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On Foundation of the Generalized Nambu Mechanics

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Abstract: We outline basic principles of a canonical formalism for the Nambu mechanics – a generalization of Hamiltonian mechanics proposed by Yoichiro Nambu in 1973. It is based on the notion of a Nambu bracket, which generalizes the Poisson bracket – a "binary" operation on classical observables on the phase space – to the "multiple" operation of higher order $n \geqq 3$. Nambu dynamics is described by the phase flow given by Nambu–Hamilton equations of motion – a system of ODE's which involves $n-1$ "Hamiltonians." We introduce the fundamental identity for the Nambu bracket – a generalization of the Jacobi identity – as a consistency condition for the dynamics. We show that Nambu bracket structure defines a hierarchy of infinite families of "subordinated" structures of lower order, including Poisson bracket structure, which satisfy certain matching conditions. The notion of Nambu bracket enables us to define Nambu–Poisson manifolds – phase spaces for the Nambu mechanics, which turn out to be more "rigid" than Poisson manifolds – phase spaces for the Hamiltonian mechanics. We introduce the analog of the action form and the action principle for the Nambu mechanics. In its formulation, dynamics of loops ($n-2$-dimensional chains for the general n-ary case) naturally appears. We discuss several approaches to the quantization of Nambu mechanics, based on the deformation theory, path integral formulation and on Nambu–Heisenberg "commutation" relations. In the latter formalism we present an explicit representation of the Nambu–Heisenberg relation in the $n = 3$ case. We emphasize the role ternary and higher order algebraic operations and mathematical structures related to them play in passing from Hamilton's to Nambu's dynamical picture.

1. Introduction

In 1973 Nambu proposed a profound generalization of classical Hamiltonian mechanics [1]. In his formulation a triple (or, more generally, n-tuple) of "canonical" variables replaces a canonically conjugated pair in the Hamiltonian formalism and ternary (or, more generally, n-ary) operation – the Nambu bracket – replaces the usual Poisson bracket. Dynamics, according to Nambu, is determined by

Nambu–Hamilton equations of motion, which use two (or, more generally, $n-1$) "Hamiltonians" and replace canonical Hamilton equations. Corresponding phase flow preserves the phase volume so that the analog of the Liouville theorem is still valid, which is fundamental for the formulation of statistical mechanics [1].

Nambu's proposal was partially analyzed in papers [2, 3]. In [2] it was shown that the particular example of Nambu mechanics can be treated as a six-dimensional degenerate Hamiltonian system with three constraints and a Lagrangian being linear in velocities. In [3] it was shown that one can use a four-dimensional phase space as well. However, until recently, there were no attempts to formulate the basic principles of Nambu mechanics in the invariant geometrical form similar to that of Hamiltonian mechanics [5]. In this paper we develop the basics of such a formalism and display novel mathematical structures which might have physical significance.

We start by formulating the fundamental identity (FI) for the Nambu bracket as a consistency condition for the Nambu's dynamics. As a corollary, it yields the analog of the Poisson theorem that the Poisson bracket of integrals of motion is again an integral of motion. Based on FI, we introduce the notion of Nambu–Poisson manifolds, which play the same role in Nambu mechanics that Poisson manifolds play in Hamiltonian mechanics.

We show that the Nambu bracket structure contains an infinite family of "subordinated" Nambu structures of lower degree, including Poisson bracket structure, with certain matching conditions. This implies that Nambu bracket structure is, in a certain sense, more "rigid" than the Poisson bracket structure. It can be seen explicitly by showing that FI imposes rather strong constraints on possible forms of Nambu bracket. In addition to quadratic differential equations, it also introduces an overdetermined system of quadratic algebraic equations for the Nambu bracket tensor. This is the novel feature in comparison with the Poisson bracket case, where we have differential constraints only. Additional algebraic requirements are "responsible" for the above-mentioned "rigidity" of the Nambu bracket structure in comparison with the Poisson bracket structure. Specifically, whereas any skew-symmetric constant 2-tensor yields a Poisson bracket, this is no longer true for the Nambu bracket. This manifests the rather specific nature of Nambu mechanics. We consider such exclusiveness as an advantage of the theory.

Next, we develop the canonical formalism for the Nambu mechanics. It is based on the analog of Poincaré–Cartan integral invariant – action form, which is a differential form of degree $n-1$ on the extended phase space. It has properties similar to those of the usual Poincaré–Cartan invariant and enables us to formulate the principle of least action for Nambu mechanics. Instead of all possible "histories" of the Hamiltonian system considered in action principle for classical mechanics (all paths connecting initial and final points in the configuration space), in Nambu mechanics one should consider all $n-1$-chains in the extended phase space which "time-slices" are closed $n-2$-chains satisfying certain boundary conditions. We define classical action as an integral of generalized Poincaré–Cartan invariant over such $n-1$-chains and prove the principle of least action: "world-sheets" of a given $n-2$-chains under the Nambu–Hamilton phase flow are extremals of the action.

Finally, we briefly discuss quantization of Nambu mechanics. This problem, first considered by Nambu [1], is still outstanding. We indicate several possible approaches towards its solution. In particular, we construct a special representation of Nambu–Heisenberg "commutation" relations, which is similar in spirit to the representation of canonical Heisenberg commutation relations by creation-annihilation operators in the space of states of the harmonic oscillator. In our realization, states are parametrized by a lattice of algebraic integers in a cyclotomic field for the cubic root of unity, whereas states of the harmonic oscillator are parametrized by non-negative rational integers.

Now let us explain the main ideas of the paper in more detail. We start with the simplest phase space for Hamiltonian mechanics – a two-dimensional Euclidean space $\mathbb{R}^2$ with coordinates x, y and canonical Poisson bracket

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x} = \frac{\partial(f_1, f_2)}{\partial(x, y)} .$$

This bracket satisfies the Jacobi identity

$$\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0 ,$$

and gives rise to the Hamilton equations of motion

$$\frac{df}{dt} = \{H, f\} ,$$

where f is a classical observable – smooth function on the phase space – and H is a Hamiltonian.

Generalization of this example leads to a concept of Poisson manifolds – smooth manifolds equipped with a Poisson bracket structure satisfying the skew-symmetry condition, Leibniz rule and Jacobi identity.

The canonical Nambu bracket [1] is defined for a triple of classical observables on the three-dimensional phase space $\mathbb{R}^3$ with coordinates x, y, z by the following beautiful formula:

$$\{f_1, f_2, f_3\} = \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} ,$$

where the right-hand side stands for the Jacobian of the mapping $f = (f_1, f_2, f_3)$: $\mathbb{R}^3 \mapsto \mathbb{R}^3$. This formula naturally generalizes the usual Poisson bracket from binary to ternary operation on classical observables.[1] Generalized Nambu–Hamilton equations of motion involve two "Hamiltonians" H_1 and H_2 and have the form

$$\frac{df}{dt} = \{H_1, H_2, f\} .$$

The corresponding phase flow on the phase space is divergence-free and preserves the standard volume form $dx \wedge dy \wedge dz$ – analog of Liouville theorem for Nambu mechanics [1].

In Sect. 2 we prove that the canonical Nambu bracket satisfies the following fundamental identity (and its generalizations for the n-ary case)[2]

$$\begin{aligned}\{\{f_1, f_2, f_3\}, f_4, f_5\} + \{f_3, \{f_1, f_2, f_4\}, f_5\} + \{f_3, f_4, \{f_1, f_2, f_5\}\} \\ = \{f_1, f_2, \{f_3, f_4, f_5\}\} .\end{aligned}$$

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[1] M. Flato [6] informed me that, apparently, Nambu introduced this bracket in order to develop a "toy model" for quarks considered as triples.

[2] This relation was also independently introduced by M. Flato and C Fronsdal [7].

