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Existence of Self-Similar Blow-Up Solutions for Zakharov Equation in Dimension Two. Part I

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Abstract: We consider the Zakharov equation in space dimension two

$$\begin{cases} iu_t = -\Delta u + nu\,, \\ \dfrac{1}{c_0^2}\, n_{tt} = \Delta n + \Delta |u|^2\,. \end{cases}$$

We prove the existence of blow-up solutions (stable "self-similar" blow-up solutions) for this problem and we study various properties of these solutions.

I. Introduction

In this paper, we consider the Zakharov system in space dimension two:

$$(\mathrm{I}_{c_0}) \qquad \begin{cases} iu_t = -\Delta u + nu\,, & (1.1) \\ \dfrac{1}{c_0^2}\, n_{tt} = \Delta n + \Delta |u|^2\,, & (1.2) \\ u(0) = \phi_0\,, \quad n(0) = n_0\,, \quad n_t(0) = n_1\,, \end{cases}$$

where $c_0 > 0$, Δ is the Laplace operator on $\mathbb{R}^2$, $u: [0, T)\times\mathbb{R}^2 \to \mathbb{C}$, $n: [0, T)\times\mathbb{R}^2 \to \mathbb{R}$ and ϕ_0, n_0, n_1 are initial data.

This model is often used to describe Langmuir waves in plasmas when the electric field is one dimensional. u represents the envelope of the electric field and n is the large scale flucatuation of the ionic density. We remark that the subsonic limit of these equations ($c_0 \to +\infty$) is formally

$$(\mathrm{I}_\infty) \qquad \begin{aligned} iu_t &= -\Delta u - |u|^2 u\,, & (1.3) \\ u(0) &= \phi_0\,. & (1.4) \end{aligned}$$

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This is the Schrödinger Equation in dimension two with critical exponent. This power is critical in the following sense: It is the smaller power for which blow-up occurs in finite time for some class of initial data ($\phi_0 \in H^1$).

From the mathematical point of view, little is known on equation (I_{c_0}) (with a finite c_0). The only existing results are related
– on the one hand to the local existence in time of regular solutions
– on the other hand to the limit as $c_0 \to +\infty$ of uniformly bounded regular solutions of (I_{c_0}).

In particular, there are no results on the existence and behavior of solutions of equation (I_{c_0}) which become singular in finite time (except some strong numerical evidences: see Landman, Papanicolaou, C. Sulem, P.L. Sulem, Wang [10]). The aim of this paper is to prove some existence results of singular solutions.

Existence of strong solutions for regular initial data has been investigated by several authors (C. and P.L. Sulem [21], Schochet and Weinstein [18], Ozawa and Tsutsumi [14]). One can show that, for initial data $(\phi_0, n_0, n_1) \in H^2 \times H^1 \times L^2$, there is a unique solution (u, n, n_t) in $H_2 = H^2 \times H^1 \times L^2$ on $[0, T_2)$ and
– $T_2 = +\infty$ or
– $|u(t)|_{H^2} + |n(t)|_{H^1} + |n_t(t)|_{L^2} \to +\infty$ as $t \to T_2$.

This result was first obtained by C. and P.L. Sulem [21] and by Schochet and Weinstein [18] with an extra assumption on $n_t : n_t \in \hat{H}^{-1}$, where $\hat{H}^{-1}$ is the space of functions u such that $\exists v : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$n = -\nabla \cdot v \quad \text{and} \quad v \in L^2$$

and

$$|u|_{\hat{H}^{-1}} = |v|_{L^2} .$$

This condition was later removed by Ozawa, Tsutsumi [14].

The question to know if this space is optimal for local existence is open. For example the case of the energy space for the Cauchy problem [see below (1.6)–(1.8)] $\hat{H}_1 = H^1 \times L^2 \times \hat{H}^{-1}$, (or even $H_1 = H^1 \times L^2 \times H^{-1}$) is unknown.

For a solution of (I_{c_0}) in H_1 or H_2, the conservation of mass gives

$$\forall t , \quad |u(t)|_{L^2} = |\phi_0|_{L^2} . \tag{1.5}$$

In addition, if $n_t(0) \in \hat{H}^{-1}$ then $\forall t$, $n_t(t) \in \hat{H}^{-1}$ and (1.2) may be written in the form

$$\begin{aligned} & n_t + \nabla \cdot v = 0 , \\ & \frac{1}{c_0^2} v_t + \nabla n = -\nabla |u|^2 \end{aligned} \tag{1.6}$$

In this case ($n_t(t) \in \hat{H}^{-1}$), we have an another law (Energy conservation), that is

$$\mathscr{H}(t) = \mathscr{H}(0) , \tag{1.7}$$

where

$$\begin{aligned} \mathscr{H} &= \mathscr{H}(u, n, v) \\ &= \int_{\mathbb{R}^2} \left(|\nabla u(x)|^2 + n(x) \, |u(x)|^2 + \frac{1}{2c_0^2} \, |v(x)|^2 + \frac{1}{2} \, n^2(x) \right) dx . \end{aligned} \tag{1.8}$$

For large initial data, heuristic arguments and numerical simulations suggest a finite time blow-up (Landman, Papanicolaou, C. Sulem, P.L. Sulem, Wang [10]). Nevertheless, no rigorous results of collapse are presently available. In contrast, a lot of work is done for the formal limit as $c_0 \to \infty$, that is the Nonlinear Schrödinger equation with critical exponent. Unfortunately, up to now, the limit as $c_0 \to +\infty$ has been proved rigorously only for uniformly bounded solutions.

In [1], H. Added and S. Added (see also Schochet and Weinstein [18]), if the initial data are very regular (belong to the Schwarz space) and compatible ($n_0 + |u_0|^2 = 0$), then (u_{c_0}, n_{c_0}) converges uniformly in time as $c_0 \to +\infty$ to $(u_\infty, -|u_\infty|^2)$ on compact sets of $[0, T_\infty)$, where u_∞ is the solution of (I_∞) with the same initial data, and T_∞ its blow-up time.

Since the convergence is on compact sets of $[0, T_\infty)$, we do not have by this kind of techniques any information on the singular behavior of (u_{c_0}, n_{c_0}) for c_0 fixed and large.

