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Fermion Current Algebras and Schwinger Terms in (3 + 1)-Dimensions

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Abstract: We discuss the restricted linear group in infinite dimensions modeled by the Schatten class of rank $2p = 4$ which contains the $(3 + 1)$-dimensional analogs of the loop groups and is closely related to Yang–Mills theory with fermions in $(3 + 1)$-dimensions. We give an alternative to the construction of the "highest weight" representation of this group found by Mickelsson and Rajeev. Our approach is close to quantum field theory, with the elements of this group regarded as Bogoliubov transformations for fermions in an external Yang–Mills field. Though these cannot be unitarily implemented in the physically relevant representation of the fermion field algebra, we argue that they can be implemented by sesquilinear forms, and that there is a (regularized) product of forms providing an appropriate group structure. On the Lie algebra level, this gives an explicit, non-perturbative construction of fermion current algebras in $(3 + 1)$ space-time dimensions which explicitly shows that the "wave function renormalization" required for a consistent definition of the currents and their Lie bracket naturally leads to the Schwinger term identical with the Mickelsson–Rajeev cocycle. Though the explicit form of the Schwinger term is given only for the case $p = 2$, our arguments apply also to the restricted linear groups modeled by Schatten classes of rank $2p = 6, 8, \ldots$ corresponding to current algebras in $(d + 1)$-dimensions, $d = 5, 7, \ldots$.

1. Introduction

In recent years it has become obvious that the representation theory of certain infinite dimensional Lie groups and algebras can contribute in an essential way to the understanding of quantum field theory. Well-known examples are the Virasoro algebra and the affine Kac–Moody algebras which have been of crucial importance for recent spectacular progress in two dimensional conformal field theory.

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The groups $\mathrm{Map}(M^d;\, G)$ of maps from a d-dimensional manifold M^d to some (compact, semisimple) Lie group G naturally arise as gauge groups for Yang–Mills theories on space-time $M^d \times \mathbb{R}$ in the Hamiltonian formalism. General principles of quantum theory imply that the Hilbert space of such models should carry a highest-weight representation of the gauge group. This strongly suggests that such representations of $\mathrm{Map}(M^d;\, G)$ and its Lie algebra $\mathrm{map}(M^d;\, g)$ (g the Lie algebra of G) should be of crucial importance for quantum gauge theories, and, on the other hand, that quantum field theory should give a natural guide for finding interesting representations of these groups and algebras.

Indeed, the by-now well-understood representation theory of the loop groups $\mathrm{Map}(S^1;\, G)$ and loop algebras $\mathrm{map}(S^1;\, g)$ [PS, KR] has provided a general, rigorous basis for $(1+1)$-dimensional quantum gauge theories. Especially, the wedge representations of $\mathrm{Map}(S^1;\, G)$ and $\mathrm{map}(S^1;\, g)$ allows for a complete understanding and accounting for all ultra-violet divergences arising in the fermion sector of dimensional Yang–Mills theories with fermions[1] on a cylinder (= space-time $S^1 \times \mathbb{R}$) [M2]. The chiral (left- and/or right-handed) fermion currents arising naturally in such models with massless fermions, are identical with the ones proving the standard wedge representation of the affine Kac–Moody algebra $\widehat{\mathrm{map}(S^1;\, g)}$ (= central extension of $\mathrm{map}(S^1;\, g)$), and the cocycle giving the central extension has a natural interpretation as a Schwinger term. Moreover, it is such Schwinger terms which are responsible for anomalies: For gauge theories with Weyl (= chiral) fermions they lead to the gauge anomaly in the commutators of Gauss' law generators, though there is no such gauge anomaly for Dirac (= non-chiral) fermions (the Gauss' law generators involve a sum of the left- and right-handed chiral currents leading to Schwinger terms with opposite signs which can be arranged such that they cancel), these Schwinger terms show up in the commutators of the temporal and the spatial components of the vector current and lead to the chiral anomaly [J, LS2].

A central role in these developments has been played by the infinite dimensional group $\mathbf{G}_1$ modeled by the Hilbert–Schmidt class and for which a well-developed representation theory exists [PS, M2]. This group contains all loop groups as subgroups, and representations of the latter are naturally obtained by restriction from the ones of $\mathbf{G}_1$ [PS]. The interesting representations of $\mathbf{G}_1$ are not true but projective ones and correspond to true representations of a central extensions $\widehat{\mathbf{G}_1}$ of $\mathbf{G}_1$. The most important of these is the wedge representation which is a highest weight representation of $\widehat{\mathbf{G}_1}$ on the fermion Fock space [M2].

From a quantum field theory point of view, $\mathbf{G}_1$ is identical with the group of all Bogoliubov transformations which are unitarily implementable in the physical Hilbert space of some arbitrary model of the relativistic fermion in an external field, and the construction of the wedge representation of $\widehat{\mathbf{G}_1}$ is equivalent to constructing the implementers of these Bogoliubov transformations [CR, R1, M2]. This requires some regularization which, on the Lie algebra level, corresponds to normal ordering of fermion bilinears and which can be done with mathematical rigor [CR]. The very definitions of $\mathbf{G}_1$ and its Lie algebra $\mathbf{g}_1$ can be regarded as a general characterization of the degree of divergence where this kind of regularization is sufficient. It is this regularization which leads to the non-trivial 2-cocycle providing the central extension [L3]. On the Lie algebra level one obtains a highest weight representation

[1] In this paper I am discussing only quantum field theories with fermions. I am planning to report on corresponding results for bosons in a forthcoming publication.

of a central extension $\widehat{\mathbf{g}_1}$ of $\mathbf{g}_1$. We refer to the latter as *abstract current algebra in* $(1+1)$-*dimensions* as it contains all currents referred to above and also all other operators of interest for gauge theories with fermions on a cylinder. This construction also shows on a very general, abstract level how regularizations can lead to Schwinger terms implying anomalies [CR, GL].

There are two approaches to these current algebras in $(1+1)$-dimensions: the original one based on the theory of quasi-free representations of fermion field algebras was pioneered by Lundberg [Lu] and worked out in all mathematical detail by Carey and Ruijsenaars [CR]. It is conceptually very close to quantum field theory. The other approach is by means of determinant bundles over infinite dimensional Grassmannians [PS, M2] and seems to be preferred by mathematicians.

In higher dimensions, the situation is much more difficult. This can be traced back to that fact that the gauge groups Map(M^d; G) for $d = 3, 5, \ldots$ are *not* contained in $\mathbf{G}_1$ in any natural way – ultra-violet divergencies are worse in higher dimensions. There is, however, an infinite dimensional group $\mathbf{G}_p, p = (d+1)/2$, modeled on a Schatten class and containing Map(M^d; G) as subgroups [PS, MR]. These groups are the natural starting point for a generalization of the theory of loop groups and affine Kac–Moody algebras to higher dimensions. Similar to $(1+1)$-dimensions, it is natural to regard the definition of $\mathbf{G}_p$ as a general classification of the degree of ultra-violet divergencies one encounters in the fermion sector of gauge theories with fermions in $(d+1)$-dimensions.

A generalization of the "Grassmannian approach" [PS] to the abstract current algebras from $(1+1)$- to $(3+1)$- and higher even dimensional space time was developed by Mickelsson and Rajeev [MR] and Mickelsson [M1, M3] (see also [M2]). They were able to construct a "highest weight" representation[2] of an *Abelian* extension $\widehat{\mathbf{G}_p}$ of $\mathbf{G}_p$ for $p > 1$ on a bundle of fermion Fock spaces. On the Lie algebra level this leads to a fermion current algebra with a cocycle corresponding to an operator-valued Schwinger term depending on a variable in a set Gr_p carrying a non-trivial representation of $\mathbf{G}_p$. Moreover, they tried – without success – to find a Hilbert space $\mathcal{H}$ such that the currents are selfadjoint generators of one-parameter families of unitary operators on $\mathcal{H}$. Later on it was shown by Mickelsson [M1] and Pickrell [P] that this is impossible: in higher dimensions than $(1+1)$, the abstract current algebras of Mickelsson and Rajeev [MR] do not allow for a faithful, unitary representation on a separable Hilbert space.

From the physical point of view these Abelian extensions of $\mathbf{G}_p$ have a natural interpretation: restricting $\mathbf{g}_p$ to map(M^d; g), the fermion currents of Mickelsson and Rajeev [MR, M1, M3] can be regarded as generators of the gauge transformations in the fermion sector of Yang–Mills theory with Weyl fermions. The corresponding restriction of Gr_p can be naturally identified with the space $\mathcal{A}$ of all static Yang–Mills field configuration. The dependence of the Schwinger term on the Yang–Mills field reflects the fact that in higher dimensions it is not possible to regularize fermion bilinears independent of the Yang–Mills field. Indeed, it has been argued already some time ago by Faddeev [F] and Faddeev and Shatiashvili [FS] that in $(3+1)$-dimensions such an operator-valued Schwinger term should be present. It is natural to expect (though to our knowledge it has not yet been proved) that the Schwinger

[2] The quotation marks indicate that it is not a highest weight representation in the standard terminology but something similar [M1]

term of Mickelsson and Rajeev for $\mathbf{g}_2$ [MR] when restricted to map$(M^3; g)$ is identical with the one of Faddeev and Shatiashvili [F, FS].

In this paper we give an alternative construction of "highest weight" representations of the Abelian extensions $\widehat{\mathbf{g}_p}$ of $\mathbf{g}_p$ for $p > 1$ [MR] by means of the theory of quasi-free representations of fermion field algebras [CR], thus generalizing the construction of abstract current algebras by Lundberg [Lu] from $(1 + 1)$- to higher even dimensional space-time.

We hope that our approach is more transparent than the original one (at least for physicists); moreover, it shows very explicitly "how the Hilbert space is lost" in higher dimensions, i.e. for the Lie groups $\mathbf{G}_p$ with $p > 1$: as mentioned above, $\mathbf{G}_1$ is identical with the group of the Bogoliubov transformations which are unitarily implementable in the physically relevant quasi-free representations of the fermion field operators [R1], and the implementers with the usual operator product as group multiplication provide a representation of a central extension of $\mathbf{G}_1$ [PS]. For $p > 1$, the elements in $\mathbf{G}_p$ correspond to Bogoliubov transformations which are *not* unitarily implementable. However, it was shown by Ruijsenaars [R2, R4] that they can be implemented by sesquilinear forms [K], and these forms can be constructed from the implementers of the Bogoliubov transformations in $\mathbf{G}_1$ by some multiplicative regularization. The operator product of two such forms does not exist in general, and therefore, in order to get a group structure for these implementers, one has to regularize also the group multiplication. The problem is that the usual operator product does not allow for a multiplicative regularization maintaining associativity. We show that one can define another product, $\bigstar$, for the implementers of the Bogoliubov transformations in $\mathbf{G}_1$, and this naturally leads to the group strucuture of an Abelian extension of $\mathbf{G}_1$. We demonstrate that $\bigstar$ is (not identical but) equivalent to the operator product, and the resulting Abelian extension of $\mathbf{G}_1$ is trivial insofar as it is equivalent to the central one obtained with the operator product. We then argue that for each $p \in \mathbb{N}$, there is a unique "minimal" regularization of the $\bigstar$-product maintaining associativity, and this provides a group structure for the regularized implementers corresponding to the Abelian extension of $\mathbf{G}_p$ introduced in [MR] (in fact, we are able to show this explicitly only for $p = 2$; however, the results of [MR, FT] indicate that this is true for all $p \in \mathbb{N}$). Moreover, for each $p \geqq 1$, the (regularized) implementers together with the (unique regularized) $\bigstar$-product can be naturally interpreted as a representation of $\widehat{\mathbf{G}_p}$ on a space of sections in a Fock space bundle which has the mathematical structure of a module [H] over a ring of functionals. For $p = 1$, the Fock spaces above different points of the base space of this bundle are all unitarily equivalent and can be naturally identified. However, this is not true for $p > 1$. On the Lie algebra level, this implies that the construction of abstract current algebras in $2p$-dimensions, $p > 1$, requires not only a regularization of the currents, but one has to regularize their commutators as well. Our results show that there is no proper regularization of the usual commutator of the currents maintaining the Jakobi identity, but there is (an essentially unique) one for another, non-trivial Lie bracket of the currents which arises from the $\bigstar$-product.

The plan of the paper is as follows. Section 2 is preliminary: we define our notation and motivate the groups $\mathbf{G}_p$ and their Lie algebras $\mathbf{g}_p$ by discussing Weyl fermions in external Yang–Mills fields, and (for the convenience of the reader) summarize the basic facts about generalized determinants which are essential for our construction. The definitions of the Abelian extensions $\widehat{\mathbf{G}_p}$ and $\widehat{\mathbf{g}_p}$ introduced in [MR], together with the related Lie group and Lie algebra cohomology of $\mathbf{G}_p$ and

$\mathbf{g}_p$ are summarized in Sect. 3. A review of the formalism of quasi-free representations of fermion field algebras is given in Sect. 4. The ★-product for the unitary implementers of Bogoliubov transformations, and the corresponding Lie bracket for the abstract current algebra in 2-dimensions are introduced and discussed in Sects. 5 and 6, respectively. In Sect. 7 we show how the representations of $\widehat{\mathbf{G}_p}$, $p > 1$, are obtained by an essentially unique multiplicative regularization, and we derive from this the abstract current algebras in $(d+1)$-dimensions, $d = (2p-1)$, in Sect. 8. We end with a few comments in Sect. 9. Details of the calculations are left to five Appendices.

A short account on the results of this paper appeared in [L1, L2] and was discussed in [M5].

2. 1-Particle Formalism

2.1 Fermions in External Yang–Mills Fields. We consider Weyl fermions in $(d+1)$-dimensional space-time, coupled to an external Yang–Mills field A with the structure group $G = \mathrm{U}(N)$ or $\mathrm{SU}(N)$ in the fundamental representation. The space M^d is assumed to be a smooth (i.e. C^∞), oriented, riemannian spin manifold with a given spin structure, and it is convenient to assume that the space M^d is compact. (In the case of a non-compact space) (e.g. $M^d = \mathbb{R}^d$), we restrict ourselves to Yang–Mills fields and gauge transformations with compact support). For x^j some local coordinates on M^d and $d = \sum_{j=1}^d \partial_j \mathrm{d}x^j$ the usual exterior derivative, the Yang–Mills field is a g-valued 1-form on M^d: $A = \sum_{j=1}^d A_j \mathrm{d}x^j$, with g the Lie algebra of G.

