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Dilogarithm Identities in Conformal Field Theory and Group Homology

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To Professor C. N. Yang for his 70[th] birthday

Abstract: Recently, Rogers' dilogarithm identities have attracted much attention in the setting of conformal field theory as well as lattice model calculations. One of the connecting threads is an identity of Richmond–Szekeres that appeared in the computation of central charges in conformal field theory. We show that the Richmond–Szekeres identity and its extension by Kirillov–Reshetikhin (equivalent to an identity found earlier by Lewin) can be interpreted as a lift of a generator of the third integral homology of a finite cyclic subgroup sitting inside the projective special linear group of all 2×2 real matrices viewed as a *discrete* group. This connection allows us to clarify a few of the assertions and conjectures stated in the work of Nahm–Recknagel–Terhoven concerning the role of algebraic K-theory and Thurston's program on hyperbolic 3-manifolds. Specifically, it is not related to hyperbolic 3-manifolds as suggested but is more appropriately related to the group manifold of the universal covering group of the projective special linear group of all 2×2 real matrices viewed as a topological group. This also resolves the weaker version of the conjecture as formulated by Kirillov. We end with a summary of a number of open conjectures on the mathematical side.

0. Introduction

Very recently, much has been written about the Rogers' dilogarithm identities and its role in conformal field theory, see [BR, KKMM, FS, K, KR, KP, KN, KNS, NRT]. For an excellent general survey for mathematicians concerning hypergeometric functions algebraic K-theory, algebraic geometry and conformal field theory, see [V] and its extensive section of references. For a recent review from the physics side, see [DKKMM]. In the present work, we limit our attention to the special case of dilogarithm identities. In spirit, it fits into the program surveyed by

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Varchenko [V]. Some, though not all, of the relevant calculations have been carried out on both sides of the fence. Conjectures abound even in this case. Most of our task consists of pulling together items that are scattered in the literature in various forms. The new ingredient is to give a direct interpretation in terms of group homology to account for the Richmond–Szekeres identity, see [RS], and its extension by Kirillov–Reshetikhin, see [KR, II, (2.33) and Appendix 2], see also Lewin [L1, (5.117) and (5.119) and L2, p. 19–20]. What we show is that the basic identities are those found by Rogers in [R]. Rogers' dilogarithm function then leads to a real valued cohomology class defined on the third integral homology of the universal covering group of **PSL**(2, **R**), viewed as a *discrete* group. The Richmond–Szekeres identity, see [RS and L1, (5.119)], and the Kirillov–Reshetikhin (or the equivalent Lewin) identities, see [KR II, (2.33) and Appendix 2; L1, (5.117)], are the reesults of restricting the evaluation of this cohomology class (the real part of the second Cheeger–Chern–Simons class) to the inverse image of a suitable homology class that covered the generator of a suitable finite cyclic subgroup. This will then provide partial clarification of some of the assertions and conjectures made by Nahm–Recknagel–Terhoven [NRT] related to algebraic K-theory [B1] and Thurston's program on hyperbolic 3-manifolds, [Th]. Specifically, we show that it is more appropriately related to the group manifold underlying the universal covering group of **PSL**(2, **R**).

1. Rogers' Dilogarithm

Rogers' dilogarithm (also called Rogers' L-function) was defined in [R]:

$$L(x) = -\frac{1}{2}\left\{\int_0^x \frac{\log x}{1-x}\,dx + \int_0^x \frac{\log(1-x)}{x}\,dx\right\}$$

$$= \sum_{n>0} \frac{x^n}{n^2} + \frac{1}{2}\cdot(\log x)\cdot(\log(1-x)), \quad 0 < x < 1\,. \tag{1.1}$$

$L(x)$ is real analytic, strictly increasing and $\lim_{x\to 1} L(x) = \pi^2/6$.

Rogers showed that L satisfied the following two basic identities:

$$L(x) + L(1-x) = \pi^2/6, \quad 0 < x < 1\,, \tag{1.2}$$

$$L(x) + L(y) = L(xy) + L\left(\frac{x-xy}{1-xy}\right) + L\left(\frac{y-xy}{1-xy}\right), \quad 0 < x, y < 1\,. \tag{1.3}$$

If we use (1.2), take $s_1 = (1-x)/(1-xy)$ and $s_2 = y(1-x)/(1-xy)$ so that $y = s_2/s_1$ and $x = (1-s_1)/(1-s_2)$ with $0 < s_2 < s_1 < 1$, then (1.3) is seen to be equivalent to:

$$L(s_1) - L(s_2) + L\left(\frac{s_2}{s_1}\right) - L\left(\frac{1-s_1^{-1}}{1-s_2^{-1}}\right) + L\left(\frac{1-s_1}{1-s_2}\right) = \frac{\pi^2}{6}, \quad 0 < s_2 < s_1 < 1\,. \tag{1.4}$$

If we set $r_i = s_i^{-1}$, and define $L(r) = -L(r^{-1})$ for $r > 1$, then (1.4) can be rewritten in the form:

$$L(r_1) - L(r_2) + L\left(\frac{r_2}{r_1}\right) - L\left(\frac{r_2 - 1}{r_1 - 1}\right) + L\left(\frac{1 - r_2^{-1}}{1 - r_1^{-1}}\right) = \frac{\pi^2}{6}, \quad 1 < r_1 < r_2 . \tag{1.5}$$

Motivated by [DS1], Rogers' dilogarithm was shifted in [PS] to:

$$L^{\mathrm{PS}}(x) = L(x) - \pi^2/6 = -L(1 - x), \quad 0 < x < 1 . \tag{1.6}$$

If we replace L by L^{PS} throughout, then (1.2) and (1.4) become:

$$L^{\mathrm{PS}}(x) + L^{\mathrm{PS}}(1 - x) = -\pi^2/6 , \tag{1.7}$$

$$L^{\mathrm{PS}}(x) - L^{\mathrm{PS}}(y) + L^{\mathrm{PS}}\left(\frac{y}{x}\right) - L^{\mathrm{PS}}\left(\frac{1 - x^{-1}}{1 - y^{-1}}\right) + L^{\mathrm{PS}}\left(\frac{1 - x}{1 - y}\right) = 0, \quad 0 < y < x < 1 . \tag{1.8}$$

A huge number of identities have been found in connection with Rogers' dilogarithm, see [L1]. The situation is somewhat similar, and is often, related to trigonometry, where the basic identities are the two additional formulae for the sine and cosine function, which are just the coordinate description of the group law for $\mathbf{SO}(2)$ or $\mathbf{U}(1)$. This analogy can be made more precise. Namely, $\mathbf{U}(1)$, more appropriately, $\mathbf{GL}(1, \mathbf{C}) \cong \mathbf{C}^{\times}$ is just the first Cheeger–Chern–Simons characteristic class in disguise. This is well-known and tends to be overlooked.

Richmond–Szekeres [RS] obtained the following identity (in a slightly different form) from evaluating the coefficients of certain Rogers–Ramanujan partition identities as generalized by Andrews–Gordon, cf. [L1; (5.119) and L2; p. 19]:

$$\sum_{1 \leqq i \leqq r} L(d_i) = \frac{\pi^2}{6} \cdot \frac{2r}{2r + 3}, \quad d_j = \frac{\sin^2 \theta}{\sin^2 (j + 1)\theta}, \quad \theta = \frac{\pi}{2r + 3} . \tag{1.9}$$

This has been extended by Kirillov–Reshetikhin [KR], cf. [L1; (5.117)], to:

$$\sum_{1 \leqq j \leqq n-2} L(d_j) = \frac{\pi^2}{6} \cdot \frac{3(n - 2)}{n}, \quad d_j = \frac{\sin^2 \theta}{\sin^2 (j + 1)\theta}, \quad \theta = \frac{\pi}{n} . \tag{1.10}$$

Apparently, identity (1.9) arose in the study of low-temperature asymptotics of entropy in the RSOS-models, see [ABF, BR, and KP] while (1.10) arose in the calculation of magnetic susceptibility in the XXZ model at small magnetic field, see [KR]. They are connected to conformal field theory in terms of the identification of the right-hand sides as the central charges of the non-unitary Virasoro minimal model and with the level ℓ $A_1^{(1)}$ WZW model respectively, see [BPZ, Z2, K, KN, KNS, DKKMM, KKMM, Te], ... Our goal is to show that these identities can be understood in terms of the evaluation of a Cheeger–Chern–Simons characteristic class on a generator of the third integral homology of a finite cyclic group of order $2r + 3$ and n respectively.

2. Geometry and Algebra of Volume Calculations

In any sort of volume computation, the volume is additive with respect to division of the domain into a finite number of admissible pieces. Depending on the coordinates used to describe the domain, the volume function must then satisfy some sort of "functional equation." This is the geometric content behind the Rogers' dilogarithm identity. The geometric aspect was described in [D1] while some of the relevant algebraic manipulations were carried out in [PS] (up to some sign factors that only became important in [D1]). To get a precise description, it is necessary to examine [DS1, DPS and Sa3]. These involved use of algebraic K-theory. We review the ideas and results but omit the technical details.