Equation (I_∞) has a unique solution in the space H^1 and there is $T > 0$ such that for all $t \in [0, T)$, either $T = +\infty$ or $\lim\limits_{t\to T} |u(t)|_{H^1} = +\infty$ (see Ginibre and Velo [5], Kato [8]). In addition we have for all $t \in [0, T)$,

$$\int\limits_{\mathbb{R}^2} |u(t,x)|^2\, dx = \int\limits_{\mathbb{R}^2} |\phi_0(x)|^2\, dx\,, \tag{1.9}$$

$$\mathscr{E}(u(t)) = \frac{1}{2} \int\limits_{\mathbb{R}^2} |\nabla u(t,x)|^2\, dx - \frac{1}{4} \int\limits_{\mathbb{R}^2} |u(t,x)|^4\, dx = \mathscr{E}(\phi_0)\,. \tag{1.10}$$

If $\phi_0 \in \Sigma = \{|x|\, u \in L^2\} \cap H^1$, then $\forall t \in [0, T)$, $u(t) \in \Sigma$ and

$$\frac{d}{dt} \int\limits_{\mathbb{R}^2} |x|^2\, |u(t,x)|^2\, dx = 4\, \mathrm{Im} \int\limits_{\mathbb{R}^2} r u(t,x)\, \bar{u}_r(t,x)\, dx\,, \tag{1.11}$$

$$\frac{d^2}{dt^2} \int\limits_{\mathbb{R}^2} |x|^2\, |u(t,x)|^2\, dx = 16 \mathscr{E}(\phi_0)\,, \tag{1.12}$$

where $r = \dfrac{x}{|x|}$ and $u_r = \dfrac{\partial u}{\partial r}$.

From the last two identities, Zakharov, Sobolev, Synakh [19] and Glassey [7] derived the existence of singular solutions of (I_∞). That is, solutions for which there is a $T > 0$ such that

$$\lim_{t \to T} |u(t)|_{H^1} = +\infty\,.$$

Indeed, if the energy of the initial data $\mathscr{E}(\phi_0)$ is negative, then $P(t) = \int\limits_{\mathbb{R}^2} |x|^2\, |u(t,x)|^2\, dx$ is a polynomial in time of degree two on $[0, T)$ such that the coefficient director is a fixed negative number. If $P(t)$ were defined for all t, there would be a $t_0 > 0$ such that $P(t_0) < 0$, which is impossible. Therefore the solution must blow up.

This equation has two important sets of explicit solutions:
- Periodic solutions
- Blow-up self similar solutions (with respect to a conformal invariance).

Let us consider solutions of the form

$$u(t,x) = e^{i\omega t} V(x)\,,$$

where ω is a positive parameter, $u(t)$ will be a solution of (I_∞) if and only if V satisfies the following elliptic equation

$$\omega V = \Delta V + |V|^2 V \quad \text{in} \quad \mathbb{R}^2\,. \tag{1.13ω}$$

Existence of solutions of such equations is well known (see Berestycki and Lions [3] and also Strauss [20]). Moreover, the set of solutions of (1.13ω) with $\omega > 0$ has a "minimal" element in the L^2 sense which is called a ground state. More precisely, there exists a unique radial solution in H^1 of the problem

$$V = \Delta V + |V|^2 V \quad \text{in} \quad \mathbb{R}^2\,, \qquad V > 0 \tag{1.14}$$

denoted Q (see Kwong [9] and the other references in [9]). If $V \not\equiv 0$ is solution of (1 13ω_0) for some $\omega_0 > 0$ (it is easy to see by Pohozaev identity that there are no solutions $V \not\equiv 0$ in H^1 with a "good" decay at infinity for $\omega \le 0$), then
- $|V|_{L^2} \ge |Q|_{L^2}$.
- if $|V|_{L^2} = |Q|_{L^2}$, then tere is $x_0 \in \mathbb{R}^2$ such that $V(x) = \omega_0^{1/2} Q(\omega_0^{1/2}(x - x_0))$ (see Cazenave and Lions [4] and Weinstein [23]).

These periodic solutions yield to explicit blow-up solutions. Indeed, equation (I_∞) has a conformal invariance: if $u(x,t)$ is a solution of (I_∞) then

$$\frac{1}{T-t}\, e^{\frac{i|x|^2}{4(-T+t)}}\, \bar{u}\left(\frac{1}{T-t}, \frac{x}{T-t}\right) \tag{1.15}$$

is also a solution of (I_∞) (see for example Ginibre and Velo [5] and the references therein). Thus if $V(x)$ is a solution of (1.13ω), then

$$\frac{1}{T-t}\, e^{i\left(\frac{|x|^2}{4(-T+t)} - \frac{\omega}{(-T+t)}\right)}\, V\left(\frac{x}{T-t}\right) \tag{1.16}$$

is a blow-up solution of (I_∞).

In particular, for $\theta \in S^1$, x_0, $x_1 \in \mathbb{R}^2$, $T > 0$, $\omega > 0$,

$$S_{\theta,\omega,x_0,x_1}(t,x) = \frac{\omega}{T-t}\, e^{i\left(\theta + \frac{|x-x_1|^2}{4(-T+t)} - \frac{\omega^2}{(-T+t)}\right)}\, Q\left(\frac{(x-x_1)\omega - (T-t)x_0}{T-t}\right) \tag{1.17}$$

is a blow-up solution.

These solutions $S_{\theta,\omega,x_0,x_1}$ are important in the following sense. They are minimal blow-up solutions. Indeed,
- If $|\phi_0|_{L^2} < |Q|_{L^2}$, then the solution $u(t)$ is globally defined in time (see Weinstein [23]).
- If $|\phi_0|_{L^2} = |Q|_{L^2}$ then $u(t)$ blows up in finite time if and only if there are $T > 0$, $\theta \in S^1$, x_0, $x_1 \in \mathbb{R}^2$, $T > 0$, $\omega > 0$ such that $u(t) = S_{\theta,\omega,x_0,x_1}(t)$.

The proof of this result has been done in two steps
- in the case where ϕ_0 is radial and in Σ (see Merle [12])
- in the general case (see Merle [11]).

How are these properties transformed for the Zakharov equation (I_{c_0}) where $c_0 \in (0, +\infty)$? Clearly, the quantity $\int_{\mathbb{R}^2} |x|^2 \,|(u(t,x)|^2\, dx$ does not anymore satisfy a simple relation, which does not allow us to prove blow-up theorems for the Zakharov equation.

The existence of periodic solutions of this form is still true. Indeed, if $V(x)$ is a solution of (1.13ω) for $\omega > 0$, we can remark that $(u(t,x), n(t,x)) = (e^{\imath\omega t}V(x), -|V(x)|^2)$ is a periodic solution of equation (I_{c_0}).

There is not conformal invariance of the equation. We can check by hand that

$$u(t,x) = \frac{\omega}{T-t}\, e^{\imath\left(\theta + \frac{|x-x_1|^2}{4(-T+t)} - \frac{\omega^2}{(-T+t)}\right)} V\left(\frac{(x-x_1)\omega - (T-t)x_0}{T-t}\right),$$

$$n(t,x) = -\left(\frac{\omega}{T-t}\right)^2 \left[V\left(\frac{(x-x_1)\omega - (T-t)x_0}{T-t}\right)\right]^2,$$

is not an exact solution of (I_{c_0}), where V is a solution of (1.13ω) for $\omega > 0$.

The idea is to look for the natural extension of the form

$$u(t,x) = \frac{\omega}{T-t}\, e^{\imath\left(\theta + \frac{|x-x_1|^2}{4(-T+t)} - \frac{\omega^2}{(-T+t)}\right)} P\left(\frac{(x-x_1)\omega}{T-t} - x_0\right),$$

$$n(t,x) = \left(\frac{\omega}{T-t}\right)^2 N\left(\frac{(x-x_1)\omega}{T-t} - x_0\right),$$

where $P:\mathbb{R}^2 \to \mathbb{R}$, $N:\mathbb{R}^2 \to \mathbb{R}$.