The quantum description of *one* fermion can be given in the Hilbert space $h = L^2(M^d) \otimes V$ of square integrable functions with values in the finite dimensional Hilbert space V which carries representations of the spin structure and of G and g (to simplify notation, we do not distinguish the elements in G and g from their representatives on V). On this 1-particle level, the time evolution is generated by the usual Weyl Hamiltonian D_A [M2] which is a self-adjoint operator on h, and gauge transformations are given by smooth, G-valued functions U on the space M^d:

$$D_A \to U^{-1} D_A U = D_{A^U} \tag{1}$$

with $A^U = U^{-1}AU + \mathrm{i}U^{-1}\mathrm{d}U$ the gauge transformed Yang–Mills field as usual. We will be mainly concerned with gauge transformations of the form $U = \exp(\mathrm{i}tu)$ with $t \in \mathbb{R}$, and u some smooth, g-valued function on M^d; we denote such an u as *infinitesimal gauge transformation.* Obviously, every gauge transformation can be identified with a unitary operator on h (which we denote by the same symbol), and the group multiplication in the group $\mathrm{Map}(M^d;\, G)$ of all gauge transformations is identical with the product as operators on h. Similarly, the infinitesimal gauge transformations u can be identified with self-adjoint, bounded operators u on h, and the Lie bracket in the Lie algebra $\mathrm{map}(M^d;\, g)$ of all infinitesimal gauge transformations (= pointwise commutator) is identical with the commutator as operators on h.[3]

[3] Strictly speaking, the Lie bracket is $\mathrm{i}^{-1} \times$ commutator. We find it convenient, however, to always omit the factor i^{-1}.

2.2. Linear Groups and Lie Algebras Modelled on Schatten Classes. For $p \in \mathbb{N}$, we denote as $B_p(h)$ the (so-called) Schatten class of all bounded operators a on h with a finite norm

$$\|a\|_p \equiv [\mathrm{tr}((a^* a)^{p/2})]^{1/p} \tag{2}$$

($\mathrm{tr}(\cdot)$ is the trace in h). Thus $\mathrm{tr}(a^r)$ can be defined only if $a \in B_p(h)$ with $p \leqq r$. Especially, $B_1(h)$ and $B_2(h)$ are the trace class and the Hilbert–Schmidt class, respectively. By definition, $B_\infty(h)$ is the set of all compact operators on h, and $\|\cdot\|_\infty = \|\cdot\|$ (= operator norm). Note that $a \in B_p(h)$, $b \in B_q(h)$ implies $ab \in B_r(h)$ for all $r \geqq \left(\frac{1}{p} + \frac{1}{q}\right)^{-1}$, especially $a^p \in B_1(h)$ for $a \in B_p(h)$.

The essential ingredient for constructing the appropriate multiparticle theory is $\varepsilon = \mathrm{sign}(D_A)$ (with $\mathrm{sign}(x) = 1(-1)$ for $x \geqq 0$ $(x < 0)$ and ε defined via the spectral theorem for self-adjoint operators [RS1]) which is a grading operator[4] on h. Physically, ε characterizes the splitting of the 1-particle Hilbert space h in the subspaces $h_\pm = \frac{1}{2}(1 \pm \varepsilon)h$ of positive (+) and negative (−) energy states, and it determines the appropriate quasi-free representation of the fermion field algebra corresponding to "filling up the Dirac sea" (see Sect. 4).

It can be shown that gauge transformations $U \in \mathrm{map}(M^d;\ G)$ have the following crucial property [MR]:

$$[U, \varepsilon] \in B_{2p}(h)\ , \tag{3}$$

and the rank $2p$ of the Schatten class is determined by the dimension of M^d: [5]

$$2p = (d+1)\ . \tag{4}$$

This allows us to embed $\mathrm{Map}(M^d;\ G)$ in the larger group $\mathbf{G}_p(h;\ \varepsilon)$ of all unitary operators U on h obeying the condition (3). Similarly, $\mathrm{map}(M^d;\ g)$ can be embedded in the Lie algebra $\mathbf{g}_p(h;\ \varepsilon)$ of all bounded, self-adjoint operators U on h obeying this condition (3). These definitions naturally extend to $p = \infty$. Note that for all $p \in \mathbb{N} \cup \{\infty\}$, $\mathbf{G}_p(h;\ \varepsilon)$ and $\mathbf{g}_p(h;\ \varepsilon)$ both are Banach algebras with the norm $|||\cdot|||_p$ given by

$$|||U|||_p = \frac{1}{2}\|\{U, \varepsilon\}\| + \frac{1}{2}\|[U, \varepsilon]\|_{2p} \tag{5}$$

($\{\cdot, \cdot\}$ is the anticommutator as usual), and $\mathbf{g}_p(h;\ \varepsilon)$ is the Lie algebra of $\mathbf{G}_p(h;\ \varepsilon)$. In addition, we introduce the group $\mathbf{G}_0(h;\ \varepsilon)$ of all unitary operators U on h with $(U-1) \in B_1(h)$, and its Lie algebra $\mathbf{g}_0(h;\ \varepsilon)$ containing all self-adjoint operators in $B_1(h)$.

Every gauge transformed Yang–Mills field A^U gives rise to another grading operator $F = \mathrm{sign}(D_{A^U}) = U^{-1}\varepsilon U$. This suggests to introduce the set $\mathrm{Gr}_p(h;\ \varepsilon)$ of all grading operators of the form $F = T^{-1}\varepsilon T$ for some $T \in \mathbf{G}_p(h;\ \varepsilon)$, and due to (3)

[4] I.e., $\varepsilon^2 = 1$ (identity) and $\varepsilon^* = \varepsilon$, where $*$ denotes the Hilbert space adjoint

[5] The dimension comes in as follows: by direct calculation one can estimate [MR]

$$\mathrm{tr}([U, \varepsilon]^*[U, \varepsilon])^p \leqq \text{const.} \int_c^\infty dk k^{d-1} k^{-2p}$$

(const. always finite for smooth functions. $U: M^d \mapsto G$) for some positive c, the dkk^{d-1} resulting from the volume element $d^d k$

$$(F - \varepsilon) \in B_{2p}(h) \tag{6}$$

$\forall F \in \mathrm{Gr}_p(h;\, \varepsilon)$ (it is also easy to see that every grading operator satisfying this relation is in $\mathrm{Gr}_p(h;\, \varepsilon)$). This set carries a natural representation $F \to F^U$ of the group $\mathbf{G}_p(h;\, \varepsilon)$ with

$$F^U \equiv U^{-1} F U \,, \tag{7}$$

and it is a metric space with the metric

$$d_p(F, F') \equiv \|F - F'\|_p \tag{8}$$

$\forall U \in \mathbf{G}_p(h;\, \varepsilon), F, F' \in \mathrm{Gr}_p(h;\, \varepsilon)$. Note that $\mathbf{G}_p(h;\, F) = \mathbf{G}_p(h;\, \varepsilon)$ and $\mathbf{g}_p(h;\, F) = \mathbf{g}_p(h;\, \varepsilon)$ for all $F \in \mathrm{Gr}_p(h;\, \varepsilon)$.[6] Moreover,

$$\mathbf{G}_0(h;\, \varepsilon) \subseteq \mathbf{G}_1(h;\, \varepsilon) \subseteq \mathbf{G}_2(h;\, \varepsilon) \subseteq \cdots \subseteq \mathbf{G}_\infty(h;\, \varepsilon) \tag{9}$$

and $\mathbf{G}_p(h;\, \varepsilon)$ is dense in $\mathbf{G}_q(h;\, \varepsilon)$ for $p < q$, and similarly for $\mathbf{g}_p(h;\, \varepsilon)$ and $\mathrm{Gr}_p(h;\, \varepsilon)$.

2.3. Generalized Determinants [S]. For a a linear operator on a Hilbert space h, the (Fredholm) determinant $\det(a)$ exists if and only if $(a-1) \in B_1(h)$, and $\det(ab) = \det(a)\det(b)$ for all $(a-1), (b-1) \in B_1(h)$. For $\|a - 1\| < 1$ we have then

$$\det(a) = \exp(\mathrm{tr}(\log a)) = \exp\left\{ \sum_{j=1}^{\infty} (-)^{j-1} \frac{\mathrm{tr}((a-1)^j)}{j} \right\} ,$$

thus suggesting to define the generalized determinant $\det_p(a)$ for $(a-1) \in B_p(h)$, $p > 1$, by just omitting in this expression the non-existing traces:

$$\det_p(a) = \exp\left\{ \sum_{j=p}^{\infty} (-)^{j-1} \frac{\mathrm{tr}((a-1)^j)}{j} \right\} .$$

Indeed, one can show that for $(a-1) \in B_p(h)$, the operator

$$R_p(a) \equiv -1 + a \exp\left\{ \sum_{j=1}^{p-1} (-)^j \frac{(a-1)^j}{j} \right\} \tag{10}$$

is in $B_1(h)$, hence

$$\det_p(a) \equiv \det(1 + R_p(a)) \tag{11}$$

exists, and for $\|a - 1\| < 1$ it coincides with the expression given above. Moreover, the mapping $B_p(h) \ni a \mapsto \det_p(a)$ is continuous in the norm $\|\cdot\|_p$, and $a \in B_p(h)$ is invertible if and only if $\det_p(a) \neq 0$.

It is natural to regard $\det_p(\cdot)$ as the *minimal multiplicative regularization* of $\det(\cdot)$ appropriate for $B_p(h)$. Note that $\det_1(\cdot) = \det(\cdot)$, and $\det_p(ab) \neq \det_p(a)\det_p(b)$ for $p > 1$.

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[6] The norms $|||\cdot|||_p$ defined above obviously *depend* on F; however, it can be easily shown that they are equivalent (give rise to the same topology) for all $F \in \mathrm{Gr}_p(h;\, \varepsilon)$

3. Cohomology and Abelian Extensions

In this section we introduce some terminology needed in the sequel.

3.1. Abelian Extensions of Lie Groups. Let $p \in \mathbb{N}_0 \cup \{\infty\}$. We introduce the set

$$\mathbf{C}_p(h;\, \varepsilon) \equiv \mathrm{Map}(\mathrm{Gr}_p(h;\, \varepsilon);\, \mathrm{U}(1)) \tag{12}$$

of "smooth"[7] functions $\mu\colon \mathrm{Gr}_p(h;\, \varepsilon) \to \mathrm{U}(1);\ F \mapsto \mu(F)$; $\mathbf{C}_p(h;\, \varepsilon)$ is an Abelian group under point-wise multiplication:

$$(\mu\nu)(F) = \mu(F)\nu(F)\,, \tag{13}$$

and it carries a natural representation $\mu \mapsto \mu^U$ of the group $\mathbf{G}_p(h;\, \varepsilon)$ with

$$(\mu^U)(F) \equiv \mu(F^U) = \mu(U^{-1}FU) \tag{14}$$

$\forall \mu \in \mathbf{C}_p(h;\, \varepsilon),\ U \in \mathbf{G}_p(h;\, \varepsilon),\ F \in \mathrm{Gr}_p(h;\, \varepsilon)$.

"Smooth", $\mathbf{C}_p(h;\, \varepsilon)$-valued functions β and χ on $\mathbf{G}_p(h;\, \varepsilon)$ and $\mathbf{G}_p(h;\, \varepsilon) \times \mathbf{G}_p(h;\, \varepsilon)$ satisfying

$$\beta(U^*) = \overline{\beta^{U^*}(U)}, \quad \beta(1) = 1 \tag{15}$$

(the bar denotes complex conjugation) and

$$\chi(V^*,\, U^*) = \overline{\chi^{V^*U^*}(U,\, V)}, \quad \chi(U,\, U^*) = 1 \tag{16}$$

$\forall U, V \in \mathbf{G}_p(h;\, \varepsilon)$ are called *1-cochains* and *2-cochains* of the group $\mathbf{G}_p(h;\, \varepsilon)$, respectively.[8] We consider the set

$$\widehat{\mathbf{G}_p}(h;\, \varepsilon) \equiv \mathbf{G}_p(h;\, \varepsilon) \times \mathbf{C}_p(h;\, \varepsilon) \tag{17}$$

with a multiplication $\cdot$ given by

$$(U, \mu)\cdot(V, \nu) \equiv (UV, \mu\nu^U\chi(U, V)) \tag{18}$$

for all $(U, \mu),\ (V, \nu) \in \widehat{\mathbf{G}_p}(h;\, \varepsilon)$, with χ some 2-cochain. It is easy to see that this multiplication is associative and allows for an inverse if and only if χ satisfies

$$\chi(U, V)\chi(UV, W) = \chi^U(V, W)\chi(U, VW) \tag{19}$$

for all $U, V, W \in \mathbf{G}_p(h;\, \varepsilon)$. Moreover, these relations imply that we can define an involution

$$(U, \mu) \mapsto (U, \mu)^* \equiv (U^*, \overline{\mu^{U^*}}) \tag{20}$$

making $\widehat{\mathbf{G}_p}(h;\, \varepsilon)$ to an unitary group (i.e. the inversion is equal to the involution). Equation (19) is a *2-cocycle relation*, and a 2-cochain χ satisfying it is a *2-cocycle* of $\mathbf{G}_p(h;\, \varepsilon)$ [F, FS, M2].

