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Superderivations of C^*-Algebras Implemented by Symmetric Operators

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Abstract: The paper studies unbounded symmetric and dissipative implementations (S, G) of *-superderivations δ of C^*-algebras $\mathfrak{A}$. It associates with them representations π_S^δ of the domains $D(\delta)$ of δ on the deficiency spaces $N(S)$ of the symmetric operators S. A link is obtained between the deficiency indices $n_\pm(S)$ of S and the dimensions of irreducible representations of $\mathfrak{A}$. For the case when (S, G) is a maximal implementation and $\max(n_\pm(S)) < \infty$, some conditions are given for the representation π_s^δ to be semisimple and to extend to a bounded representation of $\mathfrak{A}$.

1. Introduction

Let $\mathfrak{A}$ be a C^*-algebra and ϱ be a *-representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$. Let δ be a linear closed mapping from a dense *-subalgebra $D(\delta)$ of $\mathfrak{A}$ into the algebra $B(\mathfrak{H})$ of all bounded operators on $\mathfrak{H}$ such that, for $A \in D(\delta)$,

(i) $\delta(AB) = \delta(A)\,\varrho(B) + \varrho(\varphi(A))\,\delta(B)$,

(ii) $\delta(\varphi(A)^*) = \delta(A)^*$,

where φ is an automorphism of $D(\delta)$. Then δ a closed *-*superderivation of* $\mathfrak{A}$ *relative to the pair* (ϱ, φ). A pair (S, G), where S is a densely defined closed operator on $\mathfrak{H}$, S^* is its adjoint and G is a bounded operator on $\mathfrak{H}$ such that $G^{-1} \in B(\mathfrak{H})$, implements δ if, for $A \in D(\delta)$,

$$\varrho(\varphi(A)) = G^{-1}\varrho(A)\,G\,, \tag{1}$$

$$GD(S) = D(S) \quad \text{and} \quad GD(S^*) = D(S^*)\,, \tag{2}$$

$$\varrho(A)\,D(S) \subseteq D(S) \quad \text{and} \quad \delta(A)|_{D(S)} = i(S\varrho(A) - G^{-1}\varrho(A)GS)|_{D(S)}\,. \tag{3}$$

If a pair (T, G) also implements δ and T extends S, then (T, G) is a δ-extension of (S, G). If S is symmetric and G is selfadjoint, (S, G) is a symmetric implementation of δ. If (S, G) has no symmetric δ-extensions, it is a maximal symmetric implementation of δ.

Remark. (i) If $\varphi = \text{id}$ and $G = \mathbf{1}_{\mathfrak{H}}$, then δ is a *-derivation of $\mathfrak{A}$ into $B(\mathfrak{H})$ relative to ϱ and S is an implementation of δ.
(ii) Changing condition (ii) in the above definition to the condition:

$$\delta(\varphi(A)^*) = e^{i\pi\lambda}\delta(A)^*\,, \qquad A \in D(\delta)\,, \qquad 0 \le \lambda < 2\,,$$

we obtain λ-symmetric superderivations of $\mathfrak{A}$. If, however, δ is λ-symmetric, then $\tau(A) = e^{-i\pi\lambda/2}\delta(A)$ is a *-superderivation of $\mathfrak{A}$. Moreover, if (S, G) implements δ, then $(e^{-i\pi\lambda/2}S, G)$ implements τ.

Davies and Lindsay [2] introduced 1-symmetric superderivations δ, i.e.,

$$\delta(\varphi(A)^*) = e^{i\pi}\delta(S)^* = -\,\delta(A)^*\,, \qquad A \in D(\delta)\,,$$

for the case when $\varrho = \text{id}\,.$, $\varphi^2 = \text{id}\,.$ and $\delta(\varphi(A)) = -\,\varphi(\delta(A))$. By establishing a Dirichlet property for a class of superderivations they were able to apply the theory of non-commutative symmetric Markov semigroups to the construction of dynamical semigroups on $\mathbb{Z}_2$-graded algebras of quantum observables.

This paper studies unbounded symmetric and dissipative implementations of *-superderivations. For derivations this was done in [3–5]. As in the case of derivations, with every symmetric implementation (S, G) of δ we associate a representation π_S^δ of $D(\delta)$ on the deficiency space $N(S)$ of the symmetric operator S. Making use of the fact that $D(\delta)$ is a Q-subalgebra of $\mathfrak{A}$ [6], Theorem 3 obtains the link between the deficiency indices of S and the dimensions of irreducible finite-dimensional representations of $\mathfrak{A}$.

The space $N(S)$ has a natural indefinite form which converts it into a Krein space. However, unlike the case of derivations, π_S^δ is not symmetric with respect to this form. To make up for this shortcoming, three new indefinite forms on $N(S)$ are introduced with respect to which π_S^δ is symmetric. Although the geometry of $N(S)$ supplied with these forms becomes even more complicated than the geometry of $N(S)$ as a Krein space, they play a crucial role in the proof of the fact that δ always has a maximal symmetric implementation. Theorem 5 also uses them to show that there is a one-to-one correspondence between δ-extensions of (S, G) and invariant subspaces in $N(S)$ neutral with respect to the forms.

It was established in [3] that if δ is a derivation, S is a maximal implementation of δ and $\max(n_\pm(S)) < \infty$, then the representation π_S^δ of $D(\delta)$ is semisimple and extends to a bounded representation of $\mathfrak{A}$ on $N(S)$. Under some conditions on the implementations, Theorems 6 and 7 prove this result for the case when δ is a superderivation. Section 4 considers examples of symmetric implementations for which Theorems 6 and 7 hold.

2. Maximal Symmetric Implementations of Superderivations

Let δ be a closed *-superderivation of $\mathfrak{A}$. The algebra $D(\delta)$ is a Banach *–algebra with respect to the norm $\|A\|_\delta = \|A\| + \|\delta(A)\|$. Let (S, G) be a symmetric implementation of δ. Set

$$\sigma(A) = G\delta(A)\,, \qquad \tau(A) = \delta(A^*)^*G \quad \text{and} \quad \Delta(A) = \tfrac{1}{2}\left(\sigma(A) + \tau(A)\right), \qquad A \in D(\delta)\,.$$

From (1) it follows that σ, τ and Δ are closed derivations of $\mathfrak{A}$. We also have that

$$D(\sigma) = D(\tau) = D(\delta)\,, \qquad \sigma(A^*)^* = \delta(A^*)^*G = \tau(A) \quad \text{and} \quad \Delta(A^*) = \Delta(A)^*\,,$$

so that Δ is a *-derivation. Set

$$U = GS\,, \qquad V = SG \quad \text{and} \quad W = \tfrac{1}{2}(U+V)\,.$$

Then U and V are closed operators, W is a symmetric but not necessarily a closed operator and

$$D(U) = D(V) = D(W) = D(S)\,, \qquad D(U^*) = D(V^*) = D(S^*) \subseteq D(W^*)$$

and

$$U^* = S^*G\,, \qquad V^* = GS^*\,, \qquad W^*|_{D(S^*)} = \tfrac{1}{2}(U^* + V^*)\,.$$

Therefore

$$\begin{aligned}\sigma(A)|_{D(S)} &= i(V\varrho(A) - \varrho(A)V)|_{D(S)}\,,\\ \tau(A)|_{D(S)} &= i(U\varrho(A) - \varrho(A)U)|_{D(S)}\,,\\ \Delta(A)|_{D(S)} &= i(W\varrho(A) - \varrho(A)W)|_{D(S)}\,.\end{aligned}$$

Since W is symmetric, Δ is closable. From this and from Theorem 5 [6] we obtain the following lemma.

