \langle l_g, \theta_g\rangle = \langle l_0, \mu_g\rangle \tag{38}$$

satisfies the equation

$$d\alpha = \Omega^G_{l_0} . \tag{39}$$

In physical applications the form α defines an action for a hamiltonian system on the orbit:

$$S = \int \alpha . \tag{40}$$

Returning to the formula (36) we shall speculate with the definition of G^*. In our case $G^* = \mathscr{G}^*$ and we may treat l_g as an element of G^*. For an abelian group the Maurer-Cartan forms θ and μ coincide with the differential of the group element:

$$\theta_l = \mu_l = dl . \tag{41}$$

Using (41) we rewrite (36):

$$\Omega^G_{l_0} = \tfrac{1}{2}\langle \theta_l, \theta_g\rangle , \tag{42}$$

where l is the function of g given by formula (35). Expression (42) admits a straightforward generalization for Lie-Poisson case.

The rest of this section is devoted to the cotangent bundle T^*G of the group G. Actually, the bundle T^*G is trivial. The group G acts on itself by means of right and left multiplications. Both these actions may be used to trivialize T^*G. So we have two parametrizations of

$$T^*G = G \times \mathscr{G}^* \tag{43}$$

by pairs (g, l) and (g, m), where l and m are elements of $\mathscr{G}^*$. In the left parametrization G acts on T^*G as follows:

$$\text{L} \quad h:(g, m) \to (hg, m) , \tag{44}$$

$$\text{R} \quad h:(g, m) \to (gh^{-1}, \mathrm{Ad}^*(h)m) . \tag{45}$$

In the right parametrization left and right multiplications change roles:

$$\text{L} \quad h:(g, l) \to (hg, \mathrm{Ad}^*(h)l) , \tag{46}$$

$$\text{R} \quad h:(g, l) \to (gh^{-1}, l) . \tag{47}$$

The two coordinates l and m are related:

$$l = \mathrm{Ad}^*(g)m\,. \tag{48}$$

The cotangent bundle T^*G carries the canonical symplectic structure Ω^{T^*G} [2]. Using coordinates (g, l, m), we write a formula for Ω^{T^*G} without the proof:

$$\Omega^{T^*G} = \tfrac{1}{2}\left(\langle dm \overset{\wedge}{,} \mu_g\rangle + \langle dl \overset{\wedge}{,} \theta_g\rangle\right). \tag{49}$$

The symplectic structure on T^*G is a sort of universal one. We can recover the Kirillov two-form (36) for any orbit starting from (49). More exactly, let us impose in (49) the condition:

$$m = m_0 = \text{const}\,. \tag{50}$$

It means that instead of T^*G we consider a reduced symplectic manifold with the symplectic structure

$$\Omega_r = \tfrac{1}{2}\langle dl, \theta_g\rangle\,, \tag{51}$$

where l is subject to constraint

$$l = \mathrm{Ad}^*(g)m_0\,. \tag{52}$$

Formulae (51), (52) reproduce formulae (35), (36) and we can conclude that the reduction leads to the orbit O_{m_0} of the point m_0 in $\mathscr{G}^*$.

The aim of this paper is to present Lie-Poisson analogues of formulae (36) and (49). Having finished our sketch of the classical theory, we pass to the deformed case.

2. Heisenberg Double of Lie Bialgebra

One of the ways to introduce a deformation leading to Lie-Poisson groups is to consider the bialgebra structure on $\mathscr{G}$. Following [5], we consider a pair $(\mathscr{G}, \mathscr{G}^*)$, where we treat $\mathscr{G}^*$ as another Lie algebra with the commutator $[\,,]^*$. For a given commutator $[\,,]$ in $\mathscr{G}$ we cannot choose an arbitrary commutator $[\,,]^*$ in $\mathscr{G}^*$. The axioms of the bialgebra can be reformulated as follows. The linear space

$$\mathscr{D} = \mathscr{G} + \mathscr{G}^* \tag{53}$$

with the commutator $[\,,]_{\mathscr{D}}$:

$$[\varepsilon, \eta]_{\mathscr{D}} = [\varepsilon, \eta]\,, \tag{54}$$

$$[x, y]_{\mathscr{D}} = [x, y]^*\,, \tag{55}$$

$$[\varepsilon, x]_{\mathscr{D}} = \mathrm{ad}^*(\varepsilon)x - \mathrm{ad}^*(x)\varepsilon\,. \tag{56}$$

must be a Lie algebra. In the last formula (56) $\mathrm{ad}^*(\varepsilon)$ is the usual ad^*-operator for the Lie algebra $\mathscr{G}$ acting on $\mathscr{G}^*$. The symbol $\mathrm{ad}^*(x)$ corresponds to the coadjoint action of the Lie algebra $\mathscr{G}^*$ on its dual space $\mathscr{G}$.

The only thing we have to check is the Jacobi identity for the commutator $[\,,]_{\mathscr{D}}$. If it is satisfied, we call the pair $(\mathscr{G}, \mathscr{G}^*)$ a Lie bialgebra. Algebra $\mathscr{D}$ is called a Drinfeld double. It has the nondegenerate scalar product $\langle\,,\rangle_{\mathscr{D}}$:

$$\langle(\varepsilon, x), (\eta, y)\rangle_{\mathscr{D}} = \langle y, \varepsilon\rangle + \langle x, \eta\rangle\,, \tag{57}$$

where in the r.h.s. $\langle\,,\rangle$ is the canonical pairing of $\mathscr{G}$ and $\mathscr{G}^*$. It is easy to see that

$$\langle \mathscr{G}, \mathscr{G}\rangle_{\mathscr{D}} = 0\,, \qquad \langle \mathscr{G}^*, \mathscr{G}^*\rangle_{\mathscr{D}} = 0\,. \tag{58}$$

In other words, $\mathscr{G}$ and $\mathscr{G}^*$ are isotropic subspaces in $\mathscr{D}$ with respect to the form $\langle\,,\rangle_{\mathscr{D}}$. We call the form $\langle\,,\rangle_{\mathscr{D}}$ on the algebra $\mathscr{D}$ standard product in $\mathscr{D}$.

We shall need two operators P and P^* acting in $\mathscr{D}$. P is defined as a projector onto the subspace $\mathscr{G}$:

$$P(x+\varepsilon) = \varepsilon\,. \tag{59}$$

The operator P^* is its conjugate with respect to form (57). It appears to be a projector onto the subspace $\mathscr{G}^*$:

$$P^*(x+\varepsilon) = x\,. \tag{60}$$

The standard product in $\mathscr{D}$ enables us to define the canonical isomorphism $J:\mathscr{D}^* \to \mathscr{D}$ by means of the formula

$$\langle J(a^*), b\rangle_{\mathscr{D}} = \langle a^*, b\rangle\,, \tag{61}$$

where a^* is an element of $\mathscr{D}^*$ and b belongs to $\mathscr{D}$. In the r.h.s. we use the canonical pairing of $\mathscr{D}$ and $\mathscr{D}^*$. The standard product can be defined on the space $\mathscr{D}^*$:

$$\langle a^*, b^*\rangle_{\mathscr{D}^*} = \langle J(a^*), J(b^*)\rangle_{\mathscr{D}}\,, \tag{62}$$

where a^* and b^* belong to $\mathscr{D}^*$. The scalar product $\langle\,,\rangle_{\mathscr{D}}$ is invariant with respect to the commutator in $\mathscr{D}$:

$$\langle [a,b], c\rangle_{\mathscr{D}} + \langle b, [a,c]\rangle_{\mathscr{D}} = 0\,. \tag{63}$$

It is easy to check that the operator J converts ad^* into ad:

$$J\,\mathrm{ad}^*(a)J^{-1} = \mathrm{ad}(a)\,. \tag{64}$$

Using the standard scalar product in $\mathscr{D}$, one can construct elements r and r^* in $\mathscr{D}\otimes\mathscr{D}$ which correspond to the operators P and P^*:

$$\langle a\otimes b, r\rangle_{\mathscr{D}\otimes\mathscr{D}} = \langle a, Pb\rangle_{\mathscr{D}}\,, \tag{65}$$

$$\langle a\otimes b, r^*\rangle_{\mathscr{D}\otimes\mathscr{D}} = -\langle a, P^*b\rangle_{\mathscr{D}}\,. \tag{66}$$

In terms of dual bases $\{\varepsilon^a\}$ and $\{l_a\}$ in $\mathscr{G}$ and $\mathscr{G}^*$,

$$r = \sum_a \varepsilon^a \otimes l_a\,, \qquad r^* = -\sum_a l_a \otimes \varepsilon^a\,. \tag{67}$$

The Lie algebra $\mathscr{D}$ may be used to construct the Lie group D. We suppose that D exists (for example, for finite dimensional algebras it is granted by the Lie theorem) and we choose it to be connected. Originally the double is defined as a connected and simply connected group. However, we may use any connected group D corresponding to Lie algebra $\mathscr{D}$. Property (64) can be generalized for Ad and Ad*:

$$J\,\mathrm{Ad}^*(d)J^{-1} = \mathrm{Ad}(d)\,, \tag{68}$$

where d is an element of D.

Let us denote by G and G^* the subgroups in D corresponding to subalgebras $\mathscr{G}$ and $\mathscr{G}^*$ in $\mathscr{D}$. In the vicinity of the unit element of D the following two decompositions are applicable:

$$d = gg^* = h^*h\,, \tag{69}$$

where d is an element of D, coordinates g, h belong to the subgroup G, coordinates g^*, h^* belong to the subgroup G^*.