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[7] The quotation marks in "smooth" (here and elsewhere in the paper) are to indicate that we do not give a precise definition of the meaning of this term and use it only in a sloppy way as we do not really need all its implications and want to avoid irrelevant technical discussions. In fact, all we require is that the differentiations necessary to go from the group to the Lie algebra level (see the next section) are well-defined

[8] Note that our definition adequate for groups with involution is slightly more restrictive than the usual one [M2]

For every 1-cochain β of $\mathbf{G}_p(h;\ \varepsilon)$, there is an automorphism σ_β of $\widehat{\mathbf{G}_p}(h;\ \varepsilon)$ given by

$$\sigma_\beta((U,\mu)) \equiv (U, \beta(U)\mu) \tag{21}$$

for all $(U,\mu) \in \widehat{\mathbf{G}_p}(h;\ \varepsilon)$. It is easy to see that the $\sigma_\beta((U,\mu))$ have product relations similar to (18) with χ replaced by $\delta\beta\chi$, $\delta\beta$ the 2-cochain given by

$$\delta\beta(U,V) \equiv \frac{\beta^U(V)\beta(U)}{\beta(UV)}\ , \tag{22}$$

and satisfying the 2-cocycle relation (19) trivially. A 2-cochain of the form $\delta\beta$ (22), β some 1-cochain, is called a *2-coboundary* of $\mathbf{G}_p(h;\ \varepsilon)$. Thus it is natural to regard the product (18) equivalent to all those obtained by replacing χ with $\delta\beta\chi$, $\delta\beta$ any 2-coboundary of $\mathbf{G}_p(h;\ \varepsilon)$, and the Abelian extensions $\widehat{\mathbf{G}_p}(h;\ \varepsilon)$ are in one-to-one correspondence to equivalence classes of 2-cocycles of $\mathbf{G}_p(h;\ \varepsilon)$ which are equal up to a 2-coboundary.

Remark. Obviously the group $\widehat{\mathbf{G}_p}(h;\ \varepsilon)$ is the *semi-direct product* of $\mathbf{C}_p(h;\ \varepsilon)$ by $\mathbf{G}_p(h;\ \varepsilon)$ with the action of the latter on the former given by Eq. (14) [M2]. There is another Abelian extension which is the *direct product* of $\mathbf{C}_p(h;\ \varepsilon)$ with $\mathbf{G}_p(h;\ \varepsilon)$, and a corresponding cohomology which is similar (but simpler) than the one discussed above (see e.g. [L3]). A similar remark applies to the Lie algebra cohomology discussed below.

Note that the ★-product introduced in Sect. 5 is naturally associated with this semidirect product cohomology of $\mathbf{G}_1(h;\ \varepsilon)$, whereas the usual operator product is associated with its direct product cohomology [L3].

3.2. Abelian Extensions of Lie Algebras. The Lie algebra of $\mathbf{C}_p(h;\ \varepsilon)$ (12) is the set

$$\mathbf{c}_p(h;\ \varepsilon) = \mathrm{map}(\mathrm{Gr}_p(h;\ \varepsilon);\ \mathbb{R}) \tag{23}$$

of "smooth" maps $m\colon \mathrm{Gr}_p(h;\ \varepsilon) \to \mathbb{R};\ F \mapsto m(F)$ with the Lie bracket given by

$$[m,n](F) = [m(F), n(F)] = 0\ . \tag{24}$$

Corresponding to (14), the natural representation $m \mapsto \mathcal{L}_u m$ of $\mathbf{g}_p(h;\ \varepsilon)$ on $\mathbf{c}_p(h;\ \varepsilon)$ is given by the Lie derivative,

$$(\mathcal{L}_u m)(F) \equiv \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t} m(\mathrm{e}^{-\mathrm{i}tu} F \mathrm{e}^{\mathrm{i}tu})|_{t=0} \tag{25}$$

$\forall m \in \mathbf{c}_p(h;\ \varepsilon),\ u \in \mathbf{g}_p(h;\ \varepsilon),\ F \in \mathrm{Gr}_p(h;\ \varepsilon)$.

Similar to the group case, *1-cochains* and *2-cochains* of $\mathbf{g}_p(h;\ \varepsilon)$ are "smooth," linear, antisymmetric, $\mathbf{c}_p(h;\ \varepsilon)$-valued maps on $\mathbf{g}_p(h;\ \varepsilon)$ and $\mathbf{g}_p(h;\ \varepsilon) \times \mathbf{g}_p(h;\ \varepsilon)$, respectively. The Lie algebra corresponding to $\widehat{\mathbf{G}_p}(h;\ \varepsilon)$ is

$$\widehat{\mathbf{g}_p}(h;\ \varepsilon) \equiv \mathbf{g}_p(h;\ \varepsilon) \oplus \mathbf{c}_p(h;\ \varepsilon) \tag{26}$$

with the Lie bracket given by

$$\begin{aligned}[(u,m),(v,n)] \equiv &\frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{i}\mathrm{d}s}(\mathrm{e}^{\mathrm{i}tu}, \mathrm{e}^{\mathrm{i}tm}) \cdot (\mathrm{e}^{\mathrm{i}sv}, \mathrm{e}^{\mathrm{i}sn}) \\ &\cdot (\mathrm{e}^{-\mathrm{i}tu}, \mathrm{e}^{-\mathrm{i}tm}) \cdot (\mathrm{e}^{-\mathrm{i}sv}, \mathrm{e}^{-\mathrm{i}sn})|_{s=t=0}\ .\end{aligned} \tag{27}$$

Using (18), a simple calculation gives

$$[(u,m),(v,n)] = ([u,v],\ \mathcal{L}_u n - \mathcal{L}_v m + c(u,v)) \tag{28}$$

with

$$c(u,v) = \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{i}\mathrm{d}s}\chi(\mathrm{e}^{\mathrm{i}tu},\mathrm{e}^{\mathrm{i}sv})\chi(\mathrm{e}^{-\mathrm{i}tu},\mathrm{e}^{-\mathrm{i}sv})\chi(\mathrm{e}^{\mathrm{i}tu}\mathrm{e}^{\mathrm{i}sv},\mathrm{e}^{-\mathrm{i}tu}\mathrm{e}^{-\mathrm{i}sv})|_{s=t=0}\ , \tag{29}$$

a 2-cochain as (27) implies antisymmetry: $c(u,v) = -c(v,u)$. It follows from (19) that c obeys the relation

$$\begin{aligned} c(u,[v,w]) + \mathcal{L}_u c(v,w) &= c([u,v],w) - \mathcal{L}_w c(u,v) \\ &\quad + c(v,[u,w]) + \mathcal{L}_v c(u,w) \end{aligned} \tag{30}$$

for all $u,v,w \in \mathbf{g}_p(h;\ \varepsilon)$, which is equivalent to $[\cdot,\cdot]$ fulfilling the Jacobi identity. Equation (30) is a *2-cocycle relation*, and a 2-cochain satisfying it is a *2-cocycle* of $\mathbf{g}_p(h;\ \varepsilon)$.

The analog of the automorphism σ_β (21) is the map

$$s_b((u,m)) \equiv (u, m + b(u)) \tag{31}$$

with b the 1-cochain

$$b(u) = \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\beta(\mathrm{e}^{\mathrm{i}tu})|_{t=0}\ ; \tag{32}$$

this changes c to $c - \mathrm{d}b$ with $\mathrm{d}b$ the 2-cochain given by

$$\mathrm{d}b(u,v) = b([u,v]) - \mathcal{L}_u b(v) + \mathcal{L}_v b(u) \tag{33}$$

and satisfying (30) trivially; such a $\mathrm{d}b$ is denoted as *2-coboundary* of $\mathbf{g}_p(h;\ \varepsilon)$. Similar as in the Lie group case, we regard the Lie bracket (28) equivalent to the ones obtained by replacing c with $c - \mathrm{d}b$, b any 1-cochain.

4. Quantization of Fermions in External Fields

4.1. Fermion Field Algebras and Fock Spaces. In the spirit of the algebraic approach to quantum field theory [HK], we start with the fermion field algebra $\mathcal{A}(h)$ over h which contains the observable algebra as subalgebra. $\mathcal{A}(h)$ is defined as C^*-algebra with involution $a \mapsto a^*$, generated by the elements $\psi^*(f), f \in h$, such that the mapping $f \mapsto \psi^*(f)$ is linear and the following CAR are fulfilled:

$$\begin{aligned} \{\psi(f),\psi^*(g)\} &= (f,g)\ , \\ \{\psi(f),\psi(g)\} &= 0 \quad \forall, g \in h\ , \end{aligned} \tag{34}$$

with

$$\psi(f) = \psi^*(f)^* \quad \forall f \in h \tag{35}$$

($(\cdot,\cdot)$ denotes the inner product in h).

In this paper we consider only unitary representations of $\mathcal{A}(h)$ on the Fock space $\mathcal{F}(h)$ over h, where the involutuion $*$ can be identified with taking the Hilbert space adjoint [BR2] (note that we use the same symbol $*$ to denote the Hilbert space adjoint in h and in $\mathcal{F}(h)$).

Let $a^{(*)}(f), f \in h$, be the annihilation (creation) operators on $\mathcal{F}(h)$ satisfying the CAR and

$$a(f)\Omega = 0 \quad \forall f \in h \tag{36}$$

with Ω the vacuum in $\mathcal{F}(h)$. We denote as $\mathcal{D}_{at}(h)$ the set of algebraic tensors in $\mathcal{F}(f)$ which is the linear span of the monomials

$$a^*(f_1)a^*(f_2)\dots a^*(f_n)\Omega$$

with $f_1, f_2, \cdots f_n \in h$ and $n \in \mathbb{N}_0$. Note that $\mathcal{D}_{at}(h)$ is dense in $\mathcal{F}(h)$.

The free (= Fock–Cook) representation Π_1 of $\mathcal{A}(h)$ on $\mathcal{F}(h)$ is given by

$$\Pi_1(\psi^{(*)}(f)) \equiv a^{(*)}(f) \quad \forall f \in h\ . \tag{37}$$

For all unitary operators U on h, there is a unique unitary operator $\Gamma(U)$ on $\mathcal{F}(h)$ such that

$$\Gamma(U)a^{(*)}(f)\Gamma(U)^* = a^{(*)}(Uf) \quad \forall f \in h, \quad \Gamma(U)\Omega = \Omega\ , \tag{38}$$

and one can show that

$$\Gamma(U)\Gamma(V) = \Gamma(UV), \quad \Gamma(U^*) = \Gamma(U)^* \tag{39}$$

for all unitary operators U, V on h [BR2]. Hence $\Gamma(\cdot)$ provides a unitary representation of the group $\mathbf{G}(h)$ of all unitary operators on h. Moreover, for u a self-adjoint operator on $h, \Gamma(\mathrm{e}^{\mathrm{i}tu})$ is a strongly continuous 1-parameter family of unitary operators on $\mathcal{F}(h)$, hence (Stone's theorem [RS2])

$$\Gamma(\mathrm{e}^{\mathrm{i}tu}) = \mathrm{e}^{\mathrm{i}tJ(u)} \tag{40}$$

with

$$J(u) = \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\Gamma(\mathrm{e}^{\mathrm{i}tu})|_{t=0} \tag{41}$$

a self-adjoint operator on $\mathcal{F}(h)$ [BR2]. The operators $J(u)$ are unbounded in general, even if u is bounded. However, the set

$$\mathcal{D}^\infty(h) \equiv \{\psi \in \mathcal{F}(h) | \|N^k\psi\| < \infty \quad \forall k \in \mathbb{N}\} \tag{42}$$

($\|\cdot\|$ the Hilbert space norm) with $N \equiv J(1)$ the particle number operator, can be shown to be a common, dense, invariant domain of definition for all $J(u)$, u a bounded, self-adjoint operator on h. Moreover, it follows from (39) that

$$[J(u), J(v)] = J([u, v]) \tag{43}$$

on $\mathcal{D}^\infty(h)$, i.e. that $u \mapsto J(u)$ provides a unitary representation of the Lie algebra $\mathbf{g}(h)$ of bounded, self-adjoint operators on h [BR2].

Especially, $\Gamma(\cdot)$ and $J(\cdot)$ by restriction give representations for all the Lie groups and Lie algebras Map(M^d; G) and map(M^d; g), respectively. However, these are no highest weight representations, hence they are not very interesting from a mathematical point of view [KR]. This corresponds to the fact that the free representation Π_1 (37) of the fermion field algebra $\mathcal{A}(h)$ is not the one of interest to quantum field theory: The time evolution is implemented in this representation by $\Gamma(\mathrm{e}^{-itD_A})$, and its generator $J(D_A)$, the multi-particle Hamiltonian, is not bounded from below [CR, GL].

4.2 Quasi-Free Representations. A physically relevant representation for the model can be constructed by "filling up the Dirac sea" [CR]: Let

$$h = h_+ \oplus h_-, \quad h_\pm = P^0_\pm h, \quad P^0_\pm = \frac{1}{2}(1 \pm \varepsilon), \quad \varepsilon = \mathrm{sign}(D_A)$$

be the splitting of the 1-particle Hilbert space in positive and negative energy subspaces as discussed in Sect. 2.2. Then the quasi-free representation Π_ε of $\mathcal{A}(h)$ is given by $\Pi_\varepsilon(\psi^{(*)}(\cdot)) \equiv \hat{\psi}^{(*)}(\cdot;\,\varepsilon)$,

$$\hat{\psi}^*(f;\,\varepsilon) = a^*(P^0_+ f) + a(C_\varepsilon P^0_- f) \qquad \forall f \in h \tag{44}$$

with C_ε a conjugation[9] on h commuting with ε. Indeed, one can easily check the multi-particle Hamiltonian in this representation is $J(|D_A|)$ (with $|D_A| = D_A\varepsilon$) and positive [GL].

Obviously, we can construct a quasi-free representation Π_F of $\mathcal{A}(h)$ for any grading operator F on h. Π_F is called unitarily equivalent to Π_ε if there is a unitary operator $\mathcal{U}(F,\varepsilon)$ on $\mathcal{F}(h)$ such that

$$\mathcal{U}(F,\varepsilon)\hat{\psi}^*(f;\,\varepsilon) = \hat{\psi}^*(f;\,F)\mathcal{U}(F,\varepsilon) \quad \forall f \in h\,. \tag{45}$$

The well-known necessary and sufficient condition for this to be the case is [KS]

$$(F - \varepsilon) \in B_2(h)\,, \tag{46}$$

hence $\mathrm{Gr}_1(h;\,\varepsilon)$ introduced in the last section is just the set of grading operators F with Π_F unitarily equivalent to Π_ε.