Lemma 1. *Let δ be a closed *-superderivation of a unital C^*-algebra $\mathfrak{A}$.*
(i) *The *-derivation $\Delta = \sigma + \tau$ is closable and implemented by the symmetric operator W.*
(ii) [6] *$D(\delta)$ is a Q-subalgebra of $\mathfrak{A}$,* i.e., $\mathbf{1} \in D(\delta)$ *and* $Sp_{\mathfrak{A}}(A) = Sp_{D(\delta)}(A)$, $A \in D(\delta)$.

For any operator B on $\mathfrak{H}$ and linear manifold $L \subseteq \mathfrak{H}$, $BL = \{Bx : x \in L\}$.

Lemma 2. *Let (S,G) be a symmetric implementation of a closed *-superderivation δ of a C^*-algebra $\mathfrak{A}$ relative to (ϱ,φ). Then, for $A \in D(\delta)$,*

$$\varrho(A)D(S^*) \subseteq D(S^*)\,, \qquad \delta(A)^*|_{D(S^*)} = i(S^*G\varrho(A^*)G^{-1} - \varrho(A^*)S^*)|_{D(S^*)}\,,$$
$$\delta(A)|_{D(S^*)} = i(S^*\varrho(A) - G^{-1}\varrho(A)GS^*)|_{D(S^*)}\,.$$

Proof Let $x \in D(S)$ and $y \in D(S^*)$. Then, for $A \in D(\delta)$, by (3),

$$\begin{aligned}(Sx, G\varrho(A^*)G^{-1}y) &= (G^{-1}\varrho(A)GSx, y) = ((i\delta(A) + S\varrho(A))x, y)\\ &= (x, (-i\delta(A)^* + \varrho(A^*)S^*)y)\,.\end{aligned}$$

Therefore $G\varrho(A^*)G^{-1}D(S^*) \subseteq D(S^*)$ and

$$\delta(A)^*|_{D(S^*)} = i(S^*G\varrho(A^*)G^{-1} - \varrho(A^*)S^*)|_{D(S^*)}\,.$$

From (1) it follows that

$$G\varrho(A^*)G^{-1} = \varrho(\varphi^{-1}(A^*))\,. \tag{4}$$

Hence, since φ is an automorphism of $D(\delta)$ and $D(\delta)$ is a *-algebra, it follows from (4) that, for all $A \in D(\delta)$,

$$\varrho(A)D(S^*) \subseteq D(S^*) \quad \text{and} \quad \delta(A)^*|_{D(S^*)} = i(S^*\varrho(\varphi^{-1}(A^*)) - \varrho(A^*)S^*)|_{D(S^*)}\,.$$

Setting $B = \varphi^{-1}(A^*)$ and making use of (1), we obtain that

$$\delta(\varphi(B)^*)^*|_{D(S^*)} = i(S^*\varrho(B) - G^{-1}\varrho(B)GS^*)|_{D(S^*)}\,.$$

Since δ is a *-superderivation, $\delta(\varphi(B)^*)^* = \delta(B)$. Hence

$$\delta(B)|_{D(S^*)} = i(S^*\varrho(B) - G^{-1}\varrho(B)GS^*)|_{D(S^*)} . \qquad \square$$

Let (S,G) be a symmetric implementation of a *-superderivation δ. Since $D(S)$ and $D(S^*)$ are invariant for all operators $\varrho(A)$, $A \in D(\delta)$, and can be considered as Hilbert spaces (see below), we can define a representation π_S^δ of $D(\delta)$ on the deficiency space $N(S)$ of S by the formula:

$$\pi_S^\delta(A)x = Q\varrho(A)x\,, \qquad x \in N(S)\,, \tag{5}$$

where Q is the orthoprojection on $N(S)$.

The following result is similar to the result of Theorem 3.11(i) [3] about the deficiency indices of symmetric implementations of *-derivations.

Theorem 3. *Let δ be a *-superderivation of a unital C^*-algebra $\mathfrak{A}$ and (S,G) be a symmetric implementation of δ If* $\max n_\pm(S) < \infty$, *then there are irreducible representations $\{\varrho_i\}_{i=1}^m$ of $\mathfrak{A}$ such that $n_+(S) + n_-(S) = \sum_{i=1}^m \dim \varrho_i$. If $\mathfrak{A}$ has no finite-dimensional representations, then either S is selfadjoint or* $\max n_\pm(S) = \infty$.

Proof. Since $\max n_\pm(S) < \infty$, $N(S)$ is finite-dimensional. Using the standard techniques of linear algebra, we obtain that there is a finite nest $\{0\} = L_0 \subset L_1 \subset \ldots \subset L_m = N(S)$ of subspaces invariant for π_s^δ such that the representations π_i of the algebra $D(\delta)$ in the quotient subspaces L_i/L_{i-1} are irreducible and $\dim N(S) = n_+(S) + n_-(S) = \sum_{i=1}^m \dim \pi_i$. From Lemma 1(ii) it follows that $\mathbf{1} \in D(\delta)$ and that $D(\delta)$ is a Q-subalgebra of $\mathfrak{A}$. Therefore all π_i are non-trivial and it follows from Theorem 2.2 [3] (cf. [6]) that every π_i extends to an irreducible representation ϱ_i of $\mathfrak{A}$ on L_i/L_{i-1}. $\square$

Remark In Lemma 2 and Theorem 3 the conditon (2) was not used.

We shall now consider briefly the link between symmetric implementations of *-superderivations of C^*-algebras and J-symmetric representations of *-algebras on Krein spaces. Let H be a Hilbert space with a scalar product (x,y) and $H = H_- \oplus H_+$ be an orthogonal decomposition of H. The involution $J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ defines an indefinite form $[x,y] = (Jx,y)$ on H. With this form H is called a *Krein space*. Let $k_d = \dim H_d$, $d = \pm$. If $k = \min(k_-, k_+) < \infty$, H is called a *Π_k-space*.

A subspace L in H is called *neutral* if $[x,y] = 0$, $x,y \in L$. The subspace $L^{[\perp]} = \{y \in H : [x,y] = 0, x \in L\}$ is called the *J-orthogonal complement of* L. If L is uniformly definite, i.e., there is $r > 0$ such that $|[x,x]| \geq r(x,x)$ for $x \in L$, then H can be decomposed in the direct and J-orthogonal sum

$$H = L[+]L^{[\perp]}\,.$$

A representation π of a *-algebra $\mathscr{A}$ on a Krein space H is called

- *J-symmetric* if $[\pi(A)x,y] = [x,\pi(A^*)y]$, $x,y \in H$, $A \in \mathscr{A}$;
- *non-degenerate* if π has no neutral invariant subspaces.

If a subspace L is invariant for π, $L^{[\perp]}$ is also invariant for π.

Let S be a symmetric operator on a Hilbert space $\mathfrak{H}$. The scalar product

$$\langle x,y\rangle = (x,y) + (S^*x, S^*y)\,, \qquad x,y \in D(S^*)\,,$$

converts $D(S^*)$ into a Hilbert space with the norm

$$|x| = (\|x\|^2 + \|S^*x\|^2)^{1/2}$$

and

$$D(S^*) = D(S)\langle+\rangle N_+(S)\langle+\rangle N_-(S)$$

is the orthogonal sum of $D(S)$ and the *deficiency spaces*

$$N_\pm(S) = \{x \in D(S^*) : S^*x = \pm ix\}$$

of S. Let $N(S) = N_+(S)\langle+\rangle N_-(S)$ and let Q be the projection on $N(S)$ and Q_+ be the projection on $N_+(S)$ in $D(S^*)$. Set $J = 2Q_+ - Q$. Then J is an involution on $N(S)$, i.e., $J^* = J$ and $J^2 = \mathbf{1}_{N(S)}$.