To begin the review, we recall the definition of some commutative groups (called the "scissors congruence groups", cf. [DPS]). Let **F** denote a division ring (we are only interested in three classical cases: $\mathbf{R}$ = real number, $\mathbf{C}$ = complex numbers, $\mathbf{H}$ = quaternions). The abelian group $P_{\mathbf{F}}$ is generated by symbols: $[x]$, x in $\mathbf{F}$, $x \neq 0, 1$ and satisfies the following identity for $x \neq y$:

$$[xyx^{-1}] = [y], \text{ (this is automatic for fields)}\,, \tag{2.1a}$$

$$[x] - [y] + [x^{-1}y] - [(x-1)^{-1}(y-1)] + [(x^{-1}-1)^{-1}(y^{-1}-1)] = 0\,. \tag{2.1b}$$

This group was studied in [DS1] for the case of $\mathbf{F} = \mathbf{C}$. It is closely related to, but not identical to, the Bloch group that was studied in [B1]. A second abelian group $P(\mathbf{F})$ is defined by using generating symbols $[[x]]$, x in $\mathbf{F} - \{0,1\}$, with defining relations:

$$\text{same as (2.1) with } [[z]] \text{ in place of } [z]\,, \tag{2.2}$$

$$[[x]] + [[x^{-1}]] = 0\,, \tag{2.3}$$

$$[[x]] + [[1-x]] = \operatorname{cons}(\mathbf{F}) \text{ (depending on } \mathbf{F})\,. \tag{2.4}$$

The following result can be found in [DPS]:

$$0 \to \mathbf{F}^\times/(\mathbf{F}^\times)^2 \to P_{\mathbf{F}} \to P(\mathbf{F}) \to 0 \text{ is exact for } \mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}\,. \tag{2.5}$$

The first map in (2.5) is defined by sending x in $\mathbf{F} - \{0, 1\}$ to $[x] + [x^{-1}]$. The second map then sends $[x]$ to $[[x]]$. In particular, when $\mathbf{F} = \mathbf{C}$, we may set $[x] = 0$ for $x = \infty, 0, 1$ and remove the restriction $x \neq y$ in (2.1) by adopting the convention: meaningless symbols are taken to be zero, see [DS1]. For the division ring $\mathbf{H}$, we observe that every element of $\mathbf{H}$ is conjugate to an element of $\mathbf{C}$, thus $P(\mathbf{H})$ is quotient of $P(\mathbf{C})$.

The geometric content of (2.1b) is best seen by thinking in terms of a Euclidean picture. Suppose we have 5 points in Euclidean 3-space so that p_1, p_2, p_3 form a horizontal triangle while p_0, p_4 are respectively above and below the triangle. The convex closure is divided by the triangle into two tetrahedra and also divided into three tetrahedra by the line joining p_0 and p_4, see (Fig. 1). Thus, if any function of a tetrahedron is additive with respect to finite decompositions, it would follow from (Fig. 1) that there should be a 5 term identity to be satisfied by such a function.

We examine the special case of $\mathbf{F} = \mathbf{C}$. Here $P_{\mathbf{C}} = P(\mathbf{C})$ is known to be a $\mathbf{Q}$-vector space of continuum dimension, see [DS1]. It is best to consider the (-1)-eigenspace $P(\mathbf{C})^-$ of $P(\mathbf{C})$ under the action of complex conjugation. It is classically known that the projective line $P^1(\mathbf{C})$ can be viewed as the boundary of the hyperbolic 3-space. An ordered set of 4 non-coplanar points on $P^1(\mathbf{C})$ (in terms of the extended hyperbolic 3-space) determines a unique ideal (or totally asymptotic) tetrahedron of finite invariant volume (by using the constant negative

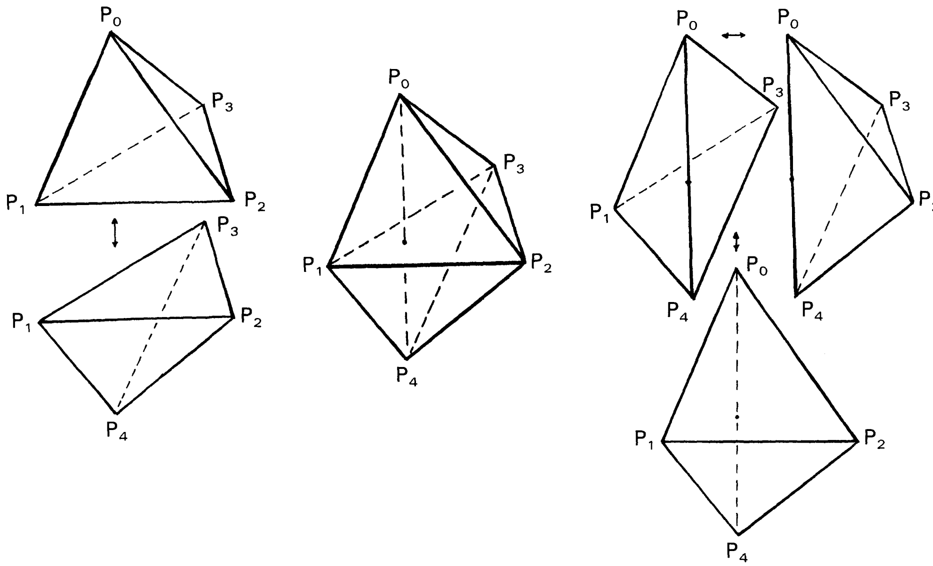

Fig. 1. Dividing a polytope in two different ways

curvature of hyperbolic 3-space). Since the orientation preserving isometry group is $\mathbf{PSL}(2, \mathbf{C})$, we can take 3 of the 4 vertices to be ∞, 0, 1 the 4$^{\text{th}}$ point is then defined to be the "cross-ratio" of the 4 distinct points (which may determine a degenerate tetrahedron when they are coplanar). Equation (2.1b) is the result of taking 5 distinct points: ∞, 0, 1, x and y as pictured in (Fig. 1). For a general division ring $\mathbf{F}$, $P_{\mathbf{F}}$ merely formalizes the discussion. The difference between $P(\mathbf{F})$ and $P_{\mathbf{F}}$ amounts to permitting some of the vertices to be duplicated. Equations (2.3) and (2.4) express the fact that oriented volume changes sign when the exchange of two vertices reverses the orientation. The equality $P_{\mathbf{C}} = P(\mathbf{C})$ simply means that the introduction of degenerate tetrahedra with duplicated vertices does not make any difference (it does make a difference in the case of $\mathbf{F} = \mathbf{R}$). With (2.3) in place, it is now evident that (1.7) and (1.8) are directly related to (2.4) and (2.1b). The problem is that our explanation so far is based on $\mathbf{F} = \mathbf{C}$ while L^{PS} dealt with $\mathbf{F} = \mathbf{R}$. This will be reviewed in the next section. It should be noted that the volume calculation makes perfectly good sense for tetrahedra with vertices in the finite part of the hyperbolic 3-space. It is known that any such tetrahedron can be written in many different ways as a sum and difference of ideal tetrahedra, see [DS1]. A general volume formula for a tetrahedron is quite complicated. However, the volume of an ideal tetrahedron is quite simple. It is given by the *imaginary* part of the complexified Rogers dilogarithm function (up to normalization) evaluated at the cross-ratio.

We end the present section by giving the structures and inter-relations of the groups $P(\mathbf{F})$, $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ with $\mathbf{R} \subset \mathbf{C} \subset \mathbf{H}$: (The details can be found in [DPS] and [Sa3].)

$$P(\mathbf{C}) = P(\mathbf{C})^+ \oplus P(\mathbf{C})^- . \tag{2.6}$$

This is a **Q**-vector space direct sum in terms of its ± 1 eigenspaces under the action of complex conjugation. Both summands have continuum dimension.

$$0 \to \mathbf{Q}/\mathbf{Z} \to P(\mathbf{R}) \to P(\mathbf{C})^{+} \to \Lambda^2_{\mathbf{Z}}(\mathbf{R}/\mathbf{Z}) \to 0 \quad \text{is exact .} \tag{2.7}$$

$P(\mathbf{R})$ is the direct sum of $\mathbf{Q}/\mathbf{Z}$ and a **Q**-vector space of continuum dimension,

$$P(\mathbf{C})^{+} \to P(\mathbf{H}) \to 0 \text{ is exact and } P(\mathbf{H}) \cong \Lambda^2_{\mathbf{Z}}(\mathbf{R}^{+})\,. \tag{2.8}$$

The group $P(\mathbf{C})^{-}$ is the "scissors congruence group" in hyperbolic 3-space, see [DS1]. The kernel of the homomorphism in (2.8) is related to the "scissors congruence group modulo decomposables" in spherical 3-space and is conjecturally equal to it, see [DPS]. These results depend on algebraic K-theory and use, in particular, a special case of Suslin's celebrated solution of the conjecture of Lichtenbaum–Quillen, see [Su2].