This formula might be considered as the most natural (at least from a "dynamical" point of view) generalization of the Jacobi identity. It yields (see Theorem 3) an analog of classical Poisson theorem that Poisson bracket of two integrals of motion is again an integral of motion. The generalized version of FI for the n-ary case (where Jacobian of the mapping $f = (f_1, \ldots, f_n)$: $\mathbb{R}^n \mapsto \mathbb{R}^n$ defines the canonical Nambu bracket of order n) enables us to introduce Nambu–Poisson manifolds as smooth manifolds with Nambu bracket structure – an n-ary operation on classical observables satisfying the skew-symmetry condition, Leibniz rule and FI. We show that by fixing some of the arguments in the Nambu bracket of order n one can get brackets of lower order which still satisfy the fundamental identity (i.e. are Nambu brackets) and, in addition, satisfy certain matching conditions for different choices of fixed arguments. This hierarchical structure of the Nambu bracket shows that it is a more "rigid" concept than that of the Poisson bracket. We discuss explicit conditions FI imposes on the corresponding n-tensor of the Nambu bracket. We show, contrary to the Poisson case, that the algebraic part of FI substantially reduces possible Nambu structures with constant skew-symmetric n-tensor. Specifically, we show that the bracket on the phase space $\mathbb{R}^{3n} = \oplus_{i=1}^{n} \mathbb{R}^3$, defined in [1] (and used in [4]) as a direct sum of canonical Nambu brackets on $\mathbb{R}^3$, does not satisfy FI and, therefore, is not a Nambu bracket. Next, we discuss linear Nambu brackets. We show that they naturally lead to a new notion of Nambu–Lie "gebras," which generalizes Lie algebras for the n-ary case. We close Sect. 2 by presenting several simple examples of evolution equations which admit Nambu formulation.

In Sect. 3 we introduce the analog of the Poincaré–Cartan integral invariant for Nambu mechanics – a differential form of degree $n-1$ on the extended phase space. In the simplest three-dimensional example described above it is given by the following 2-form on $\mathbb{R}^4$,

$$\omega^{(2)} = x dy \wedge dz - H_1 dH_2 \wedge dt \, .$$

We prove in Theorem 6 that the vector field of Nambu–Hamilton phase flow in the extended phase space is the line field of the 3-form $d\omega^{(2)}$, so that integral curves are its characteristics. We define classical action as an integral of the Poincaré–Cartan action form over $n-1$-chains, and in Theorem 7 prove the principle of least action. It states that $n-1$-chains – "tubes" of integral curves of Nambu–Hamilton phase flow "passing through" a given $n-2$-chains – are extremals of the action. These results generalize the basic facts lying in the foundation of Hamiltonian mechanics (see, e.g., [5]).

In Sect. 4 we discuss possible approaches to the quantization of Nambu mechanics. Though we only mention those based on the deformation theory and Feynman path integral, our main result is the explicit construction of a special representation of the Nambu–Heisenberg commutation relation. For the ternary case this relation was introduced in [1] and has the form

$$\begin{aligned}[A_1, A_2, A_3] = A_1A_2A_3 - A_1A_3A_2 + A_3A_1A_2 - A_3A_2A_1 + A_2A_3A_1 \\ - A_2A_1A_3 = cI \, ,\end{aligned}$$

where A_1, A_2, A_3 are linear operators, I is a unit operator and c is a constant. Let $\mathbf{Z}[\rho]$ be a lattice of algebraic integers in quadratic number field $\mathbf{Q}[\rho]$, where $\rho^3 = 1$. We prove in Theorem 8 that Nambu–Heisenberg relation can be represented by operators A_i acting in the linear space $\mathbf{H} = \{|\omega\rangle, \omega \in \mathbf{Z}[\rho]\}$ by the following simple formulas:

$$A_1|\omega\rangle = (\omega + 1 + \rho)|\omega + 1\rangle, \quad A_2|\omega\rangle = (\omega + \rho)|\omega + \rho\rangle, \quad A_3|\omega\rangle = \omega|\omega + \rho^2\rangle .$$

This realization should be compared with canonical representation of Heisenberg commutation relations in the space of states of the harmonic oscillator given by creation-annihilation operators. Contrary to the latter case, our representation does not have a vacuum vector (at least in the conventional sense) and we do not know whether the analog of the Stone–von Neumann theorem (unitary equivalence of irreducible representations) is still true.

2. Nambu–Poisson Manifolds

In Hamiltonian mechanics a smooth manifold X is called a *Poisson manifold* and its function ring $A = C^\infty(X)$ – *algebra of observables*, if there exists a map $\{,\}: A \otimes A \mapsto A$ with the following properties.

1. Skew-symmetry

$$\{f_1, f_2\} = -\{f_2, f_1\} ,$$

for all $f_1, f_2 \in A$.

2. Leibniz rule (derivation property)

$$\{f_1 f_2, f_3\} = f_1\{f_2, f_3\} + f_2\{f_1, f_3\} .$$

3. Jacobi identity

$$\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0 ,$$

for all $f_1, f_2, f_3 \in A$.

The corresponding "binary operation" $\{,\}$ on A is called the *Poisson bracket* and plays a fundamental role in classical mechanics. Namely, according to Hamilton, dynamics on the *phase space* X is determined by a distinguished function $H \in A$ called the *Hamiltonian*, and is described by Hamilton equations of motion

$$\frac{df}{dt} = \{H, f\}, \quad f \in A .$$

When solution to Hamilton equations exists for all times $t \in \mathbb{R}$ and all initial data (this is so when X is compact), it defines Hamilton phase flow $x \mapsto g^t(x)$, $x \in X$, and evolution operator U_t: $A \mapsto A$,

$$U_t(f)(x) = f(g^t(x)), \quad x \in X, \quad f \in A .$$

Hamilton's dynamical picture is consistent if and only if the evolution operator U_t is an isomorphism of the algebra of observables A. This means that U_t is an algebra isomorphism, i.e. $U_t(f_1 f_2) = U_t(f_1)U_t(f_2)$ and, in addition, preserves the Poisson structure on A, i.e. $U_t(\{f_1, f_2\}) = \{U_t(f_1), U_t(f_2)\}$. It is easy to see (using the standard uniqueness theorem for ODE's) that the first property is equivalent to the Leibniz rule and the second one is equivalent to the Jacobi identity. We summarize these well known results in the following statement.

Theorem 1. *The evolution operator in Hamilton's dynamical picture is an isomorphism of algebra of observables $A = C^\infty(X)$ if and only if the phase space X is a Poisson manifold.*

The basic examples of Poisson manifolds are given by two-dimensional phase space $X = \mathbb{R}^2$ with coordinates x, y and Poisson bracket

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x} = \frac{\partial(f_1, f_2)}{\partial(x, y)},$$

and by its generalization – $X = \mathbb{R}^{2N}$ with coordinates $x_1, \ldots, x_N, y_1, \ldots, y_N$ and Poisson bracket

$$\{f_1, f_2\} = \sum_{i=1}^{N}\left(\frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial y_i} - \frac{\partial f_1}{\partial y_i}\frac{\partial f_2}{\partial x_i}\right).$$

Geometrically, the Poisson manifold X is characterized by a *Poisson tensor* η – a section of the exterior square $\bigwedge^2 TX$ of a tangent bundle TX of X, which defines Poisson structure by the formula

$$\{f_1, f_2\} = \eta\{df_1, df_2\}.$$

Jacobi identity is equivalent to the property that η has a vanishing Schouten bracket with itself. In local coordinates $(x_1, \ldots, x_N)$ on X Poisson tensor η is given by

$$\eta = \sum_{i,j=1}^{N} \eta_{ij}(x)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j},$$

and the Jacobi identity takes the form

$$\sum_{l=1}^{N}\left(\eta_{il}\frac{\partial \eta_{jk}}{\partial x_l} + \eta_{jl}\frac{\partial \eta_{ki}}{\partial x_l} + \eta_{kl}\frac{\partial \eta_{ij}}{\partial x_l}\right) = 0,$$

for all $i, j, k = 1, \ldots, N$.