Set

$$\{x, y\} = i((x, S^*y) - (S^*x, y)), \qquad x, y \in D(S^*).$$

Then $\{\,,\}$ is an indefinite form on $D(S^*)$ and

$$\{x, y\} = \overline{\{y, x\}}, \qquad x, y \in D(S^*), \tag{6}$$

$$\{x, y\} = 0 \quad \text{if} \quad x \in D(S) \quad \text{or if} \quad y \in D(S), \tag{7}$$

$$\{x, y\} = 0 \quad \text{if} \quad x \in N_d(S) \quad \text{and} \quad y \in N_{-d}(S), \qquad d = \pm, \tag{8}$$

$$\{x, y\} = 2d(x, y) = d\langle x, y\rangle \quad \text{if} \quad x, y \in N_d(S), \qquad d = \pm. \tag{9}$$

We denote the restriction of $\{\,,\}$ to $N(S)$ by $[\,,]$, i.e.

$$[x, y] = \{x, y\}, \qquad x, y \in N(S).$$

It follows from (8) and (9) that

$$[x, y] = \langle Jx, y\rangle, \qquad x, y \in N(S), \tag{10}$$

so that $N(S)$ is a Krein space and $N(S) = N_+(S) + N_-(S)$ is the orthogonal and J-orthogonal sum. The numbers $n_\pm(S) = \dim N_\pm(S)$ are the *deficiency indices* of S. If $k = \min n_\pm(S) < \infty$, then $N(S)$ is a Π_k-space.

From (2) we have that $G^{-1}D(S) = D(S)$ and $G^{-1}D(S^*) = D(S^*)$. Set

$$G_S = G|_{D(S^*)}.$$

Then $G_S^{-1} = G^{-1}|_{D(S^*)}$ is the inverse of G_S. From (2) it follows that

$$QG_SQ = QG_S \quad \text{and} \quad QG_S^{-1}Q = QG_S^{-1}. \tag{11}$$

We define now new indefinite forms on $D(S^*)$:

$$\begin{aligned}
\{x, y\}_l &= \{G_Sx, y\} = i((Gx, S^*y) - (S^*Gx, y)) \\
&= i((x, V^*y) - (U^*x, y)), \\
\{x, y\}_r &= \{x, G_Sy\} = i((x, S^*Gy) - (S^*x, Gy)) \\
&= i((x, U^*y) - (V^*x, y)), \\
\{x, y\}_t &= \tfrac{1}{2}(\{x, y\}_l + \{x, y\}_r = \tfrac{1}{2}(\{G_Sx, y\} + \{x, G_Sy\}) \\
&= i((x, W^*y) - (W^*x, y)).
\end{aligned} \tag{12}$$

Since $GD(S) = D(S)$, it follows from (7) that

$$\{x,y\}_l = \{x,y\}_r = \{x,y\}_t = 0 \quad \text{if} \quad x \in D(S) \quad \text{or if} \quad y \in D(S). \qquad (13)$$

From (6) and (12) it follows that

$$\{y,x\}_l = \{G_S y, x\} = \overline{\{x, G_S y\}} = \overline{\{x,y\}_r} \quad \text{and} \quad \{y,x\}_t = \overline{\{x,y\}_t}. \qquad (14)$$

Set

$$F = QG_SQ \quad \text{and} \quad [x,y]_d = \{x,y\}_d, \qquad x,y \in N(S) \quad \text{and} \quad d = l,r,t.$$

Then F is an operator on $N(S)$ and, by (11), $F^{-1} = QG_S^{-1}Q$ is the inverse of F.

For every operator B on $N(S)$ we denote by B^+ its adjoint with respect to $\langle\, , \rangle$ and B^J its J-adjoint:

$$[Bx,y] = [x, B^J y], \qquad x,y \in N(S), \quad \text{i.e.,} \quad B^J = (JBJ)^+ = JB^+J, \qquad (15)$$

since $J^+ = J$. We have that $(B^J)^J = B$.

Lemma 4. (i) *The operators G_S and G_S^{-1} on $D(S^*)$ are bounded with respect to the norm $|\ |$, so that the operators F and F^{-1} are bounded, and $\{G_s x, y\}_r = \{x, G_S y\}_l$.*
(ii) *Set $R = \frac{1}{2}(F + F^J)$. For $x,y \in N(S)$,*

$$[x,y]_l = [Fx,y] = \langle JFx,y\rangle, \qquad [x,y]_r = [x,Fy] = \langle F^+Jx,y\rangle, \qquad (16)$$

$$[x,y]_t = \tfrac{1}{2}([Fx,y] + [x,Fy]) = [Rx,y], \qquad (17)$$

$$[Fx,y]_r = [x,Fy]_l, \qquad [F^Jx,y]_r = [x,Fy]_r; \qquad (18)$$

$$[Fx,y]_l = [x,F^Jy]_l, \qquad [F^Jx,y]_l = [x,F^Jy]_r; \qquad (19)$$

$$[Rx,y]_t = [x,Ry]_t, \qquad [Fx,y]_t = [x,Ry]_l, \qquad [F^Jx,y]_t = [x,Ry]_r. \quad (20)$$

The forms $[\,,]_l$ and $[\,,]_r$ are not degenerate on $N(S)$.
(iii) *If W is closed and $D(W^*) = D(S^*)$, then the form $[\,,]_t$ is not degenerate on $N(S)$ If, in addition, $\max(n_\pm(S)) < \infty$, then R has the inverse.*
(iv) *$\{\varrho(A)x,y\}_d = \{x,\varrho(A^*)y\}_d$, $A \in D(\delta)$ and $x,y \in D(S^*)$, where $d = l,r,t$.*
(v) *The representation π_S^δ of $D(\delta)$ on $N(S)$ is symmetric with respect to the forms $[\,,]_d$, $d = l,r,t$, i.e.,*

$$[\pi_S^\delta(A)x,y]_d = [x,\pi_S^\delta(A^*)y]_d, \qquad A \in D(\delta), \qquad x,y \in N(S),$$

and bounded: $|\pi_S^\delta(A)x|^2 \le 2\|G\|\,\|G^{-1}\|\,\|A\|_\delta^2\,|x|^2$, where $\|A\|_\delta = \|A\| + \|\delta(A)\|$.
(vi) *A subspace in $N(S)$ is neutral with respect to $[\,,]_l$ if and only if it is neutral with respect to $[\,,]_r$. A subspace in $N(S)$ invariant for F and F^{-1}, is neutral with respect to $[\,,]$ if and only if it is neutral with respect to $[\,,]_l$, ($[\,,]_r$).*

Proof. Let $|x_n| \to 0$ and $|y - G_S x_n| \to 0$. Then $\|x_n\| \to 0$ and $\|y - Gx_n\| \to 0$. Since G is bounded on $\mathfrak{H}$, $y = 0$. Thus G_S is closed with respect to the norm $|\ |$. Since it is defined everywhere on $D(S^*)$, it is bounded. Similarly, G_S^{-1} is bounded. By (12), $\{G_S x, y\}_r = \{G_S x, G_S y\} = \{x, G_S y\}_l$. Part (i) is proved.

For $x,y \in N(S)$, $(\mathbf{1}_{D(S^*)} - Q)G_S x \in D(S)$, so that, by (7) and (10),

$$[x,y]_l = \{G_S x, y\} = \{QG_S x, y\} + \{(\mathbf{1}_{D(S^*)} - Q)G_S x, y\} = [Fx,y] = \langle JFx, y\rangle.$$

By (6) and (14),

$$[x,y]_r = \overline{[y,x]_l} = \overline{[Fy,x]} = [x,Fy] = \langle F^+Jx, y\rangle.$$

Therefore (16) and (17) hold and

$$[Fx, y]_r = [Fx, Fy] = [x, Fy]_l$$

and

$$[F^J x, y]_r = [F^J x, Fy] = [x, F^2 y] = [x, Fy]_r\,,$$

so that (18) holds. Similarly, one can prove (19). Then (20) follows immediately from (18) and (19).