3. Rogers' Dilogarithm and Characteristic Classes

As reviewed in the preceding sections, there is a formal resemblance between the Rogers' dilogarithm identities and volume calculation in hyperbolic 3-space. However, the underlying space is quite different. The explanations were carried out in [D1]. For the convenience of the reader, we review the results. The relevant characteristic class is that of the Cheeger–Chern–Simons characteristic class $\hat{c}_2$ which lies in the third cohomology of $\mathbf{SL}(2, \mathbf{C})$ viewed as a *discrete* group and where the coefficients lie in $\mathbf{C}/\mathbf{Z}$. In general, one has $\hat{c}_n$ which lies in the $(2n-1)^{\text{th}}$ cohomology group of $\mathbf{GL}(m, \mathbf{C})$, $m \geqq n$, viewed as a discrete group, where the coefficients lie in $\mathbf{C}/\mathbf{Z}$. The standard mathematical notation for this cohomology group is $\mathbf{H}^{2n-1}(\mathbf{BGL}(m, \mathbf{C})^{\delta}, \mathbf{C}/\mathbf{Z})$, this is the group cohomology where $\mathbf{GL}(m, \mathbf{C})$ is given the discrete topology (the superscript δ emphasizes this fact). $\hat{c}_1$ is nothing more than the determinant map with kernel $\mathbf{SL}(m, \mathbf{C})$. With the replacement of **GL** by **SL**, $\hat{c}_1$ becomes 0. The replacement of $\mathbf{GL}(m, \mathbf{C})$ by $\mathbf{GL}(n, \mathbf{C})$ arises from homological stability theorems, see [Su1] (a simplified version can be found in [Sa2]). In general, $\hat{c}_n$ is conjectured to be connected to the n-polylogarithm, see [D2 and D3]. Although we are only interested in $\hat{c}_2$, we will state the results for general n. The construction arises by starting with the Chern form c_n (a $2n$-form) which represents an integral cohomology class of the classifying space $\mathbf{BGL}(n, \mathbf{C})$, where $\mathbf{GL}(n, \mathbf{C})$ is now given the usual topology. Since we have replaced the usual topology by the discrete topology (this amounts to "zero curvature conditions"), it follows from Chern–Weil theory (where closed forms are viewed as complex cohomology classes) that c_n can be written as the differential of a $(2n-1)$-form, (for $n = 2$ this is the Chern–Simons form that appears ubiquitously in physics). When the coefficients are taken in $\mathbf{C}/\mathbf{Z}$, this $(2n-1)$-form is closed and leads to the class $\hat{c}_n$ in $\mathbf{H}^{2n-1}(\mathbf{B}GL(n, \mathbf{C})^{\delta}, \mathbf{C}/\mathbf{Z})$ through the exact sequence:

$$0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}/\mathbf{Z} \to 0\,. \tag{3.1}$$

We now concentrate on $n = 2$. If we take the coefficients to be $\mathbf{C}/\mathbf{Z}$, then the characteristic class $\hat{c}_2$ has a purely imaginary part and a real part. The purely imaginary part has values in $\mathbf{R}$ and is related to volume calculation in hyperbolic

3-space while the real part lies in $\mathbf{R}/\mathbf{Z}$ and is related to volume calculation in spherical 3-space. These volume calculations are classically known to involve the dilogarithm function. See [C] for the details related to the work of Lobatchevskii and Schläfli respectively. The integer ambiguity in the spherical case arises from the fact that a large tetrahedron can be viewed as a small tetrahedron on the "back side" of the sphere with a reversed orientation. Thus its volume is only unique up to an integer multiple of the total volume of the spherical 3-space.

For the Rogers' dilogarithm, the space is actually the group-space $\tilde{\mathbf{S}}$ of the universal covering group $\mathbf{PSL}(2, \mathbf{R})\tilde{}$. The task of defining a tetrahedron and calculating its volume becomes more delicate. If we select a base point p in $\tilde{\mathbf{S}}$, then any point can be written as $g(p)$ for a uniquely determined group element g of $\mathbf{PSL}(2, \mathbf{R})\tilde{}$. We first define a left invariant "geodesic" in the group that joins 1 to g (this definition is asymmetric). This can be accomplished by exponentiating a Cartan decomposition of the Lie algebra of $\mathbf{PSL}(2, \mathbf{R})\tilde{}$. In essence, we coordinate $\mathbf{PSL}(2, \mathbf{R})\tilde{}$ by $\mathbf{R} \times \mathbf{H}^2$, where $\mathbf{H}^2$ denotes the hyperbolic plane. Inductively, we can than define a "geodesic cone" for any ordered set of $n + 1$ points, $n \geqq 0$, see (Fig. 2). This is similar to [GM] where Rogers dilogarithm appeared in terms of volumes in Grassmann manifolds of 2-planes in $\mathbf{R}^4$. Our interpretation is dual to [GM] since the transpose of a 4×2 matrix is a 2×4 matrix. Namely, for the ordered set $(p_0, \ldots, p_n)$, the cone is the collection of all points on the "geodesics" from p_0 to the "geodesic cone" inductively defined for $(p_1, \ldots, p_n)$. For the definition of volume $(n = 3)$, the next step is to show that it is enough to consider the case where the 4 vertices are close to each other. In fact, in terms of the Cartan coordinates of the group elements, one may assume that the θ-coordinates are strictly positive and small (this involves changing by a boundary which causes no problem because the volume is obtained by evaluating a 3-cocycle on the chain, in essence we invoke Stokes' Theorem). We next form the boundary $\mathbf{R} \times \partial\mathbf{H}^2$, where $\partial\mathbf{H}^2 = P^1(\mathbf{R})$ is the projective line over the real numbers (which can be identified with $\{-\infty\} \cup \mathbf{R}$ by using the slopes in the right half plane as in [PS]). At this point, we begin to mimic the hyperbolic 3-space and move p continuously towards $\{0\} \times P^1(\mathbf{R})$ (this amounts to right multiplication). When p lands on $\{0\} \times P^1(\mathbf{R})$, so will all four vertices so that we have the analog of an ideal hyperbolic tetrahedron. The volume (up to a normalizing factor) is just the value of the Rogers dilogarithm evaluated on the "cross ratio" of the ordered set of vertices viewed as points of $P^1(\mathbf{R})$ (adjustments

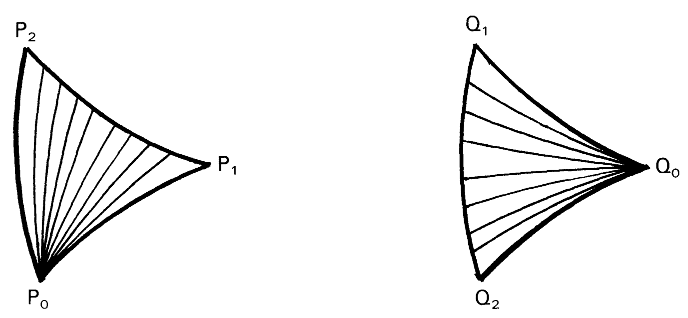

Fig. 2. Ordered "geodesic triangles"

are needed for the degenerate cases). The situation now resembles the case of spherical 3-space. Namely, the final volume will involve an integer (after normalization) ambiguity which depends on the path of p. We ignore the question of representing the original tetrahedron as sums and differences of these "ideal tetrahedra" since our concern is to interpret the value of the Rogers' dilogarithm as a volume.

We summarize this discussion in the form, cf. [D1, Th. 1.11]:

Theorem 3.2. *The restriction of the second Cheeger–Chern–Simons characteristic class $\hat{c}_2$ to $\mathbf{PSL}(2, \mathbf{R})$ can be lifted to the universal covering group $\mathbf{PSL}(2, \mathbf{R})^{\sim}$ and is then given by the Rogers dilogarithm (more precisely, by L^{PS} through L).*

A more detailed discussion will be given in the following sections.

4. Homology of Abstract Groups

The basic reference is [Br]. Let $\mathbf{G}$ be an abstract group. We consider the non-homogeneous formulation of the integral homology of $\mathbf{G}$ with integer coefficients $\mathbf{Z}$. The j^{th} chain group $\mathbf{C}_j(\mathbf{G})$ is the free abelian group generated by all j-tuples $[g_1|\cdots|g_j]$ with g_i ranging over $\mathbf{G}$, $j \geqq 1$. $\mathbf{C}_0(\mathbf{G})$ is the infinite cyclic group generated by $[\cdot]$. Such a j-cell should be identified with each of the formal j-simplices $(g_0, g_0g_1, g_0g_1g_2, \ldots, g_0g_1, \ldots, g_j)$ as g_0 ranges over $\mathbf{G}$. The boundary homomorphism: $\partial_j : \mathbf{C}_j(\mathbf{G}) \to \mathbf{C}_{j-1}(\mathbf{G})$ is defined by translating the usual boundary of the formal j-simplex. For example, $\partial_3[g_1|g_2|g_3] = [g_2|g_3] - [g_1g_2\,g_3] + [g_1|g_2\,g_3] - [g_1|g_2]$. *The* j^{th} integral homology group of $\mathbf{G}$, $\mathbf{H}_j(\mathbf{G}, \mathbf{Z})$, or simply $\mathbf{H}_j(\mathbf{G})$, is defined to be $\ker \partial_j/\mathrm{im}\,\partial_{j+1}$. $\mathbf{H}_0(\mathbf{G})$ is just $\mathbf{Z}$ while $\mathbf{H}_1(\mathbf{G})$ is canonically the commutator quotient group of G with the class of $[g]$ mapped onto the coset of g in the commutator quotient group. We note that homology groups can also defined for any $\mathbf{G}$-module $\mathbf{M}$ (e.g. any vector space on which $\mathbf{G}$ acts by means of linear transformations). This generalization is often needed for computational purposes and requires more care.

