Commun. Math. Phys. 161, 215–232 (1994)

Communications in Mathematical Physics

String Equations for the Unitary Matrix Model and the Periodic Flag Manifold

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Received: 27 March 1993/in revised form: 17 June 1993

Abstract: The periodic flag manifold (in the Sato Grassmannian context) description of the modified Korteweg–de Vries hierarchy is used to analyse the translational and scaling self-similar solutions of this hierarchy. These solutions are characterized by the string equations appearing in the double scaling limit of the symmetric unitary matrix model with boundary terms. The moduli space is a double covering of the moduli space in the Sato Grassmannian for the corresponding self-similar solutions of the Korteweg–de Vries hierarchy, i.e. of stable 2D quantum gravity. The potential modified Korteweg–de Vries hierarchy, which can be described in terms of a line bundle over the periodic flag manifold, and its self-similar solutions corresponds to the symmetric unitary matrix model. Now, the moduli space is in one-to-one correspondence with a subset of codimension one of the moduli space in the Sato Grassmannian corresponding to self-similar solutions of the Korteweg–de Vries hierarchy.

1. Introduction

In the last few years matrix models have received much attention as a non-perturbative formulation of string theory. These models can be described in the double scaling limit in terms of solutions to certain integrable systems. For the Hermitian matrix model (HMM) it was found [3] that in the double scaling limit the specific heat of the theory is a solution to the Korteweg–de Vries (KdV) hierarchy. This solution must satisfy also the string equation which is a self-similarity condition under Galilean symmetry transformations. This result was achieved by the use of orthogonal polynomials on the real line. The string equation can be written in terms of two scalar differential operators P, Q as

$$[P, Q] = \mathrm{id}\ .$$

★ Research supported by MEC/FLEMING postdoctoral fellowship GB92 00411668
★★ Research supported by British Council

The case of pure gravity leads to the Painlevé I equation. This case corresponds to the 2-multicritical point of the theory; when one considers the k-multicritical points one finds $2k-2$ order non-linear ODE. Because of some anomalous behaviour of the solutions to the string equation in [4], an alternative string equation that contains the former one was proposed. This string equation is the self-similarity condition under the local symmetries of the KdV hierarchy. This theory is called 2D stable quantum gravity.

Later it was shown in [15], with the use of orthogonal polynomials in the circle, that for the symmetric unitary matrix model (UMM) in the double scaling limit the specific heat satisfies the modified KdV hierarchy and a string equation. The solutions to this string equation are scaling self-similar solutions to the modified KdV hierarchy. In the case of the k-multicritical point one has a $2k$ order ODE which again can be recasted as

$$[L, T] = \text{constant} ,$$

where L, T are 2×2 matrices of differential operators of order $2k$ and 1, respectively. In [11] some boundary terms, modelling the presence of quarks, were added to the model; the corresponding string equation turns out to be the scaling self-similar condition for the modified KdV hierarchy. This was connected with 2D stable quantum gravity in [5] where the Miura map was extensively used.

The Sato Grassmannian description [17] of the solutions was used in [9] to characterize the solution to the string equation connected with the moduli space of complex curves [21, 10]. In the papers [18] a description of the moduli space for the Galilean self-similar solutions of the KdV hierarchy in the Sato Grassmannian was given, and in [12] one can find a more analytical treatment in terms of Stokes parameters. In [8] a complete description, in terms of the initial data for the zero-curvature 1-form, of the moduli space of self-similar solutions under local symmetries of the potential KdV hierarchy can be found. Finally in [2] one can find a description of the moduli space for the UMM.

In this paper, following closely the methods of [8], we analyse the geometrical description of the solutions to the double scaling limit of the UMM with boundary terms and without them. It will turn out that the description is completely different in each case. Our aim is to describe the moduli space of solutions as a subset of the periodic flag manifold [19, 20] in the Sato context. We find that the UMM string equation corresponds to the scaling self-similarity condition for the potential modified KdV hierarchy. When border terms are added the self-similar condition is for the modified KdV hierarchy. We characterize the moduli space in terms of the initial data for the corresponding zero-curvature 1-forms giving in this way a coordinate chart, that happens to be closely connected to certain algebraic varieties. The flag manifold is fibered over the Grassmannian and the moduli space when boundary terms are present is a double covering of the moduli space for 2D stable gravity. When no boundary terms are present the moduli space for the UMM is a subspace of codimension one of the former.

Our geometrical description in terms of homogeneous spaces and local symmetries complements that of [2] where an analysis, based on the fermionic approach, of the moduli space of the string equation of the UMM with no boundary terms is given.

In the second section we define the modified KdV and potential modified KdV hierarchies and we give its zero-curvature formulation. We also analyse there the local symmetries and the corresponding self-similar conditions, giving a zero-curvature type formulation of it. In Sect. 3 we introduce the factorization problem and the description of these integrable hierarchies in certain homogeneous spaces. This allows us to study the set of solutions to the string equations in terms of these homogeneous spaces, essentially a periodic flag manifold. In Sect. 4 we analyse the moduli space of string equations using Sato's periodic flag manifold corresponding to the scaling self-similar solutions of the modified KdV hierarchy, and a line bundle over this homogeneous space corresponding to the potential modified KdV hierarchy. In the final section we analyse the relation between these moduli spaces of string equations for the UMM and that of 2D stable quantum gravity.

2. Modified KdV Hierarchy and String Equations

We begin this section with the definition of the integrable hierarchies known as the modified KdV (mKdV) and the potential mKdV hierarchies. They are defined for scalar functions v, w that depends on an infinite number of variables $\mathbf{t} := \{t_{2n+1}\}_{n \geqq 0}$, the local coordinates for the time manifold $\mathscr{T}$. In this convention we adopted t_1 to be the space coordinate, usually denoted by x, and t_{2n+1} with $n > 0$ corresponds to time variables, for example t_3 is usually denoted by t. For its construction it is very convenient to use the so-called Gel'fand–Dikii potentials $R_n[u]$, [6], which are the coefficients for the expansion of the kernel of the resolvent of the associated Schrödinger equation with potential u.