We consider solutions with radial symmetry, that is

$$u(t,x) = \frac{\omega}{T-t}\, e^{\imath\left(\theta + \frac{|x|^2}{4(-T+t)} - \frac{\omega^2}{(-T+t)}\right)} P\left(\frac{x\omega}{T-t}\right), \tag{1.18}$$

$$n(t,x) = \left(\frac{\omega}{T-t}\right)^2 N\left(\frac{x\omega}{T-t}\right), \tag{1.19}$$

where $P(x) = P(|x|)$ and $N(x) = N(|x|)$.

After explicit computations, we find that (u,n) is a solution of (I_{c_0}) if and only if (P,N) satisfies the following system:

$$(\mathrm{II}_\lambda) \quad \begin{cases} \Delta P - P = NP\,, & (1.20) \\ \lambda^2(r^2N_{rr} + 6rN_r + 6N) - \Delta N = \Delta|P|^2\,, & (1.21) \end{cases}$$

where

$$\lambda = \frac{1}{c_0\omega}$$

and $r = |x|$, $W_r = \dfrac{\partial W}{\partial r}$, $\Delta W = W_{rr} + \dfrac{1}{r}\, W_r$.

Our purpose in this paper is to study this system. That is to find a range of parameters λ (or equivalently ω) such that (II_λ) has a solution (P_λ, N_λ). Consequently

$$u_{\theta,\lambda}(t,x) = \frac{1}{c_0\lambda(T-t)}\, e^{i\left(\theta + \frac{|x|^2}{4(-T+t)} - \frac{1}{c_0^2\lambda^2(-T+t)}\right)} P_\lambda\left(\frac{x}{c_0\lambda(T-t)}\right),$$

$$n_{\theta,\lambda}(t,x) = \left(\frac{1}{c_0\lambda(T-t)}\right)^2 N_\lambda\left(\frac{x}{c_0\lambda(T-t)}\right)$$

will be a blow-up solution of the Zakharov equation (I_{c_0}). In this direction, we are not able to use variational methods because of the lack of variational structure of equation (II_λ). We use a different approach: perturbation methods for λ small.

Indeed, for $\lambda = 0$ (or equivalently $\omega = +\infty$ or $c_0 = +\infty$), (II_λ) is reduced to

$$\Delta P - P = NP\,,$$
$$\Delta(N + |P|^2) = 0\,,$$

that is

$$P = \Delta P + |P|^2 P\,,$$
$$N = -|P|^2\,,$$

which is (1.13ω) with $\omega = 1$.

Thus, the idea is to use a fixed point argument to find for λ small solutions (P, N) of (II_λ) near $(V, -|V|^2)$, where V is a solution of the equation

$$V = \Delta V + |V|^2 V \tag{1.22}$$

on $\mathbb{R}^2$ and then a contiuation argument (degree theory) to get an unbounded (in a certain sense) branch of solutions of (1.22), under a condition of nondegeneracy (which is only known for minimal solutions of (1.22) (that is the positive solution). Most of the method applies for a radial general solution except for one fundamental part of the proof: it is only known that minimal solutions of Eq. (1.14) (that is positive solutions) are in a certain class of nondegenerate functions.

For that reason, we end up finding a branch of solutions of the system

$$(\mathrm{II}_\lambda^+) \qquad \begin{cases} \Delta P - P = NP\,, \\ \lambda^2(r^2 N_{rr} + 6rN_r + 6N) - \Delta N = \Delta |P|^2\,, \\ P > 0\,, \end{cases}$$

where $r = |x|$, $W_r = \dfrac{\partial W}{\partial r}$.

We have the following results for solutions of (II_λ^+). The first result is a regularity result on solutions of system (II_λ).

Proposition 1. *Let* $(P, N) \in H^1 \times L^2$ *be a radial solution of* (II_λ) *in the distributions sense. Then for all* $k \geq 0$, $(P, N) \in H^k \times H^k$ *In addition there is a* $\delta > 0$, $c_k > 0$ *such that*

$$|P^{(k)}(x)| \leq c_k e^{-\delta|x|}\,, \qquad |N^{(k)}(x)| \leq \frac{c_k}{1 + |x|^{3+k}}\,.$$

Remark. The space $H^1 \times L^2$ is not optimal for elliptic theory but it is the one which is important in terms of $(u(t), n(t))$ for the time evolution problem. We say that (P, N) is a solution of (II_λ) if $(P, N) \in H^1 \times L^2$ and satisfies (II_λ) in the distributions sense. From Proposition 1 we have that $(P, N) \in H^2 \times H^2$ and (P, N) is a classical solution of (II_λ).

Theorem 1. i) *(Existence and uniqueness property for λ small). There exists* $\lambda^+ > 0$ *such that for* $0 < \lambda < \lambda^+$ *there is a solution* (P_λ, N_λ) *of the system* (II_λ^+) *such that*

$$(P_\lambda, N_\lambda) \to (Q, -Q^2) \quad \text{as} \quad \lambda \to 0$$

in $H^1 \times L^2$. *Moreover, for all* $c > |Q|_{L^2}$, *there exists* $\lambda_c > 0$ *such that for* $0 < \lambda < \lambda_c$ *there is a unique solution* (P_λ, N_λ) *of* (II_λ^+) *such that* $|P_\lambda|_{L^2} < c$.

ii) *(Existence of a branch of solutions of* (II_λ^+)*). There is a branch of solutions of* (II_λ^+) $(\lambda, (P_\lambda, N_\lambda))$ *for* $\lambda > 0$ *which is unbounded in* $\mathbb{R}^+ \times (H^1 \times L^2)$.

Remark We do not know whether there exists a λ^* such that for $0 < \lambda < \lambda^*$ there is a unique solution of (II_λ^+).

Remark From ii), there exists $\lambda^* > 0$ such that
– for $0 < \lambda < \lambda^*$ there is a solution (P_λ, N_λ) of (II_λ^+)
– if $\lambda^* < +\infty$, there is a sequence $(P_{\lambda_n}, N_{\lambda_n})$ of solutions of (II_λ^+) such that

$$\lambda_n \to \lambda^{**} < +\infty\,,$$
$$|(P_{\lambda_n}, N_{\lambda_n})|_{H^1\times L^2} \to +\infty\,,$$

as $n \to +\infty$.

It is an open problem to know whether $\lambda^* = +\infty$ or not. We conjecture that $\lambda^* = +\infty$.

Remark In Part II [6], we give results on the behavior of solutions of (II_λ) uniformly bounded in L^2 as $\lambda \to 0$.

From Theorem 1, we have the following result of existence of blow-up solutions of (I_{c_0}).