To generalize formula (69), let us consider the set $\mathfrak{I}$ of classes $G\backslash D/G^*$. We denote individual classes by small letters $i, j, \ldots$. Let us pick up a representative d_i in each class i. If an element d belongs to the class i, it can be represented in the form

$$d = g d_i g^* \tag{70}$$

for some g and g^*. In the general case the elements g and g^* in decomposition (70) are not defined uniquely. If $S(d_i)$ is a subgroup in G,

$$S(d_i) = \{h \in G, d_i^{-1} h d_i \in G^*\}, \tag{71}$$

we can take a pair $(gh, d_i^{-1}h^{-1}d_i g^*)$ instead of (g, g^*), where h is an arbitrary element of $S(d_i)$. We denote $T(d_i)$ the corresponding subgroup in G^*:

$$T(d_i) = d_i^{-1} S(d_i) d_i . \tag{72}$$

So we have the following stratification of the double D:

$$D = \bigcup_{i\in\mathfrak{I}} G d_i G^* = \bigcup_{i\in\mathfrak{I}} C_i . \tag{73}$$

Each cell

$$C_i = G d_i G^* \tag{74}$$

in this decomposition is isomorphic to the quotient of the direct product $G \times G^*$ over $S(d_i)$, where

$$(g.g^*) \sim (g', g^{*\prime}) \quad \text{if} \tag{75}$$

$$g' = gh, \qquad g^{*\prime} = d_i^{-1} h^{-1} d_i g, \qquad h \in S(d_i). \tag{76}$$

For the inverse element d^{-1} in the relation (70) we get another stratification of D in which G and G^* replace each other:

$$D = \bigcup_{i\in\mathfrak{I}} G^* d_i^{-1} G = \bigcup_{i\in\mathfrak{I}} c_i . \tag{77}$$

Now we turn to the description of the Poisson brackets on the manifold D. Double D admits two natural Poisson structures. The first was proposed by Drinfeld [5]. For two functions f and h on D the Drinfeld bracket is equal to

$$\{f, h\} = \langle \nabla_{\mathrm{L}} f \otimes \nabla_{\mathrm{L}} h, r\rangle - \langle \nabla_{\mathrm{R}} f \otimes \nabla_{\mathrm{R}} h, r\rangle , \tag{78}$$

where $\langle\, ,\rangle$ is the canonical pairing between $\mathscr{D}\otimes\mathscr{D}$ and $\mathscr{D}^*\otimes\mathscr{D}^*$. The Poisson bracket (78) defines a structure of a Lie-Poisson group on D. However, the most important for us is the second Poisson structure on D suggested by Semenov-Tian-Shansky [13] (see also [11]):

$$\{f, h\} = -(\langle \nabla_{\mathrm{L}} f \otimes \nabla_{\mathrm{L}} h, r\rangle + \langle \nabla_{\mathrm{R}} f \otimes \nabla_{\mathrm{R}} h, r^*\rangle) . \tag{79}$$

The brackets (78) and (79) are skew-symmetric because the symmetric parts of both r and r^* are Ad-invariant.

The manifold D equipped with bracket (79) is called the Heisenberg double or D_+. It is a natural analogue of T^*G in the Lie-Poisson case. When $\mathscr{G}^*$ is abelian,

$G^* = \mathscr{G}^*$ and $D_+ = T^*G$. If the double D is a matrix group, we can rewrite the basic formula (79) in the following form:

$$\{d^1, d^2\} = -(rd^1d^2 + d^1d^2r^*), \tag{80}$$

where $d^1 = d \otimes I$, $d^2 = I \otimes d$.

The problem which appears immediately in the theory of D_+ is the possible degeneracy of the Poisson structure (79) in some points of D. It is important to describe the stratification of D_+ into the set of symplectic leaves. The answer is given by the following

Theorem 2. *Symplectic leaves of D_+ are connected components of nonempty intersections of left and right stratification cells:*

$$D_{ij} = C_i \cap c_j = Gd_iG^* \cap G^*d_j^{-1}G. \tag{81}$$

Remark. The double D_+ is a symplectic manifold if the product GG^* provides a global decomposition of D [12].

Proof. The tangent space T_d^S to the symplectic leaf at the point d coincides with the space of values of all hamiltonians vector fields at this point. For concrete calculations let us choose the left identification of the tangent space to D with $\mathscr{D}$. We can rewrite the Poisson bracket (79) in terms of left derivatives ∇_L:

$$\begin{aligned}\{f, h\}(d) &= -(\langle \nabla_L f \otimes \nabla_L r\rangle + \langle \mathrm{Ad}^*(d^{-1})\nabla_L f \otimes \mathrm{Ad}^*(d^{-1})\nabla_L h, r^*\rangle) \\ &= -\langle \nabla_L f \otimes \nabla_L h, r + \mathrm{Ad}(d) \otimes \mathrm{Ad}(d) r^*\rangle .\end{aligned} \tag{82}$$

Here we use relation (22) between left and right derivatives on a group.

A hamiltonian h produces the hamiltonian vector field v_h so that the formula

$$\langle df, v_h\rangle = \{h, f\} \tag{83}$$

holds for any function f. Using (82), (83) we can reconstruct the field v_h:

$$v_h = \langle \nabla_L h, r + \mathrm{Ad}(d) \otimes \mathrm{Ad}(d) r^*\rangle_2 . \tag{84}$$

Here the subscript 2 in the r.h.s. means that the pairing is applied only to the second component of the r-matrix expression $r + \mathrm{Ad}(d) \otimes \mathrm{Ad}(d)r^*$. Having identified $\mathscr{D}$ and $\mathscr{D}^*$ by means of the operator J, we can rewrite the r.h.s. of (84) as follows:

$$v_h|_d = \mathscr{P}dh = (P - \mathrm{Ad}(d)P^* \,\mathrm{Ad}(d^{-1}))J(\nabla_L h(d)), \tag{85}$$

where $\mathscr{P}$ acts in $\mathscr{D}$:

$$\mathscr{P} = P - \mathrm{Ad}(d)P^* \,\mathrm{Ad}(d^{-1}). \tag{86}$$

It is called a Poisson operator. Using the fact that the value of $\nabla_L h$ at the point d is an arbitrary vector from $\mathscr{D}^*$, we conclude that T_d^S coincides with the image of the operator $\mathscr{P}$:

$$T_d^S = \mathrm{Im}\,\mathscr{P}. \tag{87}$$

The most simple way to describe the image of $\mathscr{P}$ is to use the property:

$$\mathrm{Im}\,\mathscr{P} = (\mathrm{Ker}\,\mathscr{P}^*)^\perp . \tag{88}$$

Here conjugation and the symbol $\perp$ correspond to the standard product in $\mathscr{D}$. The operator $\mathscr{P}^*$ is given by the formula

$$\mathscr{P}^* = P^* - \mathrm{Ad}(d)P\,\mathrm{Ad}(d^{-1}). \tag{89}$$

Suppose that a vector $a = x + \varepsilon$ belongs to $\mathrm{Ker}\mathscr{P}^*$:

$$\mathscr{P}^*(x+\varepsilon) = 0\,. \tag{90}$$

Let us rewrite the condition (90) in the following form:

$$(\mathrm{Ad}(d^{-1})P^* - P\,\mathrm{Ad}(d^{-1}))\,(x+\varepsilon) = 0\,, \tag{91}$$

or, equivalently,

$$\mathrm{Ad}(d^{-1})x = P(\mathrm{Ad}(d^{-1})x + \mathrm{Ad}(d^{-1})\varepsilon)\,. \tag{92}$$

Using the property

$$P + P^* = id \tag{93}$$

of the projectors P and P^*, one can get from (92):

$$P^*(\mathrm{Ad}(d^{-1})x) = P(\mathrm{Ad}(d^{-1})\varepsilon)\,. \tag{94}$$

The l.h.s. of (94) is a vector from $\mathscr{G}^*$ whereas the r.h.s. belongs to $\mathscr{G}$. So Eq. (94) implies that both the l.h.s. and the r.h.s. are equal to zero.

Let $V(d)$ be the subspace in $\mathscr{G}$ defined by the following condition:

$$V(d) = \{\varepsilon \in \mathscr{G}, \mathrm{Ad}(d^{-1})\varepsilon \in \mathscr{G}^*\}\,. \tag{95}$$

In the same way we define the subspace $V^*(d)$ in $\mathscr{G}^*$:

$$V^*(d) = \{\varepsilon \in \mathscr{G}^*, \mathrm{Ad}(d^{-1})\varepsilon \in \mathscr{G}\}\,. \tag{96}$$

It is not difficult to check that $V(d)$ and $V^*(d)$ are actually Lie subalgebras in $\mathscr{G}$ and $\mathscr{G}^*$. The kernel of the operator $\mathscr{P}^*$ may be represented as a direct sum of $V(d)$ and $V^*(d)$:

$$\mathrm{Ker}\mathscr{P}^* = V(d) \oplus V^*(d)\,. \tag{97}$$

The tangent space T^S_d to the symplectic leaf at the point d acquires the form

$$T^S_d = (V(d) \oplus V^*(d))^{\perp}. \tag{98}$$

The result (98) can be rewritten:

$$T^S_d = V(d)^{\perp} \cap V^*(d)^{\perp} = (V(d)^{\perp} \cap \mathscr{G}^*) \oplus (V^*(d)^{\perp} \cap \mathscr{G})\,. \tag{99}$$

Here the last expression represents T^S_d as a direct sum of its intersections with $\mathscr{G}$ and $\mathscr{G}^*$.