In general, the procedure described in the preceding paragraph is not very revealing. Somewhat more revealing is to use the action of $\mathbf{G}$ of a suitably selected set $\mathbf{X}$. Typically, we end up describing the homology groups through a spectral sequence that reveals a composition series. If X is the underlying set of G under the left multiplication action and the spectral sequence "degenerates." In the case of $\mathbf{PSL}(2, \mathbf{R})^{\sim}$, we can take the space $\mathbf{X}$ to be that of $P^1(\mathbf{R}) = \{0\} \times P^1(\mathbf{R})$ which is viewed as part of the boundary of the group space $\tilde{\mathbf{S}}$. The spectral sequence is the algebraic procedure to keep track of the geometry. If p is a base point in the group space $\tilde{\mathbf{S}}$, the 3-cell $[g_1|g_2|g_3]$ is an abstraction of the "geodesic" 3-simplex $(p, g_1(p), g_1g_2(p), g_1g_2g_3(p))$ in the group space $\tilde{\mathbf{S}}$. If p is moved to $\infty = \mathbf{R}\binom{1}{0}$ in $P^1(\mathbf{R})$, then we have an "ideal" 3-simplex. Although the action of $\mathbf{PSL}(2, \mathbf{R})^{\sim}$ on $\tilde{\mathbf{S}}$ is faithful, its action on $P^1(\mathbf{R})$ is not. In fact, it factors through $\mathbf{PSL}(2, \mathbf{R})$ by way of the following exact sequence:

$$0 \to \mathbf{Z} \cdot c \to \mathbf{PSL}(2, \mathbf{R})^{\sim} \to \mathbf{PSL}(2, \mathbf{R}) \to 1\,. \tag{4.1}$$

The results in [PS] and [DPS] can be recast and summed up by the following commutative diagram of maps where the rows and columns are exact:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathbf{Z} \arrow[r, "\cong"] \arrow[d] & \mathbf{Z}\cdot c \arrow[d] & \\
 & 0 \arrow[r] & \mathbf{H}_3(\tilde{\mathbf{S}}, \mathbf{Z}) \arrow[r] \arrow[d] & PS(\mathbf{R}) \arrow[d, "\eta"] & \\
0 \arrow[r] & \mathbf{Z}_2 \arrow[r] & \mathbf{H}_3(\mathbf{S}, \mathbf{Z}) \arrow[r, "\sigma"] \arrow[d] & P(\mathbf{R}) \arrow[r, "d2"] \arrow[d] & \Lambda^2_{\mathbf{Z}}(\mathbf{R}^+) \\
 & & 0 & 0 &
\end{tikzcd} \tag{4.2}$$

In (4.2), we abuse the notation and set $\mathbf{S} = \mathbf{PSL}(2, \mathbf{R})$. PS($\mathbf{R}$) is the abelian group generated by all cross-ratio symbols $\{r\} = (\infty, 0, 1, r)$, $r \in \mathbf{R}^\times \cup \{\infty\}$, and subjected to the defining relations, cf. (1.5), (1.8):

$$\{r_1\} - \{r_2\} + \left\{\frac{r_2}{r_1}\right\} - \left\{\frac{r_2 - 1}{r_1 - 1}\right\} + \left\{\frac{1 - r_2^{-1}}{1 - r_1^{-1}}\right\} = 0, \quad 1 < r_1 < r_2\,, \tag{4.3}$$

$$\{r\} + \{r^{-1}\} = 0, \quad r > 1\,, \tag{4.4}$$

$$\{\infty\} = 2\{2\} = -2\{1/2\} \quad \text{and} \quad \{1\} = 0\,, \tag{4.5}$$

$$\{-r\} = \{1 + r^{-1}\} + \{\infty\}, \quad r > 0\,. \tag{4.6}$$

These involve slight modifications of the results in [PS]. The group PS($\mathbf{R}$) is isomorphic to the group $\mathbf{H}_3(W/\mathbf{S})$ of [PS] if we simply view (4.4) through (4.6) as the definition of $\{s\}$ for $0 < s < 1$, $s = \infty$ or 1 and $s < 0$ respectively. More precisely, we take as j-cells the *ordered* $(j + 1)$-tuples of elements of the universal covering group $\mathbf{R}$ of $\mathbf{PSO}(2, \mathbf{R})$ so that the convex closure of these points cover an interval of length less than π (length of $\mathbf{PSO}(2, \mathbf{R})$)). Moreover, we also enlarge the action to the "universal covering group" of $\mathbf{PGL}(2, \mathbf{R})$. We note that in general, the universal covering group of a disconnected Lie group is not well defined. In the present case, it is well defined and happens to be a semi-direct product of the universal covering group of $\mathbf{PSL}(2, \mathbf{R})$ by an element of order 2 that inverts its infinite cyclic center. The later results in [DPS] and [Sa3] showed that $\mathbf{H}_3(W/\mathbf{S})$ is a $\mathbf{Q}$-vector space. In [PS], it was shown that $\mathbf{H}_3(W/\mathbf{S})/\mathbf{Z}\cdot 48\{2\} \subset \mathbf{H}_3(\mathbf{SL}(2, \mathbf{R}), \mathbf{Z})$ and that $\mathbf{H}_3(W/\mathbf{S})/\mathbf{Z}\cdot 12\{2\} \cong P_{\mathbf{R}} \supset \mathbf{H}_3(\mathbf{PSL}(2, \mathbf{R}), \mathbf{Z})$. The first arose by showing that a certain element $c(-1, -1) = 8c$ is mapped onto $\pm 48\{2\}$ (with a little care, the image is $-48\{2\}$). The second involves a direct argument. We note that $\mathbf{H}_3(\mathbf{SL}(2, \mathbf{R}), \mathbf{Z})$ maps surjectively to $\mathbf{H}_3(\mathbf{PSL}(2, \mathbf{R}), \mathbf{Z})$ with kernel $\mathbf{Z}_4$. This accounts for various $\mathbf{Z}_2$'s. Equation (4.2) now results from (2.5) with c mapped by η onto $-6[[2]]$ in $P(\mathbf{R})$, namely, $P(\mathbf{R}) \cong \mathbf{H}_3(W/\mathbf{S})/\mathbf{Z}\cdot 6\{2\}$. From Sect. 1, we have a subjective homomorphism:

$$L^{\mathrm{PS}}\colon \mathrm{PS}(\mathbf{R}) \twoheadrightarrow \mathbf{R}, \quad L^{\mathrm{PS}}(\{s\}) = L(s) - \frac{\pi^2}{6} = -L(1 - s), \quad 0 < s \leqq 1\,. \tag{4.7}$$

In particular, $L^{PS}(\{1/2\}) = -\pi^2/12$ and $L^{PS}(\{r\}) = L(1-r^{-1})$, for $1 \leqq r \leqq \infty$. This leads to surjective homomorphisms:

$$L^{PS}_{\mathbf{R}}: P_{\mathbf{R}} \twoheadrightarrow \mathbf{R} \bmod \mathbf{Z}\cdot(\pi^2)\,,$$

$$L^{PS}(\mathbf{R}): P(\mathbf{R}) \twoheadrightarrow \mathbf{R} \bmod \mathbf{Z}\cdot\left(\frac{\pi^2}{2}\right). \tag{4.8}$$

Using (2.5) and (2.7) we then have:

$$L^{PS}_{\mathbf{R}}: \mathbf{H}_3(\mathbf{PSL}(2,\mathbf{R}),\mathbf{Z}) \longrightarrow \mathbf{R} \bmod \mathbf{Z}\cdot(\pi^2),$$

$$L^{PS}(\mathbf{R}): \mathbf{H}_3(\mathbf{PSL}(2,\mathbf{R}),\mathbf{Z}) \longrightarrow \mathbf{R} \bmod \mathbf{Z}\cdot\left(\frac{\pi^2}{2}\right). \tag{4.9}$$

$L^{PS}_{\mathbf{R}}$ is injective on torsion elements and $L^{PS}(\mathbf{R})$ maps an element of order m to one of order m or $m/2$ according to m is odd or even.