Theorem 2 (Existence of Self Similar Blow-Up Solutions of Equation (I_{c_0})). *Let* (P_λ, N_λ) *be a solution of* (II_λ) *in* $H^1 \times L^2$. *Then* $\forall T > 0$, $\forall \theta \in S^1$,

$$u_{\theta,\lambda}(t,x) = \frac{1}{c_0\lambda(T-t)}\, e^{i\left(\theta + \frac{|x|^2}{4(-T+t)} - \frac{1}{c_0^2\lambda^2(-T+t)}\right)} P_\lambda\left(\frac{x}{c_0\lambda(T-t)}\right),$$
$$n_{\theta,\lambda}(t,x) = \left(\frac{1}{c_0\lambda(T-t)}\right)^2 N_\lambda\left(\frac{x}{c_0\lambda(T-t)}\right)$$

is a blow-up solution of equation (I_{c_0}) *with the following properties*·
– $\forall t \in [0,T)$, $\forall k \geq 1$, $(u, n, n_t) \in H_k$ *and in particular* $n_t \in \hat{H}^{-1}$
– $|(u_{\theta,\lambda}, n_{\theta,\lambda})|_{\hat{H}_1} = |u_{\theta,\lambda}|_{H^1} + |n_{\theta,\lambda}|_{L^2} + \left|\dfrac{\partial n_{\theta,\lambda}}{\partial t}\right|_{\hat{H}^{-1}} \to +\infty$ *as* $t \to T$.

Remarks. i) For the existence of blow-up solutions the value of c_0 does not play a role. Indeed, if $(u(t), n(t))$ is a blow-up solution of equation (I_{c_0}) then $\forall \alpha_0 > 0$, $(\alpha_0 u(\alpha_0^2 t, \alpha_0 x), \alpha_0^2 n(\alpha_0^2 t, \alpha_0 x))$ is also a blow-up solution of $(\mathrm{I}_{c_0\alpha_0})$.[1]
ii) We can see that as $t \to T$,

$$|u(t)|^2 \rightharpoonup |P|_{L^2}^2 \delta_{x=0}\,,$$
$$|n(t)| \rightharpoonup |P|_{L^2}^2 \delta_{x=0}\,,$$

in the distributions sense. We will see in Part II [6] that it is in a certain sense a general behavior for blow-up solutions at the blow-up time.

[1] From this result, we derive the existence of blow-up solutions of vector value Zakharov equations by considering solutions of the form $U(t,x) = u(t,x)U_0$ where U_0 is a fixed vector and $u(t,x)$ is complex

ii) It is important to remark that the form

$$u_{\theta,\lambda}(t,x)=\frac{1}{c_0\lambda(T-t)}e^{i\left(\theta+\frac{|x|^2}{4(-T+t)}-\frac{1}{c_0^2\lambda^2(-T+t)}\right)}P_\lambda\left(\frac{x}{c_0\lambda(T-t)}\right),$$

$$n_{\theta,\lambda}(t,x)=\left(\frac{1}{c_0\lambda(T-t)}\right)^2N_\lambda\left(\frac{x}{c_0\lambda(T-t)}\right)$$

is the stable form which is numerically observed (stability of the profil at the blow-up with respect to the initial data, see [10, 15]). This fact points out the importance of these solutions for Zakharov system and of the solutions of the form

$$u_{\theta,\lambda}(t,x)=\frac{1}{c_0\lambda(T-t)}e^{i\left(\theta+\frac{|x|^2}{4(-T+t)}-\frac{1}{c_0^2\lambda^2(-T+t)}\right)}Q\left(\frac{x}{c_0\lambda(T-t)}\right),$$

for the Schrödinger equation which are in some sense limit of stable solutions (of Zakharov system).

The paper is organized as follows:
– In Sect. II, we study the regularity and basic properties of a given solution of (II_λ).
– In Sect. III, we precise the asymptotic behavior of a solution of $(\mathrm{II}_\varepsilon)$ as $\varepsilon\to 0$.
– In Sect. IV, we prove existence and uniqueness of a solution of $(\mathrm{II}_\varepsilon^+)$ for $\varepsilon>0$ small enough by a fixed point argument.
– In Sect. V, we prove the existence of an unbounded (in a certain sense) branch of solutions of (II_λ^+).

II. Properties of a Given Radial Solution (P,N) of (II_λ)

In this section, we consider a *radial* solution $(P,N)\not\equiv 0$ in $H^1\times L^2$ satisfying in the sense of distributions the system

$$(\mathrm{II}_\lambda)\qquad\begin{cases}\Delta P-P=NP\,, & (2.1)\\ \lambda^2(r^2N_{rr}+6rN_r+6N)-\Delta N=\Delta|P|^2\,, & (2.2)\end{cases}$$

and the associated blow-up solutions of (I_{c_0}) (solution in a weak sense)

$$u_\lambda(t,x)=\frac{1}{c_0\lambda(T-t)}e^{i\left(\frac{|x|^2}{4(-T+t)}-\frac{1}{c_0^2\lambda^2(-T+t)}\right)}P\left(\frac{x}{c_0\lambda(T-t)}\right),$$

$$n_\lambda(t,x)=\left(\frac{1}{c_0\lambda(T-t)}\right)^2N\left(\frac{x}{c_0\lambda(T-t)}\right),$$

where $T\geq 0$.

Assuming existence of such solutions (we always in this section assume that these solutions are radial), we look for their various properties. We do it from two points of view.
– In II.1 we show properties of (P,N) from the point of view of the elliptic theory. More precisely, we will present:
a) Equivalent formulations of (II_λ) and local regularity of (P,N).
b) Decay at infinity of (P,N).
c) Various identities for (P,N).

– In II.2 we give properties of (u_λ, n_λ) and thus of (P, N) from the point of view of the evolution equations:

Computations of invariants, related properties and uniform lowerbound in L^2 on P.

II.1 Elliptic Properties of (P, N)

a) Local regularity of (P, N) and equivalent formulations. We first prove that if (P, N) are solutions of (II_λ), then (P, N) are regular functions. We then derive a simpler verison of (II_λ) where N is an explicit function of P.

In fact, we show that $(P, N) \in H^1 \times L^2$ solution of (II_λ) is equivalent to (P, N) solution of (III_λ) or (IV_λ), where

$$(\mathrm{III}_\lambda) \quad \begin{cases} \Delta P - P = NP\,, \\ (\lambda^2 r^2 - 1)N'(r) + 3\lambda^2 r N(r) = 2P(r)P'(r)\,, \end{cases} \tag{2.3}$$

and

$$(\mathrm{IV}_\lambda) \quad \begin{cases} \Delta P - P = NP\,, \\ N(r) = \dfrac{1}{(\lambda^2 r^2 - 1)^{3/2}} \displaystyle\int_{1/\lambda}^{r} 2P(s)P'(s)(\lambda^2 s^2 - 1)^{1/2}\, ds\,, \end{cases} \tag{2.4}$$

where $(\lambda^2 x^2 - 1)^{k/2} = \dfrac{\lambda^2 x^2 - 1}{|\lambda^2 x^2 - 1|} |\lambda^2 x^2 - 1|^{k/2}$ for $k = 1, 2, 3$.