Now we must compare subspace (99) with the tangent space T'_d of the intersection of the stratification cells (Theorem 2). Suppose that the point d belongs to the cell D_{ij} of the stratification. We can rewrite the definition of D_{ij} as follows:

$$D_{ij} = GdG^* \cap G^*dG = C(d) \cap c(d)\,. \tag{100}$$

The tangent space to D_{ij} may be represented as an intersection of tangent spaces to left and right cells $C(d)$ and $c(d)$:

$$T'_d = T_d(C(d)) \cap T_d(c(d))\,. \tag{101}$$

For the latter the following formulae are true:

$$T_d(C(d)) = \mathscr{G} + \mathrm{Ad}(d)\mathscr{G}^*\,, \tag{102}$$

$$T_d(c(d)) = \mathscr{G}^* + \mathrm{Ad}(d)\mathscr{G}\,. \tag{103}$$

The space $T_d(C(d))$ coincides with $V(d)^\perp$. Indeed, $T_d(C(d))^\perp$ lies in $\mathcal{G}$ because $T_d(C(d))^\perp \subset \mathcal{G}^\perp = \mathcal{G}$. On the other hand

$$\langle T_d(C(d))^\perp, \mathrm{Ad}(d)\mathcal{G}^*\rangle_{\mathcal{D}} = 0\,. \tag{104}$$

Formula (104) implies that $\mathrm{Ad}(d^{-1})T_d(C(d))^\perp \subset \mathcal{G}^{*\perp} = \mathcal{G}^*$. So $T_d(C(d))^\perp$ is the subspace in $\mathcal{G}$ which is mapped by $\mathrm{Ad}(d^{-1})$ into $\mathcal{G}^*$. It is the subspace $V(d)$ that satisfies these conditions. So we have

$$T_d(C(d))^\perp = V(d)\,, \qquad T_d(C(d)) = V(d)^\perp\,. \tag{105}$$

Similarly,

$$T_d(c(d)) = V^*(d)^\perp\,. \tag{106}$$

Comparing (99), (101), (105), (106), we conclude that the tangent space T_d' to the cell D_{ij} coincides with the tangent space T_d^S to the symplectic leaf. Thus the symplectic leaf coincides with a connected component of the cell D_{ij}.

We have proved Theorem 2. The next question concerns the symplectic structure on the leaves D_{ij}.

3. Symplectic Structure of the Heisenberg Double

Each symplectic leaf D_{ij} introduced in the last section carries a nondegenerate Poisson structure and hence the corresponding symplectic form Ω_{ij} can be defined. To write down the answer we need several new objects. Let us denote by L_{ij} the subset in $G \times G^*$ defined as follows:

$$L_{ij} = \{(g, g^*) \in G \times G^*, g d_i g^* \in D_{ij}\}\,. \tag{107}$$

In the same way we construct the subset M_{ij} in $G^* \times G$:

$$M_{ij} = \{(h^*, h) \in G^* \times G, h^* d_j^{-1} h \in D_{ij}\}\,. \tag{108}$$

Finally let N_{ij} be the subset in $L_{ij} \times M_{ij}$:

$$N_{ij} = \{[g, g^*), (h^*, h)] \in L_{ij} \times M_{ij}, g d_i g^* = h^* d_j^{-1} h\}\,. \tag{109}$$

We can define the projection

$$p_{ij} : N_{ij} \to D_{ij}\,, \tag{110}$$

$$p_{ij} : [(g, g^*), (h^*, h)] \to d = g d_i g^* = h^* d_j^{-1} h\,, \tag{111}$$

and consider the form $p_{ij}^* \Omega_{ij}$ on N_{ij} instead of the original form Ω_{ij} on D_{ij}. It is parallel to the construction of the Kirillov form on the coadjoint orbit (see Sect. 1). Parametrizations (107), (108) provide us with the coordinates (g, g^*) and (h^*, h) on N_{ij}. We can use them to write down the answer:

Theorem 3. *The symplectic form $p_{ij}^* \Omega_{ij}$ on N_{ij} can be represented as follows:*

$$p_{ij}^* \Omega_{ij} = \tfrac{1}{2}\left(\langle \theta_{h*} \overset{\wedge}{,} \theta_g \rangle + \langle \mu_{g*} \overset{\wedge}{,} \mu_h \rangle\right). \tag{112}$$

In formula (112) $\theta_g, \theta_{h*}, \mu_h, \mu_{g*}$ are restrictions of the corresponding one-forms from $(G \times G^*) \times (G^* \times G)$ to N_{ij}. The pairing $\langle\,,\rangle$ is applied to values of Maurer-Cartan forms, which can be treated as elements of $\mathscr{G}$ and $\mathscr{G}^*$ embedded to $\mathscr{D} = \mathscr{G} + \mathscr{G}^*$. So we can use $\langle\, ,\rangle_{\mathscr{D}}$ as well as $\langle\, ,\rangle$.

Proof of Theorem 3. The strategy of the proof is quite straightforward. We consider the Poisson bracket (79) on the symplectic leaf D_{ij}. If we use dual bases $\{e_a\}$ and $\{e^a\}$ $(a = 1, \ldots, n = \dim D)$ of right-invariant vector fields and one-forms on D, formula (79) acquires the following form:

$$\begin{aligned} \{f,h\}(d) &= -\langle \nabla_{\rm L} f \otimes \nabla_{\rm L} h, r + \mathrm{Ad}(d) \otimes \mathrm{Ad}(d) r^* \rangle \\ &= -\sum_{a,b=1}^{n} \langle \nabla_{\rm L} f, e_a \rangle \langle \nabla_{\rm L} h, e_b \rangle \langle e^a, \mathscr{P} J e^b \rangle . \end{aligned} \tag{113}$$

The last multiplier in (113) is the Poisson matrix corresponding to the bracket (79):

$$\mathscr{P}^{ab} = \langle e^a, \mathscr{P} J e^b \rangle . \tag{114}$$

Here $\mathscr{P}$ is the same as in (86). The matrix $\mathscr{P}^{ab}$ may be degenerate. Let us choose vectors $\{e_a, a \in s_{ij} = \{1, \ldots, n_{ij} = \dim D_{ij}\}\}$ so that they form a basis in the space T_d tangent to D_{ij}. $\mathscr{P}^{ab}$ is not zero only if both a and b belong to s_{ij}. The symplectic form Ω_{ij} on the cell D_{ij} can be represented as follows (see Sect. 1):

$$\Omega_{ij} = \sum_{a,b=1}^{n_{ij}} \Omega_{ab} e^a \otimes e^b , \tag{115}$$

where the matrix Ω satisfies the following condition:

$$\sum_{c=1}^{nij} \Omega_{ac} \mathscr{P}^{cb} = \delta_a^b . \tag{116}$$

So what we need is the inverse matrix $\mathscr{P}^{-1}$ for $\mathscr{P}^{ab}$. To make the symbol $\mathscr{P}^{-1}$ meaningful we introduce two operators $\mathscr{P}_1$ and $\mathscr{P}_2$:

$$\mathscr{P}_1 = (P + \mathrm{Ad}(d) P^*) , \tag{117}$$

$$\mathscr{P}_2 = (P^* - \mathrm{Ad}(d) P) . \tag{118}$$

$\mathscr{P}$ may b decomposed in two ways, using $\mathscr{P}_1$ and $\mathscr{P}_2$:

$$\mathscr{P} = \mathscr{P}_1 \mathscr{P}_2^* = -\mathscr{P}_2 \mathscr{P}_1^* . \tag{119}$$

Some useful properties of the operators $\mathscr{P}_1$ and $\mathscr{P}_2$ are collected in the following lemma.

Lemma 1.

$$\begin{aligned} \mathrm{Im}\,\mathscr{P}_1 = V(d)^{\perp}, &\qquad \mathrm{Im}\,\mathscr{P}_2 = V^*(d)^{\perp}, \\ P(\mathrm{Ker}\,\mathscr{P}_1) = V(d) , &\qquad P^*(\mathrm{Ker}\,\mathscr{P}_2) = V^*(d) . \end{aligned} \tag{120}$$

Proof. First let us consider the formula

$$\mathrm{Im}\,\mathscr{P}_1 = (\mathrm{Ker}\,\mathscr{P}_1^*)^{\perp} . \tag{121}$$

The operator $\mathscr{P}_1^*$ looks as follows:

$$\mathscr{P}_1^* = P^* + P\,\mathrm{Ad}(d^{-1}) . \tag{122}$$

The equation for $\operatorname{Ker}\mathscr{P}_1^*$,

$$(P^* + P\operatorname{Ad}(d^{-1}))\,(x+\varepsilon) = 0\,, \tag{123}$$

leads immediately to the following restrictions for x and ε:

$$x = 0\,, \qquad \operatorname{Ad}(d^{-1})\varepsilon \in \mathscr{G}^*\,. \tag{124}$$

Comparing (124) with definition (95), we see that $\operatorname{Ker}\mathscr{P}_1^* = V(d)$ and hence $\operatorname{Im}\mathscr{P}_1 = V(d)^\perp$.

If a vector $x+\varepsilon$ belongs to the kernel of the operator $\mathscr{P}_1$, it satisfies the following equation:

$$(P + \operatorname{Ad}(d)P^*)\,(x+\varepsilon) = 0\,. \tag{125}$$

It can be rewritten as a set of conditions for the components x, ε:

$$\operatorname{Ad}(d^{-1})\varepsilon \in \mathscr{G}^*\,, \qquad x = -\operatorname{Ad}(d^{-1})\varepsilon\,. \tag{126}$$

ε again appears to be an element of $V(d)$. This fact may be represented as the equation $P(\operatorname{Ker}\mathscr{P}_1) = V(d)$.

We omit the proofs of formulae (120) concerning the operator $\mathscr{P}_2$ because they are parallel to the proofs given above.