Remarks 4.10. (i) In using the extension to $\mathbf{PGL}(2,\mathbf{R})$ and its universal covering group, $[[r]]$ is the usual cross-ratio symbol associated to $(\infty, 0, 1, r)$ for r in $\mathbf{R} - \{0, 1\}$, see [PS]. Thus, $\{r\}$ is mapped to $[[r]]$. (ii) $\mathbf{H}_3(\mathbf{PSL}(2,\mathbf{R}),\mathbf{Z})$ is conjectured to be equal to $\mathbf{H}_3(\mathbf{PSL}(2,\mathbf{R}^{\mathrm{alg}}),\mathbf{Z})$, where $\mathbf{R}^{\mathrm{alg}}$ denote the field of all real algebraic numbers. This follows from a similar conjecture for $\mathbf{C}$ in place of $\mathbf{R}$. Thus, the two maps in (4.9) are not expected to be surjective. So far, all the non-trivial elements in the image are obtained by using algebraic numbers. (iii) It is both convenient and essential to consider the group $\mathbf{H}_3(\mathbf{PSL}(2,\mathbf{C}),\mathbf{Z})$ or $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}),\mathbf{Z})$. Namely, $\mathbf{C}$ admits a huge group of automorphisms while $\mathbf{R}$ has only the trivial automorphism. While we do not know the injectivity of $\hat{c}_2: \mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}),\mathbf{Z}) \to \mathbf{C}/\mathbf{Z}$, we do know that each non-zero element of $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}^{\mathrm{alg}}),\mathbf{Z})$ can be detected by a composition $\hat{c}_2 \circ \tau$ for a suitable automorphism τ of $\mathbf{C}$. This is a theorem of Borel, see [B]. Except when τ is the identity or the complex conjugation map, the image $\tau(\mathbf{R})$ is everywhere dense in $\mathbf{C}$. It is the use of the hyperbolic volume interpretation that ultimately leads to conclusion that $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}),\mathbf{Z})$ and $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{R}),\mathbf{Z})$ both contain a $\mathbf{Q}$-vector subspace of infinite dimension.

5. Connection with Richmond–Szekeres and Kirillov–Reshetikhin Identities

Granting the assertions in the preceding reviews, we can now describe the relation of the above discussions with the Richmond–Szekeres identity (1.9) and the extension by Kirillov–Reshetikhin (1.10). As described in [PS], if $\mathbf{G}$ is a cyclic group of order m with generator g, then the following chain is a $(2j-1)$-cycle and its class generates $\mathbf{H}_{2j-1}(\mathbf{G},\mathbf{Z}) \cong \mathbf{Z}_m$, $j > 0$:

$$c_m^{(j)} = \sum [g|x_1|g|\ldots|x_{j-1}|g], \quad x_i \text{ range over } G \text{ independently}\,. \tag{5.1}$$

More generally, $\sum [g^{i(1)}|x_1|\ldots|x_{j-1}|g^{i(j)}]$ is homologous to $i(1)\ldots i(j)\cdot c_m^{(j)}$. The superscript is used to remind us that the class behaves as a j^{th} power character on the cyclic groups. We now map $\mathbf{G}$ into $\mathbf{S} = \mathbf{PSL}(2,\mathbf{R})$ by sending g to the following matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \theta = \frac{\pi}{m}\,.$$

The map σ in (4.2) sending $\mathbf{H}_3(\mathbf{S}, \mathbf{Z})$ into $P(\mathbf{R})$ is obtained by sending the 3-cell $[g_1|g_2|g_3]$ to the cross-ratio symbol of $(\infty, g_1(\infty), g_1g_2(\infty), g_1g_2g_3(\infty))$. Here $\infty = \mathbf{R}\binom{0}{1}$, $r = \mathbf{R}\binom{1}{r}$, more generally, $y/x = \mathbf{R}\binom{x}{y}$, $x \geqq 0$ and $\mathbf{PGL}(2, \mathbf{R})$ acts on these lines through matrix multiplication. However, as discussed in Sect. 3, in the evaluation of volume, chains may be modified by boundaries. For the special form of the 3-cells that appears in $c_m^{(2)}$, this is not a serious problem. In any event, we have a canonical identification of the torsion subgroup:

$$\mathrm{tor}(\mathbf{H}_3(\mathbf{PSL}(2, \mathbf{R}), \mathbf{Z})) \cong \mathbf{Q}\pi/\mathbf{Z}\pi, \text{ the rational rotations in } \mathbf{PSO}(2, \mathbf{R})\,. \quad (5.2)$$

We now consider $c_m = c_m^{(2)}$ and note that $\sigma(c_m)$ is of order m or $m/2$ in $P(\mathbf{R})$ according to m is odd or even. Thus, we will restrict ourselves to $m > 2$. $[g|g^j|g]$ corresponds to $(\infty, g(\infty), g^{j+2}(\infty))$. Except when $j = 0, m-2, m-1$, this is just $[[Q_j^2/Q_{j-1}Q_{j+1}]]$, where $Q_j = Q_j(\theta) = \sin(j+1)\theta/\sin\theta$, $\theta = \pi/m$.

When $j = 0$, $[g|1|g]$ is 0 under the usual normalization. The corresponding formal 3-cell has two identical adjacent vertices and represents 0.

When $j = m-2 > 0$. We have the formal 3-cell $(\infty, -1, 1, \infty)$ independent of m. It is the same as $(\infty, 0, 1, \infty)$ and is assigned the cross ratio symbol $\{\infty\}$. By taking the boundary of $\{\infty, 0, 1, 2, \infty\}$, $\{\infty\}$ is seen to be homologous to $2\{2\} = -2\{1/2\}$ as in (4.5).

When $j = m-1 \geqq 2$. We have the formal 3-cell $(\infty, 0, \infty, 0)$ independent of m. It is the boundary of $(\infty, 0, \infty, 0, 1)$. Thus, we set it to 0.

To see how the preceding assignments work, we consider the cases: $m = 3$ and 4.

When $m = 3$, $\sigma(c_3) = [[\infty]]$ and $L^{\mathrm{PS}}(\{\infty\}) = \pi^2/6$. This represents an element of order 3 in $\mathbf{R}$ mod $Z\cdot(\pi^2/2)$.

When $m = 4$, $\sigma(c_4) = [[\infty]] + [[2]]$ and $L^{\mathrm{PS}}(\{\infty\}) + L^{\mathrm{PS}}(\{2\}) = \pi^2/6 + \pi^2/12 = \pi^2/4$. This represents an element of order 2 in $\mathbf{R}$ mod $\mathbf{Z}(\pi^2/2)$.

We now go to the general case. For $m > 2$, we have:

$$\sigma(c_m) = [[\infty]] + \sum_{1 \leqq j \leqq m-3} \left[\left[\frac{Q_j^2}{Q_{j-1}Q_{j+1}}\right]\right],$$

$$Q_j = Q_j(\theta) = \frac{\sin(j+1)\theta}{\sin\theta}, \quad \theta = \frac{\pi}{m}\,. \quad (5.3)$$

The above calculatioin is purely formal and the only reason that θ is chosen to be π/m arises from the fact that the expression in (5.1) represents the image of an element of order m or $m/2$ in $\mathbf{H}_3(\mathbf{S}, \mathbf{Z})$. The expression for Q_j is well known in terms of representation theory. Namely, consider the irreducible representations of $\mathbf{SL}(2, \mathbf{C})$ of finite dimension. It is well known that there is exactly one in each dimension $n + 1 \geqq 1$. It is realized in the n^{th} symmetric powers of the fundamental representation of $\mathbf{SL}(2, \mathbf{C})$ on $\mathbf{C}^2$. This is the spin $n/2$ representation in physics. Evidently, the matrix $\mathrm{diag}(z, z^{-1})$ is represented by $\mathrm{diag}(z^n, z^{n-2}, \ldots, z^{-n})$. $Q_j(\theta)$ is just the trace of $\mathrm{diag}(z, z^{-1})$ in the spin $j/2$ representation where $z = \exp(\iota\theta)$. The following lemma results from looking at the character of the representation theory $\mathbf{SL}(2, \mathbf{C})$:

Lemma 5.4. *Let $S(i)$ denote the i^{th} symmetric tensor representation of* $\mathbf{SL}(2, \mathbf{C})$, *$i > 0$. Let j, p, $q > 0$. Then $S(p+j-1) \otimes S(q+j-1) \cong S(p-1) \otimes S(q-1) \oplus S(p+q+j-1) \otimes S(j-1)$ holds. (Note: the representation $S(i)$ has degree $i+1$.)*

For the proof it is enough to look at $\mathrm{diag}(z, z^{-1})$. If we consider the special case of $z = \exp(\iota\theta)$, $p = q = 1$, we get $Q_i^2 = Q_{i-1}Q_{i+1} + 1$. Since $Q_j^2 = 1/d_j$ by definition, we have:

$$\sigma(c_m) = [[\infty]] + \sum_{1 \leqq j \leqq m-3} [[(1 - d_j)^{-1}]], \quad 1 \leqq j \leqq m - 3 . \tag{5.5}$$

The right-hand side of (5.5) is $[[\infty]] + 2.\sum_{1 \leqq j \leqq k-1} [[(1 - d_j)^{-1}]]$ for $m = 2k + 1$ and is $[[\infty]] + [[(1 - d_k)^{-1}]] + 2.\sum_{1 \leqq j \leqq k-1} [[(1 - d_j)^{-1}]]$ for $m = 2k + 2$.

We next have the following elementary result:

Lemma 5.6. *Let $F: \mathbf{Q} \to \mathbf{Q}$ be an additive homomorphism so that $F(\mathbf{Z}) \subset \mathbf{Z}$ and so that $F: \mathbf{Q}/\mathbf{Z} \cong \mathbf{Q}/\mathbf{Z}$. Then $F = \pm \mathrm{Id}$. If $F(1/3) \equiv -1/3 \bmod \mathbf{Z}$, then $F = -\mathrm{Id}$.*

Proof. Recall that F is just multiplication by a rational number because division by integers is unique. The two restrictions on F force F to be multiplication by ± 1. The final restriction forces F to be minus identity.