Remark. The conjugation C_ε required for the construction of the quasi-free representation Π_ε (44) is not unique. Indeed, if U is a unitary operator on h commuting with ε, then

$$C^U_\varepsilon \equiv U^{-1}C_\varepsilon U$$

is a conjugation on h commuting with ε if (and only if) C_ε is. However, this ambiguity is harmless as the representations obtained with different choices for this conjugation are all unitarily equivalent: Indeed, one easily sees that

$$\begin{aligned} a^*(P^0_+ f) + a(C^U_\varepsilon P^0_- f) &= \Gamma(P^0_+ \oplus C^U_\varepsilon C_\varepsilon P^0_-)[a^*(P^0_+ f) + a(C_\varepsilon P^0_- F)] \\ &\quad \times \Gamma(P^0_+ \oplus C^U_\varepsilon C_\varepsilon P^0_-)^* \quad \forall f \in h \end{aligned}$$

with $P^0_+ \oplus C^U_\varepsilon C_\varepsilon P^0_-$ the unitary operator on h which is 1 on $h = P^0_+ h$ and $C^U_\varepsilon C_\varepsilon$ on $h_- = P^0_- h$.

In the following, we find it convenient to fix this ambiguity as follows: for a given ε we assume that some conjugation C_ε is chosen. Then for any unitary operator U, the grading operator $F = U^{-1}\varepsilon U$ has the conjugation $C_F = U^{-1}C_\varepsilon U$.

[9] I.e., C_ε is anti-linear and obeys

$$(C_\varepsilon f, C_\varepsilon g) = (g, f) \quad \forall f, g \in h$$

4.3. Bogoliubov Transformations. Let F be any grading operator on h.

Every unitary operator U on h defines an automorphism α_U of the fermion field algebra $\mathcal{A}(h)$:

$$\alpha_U(\psi^{(*)}(f)) \equiv \psi^{(*)}(Uf) \quad \forall f \in h \ . \tag{47}$$

Such an α_U is called a Bogoliubov transformation. It is called unitarily implementable in the quasi-free representation Π_F if there is a unitary operator $\hat{\Gamma}(U;\, F)$ on $\mathcal{F}(h)$, a (so-called) implementer, such that

$$\hat{\Gamma}(U;\, F)\hat{\psi}^*(f;\, F) = \hat{\psi}^*(Uf;\, F)\hat{\Gamma}(U;\, F) \quad \forall f \in h \ . \tag{48}$$

If the implementer $\hat{\Gamma}(U;\, F)$ exists it is unique up to a phase [R1]. The well-known necessary and sufficient criterion for this to be the case is the Hilbert–Schmidt condition [R1]

$$[U, F] \in B_2(h) \ , \tag{49}$$

hence $\mathbf{G}_1(h;\, F)$ introduced in the last section can be identified with the group of all unitarily implementable Bogoliubov transformations in Π_F.

5. Group Structure of the Implementers

5.1. General Discussion. Let $F \in \mathrm{Gr}_1(h;\, \varepsilon)$. It is known that then the implementers $\hat{\Gamma}(\cdot;\, F)$ provide a projective representation of the Lie group $\mathbf{G}_1(h;\, \varepsilon)$ with the usual operator product as group multiplication [L3]. However, it is possible to define another product, ★, also providing a group structure of the implementers: From the defining relations (44) of the quasi-free representation Π_F one can see that every Bogoliubov transformation α_U in Π_F can be written as

$$\hat{\psi}^*(Uf;\, F) = \Gamma(U)\hat{\psi}^*(f;\, F^U)\Gamma(U)^* \quad \forall f \in h \tag{50}$$

(we just used $UU^* = 1, U^*FU = F^U, U^*C_FU = C_{FU}$, and Eq. (38)). Using this formula, Eq. (48), and assuming $U, V \in \mathbf{G}_1(h;\, \varepsilon)$, we rewrite the Bogoliubov transformation α_{UV} in Π_F as follows:

$$\begin{aligned}\hat{\psi}^*(UVf;\, F) =& \Gamma(U)\hat{\psi}^*(Vf;\, F^U)\Gamma(U)^* = \Gamma(U)\hat{\Gamma}(V;\, F^U)\Gamma(U)^*\hat{\Gamma}(U;\, F)\\ &\times \hat{\psi}^*(f;\, F)\hat{\Gamma}(U;\, F)^*\Gamma(U)\hat{\Gamma}(V;\, F^U)^*\Gamma(U)^* \ .\end{aligned}$$

From this we can see that the unitary operators $\Gamma(U)\hat{\Gamma}(V;\, F^U)\Gamma(U)^*\hat{\Gamma}(U;\, F)$ and $\hat{\Gamma}(UV;\, F)$ both implement the same Bogoliubov transformation α_{UV} in Π_F. As the implementer of a Bogoliubov transformation is unique up to the phase [R1], we conclude that

$$\begin{aligned}\hat{\Gamma}(U;\, F) \bigstar \hat{\Gamma}(V;\, F) &\equiv \Gamma(U)\hat{\Gamma}(V;\, F^U)\Gamma(U)^*\hat{\Gamma}(U;\, F)\\ &= \hat{\Gamma}(UV;\, F)\chi(U, V;\, F)\end{aligned} \tag{51}$$

with χ some U(1)-valued function on $\mathbf{G}_1(h;\, \varepsilon) \times \mathbf{G}_1(h;\, \varepsilon) \times \mathrm{Gr}_1(h;\, \varepsilon)$. It is natural to regard this as *definition of a product* ★ of two implementers. Associativity of this product

$$\begin{aligned}&(\hat{\Gamma}(U;\, F)\bigstar\hat{\Gamma}(V;\, F))\bigstar\hat{\Gamma}(W;\, F)\\&\quad=\Gamma(UV)\hat{\Gamma}(W;\, F^{UV})\Gamma(UV)^*(\hat{\Gamma}(U;\, F)\bigstar\hat{\Gamma}(V;\, F))\\&\quad\overset{!}{=}\hat{\Gamma}(U;\, F)\bigstar(\hat{\Gamma}(V;\, F)\bigstar\hat{\Gamma}(W;\, F))\\&\quad=\Gamma(U)(\hat{\Gamma}(V;\, F^U)\bigstar\hat{\Gamma}(W;\, F^U))\Gamma(U)^*\hat{\Gamma}(U;\, F)\end{aligned}$$

is equivalent to the following relation:

$$\chi(U,V;\, F)\chi(UV,W;\, F)=\chi(V,W;\, F^U)\chi(U,VW;\, F) \tag{52}$$

$\forall U,V,W\in\mathbf{G}_1(h;\,\varepsilon)$ and $F\in\mathrm{Gr}_1(h;\,\varepsilon)$. Similarly, it follows from (50) and (48) that for $U\in\mathbf{G}_1(h;\,\varepsilon), F\in\mathrm{Gr}_1(h;\,\varepsilon)$, the unitary operators $\Gamma(U)^*\hat{\Gamma}(U;\, F)^*\Gamma(U)$ and $\hat{\Gamma}(U^*;\, F^U)$ both implement the same Bogoliubov transformation α_{U^*} in Π_{FU} and therefore are equal up to a phase. The phases of the implementers are arbitrary, and we assume them to be chosen in a "smooth" way (this is possible at least for U in some neighborhood of the identity – see below) and such that

$$\hat{\Gamma}(U^*;\, F^U)=\Gamma(U)^*\hat{\Gamma}(U;\, F)^*\Gamma(U),\quad \hat{\Gamma}(1;\, F)=1\,. \tag{53}$$

This and (51) imply that

$$\overline{\chi(U,V;\, F)}=\chi(V^*,V^*;\, F^{UV}),\quad \chi(U,U^*;\, F)=1 \tag{54}$$

$\forall U,V\in\mathbf{G}_1(h;\,\varepsilon), F\in\mathrm{Gr}_1(h;\,\varepsilon)$. One can change the phase convention:

$$\hat{\Gamma}(U;\, F)\to\hat{\Gamma}(U;\, F)\beta(U;\, F) \tag{55}$$

with β some "smooth" U(1)-valued function on $\mathbf{G}_1(h;\,\varepsilon)\times\mathrm{Gr}_1(h;\,\varepsilon)$ satisfying

$$\overline{\beta(U;\, F)}=\beta(U^*;\, F^U),\quad \beta(1;\, F)=1 \tag{56}$$

$\forall U\in\mathbf{G}_1(h;\,\varepsilon), F\in\mathrm{Gr}_1(h;\,\varepsilon)$. This amounts to changing

$$\chi\to\delta\beta\chi \tag{57}$$

with

$$\delta\beta(U,V;\, F)\equiv\frac{\beta(V;\, F^U)\beta(U;\, F)}{\beta(UV;\, F)}\,. \tag{58}$$

Moreover, the formula (59) for χ below shows explicitly that the mapping $(U,V,F)\mapsto\chi(U,V;\, F)$ is "smooth" (at least *locally*, i.e. for U,V in some neighborhood of the identity) and satisfies all the relations given above. Thus the cohomology as discussed in Sect. 3 naturally emerges here: writing $\chi(U,V;\, F)=\chi(U,V)(F)$, one can see that χ is a 2-cocycle of the group $\mathbf{G}_1(h;\,\varepsilon)$ as (52) are just the 2-cocycle relations (19), and (54) is equivalent to (16). Similarly, $\delta\beta$ is a 2-coboundary and (58) and (56) are equivalent to (22) and (15), respectively.

Remark. It is easy to see that the ★- and the operator product are related to each other: As $\hat{\Gamma}(U;\, F)\hat{\Gamma}(V;\, F)$ also implements the Bogoliubov transformation α_{UV} in Π_F, there must be a phase $\eta(U,V;\, F)$ such that

$$\hat{\Gamma}(U;\ F) \bigstar \hat{\Gamma}(V;\ F) = \eta(U,V;\ F)\hat{\Gamma}(U;\ F)\hat{\Gamma}(V;\ F)\ .$$

This allows us to identify

$$\Gamma(U)\hat{\Gamma}(V;\ F^U)\Gamma(U)^* = \eta(U,V;\ F)\hat{\Gamma}(U;\ F)\hat{\Gamma}(V;\ F)\hat{\Gamma}(U;\ F)^*$$

for all $U,V \in \mathbf{G}_1(h;\ \varepsilon), F \in \mathrm{Gr}_1(h;\ \varepsilon)$. One can regard η as an intertwiner relating the semidirect-product and the direct-product cohomologies of $\mathbf{G}_1(h;\ \varepsilon)$ (see the remark in Sect. 3.1). Under a change of phases of the implementers (55) it obviously transforms as

$$\eta(U,V;\ F) \to \frac{\beta(V;\ F)}{\beta(V;\ F^U)}\eta(U,V;\ F)\ .$$

It is even possible to choose the phases of the implementers such that

$$\eta(U,V;\ F) = 1\ ,$$

i.e. that the ★- and the operator product of the implementers coincide (below we show this explicitly for $U,V \in \mathbf{G}_1(h;\ \varepsilon)$ in some neighborhood of the identity).

Hence at this stage, the two products are essentially the same. However, when introducing the implementers $\hat{\Gamma}_p(U;\ F)$ for $U \in \mathbf{G}_p(h;\ \varepsilon), F \in \mathrm{Gr}_p(h;\ \varepsilon), p > 1$, this will be no longer the case: it is only the ★-product that allows for a multiplicative regularization leading to an *associative* product of these implementers, whereas all non-trivial regularizations of the operator product are non-associative and give rise to a non-trivial 3-cocycle [C]. Indeed, a non-trivial multiplicative regularization of the operator product would provide a "highest weight" representation of a non-trivial *central* extension of $\mathbf{G}_p(h;\ \varepsilon)$ which does not exist for $p > 1$ [P].

5.2. Explicit Formulas. The explicit form of the implementers $\hat{\Gamma}(U;\ F)$ for $U \in \mathbf{G}_1(h;\ F)$, F any grading operator on h, was worked out by Ruijsenaars [R1], and is quite complicated in general. Assuming, however, that P_-UP_- has a bounded inverse on P_-h, it simplifies and can be used to explicitly evaluate the 2-cocycle χ in (51).

We denote the set of all unitary operators U with P_-UP_- bijective on P_-h as $\mathcal{U}(h;\ F)$, and $\mathcal{U}^{(F)}(h;\ \varepsilon) = \mathcal{U}(h;\ \varepsilon) \cap \mathcal{U}(h;\ F)$. Note that for all $F \in \mathrm{Gr}_p(h;\ \varepsilon), \mathbf{G}_p(h;\ \varepsilon) \cap \mathcal{U}^{(F)}(h;\ \varepsilon)$ is a neighborhood (but no subgroup) of $\mathbf{G}_p(h;\ \varepsilon)$ for all $p \in \mathbb{N}_0$.

In Appendix A we prove that for all $F \in \mathrm{Gr}_1(h;\ \varepsilon)$, $U,\ V \in \mathbf{G}_1(h;\ \varepsilon) \cap \mathcal{U}^{(F)}(h;\ \varepsilon)$, the phases of the implementers can be chosen such that

$$\chi(U,V;\ F) = \left(\frac{\det(1 + [P_-^0 UP_-^0]^{-1} P_-^0 UP_+^0 VP_-^0 [P_-^0 VP_-^0]^{-1}}{\det(1 + [P_-^0 V^* P_-^0]^{-1} P_-^0 V^* P_+^0 U^* P_-^0 [P_-^0 U^* P_-^0]^{-1})}\right)^{1/2} \tag{59}$$

is independent of F. Moreover, we show that if in addition $U,V \in \mathbf{G}_0(h;\ \varepsilon)$, one can write this locally as 2-coboundary: $\chi = \delta\beta_1$, with

$$\beta_1(U;\ F) = \left(\frac{\det(P_+^0 + [P_-^0 UP_-^0]^{-1})}{\det(P_+^0 + [P_-^0 U^* P_-^0]^{-1})}\right)^{1/2} \tag{60}$$

demonstrating that χ is locally a trivial 2-cocycle for $\mathbf{G}_0(h;\ \varepsilon)$ but (as $\beta_1(U;\ F)$ does not exist for general $U \in \mathbf{G}_1(h;\ \varepsilon) \cap \mathcal{U}^{(F)}(h;\ \varepsilon)$) non-trivial for $\mathbf{G}_1(h;\ \varepsilon)$.