Dynamics according to Nambu, consists in replacing the Poisson bracket by a ternary (n-ary) operation on algebra of observables A and requires two $(n-1)$ "Hamiltonians" H_1, H_2 $(H_1, \ldots, H_{n-1})$ to describe the evolution. This dynamical picture is consistent if and only if the evolution operator is an isomorphism of algebra of observables. Therefore, we propose the following definition.

Definition 1. *Manifold X is called a Nambu–Poisson manifold of order n if there exists a map $\{, \ldots, \}: A^{\otimes^n} \mapsto A$ – generalized Nambu bracket of order n, satisfying the following properties.*

1. Skew-symmetry

$$\{f_1, \ldots, f_n\} = (-1)^{\varepsilon(\sigma)}\{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\},$$

for all $f_1, \ldots, f_n \in A$ and $\sigma \in \mathrm{Symm}(n)$, where $\mathrm{Symm}(n)$ is a symmetric group of n elements and $\varepsilon(\sigma)$ is the parity of permutation σ.

2. Leibniz rule

$$\{f_1 f_2, f_3, \ldots, f_{n+1}\} = f_1\{f_2, f_3, \ldots, f_{n+1}\} + f_2\{f_1, f_3, \ldots, f_{n+1}\},$$

for all $f_1, \ldots, f_{n+1} \in A$.

3. Fundamental identity (FI)

$$\{\{f_1, \ldots, f_{n-1}, f_n\}, f_{n+1}, \ldots, f_{2n-1}\} + \{f_n, \{f_1, \ldots, f_{n-1}, f_{n+1}\}, f_{n+2}, \ldots, f_{2n-1}\}$$
$$+ \ldots + \{f_n, \ldots, f_{2n-2}, \{f_1, \ldots, f_{n-1}, f_{2n-1}\}\} = \{f_1, \ldots, f_{n-1}, \{f_n, \ldots, f_{2n-1}\}\},$$

for all $f_1, \ldots, f_{2n-1} \in A$.

Remark 1. Nambu bracket structure of order n on phase space X induces infinite family of "subordinated" Nambu structures of orders $n-1$ and lower, including the family of Poisson structures. Indeed, consider the $n = 3$ case and for any $H \in A$ define the bracket $\{\,,\}_H$ on X as

$$\{\psi, \phi\}_H = \{H, \psi, \phi\}$$

for all $\psi, \phi \in A$. Setting in FI $f_1 = f_3 = H$, we see that it turns into the Jacobi identity for the family of brackets $\{\,,\}_H$ parametrized by observables H. Conversely, such a family of Poisson brackets give rise to the Nambu bracket if certain matching conditions are satisfied. Namely, for any $\phi \in A$ define

$$D_\phi^H(f) = \{\phi, f\}_H, \quad f \in A,$$

which is the derivation of the Poisson bracket $\{\,,\}_H$ (Jacobi identity). Then the family $\{\,,\}_H$, $H \in A$ of Poisson brackets on X give rise to a Nambu bracket, defined as

$$\{f_1, f_2, f_3\} = \{f_2, f_3\}_{f_1},$$

if and only if

$$D_\phi^{H_1}(\{\psi, \chi\}_{H_2}) = \{D_\phi^{H_1}(\psi), \chi\}_{H_2} + \{\psi, D_\phi^{H_1}(\chi)\}_{H_2} + \{\psi, \chi\}_{D_\phi^{H_1}(H_2)}, \quad \phi, \psi, \chi \in A,$$

for all $H_1, H_2 \in A$. Indeed, it is easy to see that this equation – a derivation property of $D_\phi^{H_1}$ with respect to the whole family of Poisson brackets $\{\,,\}_H$ – is equivalent to FI (if one identifies $f_1 = H_1$, $f_2 = \phi$, $f_3 = \psi$, $f_4 = \chi$ and $f_5 = H_2$). Moreover, this condition for the case $H_1 = H_2 = H$ is equivalent to the Jacobi identity for the bracket $\{\,,\}_H$. The same is true for the general case, where for all $H_1, \ldots, H_{n-k} \in A$ the assignment

$$\{f_1, \ldots, f_k\}_{H_1 \quad H_{n-k}} = \{H_1, \ldots, H_{n-k}, f_1, \ldots f_k\}$$

defines a hierarchy of subordinated Nambu structures of orders $k = 2, \ldots, n-1$, parametrized by the elements in $\bigwedge^{n-k} A$. They all satisfy FI (which follows from FI for the "basic" structure of order n) and matching conditions of the same type as above.

Dynamics on the Nambu–Poisson manifold is determined by $n-1$ functions $H_1, \ldots, H_{n-1}$ and is described by generalized Nambu–Hamilton equations of motion

$$\frac{df}{dt} = \{H_1 \ldots, H_{n-1}, f\}, \quad f \in A. \tag{1}$$

The corresponding Nambu–Hamilton phase flow g^t defines evolution operator U_t, $U_t(f)(x) = f(g^t x)$, $x \in X$, for all $f \in A$.

The following theorem clarifies "dynamical" meaning of the concept of Nambu–Poisson manifolds.

Theorem 2. *Evolution operator in Nambu's dynamical picture is an isomorphism of algebra of observables $A = C^\infty(X)$ if and only if the phase space X is a Nambu–Poisson manifold.*

Proof. We need to prove that

$$U_t(\{f_1, \ldots, f_n\}) = \{U_t(f_1), \ldots, U_t(f_n)\} . \tag{2}$$

Since (2) is obviously valid at time $t = 0$, it is sufficient to show that both sides satisfy the same evolution differential equation. Denoting by $L \in \mathrm{Vect}(X)$ the vector field of the Nambu–Hamilton flow g^t, i.e.

$$L(f) = \{H_1, \ldots, H_{n-1}, f\} ,$$

we can express t-derivative of (2) as

$$L(\{f_1, \ldots, f_n\}) = \{L(f_1), f_2, \ldots, f_n\} + \{f_1, L(f_2), \ldots, f_n\} + \{f_1, f_2, \ldots, L(f_n)\} ,$$

which is nothing but FI specialized for the functions $H_1, \ldots, H_{n-1}$, $f_1, \ldots, f_n$. $\square$

Remark 2. Equivalent formulation of FI used in the previous proof can be stated explicitly that for any elements $H_1, \ldots, H_{n-1}$ the mapping $L\colon A \mapsto A$ is a derivation of the Nambu bracket.

Definition 2. *The observable $F \in A$ is called an integral of motion for Nambu–Hamilton system with Hamiltonians $H_1, \ldots, H_{n-1}$ if*

$$\{H_1, \ldots, H_{n-1}, F\} = 0 .$$

As an obvious corollary of FI we get the following result.