The following step is to define inverse operators:

$$\mathscr{P}_1^{-1}: \operatorname{Im}\mathscr{P}_1 \to \mathscr{D}/\operatorname{Ker}\mathscr{P}_1\,, \tag{127}$$

$$\mathscr{P}_2^{-1}: \operatorname{Im}\mathscr{P}_2 \to \mathscr{D}/\operatorname{Ker}\mathscr{P}_2\,. \tag{128}$$

The solution of the equation

$$\mathscr{P}_{1,2}^{-1}a = b \tag{129}$$

exists if and only if $a \in \operatorname{Im}\mathscr{P}_{1,2}$ and b is defined up to an arbitrary vector from $\operatorname{Ker}\mathscr{P}_{1,2}$.

Now we are ready to write down the answer for Ω_{ab}:

$$\Omega_{ab} = \langle e_a, \Omega e_b\rangle_{\mathscr{D}}\,, \qquad \Omega = P\mathscr{P}_1^{-1} - P^*\mathscr{P}_2^{-1}. \tag{130}$$

First of all let us check that matrix elements Ω_{ab} are well-defined. Vectors e_b form the basis in the space $T_d = (V(d)\oplus V^*(d))^\perp$. Both $\mathscr{P}_1^{-1}$ and $\mathscr{P}_2^{-1}$ are defined on T_d because $T_d \subset V(d)^\perp = \operatorname{Im}\mathscr{P}_1$ and also $T_d \subset V^*(d)^\perp = \operatorname{Im}\mathscr{P}_2$. So the vector Ωe_b exists but it is not unique. It is defined up to an arbitrary vector,

$$\delta \in P(\operatorname{Ker}\mathscr{P}_1) + P^*(\operatorname{Ker}\mathscr{P}_2) = V(d) + V^*(d)\,. \tag{131}$$

Fortunately the vector $e_a \in T_d$ and $\langle e_a, \delta\rangle = 0$ for any δ of the form (131). We conclude that the ambiguity in the definition of the operator Ω does not lead to an ambiguity for matrix elements Ω_{ab}.

Now we must check condition (116):

$$\delta_a^b = \sum_{c=1}^{n_{ij}} \Omega_{ac}\mathscr{P}^{cb} = \sum_{c=1}^{n_{ij}} \langle e_a, \Omega e_c\rangle\,\langle e^c, \mathscr{P}J(e^b)\rangle = \langle e_a, \Omega\mathscr{P}J(e^b)\rangle_{\mathscr{D}}\,. \tag{132}$$

The product $\Omega\mathscr{P}$ can be easily calculated using (119), (130):

$$\begin{aligned}\Omega\mathscr{P} &= P\mathscr{P}_1^{-1}\mathscr{P}_1\mathscr{P}_2^* + P^*\mathscr{P}_2^{-1}\mathscr{P}_2\mathscr{P}_1^* \\ &= P(P - P^*\operatorname{Ad}(d^{-1})) + P^*(P^* + P\operatorname{Ad}(d^{-1})) = P + P^* = I\,.\end{aligned} \tag{133}$$

We must remember that the vector $\Omega\mathscr{P}J(e^b)$ is defined up to an arbitrary vector from $V(d) \oplus V^*(d)$ because in (133) we used the "identities"

$$\mathscr{P}_1^{-1}\mathscr{P}_1 \approx \mathscr{P}_2^{-1}\mathscr{P}_2 \approx id\,. \tag{134}$$

The ambiguity in (134) does not influence the answer:

$$\langle e_a, \Omega\mathscr{P}J(e^b)\rangle_{\mathscr{D}} = \langle e^b, e_a\rangle = \delta_a^b \tag{135}$$

as it is required by (116).

We can rewrite formula (130) in more invariant way:

$$\Omega_{ij} = \langle \theta_d^{ij} \overset{\otimes}{,} \Omega\theta_d^{ij}\rangle_{\mathscr{D}}\,, \tag{136}$$

where θ_d^{ij} is the restriction of the Maurer-Cartan form to the cell D_{ij}. Expression (130) for the operator Ω still includes inverse operators $\mathscr{P}_{1,2}^{-1}$ implying that some equations must be solved. To this end we consider the pull-back of the form Ω_{ij}:

$$p_{ij}^*\Omega_{ij} = \langle p_{ij}^*\theta_d^{ij} \overset{\otimes}{,} \Omega p_{ij}^*\theta_d^{ij}\rangle_{\mathscr{D}}\,. \tag{137}$$

There are coordinates (g, g^*) and (h^*, h) on N_{ij}. The Maurer-Cartan form $p_{ij}^*\theta_d^{ij}$ can be rewritten in two ways:

$$p_{ij}^*\theta_d^{ij} = \theta_g + \mathrm{Ad}(d)\mu_{g^*}\,, \tag{138}$$

$$p_{ij}^*\theta_d^{ij} = \theta_{h^*} + \mathrm{Ad}(d)\mu_h\,. \tag{139}$$

Representations (138), (139) allow us to calculate $\mathscr{P}_{1,2}^{-1}p_{ij}^*\theta_d^{ij}$ explicitly:

$$\mathscr{P}_1^{-1}p_{ij}^*\theta_d^{ij} = \theta_g + \mu_{g^*}\,, \tag{140}$$

$$\mathscr{P}_2^{-1}p_i^*\theta_d^{ij} = \theta_{h^*} - \mu_h\,. \tag{141}$$

Let us mention again that solutions (140), (141) are not unique. We can take any possible value of $\Omega\theta_d^{ij}$. The answer for the form Ω_{ij} is independent of this choice.

Putting together (130), (137), (140) and (141), we obtain the following formula for the symplectic form:

$$\begin{aligned} p_{ij}^*\Omega_{ij} &= \langle(\theta_g + \mathrm{Ad}(d)\mu_{g^*})\overset{\otimes}{,}\theta_g\rangle_{\mathscr{D}} - \langle(\theta_{h^*} + \mathrm{Ad}(d)\mu_h)\overset{\otimes}{,}\theta_{h^*}\rangle_{\mathscr{D}} \\ &= \langle \mathrm{Ad}(d)\mu_{g^*} \overset{\otimes}{,} \theta_g\rangle_{\mathscr{D}} - \langle \mathrm{Ad}(d)\mu_h \overset{\otimes}{,} \theta_{h^*}\rangle_{\mathscr{D}}\,. \end{aligned} \tag{142}$$

Actually, the form (142) is antisymmetric. To make it evident, let us consider the identity

$$\begin{aligned} \langle p_{ij}^*\theta_d^{ij} \overset{\otimes}{,} p_{ij}^*\theta_d^{ij}\rangle_{\mathscr{D}} &= \langle \mathrm{Ad}(d)\mu_{g^*} \overset{\otimes}{,} \theta_g\rangle_{\mathscr{D}} + \langle \theta_g \overset{\otimes}{,} \mathrm{Ad}(d)\mu_{g^*}\rangle_{\mathscr{D}} \\ &= \langle \mathrm{Ad}(d)\mu_h \overset{\otimes}{,} \theta_{h^*}\rangle_{\mathscr{D}} + \langle \theta_{h^*} \overset{\otimes}{,} \mathrm{Ad}(d)\mu_h\rangle_{\mathscr{D}}\,. \end{aligned} \tag{143}$$

Or, equivalently

$$\begin{aligned} &\langle \mathrm{Ad}(d)\mu_{g^*} \overset{\otimes}{,} \theta_g\rangle_{\mathscr{D}} - \langle \mathrm{Ad}(d)\mu_h \overset{\otimes}{,} \theta_{h^*}\rangle_{\mathscr{D}} \\ &\quad = -\langle \theta_g \overset{\otimes}{,} \mathrm{Ad}(d)\mu_{g^*}\rangle_{\mathscr{D}} + \langle \theta_{h^*} \overset{\otimes}{,} \mathrm{Ad}(d)\mu_h\rangle_{\mathscr{D}}\,. \end{aligned} \tag{144}$$

Applying (144) to make (142) manifestly antisymmetric, one gets:

$$p_{ij}^*\Omega_{ij} = \tfrac{1}{2}\left(\langle \mathrm{Ad}(d)\mu_{g^*} \overset{\wedge}{,} \theta_g\rangle_{\mathscr{D}} + \langle \theta_{h^*} \overset{\wedge}{,} \mathrm{Ad}(d)\mu_h\rangle_{\mathscr{D}}\right). \tag{145}$$

Using representation (111) of d in terms of (g, g^*) and (h^*, h), it is easy to check that formula (145) coincides with

$$p_{ij}^* \Omega_{ij} = -\tfrac{1}{2}\left(\langle \mu_g \overset{\wedge}{,} \mathrm{Ad}(d_i)\theta_{g^*}\rangle_{\mathscr{D}} + \langle \theta_h \overset{\wedge}{,} \mathrm{Ad}(d_j)\mu_{h^*}\rangle_{\mathscr{D}}\right). \tag{146}$$

To obtain formula (112) one can use (138), (139):

$$p_{ij}^* \theta_d^{ij} = \theta_g + \mathrm{Ad}(d)\mu_{g^*} = \theta_{h^*} + \mathrm{Ad}(d)\mu_h . \tag{147}$$

Or, equivalently,

$$\theta_g - \mathrm{Ad}(d)\mu_h = \theta_{h^*} - \mathrm{Ad}(d)\mu_{g^*} . \tag{148}$$

Due to antisymmetry we have

$$\langle (\theta_g - \mathrm{Ad}(d)\mu_h) \overset{\wedge}{,} (\theta_{h^*} - \mathrm{Ad}(d)\mu_{g^*})\rangle_{\mathscr{D}} = 0 . \tag{149}$$

Therefore,

$$\begin{aligned} &\tfrac{1}{2}\left(\langle \theta_{h^*} \overset{\wedge}{,} \theta_g\rangle_{\mathscr{D}} + \langle \mu_{g^*} \overset{\wedge}{,} \mu_h\rangle_{\mathscr{D}}\right) \\ &\quad = \tfrac{1}{2}\left(\langle \mathrm{Ad}(d)\mu_{g^*} \overset{\wedge}{,} \theta_g\rangle_{\mathscr{D}} + \langle \theta_{h^*} \overset{\wedge}{,} \mathrm{Ad}(d)\mu_h\rangle_{\mathscr{D}}\right) = p_{ij}^* \Omega_{ij} , \end{aligned} \tag{150}$$

which coincides with (112).