Proposition 2.1 (Regularity of (P, N) and Equivalence of the Systems). i) *Let* $(P, N) \in H^1 \times L^2$ *a radially symmetric solution of* (II_λ) *in the sense of distributions. Then* P, N *are* C^∞ *on* $\mathbb{R}^2$ *and are classical solutions of* (II_λ).
ii) *Let* $(P, N) \in H^1 \times L^2$ *radially symmetric and* C^∞ *Then systems* (II_λ), (III_λ), (IV_λ) *are equivalent.*

Remark Therefore the degeneracy of the differential operator of (2.2) does not imply for $(P, N) \in H^1 \times L^2$ solution of (II_λ) a weak singularity.

Proof of i). Consider $(P, N) \in H^1 \times L^2$ solution of (II_λ). Let us prove that (P, N) are C^∞. We consider the cases outside and inside the nondegeneracy point.

Case $|x| \neq \dfrac{1}{\lambda}$

Let us first prove L^∞ and H^2_{loc} estimates of P.

Lemma 2.2. *Let* $(P, N) \in H^1 \times L^2$ *a solution of* (II_λ). *Then* $P \in H^2_{\mathrm{loc}} \cap L^\infty$ *and there exists a constant c which depends only on* $|P|_{H^1}$ *and* $|N|_{L^2}$ *such that*

$$|P|_{L^\infty} \leq c\,.$$

Remark. The constant is independent of λ.

Proof. We prove $P \in L^\infty$. Let us consider the elliptic problem in the unit ball B_1:

$$\Delta P - P = NP\,. \tag{2.5}$$

P is in H^1 and thus P is in L^4 by Sobolev embedding. Since N is in L^2 we deduce from regularity of the elliptic problem that P is in $W^{2,3/2}(B_1)$, therefore P is in $L^\infty(B_1)$.

For $|x| \geq 1$, we have the classical inequality:

Lemma 2.3. *For $u \in H^1$ radially symmetric we have*

$$|u|_{L^\infty[A,+\infty)} \leq \frac{1}{\sqrt{A}} |u|_{H^1} . \tag{2.6}$$

Proof Considering $u^2(r) = \int\limits_r^{+\infty} 2u(s)u'(s)ds$, we have for $r \geq A$,

$$u^2(r) \leq \int\limits_r^{+\infty} 2|u(s)|\,|u'(s)| \frac{s}{A}\, ds \leq \frac{1}{A} \int\limits_r^{+\infty} (|u(s)|^2 + |u'(s)|^2)\, s\, ds ,$$

which yields to the result.

Therefore $|P|_{L^\infty(|x|\geq 1)} \leq |u|_{H^1}$ and there exists a constant c which depends only on $|P|_{H^1}$ and $|N|_{L^2}$ such that $|P|_{L^\infty} \leq c$.

Let us prove $P \in H^2_{\text{loc}}$. For $A > 0$ we consider again the elliptic problem (2.5) on the ball B_A of radius A in $\mathbb{R}^2$. NP is L^2 on B_A and then from elliptic theory $P \in H^2_{\text{loc}}$. This concludes the proof of the lemma.

Let us show that (P, N) are C^∞ on $\mathbb{R}^2 \Big\backslash \left\{ |x| = \frac{1}{\lambda} \right\}$ using classical regularity theory for elliptic equations. We first need regularity results for the linear operator appearing in (II_λ): Let

$$L_1 P = \Delta P - P ,$$
$$L_2 N = \lambda^2 (r^2 N_{rr} + 6 r N_r + 6N) - \Delta N .$$

Therefore system (II_λ) can be written

$$L_1 P = NP ,$$
$$L_2 N = \Delta P^2 .$$

Lemma 2.4 $\left(\text{Uniform Ellipticity of } L_1,\ L_2 \text{ for } x \neq \frac{1}{\lambda}\right)$.

1) *L_1 is uniformly elliptic on compacts of $\mathbb{R}^2$ and L_2 is uniformly elliptic on compacts of $\mathbb{R}^2 \Big\backslash \left\{ |x| \neq \frac{1}{\lambda} \right\}$.*

2) *If P is a $L^2(\mathbb{R}^2)$ radial fucntion such that $L_1 P \in H^k_{\text{loc}}\left(\mathbb{R}^2 \Big\backslash \left\{ |x| = \frac{1}{\lambda} \right\}\right)$, then $P \in H^{k+2}_{\text{loc}}\left(\mathbb{R}^2 \Big\backslash \left\{ |x| = \frac{1}{\lambda} \right\}\right)$.*

3) *If N is a $L^2(\mathbb{R}^2)$ radial function such that $L_2 N \in H^k_{\text{loc}}\left(\mathbb{R}^2 \Big\backslash \left\{ |x| = \frac{1}{\lambda} \right\}\right)$, then $N \in H^{k+2}_{\text{loc}}\left(\mathbb{R}^2 \Big\backslash \left\{ |x| = \frac{1}{\lambda} \right\}\right)$.*

Proof of the lemma. 2) and 3) follows from 1) and classical regularity results. Clearly L_1 is uniformly elliptic. Let us prove that L_2 is uniformly elliptic on the open set of the form $C_{A,\delta} = \{|x| < A\} \cap \left\{ \left| |x|^2 - \frac{1}{\lambda^2} \right| > \delta^2 \right\}$, where $\delta > 0$ and $A > 0$.

By computations we have on $C_{A,\delta}$,

$$L_2N = (\lambda^2|x|^2 - 1)\Delta N + 5x \cdot \nabla N + 6\lambda^2 N$$

and $\lambda^2\delta^2 \le |\lambda^2|x|^2 - 1| \le A^2\lambda^2 + 1$. Therefore L_2N is uniformly elliptic on $C_{A,\delta}$. This concludes the proof of Lemma 2.4.

Consequently,

Proposition 2.5. *Let* $(P, N) \in H^1 \times L^2$ *a solution of* (II_λ). *We then have* $P, N \in C^\infty\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right)$.

Proof. We prove by induction on k that $P, N \in H^k_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right)$ and conclude using Sobolev inequalities.

Let us prove $P, N \in H^2_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right)$. We have that $P \in H^2_{\mathrm{loc}}$ and thus $\Delta P^2 \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. Thus $L_2(N) \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ and from Lemma 2.4, $N \in H^2_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right)$ thus $N \in L^\infty_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right)$.

Let us suppose $P, N \in H^k_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right)$ for $k \ge 2$. We have

$$P, N \in H^{k+2}_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right).$$

Since $k \ge 2$, $N, P \in L^\infty_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right)$ and then $(NP) \in H^k_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right)$. Then $P \in H^{k+2}_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right)$ and $\Delta P^2 \in H^k_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right)$. Thus again from Lemma 2.4 $N \in H^{k+2}_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right)$.