If $x \in N(S)$ is such that $[x, y]_l = 0$, for all $y \in N(S)$, then, by (13), $\{x, z\}_l = 0$ for all $z \in D(S^*)$. Therefore, by (12),

$$(Gx, S^* z) = (S^* Gx, z)\,,$$

so that $Gx \in D(S^{**}) = D(S)$, since S is closed. Thus $x \in D(S)$. Similarly, if $[y, x]_l = 0$, for $y \in N(S)$, then, by (13), $\{z, x\}_l = 0$ for all $z \in D(S^*)$. Hence

$$(Gz, S^* x) = (S^* Gz, x)\,.$$

Since $GD(S^*) = D(S^*)$, $x \in D(S)$. This contradiction shows that $[\,,\,]_l$ is not degenerate. From this and from (14) it follows that $[\,,\,]_r$ also is not degenerate. Part (ii) is proved.

If $[\,,\,]_t$ is degenerate, there is $x \in N(S)$ such that $[x, y]_t = [Rx, y] = 0$, for all $y \in N(S)$. By (12) and (13),

$$i((x, W^* z) - (W^* x, z)) = \{x, z\}_t = 0 \quad \text{for all } z \in D(S^*)\,.$$

Thus $(x, W^* z) = (W^* x, z)$, $z \in D(S^*)$. Since $D(S^*) = D(W^*)$, $x \in D(W^{**})$. Since W is closed, $W^{**} = W$, so that $x \in D(W) = D(S)$ which contradicts the assumption that $x \in N(S)$. Thus $[\,,\,]_t$ is non-degenerate. If $\dim(n_\pm(S)) < \infty$, $N(S)$ is finite-dimensional. If R does not have the inverse, there is $x \in N(S)$ such that $Rx = 0$, so that $[x, y]_t = [Rx, y] = 0$, $y \in N(S)$. Since $[\,,\,]_t$ is non-degenerate, R has the inverse. Part (iii) is proved.

Since $GD(S^*) = D(S^*)$, it follows from Lemma 2 that

$$S^* G\varrho(A)|_{D(S^*)} = (-i\delta(A^*)^* G + \varrho(A) S^* G)|_{D(S^*)}\,.$$

From this, from (12) and from Lemma 2 we obtain that

$$\begin{aligned}\{\varrho(A)x, y\}_l &= i((G\varrho(A)x, S^* y) - (S^* G\varrho(A)x, y)) \\ &= i((G\varrho(A)x, S^* y) + i(\delta(A^*)^* Gx, y) - (\varrho(A) S^* Gx, y)) \\ &= i((G\varrho(A)x, S^* y) - (\varrho(A) S^* Gx, y) + i(Gx, \delta(A^*) y)) \\ &= i((G\varrho(A)x, S^* y) - (\varrho(A) S^* Gx, y) + (Gx, S^* \varrho(A^*) y) \\ &\quad - (Gx, G^{-1}\varrho(A^*) G S^* y)) \\ &= i((Gx, S^* \varrho(A^*) y) - (S^* Gx, \varrho(A^*) y)) = \{x, \varrho(A^*) y\}_l\,.\end{aligned}$$

By (14),

$$\{\varrho(A)x, y\}_r = \overline{\{y, \varrho(A)x\}_l} = \overline{\{\varrho(A^*) y, x\}_l} = \{x, \varrho(A^*) y\}_r\,.$$

Thus also $\{\varrho(A)x, y\}_t = \{x, \varrho(A^*) y\}_t$. Part (iv) is proved.

By (5), for $x, y \in N(S)$,

$$[\pi_S^\delta(A)x, y]_l = \{\varrho(A)x, y\}_l - \{(\mathbf{1}_{D(S^*)} - Q)\varrho(A)x, y\}_l\,.$$

Since $(\mathbf{1}_{D(S^*)} - Q)\varrho(A)x \in D(S)$, it follows from (13) and (iv) that

$$[\pi_S^\delta(A)x, y]_l = \{\varrho(A)x, y\}_l = \{x, \varrho(A^*)y\}_l$$
$$= \{x, Q\varrho(A^*)y\}_l + \{x, (\mathbf{1}_{D(S^*)} - Q)\varrho(A^*)y\}_l = [x, \pi_S^\delta(A^*)y]_l \,.$$

Thus π_S^δ is symmetric with respect to $[\,,]_l$. From (14) it follows that π_S^δ is also symmetric with respect to $[\,,]_r$ and $[\,,]_t$.

From (10) and Lemma 2(ii) we obtain that

$$|\pi_S^\delta(A)x|^2 = |Q\varrho(A)x|^2 \le |\varrho(A)x|^2 = \|\varrho(A)x\|^2 + \|S^*\varrho(A)x\|^2$$
$$= \|\varrho(A)x\|^2 + \|(-i\delta(A) + G^{-1}\varrho(A)GS^*)x\|^2$$
$$\le \|\varrho(A)x\|^2 + 2\|\delta(A)x\|^2 + 2\|G^{-1}\varrho(A)GS^*x\|^2$$
$$\le 2\|G\|\,\|G^{-1}\|(\|\varrho(A)\|^2 + \|\delta(A)\|^2)\,|x|^2 \,.$$

Since ϱ is a *-representation, $\|\varrho(A)\| \le \|A\|$. Part (v) is proved.

From (14) it follows that $[x, y]_l = 0$ for all $x, y \in L \subset N(S)$ if and only if $[x, y]_r = 0$ for $x, y \in L$. Let L be a subspace in $N(S)$ neutral with respect to $[\,,]$ and invariant for F and F^{-1}. By (16), for $x, y \in L$, $[x, y]_l = [Fx, y] = 0$, since $Fx \in L$. Conversely, if L is neutral with respect to $[\,,]_l$, then, for $x, y \in L$, $[x, y] = [F^{-1}x, y]_l = 0$, since $F^{-1}x \in L$. □

The following theorem extends some results about *-derivations of C^*-algebras (see Theorems 3.6 and 3.7 [3]), to the case of *-superderivations. It establishes a link between symmetric δ-extensions of a symmetric implementation (S, G) of a *-superderivation δ and neutral invariant subspaces in $N(S)$ and proves the existence of a maximal symmetric implementation of δ.

Theorem 5. *Let (S, G) be a symmetric implementation of a closed *-superderivation δ relative to (ϱ, φ).*

(i) *There is a one-to-one correspondence between closed symmetric δ-extensions of (S, G) and subspaces L in $N(S)$ neutral with respect to $[\,,]_l$ ($[\,,]_r$), invariant for π_S^δ and such that $FL = L$ and $F^J L = L$.*

(ii) *There is a maximal symmetric implementation (T, G) of δ such that T extends S.*

(iii) *If (S, G) is a maximal symmetric implementation of δ, $N(S)$ has no subspaces L neutral with respect to $[\,,]_l$ ($[\,,]_r$), invariant for π_S^δ and such that $FL = L$ and $F^J L = L$.*

Proof. There is a one-to-one correspondence (see [1]) between closed symmetric extensions T of the operator S and subspaces M, $D(S) \subset M \subset D(S^*)$, neutral with respect to $\{\,,\}$: $M(T) = D(T)$ and $T(M) = S^*|_M$. Since $D(S^*)$ is a Hilbert space, $D(T) = D(S)\langle+\rangle L(T)$, where $L(T) \subseteq N(S)$ and $L(T)$ is neutral with respect to $[\,,]$. From Lemma 15 [1] it follows that

$$D(T^*) = D(S)\langle+\rangle L(T)^{[\perp]} \,,$$

where $L(T)^{[\perp]}$ is the J-orthogonal complement of $L(T)$ in $N(S)$ with respect to $[\,,]$.