Definition 2.1. *The modified Korteweg–de Vries hierarchy for v is the following collection of compatible equations*:

$$\partial_{2n+1} v = \partial_1 S_n[v], \quad n \geqq 0 ,$$

where $\partial_{2n+1} := \partial/\partial t_{2n+1}$ and

$$S_n[v] := (\partial_1 + 2v) R_n[u] ,$$

where the Gel'fand–Dikii potentials are evaluated on the Miura transformation of v,

$$u = \partial_1 v - v^2 . \tag{2.1}$$

Notice that the potential u, given by the Miura map (2.1), satisfies the KdV hierarchy

$$\partial_{2n+1} u = 4\partial_1 R_{n+1}[u], \quad n \geqq 0 .$$

The KdV equation $4\partial_3 u = \partial_1^3 u + 6u\partial_1 u$ follows from the first of its equations. The first equation of the mKdV hierarchy is the mKdV equation $4\partial_3 v = \partial_1^3 v - 6v^2(\partial_1 v)$.

Definition 2.2. *The potential modified Korteweg–de Vries hierarchy for the function w is the following set of equations*:

$$\partial_{2n+1} w = S_n[v], \quad n \geqq 0 ,$$

where

$$v := \partial_1 w .$$

Observe that if w is a solution to the potential mKdV hierarchy then $v = \partial_1 w$ is a solution to the mKdV hierarchy. The potential mKdV equation is $4\partial_3 w = \partial_1^3 w - 2(\partial_1 w)^3$.

These integrable hierarchies are equivalent to zero-curvature conditions, which turn out to be an essential feature of its integrability condition. Novikov [14] gave for the KdV equation a zero-curvature representation in terms of a differential 1-form $\chi(\lambda)$ that depends on a complex spectral parameter $\lambda \in \mathbb{C}$. The KdV hierarchy has a similar formulation. Let χ be the 1-form on $\mathscr{T}$ defined by

$$\chi(\lambda) := \sum_{n \geqq 0} L_{2n+1}(\lambda)\, dt_{2n+1} ,$$

where

$$L_{2n+1}(\lambda) := \begin{pmatrix} -\frac{1}{2}\partial_1 \rho_n(\lambda) & \rho_n(\lambda) \\ (\lambda - u)\rho_n(\lambda) - \frac{1}{2}\partial_1^2 \rho_n(\lambda) & \frac{1}{2}\partial_1 \rho_n(\lambda) \end{pmatrix} ,$$

with

$$\rho_n(\lambda) := 2 \sum_{m=0}^{n} \lambda^m R_{n-m}[u] . \tag{2.2}$$

Then, the KdV hierarchy is equivalent to the zero-curvature condition,

$$[d - \chi, d - \chi] = 0 ,$$

where d is the exterior derivative $d := \sum_{n \geqq 0} dt_{2n+1} \partial_{2n+1}$. For the mKdV hierarchy there is an equivalent statement.

Proposition 2.1. *The* 1*-form*

$$\xi := da \cdot a^{-1} + \mathrm{Ad}\, a\chi ,$$

where

$$a := \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} ,$$

has zero-curvature if and only if v is a solution of the mKdV *hierarchy. This* 1*-form can be represented as*

$$\xi(\lambda) = \sum_{n \geqq 0} \ell_{2n+1}(\lambda)\, dt_{2n+1} ,$$

and

$$\ell_{2n+1}(\lambda) := \begin{pmatrix} -(\partial_1 + 2v)\rho_n(\lambda)/2 & \rho_n(\lambda) \\ \lambda\rho_n(\lambda) - \partial_1(\partial_1 + 2v)(\rho_n(\lambda)/2 - R_n) & (\partial_1 + 2v)\rho_n(\lambda)/2 \end{pmatrix} , \tag{2.3}$$

with ρ given in Eq. (2.2).

Proof. It follows from the equation

$$\partial_{2n+1}\ell_1 - \partial_1 \ell_{2n+1} + [\ell_1, \ell_{2n+1}] = 0 .$$

□

One can equally proof the following

Proposition 2.2. *The potential* mKdV *hierarchy is equivalent to the zero-curvature condition for the* 1*-form*

$$\eta := db \cdot b^{-1} + \mathrm{Ad}\, b\xi ,$$

where

$$b := \begin{pmatrix} \exp(w) & 0 \\ 0 & \exp(-w) \end{pmatrix} .$$

Also

$$\eta(\lambda) = \sum_{n \geqq 0} \tilde{\ell}_{2n+1}(\lambda)\, dt_{2n+1} ,$$

and

$$\tilde{\ell}_{2n+1}(\lambda) := \begin{pmatrix} -(\partial_1 + 2v)(\rho_n(\lambda)/2 - R_n) & e^{2w}\rho_n(\lambda) \\ e^{-2w}(\lambda\rho_n(\lambda) - \partial_1(\partial_1 + 2v)(\rho_n(\lambda)/2 - R_n)) & (\partial_1 + 2v)(\rho_n(\lambda)/2 - R_n) \end{pmatrix} . \tag{2.4}$$

Let us now consider the symmetries defined by translations and scaling transformations.

The infinite set of translational symmetries are the isospectral symmetries of these hierarchies in the sense that they preserve the associated spectral problem, i.e. the Schrödinger equation for u. In fact the flows in the hierarchies are defined by the generators ∂_{2n+1} of translations.

Definition 2.3. *Let*

$$\vartheta(\mathbf{t}) := \mathbf{t} + \boldsymbol{\theta} ,$$

be the action of translations, where

$$\boldsymbol{\theta} := \{\theta_{2n+1}\}_{n \geqq 0} \in \mathbb{C}^\infty ,$$

are the shifts of the time variables.

We have a local action of the abelian group $\mathbb{C}^\infty$ over the time manifold $\mathscr{T}$. The following is obvious.

Proposition 2.3. *If v, w are solutions to the* mKdV *and potential* mKdV *hierarchies respectively then so are $\vartheta^* v$, $\vartheta^* w$.*

For the scaling symmetry we have

Definition 2.4. *The scaling transformation is*

$$\zeta_\sigma(\mathbf{t}) := \{e^{\sigma(n+1/2)} t_{2n+1}\}_{n \geqq 0} ,$$

where $\sigma \in \mathbb{C}$.