Theorem 3. *Nambu bracket of n integrals of motion is an integral of motion.*

Remark 3. As we have seen in Remark 1, the Nambu bracket structure of order n contains an infinite family of subordinated structures of lower degree, including Poisson bracket structure, with matching conditions between them. Therefore one might suspect that Nambu structure should be more "rigid" than its Poisson counterpart. This can be seen by comparing FI of order n with Jacobi identity – its special case when $n = 2$. As we know, the left-hand side of Jacobi identity, considered as a map from $A \otimes A \otimes A$ into A, is a derivation with respect to every argument. However, for the general case $n \geqq 3$ the difference between the left- and right-hand sides of FI, considered as a map from $2n - 1$-fold tensor product $A \otimes \ldots \otimes A$ into A, is a derivation only with respect to arguments $f_n, \ldots, f_{2n-1}$, and not to $f_1, \ldots, f_{n-1}$. This is because each of these groups appear in FI in their own way. Namely, analyzing FI, it is easy to see that observables from the first group appear only twice under the "double Nambu bracket," whereas there are n such terms for every member in the second group[3]. Consider for instance, such terms for argument f_{2n-1}. Denoting by $L_{i_1 \quad i_{n-1}}$ the vector field corresponding to

[3] This is not surprising since, according to interpretation in Remark 2, members from the second group may be considered as Hamiltonians for Nambu–Hamilton equations of motion, whereas members from the first group are "just" observables.

the Nambu–Hamilton phase flow with Hamiltonians $f_{i_1}, \ldots, f_{i_{n-1}}$, we can arrange these terms as commutator $[L_{n \quad 2n-2}, L_{1 \quad n-1}]\,(f_{2n-1})$, so they are still given by vector field action. This proves the derivation property with respect to f_{2n-1} (cf. standard arguments in Hamiltonian mechanics [5]). The same arguments apply to all members of the second group, but not to those of the first group. As we shall see later, this feature of FI implies strong constraints on the possible forms of the Nambu bracket.

Geometrically, Nambu structure of order n can be realized as

$$\{f_1, \ldots, f_n\} = \eta(df_1, \ldots, df_n)\,,$$

where η is a section of the n-fold exterior power $\bigwedge^n TX$ of the tangent bundle TX. In local coordinates $(x_1, \ldots, x_N)$ on X *Nambu tensor* η is given by

$$\eta = \sum_{t_1, \quad , i_n = 1}^{N} \eta_{i_1 \quad , i_n}(x) \frac{\partial}{\partial x_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_n}}$$

and should satisfy FI. As we mentioned earlier, FI implies strong constraints on n-tensor η.

First, taking into account Remark 3, we see that all terms containing second derivatives of $f_1, \ldots, f_{n-1}$ should vanish. This results in the following system of quadratic algebraic equations

$$N_{ij} + P(N)_{ij} = 0\,, \tag{3}$$

for all multi-indices $i = \{i_1, \ldots, i_n\}$ and $j = \{j_1, \ldots, j_n\}$ from set $\{1, \ldots, N\}$, where

$$\begin{aligned} N_{ij} = {} & \eta_{i_1 \quad i_n}\eta_{j_1 \quad j_n} + \eta_{j_n i_1 i_3 \quad i_n}\eta_{j_1 \quad j_{n-1} i_2} + \ldots + \eta_{j_n i_2 \quad i_{n-1} i_1}\eta_{j_1 \quad j_{n-1} i_n} \\ & - \eta_{j_n i_2 \quad i_n}\eta_{j_1 \quad j_{n-1} i_1}\,, \end{aligned} \tag{4}$$

and P is a permutation operator which interchanges first and $n+1^{\text{th}}$ indices (i.e. i_1 and j_1) of a $2n$-tensor N.

Second, all terms containing first derivatives of $f_1, \ldots, f_{2n-1}$ must vanish. This yields the following system of quadratic differential equations

$$\sum_{l=1}^{N} \left(\eta_{l i_2 \quad i_n} \frac{\partial \eta_{j_1 \quad j_n}}{\partial x_l} + \eta_{j_n l i_3 \quad i_n} \frac{\partial \eta_{j_1 \quad j_{n-1} i_2}}{\partial x_l} + \ldots + \eta_{j_n i_2 \quad i_{n-1} l} \frac{\partial \eta_{j_1 \quad j_{n-1} i_n}}{\partial x_l} \right) = 0\,, \tag{5}$$

for all indices $i_2, \ldots, i_n, j_1, \ldots, j_n = 1, \ldots, N$.

Thus the skew-symmetric n-tensor η defines Nambu bracket of order n if and only if it satisfies Eqs. (3)–(5).

This shows a significant difference between Nambu and Hamiltonian formulations – a constant skew-symmetric n-tensor η (which obviously satisfies (5)) for $n \geqq 3$ no longer "automatically" defines the Nambu bracket! To do so, it must satisfy the algebraic constraints (3)–(4). One may wonder whether there are any solutions at all to the algebraic-differential system (3)–(5)?

The following geometric interpretation of the tensor N provides the simplest examples. Let V be the N-dimensional linear space and V^* be its dual space. Any constant skew-symmetric n-tensor η can be interpreted as an element in the linear space $\bigwedge^n V$, which we also will denote by η. With every $\eta \in \bigwedge^n V$ one can associate a map N: $\bigwedge^{n-1} V^* \mapsto \bigwedge^{n+1} V$, defined by the formula

$$Na = i_a(\eta) \wedge \eta \in \bigwedge^{n+1} V\,,$$

for all $a \in \bigwedge^{n-1} V^*$. Here $i_a(\eta) \in V$ is given by $(i_a(\eta), v^*) = (\eta, a \wedge v^*)$ for any $v^* \in V^*$ and $(,)$ stands for the pairing between V and V^*. It is well known that equation $N = 0$ is equivalent to the condition that element $\eta \in \bigwedge^n V$ is decomposable, i.e. there exist $v_1, \ldots, v_n \in V$ such that $\eta = v_1 \wedge \ldots \wedge v_n$, and it is easy to verify that in coordinates map N is represented by $2n$-tensor N given by formula (4).

Thus we have proved the following result.

Theorem 4. *Let V be a linear space. Any decomposable element in $\bigwedge^n V$ endows V with the structure of Nambu–Poisson manifold of order n.*

In particular, consider Nambu's original example [1], when $X = \mathbb{R}^n$ is a phase space with coordinates $x_1, \ldots, x_n$ and "canonical" Nambu bracket is given by

$$\{f_1, \ldots, f_n\} = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}\,, \tag{6}$$

where the right-hand-side stands for the Jacobian of the mapping $f = (f_1, \ldots, f_n)$: $\mathbb{R}^n \mapsto \mathbb{R}^n$. From Theorem 4 we get

Corollary. *Euclidean space $\mathbb{R}^n$ with canonical Nambu bracket of order n is a Nambu–Poisson manifold.*

Proof. The canonical Nambu bracket of order n is given by the totally anti-symmetric n-tensor $\eta_{i_1 \ldots i_n} = \varepsilon_{i_1 \ldots i_n}$, which corresponds to the volume element of $\mathbb{R}^n$ and, therefore, is decomposable. □

Remark 4. We define *canonical transformations* as diffeomorphisms of the phase space which preserve Nambu bracket structure. For Nambu's example, linear canonical transormations form a group $SL(n, \mathbb{R})$.

Remark 5. In [1] Nambu also considered another example – the "direct sum" of canonical brackets. In this case $X = \mathbb{R}^{3n} = \bigoplus_{i=1}^n \mathbb{R}^3$ and the bracket is given by the following element:

$$\eta = e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6 + \ldots + e_{3n-2} \wedge e_{3n-1} \wedge e_{3n} \in \bigwedge^3 \mathbb{R}^{3n}\,,$$

where e_i, $i = 1, \ldots, 3n$ is a basis in $\mathbb{R}^{3n}$ induced by a standard basis in $\mathbb{R}^3$. (This bracket was used in [4] to write equations of motion of a particle interacting with the $SU(2)$ monopole.) However, it is easy to see that such a tensor η does not satisfy system (3) and, therefore, does not define Nambu bracket! This "explains" Nambu's observation [1] that linear canonical transformations for this bracket "decouple," i.e. form a direct product of n copies of $SL(3, \mathbb{R})$ – a fact he considered to be rather disappointing.