If (T, G) implements δ, it follows from (2) that $GD(T) = D(T)$ and $GD(T^*) = D(T^*)$. By (11),

$$FL(T) = QG_SQD(T) = QGD(T) = QD(T) = L(T) \,.$$

Hence $F^{-1}L(T) = L(T)$. By Lemma 4(vi), $L(T)$ is neutral with respect to $[\,,]_l$.

Similarly, we obtain from (11) that

$$F(L(T)^{[\perp]}) = L(T)^{[\perp]} \quad \text{and} \quad F^{-1}(L(T)^{[\perp]}) = L(T)^{[\perp]}.$$

Let $x \in L(T)$ and $y \in L(T)^{[\perp]}$. Then $Fy \in L(T)^{[\perp]}$ and, by (15), $0 = [x, Fy] = [F^J x, y]$. Therefore, $F^J x \in (L(T)^{[\perp]})^{[\perp]} = L(T)$ (see [7, Lemma 2.1]). Hence $F^J L(T) \subseteq L(T)$. Similarly, $(F^{-1})^J L(T) \subseteq L(T)$. Since $(F^J)^{-1} = (F^{-1})^J$, we obtain that $F^J L(T) = L(T)$. We also have that $\varrho(A)D(T) \subseteq D(T)$, $A \in D(\delta)$. Hence from (3) and (5) it follows that $L(T)$ is invariant for π_S^δ.

Conversely, let L be a subspace in $N(S)$ neutral with respect to $[\,,\,]_l$, invariant for π_S^δ and such that $FL = L = F^J L$. Then $F^{-1}L = L$. By Lemma 4(vi), L is neutral with respect to $[\,,\,]$. Set $M = D(S) + L$. By (7), M is a subspace in $D(S^*)$ neutral with respect to $\{\,,\,\}$. Hence $T = S^*|_M$ is symmetric,

$$\varrho(A)D(T) \subseteq D(T)\,, A \in D(\delta), \quad \text{and} \quad GD(T) = GM = D(S) + FL = M = D(T).$$

Since $F^J L = L$, $(F^J)^{-1}L = L$ and, for $x \in L$, $y \in L^{[\perp]}$, it follows from (15) that

$$0 = [F^J x, y] = [x, Fy]\,.$$

Hence $Fy \in L^{[\perp]}$, so that $FL^{[\perp]} \subseteq L^{[\perp]}$. Similarly, $F^{-1}L^{[\perp]} \subseteq L^{[\perp]}$. Therefore $FL^{[\perp]} = L^{[\perp]}$. Thus $GD(T^*) = D(T^*)$.

From Lemma 2 it follows that

$$\delta(A)|_{D(T)} = i(S^*\varrho(A) - G^{-1}\varrho(A)GS^*)|_{D(T)} = i(T\varrho(A) - G^{-1}\varrho(A)GT)|_{D(T)}\,,$$

so that the pair (T, G) implements δ. Part (ii) is proved.

Let $\{L_\alpha\}$ be a set of subspaces in $N(S)$ neutral with respect to $[\,,\,]_l$, invariant for π_S^δ, ordered by inclusion and such that $FL_\alpha = L_\alpha = F^J L_\alpha$. Let $L = \overline{\bigcup L_\alpha}$. By Lemma 4, the operators F, F^{-1}, and $\pi_S^\delta(A)$, $A \in D(\delta)$, are bounded on $N(S)$ with respect to $|\!|\;|\!|$. Hence L is invariant for π_S^δ and $FL = L = F^J L$. From Lemma 4 it also follows that $|[x,y]_l| \le |\!|G_S|\!|\,|\!|x|\!|\,|\!|y|\!|$, $x, y \in N(S)$. Therefore L is neutral with respect to $[\,,\,]_l$. Hence by Zorn's theorem, there exists a maximal subspace L_0 in $N(S)$ neutral with respect to $[\,,\,]_l$, invariant for π_S^δ and such that $FL = L = F^J L$. Thus, by (i), the corresponding pair (T, G) is a maximal symmetric implementation of δ such that T extends S. Part (ii) is proved. Part (iii) follows immediately from (i). □

3. Extensions of π_S^δ to Representations of the C^*-Algebra $\mathfrak{U}$

If δ is a *-derivation of $\mathfrak{U}$, S is a maximal implementation of δ and $\max(n_\pm(S)) < \infty$, then π_S^δ is a non-degenerate representation of $D(\delta)$ on a finite-dimensional space $N(S)$. It was proved in [3] that π_S^δ is semisimple and extends to a bounded representation of $\mathfrak{U}$. Theorems 6 and 7 prove this result for some maximal symmetric implementations of *-superderivations.

Theorem 6. *Let $\mathfrak{U}$ be a unital C^*-algebra and the operator $R = \frac{1}{2}(F + F^J)$ have a bounded inverse on $N(S)$.*

(i) *There are a new scalar product $(\,,\,)_1$ and a new involution I on $N(S)$ such that the norm $\|\;\|_1 = (\,,\,)_1^{1/2}$ is equivalent to the norm $|\!|\;|\!|$ on $N(S)$ and that $[x,y]_t = (Ix, y)_1$. Thus $N(S)$ is a Krein space with respect to $(\,,\,)_1$ and $[\,,\,]_t$.*

(ii) *If* (S, G) *is a maximal symmetric implementation of* δ *and* $\max(n_\pm(S)) < \infty$, *then* π_S^δ *is semisimple, bounded and extends to a bounded representation of* $\mathfrak{U}$ *on* $N(S)$ *which is symmetric with respect to* $[\,,]_d$, $d = l, r, t$

Proof. The operator $JR = \frac{1}{2}(JF + F^+J)$ is selfadjoint on $N(S)$ and, by (10) and (17), $[x, y]_t = [Rx, y] = \langle JRx, y\rangle$. If R has a bounded inverse, JR also has a bounded inverse and part (i) follows from [7].

Let K be a subspace in $N(S)$ invariant for π_S^δ, F and F^J. Since F and F^J have inverses and since $N(S)$ is finite-dimensional, $FK = K = F^JK$. Set

$$M = K^{[\perp]_t} = \{y \in N(S) : [x, y]_t = 0, \text{ for all } x \in K\}.$$

By Lemma 4(v), M is invariant for π_S^δ. We claim that

$$FM = M = F^JM, \qquad K \cap M = \{0\} \quad \text{and} \quad N(S) = K[+]_tM. \tag{21}$$

From (20) it follows that

$$[x, Ry]_l = [Fx, y]_l = 0 \ \text{ and } \ [x, Ry]_r = [F^Jx, y]_t = 0, \quad x \in K, y \in M. \tag{22}$$

Therefore $[x, Ry]_t = 0$. Hence $RM \subseteq M$. Since R has a bounded inverse and M is finite-dimensional, $RM = M$. Therefore, by (22),

$$[x, y]_l = [x, y]_r = 0, \qquad x \in K, y \in M. \tag{23}$$

Since $FK = K = F^JK$, from (18) and (19) it follows that

$$[x, Fy]_l = [x, Fy]_r = [x, F^Jy]_l = [x, F^Jy]_r = 0.$$

Hence, by (12),

$$[x, Fy]_t = [x, F^Jy]_t = 0, \qquad x \in K, y \in M,$$

so that $FM \subseteq M$ and $F^JM \subseteq M$. Therefore $FM = M$ and $F^JM = M$.