We have an additive local action of $\mathbb{C}$ over $\mathscr{T}$. One can easily show that

Proposition 2.4. *If v, w are solutions of the* mKdV *an potential* mKdV *hierarchies respectively then so are $e^{\sigma/2}\zeta_\sigma^* v$, $\zeta_\sigma^* w$.*

The related fundamental vector fields, infinitesimal generators of the action of translation and scaling transformations are given by

$$\partial_{2n+1}, n \geqq 0, \quad \zeta := \sum_{n \geqq 0} (n + 1/2) t_{2n+1} \partial_{2n+1}$$

respectively. They generate the linear space $\mathbb{C}\{\partial_{2n+1}, \zeta\}_{n \geqq 0}$ which is the Lie algebra of local symmetries of the mKdV and potential mKdV hierarchies, the non-vanishing Lie brackets are

$$[\partial_{2n+1}, \zeta] = (n + 1/2)\partial_{2n+1} .$$

We have the very important notion

Definition 2.5. *A self-similar solution under any of the mentioned symmetries is a solution which remains invariant under the corresponding transformation.*

Consider the following vector field belonging to this Lie algebra

$$X := \vartheta + \sigma\zeta , \tag{2.5}$$

with

$$\vartheta := \sum_{n \geqq 0} \theta_{2n+1}\partial_{2n+1} ,$$

defining a superposition of translational and scaling transformations. If v is a solution of the mKdV hierarchy then the function

$$\exp(\sigma/2 + X)v$$

is a solution as well. In what follows it will be convenient to use

Definition 2.6. *Let us denote*

$$\mathscr{R} := \sum_{n \geqq 0} (n + 1/2)t_{2n+1}R_n .$$

Then we have,

Theorem 2.1. *A solution v of the* mKdV *hierarchy is self-similar under the action of the vector field X if and only if it satisfies the generalized string equation*

$$\partial_1(\partial_1 + 2v)\left(\sum_{n \geqq 0} \theta_{2n+1}R_n + \sigma\mathscr{R}\right) = 0 . \tag{2.6}$$

Proof. A solution v of the mKdV hierarchy is self-similar under X if

$$\left(\vartheta + \sigma\zeta + \frac{\sigma}{2}\right)v = 0 . \tag{2.7}$$

Recalling the mKdV hierarchy one can show that this equation is actually equivalent to (2.6). □

The theorem above implies

$$(\partial_1 + 2v)\left(\sum_{n \geqq 0} \theta_{2n+1}R_n + \sigma\mathscr{R}\right) = c(t_3, t_5, \ldots) + \frac{\sigma}{4} ,$$

but

$$\partial_{2m+1}c = \sum_{n \geqq 0} (\theta_{2n+1} + \sigma(n + 1/2)t_{2n+1})\partial_{2m+1}S_n + \sigma(m + 1/2)S_m .$$

Using the commuting flow condition $[\partial_{2n+1}, \partial_{2m+1}]w = 0$ one realizes that the above equation can be written as

$$\partial_{2m+1}c = (X + \sigma(m + 1/2))S_m .$$

Because v is self-similar the right-hand side of this equation vanishes, hence c is a constant.

Corollary 2.1. *The solution v of the* mKdV *hierarchy is self-similar under the action of the vector field X if and only if*

$$(\partial_1 + 2v)\left(\sum_{n \geqq 0} \theta_{2n+1} R_n + \sigma \mathscr{R} \right) = c + \frac{\sigma}{4}, \tag{2.8}$$

for some complex number c.

For the potential mKdV hierarchy we have

Theorem 2.2. *A solution w of the potential* mKdV *hierarchy is self-similar under the action of the vector field X if and only if $v = \partial_1 w$ satisfies Eq.* (2.8) *with $c = 0$.*

Proof. The self-similar condition is

$$(\vartheta + \sigma\zeta)w = 0$$

which, using the hierarchy equations, gives the desired result. □

Notice that when $\theta(\lambda) = a(N + 1/2)\lambda^N$ the translation term in the string equation is removed if we transform the time coordinates as follows: $t_{2n+1} \mapsto t_{2n+1} + a\delta_{nN}$. This allows us to study the solutions away from the singularities. Observe also that given a self-similar solution w of the potential mKdV hierarchy then $v = \partial_1 w$ is self-similar with $c = 0$. But given a self-similar solution v of the mKdV hierarchy there is no self-similar w solution of the potential mKdV hierarchy such that $v = \partial_1 w$, unless $c = 0$. The point is that the string equation for the mKdV hierarchy only implies $\zeta_\sigma^* w(\mathbf{t}) = w(\mathbf{t}) + w_0(\sigma)$.

The self-similar condition for the potential mKdV hierarchy is the string equation that appears in [15] for the double scaling limit of the UMM. In this case $c = 0$, but when the self-similar condition is required for the mKdV hierarchy there is no need to confine $c = 0$; this is the case for the double scaling limit of the UMM with an additional boundary term that models the presence of c flavours of quarks [11, 5].

The general self-similarity condition can be reformulated as a zero-curvature type condition. This approach is closely connected with the isomonodronic technique employed in [12]. We define the outer derivative

$$\delta := \sigma\left(\lambda \frac{d}{d\lambda} + \frac{1}{4} \operatorname{ad} H\right), \tag{2.9}$$

where $H = \sigma_3$ is the diagonal Pauli matrix – observe that δ is proportional to the derivation defining the principal grading of the affine Lie algebra $A_1^{(1)}$ – and

$$M := \langle \xi, X \rangle, \quad \tilde{M} := \langle \eta, X \rangle . \tag{2.10}$$

Here $\langle \cdot, \cdot \rangle$ is the standard pairing between 1-forms and vector fields. Then one has,

Theorem 2.3.

1. *The zero-curvature type condition*

$$[d - \xi, \delta - M] = 0 \tag{2.11}$$

is equivalent to the generalized string equation (2.6).

2. *The equation*

$$[d - \eta, \delta - \tilde{M}] = 0$$

is equivalent to Eq. (2.8) *with $c = 0$.*

Proof. For the 1-form ξ this follows from the condition

$$\exp(t\delta)\xi = \exp(tX)\xi ,$$

that is equivalent to

$$\delta\xi = L_X \xi ,$$

where L_X denotes the Lie derivative along the vector field X. But

$$L_X \xi = (i_X d + d i_X)\xi ,$$

and recalling the zero-curvature condition for ξ, we obtain the desired result. The same argumentation holds for η. □

This theorem is the key for the analysis of the moduli space of the string equation.

3. Homogeneous Spaces and the String Equations

In this section we use the periodic flag manifold $\mathrm{Fl}^{(2)}$ description of the mKdV flows, [20, 16], in order to characterize geometrically the string equations for the self-similar solutions of the mKdV hierarchy. We also analyse the string equation for the potential mKdV hierarchy, not in the periodic flag manifold but in a line bundle over $\mathrm{Fl}^{(2)}$. These manifolds appear when one considers certain factorization problems in loop groups.