Now we have to check that the r.h.s. of formula (112) does represent the pullback of some two-form on D_{ij}. The problem is in the ambiguity of formula (70). Coordinates g and g^* are defined only up to the following change of variables:

$$g' = gs , \qquad g^{*\prime} = tg^* , \tag{151}$$

where

$$sd_i t = d_i . \tag{152}$$

Here s is an element of $S(d_i)$ and t belongs to $T(d_i)$. The parameter s determines t by means of formula (152). Similar ambiguity exists in the definition of h and h^*. We can construct an infinitesimal analogue of formula (151). The vector field v_ε on N_{ij},

$$v_\varepsilon = (\mathrm{Ad}(g)\varepsilon, -\mathrm{Ad}(d_i^{-1})\varepsilon) , \tag{153}$$

does not correspond to any nonzero vector field on D_{ij}. Here we use coordinates (g, g^*) on N_{ij} and left identification of vector fields on $G \times G^*$ and $\mathscr{G} + \mathscr{G}^*$. So the first term is an element of $\mathscr{G}$ and the second one belongs to $\mathscr{G}^*$. Therefore $\mathrm{Ad}(g)\varepsilon$ belongs to $V(d_i)$ (see Sect. 2).

Actually we must check two nontrivial statements:

i. Form $p_{ij}^* \Omega_{ij}$ is invariant with respect to change of variables (151). It follows from the definition of the Maurer-Cartan forms θ and μ.

ii. Tangent vectors (153) belong to the kernel of $p_{ij}^* \Omega_{ij}$.

It is convenient to use expression (146) for $p_{ij}^* \Omega_{ij}$:

$$p_{ij}^* \Omega_{ij} = -\tfrac{1}{2}\left(\langle \mu_g \overset{\wedge}{,} \mathrm{Ad}(d_i)\theta_{g^*}\rangle_{\mathscr{D}} + \langle \theta_h \overset{\wedge}{,} \mathrm{Ad}(d_j)\mu_{h^*}\rangle_{\mathscr{D}}\right) = -\tfrac{1}{2}(\omega_1 + \omega_2) , \tag{154}$$

where

$$\omega_1 = \langle \mu_g \overset{\wedge}{,} \mathrm{Ad}(d_i)\theta_{g^*}\rangle_{\mathscr{D}} , \tag{155}$$

$$\omega_2 = \langle \theta_h \overset{\wedge}{,} \mathrm{Ad}(d_j)\mu_{g^*}\rangle_{\mathscr{D}} . \tag{156}$$

We have to consider $\omega_1(\,.\,,v_\varepsilon)$ and $\omega_2(\,.\,,v_\varepsilon)$,

$$\begin{aligned}\omega_1(\,.\,,v_\varepsilon) &= \langle \mu_g, \mathrm{Ad}(d_i)\theta_{g^*}(v_\varepsilon)\rangle_{\mathscr{D}} - \langle \mu_g(v_\varepsilon), \mathrm{Ad}(d_i)\theta_{g^*}\rangle_{\mathscr{D}} \\ &= \langle \mu_g, \mathrm{Ad}(d_i)\,\mathrm{Ad}(d_i^{-1})\varepsilon\rangle_{\mathscr{D}} + \langle \mathrm{Ad}(g^{-1})\,\mathrm{Ad}(g)\varepsilon, \mathrm{Ad}(d_i)\theta_{g^*}\rangle_{\mathscr{D}} \\ &= \langle \mu_g, \varepsilon\rangle_{\mathscr{D}} + \langle \theta_{g^*}, \mathrm{Ad}(d_i^{-1})\varepsilon\rangle_{\mathscr{D}}\,. \end{aligned} \tag{157}$$

Here we use properties (16), (17) of the Maurer-Cartan forms. It is easy to see that both terms in the last expression (157) are equal to zero. The first of them

$$\langle \mu_g, \varepsilon\rangle_{\mathscr{D}} = 0\,, \tag{158}$$

because both ε and a value of μ_g belong to $\mathscr{G}$. All is the same with the second term:

$$\langle \theta_{g^*}, \mathrm{Ad}(d_i^{-1})\varepsilon\rangle_{\mathscr{D}} = 0 \tag{159}$$

because for $\mathrm{Ad}(g)\varepsilon \in V(d_i)$ the combination $\mathrm{Ad}(d_i^{-1})\varepsilon$ belongs to $\mathscr{G}^*$. We remind that both $\mathscr{G}$ and $\mathscr{G}^*$ are isotropic subspaces in $\mathscr{D}$.

We omit the proof for the second term ω_2 in (154) because it is quite parallel to the one described above. We conclude that form (112) indeed corresponds to some two-form on the symplectic leaf D_{ij}.

It is known from general Poisson theory that

$$d\Omega = 0\,, \tag{160}$$

but it is interesting to check that form (112) is closed by direct calculations. Rewriting Eq. (148) we get:

$$\theta_g - \theta_{h^*} = \mathrm{Ad}(d)\mu_h - \mathrm{Ad}(d)\mu_{g^*}\,. \tag{161}$$

Taking the cube of the last equation we get:

$$\begin{aligned} &\langle \theta_g \overset{\wedge}{,} \theta_g \wedge \theta_g\rangle_{\mathscr{D}} - \langle \theta_{h^*} \overset{\wedge}{,} \theta_{h^*} \wedge \theta_{h^*}\rangle_{\mathscr{D}} \\ &\quad + 3\langle \theta_g \overset{\wedge}{,} \theta_{h^*} \wedge \theta_{h^*}\rangle_{\mathscr{D}} - 3\langle \theta_g \wedge \theta_g \overset{\wedge}{,} \theta_{h^*}\rangle_{\mathscr{D}} \\ &= \langle \mu_h \overset{\wedge}{,} \mu_h \wedge \mu_h\rangle_{\mathscr{D}} - \langle \mu_{g^*} \overset{\wedge}{,} \mu_{g^*} \wedge \mu_{g^*}\rangle_{\mathscr{D}} \\ &\quad + 3\langle \mu_h \overset{\wedge}{,} \mu_{g^*} \wedge \mu_{g^*}\rangle_{\mathscr{D}} - 3\langle \mu_h \wedge \mu_h \overset{\wedge}{,} \mu_g\rangle_{\mathscr{D}}\,. \end{aligned} \tag{162}$$

As $\theta_g \wedge \theta_g = \frac{1}{2}[\theta_g \overset{\wedge}{,} \theta_g]$ and $\mu_h \wedge \mu_h = \frac{1}{2}[\mu_h \overset{\wedge}{,} \mu_h]$ take values in $\mathscr{G}$, $\theta_{h^*} \wedge \theta_{h^*} = \frac{1}{2}[\theta_{h^*} \overset{\wedge}{,} \theta_{h^*}]$ and $\mu_{g^*} \wedge \mu_{g^*} = \frac{1}{2}[\mu_{g^*} \overset{\wedge}{,} \mu_{g^*}]$ take values in $\mathscr{G}^*$ we may use the pairing $\langle\,,\rangle_{\mathscr{D}}$ for them. Moreover, as both $\mathscr{G}$ and $\mathscr{G}^*$ are isotropic subspaces in $\mathscr{D}$, we rewrite (162) as follows:

$$\begin{aligned} &\langle \theta_g \overset{\wedge}{,} \theta_{h^*} \wedge \theta_{h^*}\rangle_{\mathscr{D}} - \langle \theta_g \wedge \theta_g \overset{\wedge}{,} \theta_{h^*}\rangle_{\mathscr{D}} \\ &\quad - \langle \mu_h \overset{\wedge}{,} \mu_{g^*} \wedge \mu_{g^*}\rangle_{\mathscr{D}} + \langle \mu_h \wedge \mu_h \overset{\wedge}{,} \mu_{g^*}\rangle_{\mathscr{D}} = 0\,. \end{aligned} \tag{163}$$

We remind that $d\theta_g = \theta_g \wedge \theta_g$ and $d\mu_g = -\mu_g \wedge \mu_g$. Thus,

$$\begin{aligned} dp_{ij}^*\Omega_{ij} &= -\langle d\theta_g \overset{\wedge}{,} \theta_{h^*}\rangle_{\mathscr{D}} + \langle \theta_g \overset{\wedge}{,} d\theta_{h^*}\rangle_{\mathscr{D}} \\ &\quad - \langle d\mu_h \overset{\wedge}{,} \mu_{g^*}\rangle_{\mathscr{D}} + \langle \mu_h \overset{\wedge}{,} d\mu_{g^*}\rangle_{\mathscr{D}} = 0\,. \end{aligned} \tag{164}$$