5.3. Interpretation. For each $F \in \mathrm{Gr}_1(h;\, \varepsilon)$ we have a quasi-free representation Π_F of the fermion field algebra $\mathcal{A}(h)$ on the Fock space $\mathcal{F}(h)$, and this can also be regarded as a representation of $\mathcal{A}(h)$ on a Fock space bundle with the base space $\mathrm{Gr}_1(h;\, \varepsilon)$ and fibres $\mathcal{F}(h)$. Equation (50) shows that it is natural to interpret $\hat{\Gamma}(U;\, F)^*$ as transformation from Π_{FU} to Π_U. This suggests to define a mapping $\hat{\Gamma}(U)$ on the space $\mathcal{S}_1(h;\, \varepsilon)$ of sections in the above-mentioned bundle. To be specific, $\mathcal{S}_1(h;\, \varepsilon)$ is the vector space of "smooth" mappings

$$\Psi\colon \mathrm{Gr}_1(h;\, \varepsilon) \to \mathcal{F}(h), \quad F \mapsto \Psi(F)\,,$$

and it is a $\mathbf{C}^c_1(h;\, \varepsilon)$-module [H] with

$$\mathbf{C}^c_1(h;\, \varepsilon) \equiv \mathrm{Map}(\mathrm{Gr}_1(h;\, \varepsilon); \mathbb{C}) \tag{61}$$

the ring of "smooth," $\mathbb{C}$-valued functions on $\mathrm{Gr}_1(h;\, \varepsilon)$:

$$(v\Psi)(F) = (\Psi v)(F) \equiv v(F)\Psi(F) \quad \forall v \in \mathbf{C}^c_1(h;\, \varepsilon), \Psi \in \mathcal{S}_1(h;\, \varepsilon)\,. \tag{62}$$

Moreover, one can define an "inner product"[10],

$$\langle\langle \mathcal{L}_1 \Psi_1, \Psi_2\rangle\rangle(F) \equiv \langle \Psi_1(F), \Psi_2(F)\rangle \quad \forall \Psi_1, \Psi_2 \in \mathcal{S}_1(h;\, \varepsilon)\,. \tag{63}$$

Then if $\hat{\Gamma}(U)$ is defined as[11]

$$(\hat{\Gamma}(U)\Psi)(F) \equiv \hat{\Gamma}(U;\, F)^* \Gamma(U) \Psi(F^U) \quad \forall \Psi \in \mathcal{S}_1(h;\, \varepsilon)\,, \tag{64}$$

the definition (51) of the $\bigstar$-product implies

$$(\hat{\Gamma}(U) \bigstar \hat{\Gamma}(V))\Psi \equiv \hat{\Gamma}(U)(\hat{\Gamma}(V)\Psi) = \overline{\chi(U, V)}\hat{\Gamma}(UV)\Psi \tag{65}$$

$\forall \Psi \in \mathcal{S}_1(h;\, \varepsilon), U, V \in \mathbf{G}_1(h;\, \varepsilon)$. Moreover, it follows from (53) the involution $*$ is identical with the adjungation

$$\langle\langle \Psi_1, \hat{\Gamma}(U)\Psi_2\rangle\rangle(F) = \langle\langle \hat{\Gamma}(U)^* \Psi_1, \Psi_2\rangle\rangle(F^U) \quad \forall \Psi_1, \Psi_2 \in \mathcal{S}_1(h;\, \varepsilon)\,, \tag{66}$$

and the $\hat{\Gamma}(U)$ are unitary with respect to $\langle\langle \cdot, \cdot \rangle\rangle$. Thus $(U, v) \mapsto \hat{\Gamma}(U)v$ provides a "unitary" representation of an Abelian extension $\widehat{\mathbf{G}_1}(h;\, \varepsilon)$ of $\mathbf{G}_1(h;\, \varepsilon)$ on the space $\mathcal{S}_1(h;\, \varepsilon)$ of sections in a Fock-space bundle, and the corresponding 2-cocycle is *determined* by (51) (and equal to $\overline{\chi(U, V;\, F)}$ (59) in some neighborhood of the identity).

Remark. As we can choose the 2-cocycle χ in (51) F-independent, $\widehat{\mathbf{G}_1}(h;\, \varepsilon)$ is equivalent a central extension of $\mathbf{G}_1(h;\, \varepsilon)$. The latter is the same playing a prominent role in the theory of loop groups as mentioned in the introduction (this can be explicitly seen by combining our results here with the ones from Ref. [L3]).

6. Current Algebras in (1 + 1)-Dimensions

6.1. General Discussion. From the results of the last section one can easily obtain the corresponding formulas on the Lie algebra level.

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[10] To be precise, a $C^c_1(h;\, \varepsilon)$-valued Hermitian sesquilinearform

[11] I am grateful to S N. M. Ruijsenaars for suggesting this modification of my original argument

For $F \in \mathrm{Gr}_1(h; \varepsilon)$ and $u \in \mathbf{g}_1(h; \varepsilon)$, the current $\hat{J}(u; F)$ can be defined as

$$\hat{J}(u; F) = \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\hat{\Gamma}(\mathrm{e}^{\mathrm{i}tu}; F)|_{t=0} \ . \tag{67}$$

Indeed, it was shown by Carey and Ruijsenaars [CR] that for any grading operator F on h and all $u \in \mathbf{g}_1(h; F)$, there is a self-adjoint operator $\hat{J}(u; F)$ on $\mathcal{F}(h)$ such that $\mathrm{e}^{\mathrm{i}t\hat{J}(u;F)}$ for all $t \in \mathbb{R}$ implements the Bogoliubov transformation $\alpha_{\mathrm{e}^{\mathrm{i}tu}}$ in Π_F, and it therefore must be equal to $\hat{\Gamma}(\mathrm{e}^{\mathrm{i}tu}; F)$ up to a "smooth" function $\tilde{\eta}_F: \mathbb{R} \to \mathrm{U}(1); t \mapsto \tilde{\mu}_F(\mathrm{e}^{\mathrm{i}tu})$. Moreover, $\mathcal{D}^\infty(h)$ (42) is a common, dense, invariant domain for all $\hat{J}(u; F), u \in \mathbf{g}_1(h; \varepsilon)$, hence by Stone's theorem [RS2], the differentiation in (67) is defined in the strong sense on $\mathcal{D}^\infty(h)$. We define the Lie bracket of the currents as[12]

$$[\hat{J}(u; F), \hat{J}(v; F)]_1 \equiv \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{i}\mathrm{d}s}\hat{\Gamma}(\mathrm{e}^{\mathrm{i}tu}; F) \star \hat{\Gamma}(\mathrm{e}^{\mathrm{i}sv}; F) \star \hat{\Gamma}(\mathrm{e}^{-\mathrm{i}tu}; F) \star \hat{\Gamma}(\mathrm{e}^{-\mathrm{i}sv}; F)\,|_{s=t=0} \ , \tag{68}$$

and with (51) we obtain

$$[\hat{J}(u; F), \hat{J}(v; F)]_1 = \hat{J}([u, v]; F) + c(u, v; F) \tag{69}$$

with the Schwinger term c a 2-cochain of the Lie algebra $\mathbf{g}_1(h; \varepsilon)$ given by Eq. (29) with χ (59), and therefore satisfying the 2-cocycle relation (30). In Appendix C we show that

$$c(u, v; F) = \frac{1}{4}\mathrm{tr}([u, \varepsilon][v, \varepsilon]\varepsilon) \ . \tag{70}$$

This is the cocycle derived originally by Lundberg [Lu] (in a different but equivalent form) and which is usually referred to as the Kac–Peterson cocycle [PS].

Changing the phases of the implementers (55) changes

$$\hat{J}(u; F) \to \hat{J}(u; F) + b(u; F) \tag{71}$$

with b the 1-cochain of $\mathbf{g}_1(h; \varepsilon)$ given by (32), and by an explicit calculation one can check that this amounts to changing

$$c \to c - \mathrm{d}b \tag{72}$$

with $\mathrm{d}b$ the 2-coboundary (33). Thus $[\cdot, \cdot]_1$ is not just the commutator. It is natural to set

$$[\hat{J}(u; F), n(F)]_1 \equiv (\mathcal{L}_u n)(F) \tag{73}$$

for all "smooth," $\mathbb{C}$-valued functions n on $\mathrm{Gr}_1(h; \varepsilon)$ as this definition naturally accounts for (71), (72) and the Jacobi identity in (30).

Remark. As discussed above, in the phase convention leading to the F-independent 2-cocycle (59), the $\star$-product coincides with the operator product; therefore, the Lie bracket of the currents in (69) is equivalent to (but not identical with) the commutator of these as operators on $\mathcal{F}(h)$.

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[12] Note that this is well-defined on $\mathcal{D}^\infty(h)$

6.2. Interpretation. Corresponding to the interpretation of $\hat{\Gamma}(U;\, F)$ as mapping $\hat{\Gamma}(U)$ on the space $\mathcal{S}_1(h;\, \varepsilon)$ of sections in a Fock space bundle in Sect. 5.3, it is natural to set

$$\hat{J}(u)\Psi \equiv \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\hat{\Gamma}(\mathrm{e}^{\mathrm{i}tu})\Psi\,|_{t=0} \quad \forall\Psi \in \mathcal{S}_1(h;\, \varepsilon) \tag{74}$$

for all $u \in \mathbf{g}_1(h;\, \varepsilon)$. This defines the currents $\hat{J}(u)$ as mappings on $\mathcal{S}_1(h;\, \varepsilon)$. Corresponding to (65), the Lie bracket of two such currents is

$$[\hat{J}(u), \hat{J}(v)]_1\Psi = [\hat{J}(u), \hat{J}(v)]\Psi \quad \forall\Psi \in \mathcal{S}_1(h;\, \varepsilon) \tag{75}$$

with $[\cdot,\cdot]$ the usual commutator. With (65) we obtain[13]

$$[\hat{J}(u), \hat{J}(v)]_1 = \hat{J}([u,v]) - c(u,v) \tag{76}$$

$\forall u, v \in \mathbf{g}_1(h;\, \varepsilon)$, showing that $(u,n) \mapsto \hat{J}(u) + n$ provides a "unitary" representation of the Abelian extension $\widehat{\mathbf{g}_1}(h;\, \varepsilon)$ associated with the 2-cocycle c (70) on $\mathcal{S}_1(h;\, \varepsilon)$.

Equations (74) and (75) give a simple interpretation to the Lie bracket $[\cdot,\cdot]_1$ in Eq. (69) (and its seemingly strange property (73)): From (74) and (64) we obtain

$$(\hat{J}(u)\Psi)(F) = \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\hat{\Gamma}(\mathrm{e}^{\mathrm{i}tu};\, F)^*\Gamma(\mathrm{e}^{\mathrm{i}tu})\Psi(\mathrm{e}^{-\mathrm{i}tu}F\mathrm{e}^{\mathrm{i}tu})\,|_{t=0}\ ,$$

hence

$$(\hat{J}(u)\Psi)(F) = (\mathcal{L}_u + \tilde{J}(u;\, F))\Psi(F) \tag{77}$$

with

$$\tilde{J}(u;\, F) \equiv \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\tilde{\Gamma}(\mathrm{e}^{\mathrm{i}tu};\, F)^*\Gamma(\mathrm{e}^{\mathrm{i}tu})\,|_{t=0} = J(u) - \hat{J}(u;\, F)\ . \tag{78}$$

This shows that it is natural to regard $\hat{J}(u)$ as Gauss's law generators rather than as currents, and to interpret (76) as an abstract version of the algebra of the Gauss' law constraint operators in $(1+1)$-dimensions [J].

7. Multiplicative Regularization

7.1. General Discussion. For $U \in \mathbf{G}_1(h;\, \varepsilon)$, the implementer $\hat{\Gamma}(U;\, F)$ can be written as [R1]

$$\hat{\Gamma}(U;\, F) = E(U;\, F)N(U;\, F) \tag{79}$$

with $E(U;\, F)$ an operator on $\mathcal{F}(h)$ evaluated such that it implements the Bogoliubov transformation α_U in Π_F (i.e. obeys (48)) and $N(U;\, F) \in \mathbb{C}^\times$ a normalization constant needed to make the implementer unitary. It is given (up to a phase) by the condition that $\hat{\Gamma}(U;\, F)\hat{\Gamma}(U;\, F) = 1$, i.e.

$$|\, N(U;\, F)|^{-2} = \langle \Omega, E(U;\, F)E(U^*;\, F)\Omega\rangle\ .$$

By the explicit formula for $E(U;\, F)$ given in Eq. (A2) for $U \in \mathbf{G}_1(h;\, \varepsilon) \cap \mathcal{U}^{(F)}(h;\, \varepsilon)$ one finds

[13] The minus sign arises due to the complex conjugation

$$N(U;\,F) = \left(\det(1 - P_- U P_+ U^* P_-)\frac{\det_2(1 + a(U;\,F))}{\det_2(1 + a(U;\,F))^*}\right.$$
$$\left.\times\,\frac{\exp(\mathrm{tr}(P_+^0 a(U;\,F)P_+^0 + P_-^0 a(U;\,F)P_-^0))}{\exp(\mathrm{tr}(P_+^0 a(U;\,F)P_+^0 + P_-^0 a(U;\,F)P_-^0)^*)}\right)^{1/2} \tag{80}$$

with $a(U;\,F)$ defined in Appendix A, Eq. (A19) (we also show there that (80) is well-defined).

The crucial point is that $E(U;\,F)$ does not only exist for $U \in \mathbf{G}_1(h;\,\varepsilon)$, but that it can be defined in as sesquilinear form [K] on the set $\mathcal{D}_{at}(h)$ of algebraic tensors in $\mathcal{F}(h)$ (cf. Sect. 4.1) for all $U \in \mathbf{G}_\infty(h;\,F)$. Hence the non-existence of $\hat{\Gamma}(U;\,F)$ for general $U \in \mathbf{G}_p(h;\,\varepsilon)$, $p > 1$, is only due to the non-existence of the normalization constant $N(U;\,F)$ (this observation is due to Ruijsenaars [R2, R4]). As $E(U;\,F)$ for $U \in \mathbf{G}_\infty(h;\,F)$ obeys (48) in the form sense on $\mathcal{D}_{at}(h)$, it is natural to perform a multiplicative regularization of the implementers

$$\hat{\Gamma}(U;\,F) \to \hat{\Gamma}_p(U;\,F) \tag{81}$$

and to define[14]

$$\hat{\Gamma}_p(U;\,F) \equiv \hat{\Gamma}(U;\,F)\beta_p^N(U;\,F) \tag{82}$$

with $\beta_p^N(U;\,F)$ such that it cancels the divergency in $N(U;\,F)$ for all $U \in \mathbf{G}_p(h;\,\varepsilon), F \in \mathrm{Gr}_p(h;\,\varepsilon)$. This regularization, however, is not sufficient due to the fact that the operator product of forms cannot be defined in general. Explicitly we find for $F \in \mathrm{Gr}_1(h;\,\varepsilon), U, V \in \mathbf{G}_1(h;\,\varepsilon) \cap \mathcal{U}^{(F)}(h;\,\varepsilon)$,

$$E(U;\,F) \bigstar E(V;\,F) = \mathcal{E}(U, V;\,F)E(UV;\,F)\,, \tag{83}$$

with (see Appendix A, Eq. (A10))

$$\mathcal{E}(U, V;\,F) = \det(1 + [P_- UVU^* P_-]^{-1} P_- UVU^* P_+ UP_- [P_- UP_-]^{-1})\,. \tag{84}$$

This does not exist (diverges) for $U, V \notin \mathbf{G}_1(h;\,\varepsilon)$.[15] Therefore, $\bigstar$ is not the appropriate group multiplication for the implementers $\hat{\Gamma}_p(U;\,F)$, and we are forced to regularize it as well:

$$\bigstar \to \hat{\bigstar}_p\,. \tag{85}$$

From a quantum field theoretic point of view, it is natural to demand that the regularizations (82) and (85) are *minimal* in a sense made precise below.