Recall that ξ defines a 1-form with values in the loop algebra $L\mathfrak{sl}(2, \mathbb{C})$ of smooth maps from the circle $S^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ to the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, traceless 2×2 complex matrices. We define an infinite set of commuting flows in the corresponding loop group $LSL(2, \mathbb{C})$,

$$\psi(\mathbf{t}, \lambda) := S(\mathbf{t}, \lambda) \cdot g(\lambda) , \tag{3.1}$$

where g is the initial condition and

$$S(\mathbf{t}, \lambda) := \exp\left(\sum_{n \geqq 0} t_{2n+1} \lambda^n J(\lambda) \right) , \tag{3.2}$$

where

$$J(\lambda) := \lambda F + E \tag{3.3}$$

in terms of the standard Cartan–Weyl basis $\{E, H, F\}$ for $\mathfrak{sl}(2, \mathbb{C})$, i.e.

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ;$$

these notations will be used in the rest of the paper.

Now, we introduce some definitions; the notation is that of [16]. Denote by $L^+SL(2, \mathbb{C})$ those loops which have a holomorphic extension to the interior of S^1, by $L^-SL(2, \mathbb{C})$ those which extend analytically to the exterior of the circle, and by $L_1^- SL(2, \mathbb{C}) \subset L^-SL(2, \mathbb{C})$ the subset of those extensions which are normalized by the identity at ∞. Consider the subgroup $B^+SL(2, \mathbb{C})$ of loops of $L^+SL(2, \mathbb{C})$ such that its holomorphic extensions to the interior of S^1 when evaluated at the origin belongs to the standard Borel group of $SL(2, \mathbb{C})$, that is the upper triangular 2×2 matrices with unity determinant. The group $N^+SL(2, \mathbb{C})$ is defined analogously, but now the Borel subgroup is replaced by the standard nilpotent group, i.e. upper triangular 2×2 matrices with 1 in the diagonal. The subgroup $B^- SL(2, \mathbb{C})$ is the set of loops of $L^- SL(2, \mathbb{C})$ such that its holomorphic extension to the exterior of S^1 when evaluated at infinity belongs to the set of 2×2 lower triangular matrices with unity determinant; when we ask to the elements of the diagonal to be equal to 1 we have the subgroup $N^- SL(2, \mathbb{C})$.

The factorization problem

$$\psi = \psi_-^{-1} \cdot \psi_+ , \tag{3.4}$$

where $\psi_- \in N^- SL(2, \mathbb{C})$ and $\psi_+ \in B^+ SL(2, \mathbb{C})$, for $\psi(\mathbf{t})$ is connected with the mKdV hierarchy. The element ψ_- can be parametrized by a function v, in such a way that ψ_- is a solution to the factorization problem if and only if v is a solution to the mKdV hierarchy. Therefore

$$\xi := d\psi_+ \cdot \psi_+^{-1} = P_+ \operatorname{Ad} \psi_- \left(\sum_{n \geqq 0} \lambda^n J(\lambda) dt_{2n+1} \right) \tag{3.5}$$

is the zero-curvature 1-form for the mKdV equation [7]. Here $\mathrm{id} = P_+ + P_-$ is the resolution of the identity related to the splitting

$$L\mathfrak{sl}(2, \mathbb{C}) = B^+ \mathfrak{sl}(2, \mathbb{C}) \oplus N^- \mathfrak{sl}(2, \mathbb{C}) .$$

Similarly, if we consider the factorization problem

$$\psi = \tilde{\psi}_-^{-1} \cdot \tilde{\psi}_+ ,$$

with $\tilde{\psi}_- \in B^- SL(2, \mathbb{C})$ and $\tilde{\psi}_+ \in N^+ SL(2, \mathbb{C})$, for $\psi(\mathbf{t})$ we find the potential mKdV hierarchy. Now, $\tilde{\psi}_-$ can be parametrized by a function w, such that $\tilde{\psi}_-$ is a solution to the factorization problem if and only if w is a solution to the potential mKdV hierarchy. Thus

$$\eta := d\tilde{\psi}_+ \cdot \tilde{\psi}_+^{-1} = \tilde{P}_+ \operatorname{Ad} \tilde{\psi}_- \left(\sum_{n \geqq 0} \lambda^n J(\lambda)\, dt_{2n+1} \right)$$

is the zero-curvature 1-form for the potential mKdV equation [7]. The resolution $\mathrm{id} = \tilde{P}_+ + \tilde{P}_-$ is associated with the decomposition

$$L\mathfrak{sl}(2, \mathbb{C}) = N^+ \mathfrak{sl}(2, \mathbb{C}) \oplus B^- \mathfrak{sl}(2, \mathbb{C}) .$$

One can conclude from these considerations that the projection of the commuting flows $\psi(\mathbf{t})$ on the periodic flag manifold [16, 20]

$$LSL(2, \mathbb{C})/B^+ SL(2, \mathbb{C}) \cong \mathrm{Fl}^{(2)} ,$$

can be described in terms of the mKdV hierarchy.

We must remark that g determines a point in the periodic flag manifold up to the gauge freedom $g \mapsto g \cdot h$, where $h \in B^+ SL(2, \mathbb{C})$. A solution of the mKdV hierarchy does not change when $g(\lambda) \mapsto \exp(\beta(\lambda) J(\lambda)) \cdot g(\lambda)$ if $\exp(\beta J) \in N^- SL(2, \mathbb{C})$. We can say that the moduli space for the KdV hierarchy contains the double coset space

$$\mathscr{M} := \Gamma_- \backslash LSL(2, \mathbb{C})/B^+ SL(2, \mathbb{C}) ,$$

where Γ_- is the abelian subgroup with Lie algebra $\mathbb{C}\{\lambda^n J(\lambda)\}_{n<0}$, [20].

The potential mKdV hierarchy describes the projection of these commuting flows over

$$LSL(2, \mathbb{C})/N^+ SL(2, \mathbb{C}) ,$$

a line bundle over the periodic flag manifold $\mathrm{Fl}^{(2)}$. Being the moduli space

$$\tilde{\mathscr{M}} := \Gamma_- \backslash LSL(2,\mathbb{C})/N^+SL(2,\mathbb{C}) .$$

Let us now try to find for which initial conditions g one gets self-similar solutions, i.e. points in these homogeneous manifolds that are connected to self-similar solutions of the mKdV hierarchy and to the potential mKdV hierarchy.