We can now apply Lemma 5.6 to obtain the following:

Theorem 5.7. *For $m \geqq 3$, $L^{\mathrm{PS}}(\sigma(c_m)) = -\pi^2/m \bmod \mathbf{Z}\cdot(\pi^2/2)$. In general, we have the congruence Kirillov–Reshetikhin identity:*

$$\sum_{1 \leqq j \leqq m-2} L\left(\frac{\sin^2\dfrac{\pi}{m}}{\sin^2\dfrac{(j+1)\pi}{m}}\right) = \frac{\pi^2}{6}\cdot\frac{3(m-2)}{m} \equiv -\frac{\pi^2}{m} \bmod \mathbf{Z}\cdot\left(\frac{\pi^2}{2}\right).$$

in particular, we have the congruence Richmond–Szekeres identity for $m = 2k + 1$:

$$\sum_{1 \leqq j \leqq k-1} L\left(\frac{\sin^2\dfrac{\pi}{2k+1}}{\sin^2\dfrac{(j+1)\pi}{2k+1}}\right) = \frac{\pi^2(2k-2)}{6\cdot(2k+1)} \bmod \mathbf{Z}\cdot\left(\frac{\pi^2}{4}\right).$$

Proof. We already know that $\mathbf{Q}\{2\}$ is the inverse image of the torsion subgroup of $P(\mathbf{R})$ in $\mathrm{PS}(\mathbf{R})$. Moreover, $L^{\mathrm{PS}}: \mathbf{Q}\{2\} \to \mathbf{Q}\pi^2$ is an isomorphism that carries $6\{2\}$ onto $\pi^2/2$. The torsion subgroup of $\mathbf{H}_3(\mathbf{PSL}(2, \mathbf{R}), \mathbf{Z})$ is identified with $\mathbf{Q}\pi/\mathbf{Z}\pi$, where the elements c_m arising from rotation by π/m in $\mathbf{PSO}(2, \mathbf{R})$ and $\sigma(c_m)$ has order m or $m/2$ in $P(\mathbf{R})$ according to whether m is odd or even. Since c_m corresponds to π/m in $\mathbf{Q}\pi/\mathbf{Z}\pi$, Lemma 5.6 shows that $L^{\mathrm{PS}}(\mathbf{R})(\sigma(c_m))$ must be $\pm\pi^2/m$ in $\mathbf{Q}\pi^2$ mod $\mathbf{Z}\cdot(\pi^2/2)$. When $m = 3$, we saw that the image is $\pi^2/6 = \pi^2/2 - \pi^2/3$. It follows that $L^{\mathrm{PS}}(\mathbf{R})(\sigma(c_m)) = -\pi^2/m \bmod \mathbf{Z}\cdot(\pi^2/2)$. This is just the general congruence identity. The more precise equality was proved in [KR-II, (2.33) and Appendix 2.] by an analytic argument.

Let $m = 2k + 1$. By (4.4), (4.7), and $\sin(\pi - \phi) = \sin\phi$, $L^{\mathrm{PS}}(\mathbf{R})(\sigma(c_m)) = \pi^2/6 + 2\sum_j L(d_j)$, $1 \leqq j \leqq k - 1$. Next $\pi^2/2 - \pi^2/(2k + 1) = (2k - 1)\pi^2/2(2k + 1) = \pi^2/6 + (4k - 4)\pi^2/6(2k + 1)$. The congruence immediately follows.

If we use the fact that L^{PS} is injective on $\mathbf{Q}\{2\}$, we have the immediate corollary:

Corollary 5.8. *In $\mathrm{PS}(R)$, $4(m-3)\cdot\{2\} = m\cdot\sum_{1 \leqq j \leqq m-3} \{(1 - d_j)^{-1}\}$, $m > 2$. Equivalently, $6(m-2)\cdot\{2\} = m\cdot\sum_{1 \leqq j \leqq m-2} \{(1 - d_j)^{-1}\}$, $m > 2$.*

We may obtain more congruence identities by computing the image in $P(\mathbf{R})$ of a representative for the class $p \cdot q \cdot c_m^{(2)}$, $0 < p, q < m$. Namely, we take $i(1) = p$ and $i(2) = q$ in the extension of (5.1). There are at most 4 exceptional symbols to consider according to j mod m. When $j = 0$, we always have 0. We therefore assume $0 < j < m$. If $j = -p$ or $-q$, depending on $p = q$ or $p \neq q$, we end up with either 0 or $-\{\infty\}$. Finally, if $j \equiv -p-q \bmod m$ (this forces $p + q \neq m$), then the symbl is $\{\infty\}$ as before. The general congruence identity then takes on the following form:

Theorem 5.9. *Let L^{PS} denote the shifted Rogers' dilogarithm as in* (4.7). *Let $m > 0$, $0 < p, q < m$. Let*

$$\delta_j(p, q; m) = \frac{\sin(p+j)\theta \cdot \sin(q+j)\theta}{\sin j\theta \cdot \sin(p+q+j)\theta}, \quad 0 < j < m, \quad \theta = \frac{\pi}{m}.$$

We then have the following congruence with the understanding that: the index j is to skip over the cases, $-p$, $-q$, $-p-q$ mod m; and $\delta_{a,b}$ is the Kronecker delta mod m:

$$\sum_{1 \leqq j \leqq m-1} L^{\mathrm{PS}}((1 - d_j^{(p)})^{-1}) \equiv -\frac{pq\pi^2}{m} + (\delta_{p,-q} - \delta_{p,q}) \cdot \frac{\pi^2}{6} \bmod \mathbf{Z} \cdot \left(\frac{\pi^2}{2}\right).$$

We note that the number $(1 - d_j^{(p)})$ lies in $\mathbf{R} - \{0, 1\}$ after we exclude the exceptional cases. It is easy to see that $\sin(x+p)\theta/\sin x$ is strictly decreasing in x. Thus, $\delta_j(p, q; m)$ can be negative. In general, it is necessary to use the defining properties (4.3)–(4.7) of L^{PS} in order to express the congruence in terms of L. If we use Lemma 5.4, it is easy to see that:

$$\delta_j(p, q; m)^{-1} = 1 - \frac{\sin p\theta \cdot \sin q\theta}{\sin(p+j)\theta \cdot \sin(q+j)\theta}.$$

In the case of $p = q = 1$, the right-hand side is strictly between 0 and 1 so that (4.4) and (4.7) recover the congruence in Theorem 5.7. However, for general, p, q, we do not have a good way to determine the "integral ambiguity" implicit in lifting the congruence to an identity. This resembles the classical treatment of Gauss' treatment of Gauss' quadratic reciprocity theorem in number theory via the use of Gauss' sums.

Remark 5.10. In Theorem 5.7, the rational numbers, $(2k-2)/(2k+1)$, are the "so-called" effective central charge of the $(2, 2k+1)$ Virasoro minimal model. Similarly, the rational number $3\ell/(\ell+2)$ is the central charge of the level ℓ $A_1^{(1)}$ WZW model. Both are models in conformal field theory. In our present setting, they are identified as specific values of the evaluation of the Cheeger–Chern–Simons characteristic class on the third integral homology of the universal covering group $\mathbf{PSL}(2, \mathbf{R})^{\sim}$ of $\mathbf{PSL}(2, \mathbf{R})$ (viewed as a discrete group). These homology classes are the lifts of the torsion classes for $\mathbf{PSL}(2, \mathbf{R})$.

In the recent work of Kirillov [K] concerning a conjecture of Nahm on the spectrum of rational conformal field theory [NRT], the following abelian subgroup $\mathbf{W}$ of $\mathbf{Q}$ was considered:

$$\mathbf{W} = \left\{ \sum_i n_i L(a_i)/L(1) \,|\, n_i \in \mathbf{Z},\, a_i \in \mathbf{R}^{\mathrm{alg}} \right\} \cap \mathbf{Q}.$$

From our discussion, it is clear that **W** contains both 1 as well as $-1/m \bmod \mathbf{Z}$ for every positive integer m. Thus, **W** is simply **Q**. In the conjecture of Nahm, one is more concerned with the set of effective central charges and n_i is assumed to be non-negative. This subset is closed under addition because one can form a tensor product of models. Our discussion only pins down the fractional part of such central charges while the integral parts apparently spread the central charges out in a way that resembled the volume distribution of hyperbolic 3-manifolds. In the present approach, these effective central charges are volumes of certain 3-cycles in a totally different space – the compactification of the universal covering group of $\mathbf{PSL}(2, \mathbf{R})$. These 3-cycles can be viewed as "orbifolds" since they arise from the finite cyclic subgroups of $\mathbf{SL}(2, \mathbf{R})$. It should also be noted that the central charge of the Virasoro algebra is the value of a degree two cohomology class while our description is on the level of degree three group cohomology, but for the Lie group viewed as a discrete group. The precise relation between these cohomologies is not too well understood. On the level of classifying spaces of topological groups, there is the well known conjecture, see [M] and [FM]:

Conjecture of Friedlander–Milnor. *Let G be any Lie group and let p be a prime. Then $\mathbf{H}_i(BG^\delta, \mathbf{Z}_p) \to \mathbf{H}_i(BG, \mathbf{Z}_p)$ is an isomorphism (it is known to be surjective).*