The subspace $P = K \cap M$ is invariant for π_S^δ, $FP = P = F^JP$ and, by (23), it is neutral with respect to $[\,,]_l$. Since (S, G) is a maximal symmetric implementation of δ, it follows from Theorem 5(iii) that $P = \{0\}$. By (17),

$$\begin{aligned} M &= \{y \in N(S) : [x, y]_t = [Rx, y] = \langle JRx, y\rangle = \langle x, R^+Jy\rangle = 0, x \in K\} \\ &= (R^+J)^{-1}K^\perp, \end{aligned}$$

where $K^\perp$ is the orthogonal complement of K with respect to $\langle\,,\rangle$. Hence $\dim M = \dim K^\perp$, so that $K[+]_tM = N(S)$. Thus (21) is proved.

From (21) it follows that $N(S)$ can be decomposed in the direct sum

$$N(S) = \sum_{i=1}^{m} [+]_t K_i \tag{24}$$

of subspaces K_i invariant for π_S^δ, for F and F^J, orthogonal with respect to $[\,,]_t$ and such that they have no subspaces invariant for π_S^δ, F and F^J.

Let Γ be the group of operators on $N(S)$ generated by F and F^J. We have that $Q\varrho(A)Q = Q\varrho(A)$, $A \in D(\delta)$. From this and from (1), (5) and (11) it follows that

$$\begin{aligned} F^{-1}\pi_S^\delta(A)F &= QG_S^{-1}Q\varrho(A)QG_SQ \\ &= QG^{-1}\varrho(A)GQ = Q\varrho(\varphi(A))Q = \pi_S^\delta(\varphi(A)). \end{aligned} \tag{25}$$

From (15), (16) and Lemma 4(iv) we have that

$$[\pi_S^\delta(A)x, y]_l = [F\pi_S^\delta(A)x, y] = [x, (\pi_S^\delta(A))^J F^J y]$$
$$= [x, \pi_S^\delta(A^*)y]_l = [Fx, \pi_S^\delta(A^*)y] = [x, F^J \pi_S^\delta(A^*)y]$$

and

$$[\pi_S^\delta(A)x, y]_r = [\pi_S^\delta(A)x, Fy] = [x, (\pi_S^\delta(A))^J Fy]$$
$$= [x, \pi_S^\delta(A^*)y]_r = [x, F\pi_S^\delta(A^*)y].$$

Since the form $[\,,]$ is not degenerate on $N(S)$,

$$(\pi_S^\delta(A))^J F^J = F^J \pi_S^\delta(A^*) \quad \text{and} \quad (\pi_S^\delta(A))^J F = F\pi_S^\delta(A^*).$$

Thus $F^J\pi_S^\delta(A^*)(F^J)^{-1} = F\pi_S^\delta(A^*)F^{-1}$ and from (25) we conclude that

$$F^J\pi_S^\delta(A)(F^J)^{-1} = \pi_S^\delta(\varphi^{-1}(A)). \tag{26}$$

Let $B = F^{m_1}(F^J)^{p_1}\ldots F^{m_n}(F^J)^{p_n} \in \Gamma$, $m_i, p_i \in \mathbb{Z}$. Set $\deg(B) = \sum_{i=1}^{n}(m_i + p_i)$. From (25) and (26) it follows that

$$B^{-1}\pi_S^\delta(A)B = \pi_S^\delta(\varphi^{\deg(B)}(A)). \tag{27}$$

Let $K = K_i$ be a subspace in decomposition (24). Since $\mathfrak{U}$ is unital, $\mathbf{1} \in D(\delta)$, by Lemma 1. Therefore there is a subspace L in K invariant for π_S^δ such that the restriction π_L of π_S^δ to L is irreducible and non-trivial. The subspace K is invariant for all $B \in \Gamma$. Hence $BL \subseteq K$ and it follows from (27) that BL is invariant for π_S^δ and the restriction of π_S^δ to BL is irreducible. Therefore if M is a subspace in K invariant for π_S^δ, then either $M \cap BL = \{0\}$ or $BL \subseteq M$. From this and from the fact that K is finite-dimensional and has no subspace invariant for π_S^δ, for F and F^J it follows that there are $B_j \in \Gamma$, $j = 1, \ldots, q$, such that K is the direct sum of the subspaces B_jL: $K = B_1L + B_2L + \ldots + B_nL$. From this and from (24) we conclude that π_S^δ decomposes in the direct sum of irreducible representations of the algebra $D(\delta)$. Hence π_S^δ is a semisimple representation. Since, by Lemma 1, $D(\delta)$ is a Q-subalgebra of $\mathfrak{U}$, it follows from Theorem 6 [6] that π_S^δ is bounded with respect to the norm on $\mathfrak{U}$ and extends to a bounded representation ψ of $\mathfrak{U}$ on $N(S)$. Since π_S^δ is symmetric with respect to $[\,,]_d$, $d = l, r, t$, ψ is also symmetric. □

In Theorem 6 we assumed that the operator R has a bounded inverse on $N(S)$. Now we assume that $R = 0$, i.e., $F^J = -F$. Then, by (16) and (17),

$$[x, y]_r = -[x, y]_l \quad \text{and} \quad [x, y]_t \equiv 0. \tag{28}$$

Set $[\![x, y]\!] = i[x, y]_l$ and $R_1 = iF$. Then, by (14),

$$[\![y, x]\!] = i[y, x]_l = -i[y, x]_r = -i\overline{[x, y]_l} = \overline{[\![x, y]\!]},$$

and, by (18) and (28),

$$[\![R_1x, y]\!] = -[Fx, y]_l = [Fx, y]_r = [x, Fy]_l = [\![x, R_1y]\!]. \tag{29}$$

Since $R = \frac{1}{2}(F + F^J) = 0$, the proof of Theorem 6 obviously fails. However, the following theorem holds which replaces Theorem 6.

Theorem 7. *Let $\mathfrak{U}$ be a unital C^*-algebra and let $F^J = -F$.*

(i) *There are a new scalar product* $(\,,\,)_1$ *and a new involution* I *on* $N(S)$ *such that the norm* $\|\ \|_1 = (\,,\,)_1^{1/2}$ *is equivalent to the norm* $|\ |$ *on* $N(S)$ *and that* $[\![x,y]\!] = (Ix,y)_1$. *Thus* $N(S)$ *is a Krein space with respect to* $(\,,\,)_1$ *and* $[\![\,,\,]\!]$.
(ii) *If* (S,G) *is a maximal symmetric implementation of* δ *and* $\max(n_\pm(S)) < \infty$, *then* π_S^δ *is semisimple and extends to a bounded representation of* $\mathfrak{U}$ *on* $N(S)$ *which is symmetric with respect to* $[\![\,,\,]\!]$.

Proof. Since $F^J = -F$, $R_1^J = R_1$. Hence $R_1^+ J = JR_1$. It follows from (16) that

$$[\![x,y]\!] = [R_1 x, y] = \langle JR_1 x, y\rangle\,.$$

The operator JR_1 is selfadjoint and has a bounded inverse, since F has a bounded inverse. Thus part (i) follows from [7].

Let K be a subspace in $N(S)$ invariant for π_S^δ and R_1. Since R_1 has a bounded inverse and since $N(S)$ is finite-dimensional, $R_1 K = K$. Set

$$M = \{y \in N(S) : [\![x,y]\!] = 0, \text{ for all } x \in K\}\,.$$

By Lemma 4(v) and by (29), M is invariant for π_S^δ and R_1. Therefore $K \cap M$ is invariant for π_S^δ and for R_1 and is neutral with respect to $[\![\,,\,]\!]$. It follows from Theorem 5(iv) that $K \cap M = \{0\}$. In the same way as in Theorem 6 we obtain that $\dim M = \dim K^\perp$, so that

$$N(S) = K[\![+]\!]M\,.$$

Making use of the above formula and repeating the argument of Theorem 6, we conclude the proof of the theorem. □

Recall that an operator T is called *dissipative* if

$$(Tx,x) + (x,Tx) \le 0\,, \qquad x \in D(T)\,,$$

and *maximal dissipative* if it is dissipative but not a proper restriction of any other dissipative operator.