Recall that we have the derivation $\delta \in \mathrm{Der}\, B^+\mathfrak{sl}(2,\mathbb{C})$, $\mathrm{Der}\, N^+\mathfrak{sl}(2,\mathbb{C})$ defined in (2.9) and the vectors $M(\mathbf{t}) \in B^+\mathfrak{sl}(2,\mathbb{C})$, $\tilde{M}(\mathbf{t}) \in N^+\mathfrak{sl}(2,\mathbb{C})$ defined in (2.10). If we denote by

$$\theta(\lambda) := \sum_{n \geq 0} \theta_{2n+1}\lambda^n , \tag{3.6}$$

then it follows

Theorem 3.1.

1. *If the initial condition g satisfies the equation*

$$\delta g \cdot g^{-1} + \mathrm{Ad}\, gK = \theta J , \tag{3.7}$$

for some $K \in B^+\mathfrak{sl}(2,\mathbb{C})$, where θ, J are given by (3.6), (3.3), *then the corresponding solution to the* mKdV *hierarchy satisfies the string equation* (2.6), i.e. *Eq.* (2.8).

2. *If g satisfies the Eq.* (3.7) *for some $K \in N^+\mathfrak{sl}(2,\mathbb{C})$ then the associated solution w to the potential* mKdV *hierarchy is self-similar under the action of the vector field X defined in* (2.5) *and so $v = \partial_1 w$ is the solution to* (2.8) *with $c = 0$.*

Proof. We prove the first statement. For $\xi = d\psi_+ \cdot \psi_+^{-1}$ we observe that Eq. (2.11) holds if and only if

$$M = \delta\psi_+ \cdot \psi_+^{-1} + \mathrm{Ad}\,\psi_+ K , \tag{3.8}$$

for some $K \in B^+\mathfrak{sl}(2,\mathbb{C})$. This, together with the factorization problem (3.4), implies the relation

$$M = \delta\psi_- \cdot \psi_-^{-1} + \mathrm{Ad}\,\psi_-(\delta S \cdot S^{-1} + \mathrm{Ad}\, S(\delta g \cdot g^{-1} + \mathrm{Ad}\, gK)) .$$

Now, $M(\mathbf{t}) \in B^+\mathfrak{sl}(2,\mathbb{C})$, and Eq. (3.7) gives

$$M = P_+ \mathrm{Ad}\,\psi_-(\delta S \cdot S^{-1} + \mathrm{Ad}\, S(\theta J)) .$$

But, as can be easily shown,

$$M = P_+ \mathrm{Ad}\,\psi_- \left(\theta(\lambda) + \sigma \sum_{n \geq 0} (n + 1/2) t_{2n+1} \lambda^n\right) J(\lambda) .$$

Taking into account Eq. (3.5) we recover (2.10) and therefore the string equation is satisfied. The second statement can be proved as above but replacing $B^+SL(2,\mathbb{C})$ by $N^+SL(2,\mathbb{C})$. □

4. Description of the Moduli Space of the String Equations

Now, we shall give a description of the points in the periodic flag manifold corresponding to self-similar solutions of the mKdV hierarchy. The periodic flag manifold $\mathrm{Fl}^{(2)}$, [16, 20] is the set of pairs (V, W) of subspaces in the Hilbert space $\mathscr{H} = L^2(S^1, \mathbb{C})$ such that they belong to the Segal–Wilson Grassmannian [19] and satisfy the periodicity condition $\lambda^2 W \subset \lambda V \subset W$. In the Segal–Wilson framework only the family of solutions related through the Miura map to the Adler–Moser rational solutions of the KdV hierarchy [1] appears as self-similar solutions. A much larger family lies in the Sato extension of the periodic flag manifold, where $\mathscr{H} = \mathbb{C}[[\lambda^{-1}, \lambda]$ and the subspaces belong to the Sato Grassmannian [17]. Therefore, we shall consider the Sato periodic flag manifold $\mathrm{Fl}^{(2)}$. The statements of the previous section which are rigorous in the Segal–Wilson case, can be extended to the Sato frame if the formal groups $N^- SL(2, \mathbb{C})$, $B^- SL(2, \mathbb{C})$ are considered only when acting by its adjoint action or by gauge transformations in the formal Lie algebra $\mathfrak{sl}(2, \mathbb{C})[[\lambda^{-1}, \lambda]$. In this context Eqs. (2.11), (3.8) and (3.7) still hold.

To connect the results of the previous section with this description we write

$$g = \begin{pmatrix} \varphi_1 & \tilde{\varphi}_1 \\ \varphi_2 & \tilde{\varphi}_2 \end{pmatrix},$$

with $\varphi_1 \tilde{\varphi}_2 - \tilde{\varphi}_1 \varphi_2 = 1$, and introduce the notation

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \tilde{\Phi} = \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix}.$$

Define also the map [16, 19] $\Phi \mapsto \varphi := T\Phi$, where $(T\Phi)(\lambda) := \lambda \varphi_1(\lambda^2) + \varphi_2(\lambda^2)$.

Notice that for each equivalence class in $\mathscr{M}$ an element g can be taken such that $\ln g \in \mathbb{C}F \oplus \mathfrak{sl}(2, \mathbb{C})[[\lambda^{-1})$, and that any element in the coset $g \cdot B^+ \mathrm{SL}(2, \mathbb{C})$ gives the same point in the periodic flag manifold.