6. The "Beta Map" and Various Conjectures

In the work of Nahm–Recknagel–Terhoeven, [NRT], speculations were made about the relevance of algebraic K-theory, Bloch groups [B1], geometry of hyperbolic 3-manifolds, [Th], as well as the "physical meaning" of a "beta map." To some extent, we have clarified the first three of these. Namely, the connection between the effective central charge in rational conformal field theory with algebraic K-theory and Bloch groups, [B1], can be made by way of the second characteristic class of Cheeger–Chern–Simons and its interpretation via volume calculation in the universal covering group space of $\mathbf{PSL}(2, \mathbf{R})$. Specifically, it is *not* connected with the volume calculation in hyperbolic 3-space. (Note: According to Thurston's work, [Th], volume of hyperbolic 3-manifolds is a topological invariant.) Roughly speaking, the difference rests with a missing factor of $(-1)^{1/2}$. We next clarify the "beta map." In terms of diagram (4.2), the "beta map" is denoted by:

$$d^2 : P(\mathbf{R}) \to \Lambda^2_{\mathbf{Z}}(\mathbf{R}^\times), \quad d^2([[r]]) = r \wedge (r-1), \quad r > 1 . \tag{6.1}$$

d^2 arises as the second differential in a spectral sequence. It is defined by solving a "descent equation." This is typical of the higher differential maps in a spectral sequence. The exactness of the rows in (4.2) showed that $\ker d^2 = \operatorname{im} \sigma$. If we move up to the level of $\mathrm{PS}(\mathbf{R})$, it is then clear that the vanishing of the d^2-invariant characterizes the elements of $\mathbf{H}_3(\widetilde{\mathbf{PSL}(2, \mathbf{R})}, \mathbf{Z})$. The origin of d^2 comes from the Dehn invariant in Euclidean 3-space. In 1900, Dehn used it to solve Hilbert's Third Problem and extended it to hyperbolic and spherical 3-space, see [DS2]. By working with $P(\mathbf{C})$, see [DS1] and [DPS], d^2 then incorporates both versions of the Dehn invariants. In the present case, we would interpret d^2 in terms of "ideal polyhedra" in $\tilde{\mathbf{S}}$. As pointed out in [PS], the following conjecture is still open:

Conjecture 6.2. L^{PS}: $\mathbf{H}_3(\mathbf{PSL}(2,\mathbf{R})\tilde{}, \mathbf{Z}) \to \mathbf{R}$ *is injective.*

We already mentioned the following conjecture along this line:

Conjecture 6.3. $\mathbf{H}_3(\mathbf{PSL}(2,\mathbf{R}^{\text{alg}})\tilde{}, \mathbf{Z}) \to \mathbf{H}_3(\mathbf{PSL}(2,\mathbf{R})\tilde{}, \mathbf{Z})$ *is bijective.*

The preceding conjecture is a special case of the more general "folklore" conjecture:

Conjecture 6.4. $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}^{\text{alg}}), \mathbf{Z}) \to \mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}), \mathbf{Z})$ *is bijective.*

More precisely, Conjecture 6.3 is equivalent to any of the corresponding conjectures for a nontrivial quotient group of $\mathbf{PSL}(2,\mathbf{R})\tilde{}$, for example $\mathbf{PSL}(2,\mathbf{R})$. $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{R}), \mathbf{Z})$ is known to be isomorphic to the fixed point set of $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}), \mathbf{Z})$, see [Sa3]. The map in Conjecture 6.4 is known to be injective, see [Su2]. Thus Conjectures 6.3 and 6.4 would follow from:

Conjecture 6.5. $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}^{\text{alg}}), \mathbf{Z}) \to \mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}), \mathbf{Z})$ *is surjective.*

It should be mentioned that the map $\mathbf{H}_3(\mathbf{SU}(2), \mathbf{Z}) \to \mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}), \mathbf{Z})$ has image equal to the image of $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{R}), \mathbf{Z})$. In this connection, we have:

Conjecture 6.6. $\mathbf{H}_3(\mathbf{SU}(2), \mathbf{Z}) \to \mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}), \mathbf{Z})$ *is injective.*

Conjecture 6.7. $\hat{c}_2$: $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}), \mathbf{Z}) \to \mathbf{C}/\mathbf{Z}$ *is injective.*

Conjecture 6.7 is equivalent to the conjunction of Conjecture 6.6 and the converse of the Hilbert's Third Problem for hyperbolic as well as spherical polytopes in dimension 3. Namely, the Dehn invariant together with volume detect the scissors congruence classes of such polytopes. The Euclidean case was solved by Dehn–Sydler, see [DS2] for discussions. The best result in this direction is the theorem of Borel, [Bo]:

Borel's Theorem. *Suppose c is non-zero in* $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}^{\text{alg}}), \mathbf{Z})$, *then* $\hat{c}_2(\tau(c))$ *is non-zero for a suitable automorphism* τ *of* $\mathbf{C}$.

We note that an illustration of the idea behind Borel's Theorem was the proof given in [PS] that $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{C}^{\text{alg}}), \mathbf{Z})$ contains a rational vector space of infinite dimension. Recall, we consider a real algebra number r_p satisfying the equation $X^p - X + 1 = 0, p$ an odd prime. $d^2(\{r_p\})$ is therefore 0 and $[[r_p]]$ then defines an element of $\mathbf{H}_3(\mathbf{SL}(2,\mathbf{R}^{\text{alg}}), \mathbf{Z})$. Since L^{PS} is strictly monotone, there is no problem showing that we have distinct elements. However, it is not obvious that these elements are $\mathbf{Q}$-linearly independent. This stronger statement was a combination of Galois theory together with the use of the hyperbolic volume.

7. Concluding Remarks

In the present work, we showed that the effective central charges for certain models in conformal field theory can be connected to the evaluation of a real valued cohomology class on a suitable degree 3 homology class for the integral group homology of the universal covering group $\mathbf{PSL}(2,\mathbf{R})\tilde{}$ of $\mathbf{PSL}(2,\mathbf{R})$. The important point is that we have replaced the usual topology by the discrete topology. In addition, instead of the hyperbolic 3-space, we use the group space of this universal covering group. The particular homology clas is a suitable lift of a homology class of finite order that generates the third integral homology of a finite cyclic subgroup of $\mathbf{PSL}(2,\mathbf{R})$. The lift is connected with the Rogers' dilogarithm identities due to

Richmond–Szekeres [RS] and Kirillov–Reshetikhin [KR]. The latter is equivalent to an identity found earlier by Lewin [L1, (5.117); L2, p. 19]; the equivalence is in exact parallel to the fact that Rogers' identity is equivalent to an earlier identity found by Abel, see the authoritative historical accounts in Lewin [L1, p. 8–18; L2, p. 1–25]. All these identities are shown to originate from the basic identities found by Rogers [R]. Our route ends in the central charge identification but there are no firm connections between any of the intermediate steps followed by us with the intermediate steps used in solvable moels in conformal field theory. A casual reading of [BPZ] and [Z2] does show that many appearances of cross-ratios. However, instead of the complex numbers or the real numbers, we see meromorphic functions. This is also the basic theme in the work of Bloch [B1]. On the mathematical side, there are efforts to build up enormous structures to explain the steps on the physics side. Our present effort does not do this. In particular, attempts have been made to relate the various identities directly with torsion elements in algebra K_3 groups of the complex numbers, see [G]. Apparently, it has not been as successful since the intermediate step of group homology appears to be essential in our present state of understanding.

Another of the principal points in the present work is the fact that Rogers' dilogarithm has long been known to be connected with the second Cheeger–Chern–Simons characteristic class which is represented by the Chern–Simons form that appears in many current theoretical physics investigations. This connection is related to the interplay between the "continuous" picture and the "discrete" picture. On the mathematical side, we have a direct map on the level of classifying spaces for groups equipped with two topologies: one discrete, the other continuous. The map is the one that goes from the discrete to the continuous. On the physics side, the passage from the discrete to the continuous is a subject of debate since there does not appear to be a specific map from the discrete to the continuous (in the mathematical sense). However, there are still a large number of unresolved issues on the mathematical side. For example, the Virasoro algebra is typically viewed as the algebraic substitute for the diffeomorphism group of the circle. (More precisely, it may be viewed as the "pseudo-group" of holomorphic maps on the sphere with two punctures.) This contains $\mathbf{PSL}(2, \mathbf{R})$ which acts as a diffeomorphism group of the circle through the identification of the circle with $P^1(\mathbf{R})$. Our procedure replaces these infinite dimensional (pseudo-) groups by the finite dimensional subgroups. However, it is also accompanied by the use of the discrete topology. Although the process of playing off one topology against another is familiar in foliation theory, it is not explored in the present work.