To be specific: Using the lemma in Appendix B, it is easy to see that *locally* (i.e. for $U, V \in \mathbf{G}_p(h;\,\varepsilon) \cap \mathcal{U}^{(F)}(h;\,\varepsilon)$ and $F \in \mathrm{Gr}_p(h;\,\varepsilon)), P_- UP_+ U^* P_- \in B_p(h), a(U;\,F) \in B_{2p}(h)$ (cf. (A19)), and the r.h.s. of Eq. (84) is formally of the form $\det(1 + (\cdots))$ with $(\cdots) \in B_p(h)$. As discussed in Sect. 2.3, a minimal regularization of the implementers (81) therefore amounts to dropping the non-existent traces in (80), especially replacing the determinants in by regularized ones as follows:

$$N(U;\,F) \to N_p(U;\,F)$$

[14] This and similar equations below have to be interpreted as follows: for each sequence in $\mathbf{G}_1(h;\,\varepsilon)$ converging to $U \in \mathbf{G}_p(h;\,\varepsilon)$ in the $\|\cdot\|_p$-norm, the r.h s has a well-defined limit depending only on U, and the l.h.s is defined as this limit

[15] One obtains a similar formula for the operator product – see [L3]

$$\equiv \left(\det_p(1 - P_- U P_+ U^* P_-) \frac{\det_{2p}(1 + a(U;\, F))}{\det_{2p}(1 + a(U;\, F))^*}\right)^{1/2} , \qquad (86)$$

and to define $\hat{\Gamma}_p(U;\, F)$ (82) with

$$\beta_p^N(U;\, F) = \frac{N_p(U;\, F)}{N(U;\, F)} . \qquad (87)$$

However, in order to regularize the ★-product, we cannot simply replace in (84) the det(·) by $\det_p(\cdot)$ as the *regularized product should be associative.* Thus the (non-trivial) problem is to find an appropriate function $\beta_p^\star : \mathbf{G}_1(h;\, \varepsilon) \times \mathrm{Gr}_1(h;\, \varepsilon) \to \mathbb{C}^\times$ diverging for general $U, V \in \mathbf{G}_p(h;\, \varepsilon)$ and $F \in \mathrm{Gr}_p(h;\, \varepsilon)$ such that

$$\hat{\Gamma}_p(U;\, F))\hat{\star}_p \hat{\Gamma}_p(V;\, F) \equiv \hat{\Gamma}_p(U;\, F) \star \hat{\Gamma}_p(V;\, F) \frac{\beta_p^\star(V;\, F^U)\beta_p^\star(U;\, F)}{\beta_p^*(UV;\, F)} \qquad (88)$$

exists and is equal to $\hat{\Gamma}_p(UV;\, F)$ up to a phase, as such (and only such) a regularization maintains associativity and leads to a projective representation of the group $\mathbf{G}_1(h;\, \varepsilon)$, Then

$$\hat{\Gamma}_p(U;\, F)\hat{\star}_p \hat{\Gamma}_p(V;\, F) = \hat{\Gamma}_p(UV;\, F)\chi_p(U, V;\, F) \qquad (89)$$

with

$$\chi_p = \delta\beta_p\chi, \quad \beta_p \equiv \beta_p^N \beta_p^\star \qquad (90)$$

and δ defined in (58).

Our discussion above can be summarized by saying that we have to find a 1-cochain β_p of the group $\mathbf{G}_1(h;\, \varepsilon)$ such that χ_p (90) extends to a (non-trivial) 2-cocycle of the group $\mathbf{G}_p(h;\, \varepsilon)$. Moreover (and this can be regarded as definition of *minimal regularization*), β_p should be such that locally

$$\chi_p(U, V;\, F) = r_p(U, V;\, F) \times \left(\frac{\det_p(1 + [P_-^0 U P_-^0]^{-1} P_-^0 U P_+^0 V P_-^0 [P_-^0 V P_-^0]^{-1})}{\det_p(1 + [P_-^0 V^* P_-^0]^{-1} P_-^0 V^* P_+^0 U^* P_-^0 [P_-^0 U^* P_-^0]^{-1})}\right)^{1/2} \qquad (91)$$

with r_p some 2-cochain of $\mathbf{G}_p(h;\, \varepsilon)$. Obviously, if such a 1-cochain β_p exists, it is locally unique up to a 1-cochain of $\mathbf{G}_p(h;\, \varepsilon)$, hence the minimal regularization and the 2-cocycle χ_p of $\mathbf{G}_p(h;\, \varepsilon)$ in (90) are locally unique up to a 2-coboundary.

Remark. It is easy to see that the $\hat{\Gamma}_p(U;\, F)$ for $p > 1$ are in general sesquilinear forms only and cannot be promoted to (unbounded) operators on $\mathcal{F}(h)$. Indeed, for $U \in \mathbf{G}_1(h;\, F)$ it follows from (82) and the unitarity of $\hat{\Gamma}(U;\, F)$ that

$$\langle \hat{\Gamma}_p(U;\, F)\psi, \hat{\Gamma}_p(U;\, F)\psi\rangle = \mid \beta_p^N(U;\, F) \mid^2 \langle \psi, \psi\rangle \quad \forall \psi \in \mathcal{F}(h) ,$$

and this does not exist for general $U \in \mathbf{G}_p(h;\, F)$.

7.2. Explicit Result for $p = 2$. For $p = 2$, the 1-cochain providing the minimal regularization as discussed above is given by

$$\beta_2(U;\, F) = \exp\left(\frac{1}{4}\mathrm{tr}(P_-^0 F P_+^0 U P_-^0 U^* P_-^0 - P_-^0 U P_-^0 (U^*) P_+^0 F P_-^0)\right) \qquad (92)$$

(by using the lemma in Appendix B, it is easy to see that this is indeed a 1-cochain of $\mathbf{G}_1(h;\,\varepsilon)$). The proof of this result is given in Appendix D.

The results of [MR, FT] show that the 1-cochains β_p of $\mathbf{G}_1(h;\,\varepsilon)$ with the properties discussed in the last section exist for all $p > 1$.

7.3. Interpretation. Similar as discussed in Sect. 5.3 for the case $p = 1$, it is natural to regard the implementers $\hat{\Gamma}_p(U;\,F)^*, U \in \mathbf{G}_p(h;\,\varepsilon), F \in \mathrm{Gr}_p(h;\,\varepsilon)$, as mapping $\hat{\Gamma}_p(U)$ on the set $\mathcal{S}_p(h;\,\varepsilon)$ of sections in the Fock space bundle with the base space $\mathrm{Gr}_p(h;\,\varepsilon)$ and the fibres $\mathcal{F}(h)$ carrying a module structure and defined completely analogous to $\mathcal{S}_1(h;\,\varepsilon)$. The $\hat{\bigstar}_p$-product can then be understood by a multiplicative regularization of (65). Then

$$\hat{\Gamma}_p(U)\hat{\bigstar}_p\hat{\Gamma}_p(V) = \overline{\chi_p(U,V)}\hat{\Gamma}_p(UV)\,, \tag{93}$$

and $(U,v) \mapsto \hat{\Gamma}_p(U)v$ provides a representation of the Abelian extension $\widehat{\mathbf{G}_p}(h;\,\varepsilon)$ of $\mathbf{G}_p(h;\,\varepsilon)$ associated with the 2-cocycle $\overline{\chi_p}$ on $\mathcal{S}_p(h;\,\varepsilon)$. In contrast to the case $p = 1$, the $\hat{\Gamma}_p(U)$ for $p > 1$ are intertwiners between unitary inequivalent representations of the fermion field algebra $\mathcal{A}(h)$.

8. Current Algebras in Higher Dimensions

8.1. Formal Construction. On the Lie algebra level, the currents $\hat{J}_p(u;\,F)$ for $F \in \mathrm{Gr}_p(h;\,\varepsilon)$ and $u \in \mathbf{g}_p(h;\,\varepsilon)$ can be formally defined as

$$\hat{J}_p(u;\,F) \equiv \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\hat{\Gamma}_p(\mathrm{e}^{itu};\,F)|_{t=0}\,, \tag{94}$$

and the (regularized) Lie bracket for such currents is

$$\begin{aligned}[][\hat{J}_p(u;\,F),\hat{J}_p(v;\,F)]_p \equiv &\frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\hat{\Gamma}_p(\mathrm{e}^{\mathrm{i}tu};\,F)\hat{\bigstar}_p\hat{\Gamma}_p(\mathrm{e}^{\mathrm{i}sv};\,F)\\ &\hat{\bigstar}_p\hat{\Gamma}_p(\mathrm{e}^{-\mathrm{i}tu};\,F)\hat{\bigstar}_p\hat{\Gamma}_p(\mathrm{e}^{-\mathrm{i}sv};\,F)|_{s=t=0}\,. \end{aligned} \tag{95}$$

Using (89), this results in (cf. Sect. 3.2)

$$[\hat{J}_p(u;\,F),\hat{J}_p(v;\,F)]_p = \hat{J}_p([u,v];\,F) + c_p(u,v;\,F) \tag{96}$$

with $c_p = c$ given by Eq. (29) with $\chi = \chi_p$ (90). It follows that

$$c_p = c - \mathrm{d}b_p \tag{97}$$

with $b_p = b$ given by Eq. (32) for $\beta = \beta_p$. Equation (96) provides the abstract current algebra in $(d + 1)$-dimensions, $d = 2p - 1$.

8.2. Explicit Results for $p = 2$. For $p = 2$ we obtain from (92)

$$b_2(u;\,F) = -\frac{1}{16}\mathrm{tr}([u,\varepsilon][F,\varepsilon])\,, \tag{98}$$

and $c_2 = c - \mathrm{d}b_2$ results in

$$c_2(u,v;\ F) = \frac{1}{8}\mathrm{tr}([[u,\varepsilon],[v,\varepsilon]](\varepsilon - F)) \tag{99}$$

(the proof is contained in Appendix E). This is exactly the cocycle of Mickelsson and Rajeev [MR].

8.3. Existence of the Currents. The discussion above is formal because we did not give a precise mathematical meaning to the differentiation in (94). In this subsection we complete our argument in this respect by showing that $\hat{J}_p(u;\ F)$ is defined as sesquilinear form on $\mathcal{D}_{at}(h)$ for all $u \in \mathbf{g}_p(h;\ \varepsilon), F \in \mathrm{Gr}_p(h;\ \varepsilon),\ p > 1$.

It follows from the result of Carey and Ruijsenaars [CR] that

$$\mathrm{d}E(u;\ F) = \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}E(\mathrm{e}^{\mathrm{i}tu};\ F)\mid_{t=0} \tag{100}$$

(E is given in Appendix A, Eq. (A2)) exists in the strong sense on $\mathcal{D}^\infty(h)$ (42) for all $u \in \mathbf{g}_1(h;\ F), F$ any grading operator on h, and it is equal to

$$\mathrm{d}E(u;\ F) = J(P_+uP_+) - J(P_-u^tP_-) + u_{+-}\,a^*a^* + u_{-+}\,aa \tag{101}$$

($u^t = C_F U^* C_F$) with $u_{\pm\mp} = P_\pm u P_\mp, J(\cdot)$ (41), and $u_{+-}\,a^*a^*, u_{-+}\,aa$ defined in Appendix A, Eqs. (A6)*ff*. Hence by (79) and (67),

$$\hat{J}(u;\ F) = \mathrm{d}E(u;\ F) + n(u;\ F) \tag{102}$$

with $n = b$ given by (32) for $\beta = N$ (80). By explicit calculation

$$n(u;\ F)) = \mathrm{tr}(u(P_- - P_-^0)) = \frac{1}{2}\mathrm{tr}(P_+^0 u(\varepsilon - F)P_+^0 + P_-^0 u(\varepsilon - F)P_-^0) \tag{103}$$

(note that $\mathrm{tr}(a) = \mathrm{tr}(P_+^0 a P_+^0 + P_-^0 a P_-^0)$ for $a \in B_1(h)$). For $u \in \mathbf{g}_1(h;\ \varepsilon), F \in \mathrm{Gr}_1(h;\ \varepsilon)$, the r.h.s. of this is obviously finite (cf. the lemma in Appendix B).

Now it is easy to see that though $u_{+-}\,a^*a^*$ and $u_{-+}\,aa$ exist as operators on $\mathcal{F}(h)$ only if $u_{\pm\mp} \in B_2(h)$ (cf. [CR]), they exist as sesquilinear forms on $\mathcal{D}_{at}(h)$ whenever $u_{\pm\mp}$ are compact operators. Hence $\mathrm{d}E(u;\ F)$ (101) exists as form on $\mathcal{D}_{at}(h)$ for all $u \in \mathbf{g}_\infty(h;\ F)$, and we *define* the r.h.s. of (100) to be equal to the r.h.s. of (101) for all $u \in \mathbf{g}_\infty(h;\ F)$. Then it follows from (94), (79) that

$$\hat{J}_p(u;\ F) = \hat{J}(u;\ F) + b_p^N(u;\ F) \tag{104}$$

with $b_p^N = b$ given by (32) for $\beta = \beta_p^N$ (87), and this exists as form on $\mathcal{D}_{at}(h)$ as $(n + b_p^N)(u;\ F)$ is (by construction) finite for all $u \in \mathbf{g}_p(h;\ \varepsilon), F \in \mathrm{Gr}_p(h;\ \varepsilon)$. Moreover, (95) is equivalent to

$$\hat{J}_p(u;\ F), \hat{J}_p(v;\ F)]_p = [\hat{J}_p(u;\ F), \hat{J}_p(v;\ F)]_1 - \mathrm{d}b_p^\star(u,v;\ F) \tag{105}$$

with $b_p^\star = b$ (32) for $\beta = \beta_p^\star$ (cf. (88)) (of course, this again gives (96) and (97) with $b_p = b_p^N + b_p^\star$).