Now it is interesting to consider the classical limit of our theory to recover the standard answer for T^*G. There is no deformation parameter in bracket (79) but it may be introduced by hand:

$$\{f,h\}_\gamma = \gamma\{f,h\}\,. \tag{165}$$

For the new bracket (165) we have the symplectic form:

$$\Omega^{\gamma}_{ij} = \frac{1}{\gamma}\,\Omega_{ij}\,. \tag{166}$$

The classical limit $\gamma \to 0$ makes sense only for the main cell corresponding to $d_i = d_j = I$. The idea is to parametrize a vicinity of the unit element in the group G^* by means of the exponential map:

$$g^* = \exp(\gamma m)\,, \tag{167}$$

$$h^* \doteq \exp(\gamma l)\,, \tag{168}$$

where m and l belong to $\mathscr{G}^*$. Coordinates m and l are adjusted in such a way that they have finite values after the limit procedure. When γ tends to zero, the formula

$$d = gg^* = h^*h \tag{169}$$

leads to the following relations:

$$g = h\,, \qquad l = \mathrm{Ad}^*(g)m\,. \tag{170}$$

Expanding the form Ω^{γ} into the series in γ we keep only the constant term (singularity γ^{-1} disappears from the answer because the corresponding two-form is identically equal to zero). The answer is the following:

$$\Omega^{\gamma} = \tfrac{1}{2}\left(\langle dm \overset{\wedge}{,} \mu_g\rangle + \langle dl \overset{\wedge}{,} \theta_g\rangle\right), \tag{171}$$

and it recovers the classical answer (49) (see Sect. 1). Deriving formula (171), we use the expansions for the Maurer-Cartan forms on G^*:

$$\theta_{g^*} = \gamma dm + O(\gamma^2)\,, \tag{172}$$

$$\mu_{h^*} = \gamma dl + O(\gamma^2)\,. \tag{173}$$

We have considered general properties of the symplectic structure on the Heisenberg double D_+ and now we turn to the theory of orbits for Lie-Poisson groups.

4. Theory of Orbits

In this section we describe reductions of the Heisenberg double D_+ which lead to Lie-Poisson analogues of coadjoint orbits. We consider quotient spaces of the double D over its subgroups G and G^*: $F_{\mathrm{R}} = D/G$, $F^*_{\mathrm{R}} = D/G^*$, $F_{\mathrm{L}} = G\backslash D$, $F^*_{\mathrm{L}} = G^*\backslash D$. They inherit the Poisson bracket from the double D_+. Indeed, let us pick up F_{R} as an example. Functions on F_{R} may be regarded as functions on D invariant with respect to the right action of G:

$$f(dg) = f(d)\,. \tag{174}$$

The right derivative $\nabla_{\mathrm{R}} f$ is orthogonal to $\mathscr{G}$ for functions on F_{R}:

$$\langle \nabla_{\mathrm{R}} f, \mathscr{G}\rangle = 0\,. \tag{175}$$

For a pair of invariant functions f and h the second term in the formula (79) vanishes because $r^* \in \mathscr{G}^* \otimes \mathscr{G}$. The first term is an invariant function because the left derivative ∇_{L} preserves the condition (174). So we conclude that the Poisson bracket

$$\{f, h\} = -\langle \nabla_{\mathrm{L}} f \otimes \nabla_{\mathrm{L}} h, r\rangle \tag{176}$$

is well-defined on invariant functions and hence it can be treated as a Poisson bracket on F_{R}. The purpose of this section is to study the stratification of the space F_{R} into symplectic leaves and describe the corresponding symplectic forms on them. One can consider $F_{\mathrm{L}}, F_{\mathrm{R}}^{*}, F_{\mathrm{L}}^{*}$ in the same way.

Using stratification (77) of the double D we can obtain the stratification of the space F_{R}:

$$F_{\mathrm{R}} = \bigcup_{j} G^{*}/T_{-j} = \bigcup_{j} G_{j}^{*} . \tag{177}$$

Each stratification cell G_j^* is just an orbit of the natural action of G^* on the quotient space $F_{\mathrm{R}} = DG$ by the left multiplication. We denote the orbit of the class of unity in D by G_0^*. It is a quotient of G^* over discrete subgroup $\mathscr{E} = G^* \cap G$, $G_0^* = G/\mathscr{E}$.

We have factorized the double D over the right action of the group G. However, the same group acts on the quotient space by the left multiplications:

$$g : dG \to gdG . \tag{178}$$

Here the class dG is mapped into the class gdG. In the vicinity of the unit element on the maximum cell $GG^* \cap G^*G$ the action (178) looks as follows:

$$gg^{*} = g^{*\prime}(g, g^{*}) g'(g, g^{*}) . \tag{179}$$

The element $g^{*\prime}(g,g^*)$ is a result of the left action of the element g on the point $g^* \in G^* \subset F_{\mathrm{R}}$. in the classical limit, when g^* and $g^{*\prime}$ are very close to the identity, formula (179) transforms into the coadjoint action of G on $\mathscr{G}^*$:

$$g^{*} = I + \gamma l + \dots , \tag{180}$$

$$g^{*\prime} = I + \gamma l' + \dots , \tag{181}$$

$$l' = \mathrm{Ad}^{*}(g) l . \tag{182}$$

For historical reasons transformations (179) are called dressing transformations. We denote them AD^* to recall their relation to the coadjoint action:

$$g^{*\prime}(g, g^{*}) = \mathrm{AD}^{*}(g) g^{*} . \tag{183}$$

As we have mentioned, the transformation AD^* is defined on the space F_{R} globally. For some values of g and g^* in (183) the element $g^{*\prime}$ does not exist and the result of the action of g on g^* belongs to some other cell G_j^* of stratification (177). So we have a correct definition of the AD^*-orbit in the Lie-Poisson case. The question is whether they coincide with symplectic leaves or not. In general the answer is negative. Characterizing the situation we shall systematically omit the proofs concerning standard Poisson theory [2, 15].

A powerful tool for studying symplectic leaves is a dual pair. By definition a pair of Poisson mappings of symplectic manifold S to different Poisson manifolds P_{L} and P_{R}:

$$\begin{array}{ccc} & S & \\ \swarrow & & \searrow \\ P_{\mathrm{L}} & & P_{\mathrm{R}} \end{array} \tag{184}$$

is called a dual pair, if the Poisson bracket of any function on S lifted from P_{R} vanishes when the second function is lifted from P_{L} and in this case only. Symplectic leaves in P_{R} can be obtained in the following way. Take a point in P_{L}, consider its preimage in S and project it into P_{R}. Connected components of the image of this projection are symplectic leaves in P_{R}.

As an example let us consider the following pair of Poisson mappings:

$$\begin{array}{ccc} & D_+ & \\ \swarrow & & \searrow \\ F_{\mathrm{L}} & & F_{\mathrm{R}} \end{array} \tag{185}$$

This pair is not a dual pair because D_+ is not a symplectic manifold. However, the pair (185) is related to a family of dual pairs:

$$\begin{array}{ccc} & D_{ij} & \\ \swarrow & & \searrow \\ F_{\mathrm{L}} & & F_{\mathrm{R}} \end{array} \tag{186}$$

Here we use symplectic leaves D_{ij} instead of D_+. One can prove [13] that the pair of mappings (186) is a dual pair by direct calculation with bracket (79). Choosing dual pairs with different indices ij, we cover all space F_{R} and find all the symplectic leaves in this space.

Let us apply the general prescription to the dual pair (186). We pick up a class $Gx \in \mathrm{Im}_{\mathrm{L}}\, D_{ij} \subset T_i \backslash G^* \subset F_{\mathrm{L}}$. Its preimage in D_{ij} is an intersection $K_{ij}(x) = Gx \cap D_{ij}$. Projecting $K_{ij}(x)$ into F_{R}, we get a symplectic leaf:

$$\mathrm{AD}^*(G)xG \cap \mathrm{Im}_{\mathrm{R}}\, D_{ij}\,. \tag{187}$$

Let us remark that $\mathrm{Im}_{\mathrm{R}}\, D_{ij}$ is an intersection $G_j^* \cap \Bigl(\bigcup\limits_{g^* \in G^*} \mathrm{AD}^*(G) d_i g^* G\Bigr)$. It implies that we may use G_j^* instead of $\mathrm{Im}_{\mathrm{R}}\, D_{ij}$ in the formula (187). So all the symplectic leaves in F_{R} are intersections of orbits of dressing transformations AD^* and orbits G_j^* of the action of G^* in F_{R}. To get all the leaves we have to use all the cells D_{ij} in D. The orbits of AD^*-action in F_{R} appear to have a complicated structure. Each orbit $O_{p_0} = \mathrm{AD}^*(G)p_0$ ($p_o \in F_{\mathrm{R}}$) may be represented as a sum of its cells:

$$O_{p_0} = \bigcup_j (\mathrm{AD}^*(G)p_0 \cap G_j^*) = \bigcup_j O_{p_0}^j\,. \tag{188}$$

Each cell of stratification (188) is a symplectic leaf in F_{R}.