In passing, we would like to indicate that Rogers' dilogarithm has appeared in various related works on the physics side. Aside from the work [BR] that led Bazhanov to ask one of us (CHS) about the connection between [BR] and [PS] in the summer 1987, there are the earlier works of Zamolodchikov [Z2] and Baxter [B]. Specifically, in the appendix of [B], Rogers' dilogarithm appeared. This has been extended recently in [BB] where they have shown that the 3-d models of Zamolodchikov can be related to the earlier 2-d chiral Potts models considered in [AMPTY, MPST, and BPA] after suitable generalizations. On the mathematics side, Atiyah and Murray [A] have identified the algebraic curves in [BPA] and [MPST] as the spectral curves of N magnetic monopoles arranged cyclically around an axis in hyperbolic 3-space. In view of the fact that our present work indicates that the group manifold $\mathbf{PSL}(2, \mathbf{R})^{\sim}$ is more appropriate than the hyperbolic 3-space, one cannot help but ask if there might be an interesting mathematical theory of monopoles in $\mathbf{PSL}(2, \mathbf{R})\tilde{}$. Evidently, the present work raises many more questions than it answers.

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Note added in proof. An interesting preprint just appeared: L.D. Faddeev, R.M. Kashaev, Quantum Dilogarithm. Preprint, hep-th/9310070.

References

[ABF] Andrews, G.E., Baxter, R.J., Forrester, P.J.: Eight vertex SOS model and generalized Rogers–Ramanujan identities. J. Stat. Phys. **35**, 193–266 (1984)

[A] Atiyah, M.F.: Magnetic monopoles and the Yang–Baxter equations. Int. J. Mod. Phys. A **6**, 2761–2774 (1991)

[AMPTY] Au-Yang, H., McCoy, B.M., Perk, J.H.H., Tang, S., Yan, M.-L.: Commuting transfer matrices in the chiral Potts models: Solutions of star-triangle equations wth genus > 1. Phys. Lett. A **123**, 219–233 (1987)

[B] Baxter, R.J.: The Yang–Baxter equations and the Zamolodchikov model. Physica **18D**, 321–347 (1986)

[BPA] Baxter, R.J., Perk, J.H.H., Au-Yang, H.: New solutions of the star triangle relations for the chiral Potts model. Phys. Lett. A **128**, 138–142 (1988)

[BB] Bazhanov, V.V., Baxter, R.J.: New solvable lattice models in three dimensions. J. Stat. Phys. **69**, 453–485 (1992); Star-triangle relation for a three dimensional model. Preprint, hep-th/9212050

[BR] Bazhanov, V.V., Reshetikhin, N.Yu.: Critical RSOS models and conformal field theory. Int. J. Mod. Phys. A4, 115–142 (1989); RSOS models connected with simply laced algebras and conformal field theory. J. Phys. A.: Math. Gen. **23**, 1477–1492 (1990)

[BPZ] Belavin, A.A., Polyakov, Zamolodchikov, A.B.: Infinite conformal symmetry in two-dimensional quantum field theory. Nucl. Phys. **B 241**, 333–380 (1984)

[Bl] Bloch, S.: Higher regulators, algebraic K-theory, and zeta functions of elliptic curves. Irvine Lecture Notes, 1978

[Bo] Borel, A.: Cohomologie de $\mathbf{SL}_n$ et valeurs de fonctions zeta aux points entiers. Ann. Scien. Norm. Super. Pisa, Cl. Sci., **4**, 613–636 (1977) Correction, ibid. **7**, 373 (1980)

[Br] Brown, K.S.: Cohomology of Groups. Heidelberg, Berlin, New York: Springer, 1982

[C] Coxeter, H.S.M.: The functions of Schläfli and Lobatschefsky. Quar. J. Math. **6**, 13–29 (1935)

[DKKMM] Dasmahapatra, S., Kedem, R., Klassen, T.R., McCoy, B.M., Melzer, E.: Quasi-particles, conformal field theory, and q-series. Preprint, hep-th/9303013

[D] Dupont, J.L.: The dilogarithm as a characteristic class for flat bundles. J. Pure Appl. Algebra **44**, 137–164 (1987); On polylogarithms. Nagoya Math. J. **114**, 1–20 (1989); Characteristic classes for flat bundles and their formulas. Preprint, Aarhus Univ. no. 23, 1992

[DPS] Dupont, J.L., Sah, C.H.: Homology of classical Lie groups made discrete, II. H_2, H_3, and relations with scissors congruences. J. Algebra **113**, 215–260 (1988)

[DS] Dupont, Sah, C.H.: Scissors congruences, II. J. Pure Appl. Algebra **25**, 159–195 (1982); Homology of Euclidean groups of motions made discrete. Acta Math. **164**, 1–27 (1990)

[FS] Frenkel, E., Szenes, A.: Dilogarithm identities, q-difference equations and the Virasoro algebra. Preprint hep-th/9212094)

[FM] Friedlander, E.M., Mislin, G.: Cohomology of classifying spaces of complex Lie groups and related discrete groups. Comm. Math. Helv. **59**, 347–361 (1984)

[GM] Gelfand, I.M., MacPherson, R.D.: Geometry in Grassmannians and a generalization of the dilogarithm. Adv. Math. **44**, 279–312 (1982)

[G] Govindachar, S.: Explicit weight two motivic cohomology complexes in algebraic K-theory. K-theory **6**, 387–430 (1992)

[KKMM] Kedem, R., Klassen, T.R., McCoy, B.M., Melzer, E.: Fermionic sum representations for conformal field theory characters. Preprint hep-th/9211102; Fermionic quasi-particle representations for characters of $(G^{(1)})_1 \times (G^{(1)})_1/(G^{(1)})_2$. Preprint hep-th/9301046

[K] Kirillov, A.N.: Identities for the Rogers dilogarithm function connected with simple Lie algebras. J. Sov. Math. **47**, 2450–2459 (1989); Dilogarithm identities and spectra in conformal field theory. Preprint hep-th/9211137, also hep-th/9212150

[KR] Kirillov, A.N., Reshetikhin, Yu.N.: Exact solutions of the integrable XZ Heisenberg model with arbitrary spin: I, II. J. Phys. A.: Math. Gen. **20**, 1565–1585, 1587–1597 (1987)

[KP] Klümper, A., Pearce, P.A.: Conformal weights of RSOS lattice models and their fusion hierarchies. Physica A **183**, 304–350 (1992)

[KN] Kuniba, A., Nakanishi, T.: Spectra in conformal field theories from the Rogers' dilogarithm. Preprint hp-th/9206034; Rogers' dilogarithm in integrable systems. Preprint hep-th/9210025

[KNS] Kuniba, A., Nakanishi, T., Suzuki, J.: Characteris in conformal field theories from thermodynamic Bethe ansatz. Preprint, hep-th/9301018

[L] Lewin, L.: Polylogarithms and Associated Functions. Amsterdam: Elsevier North Holland, 1981; *Structural Properties of Polylogarithms. Math. Surv. and Mono.* **37**, Providence, RI: Amer. Math. Soc., 1991

[MPST] McCoy, B.M., Perk, J.H.H., Tang, S., Sah, C.H.: Commuting transfer matrices for the 4-state self-dual Potts model with a genus 3 uniformizing Fermat curve. Phys. Lett. A **125**, 9–14 (1987)

[M] Milnor, J.W.: On the homology of Lie groups made discrete. Comm. Math. Helv. **58**, 72–85 (1983)

[NRT] Nahm, W., Recknagel, A., Terhoeven, M.: Dilogarithm identities in conformal field theory. Preprint, hep-th/9211034

[PS] Parry, W., Sah, C.H.: Third homology of $\mathbf{SL}(2, \mathbf{R})$ made discrete. J. Pure Appl. Algebra **30**, 181–209 (1983)

[RS] Richmond, B., Szekeres, G.: Some formulas related to dilogarithms, the zeta function and the Andrews–Gordon identities. J. Austral. Math. Soc. **31**, 262–273 (1981)

[R] Rogers, L.J.: On function sum theorem connected with the series $\Sigma_1^\infty x^n/n^2$. Proc. London Math. Soc. **4**, 169–189 (1907)

[Sa] Sah, C.H.: Scissors congruences, I. The Gauss–Bonnet map. Math. Scand. **49**, 181–210 (1981); Homology of Lie groups made discrete, I. Stability theorems and Schur multipliers. Comm. Math. Helv. **61**, 308–347 (1986); Homology of Lie groups made disrete, III. J. Pure and Appl. Algebra **56**, 269–312 (1989)

[Su] Suslin, A.A.: Homology of $\mathbf{GL}_n$, characteristic classes and Milnor K-theory. Springer Lec. Notes in Math. **1046**, 357–375 (1987); Algebraic K-theory of fields. Proc. Int. Cong. Math., Berkeley, I, 1986, pp. 222–224

[Te] Terhoeven, M.: Lift of dilogarithm to partition identities. Preprint hep-th/9211120

[Th] Thurston, W.P.: Geometry and Topology of Three-Manifolds. Lecture Notes, Princeton, 1979; Three-Dimensional Geometry and Topology. Preprint Notes, Minnesota, 1990

[V] Varchenko, A.: Multidimensional hypergeometric functions in conformal field theory, algebraic K-theory, algebraic geometry. Proc. Int. Cong. Math., Kyoto, I, 1990, pp. 281–300

[Z] Zamolodchikov, A.B.: *Tetrahedron Equations and the relativistic* S-*Matrix of* straight-strings in 2 + 1 dimensions. Commun. Math. Phys. **79**, 4898–505 (1981); Infinite additional symmetries in two-dimensional conformal quantum field theory. Theor. Math. Phys. **63**, 1205–1213 (1985)

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