To summarize this discussion, we see that there is significant difference between Nambu and Hamiltonian mechanics with Nambu formulation being "more rigid."

The problem of constructing other examples of Nambu–Poisson manifolds is of great importance. We are not intending to address this problem here, but hoping to return to it sometime later on.

Remark 6. It seems that in the constant case there should be other examples of Nambu brackets besides those given by decomposable tensors, since Eqs. (3) do not immediately imply that $N = 0$[4].

Another class of examples is provided by non-constant tensors η. As we know, the linear Poisson bracket structure (Poisson structure on linear space such that Poisson bracket of linear functions is again linear) is equivalent to the Lie algebra structure on the dual space. One can ask what kind of structure the linear Nambu bracket introduces.

Definition 3. *A vector space V is called Nambu–Lie "gebra" of order n if there exists a map – Nambu–Lie bracket – $[,\dots,]: \bigwedge^n V \mapsto V$ such that*

$$[[v_1,\dots,v_{n-1},v_n],v_{n+1},\dots,v_{2n-1}] + [v_n,[v_1,\dots,v_{n-1},v_{n+1}],v_{n+2},\dots,v_{2n-1}]$$

$$+\dots+[v_n,\dots,v_{2n-2},[v_1,\dots,v_{n-1},v_{2n-1}]] = [v_1,\dots,v_{n-1},[v_n,\dots,v_{2n-1}]]\,,$$

for all $v_1,\dots,v_{2n-1} \in V$.

Remark 7. Definition of Nambu–Lie "gebras" can be stated in equivalent form as a condition that for any $v_1,\dots,v_{n-1} \in V$ "adjoint" map $[v_1,\dots,v_{n-1},.]$: $V \mapsto V$ is a derivation with respect to the Nambu-Lie bracket $[\,,\dots,]$.

Theorem 5. *Linear Nambu structures of order n are in one-to-one correspondence with Nambu–Lie "gebras" of order n on the dual space.*

Proof. Fundamental identity for linear functions is nothing but FI in the definition of Nambu–Lie "gebras." □

In coordinates, the linear Nambu structure of order n is given by $n+1$-tensor $c^k_{i_1\dots i_n}$:

$$\{x_{i_1},\dots,x_{i_n}\} = \sum_{k=1}^{N} c^k_{i_1\dots i_n} x_k\,.$$

"Structure constants" $c^k_{i_1\dots i_n}$ satisfy the overdetermined system of quadratic algebraic equations which follows from (3)–(5). We are planning to analyze this interesting structure elsewhere.

Now we present several simple examples of dynamical systems which admit Nambu formulation.

Example 1. It goes back to Nambu [1] and is given by Euler equations for the angular momentum of a rigid body in three dimensions. These equations admit both Hamiltonian formulation with respect to the linear Poisson bracket on $\mathbb{R}^3 \cong su(2)^*$, where Hamiltonian is given by kinetic energy, and Nambu formulation with respect to canonical ternary Nambu bracket on $\mathbb{R}^3$ and two Hamiltonians – kinetic energy and total angular momentum.

---

[4] Larry Lambe, using symbolic computations technique, presents an explicit form of a Groebner basis of a polynomial ideal of all relations (3) in the case $n = 3$, $N = 6$. It follows from his analysis that in this case all solutions are represented by decomposable tensors.

Example 2. The Lagrange system (sometimes called Nahm's system in the theory of static $SU(2)$-monopoles, see, e.g., [8, 9]) on $\mathbb{R}^3$, which is given by the following equations of motion:

$$\frac{dx_1}{dt} = x_2 x_3, \quad \frac{dx_2}{dt} = x_1 x_3, \quad \frac{dx_3}{dt} = x_1 x_2 ,$$

can be written in Nambu form

$$\frac{dx_i}{dt} = \{H_1, H_2, x_i\}, \quad i = 1, 2, 3,$$

where $H_1 = x_1^2 - x_2^2, H_2 = x_1^2 - x_3^2$. Integrals H_1 and H_2 confine the phase flow to the intersection of two quadrics in $\mathbb{R}^3$ – a locus of an elliptic curve so that the system can be integrated by elliptic functions. There exists another system with quadratic nonlinearity – the so-called Halphen system, which is related to the Lagrange system (see, e.g., [8, 9]). It can be integrated in terms of modular forms (see, e.g., [9]) and does not admit global (single-valued) integrals of motion. However, this system has two multi-valued integrals which play the role of two Hamiltonians in Nambu formulation [10].

Example 3. Let $X = \mathbb{R}^n$ be the phase space with canonical Nambu bracket of order n and let the elementary symmetric functions of n variables $x_1, \ldots, x_n$ to be Hamiltonians

$$H_1 = s_1, \ldots, H_{n-1} = s_{n-1} ,$$

where $(x - x_1) \ldots (x - x_n) = x^n - s_1 x^{n-1} \ldots \pm s_n$. Nambu–Hamilton equations of motion

$$\frac{dx_i}{dt} = \{H_1, \ldots, H_{n-1}, x_i\}, \quad i = 1, \ldots, n ,$$

can be written as

$$\frac{dx_i}{dt} = \frac{\partial^n f}{\partial x_i^n}, \quad i = 1, \ldots, n ,$$

where

$$f(x_1, \ldots, x_n) = \prod_{1 \leqq i < j \leqq n} (x_i - x_j) ,$$

which might suggest a generalization of gradient flows.

3. Canonical Formalism and Action Principle

Here we extend the canonical formalism of Hamiltonian mechanics, based on Poincaré–Cartan integral invariant and on the principle of least action (see, e.g., [5]), to the case of Nambu mechanics. We illustrate essential features of our approach on the simplest example of Nambu–Poisson manifold $X = \mathbb{R}^3$ with canonical Nambu bracket.

Let $\tilde{X} = \mathbb{R}^4$ be the extended phase space with coordinates x, y, z, t.

Definition 4. *The following 2-form $\omega^{(2)}$ on $\tilde{X}$*

$$\omega^{(2)} = x dy \wedge dz - H_1 dH_2 \wedge dt \,, \tag{7}$$

is called generalized Poincaré–Cartan integral invariant-action form for Nambu mechanics (cf. [5]).

Consider Nambu–Hamilton equations with Hamiltonians H_1 and H_2

$$\frac{df}{dt} = \{H_1, H_2, f\} = L(f) \,,$$

where

$$L = L_1 \frac{\partial}{\partial x} + L_2 \frac{\partial}{\partial y} + L_3 \frac{\partial}{\partial z} \in \mathrm{Vect}(X) \,,$$

and

$$L_1 = \frac{\partial(H_1, H_2)}{\partial(y, z)}, \quad L_2 = \frac{\partial(H_1, H_2)}{\partial(z, x)}, \quad L_3 = \frac{\partial(H_1, H_2)}{\partial(x, y)} \,.$$

Denote by

$$\tilde{L} = L + \frac{\partial}{dt} \in \mathrm{Vect}(\tilde{X})$$

corresponding vector field on the extended phase space $\tilde{X}$.