Now we turn to the description of symplectic forms on the leaves (188). As usually, it is convenient to use coordinates on the orbit and on the group G at the same time. Formula

$$g h_0^* d_j^{-1} G = h^* d_j^{-1} G \tag{189}$$

for the action of AD^* on the point $h_0^* T_{-j} \in G_j^*$ provides us with the projection from the subset

$$G_j(h_0^*) = \{g \in G,\; g h_0^* d_j^{-1} \in G^* d_j^{-1} G\} \tag{190}$$

to the cell $O_{h_0^*}^j$ of the orbit:

$$p_j : G_j(h_0^*) \to O_{h_0^*}^j\,, \tag{191}$$

$$p_j : g \to h^* T_{-j}\,, \tag{192}$$

where h^* is the same as in (189). Instead of the symplectic form Ω_j on the cell $O^j_{h_0^*}$ we shall consider its pull-back $p_j^*(h_0^*)\Omega_j$ defined on $G_j(h_0^*)$. It is easy to obtain the answer, using formula (112) for the symplectic form on D_{ij}. We put the parameter of the symplectic leaf $g^* = g_0^* =$ const. It kills the second term and the rest gives us the following answer:

$$p_j^*(h_0^*)\Omega_j = \tfrac{1}{2}\left\langle \theta_{h^*} \overset{\wedge}{,} \theta g \right\rangle . \qquad (193)$$

There is no manifest dependence on d_j in (193), but one must remember that g takes values in the very special subset of G (190). The dependence is hidden there. Anyway, the final result of our investigation is quite elegant. Each orbit of the dressing transformations in F_{R} splits into the sum of symplectic leaves (188) and the symplectic form on each leaf can be represented in the uniformed way (193).

As in Sect. 3 one can check independently that two-form (193) is really a pullback of some closed form on $O^j_{h_0^*}$. We suggest this proposition as an exercise for an interested reader.

We have classified symplectic leaves in the quotient space $F_{\mathrm{R}} = D/G$ and in particular in its maximum cell $G_0^* = G^*/\mathscr{E}$. In this context the idea to find symplectic leaves in the group G^* itself arises naturally. To this end let us consider the following sequence of projections $G_U^* \to G^* \to G_0^*$, where G_U^* is a universal covering group of the group G^*. The group G_U^* is a Lie-Poisson group. The Poisson bracket on the group G_U^* is defined uniquely by the Lie commutator in $\mathscr{G}$ [5]. The covering $G_U^* \to G_0^*$ appears to be a Poisson mapping. Using this property one can check that G^* is a Lie-Poisson group and the corresponding Poisson bracket makes both projections $G_U^* \to G^*$ and $G^* \to G_0^*$ Poisson mappings. It implies that symplectic leaves in G_U^* and in G^* are connected components of preimages of symplectic leaves in G_0^*. Corresponding symplectic forms can be obtained by pull-back from (193). On the other hand, the formula (193) gives an expression for symplectic forms on the leaves in G_U^* and G^*, if we treat h^* as an element of one of these groups and g as an element of G_U, universal covering group of G. Then we define the action of G_U on G_0^* by the formula (189) (g is a projection to G of some element $g_U \in G_U$) and lift the action of G_U from G_0^* to G_U^* or G^*. It is always possible by the definition of the universal covering group. We can identify symplectic leaves in G_U^* or G^* with orbits of the action of G_U, which we have just defined.

It is remarkable that in the deformed case the groups G and G^* may be considered on the same footing. Formula (193) defines symplectic structure on the orbit of G^*-action in D/G^* as well as on the orbit of G-action in D/G. The only thing we have to change is the relation between g and h^*:

$$h^* g_0 d_i G^* = g d_i G^* . \qquad (194)$$

To consider the classical limit we can introduce a deformation parameter into the formula (193):

$$p_j^*(h_0^*)\Omega_j^\gamma = \frac{1}{2\gamma}\left\langle \theta_{h^*} \overset{\wedge}{,} \theta_g \right\rangle . \qquad (195)$$

In this way one can recover the classical Kirillov form (36) as we did it for T^*G in Sect. 3.

5. Examples

In this section we shall consider two concrete examples to clarify constructions described in Sects. 2–4.

1. The first example concerns the Borel subalgebra $\mathscr{B}_+$ of semisimple Lie algebra $\mathscr{G}$. The algebra $\mathscr{B}_+$ consists of Cartan subalgebra $\mathscr{H} \subset \mathscr{G}$ and nilpotent subalgebra $\mathscr{N}_+$ generated by the Chevalley generators corresponding to positive roots. In the simplest case $\mathscr{G} = sl(n)$ $\mathscr{B}_+$ is just an algebra of traceless upper triangular matrices. We may define the projection $p: \mathscr{B}_+ \to \mathscr{H}$. Let us call $p(\varepsilon) \in \mathscr{H}$ a diagonal part of ε and denote it ε_d.

The dual space $\mathscr{B}_+^*$ can be identified with another Borel subalgebra $\mathscr{B}_- \subset \mathscr{G}$, where $\mathscr{B}_- = \mathscr{H} + \mathscr{N}_-$ includes the nilpotent subalgebra $\mathscr{N}_-$ corresponding to negative roots. The canonical pairing of $\mathscr{B}_+$ and $\mathscr{B}_-$ is given by the Killing form $K(x, y) \equiv \mathrm{Tr}(xy)$ on $\mathscr{G}$:

$$\langle x, \varepsilon \rangle = K(x, \varepsilon) + K(x_d, \varepsilon_d) . \tag{196}$$

The natural commutator on $\mathscr{B}_+^* = \mathscr{B}_-$ defines a structure of bialgebra on $\mathscr{B}_+$. The double $\mathscr{D}$ is isomorphic to the direct sum of $\mathscr{G}$ and $\mathscr{H}$:

$$\mathscr{D}(\mathscr{B}_+) \simeq \mathscr{G} \oplus \mathscr{H} . \tag{197}$$

Isomorphism (197) looks as follows:

$$(x, \varepsilon) \to (x + \varepsilon, x_d - \varepsilon_d) . \tag{198}$$

The first component of the r.h.s. in (198) belongs to $\mathscr{G}$ and satisfies the corresponding commutation relations, while the second component is an element of $\mathscr{H}$. Elements of $\mathscr{D}$, satisfying the conditions

$$x = x_d , \qquad \varepsilon = \varepsilon_d , \qquad x_d + \varepsilon_d = 0 , \tag{199}$$

belong to the center of $\mathscr{D}$.

The group D in this case is a product of semisimple Lie group G and its Cartan subgroup H:

$$D = G \times H . \tag{200}$$

The groups B_+ and B_-, corresponding to the algebras $\mathscr{B}_+$ and $\mathscr{B}_-$, can be embedded into D as follows:

$$B_+ \to (B_+, (B_+)_d) , \tag{201}$$

$$B_- \to (B_-, (B_-)_d^{-1}) , \tag{202}$$

where $(B_+)_d, (B_-)_d$ are diagonal parts of the matrices B_+, B_-. The decomposition (73) in this case may be described more precisely:

$$D = \bigcup_{i \in W} B_+ W_i B_- , \tag{203}$$

where W is Weyl group of G and the pair $W_i = (w_i, I)$ consists of the elements w_i from W and the unit element I in H. For nontrivial w_i spaces $V(W_i)$, $V^*(W_i)$ (95), (96) are nonempty.

For the algebras $\mathscr{B}_+$ and $\mathscr{B}_-$ we can use matrix notations (18), (19) for the Maurer-Cartan forms. For example,

$$\theta_{B_+} = (dB_+B_+^{-1}, db_+b_+^{-1}), \tag{204}$$

$$\mu_{B_-} = (B_-^{-1}dB_-, -b_-^{-1}db_-). \tag{205}$$

Here b_+ and b_- are diagonal parts of B_+ and B_- correspondingly. The invariant pairing $\langle\,,\rangle_{\mathscr{D}}$ acquires the form:

$$\langle(g_1, h_1), (g_2, h_2)\rangle_{\mathscr{D}} = \mathrm{Tr}(g_1g_2 - h_1h_2). \tag{206}$$

Now we can rewrite form (112) on the cell D_{ij} in this particular case:

$$d = (B_+w_iB_-, (B_+)_d(B_-)_d^{-1}) = (B'_-w_j^{-1}B'_+, (B'_-)_d^{-1}(B'_+)_d), \tag{207}$$

$$\begin{aligned} p_{ij}^*\Omega_{ij} = \tfrac{1}{2}\,\mathrm{Tr}(&dB'_-B'^{-1}_- \wedge dB_+B_+^{-1} + db'_-b'^{-1}_- \wedge db_+b_+^{-1} \\ &+ B_-^{-1}dB_- \wedge B'^{-1}_+dB'_+ + b_-^{-1}db_- \wedge b'^{-1}_+db_+). \end{aligned} \tag{208}$$

We have the symplectic structure on D_+ and it is interesting to specialize the Poisson bracket (79) for this case. We use tensor notations and write down the Poisson bracket for matrix elements of g and h, $(g, h) \in D$:

$$\{g^1, g^2\} = -(r_+g^1g^2 + g^1g^2r_-), \tag{209}$$

$$\{g^1, h^2\} = -(\varrho g^1h^2 + g^1h^2\varrho), \tag{210}$$

$$\{h^1, h^2\} = 0. \tag{211}$$

Here r_+ and r_- are the standard classical r-matrices, corresponding to the Lie algebra $\mathscr{G}$:

$$r_+ = \tfrac{1}{2}\sum h_i \otimes h^i + \sum_{a\in\Delta_+} e_a \otimes e_{-a}, \tag{212}$$

$$r_- = -\tfrac{1}{2}\sum h_i \otimes h^i - \sum_{a\in\Delta_+} e_{-a} \otimes e_a, \tag{213}$$

and ϱ is the diagonal part of r_+:

$$\varrho = \tfrac{1}{2}\sum h_i \otimes h^i. \tag{214}$$

As a result of general consideration we have obtained the symplectic structure corresponding to the nontrivial Poisson bracket (209)–(211). At this point we leave the first example and pass to the next one.