If S is a maximal symmetric implementation of a *-derivation σ and $\max(n_\pm(S)) < \infty$, the representation π_S^σ of $D(\sigma)$ on $N(S)$ is *non-degenerate* with respect to $[\,,\,]$ and, hence, semisimple and extends to a bounded J-symmetric representation of the C^*-algebra $\mathfrak{U}$ [3]. From this it follows (see Theorem 3.2 [5]) that there exist disjoint sets of irreducible *-representations $\{\pi_i\}_{i=1}^p$ and $\{\varrho_j\}_{j=1}^m$ of $\mathfrak{U}$ such that

$$n_-(S) = \sum_{i=1}^p \dim \pi_i \quad \text{and} \quad n_+(S) = \sum_{j=1}^m \dim \varrho_j\,.$$

This fact was also used in Theorem 3.2 [4] to prove that there exist operators T_j, $j = 1,2$, such that $T_1^* = T_2$, that $S \subseteq T_j \subseteq S^*$, that iT_1 and $-iT_2$ are maximal dissipative operators and that T_j implement σ, i.e.,

$$AD(T_j) \subseteq D(T_j) \quad \text{and} \quad \sigma(A)|_{D(T_j)} = i(T_j A - AT_j)|_{D(T_j)}\,, \qquad A \in D(\sigma)\,.$$

Let (S,G) be a maximal symmetric implementation of a *-superderivation δ of a unital C^*-algebra $\mathfrak{U}$ and $\max(n_\pm(S)) < \infty$. If $W = \frac{1}{2}(GS+SG)$ is a closed operator and $D(W^*) = D(S^*)$, it follows from Lemma 4(iii) that the operator $R = \frac{1}{2}(F+F^J)$ has a bounded inverse. Although the representation π_S^δ may be degenerate with respect to $[\,,]_d$, $d = l, r, t$, nevertheless it follows from Theorem 6 that π_S^δ is semisimple and extends to a bounded representation of $\mathfrak{U}$ on $N(S)$ symmetric with respect to $[\,,]_t$. Similarly, if $F^J = -F$, i.e., $R = 0$, it follows from Theorem 7 that π_S^δ is semisimple and extends to a bounded representation of $\mathfrak{U}$ on $N(S)$ symmetric with respect to $[\![\,,]\!]$. The operator $JR = \frac{1}{2}(JF + F^+J)$ in the first case and the operator $JR_1 = iJF$ in the second case are selfadjoint on $N(S)$ and invertible. Let N_- and N_+ be the subspaces in $N(S)$ generated by all eigenvectors of JR (resp. JR_1) which correspond respectively to negative and positive eigenvalues. Set $m_\pm = \dim(N_\pm)$. Then $m_- + m_+ = \dim N(S)$. Using the same argument as in Theorems 3.2 [5] and 3.2 [4] we obtain the following corollary which refines the result of Theorem 3.

Corollary 8. *Let (S, G) be a maximal symmetric implementation of δ and* $\max(n_\pm(S)) < \infty$.

(i) *If $W = \frac{1}{2}(GS + SG)$ is a closed operator and $D(W^*) = D(S^*)$, then*

(a) *there exist disjoint sets of irreducible *-representations $\{\pi_i\}_{i=1}^p$ and $\{\varrho_j\}_{j=1}^m$ of $\mathfrak{U}$ such that $m_- = \sum_{i=1}^p \dim \pi_i$ and $m_+ = \sum_{j=1}^m \dim \varrho_j$,*

(b) *there exist operators T_j, $j = 1, 2$, such that $T_1^* = T_2$, that $W \subseteq T_j \subseteq W^*$, that iT_1 and $-iT_2$ are maximal dissipative operators and that T_j implement the *-derivation Δ associated with δ,* i.e.,

$$AD(T_j) \subseteq D(T_j) \quad \text{and} \quad \Delta(A)|_{D(T_j)} = i(T_jA - AT_j)|_{D(T_j)}, \qquad A \in D(\Delta).$$

(ii) *If $F^J = -F$, then* (i) (a) *holds.*

4. Special Type of Symmetric Implementations of Superderivations

In this section we consider examples of symmetric implementations (S, G) which satisfy Theorems 6 and 7. Assume that *there are $\lambda, \mu \in \mathbb{C}$ such that the operator*

$$B|_{D(S)} = (SG - \lambda GS - \mu S)|_{D(S)} \tag{30}$$

is bounded. Set $\nu = -\dfrac{\mu}{1+\lambda}$ if $\lambda \neq -1$.

Lemma 9. (i) *Let $(\lambda, \mu) \neq (-1, 0)$ and let $\nu \notin Sp\, G$ and $(G - \bar{\nu}\mathbf{1}_{\mathfrak{H}})D(S^*) = D(S^*)$ (for example, $\mu = 0$). Then the operator $W = \frac{1}{2}(GS + SG)$ is closed and $D(W^*) = D(S^*)$, so that the form $[\,,]_t$ on $N(S)$ is non-degenerate. If, in addition, (S, G) is a maximal implementation of δ and* $\max(n_\pm(S)) < \infty$, *then the operator $R = \frac{1}{2}(F + F^J)$ has a bounded inverse and Theorem* 6(ii) *and Corollary* 8(i) *hold.*

(ii) *The following are equivalent·* a) $|\lambda| = 1$, b) $\mu + \lambda\bar{\mu} = 0$.

(iii) *If* $|\lambda| = 1$, *then*

$$\begin{aligned} &B^* = -\bar{\lambda}B, \qquad B|_{D(S^*)} = (S^*G - \lambda GS^* - \mu S^*)|_{D(S^*)}, \\ &\nu \notin Sp\, G, \qquad [x, y]_l = \lambda[x, y]_r + \mu[x, y], \quad \text{and} \quad F^J = \bar{\lambda}F + \bar{\mu}\mathbf{1}_{N(S)}. \end{aligned} \tag{31}$$

(iv) [3] *If $|\lambda| = 1$, $\mu = 0$ and $B = \nu G$, $\nu \in \mathbb{C}$, and if $(\lambda, \nu) \neq (1, 0)$, then $n_-(S) = n_+(S)$.*

Proof. Let $\lambda \neq -1$. By (30),

$$W = \frac{1}{2}(GS + SG) = \frac{1+\lambda}{2} GS + \frac{\mu}{2} S + \frac{1}{2} B = \frac{1+\lambda}{2}(G - \nu \mathbf{1}_{\mathfrak{H}})S + \frac{1}{2} B.$$

Since $G - \nu \mathbf{1}_{\mathfrak{H}}$ has the inverse and S is closed, we have that W is closed and $D(W) = D(S)$. Let $y \in D(W^*)$. Then for $x \in D(S)$,

$$(Wx, y) = (x, W^* y) = \frac{1+\lambda}{2}(Sx, (G - \bar{\nu}\mathbf{1}_{\mathfrak{H}})y) + \frac{1}{2}(x, B^* y).$$

Hence $(G - \bar{\nu}\mathbf{1}_{\mathfrak{H}})y \in D(S^*)$. Since $(G - \bar{\nu}\mathbf{1}_{\mathfrak{H}})D(S^*) = D(S^*)$, there is $z \in D(S^*)$ such that $(G - \bar{\nu}\mathbf{1}_{\mathfrak{H}})y = (G - \bar{\nu}\mathbf{1}_{\mathfrak{H}})z$. Since G is selfadjoint and $G - \nu\mathbf{1}_{\mathfrak{H}}$ is invertible. $G - \bar{\nu}\mathbf{1}_{\mathfrak{H}}$ also has a bounded inverse. Hence $y = z \in D(S^*)$. Thus $D(W^*) = D(S^*)$.