In conclusion, $P, N \in H^k_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \left\{|x| = \frac{1}{\lambda}\right\}\right)$ for each k and the result follows from the Sobolev embedding of H^k_{loc} into C^{k-2}_{loc}.

Case $x = \frac{1}{\lambda}$.

We first show that if (P, N) solutions of (II_λ) then (P, N) is solution of (III_λ) and (IV_λ) and then conclude the proof of regularity.

We prove that (2.2) has a first integral, and in fact, that N can be written as an explicit function of P.

Proposition 2.6. *Let* $(P, N) \in H^1 \times L^2$ *satisfying* (II_λ). *Then* (P, N) *satisfies*

1) $(\lambda^2 r^2 - 1)N' + 3\lambda^2 r N = 2PP'$ *in the sense of distributions.*

2) $N(r) = \dfrac{1}{(\lambda^2 r^2 - 1)^{3/2}} \displaystyle\int_{1/\lambda}^{r} 2P(s)P'(s)(\lambda^2 s^2 - 1)^{1/2}\, ds$ for $r \ne \frac{1}{\lambda}$.

Proof. 1) Let

$$A(r) = ((\lambda^2 r^2 - 1)N' + 3\lambda^2 r N - (P^2)').$$

We remark that if (P,N) is solution of (2.2) then

$$\frac{1}{r}\frac{\partial}{\partial r}(rA(r)) = (\lambda^2(r^2 N_{rr} + 6rN_r + 6N) - \Delta N - \Delta |P|^2 = 0$$

in the sense of distributions.

Thus $rA(r)$ is a constant function on $\mathbb{R}$. But for $r \neq \frac{1}{\lambda}$, P and N are smooth functions and in particular $A(0) = 0$, which leads to the result.

2) We then write N as an explicit function of P.

Lemma 2.7. *We have for* $0 \leq r_1 \leq r_2 \leq \frac{1}{\lambda}$,

$$(1-\lambda^2 r_2^2)^{3/2} N(r_2) = (1-\lambda^2 r_1^2)^{3/2} N(r_1) - \int_{r_1}^{r_2} 2P(s)P'(s)(1-\lambda^2 s^2)^{1/2}\,ds\,, \quad (2.7)$$

and for $\frac{1}{\lambda} \leq r_1 \leq r_2$.

$$(\lambda^2 r_2^2 - 1)^{3/2} N(r_2) = (\lambda^2 r_1^2 - 1)^{3/2} N(r_1) + \int_{r_1}^{r_2} 2P(s)P'(s)(\lambda^2 s^2 - 1)^{1/2}\,ds\,. \quad (2.8)$$

Proof. Let us consider the case $\frac{1}{\lambda} < r_1 < r_2$ for example. Multiplying (2.3) by $(\lambda^2 r^2 - 1)^{1/2}$, we find

$$((\lambda^2 r^2 - 1)^{3/2} N)' = 2PP'(\lambda^2 r^2 - 1)^{1/2}\,, \quad (2.9)$$

and the result follows from integration on $[r_1, r_2]$.

Therefore (2.4) $\left(\text{in the case } r > \frac{1}{\lambda} \text{ for example}\right)$ follows from Lemma 2.7, putting $r_2 = r$ and $r_1 \to \left(\frac{1}{\lambda}\right)^+$.

This terminates the proof of Proposition 2.6.

We now conclude the proof of regularity at $|x| = \frac{1}{\lambda}$.

Lemma 2.8 $\left(\text{Regularity for } |x| = \frac{1}{\lambda}\right)$. *Let* $(P,N) \in H^2 \times L^2$ *a solution of* (II_λ). *Then* P, N *are* C^∞ *at* $|x| = \frac{1}{\lambda}$.

Proof Let $I = \left[\frac{1}{2\lambda}, \frac{2}{\lambda}\right]$. We will prove by recurrence on k the following property on the functions $P(r)$, $N(r)$ (considered here as functions from I to $\mathbb{R}$)

$$P, N \in W^{k,\infty}(I)\,.$$

Let us prove it for $k = 0$. From Lemma 2.2, $P \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$ and then $P \in W^{1,\infty}(I)$. We then deduce the regularity of N:

$$N(r) = \frac{1}{\lambda}\frac{1}{(\lambda r + 1)^{3/2}}\frac{1}{\left(r - \frac{1}{\lambda}\right)^{3/2}}\int_{1/\lambda}^{r} 2P(s)P'(s)(\lambda s + 1)^{1/2}\left(s - \frac{1}{\lambda}\right)^{1/2} ds$$

or

$$N(r)=\frac{1}{\lambda}\frac{1}{(\lambda r+1)^{3/2}}Y(r)$$

with

$$Y(r)=\frac{1}{\left(r-\frac{1}{\lambda}\right)^{3/2}}\int_{1/\lambda}^{r}Z(s)\left(s-\frac{1}{\lambda}\right)^{1/2}ds\,,$$

$$Z(s)=2P(r)P'(r)(\lambda r+1)^{1/2}\,.$$

Whe then use the following classical regularity lemma.

Lemma 2.9. *Consider I a compact interval of* $\mathbb{R}$, $r_0\in I$ *and for* $r\in I$,

$$Y(r)=\frac{1}{(r-r_0)^{3/2}}\int_{r_0}^{r}Z(s)(s-r_0)^{1/2}ds\,.$$

For each integer $k\geq 0$, *if* $Z\in W^{k,\infty}(I)$, *then* $Y\in W^{k,\infty}(I)$ *and there exists a constant* $c_{k,I}$ *which depends only on* k *and on the length of* I *such that*

$$|Y|_{W^{k,\infty}(I)}\leq c_{k,I}|Z|_{W^{k,\infty}(I)}\,.$$

Proof. It follows from classical arguments and from Taylor formula.

$PP'\in L^\infty(I)$ implies $Z\in L^\infty(I)$ and from previous lemma Y and N are in $L^\infty(I)$. Thus $P,N\in W^{0,\infty}(I)$ is true.

Let us suppose that $P,N\in W^{k,\infty}(I)$ for $k\geq 0$ and let us prove $P,N\in W^{k+1,\infty}(I)$. We first deduce $(NP)\in W^{k,\infty}(I)$ and from

$$P''+\frac{1}{r}P'-P=NP\,,$$

we deduce $P\in W^{k+2,\infty}(I)$. Therefore $Z=2PP'(\lambda r+1)^{1/2}\in W^{k+1,\infty}(I)$. From Lemma 2.9 it follows Y and $N=\frac{1}{\lambda}(\lambda r+1)^{-3/2}Y$ are in $W^{k+1,\infty}(I)$, and this concludes the proof of Lemma 2.8, thus part i) of Proposition 2.1.