8.4. Complexification. The complexification $\mathbf{g}_p^c(h;\ \varepsilon)$ of $\mathbf{g}_p(h;\ \varepsilon)$ is the Lie algebra of *all* bounded operators u on h obeying $[u, F] \in B_p(h)$. Obviously,

$$\hat{J}_p(u;\ F) \equiv \hat{J}_p\left(\frac{u + u^*}{2};\ F\right) + \mathrm{i}\hat{J}_p\left(\frac{u - u^*}{2\mathrm{i}};\ F\right) \tag{106}$$

is well-defined as form on $\mathcal{D}_{at}(h)$, and it follows that $u \mapsto \hat{J}_p(u;\ F)$ is linear and the relations (96) are fulfilled for all $u, v \in \mathbf{g}_p^c(h;\ \varepsilon)$. Hence we have in fact a representation of the complexification $\widehat{\mathbf{g}_p^c}(h;\ \varepsilon)$ of $\widehat{\mathbf{g}_p}(h;\ \varepsilon)$.

8.5. Interpretation. The interpretation of the abstract current algebra in Sect. 6.2 generalizes trivially from $(1+1)$ to $(d+1)$-dimensions, $d = 2p - 1 \geqq 3$, and we have a representation of the Abelian extension $\widehat{\mathbf{g}_p}(h;\ \varepsilon)$ of $\mathbf{g}_p(h;\ \varepsilon)$ associated with the 2-cocycle c_p on the space $\mathcal{S}_p(h;\ \varepsilon)$. Introducing

$$\mathcal{G}_p(u;\ F) \equiv \mathcal{L}_u + \tilde{J}_p(u;\ F) \tag{107}$$

with $\tilde{J}_p$ defined similarly as $\tilde{J}$ in Eq. (78), we can write (96) as

$$[\mathcal{G}_p(u;\ F), \mathcal{G}_p(v;\ F)]'_p = \mathcal{G}_p([u,v];\ F) - c_p(u,v;\ F)\ , \tag{108}$$

where $[\cdot,\cdot]'_p$ is the regularized commutator defined similarly as $[\cdot,\cdot]'_p$ in (105). We suggest to regard this as a general, abstract version of the anomalous commutator relations of the Gauss' law constraint operators in $(d+1)$-dimensions, $d = 2p - 1$, as discussed by Faddeev and Shatiashvili [F, FS].

9. Final Comments

One can expect that representations of the group $\mathbf{G}_2$ and its Lie algebra $\mathbf{g}_2$ as studied in this paper could give a non-perturbative understanding of all ultraviolet divergencies arising in the fermion sector of Yang–Mills theory with fermions in $(3+1)$-dimensions. Indeed, not only the implementers of the gauge transformations, but many other (probably all) fermion operators of interest in these theories can be regarded as a second quantization of operators in $\mathbf{G}_2$ or $\mathbf{g}_2$, no matter whether one has Weyl- or massive or massless Dirac fermions.

For example, it is known that the Dirac operator $D_{A(t)}$ for an arbitrary[16] time-dependent external Yang–Mills field $A(t)$ is in $\mathbf{g}_2$ (for $\varepsilon = \mathrm{sign}(D_A)$, A an arbitrary Yang–Mills field configuration), and the time-evolution operators $u(s,t) = T\exp(-\mathrm{i}\int_s^t \mathrm{d}rA(r))$ generated by $D_{A(t)}$ all are in $\mathbf{G}_2$ (see [M6] and references therein). Hence, using the results of this paper, one can give an explict construction of the time evolution operators $\mathcal{U}(s,t)$ for fermions interacting with $A(t)$, as a family of sesquilinear forms which are a 1-parameter group with respect to the $\hat{\bigstar}_2$-product, i.e. obey the relations

$$\mathcal{U}(t,t) = 1, \quad \mathcal{U}(r,s)\hat{\bigstar}_2\mathcal{U}(s,t) = \mathcal{U}(r,t) \quad \forall r,s,t \in \mathbb{R}\ . \tag{109}$$

Indeed, it is natural to set (cf. Sect. 5.3)

$$\mathcal{U}(s,t) \equiv \gamma(s,t)\hat{\Gamma}_2(u(s,t); F(s)) \tag{110}$$

with $F(s) = u(s,t_0)\varepsilon u(t_0,s), t_0$ is arbitrary,[17] and determine the phases $\gamma(s,t)$ such that the relations (109) are fulfilled. From (89) for $p = 2$ one then gets a condition

[16] Reasonably smooth

[17] For $A(t) \to 0$ sufficiently fast for $t \to -\infty$, a natural choice would be $t_0 = -\infty$ and $\varepsilon = \mathrm{sign}(D_0)$

with an essentially unique solution and which can be solved by a technique similar to one described in [L3]. For Dirac fermions one has, of course, $\gamma(s,t) = 1 \quad \forall s,r \in \mathbb{R}$, but for Weyl fermions one gets a non-trivial solution. The latter can also be used for a simple, direct construction of the non-trivial phase of the S-operator of Weyl fermions in external, time-dependent Yang–Mills fields alternative to the one given recently by Mickelsson [M6].

We finally remark that corresponding results for the Lie group $\mathbf{G}_1$ and Lie algebra $\mathbf{g}_1$ are already sufficient for a complete understanding of $(1+1)$-dimensional gauge theories with fermions. However, this crucially relies on the fact that Yang–Mills fields on a cylinder have only a finite number of physical degrees of freedom with an (essentially) unique Hilbert space representation and therefore do not lead to divergencies (see e.g. [M4]). Moreover, it is possible to eliminate all gauge degrees of freedom and to explicitly construct all physical states [LS1, LS2]. In higher dimension, a Yang–Mills field has an infinite number of physical degrees of freedom and associated divergencies which are highly non-trivial. Moreover, a full understanding of the Gribov problem necessary for eliminating the gauge degrees of freedom is beyond present day knowledge. Hence a non-perturbative treatment of the divergencies in the fermion sector can only be a first, though probably very important, step towards a deeper understanding of the gauge theories we are ultimately interested in, e.g. QCD_{3+1}.

Appendix A

In this appendix we prove the explicit formula (59) for the 2-cocycle $\chi(U,V;\, F)$, valid for all $F \in \mathrm{Gr}_1(h;\, \varepsilon)$ and $U, V \in \mathbf{G}_1(h;\, \varepsilon)$ in some neighborhood of the identity.

From the results of Ruijsenaars [R1] one learns that for $U \in \mathbf{G}_1(h;\, F) \cap \mathcal{U}(h;\, F)$, the implementer of the Bogoliubov transformation α_U in Π_F can be written as[18]

$$\hat{\Gamma}'(U;\, F) = E(U;\, F)N'(U;\, F) \tag{A1}$$

with

$$E(U;\, F) = \mathrm{e}^{Z_{+-}(U;\, F)a^* a^*}\,\Gamma(Z_{++}(U;\, F) \oplus Z_{--}(U;\, F)^t) \times \mathrm{e}^{-Z_{-+}(U;\, F))aa}\,, \tag{A2}$$

$((\cdot)^t \equiv C_F(\cdot)^* C_F)$ an operator on $\mathcal{F}(h)$ satisfying

$$\langle \Omega, E(U;\, F)\Omega\rangle = 1 \tag{A3}$$

($\langle \cdot , \cdot \rangle$ is the inner product in $\mathcal{F}(h)$), and

$$N'(U;\, F) = \det(1 + (Z_{+-}(U;\, F))^* Z_{+-}(U;\, F))^{-1/2} \tag{A4}$$

a normalization constant; here we introduced ($P_\pm = \frac{1}{2}(1 \pm F)$),

[18] The prime is used to indicate that the phase convention for these implementers differs from the one leading to the F-independent 2-cocycle (59)

$$
\begin{aligned}
Z_{++}(U;\,F) &= P_+UP_+ - P_+UP_-[P_-UP_-]^{-1}P_-UP_+\ ,\\
Z_{+-}(U;\,F) &= P_+UP_-[P_-UP_-]^{-1}\ ,\\
Z_{-+}(U;\,F) &= -[P_-UP_-]^{-1}P_-UP_+\ ,\\
Z_{--}(U;\,F) &= [P_-UP_-]^{-1}\ ,
\end{aligned}
\tag{A5}
$$

and used the notation

$$
\begin{aligned}
A_{+-}\,a^*a^* &= \sum_{n,m=1}^{\infty} (f_n^+, A_{+-}f_m^-)a^*(f_n^+)a^*(C_F f_n^-)\ ,\\
A_{-+}\,aa &= \sum_{n,m=1}^{\infty} (f_n^-, A_{-+}f_m^+)a(C_F f_n^-)a(f_n^+)\ ,
\end{aligned}
\tag{A6}
$$

well-defined for Hilbert– Schmidt operators $A_{\pm\mp} = P_\pm A_{\pm\mp} P_\mp$ with $\{f_n^\pm\}_{n=1}^\infty$ some complete, orthonormal bases in $P_\pm h$[19] (see Ref. [R2], Eqs. (2.15) and (3.8) and [R4], Appendix D).

Let $U, V \in \mathbf{G}_1(h;\,F) \cap \mathcal{U}(h;\,F)$. From

$$
\hat{\Gamma}'(U;\,F) \bigstar \hat{\Gamma}'(V;\,F) = \chi'(U,V;\,F)\hat{\Gamma}'(UV;\,F) \tag{A7}
$$

and (A1), (A3) one can deduce that

$$
\chi'(U,V;\,F) = \frac{N'(V;\,F^U)N'(U;\,F)}{N'(UV;\,F)}\mathcal{E}(U,V;\,F) \tag{A8}
$$

with

$$
\begin{aligned}
\mathcal{E}(U,V;\,F) &= \langle \Omega, \Gamma(U)E(V;\,F^U)\Gamma(U)^*E(U;\,F)\Omega\rangle\\
&= \langle \Omega, \Gamma(U)\mathrm{e}^{-Z_{-+}(V;\,F^U)aa}\Gamma(U)^*\mathrm{e}^{Z_{+-}(U;\,F)a^*a^*}\,\Omega\rangle
\end{aligned}
\tag{A9}
$$

(we used $\Gamma(\cdots)\Omega = \mathrm{e}^{(\ \ )aa}\Omega = \Omega$).

By (A5), (A6) and (38),

$$
\Gamma(U)(Z_{-+}(V;\,F^U)aa)\Gamma(U)^* = (Z_{-+}^U(V;\,F)aa)
$$

with

$$
Z_{-+}^U(V;\,F) = UZ_{-+}(V;\,F^U)U^* = Z_{-+}(UVU^*;\,F)\ ,
$$

hence with the formula

$$
\langle \Omega, \mathrm{e}^{A_{-+}aa}\mathrm{e}^{B_{+-}a^*a^*}\,\Omega\rangle = \det(1 + A_{-+}B_{+-})
$$

(see Theorem 3.2 in Ref. [R3]) one obtains

$$
\mathcal{E}(U,V;\,F) = \det(1 + [{}^UV_{--}]^{-1}({}^UV_{-+})U_{+-}[U_{--}]^{-1})\ , \tag{A10}
$$

where we introduced the notation

$$
A_{\varepsilon\varepsilon'} \equiv P_\varepsilon A P_{\varepsilon'} \quad \forall \varepsilon, \varepsilon' \in \{+,-\} \tag{A11}
$$

[19] See Ref. [CR]; note that $Z_{\pm\mp}(U;\,F) \in B_2(h)$ (cf. Appendix B)

and

$$ {}^{U}V \equiv UVU^* \ . \tag{A12} $$

Using $({}^{U}V_{-+}U_{+-}) = ({}^{U}VU)_{--} - ({}^{U}V_{--}U_{--})$ and introducing the determinant

$$ \det{}'((\cdots)_{--}) \equiv \det(P_+ + (\cdots)_{--}) \tag{A13} $$

in P_-h, this can be written as

$$ \mathcal{E}(U,V;\,F) = \det{}'([{}^{U}V_{--}]^{-1}({}^{U}VU)_{--}[U_{--}]^{-1}) \ . \tag{A14} $$

Similarly, we obtain for (A4)

$$ \begin{aligned} N'(U;\,F) &= (\det(1 + [U^*_{--}]^{-1}(U^*)_{-+}U_{+-}[U_{--}]^{-1}))^{-1/2} \\ &= (\det{}'([U^*_{--}]^{-1}[U_{--}]^{-1}))^{-1/2} \ . \end{aligned} \tag{A15} $$

As ${}^{U}VU = UV$ and

$$ \det(U^* \cdots U) = \det(\cdots) \ , $$

one has $N'(UV;\,F) = N'({}^{U}VU;\,F)$ and $N'(V;\,F^U) = N'({}^{U}V;\,F)$, and (A8) results in

$$ \chi'(U,V;\,F) = \left(\frac{\det{}'([{}^{U}V_{--}]^{-1}({}^{U}VU)_{--}[U_{--}]^{-1})}{\det{}'([{}^{U}V_{--}]^{-1}({}^{U}VU)_{--}[U_{--}]^{-1})^*} \right)^{1/2} , \tag{A16} $$

where we used repeatedly some basic properties of determinants. It is easy to check that the determinants in (A16) exist for $U,V \in \mathbf{G}_1(h;\,F) \cap \mathcal{U}(h;\,F)$.[20] Moreover, if U,V are even in $\mathbf{G}_0(h;\,F) \cap \mathcal{U}^{(F)}(h;\,\varepsilon)$, we can write (A16) as

$$ \chi'(U,V;\,F) = \frac{\beta_1'({}^{U}V;\,F)\beta_1'(U;\,F)}{\beta_1'({}^{U}VU;\,F)} \tag{A17} $$

with

$$ \beta_1'(U;\,F) = \left(\frac{\det{}'(U^*_{--})}{\det{}'(U_{--})} \right)^{1/2} = \left(\frac{\det(P_+ + P_-U^*P_-)}{\det(P_+ + P_-UP_-)} \right)^{1/2} . \tag{A18} $$

As ${}^{U}VU = UV$ and $\beta_1'({}^{U}V;\,F) = \beta_1'(V;\,F^U)$, we can write (A17) locally as coboundary: $\chi' = \delta\beta_1'$ (δ cf. (22); note that β_1' obeys (15)).