Since $S|_{t=0} = \mathrm{id}$, (3.2), it follows from (3.4), (3.1) that $\psi_+|_{t=0} = \mathrm{id}$ and Eq. (3.8) gives

$$K = M|_{t=0}.$$

But, from (2.10) we have

$$K = \langle \xi|_{t=0}, \vartheta \rangle,$$

where we have taken into account that (2.5) implies

$$X|_{t=0} = \vartheta.$$

From these considerations we conclude

Theorem 4.1. *The points* (V, W) *in the Sato periodic flag manifold* $\mathrm{Fl}^{(2)}$ *corresponding to self-similar solutions of the* mKdV *hierarchy are given by*

$$V = \mathbb{C}\{\lambda^{-1}\varphi, \lambda^{2n+1}\varphi, \lambda^{2n+1}\tilde{\varphi}\}_{n \geqq 0},$$

$$W = \mathbb{C}\{\lambda^{2n}\varphi, \lambda^{2n}\tilde{\varphi}\}_{n \geqq 0},$$

where φ *and* $\tilde{\varphi}$ *are the solutions of*

$$\left\{\frac{\sigma}{2}\left(\lambda \frac{d}{d\lambda} - \frac{1}{2}(1 + H)\right) - \lambda \theta(\lambda^2) + \sum_{n \geqq 0} \theta_{2n+1} \ell^t_{2n+1}|_{t=0}(\lambda^2)\right\} \begin{pmatrix} \varphi \\ \tilde{\varphi} \end{pmatrix} = 0,$$

having the asymptotic expansion

$$\begin{pmatrix} \varphi \\ \tilde{\varphi} \end{pmatrix} \sim \begin{pmatrix} \lambda + \varphi_{20} + \varphi_{11}\lambda^{-1} + \varphi_{21}\lambda^{-2} + \cdots \\ 1 + \tilde{\varphi}_{11}\lambda^{-1} + \tilde{\varphi}_{21}\lambda^{-2} + \cdots \end{pmatrix}, \quad \lambda \to \infty.$$

Here θ *and* ℓ_{2n+1} *are given by* (3.6), (2.3) *respectively.*

Observe that the subspaces V, W in the Sato Grassmannian are characterized by the periodicity condition

$$\lambda^2 W \subset \lambda V \subset W,$$

and by

$$\mathcal{A} V \subset \lambda W, \quad \mathcal{A} W \subset \lambda V,$$

where

$$\mathcal{A} := \frac{\sigma}{2}\lambda \frac{d}{d\lambda} - \lambda\theta(\lambda^2),$$

see [2]. Similarly, one can prove that

Theorem 4.2. *The points in the line bundle over the Sato's periodic flag manifold* $\mathrm{Fl}^{(2)}$

$$LSL(2,\mathbb{C})/N^+ SL(2,\mathbb{C}) \cong B^- SL(2,\mathbb{C}),$$

corresponding to self-similar solutions of the potential mKdV *hierarchy are given by the solutions* φ, $\tilde{\varphi}$ *of*

$$\left\{\frac{\sigma}{2}\left(\lambda\frac{d}{d\lambda} - \frac{1}{2}(1+H)\right) - \lambda\theta(\lambda^2) + \sum_{n\geqq 0} \theta_{2n+1}\tilde{\ell}^t_{2n+1}|_{\mathbf{t}=0}(\lambda^2)\right\}\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix} = 0,$$

having the asymptotic expansion

$$\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix} \sim \begin{pmatrix}\lambda + \varphi_{20} + \varphi_{11}\lambda^{-1} + \varphi_{21}\lambda^{-2} + \cdots\\ \tilde{\varphi}_{20} + \tilde{\varphi}_{11}\lambda^{-1} + \tilde{\varphi}_{21}\lambda^{-2} + \cdots\end{pmatrix}, \quad \lambda \to \infty.$$

Here θ *and* $\tilde{\ell}_{2n+1}$ *are given by* (3.6), (2.4) *respectively.*

Given σ one can consider $\theta(\lambda)$ as a polynomial of degree N, then the functions φ, $\tilde{\varphi}$ defining the point in $\mathrm{Fl}^{(2)}$ associated to a self-similar solution of the mKdV hierarchy depends on the parameters

$$\{v_0, R_{n,0}, \dot{R}_{n,0}, \ddot{R}_{n,0}\}_{n=1}^{N},$$

where we denote by $\dot{f} = \partial_1 f$, $f_0 = f(\mathbf{t}=0)$. These constants are not independent, in fact they fulfill the Gel'fand–Dikii relations [6],

$$R_{n+1,0} = 2\sum_{m=0}^{n-1} R_{m,0}\ddot{R}_{n-m,0} - \sum_{m=1}^{n-1} \dot{R}_{m,0}\dot{R}_{n-m,0} + 4u_0 \sum_{m=0}^{n} R_{m,0}R_{n-m,0}$$
$$- 4\sum_{m=1}^{n} R_{m,0}R_{n-m+1,0},$$

and the string equation gives the additional constraint

$$Av_0^2 + Bv_0 + C = 0,$$

where

$$A := 2\sum_{n\geqq 0} \theta_{2n+1} R_{n,0},$$

$$B := \frac{\sigma}{2} + 2\sum_{n\geqq 0} \theta_{2n+1}\dot{R}_{n,0},$$

$$C := \sum_{n\geqq 0} \theta_{2n+1}\ddot{R}_{n,0} + u_0 A.$$

Here we have used the Miura map (2.1) connecting u with v. Therefore, since these are all the constraints that must be satisfied by the constants we conclude that our solution is parametrized by a $2N+1$-dimensional algebraic variety $\Sigma_\theta \subset \mathbb{C}^{3M+1}$. For each point in this variety we have a subspace in the Sato periodic flag manifold $\mathrm{Fl}^{(2)}$, this map gives an inclusion $\Sigma_\theta \subsetneq \mathrm{Fl}^{(2)}$. This $2N+1$-dimensional surface intersects the Segal–Wilson periodic flag manifold $\mathrm{Fl}_0^{(2)}$ in a discrete set, that can be labeled by $\mathbb{N}$, in fact this intersection set corresponds to an Adler–Moser rational solution to the KdV hierarchy.

Observe that

$$c + \frac{\sigma}{4} = Av_0 + \frac{B}{2},$$

and when $c = 0$ we have the additional constraint

$$\frac{\sigma}{4} = Av_0 + \frac{B}{2}.$$

This must be satisfied if we are looking for self-similar solutions to the potential mKdV hierarchy. The functions φ, $\tilde{\varphi}$ giving these self-similar solutions depends on the above parameters and on w_0, but this parameter is irrelevant; if w is self-similar, then so is any $w + \text{const}$. We can therefore fix the value of w_0. This analysis implies that the moduli space is $2N$ dimensional.

The correct number of parameters can be found directly from the string equation. Supposing that the solution is defined at the origin we find solutions to the string equation when $0 = t_3 = t_5 = \cdots$. The number of parameters needed to describe them is the dimension of the moduli. The solutions are obtained from these initial data by applying the commuting flows on the integrable hierarchy. Also it can be obtained by an analysis of Stokes' parameters associated to the string equation, this is the approach of [12]. Nevertheless, in our description the dimension of the moduli is obtained as the number of parameters necessary to describe the points of the homogeneous space associated with self-similar solutions. Therefore they have a clear geometrical interpretation.