Proof of ii) *of Proposition 2.1* We had shown in the proof i), that is if $(P,N)\in H^1\times L^2$ is a solution of (II_λ) then it is a solution of (III_λ) and (IV_λ). Reciprocally, for C^∞ functions (P,N) satisfying

$$N(r)=\frac{1}{(\lambda^2r^2-1)^{3/2}}\int_{1/\lambda}^{r}2P(s)P'(s)(\lambda^2s^2-1)^{1/2}ds$$

we have by direct calculations

$$(\lambda^2r^2-1)N'+3\lambda^2rN=2PP'$$

and

$$\lambda^2(r^2N_{rr}+6rN_r+6N)-\Delta N=\Delta|P|^2\,.$$

b) Decay of Solutions of (II_λ) at Infinity. We have for a given solution (P,N) of (II_λ) the following decay estimates at ∞:

Proposition 2.10. *Let $(P, N) \in H^1 \times L^2$ a solution of* (II_λ). *Therefore exist constants $\delta > 0$ and c_k for $k \geq 0$ such that*

$$\forall k \geq 0\,, \quad \forall x\,, \quad |P^{(k)}(x)| \leq c_k e^{-\delta|x|}\,,$$
$$|N^{(k)}(x)| \leq \frac{c_k}{1+|x|^{k+3}}\,.$$

Proof 1) We first show the property for P, P', P'', N.

lemma 2.11. *There exist $c > 0$ and $\delta > 0$ such that*

$$|P(r)| + |P'(r)| + |P''(r)| \leq c e^{-\delta r}\,,$$
$$|N(r)| \leq \frac{c}{1+r^3}\,.$$

Proof We first show that $|N(r)| \leq \dfrac{c}{1+r^3}$.

Let us consider (2.8) with $r_1 = 2/\lambda$ and $r_2 = r > \dfrac{1}{\lambda}$. We have

$$N(r) = \frac{K(r)}{(\lambda^2 r^2 - 1)^{3/2}}$$

with

$$K(r) = 3^{3/2} N(2/\lambda) + \int_{2/\lambda}^{r} 2P(s)P'(s)(\lambda^2 s^2 - 1)^{1/2}\, ds\,,$$
$$|K(r)| \leq c + \int_{1/\lambda}^{r} 2|P(s)P'(s)|\,\lambda s\, ds \leq c + \lambda |P|_{L^2} |P'|_{L^2} \leq c\,, \tag{2 10}$$

that is $|N(r)| \leq \dfrac{c}{1+r^3}$ for $r > \dfrac{2}{\lambda}$. From Proposition 2.1,

$$\forall r\,, \quad |N(r)| \leq \frac{c}{1+r^3}\,.$$

Let us fix $A > 0$ such that $|N(r)| \leq \frac{1}{2}$ for $r \geq A$. We have

$$\Delta P = (N+1)P$$

for $r \geq A$ with

$$(N(r)+1) \in \left[\tfrac{1}{2}, \tfrac{3}{2}\right].$$

We can deduce from usual techniques (see Berestycki and Lions [3], and Stauss [20]) that there exist some constants $c, \delta > 0$ such that

$$\forall r\,, \quad |P(r)| \leq c e^{-\delta r}\,.$$

Now, writing

$$(rP')' = r(N+1)P\,, \tag{2.11}$$

we first show that $rP'(r) \to c$ as $r \to +\infty$ and we remark that $c = 0$ because $P' \in L^2$. Thus integrating (2.11) on $[r, +\infty)$, we obtain

$$|rP'(r)| = \left| \int_r^{+\infty} s(N(s)+1)P(s)ds \right|$$
$$\le \int_r^{+\infty} cs\, e^{-\delta s} ds \le cr\, e^{-\delta r},$$

that is $|P'(r)| \le ce^{-\delta r}$.

Finally, writing $P'' = (N+1)P - \dfrac{P'}{r}$ and using previous estimates on P and P', we conclude

$$|P''(r)| \le ce^{-\delta r}.$$

This concludes the proof of Lemma 2.11.
2) We then conclude the proof of the proposition by proving property $(\mathscr{P}_k)$ by induction on $k \ge 0$:

There exists c_k such that

$$\begin{aligned} |P^{(l)}(r)| &\le c_k e^{-\delta r} \quad \text{for} \quad 0 \le l \le k+2, \\ |N^{(l)}(r)| &\le \frac{c_k}{1+r^{l+3}} \quad \text{for} \quad 0 \le l \le k. \end{aligned} \qquad (\mathscr{P}_k)$$

$(\mathscr{P}_0)$ is true from part 1). Let us assume $(\mathscr{P}_k)$ is true for $k \ge 0$ and let us show $(\mathscr{P}_{k+1})$.

By Leibniz formula, for $r \ge 2/\lambda$,

$$\begin{aligned} |N^{(k+1)}(r)| &\le c \sum_{p+q=k+1} \left| \left(\frac{1}{(\lambda r^2 - 1)^{3/2}} \right)^{(p)} \right| |K^{(q)}(r)| \\ &\le \sum_{p+q=k+1} \left(\frac{1}{1+r^{p+3}} \right) |K^{(q)}(r)|, \end{aligned} \qquad (2.12)$$

where $K(r)$ is defined in (2.10). For $1 \le q \le k+1$,

$$\begin{aligned} |K^{(q)}(r)| &\le c|P(r)P'(r)(\lambda^2 r^2 - 1)^{3/2})^{(q-1)}|, \\ &\le ce^{-\delta r} \end{aligned}$$

by estimates on $P, P', \ldots, P^{k+2}$. Therefore

$$|N^{(k+1)}(r)| \le \frac{c_k}{1+r^{k+4}}.$$

Let us prove the estimate on P: Differentiating the formula $P'' = \dfrac{-P'}{r} + P + NP$, we obtain by Leibniz formula for $r > 1$ and estimates on $N, N', \ldots, N^{(k+1)}$,

$$\begin{aligned} |P^{(k+3)}(r)| &\le c\left[|P^{(k+1)}(r)| + \sum_{p+q=k+1} \left\{ \left(\frac{1}{r^{p+1}} \right) |P^{(q+1)}(r)| + |N^{(p)}(r)|\,|P^{(q)}(r)| \right\} \right] \\ &\le c \sum_{q=0}^{k+2} |P^{(q)}(r)| \le ce^{-\delta r}. \end{aligned}$$

This concludes the proof of Proposition 2.10.

c) Different Relations for (N, P). We give various relations for solutions (P, N) of (II_λ) and in particular a Pohozaev type identity.

Lemma 2.12. *Let* (P, N) *solution of the equation* (II_λ) *We then have*

$$\int_{\mathbb{R}^2} |\nabla P(x)|^2 \, dx + \int_{\mathbb{R}^2} P^2(x) \, dx = \int_{\mathbb{R}^2} -N(x) P^2(x) \, dx \, . \tag{2.13}$$

Proof. The proof follows from multiplying Eq. (2.1) by P and from integrating on $\mathbb{R}^2$.