Theorem 6. *Vector field $\tilde{L} \in \mathrm{Vect}(\tilde{X})$ is a line field of the 3-form $d\omega^{(2)}$*, i.e.

$$i_{\tilde{L}}(d\omega^{(2)}) = 0 \,.$$

Proof. Since

$$\begin{aligned} d\omega^{(2)} &= dx \wedge dy \wedge dz - dH_1 \wedge dH_2 \wedge dt \\ &= dx \wedge dy \wedge dz - L_3 dx \wedge dy \wedge dt \\ &\quad + L_2 dx \wedge dz \wedge dt - L_1 dy \wedge dz \wedge dt \,, \end{aligned}$$

direct calculation (cf. [5]) shows that

$$\begin{aligned} i_{\tilde{L}}(d\omega^{(2)}) = d\omega^{(2)}(\tilde{L}, \ldots) &= L_1 dy \wedge dz - L_2 dx \wedge dz + L_3 dx \wedge dy \\ &- L_3(L_1 dy \wedge dt - L_2 dx \wedge dt) + L_2(L_1 dz \wedge dt - L_3 dx \wedge dt) \\ &- L_1(L_2 dz \wedge dt - L_3 dy \wedge dt) - L_3 dx \wedge dy + L_2 dx \wedge dz \\ &- L_1 dy \wedge dz = 0 \,. \end{aligned}$$

□

As in the case of Hamiltonian mechanics, this theorem has an important corollary (cf. [5]). Namely, let c be a closed 2-chain in $\tilde{X}$, g^t be Nambu–Hamilton phase flow and 3-chain $J^t c = \{g^\tau c, 0 \leqq \tau \leqq t\}$ be a trace of the chain c under the isotopy g^τ.

Corollary.

$$\int_c \omega^{(2)} = \int_{g^t(c)} \omega^{(2)} \,.$$

Proof. Using formula $\partial(J^t c) = c - g^t(c)$, the Stokes theorem and Theorem 6, we get

$$\int_c \omega^{(2)} - \int_{g^t(c)} \omega^{(2)} = \int_{J^t c} d\omega^{(2)} = 0 .$$

□

Next we formulate the principle of least action. Recall that in Hamiltonian mechanics this principle states (see, e.g., [5]) that classical trajectory – the integral curve γ of Hamilton's phase flow with initial and final points (p_0, q_0, t_0) and (p_1, q_1, t_1), is an extremal of the action functional

$$A(\gamma) = \int_\gamma (pdq - Hdt)$$

in the class of all paths connecting initial and final points in given n-dimensional subspaces $(t = t_0, q = q_0)$ and $(t = t_1, q = q_1)$ in the extended phase space.

Definition 5. *The functional*

$$A(C_2) = \int_{C_2} \omega^{(2)}$$

– the integral of the action form over 2-chains in extended phase space $\tilde{X}$, is called action functional for Nambu mechanics.[5]

Let γ be a closed 1-chain in X. Define a 2-chain Γ in $\tilde{X}$ as a trace of γ under the isotopy $g^t, t_0 \leqq t \leqq t_1$, so that $\partial\Gamma = \gamma_{t_0} - \gamma_{t_1}$, where $\gamma_{t_0} = \gamma$, $\gamma_{t_1} = g^{t_1 - t_0}(\gamma)$. The following theorem states the principle of least action for Nambu mechanics.

Theorem 7. *A 2-chain Γ is an extremal of the action $A(C)$ in the class of all 2-chains C whose boundaries – 1-chains c_{t_0} and c_{t_1} – have the same projections onto the yz-planes as a given 1-chains γ_{t_0} and γ_{t_1}.*

Proof. As in the case of Hamiltonian mechanics, this statement follows from the previous theorem (cf. [5]).

Here is another proof, based on direct calculation. Assume that Γ admits a parametrization $x = x(s, t)$, $y = y(s, t)$, $z = z(s, t)$, where $0 \leqq s \leqq 1$, $t_0 \leqq t \leqq t_1$, such that its time-slices are closed curves, i.e. for all t functions x, y, z are periodic in s with period 1. Variations δx, δy, δz satisfy conditions $\delta y(s, t_0) = \delta y(s, t_1) = \delta z(s, t_0) = \delta z(s, t_1) = 0$ for all $0 \leqq s \leqq 1$. We have explicitly

$$A(C) = \iint_\Pi \left\{ x\left(\frac{\partial y}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial z}{\partial s}\right) - H_1\left(\frac{\partial H_2}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial H_2}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial H_2}{\partial z}\frac{\partial z}{\partial s}\right)\right\} dsdt.$$

where $\Pi = \{0 \leqq s \leqq 1, t_0 \leqq t \leqq t_1\}$. Using the Stokes theorem and the properties of δy and δz, we obtain the following expression for the variation of the action:

$$\delta A(\Gamma) = \iint_\Pi \left\{ \left(\left\{ \frac{\partial z}{\partial t} - \frac{\partial(H_1, H_2)}{\partial(y, z)} \right\} \frac{\partial y}{\partial s} - \left\{ \frac{\partial y}{\partial t} - \frac{\partial(H_1, H_2)}{\partial(z, x)} \right\} \frac{\partial z}{\partial s} \right) \delta x \right.$$

[5] This approach is related to the one developed by M. Fréchet in 1905 in [11]. I am grateful to Frank Nijhoff who drew my attention to this reference. M. Fréchet also acknowledged an earlier paper by M. Volterra [12], published in 1890.

$$+\left(\left\{\frac{\partial x}{\partial t}-\frac{\partial(H_1,H_2)}{\partial(y,z)}\right\}\frac{\partial z}{\partial s}-\left\{\frac{\partial z}{\partial t}-\frac{\partial(H_1,H_2)}{\partial(x,y)}\right\}\frac{\partial x}{\partial s}\right)\delta y$$

$$+\left(\left\{\frac{\partial y}{\partial t}-\frac{\partial(H_1,H_2)}{\partial(z,x)}\right\}\frac{\partial x}{\partial s}-\left\{\frac{\partial x}{\partial t}-\frac{\partial(H_1,H_2)}{\partial(y,z)}\right\}\frac{\partial y}{\partial s}\right)\delta z\Bigg\}dsdt\,,$$

which shows that $\delta A(\Gamma)=0$ for all admissible variations δx, δy, δz, if Γ consists of integral curves of Nambu–Hamilton phase flow. □

Remark 8. The converse statement is also true: extremals of the action functional are "world-sheets" $x(s,t)$, $y(s,t)$, $z(s,t)$ consisting of families of integral curves of Nambu–Hamilton phase flow parametrized by s. Indeed, these extrema are characterized by the condition that the cross product of the following vectors in $\mathbb{R}^3$:

$$\left(\frac{\partial x}{\partial t}-\{H_1,H_2,x\},\frac{\partial y}{\partial t}-\{H_1,H_2,y\},\frac{\partial z}{\partial t}-\{H_1,H_2,z\}\right)\text{ and }\left(\frac{\partial x}{\partial s},\frac{\partial y}{\partial s},\frac{\partial z}{\partial s}\right)$$

is zero, which means that there exists a function $\alpha(s,t)$ such that

$$\frac{\partial x}{\partial t}-\alpha(s,t)\frac{\partial x}{\partial s}=\frac{\partial(H_1,H_2)}{\partial(y,z)}\,,$$

$$\frac{\partial y}{\partial t}-\alpha(s,t)\frac{\partial y}{\partial s}=\frac{\partial(H_1,H_2)}{\partial(z,x)}\,,$$

$$\frac{\partial z}{\partial t}-\alpha(s,t)\frac{\partial z}{\partial s}=\frac{\partial(H_1,H_2)}{\partial(x,y)}\,.$$

Changing parametrization $s\mapsto s'=s+\alpha(s,t)t$, $t\mapsto t'=t$, we see that this system reduces to the family of Nambu-Hamilton equations parametrized by s.

Remark 9. Comparing the principles of least action for Nambu and Hamiltonian mechanics enables us to see that "configuration space" of Nambu mechanics of order 3 constitute "two-thirds" of the phase space.