5. Connection with the Moduli Space for the KdV Hierarchy

The discussion in the previous section provides us with a detailed account of the moduli space for the string equations of UMM with border terms and also when these border terms are absent. In this section we shall connect this description with that given in [8] for the moduli space of self-similar solutions to the potential KdV hierarchy. The string equation in this case is associated with the so-called 2D stable quantum gravity [4]. For the potential KdV hierarchy the Birkhoff factorization problem is essential. In fact the Birkhoff factorization

$$\psi = \hat{\psi}_-^{-1} \cdot \hat{\psi}_+ ,$$

where $\hat{\psi}_- \in L_1^- SL(2, \mathbb{C})$ and $\hat{\psi}_+ \in L^+ SL(2, \mathbb{C})$, for the commuting flows $\psi(\mathbf{t})$, can be solved in terms of a function p that parametrizes $\hat{\psi}_-$ and satisfies the potential KdV hierarchy,

$$\partial_{2n+1} p = -2R_{n+1}[u] ,$$

where $u = -2\partial_1 p$ is a solution of the KdV hierarchy, see [19, 7, 8]. This means that the projection of the commuting flows $\psi(\mathbf{t})$ in the Grassmannian manifold

$$\mathrm{Gr}^{(2)} \cong LSL(2,\mathbb{C})/L^+SL(2,\mathbb{C})$$

can be described in terms of the potential KdV hierarchy.

One can write

$$\psi_- = \exp(aF)\cdot\hat{\psi}_- ,$$

where

$$\exp(aF) = \lim_{\lambda\to\infty} \psi_- .$$

Hence, $\psi_+ = \exp(aF)\cdot\hat{\psi}_+$ and so [7]

$$\partial_1(a+p) = (a-p)^2, \quad v = a - p ,$$

or

$$a = v + p, \quad u := -2\partial_1 p = \partial_1 v - v^2 .$$

The initial condition for the mKdV hierarchy can be chosen such that

$$g = \hat{g}\cdot\exp(-a_0 F)\in N^- SL(2,\mathbb{C}) ,$$

where $\hat{g}\in L_1^- SL(2,\mathbb{C})$ is the initial condition for the corresponding solution to the potential KdV hierarchy. Thus, if φ, $\tilde{\varphi}$ are associated with g, and give the point (V, W) in $\mathrm{Fl}^{(2)}$ corresponding to a solution v of the mKdV hierarchy, and ϕ, $\tilde{\phi}$ are associated with $\hat{g}$, and thereby define a subspace $\hat{W} := \mathbb{C}\{\lambda^{2n}\phi, \lambda^{2n}\tilde{\phi}\}_{n\geqq 0}$ in $\mathrm{Gr}^{(2)}$ corresponding to a solution $u = \partial_1 v - v^2$, one has

$$\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix} = \begin{pmatrix}1 & -(p_0+v_0)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\phi\\ \tilde{\phi}\end{pmatrix} . \tag{5.1}$$

Therefore, given a solution p to the potential KdV hierarchy there is a one-dimensional space of solutions of the mKdV hierarchy that through the Miura map goes to $u = -2\partial_1 p$. A possible parameter for this family is the initial value v_0. This has a clear geometrical interpretation [20]. Observe that $W = \hat{W}$, therefore we have the projection

$$\begin{aligned}\pi:\quad \mathrm{Fl}^{(2)} &\to \mathrm{Gr}^{(2)}\\ (V, W) &\mapsto W .\end{aligned}$$

The periodic flag manifold $\mathrm{Fl}^{(2)}$ is fibered over the Grassmannian $\mathrm{Gr}^{(2)}$, the fiber being a copy of $\mathbb{C}P^1$, [16]. The fiber $\pi^{-1}(W)$ can be recovered from Eq. (5.1). The projection π can be interpreted as the Miura transformation [20]. Schematically this can be encoded in the Wilson commutative diagram

$$\begin{array}{ccc}\mathrm{Fl}^{(2)} & \to & \text{solutions of the mKdV hierarchy}\\ \pi\downarrow & & \downarrow\text{Miura map}\\ \mathrm{Gr}^{(2)} & \to & \text{solutions of KdV hierarchy .}\end{array}$$

For a given self-similar solution u of the KdV hierarchy we have a one parameter family of solutions to the mKdV hierarchy. The solution u fixes the Gel'fand–Dikii potentials. Thus if we look for a self-similar v in this family the string equation selects two possible values $v_{0,\pm}$. Hence, there are only two points (associated to self-similar solutions $v_\pm$) in the fiber corresponding to u. Let $c_\pm$ be the value of c for $v_\pm$, then

$$c_+ + c_- = -\frac{\sigma}{2},$$

and

$$c_\pm = \frac{1}{2}(-\sigma \pm \Delta),$$

where

$$\Delta = \sqrt{B^2 - 4AC}$$

is the discriminant of the equation for v_0. This Δ is essentially the parameter Γ of the first reference of [5].

Suppose as before that θ is a polynomial of degree N, then as was proof in [8] the moduli space for the self-similar to the KdV hierarchy is a $2N + 1$-dimensional surface in $\mathrm{Gr}^{(2)}$, and from the above discussion we conclude that the moduli space for the self-similar solutions of the mKdV hierarchy is double covering of this surface, see [2]. For the mKdV case we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{mKdV}} & \rightarrow & \text{solutions of the mKdV hierarchy and string equation} \\ \mathbb{Z}_2 \downarrow & & \downarrow \text{Miura map} \\ \mathcal{M}_{\mathrm{KdV}} & \rightarrow & \text{solutions of the KdV hierarchy and string equation}\,, \end{array}$$

where $\mathcal{M}_{\mathrm{mKdV}} \subset \mathrm{Fl}^{(2)}$ and $\mathcal{M}_{\mathrm{KdV}} \subset \mathrm{Gr}^{(2)}$ denote the moduli spaces for the self-similar solutions of the mKdV and KdV hierarchies, respectively.