If now $\lambda = -1$ and $\mu \neq 0$, then $W = \frac{\mu}{2} S + \frac{1}{2} B$ is closed, $W^* = \frac{\bar{\mu}}{2} S^* + \frac{1}{2} B$ and $D(W^*) = D(S^*)$. It follows from Lemma 4(iii) that in both cases, $\lambda \neq -1$ and $\lambda = -1$, $\mu \neq 0$, the form $[\,,]_t$ is non-degenerate. If $\max(n_\pm(S)) < \infty$, then, by Lemma 4(iii), the operator R has a bounded inverse. Thus Theorem 6(ii) and Corollary 8(i) hold. Part (i) is proved.

Let $x \in D(S)$ and $y \in D(S^*)$. By (30),

$$\begin{aligned}\lambda(Sx, Gy) &= (\lambda GSx, y) = (SGx, y) - ((B + \mu S)x, y) \\ &= (x, (GS^* - B^* - \bar{\mu} S^*)y).\end{aligned}$$

Therefore

$$B^*|_{D(S^*)} = (GS^* - \bar{\lambda} S^* G - \bar{\mu} S^*)|_{D(S^*)}. \tag{32}$$

Restricting (32) to $D(S)$, we obtain that $B^*|_{D(S)} = (GS - \bar{\lambda} SG - \bar{\mu} S)|_{D(S)}$. Hence

$$(B + \lambda B^*)|_{D(S)} = ((1 - |\lambda|^2)SG - (\mu + \lambda\bar{\mu})S)|_{D(S)}. \tag{33}$$

If $|\lambda| = 1$, then, since $B + \lambda B^*$ is bounded, $(\mu + \lambda\bar{\mu})S$ is bounded. Since S is unbounded, $(\mu + \lambda\bar{\mu}) = 0$. Conversely, if $\mu + \lambda\bar{\mu} = 0$, $(1 - |\lambda|^2)SG$ is bounded. If $\lambda \neq 1$, SG is bounded. Since $GD(S) = D(S)$, S is bounded. This contradiction shows that $|\lambda| = 1$. Part (ii) is proved.

If $|\lambda| = 1$, it follows from (33) that $B + \lambda B^* = 0$. Hence $B^* = -\bar{\lambda} B$. From this, from (ii) and from (32) it follows that

$$\begin{aligned}B|_{D(S^*)} &= -\lambda B^*|_{D(S^*)} = (S^* G - \lambda G S^* + \lambda\bar{\mu} S^*)|_{D(S^*)} \\ &= (S^* G - \lambda G S^* - \mu S^*)|_{D(S^*)}.\end{aligned}$$

Let $\lambda \neq -1$. If $\mu = 0$, then $\nu = 0$ and, since G has a bounded inverse, $\nu \notin Sp\,G$. If $\mu \neq 0$, it follows from (ii) that $\lambda = -\mu/\bar{\mu}$, so that $\operatorname{Im}\mu \neq 0$. Then $\nu = i|\mu|^2/2\operatorname{Im}(\mu)$. Since G is selfadjoint, $\nu \notin Sp\,G$.

By (31), $U^* = S^*G = \lambda GS^* + \mu S^* + B = \lambda V^* + \mu S^* + B$. Since $B^* = -\bar{\lambda}B$, it follows from (12) and from (ii) that

$$\begin{aligned}\{x,y\}_l &= i((x,V^*y) - (U^*x,y)) \\ &= i((x,V^*y) - (\lambda V^*x,y) - (Bx,y) - (\mu S^*x,y)) \\ &= \lambda i(\bar{\lambda}(x,V^*y) - (V^*x,y) + (x,By)) - i\mu(S^*x,y) \\ &= \lambda i((x,(\lambda V^* + B + \mu S^*)y) - (V^*x,y)) - \lambda\bar{\mu}i(x,S^*y) - \mu i(S^*x,y) \\ &= \lambda i((x,U^*y) - (V^*x,y) + \mu i((x,S^*y) - (S^*x,y)) \\ &= \lambda\{x,y\}_r + \mu\{x,y\}\,.\end{aligned}$$

Therefore $[x,y]_l = \lambda[x,y]_r + \mu[x,y]$ and it follows from (16) that

$$[x,y]_l = \langle JFx,y\rangle = \lambda[x,y]_r + \mu[x,y] = \lambda\langle F^+Jx,y\rangle + \mu\langle Jx,y\rangle\,.$$

Thus $JF = \lambda F^+J + \mu J$, so that $F^J = JF^+J = \bar{\lambda}F + \bar{\mu}\mathbf{1}_{N(S)}$. □

Let now $(\lambda,\mu) = (-1,0)$ in (30), i.e., $SG|_{D(S)} = (-GS + B)|_{D(S)}$. By Lemma 9,

$$\begin{gathered}B^* = B\,, \qquad W = B/2\,, \qquad [x,y]_l = -[x,y]_r\,, \\ [x,y]_t \equiv 0\,, \qquad F^J = -F\,.\end{gathered} \tag{34}$$

Suppose that $B = 0$. Then $SG = -GS$. If $x \in N_d(S)$, $d = \pm$, then

$$SGx = -GSx = -diGx\,.$$

Therefore $Gx \in N_{-d}(S)$, so that $FN_d(S) \subseteq N_{-d}(S)$. Since $FN(S) = N(S)$, $FN_d(S) = N_{-d}(S)$. Since $Jx = dx$, $x \in N_d(S)$, we obtain that

$$n_+(S) = n_-(S) \quad \text{and} \quad FJ = -JF\,, \tag{35}$$

Recall (see Theorem 7) that in this case, instead of the operator $R = \frac{1}{2}(F + F^J)$, we consider the operator $R_1 = iF$. Set $T = JR_1 = iJF$. Then

$$TJ = iJ(FJ) = -iJ^2F = -iF = -JT\,.$$

The operator T is selfadjoint, since, by (34), $T^+ = -iF^+J = iJF = T$. If $\lambda > 0$ is an eigenvalue of T and x is the corresponding eigenvector, then

$$TJx = -JTx = -\lambda Jx\,, \tag{36}$$

so that $(-\lambda)$ is an eigenvalue of T and Jx is the corresponding eigenvector. Let, as in Corollary 8, N_- and N_+ be the subspaces in $N(S)$ generated by all eigenvectors of T which correspond respectively to negative and positive eigenvalues. Since T is invertible, $\dim N(S) = \dim N_- + \dim N_+$. From (36) it follows that $\dim N_- = \dim N_+$. From this and from (35) we conclude that

$$n_-(S) = n_+(S) = \dim N_- = \dim N_+\,.$$

From this and from Corollary 8(i) we obtain the following lemma.

Lemma 10. *Let (S,G) be a maximal implementation of a *-superderivation δ, let $SG|_{D(S)} = (-GS + B)|_{D(S)}$ and let* $\max(n_\pm(S)) < \infty$. *Then*
(ii) $F^J = -F$ *and Theorem* 7(ii) *holds;*

(ii) *if, in addition,* $B = 0$, *then there exist disjoint sets of irreducible* **-representations* $\{\pi_i\}_{i=1}^p$ *and* $\{\varrho_j\}_{j=1}^m$ *of* $\mathfrak{U}$ *such that*

$$n_-(S) = n_+(S) = \sum_{i=1}^{p} \dim \pi_i = \sum_{j=1}^{m} \dim \varrho_j .$$

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