Generalization of the presented results for the case of Nambu bracket of order n is straightforward. The analog of Poincaré–Cartan integral invariant is defined as the following $n-1$-form

$$\omega^{(n-1)}=x_1dx_2\wedge\ldots\wedge dx_n-H_1dH_2\wedge\ldots\wedge dH_{n-1}\wedge dt\,,$$

and the action functional is given by

$$A(C_{n-1})=\int_{C_{n-1}}\omega^{(n-1)}$$

and is defined on the $n-1$-chains in the extended phase space. In this formulation admissible variations are those which do not change projections of the boundary ∂C_{n-1} on the $x_2x_3\ldots x_n$-hyperplanes; in this case the "share" of "configuration space" in a phase space is $1-1/n$.

Remark 10. This construction of action form and action functional is somewhat similar to the construction of cyclic cocycles in Connes' approach to the non-commutative differential geometry [13].

4. Quantization

There exist several different (in a certain sense, equivalent) points of view on the quantization problem.

One is based on the approach which uses *deformation theory* of associative algebras. It considers quantization as a deformation of the (commutative) algebra of classical observables on the phase space in the "direction" defined by a given Poisson (or symplectic) structure [14].

Namely, let X be a Poisson manifold with the Poisson bracket $\{\,,\}$ and algebra of classical observables $A = C^\infty(X)$. The one-parameter family $\{A_h\}$ of associative algebras is called *quantization* of commutative algebra A of classical observables, if the following conditions are satisfied.

1. Algebra A is included into this family

$$A_h|_{h=0} = A\,,$$

(or, in a formal algebraic category, $A \cong A_h/hA_h$).

2. All algebras A_h are isomorphic to A (or rather to $A[[h]]$ in formal algebraic category) as linear spaces.

3. Denoting by $*_h$ (associative) product in A_h ($*$-product in the terminology of [14]), one has the expansion

$$f_1 *_h f_2 = f_1 f_2 + \frac{h}{2} C_2(f_1, f_2) + O(h^2)\,,$$

with the property

$$C_2(f_1, f_2) - C_2(f_2, f_1) = \{f_1, f_2\}\,.$$

The latter property is often referred to as the *correspondence principle*

$$\{f_1, f_2\} = \lim_{h\to 0} \frac{1}{h}(f_1 *_h f_2 - f_2 *_h f_1)$$

between classical and quantum mechanics.

Deformation theory approach can be carried out explicitly in the case $X = \mathbb{R}^2$ (or $\mathbb{R}^{2n}$) with canonical Poisson bracket, and yields the celebrated *Hermann Weyl's quantization* scheme. Corresponding $*$-product $*_h$ – a map from $A \otimes A$ into A, is given by composition of a usual commutative point-wise product on A with the following bilinear pseudo-differential operator:

$$\exp\left(\frac{h}{2}\{\,,\}\right) = \exp\left(\frac{h}{2}\left\{\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\right\}\right) : A \otimes A \mapsto A \otimes A\,.$$

One might use the same approach towards quantization of Nambu mechanics. Namely, suppose that we try to include the "usual" ternary product – a map from $A \otimes A \otimes A$ into A, given by a point-wise multiplication of classical observables, into the "new" ternary operation $(\,,\,,)_h$ which depends on the parameter h and satisfies certain natural properties (analogous to associativity condition in the binary case) and correspondence principle

$$\{f_1, f_2, f_3\} = \lim_{h\to 0} \frac{1}{h} \mathrm{Alt}(f_1, f_2, f_3)_h\,,$$

where Alt denotes complete anti-symmetrization with respect to arguments f_1, f_2, f_3.

We can easily generalize Weyl's formula by letting the new ternary operation $(,,)_h$ be the composition of a point-wise ternary product and the following trilinear pseudo-differential operator:

$$\exp\left(\frac{h}{6}\{\ ,\ ,\ \}\right) = \exp\left(\frac{h}{6}\left\{\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\right\}\right) : A \otimes A \otimes A \mapsto A \otimes A \otimes A \ .$$

This formula generalizes verbatim for the canonical Nambu bracket of order n.

Remark 11. One should note the following.

(i) Contrary to the usual binary case, the point-wise ternary product of classical observables looks less natural.

(ii) "Natural" constraints for ternary operations generalizing the usual associativity constraint are, apparently, not well understood.

Nevertheless, the proposed deformation has certain "appeal" and should be analyzed further. It might lead to the (partial) answer to the problem in (ii).

Remark 12. Recently, R. Lawrence [15] proposed a system of axioms for linear spaces with n-ary operations (n-algebras using her terminology). It looks like the "deformed" product $(\ ,\ ,\ )_h$, as well as ternary point-wise product, does not satisfy them. However, one should have in mind that the approach in [15] generalizes certain combinatorial properties of algebraic operations (notably the Stasheff polyhedron), whereas our approach is based on "dynamics."

Another approach uses *Feynman's path integral formulation* of quantum mechanics. For the quantum particle, described classically by Hamiltonian system, it gives the probability amplitude of the transition from state $|q_0\rangle$ at time $t = t_0$ into state $|q_1\rangle$ at time $t = t_1$ as a functional integral of the exponential of the classical action

$$\exp\left\{\frac{i}{h}A(\gamma)\right\}$$

over all "histories" γ with respect to the "Liouville measure"

$$\frac{i}{h}DpDq$$

in the functional space of all "histories".

The principle of the least action for Nambu mechanics can be used to formulate a similar rule. However, the special form of the action principle (see Theorem 7) requires that quantum states $|y(s), z(s)\rangle$ should be parametrized by loops $y(s), z(s)$ rather than by points in the "configuration" space. This natural appearance of loops looks quite appealing. It suggests that one might still have particles as point-like objects in the classical picture, and dynamics of loops (or, generally, $n-2$-closed chains) in the quantum picture.

Finally, we present yet another approach to quantization – *canonical formalism.* It is based on *Heisenberg commutation relations*, which for the phase space $X = \mathbb{R}^2$ with canonical Poisson bracket, looks as follows (in a complex form):

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = I\ ,$$

where operators $a^\dagger$, a act in linear space of quantum states (Hilbert space of states). They have the following realization in space $\mathbf{H}_2$ with the basis $\{|n\rangle\}_{n\geqq 0}$ parametrized by non-negative rational integers:

$$a|n\rangle = \sqrt{n}|n-1\rangle,\ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle .$$

The vector $|0\rangle$ plays the role of a vacuum state and operators $a^\dagger$, a are called creation-annihilation operators.

Heisenberg commutation relations, being one of the fundamental principles of quantum mechanics, have remarkable mathematical properties. In particular, one has the celebrated Stone–von Neumann theorem stating that all irreducible representations of Heisenberg commutation relations are unitary equivalent. Application of the Stone–von Neumann theorem to the linear canonical transformations gives a certain projective representation of the symplectic group, which was shown by André Weil [16] to play a fundamental role in the arithmetics of quadratic forms and quadratic reciprocity for number fields.

In [1] Nambu proposed the following generalization of the Heisenberg commutation relation

$$\begin{aligned}[A_1, A_2, A_3] = A_1A_2A_3 - A_1A_3A_2 + A_3A_1A_2 - A_3A_2A_1 \\ + A_2A_3A_1 - A_2A_1A_3 = cI\ , \end{aligned} \tag{8}$$

where A_1, A_2, A_3 are linear operators, I is a unit operator and c is a constant. We will call (8) the *Nambu–Heisenberg relation.*

Remark 13. Although Nambu–Heisenberg relation satisfies the correspondence principle

$$\frac{1}{h}[\ ,\ ,\ ] \mapsto \{\ ,\ ,\ \}\ ,$$

a priori it is not clear that observables in quantum Nambu mechanics should be realized by linear operators. Therefore, form (8) might look rather *ad hoc*. It was already pointed out by Nambu [1] that, contrary to the usual commutator [,], "ternary commutator" [, ,] is not a derivation with respect to the operator product, which creates certain problems in formulating quantum dynamics (see [1] for a detailed discussion). From our point of view, it looks like there is no reason to impose such derivation property on the operator product. "Triple commutator" [, ,], should be considered to define a Nambu–Lie "gebra" structure on quantum observables. Its property of being a derivation of this structure (see Definition 3) will imply consistency of quantum Nambu dynamics.