For the potential mKdV hierarchy, the situation is rather different. The homogeneous space $LSL(2,\mathbb{C})/N^+SL(2,\mathbb{C})$ is a line bundle over $\mathrm{Fl}^{(2)}$. This fibering is a consequence of the following fact: given a solution w of the potential mKdV hierarchy any w + constant is a solution as well. Given the initial condition $\underline{g} \in B^- SL(2,\mathbb{C})$, one has the factorization

$$\underline{g} = g \cdot \exp(-w_0 H),$$

where $g \in N^- SL(2,\mathbb{C})$ is the initial condition fixing the solution $v = \partial_1 w$ of the mKdV hierarchy, and

$$\lim_{\lambda \to \infty} \underline{g} = \exp(-(p_0 + v_0)F) \cdot \exp(-w_0 H).$$

Given a self-similar solution w, any solution in the corresponding fiber is also self-similar. Thus, we can look to the corresponding point in the base manifold $\mathrm{Fl}^{(2)}$, that is to the $v = \partial_1 w$ self-similar solution of the mKdV hierarchy. In this way the periodic flag manifold contains the moduli space of self-similar solutions of the potential mKdV hierarchy. But now we have the constraint $c = 0$. In fact, we have a subset of codimension one in the $2N + 1$-dimensional moduli space for the self-similar solutions of the potential KdV hierarchy which is in a one-to-one correspondence to the self-similar solutions of the potential mKdV hierarchy. Hence, the moduli space is a $2N$-dimensional surface in $\mathrm{Fl}^{(2)}$. Summing, when $c = 0$ not only the two-folding disappears but also not every self-similar solution of the potential KdV hierarchy is connected to a self-similar solution to the potential mKdV hierarchy.

In physical terms this means that stable 2D quantum gravity [5, 4] (self-similar solutions of the potential KdV hierarchy) is covered twice by the double scaling limit of the UMM with boundary terms [11], see [5]. But there is only a subset of stable 2D quantum gravity corresponding to the double scaling limit of the UMM (no border terms) [15].

Suppose that we write $\theta_{2n+3} = \hat{\theta}_{2n+1}$, where we choose $\theta_1 = 0$. Then a possible solution to the string equation for self-similar solutions of the mKdV hierarchy is a v that satisfies

$$\sum_{n \geqq 0} \hat{\theta}_{2n+1} R_{n+1} + \mathscr{R} = 0 .$$

The corresponding u is a solution to the string equation of the double scaling limit of the HMM (translations and Galilean self-similarity in the potential KdV hierarchy). This gives a connection between the HMM and the UMM with border terms. Notice that $c = -\sigma/4$, and therefore the corresponding w is not self-similar. So the mentioned connection only exists when the border terms are present. Thus the HMM is not connected in this way with the UMM.

As an example we can analyse the case $\theta(\lambda) = -1$, $\sigma = 1$. In [8] it was found that

$$\phi = \lambda \left(\frac{1}{2} \frac{d\tilde{\phi}}{d\lambda} + \tilde{\phi} \right)$$

and

$$\tilde{\phi}(\lambda) \sim i \sqrt{\frac{4\lambda}{\pi}} e^{-2\lambda} K_\nu(-2\lambda) \sim \sum_{n \geqq 0} (-1)^n \frac{\Gamma(\nu + n + 1/2)}{4^n n! \Gamma(\nu - n + 1/2)} \lambda^{-n}, \quad \lambda \to \infty ,$$

where

$$\nu = \frac{\sqrt{1 - 16u_0}}{2} ,$$

and K_ν is the Macdonald's function [13]. We know that

$$\varphi = \phi - (p_0 + v_0)\tilde{\phi}, \quad \tilde{\varphi} = \tilde{\phi} .$$

The string equation for the potential KdV hierarchy implies

$$p_0 = -u_0 ,$$

and the string equation for the mKdV hierarchy gives

$$v_{0,\pm} = -\frac{1}{4}(-1 \pm \sqrt{1 - 16u_0}) .$$

So, for a given u, generically we have two points $(V_\pm, W)$ in the periodic flag manifold. Observe that when $u_0 = 1/16$ there is only one point $v_0 = 1/4$ that is a branch point for the double covering. Now $c = v_0$. These solutions belong to the Segal–Wilson periodic flag manifold if and only if

$$u_0 = -\frac{m(m+1)}{4}, \quad m \in \mathbb{N} \cup \{0\} , \tag{5.2}$$

which implies

$$v_{0,+} = -\frac{m}{2}, \quad v_{0,-} = \frac{m+1}{2} .$$

When (5.2) is satisfied we are dealing with the rational solutions of the mKdV hierarchy, [1], $v_+ = v_m$ and $v_- = -v_{m+1}$, where v_m is the solution of the mKdV hierarchy that for $\mathbf{t} = \{t_1, 0, 0, \ldots\}$ is of the form $v_m = m/(t_1 - 2)$. Both solutions are mapped through the Miura transformation into the rational solution of the KdV hierarchy that for $\mathbf{t} = \{t_1, 0, 0, \ldots\}$ is of the form $u = -m(m+1)/(t_1 - 2)^2$ (and $p = -m(m+1)/2(t_1 - 2)$). These are the well known rational solutions of the KdV hierarchy that vanish at $t_1 = \infty$, analysed by Adler and Moser [1]. For $m = 0$, and $u = 0$ we have $v_+ = 0$ and $v_- = -1/(t_1 - 2)$; for $m = 1$, and $u = -2/(t_1 - 2)^2$ one has $v_+ = 1/(t_1 - 2)$ and $v_- = (t_3 - 2(t_1 - 2)^3)/((t_1 - 2)((t_1 - 2)^3 + t_3))$. Observe that u only depends on t_1 and that v_- depends also upon t_3.

For an arbitrary u_0 we have two points in the Sato periodic flag manifold, so there is a one-dimensional complex curve in this space giving scaling self-similar solutions.

Observe that if (5.2) is satisfied then $v = m + 1/2$, and $\tilde{\varphi}$ is the following polynomial in λ^{-1}:

$$\tilde{\varphi}(\lambda) = \lambda^{m+1} e^{-2\lambda} \left(\frac{1}{2\lambda} \frac{d}{d\lambda} \right)^{m+1} e^{2\lambda} .$$

If we look for self-similar solutions of the potential mKdV hierarchy we need $c = 0$, hence $v_0 = 0$ and $u_0 = 0$, which gives $v = 0$ and $w = \text{cte}$. In this case the solution is unique and trivial.

6. *Acknowledgements.* One of us (MM) would like to thank Prof. Francisco Guil for initial collaboration and PG would like to thank Prof. Pierre van Moerbeke for enlightening discussions. We are also grateful to Dr. Sacha Sardo Infirri for stimulating discussions.

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Communicated by R.H. Dijkgraaf