2. Now we take a semisimple Lie algebra $\mathscr{G}$ as an object of the deformation. It is the most popular and interesting example. The dual space $\mathscr{G}^*$ may be realized as a subspace in $\mathscr{B}_+ \oplus \mathscr{B}_-$:

$$\mathscr{G}^* = \{(x, y) \in \mathscr{B}_+ \oplus \mathscr{B}_-,\ x_d + y_d = 0\}. \tag{215}$$

The pairing between $\mathscr{G}$ and $\mathscr{G}^*$ is the following:

$$\langle(x, y), z\rangle = \mathrm{Tr}\{(x - y)z\}, \tag{216}$$

and the Lie algebra structure on $\mathscr{G}^*$ is inherited from $\mathscr{B}_+ \oplus \mathscr{B}_-$. It is easy to prove that the algebra double is isomorphic to the direct sum of two copies of $\mathscr{G}$ [5]:

$$\mathscr{D} \simeq \mathscr{G} \oplus \mathscr{G}\,, \tag{217}$$

$$\{x,(y,z)\} \to (x+y, x+z) = (d,d')\,, \tag{218}$$

$$\langle (d_1,d_1'),(d_2,d_2')\rangle_{\mathscr{D}} = \mathrm{Tr}(d_1 d_2 - d_1' d_2')\,, \tag{219}$$

where $x \in \mathscr{G}$, $(y,z) \in \mathscr{G}^*$. Therefore, the group double D is a product of two copies of G:

$$D = G \times G\,. \tag{220}$$

The subgroups G and G^* can be realized in D as follows:

$$G = \{(g,g) \in D\}\,, \tag{221}$$

$$G^* = \{(L_+, L_-) \in D,\ (L_+)_d (L_-)_d = I\}\,. \tag{222}$$

Let us introduce subgroups $\mathscr{H}_i$ in the Cartan subgroup $\mathscr{H}$ as $\mathscr{H}_i = \{hw_i^{-1}hw_i, h \in \mathscr{H}\}$.

Any pair $(X,Y) \in D$ can be decomposed into the product of the elements from G^* and G by means of the Weyl group W and Cartan subgroup $\mathscr{H}$:

$$X = L_+ w_i g\,, \tag{223}$$

$$Y = L_- h g\,. \tag{224}$$

Here $(L_+, L_-) \in G^*$, $g \in G$, $h \in \mathscr{H}$ and w_i is an element of the Weyl group W. So we have the following decomposition:

$$D = \bigcup_{i\in W, b\in \mathscr{H}/\mathscr{H}_i} G^* W_i(b) G\,, \tag{225}$$

where $W_i(b) = (w_i, H)$ and h belongs to the class b in $\mathscr{H}/\mathscr{H}_i$.

In this example we do not consider the symplectic structure on D_+ and pass directly to the description of orbits. The space $F_{\mathrm{R}} = D/G$ can be decomposed as in general case:

$$F_{\mathrm{R}} = \bigcup_{i\in W, b\in \mathscr{H}/\mathscr{H}_i} (G^*/T_{-i}(b))\,, \tag{226}$$

where $T_{-i}(b)$ is the subgroup of B_+, generated by the positive roots, which transform into the negative ones by the element w_i of the Weyl group:

$$T_{-i} = \{t \in B_+,\ (hw_i^{-1} t w_i h_d^{-1)} = t_d^{-1} w_i^{-1} t w_i \in B_-\}\,. \tag{227}$$

The dressing transformations act on the space F_{R} as follows:

$$gL_+ w_i = L_+^g w_{i^g} g'\,, \tag{228}$$

$$ghL_- = L_-^g h^g g'\,, \tag{229}$$

where (L_+^g, L_-^g) is the result of the dressing action $AD^*(g)$ and i^g is the index of the cell, where it lies. By the general theory the symplectic leaves in F_{R} are intersections of the cells $(G^*/T_{-i}(b))$ and the orbits of the dressing transformations. The analogue (193) of the Kirillov two-form can be rewritten in the following form:

$$p_j^* \Omega_j = \tfrac{1}{2}\,\mathrm{Tr}(dL_+ L_+^{-1} - dL_- L_-^{-1}) \wedge dgg^{-1}\,. \tag{230}$$

It is convenient to define the matrix

$$L = L_+ w_i h^{-1} L_-^{-1} \,. \tag{231}$$

It transforms under the action of the transformations (228), (229) in a simple way [13]:

$$L^g = L_+^g w_{ig} (h_g)^{-1} (L_-^g)^{-1} = gLg^{-1} \,. \tag{232}$$

Being an element of G, the matrix L defines a mapping from F_{R} to G by means of the formula (231). On each orbit of the conjugations (232) we can find a matrix L of canonical form. Let us denote it by L_0:

$$L^g = gL_0 g^{-1} = L_+^g w_{ig} (h^g)^{-1} (L_-^g)^{-1} \,. \tag{233}$$

Using two different parametrizations of the same matrix L, we can rewrite (230):

$$p_j^* \Omega_j = \tfrac{1}{2} \operatorname{Tr}\{g^{-1} dg L_0 \wedge g^{-1} dg L_0^{-1} + L_+^{-1} dL_+ w_j h^{-1} \wedge L_-^{-1} dL_- h w_j^{-1}\} \,. \tag{234}$$

Formula (234) was obtained for $w_i = I$ in [7] as a by-product of the investigations of WZ model. The first term in (234) is rather universal. It depends neither on the choice of the Borel subalgebra in the definition of the deformation nor on the cell of F_{R}. On the contrary, the second term keeps the information about the particular choice of the (B_+, B_-) pair and it depends on the element w_i of the Weyl group characterizing the cell of the orbit.

It is instructive to write down the Poisson bracket for the matrix elements of L. Using the classical r-matrices r_+, r_- (212), (213) and tensor notations, we have [13]:

$$\{L^1, L^2\} = r_+ L^1 L^2 + L^1 L^2 r_- - L^1 r_+ L^2 - L^2 r_- L^1 . \tag{235}$$

Let us recall that the same symplectic form (230) corresponds to another Poisson structure

$$\{g^1, g^2\} = r_+ g^1 g^2 - g^1 g^2 r_+ = r_- g^1 g^2 - g^1 g^2 r_- \,, \tag{236}$$

if instead of conditions (228), (229) we impose the following set of constraints on L_+, L_- and g:

$$L_+ g w_i = g^L w_{iL} L'_+ \,, \tag{237}$$

$$L_- g h = g^L h^L L'_- \,. \tag{238}$$

6. Discussion

In this section we formulate several problems related to the symplectic structures described in the paper. The first of them concerns the quantum version of the presented formalism. In the classical case the Kirillov symplectic form appears in the content of the theory of geometric quantization. Roughly speaking, some coadjoint orbits of the group G equipped with the Kirillov form correspond to irreducible representations of the Lie algebra $\mathscr{G}$. The cotangent bundle T^*G with its canonical symplectic structure corresponds to the regular representation of $\mathscr{G}$. Actually, we may restrict ourselves to the latter case because all the particular irreducible representations can be obtained from the regular one by means of the reduction producedure. For Lie-Poisson groups the problem is not so simple even for D_+. After the quantization the Poisson algebra

(80) becomes the quantum algebra of functions on D_+. Its basic relations can be written in the following form:

$$d^1 d^2 = R d^2 d^1 R^* \,, \tag{239}$$

where we use tensor notations, R and R^* are quantum R-matrices corresponding to the classical counterparts r and r^*. The result we expect as an outcome of geometric quantization is an irreducible representation of the algebra (239) correponding to a symplectic leaf in D_+. It is easy to find such a representation for the main cell $D_{00} = GG^* \cap G^*G$. Algebra (239) $\mathrm{Funk}_q(D_+)$ acts in the space $\mathrm{Funk}_q(G)$. It is an analogue of the standard regular representation in the space of functions on the group G. The algebra $\mathrm{Funk}_q(G)$ is defined by the basic relations [6]

$$R g^1 g^2 = g^2 g^1 R \,. \tag{240}$$

On the cell D_{00} we can decompose the element d as a product

$$d = gh^* = g^* h \tag{241}$$

of elements from G and G^*. Matrix elements of G act on the space $\mathrm{Funk}_q(G)$ by means of multiplication and matrix elements of G^* generalize differential operators. The regular representation in $\mathrm{Funk}_q(G)$ was considered in [14], where the quantum analogue of the Fourier transformation was constructed.

We expect that representations corresponding to other symplectic leaves D_{ij} can be found and presented in a similar form. This would give a good basis for the geometric quantization in the direct meaning of the word, i.e. establishing of the correspondence between the orbits and the quantum group representations. For $G = SU(n)$ this correspondence has been described in [8] by means of quantization of orbits of the dressing transformations. It is a simple case because for $G = SU(n)$, $D = GG^* = G^*G$ and the orbits are symplectic leaves. It should be mentioned that this correspondence appears in a natural way in the course of investigations of the quantum group representation theory for the deformation parameter q being a root of unity. If $q^N = 1$, there exists an irreducible representation of the deformed universal enveloping algebra $U_q(\mathscr{G})$ corresponding to any orbit of dressing transformations [3].

Another problem which we would like to mention is a possible application of the machinery of Sects. 3 and 4 to physics. Having the closed form Ω, we can solve at least locally the equation

$$d\alpha = \Omega \,. \tag{242}$$

The one-form α may be treated as a lagrangian of some mechanical system so that the action looks as follows:

$$S_0 = \int \alpha \,. \tag{243}$$

If we add an appropriate hamiltonian H, we get a system with the action

$$S = \int (\alpha - H\, dt) \,. \tag{244}$$

Symplectic structure described in Sects. 3 and 4 provide a wide class of dynamical systems (244). For the classical groups one obtains many interesting examples in this way. Among them one finds the WZNW model and the gravitational WZ model [1]. Realizing the same idea for the Lie-Poisson case, one can hope to construct integrable deformations of these systems.

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