Thus for $U,V \in \mathbf{G}_0(h;\,\varepsilon) \cap \mathcal{U}^{(F)}(h;\,\varepsilon)$, we can transform χ' to a F-independent expression by multiplying it with the 2-coboundary $\delta\beta$, where locally $\beta(U;\,F) = \beta_1'(U;\,\varepsilon)/\beta_1'(U;\,F)$, i.e.

$$ \beta(U;\,F) = \left(\frac{\det(1 + a(U;\,F))}{\det(1 + a(U;\,F))^*} \right)^{1/2} $$

with

[20] Use

$$ [{}^{U}V_{--}]^{-1}({}^{U}VU)_{--}[U_{--}]^{-1} = P_- + [{}^{U}V_{--}]^{-1}({}^{U}V_{-+}U_{+-})[U_{--}]^{-1} $$

and the fact that $U_{\pm\mp} \in B_2(h)$ for $U \in \mathbf{G}_1(h;\,F)$ (see Appendix B)

$$a(U;\, F) = (P_+ + P_- U P_-)(P^0_+ + [P^0_- U P^0_-]^{-1}) - 1\ . \tag{A19}$$

It is convenient to write this as

$$\beta(U;\, F) = \left(\frac{\det_2(1 + a(U;\, F))\exp(\mathrm{tr}(P^0_+ a(U;\, F)P^0_+ + P^0_- a(U;\, F)P^0_-))}{\det_2(1 + a(U;\, F))^* \exp(\mathrm{tr}(P^0_+ a(U;\, F)P^0_+ + P^0_- a(U;\, F)P^0_-)^*)} \right)^{1/2} \tag{A20}$$

(we used $\det(1 + a) = \det_2(1 + a)\exp(\mathrm{tr}(a_{++} + a_{--}))$ which follows from $\mathrm{tr}(a_{\pm\mp}) = 0$ and the definition of $\det_2(\cdot)$). Using the lemma in Appendix B one can easily prove that for $U \in \mathbf{G}_1(h;\, \varepsilon) \cap \mathcal{U}^{(F)}(h;\, \varepsilon), F \in \mathrm{Gr}_1(h;\, \varepsilon)$, one has $a(U;\, F) \in B_2(h)$, hence $\beta(U;\, F)$ (A20) exists. It follows that $\delta\beta$ is locally a 2-coboundary of $\mathbf{G}_1(h;\, \varepsilon)$, and the implementers

$$\hat{\Gamma}(U;\, F) \equiv \hat{\Gamma}'(U;\, F)\beta(U;\, F) \tag{A21}$$

obey (51) with the 2-cocycle $\chi = \delta\beta\chi'$ which is (locally) F-independent and equal to (59). Note that

$$\hat{\Gamma}(U;\, F) = E(U;\, F)N(U;\, F) \tag{A22}$$

with

$$N(U;\, F) = N'(U;\, F)\beta(U;\, F) \tag{A23}$$

equal to (80).

Appendix B

In this appendix we summarize the basic properties of operators in $\mathbf{G}_p(h;\, \varepsilon)$, $\mathbf{g}_p(h;\, \varepsilon)$ and $\mathrm{Gr}_p(h;\, \varepsilon)$, $p \in \mathbb{N}$.

For ε a grading operator, $P^0_\pm = \frac{1}{2}(1 \pm \varepsilon)$ are orthogonal projections on h, and we introduce the notation

$$A_{\varepsilon\varepsilon'} \equiv P^0_\varepsilon A P^0_{\varepsilon'} \quad \forall \varepsilon, \varepsilon' \in \{+, -\} \tag{B1}$$

for all linear operators A on h; moreover, it is sometimes convenient to use the following 2×2-matrix notation:

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}, \quad \text{i.e.} \quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{B2}$$

Lemma *(i) A unitary operator U on h is in $\mathbf{G}_p(h;\, \varepsilon)$ if and only if*

$$U_{+-}, U_{-+} \in B_{2p}(h)\ .$$

(ii) A bounded, self-adjoint operator u on h is in $\mathbf{g}_p(h;\, \varepsilon)$ if and only if

$$u_{+-}, u_{-+} \in B_{2p}(h)\ .$$

(iii) A grading operator F on h is in $\mathrm{Gr}_p(h;\, \varepsilon)$ if and only if

$$F_{+-}, F_{-+} \in B_{2p}(h), \quad (F - \varepsilon)_{++}, (F - \varepsilon)_{--} \in B_p(h)\ .$$

Proof. By using the matrix notation (B2), (i) and (ii) immediately follow from

$$[\varepsilon, A] = 2\begin{pmatrix} 0 & A_{-+} \\ -A_{-+} & 0 \end{pmatrix} . \tag{B3}$$

To prove (iii), we note that by definition, $F \in \mathrm{Gr}_p(h;\ \varepsilon)$ if and only if $(F-\varepsilon) \in B_{2p}(h)$. This is equivalent to

$$(F-\varepsilon)_{\pm\pm}, F_{\pm\mp} \in B_{2p}(h) .$$

But as $F^2 = 1$,

$$\begin{aligned}(F-\varepsilon)^*_{\pm\pm}(F-\varepsilon)_{\pm\pm} &= F_{\pm\pm}F_{\pm\pm} \mp 2F_{\pm\pm} + P^0_\pm \\ &= P^0_\pm - F_{\pm\mp}F_{\mp\pm} \mp 2F_{\pm\pm} + P^0_\pm \\ &= \mp 2(F-\varepsilon)_{\pm\pm} - (F_{\mp\pm})^* F_{\mp\pm} ,\end{aligned}$$

hence for $F_{\pm\mp} \in B_{2p}(h), (F-\varepsilon)_{\pm\pm} \in B_{2p}(h)$ if and only if it is in $B_p(h)$.

Appendix C

In this appendix we evaluate the Kac–Peterson cocycle (70).

Assuming that $U, V \in \mathbf{G}_0(h;\ \varepsilon)$ and close to the identity, the 2-cocycle (59) can be written as 2-coboundary: $\chi = \delta\beta_1$, with β_1 (60). Thus for $u, v \in \mathbf{g}_0(h;\ \varepsilon)$, we have by our general discussion in Sect. 3.2 $c_1 = -\mathrm{d}b_1$ with

$$b_1(u;\ F) = \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\beta_1(\mathrm{e}^{\mathrm{i}tu};\ F)\ |_{t=0}$$

and d defined in (33). As

$$P^0_- \mathrm{e}^{\mathrm{i}tu} P^0_- - P^0_- = \mathrm{i}tu_{--} + \mathrm{O}(t^2)$$

(using the notation introduced in the Appendix B), we obtain

$$b_1(u;\ F) = -\mathrm{tr}(u_{--}) .$$

Therefore

$$\begin{aligned}c(u,v;\ F) &= -\mathrm{d}b_1(u,v;\ F) = -b_1([u,v];\ F) \\ &= \mathrm{tr}((uv)_{--} - (vu)_{--}) = \mathrm{tr}(u_{-+}v_{+-} - v_{-+}u_{+-})\end{aligned} \tag{C1}$$

by the cyclicity of the trace. This is finite for all $u, v \in \mathbf{G}_1(h;\ \varepsilon)$, and identical with (70) (as is easily proved by using the matrix notation (B2)).

Appendix D

In this appendix we prove that $\chi_2(U,V;\,F)$ (90) ($p=2$) with $\beta_2(U;\,F)$ (92) is of the form (91), hence can be extended to all $F\in \mathrm{Gr}_2(h;\,\varepsilon)$ and $U,V\in \mathbf{G}_2(h;\,\varepsilon)\cap \mathcal{U}^{(F)}(h;\,\varepsilon)$.

For $U,V\in \mathbf{G}_1(h;\,\varepsilon)\cap\mathcal{U}^{(F)}(h;\,\varepsilon), F\in \mathrm{Gr}_1(h;\,\varepsilon)$, the 2-cocycle (59) can be written as

$$\chi(U,V;\,F)=\left(\frac{X(U,V;\,F)}{X(U,V;\,F)^*}\right)^{1/2} \tag{D1}$$

with (we use the notation from Appendix B)

$$X(U,V;\,F)=\det(1+[U_{--}]^{-1}U_{-+}V_{+-}[V_{--}]^{-1})\,. \tag{D2}$$

Similarly, the 1-cochain (92) is

$$\beta_2(U;\,F)=\left(\frac{B_2(U;\,F)}{B_2(U;\,F)^*}\right)^{1/2} \tag{D3}$$

with

$$B_2(U;\,F)=\exp\left(\frac{1}{2}\mathrm{tr}(F_{-+}U_{+-}U^*_{--})\right)\,. \tag{D4}$$

As (D2) is equivalent to

$$X(U,V;\,F)=\exp(\mathrm{tr}([U_{--}]^{-1}U_{-+}V_{+-}[V_{--}]^{-1})$$
$$\times\det{}_2(1+[U_{--}]^{-1}U_{-+}V_{+-}[V_{--}]^{-1})\,,$$

all we have to show is that

$$\delta B_2(U,V;\,F)\exp(\mathrm{tr}([U_{--}]^{-1}U_{-+}V_{+-}[V_{--}]^{-1}))\equiv\exp(\mathrm{tr}(Y_2(U,V;\,F)))$$

allows for a continuous extension to $U,V\in\mathbf{G}_2(h;\,\varepsilon)\cap\mathcal{U}^{(F)}(h;\,\varepsilon), F\in\mathrm{Gr}_2(h;\,\varepsilon)$.

We have

$$Y_2(U,V;\,F)=[U_{--}]^{-1}U_{-+}V_{+-}[V_{--}]^{-1}+\frac{1}{2}F_{-+}U_{+-}U^*_{--}$$
$$+\frac{1}{2}(F^U)_{-+}V_{+-}V^*_{--}-\frac{1}{2}F_{-+}(UV)_{+-}(V^*U^*)_{--}\,. \tag{D5}$$

The relation $[U_{--}]^{-1}U_{--}=(U^*U)_{--}=U^*_{--}U_{--}+(U^*)_{-+}U_{+-}$ implies

$$[U_{--}]^{-1}=U^*_{--}+(U^*)_{-+}U_{+-}[U_{--}]^{-1}\,;$$

using also $(UV)_{+-}=U_{+-}V_{--}+U_{++}V_{+-}$ and a similar relation for $(F^U)_{-+}=(U^*FU)_{-+}$ and $(V^*U^*)_{--}$ we obtain (the symbol "$\sim$" below means "equal up to terms which are obviously trace-class for all $U,V\in\mathbf{G}_2(h;\,\varepsilon), F\in\mathrm{Gr}_2(h;\,\varepsilon)$")

$$[U_{--}]^{-1}U_{-+}V_{+-}[V_{--}]^{-1}\sim U^*_{--}U_{-+}V_{+-}V^*_{--}\,,$$

$$(F^U)_{-+}V_{+-}V^*_{--} \sim U^*_{--}F_{--}U_{-+}V_{+-}V^*_{--} + U^*_{--}F_{-+}U_{++}V_{+-}V^*_{--}$$

$$+U^*_{-+}F_{++}U_{++}V_{+-}V^*_{--} \sim -2U^*_{--}U_{-+}V_{+-}V^*_{--} + U^*_{--}F_{-+}U_{++}V_{+-}V^*_{--}\ ,$$

$$(\text{we used } U^*_{-+}U_{++} = -U^*_{--}U_{-+})\ ,$$

$$F_{-+}(UV)_{+-}(V^*U^*)_{--} \sim F_{-+}U_{++}V_{+-}V^*_{--}U^*_{--} + F_{-+}U_{+-}V_{--}V^*_{--}U^*_{--}$$

$$\sim F_{-+}U_{++}V_{+-}V^*_{--}U^*_{--} + F_{-+}U_{+-}U^*_{--}\ ,$$

hence

$$\mathrm{tr}(Y_2(U,V;\ F)) \sim \mathrm{tr}\left(\frac{1}{2}U^*_{--}F_{-+}U_{++}V_{+-}V^*_{--} - \frac{1}{2}F_{-+}U_{++}V_{+-}V^*_{--}U^*_{--}\right) = 0$$

due to the cyclicity of the trace.

Appendix E

In this appendix we evaluate the Mickelsson–Rajeev cocycle [MR]. From our general discussion in Sect. 3.2 we have

$$c_2(u,v;\ F) = c(u,v;\ F) - \mathrm{d}b_2(u,v;\ F)$$

with c_1 the Kac-Peterson cocycle (C1) and

$$b_2(u;\ F) = \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}\beta_2(\mathrm{e}^{itu};\ F)\ |_{t=0}\ ;$$

with β_2 (92) we obtain (we use the notation introduced in the Appendix B)

$$b_2(u;\ F) = \frac{1}{4}\mathrm{tr}(u_{-+}F_{+-} + u_{+-}F_{-+}) = -\frac{1}{16}\mathrm{tr}([u,\varepsilon][F,\varepsilon])\ ,$$

and

$$\mathrm{d}b_2(u,v;\ F) = b_2([u,v];\ F) - b_2(v;\ [F,u]) + b_2(u;\ [F,v])$$

which gives (after a lengthy but straightforward calculation)

$$\mathrm{d}b_2(u,v;\ F) = \frac{1}{2}\mathrm{tr}(F_{++}(v_{+-}u_{-+} - u_{+-}v_{-+}) - F_{--}(u_{-+}v_{+-} - v_{-+}u_{+-}))\ ,$$

hence

$$c_2(u,v;\ F) = \frac{1}{2}\mathrm{tr}((F-\varepsilon)_{--}(u_{-+}v_{+-} - v_{-+}u_{+-} - (F-\varepsilon)_{++}(v_{+-}u_{-+} - u_{+-}v_{-+}))$$

identical with (99).

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