Instead of proceeding with this discussion (which we are planning to do elsewhere), here we will only present the explicit realization of Nambu–Heisenberg relation (8).

Let $\mathbf{Q}[\rho]$ be a quadratic number field with $1 + \rho + \rho^2 = 0$, i.e.

$$\rho = \frac{-1 + \sqrt{-3}}{2}\ ,$$

and let $\mathbf{Z}[\rho]$ be a ring of algebraic integers in $\mathbf{Q}[\rho]$, i.e.

$$\omega = m_1 + m_2\rho \in \mathbf{Z}[\rho],\ m_1, m_2 \in \mathbf{Z}\ .$$

Denote by the $\mathbf{H}_3$ linear space with the basis $\{|\omega\rangle\}$ parametrized by $\mathbf{Z}[\rho]$. Direct calculation proves the following result.

Theorem 8. *The Nambu–Heisenberg relation with $c = \rho - \bar{\rho} = \sqrt{-3}$ admits the following representation:*

$$A_1|\omega\rangle = (\omega + 1 + \rho)|\omega + 1\rangle,\ A_2|\omega\rangle = (\omega + \rho)|\omega + \rho\rangle,\ A_3|\omega\rangle = \omega|\omega + \rho^2\rangle$$

in space $\mathbf{H}_3$.

The following arguments explain why this type of representation is possible. Using the analogy with $n = 2$ case, consider representation (8) of the form

$$A_1|\omega\rangle = f_1(\omega)|\omega + 1\rangle, \quad A_2|\omega\rangle = f_2(\omega)|\omega + \rho\rangle, \quad A_3|\omega\rangle = f_3(\omega)|\omega + \rho^2\rangle .$$

Because of $1 + \rho + \rho^2 = 0$, vectors $|\omega\rangle$ are eigen-vectors for all possible triple products $A_{i_1}A_{i_2}A_{i_3}$ of A_1, A_2, A_3 and Nambu–Heisenberg relation (8) reduces to functional equation for the unknowns $f_1(\omega)$, $f_2(\omega)$, $f_3(\omega)$. Theorem 8 exhibits one of its solutions. We can obtain other solutions using the following trick. Equation (8) is invariant under the similarity transformation

$$A_i \mapsto \tilde{A}_i = U^{-1}A_iU, \quad i = 1, 2, 3,$$

where $U\colon \mathbf{H}_3 \mapsto \mathbf{H}_3$ is an invertible linear operator. Choose U to be diagonal in the standard basis of $\mathbf{H}_3$, i.e.

$$U|\omega\rangle = u(\omega)|\omega\rangle, \quad u(\omega) \neq 0, \quad \omega \in \mathbf{Z}[\rho] .$$

Then operators $\tilde{A}_i$ are represented by the same formulas as A_i's with

$$\tilde{f}_1(\omega) = u^{-1}(\omega + 1)u(\omega)f_1(\omega), \quad \tilde{f}_2(\omega) = u^{-1}(\omega + \rho)u(\omega)f_2(\omega),$$

$$\tilde{f}_3(\omega) = u^{-1}(\omega + \rho^2)u(\omega)f_3(\omega).$$

Now assume that A_1 and A_2 commute and f_1, f_2 are non-zero for all $\omega \in \mathbf{Z}[\rho]$. Then one can choose $u(\omega)$ such that $\tilde{f}_1(\omega) = \tilde{f}_2(\omega) = 1$. Indeed, $u(\omega)$ can be found as the solution of the following compatible system (because of the commutativity of A_1 and A_2) of difference equations:

$$u(\omega + 1) = f_1(\omega)u(\omega) ,$$

$$u(\omega + \rho) = f_2(\omega)u(\omega) .$$

(Provided certain conditions at "infinity" for functions f_1 and f_2 are satisfied, $u(\omega)$ can be given as a semi-infinite product over the lattice $\mathbf{Z}[\rho]$.) Therefore, the original functional equation for f_i's reduces to the following simple equation:

$$\tilde{f}_3(\omega + 1) - \tilde{f}_3(\omega) = c ,$$

which can be easily solved.

"Gauging back" these solutions with $\tilde{f}_1 = \tilde{f}_2 = 1$ we obtain other solutions.

Solution presented in Theorem 8 does not belong to this class: operators A_1 and A_2 (as well as other pairs) do not commute and, since operators A_1, A_2 and A_3 annihilate vectors $|\rho^2\rangle$, $|1 + \rho^2\rangle$ and $|0\rangle$, corresponding functions f_i have zeros. This implies that in this case the corresponding invertible operator U does not exist. This shows a special role played by representation in Theorem 8:

although there is no vacuum vector in a conventional sense, there exists a triple of vectors

$$|\rho^2\rangle, \quad |1+\rho^2\rangle, \quad |0\rangle$$

with "similar" properties.

Remark 14. It should be noted that this realization of the Nambu–Heisenberg relation must be supplemented with the analog of Hermitian anti-involution, which in this case should be an operation of order 3 (since we are dealing with a triple instead of a pair).

Finally, consider briefly the general case of Nambu mechanics of order n. One can postulate the following form of the Nambu–Heisenberg relation

$$\sum_{\sigma \in \mathrm{Symm}(n)} (-1)^{\varepsilon(\sigma)} A_{\sigma(1)} \ldots A_{\sigma(n)} = cI \,. \tag{9}$$

Assuming that n is a prime (greater than 2), consider representation of (9) in linear space $\mathbf{H}_n$ with basis $\{|\omega\rangle\}$ parametrized by algebraic integers $\mathbf{Z}[\rho]$ in the cyclotomic field of n^{th} root of unity, i.e. $\rho^n = 1$, or

$$1 + \rho + \ldots + \rho^{n-1} = 0 \,.$$

Assuming that operators A_i are represented in a similar way:

$$A_i|\omega\rangle = f_i(\omega)|\omega + \rho^{i-1}\rangle, \quad i = 1, \ldots, n \,,$$

one might try to repeat the same trick: "gauge out" the coefficients $f_1, \ldots, f_{n-1}$. Namely, assume that f_i's are such that the following system of equations

$$u(\omega + \rho^{i-1}) = f_i(\omega)u(\omega), \quad i = 1, \ldots, n-1 \,,$$

is compatible and has a solution which gives rise to an invertible operator U. In this case, commuting operators $\tilde{A}_i = U^{-1}A_iU$ for $i = 1, \ldots, n-1$, will represent $n-1$ basic translations (generators) of the lattice $\mathbf{Z}[\rho]$. However, analyzing relation (9) for general n, we see that all its terms contain at least one pair of A_i's with $i \leqq n-1$ as nearest neighbors. These terms will appear twice in (9) and commutativity of $\tilde{A}_i$'s implies that they will mutually cancel each other!

This shows that in the higher order case, the Nambu–Heisenberg relation does not admit simple solutions with $n-1$ commuting operators A_i's. However it has a solution similar to that in Theorem 8.

5. Conclusion

Though this paper poses more questions than provides answers, we feel that the subject of higher order algebraic operations might be relevant for future development of mathematical structures related to physical problems. We suspect that it might clarify problems related to generalizations of the integrability concept (Yang–Baxter equation, Poisson-Lie groups, quantum groups) for the higher dimensional case (Zamolodchikov's tetrahedron equations, Frenkel–Moore solutions, 2-categories). Concluding on a highly speculative note, one might suggest that perhaps these higher order structures can be also relevant in the arithmetics of forms of higher degrees, higher reciprocity laws in number theory, etc.

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