Lemma 2.13 (Pohozaev Identity). *Let* (P, N) *be solution of the equation* (II_λ) *We then have*

$$\int_{\mathbb{R}^2} P^2(x) \, dx = \tfrac{1}{2} \int_{\mathbb{R}^2} (\lambda^2 |x|^2 + 1) N^2(x) \, dx \, . \tag{2.14}$$

Proof Let us multiply Eq. (2.2) by rP' and integrate on $[0, +\infty)$. We get

$$0 = -\int_0^{+\infty} \left(P''(r) + \frac{1}{r} P'(r) \right) (rP'(r)) \, dr + \int_0^{+\infty} (N(r) P(r)) (rP'(r)) r \, dr + \int_0^{+\infty} (P(r)) (rP'(r)) r \, dr \, . \tag{2.15}$$

We have

$$\int_0^{+\infty} (rP'(r))' (rP'(r)) \, dr = \left[\tfrac{1}{2} (rP'(r))^2 \right]_0^{+\infty} = 0 \, . \tag{2.16}$$

From (2.3) and integration by parts

$$\begin{aligned} \int_0^{+\infty} N(r) P(r) P'(r) r^2 \, dr &= \int_0^{+\infty} \tfrac{1}{2} (\lambda^2 r^2 - 1) N(r) N'(r) r^2 \, dr + \int_0^{+\infty} \tfrac{3}{2} (\lambda^2 r) N^2(r) r^2 \, dr \, , \\ &= -\int_0^{+\infty} \left(\tfrac{1}{4} (\lambda^2 r^2 - 1) r^2 \right)' N(r)^2 \, dr + \int_0^{+\infty} \tfrac{3}{2} (\lambda^2 r) N^2(r) r^2 \, dr \, , \\ &= \int_0^{+\infty} \tfrac{1}{2} (\lambda^2 r^2 + 1) N^2(r) r \, dr \, . \end{aligned} \tag{2.17}$$

By an integration by parts,

$$\int_0^{+\infty} P(r) r P'(r) r \, dr = -\int_0^{+\infty} P^2(r) r \, dr \, . \tag{2.18}$$

The result follows from (2.15)–(2.18).

Remark. For $\lambda = 0$, Eq. (II_λ) reduces to (1.22). The usual Pohozaev identity [16] gives for regular solutions in $\mathbb{R}^2$ of (1.22),

$$\int_{\mathbb{R}^2} P^2(x)\,dx = \tfrac{1}{2}\int_{\mathbb{R}^2} P^4(x)\,dx\,,$$

which is the relation (2.14) with $\lambda = 0$.

II.2 Properties of (u_λ, n_λ) *Solution of* (II_{c_0})

Let (P, N) be radially symmetric solution of (II_λ). We recall that

$$u(t,x) = \frac{\omega}{(T-t)}\, e^{i\left(\frac{|x|^2}{4(-T+t)} - \frac{\omega^2}{(-T+t)}\right)} P\left(\frac{x\omega}{T-t}\right),$$

$$n(t,x) = \left(\frac{\omega}{T-t}\right)^2 N\left(\frac{\omega x}{T-t}\right),$$

where

$$\omega = \frac{1}{\lambda c_0}$$

is a solution of (I_{c_0}) which blows up in H^1 at $t = T$.

We have the following proposition on the regularity of (u, n). Since equation (I_{c_0}) is time invariant, we assume that $T = 0$.

Proposition 2.14. *We have* $\forall t$, $\forall k \geq 0$*:*

i) $\dfrac{\partial n}{\partial t} = \nabla \cdot v$, *where* $v = \hat{v}(r)\,\dfrac{x}{r}$ *and* $\hat{v}(r) = \dfrac{\omega^2}{t^3}\, rN\left(\dfrac{r\omega}{t}\right)$.

In particular

$$\frac{\partial n}{\partial t} \in \hat{H}^{-1} \quad \textit{and} \quad \left|\frac{\partial n}{\partial t}\right|_{\hat{H}^{-1}} = |v|_{L^2} = \frac{1}{|t|}\,|rN|_{L^2}\,. \tag{2.19}$$

ii) $\left(u(t), n(t), \dfrac{\partial n(t)}{\partial t}\right) \in \hat{H}_1 = H^1 \times L^2 \times \hat{H}^{-1} \subset H_1$.

iii) $\left(u(t), n(t), \dfrac{\partial n(t)}{\partial t}\right) \in H_k = H^k \times H^{k-1} \times H^{k-2}$ *if* $k \geq 2$.

iv) *We have* $|u(t)|_{H^1} + |n(t)|_{L^2} + \left|\dfrac{\partial n(t)}{\partial t}\right|_{\hat{H}^{-1}} \to +\infty$ *as* $t \to 0$.

Proof i) Let us show that $\dfrac{\partial n}{\partial t} = \nabla \cdot v$. We look for a solution of the form $v = \hat{v}(r)\,\dfrac{x}{r}$. v must satisfy $\dfrac{\partial n}{\partial t} = \nabla \cdot v$ or equivalently, $\forall \phi \in C_0^\infty$, $\phi(x) = \phi(r)$ with $r = |x|$,

$$\int_{\mathbb{R}^2} \frac{\partial n(x)}{\partial x}\,\phi(x)\,dx = \int_{\mathbb{R}^2} -\nabla \cdot v(x)\,\phi(x)\,dx = \int_{\mathbb{R}^2} v(x) \cdot \nabla\phi(x)\,dx\,. \tag{2.20}$$

By direct calculation

$$\frac{\partial n}{\partial t}(r) = -\frac{\omega^3}{t^4}\, rN'\left(\frac{\omega r}{t}\right) - 2\frac{\omega^2}{t^3}\, N\left(\frac{\omega r}{t}\right) = -\frac{1}{r}\frac{\omega^2}{t^3}\frac{\partial}{\partial r}\left(r^2 N\left(\frac{\omega r}{t}\right)\right).$$

From (2.20)

$$\int_0^{+\infty} -\frac{1}{r}\frac{\omega^2}{t^3}\frac{\partial}{\partial r}\left(r^2 N\left(\frac{\omega r}{t}\right)\right)\phi(r)\,r\,dr = \int_0^{+\infty} \hat{v}(r)\,\frac{\partial\phi(r)}{\partial r}\,r\,dr\,,$$

or equivalently

$$\begin{aligned}\int_0^{+\infty} \hat{v}(r)\,\frac{\partial\phi(r)}{\partial r}\,r\,dr &= \frac{\omega^2}{t^3}\int_0^{+\infty} r^2 N\left(\frac{\omega r}{t}\right)\frac{\partial\phi(r)}{\partial r}\,r\,dr \\ &= \int_0^{+\infty}\left[\frac{\omega^2}{t^3}\,r^2 N\left(\frac{\omega r}{t}\right)\right]\frac{\partial\phi(r)}{\partial r}\,r\,dr\,.\end{aligned}$$

Therefore $\hat{v}(r) = \frac{\omega^3}{t^3}\,rN\left(\frac{\omega r}{t}\right)$ satisfies the previous conditions. In particular

$$\frac{\partial n}{\partial t} = \nabla\cdot v \quad\text{with}\quad v = \frac{\omega^2}{t^3}\,rN\left(\frac{\omega r}{t}\right)\frac{x}{r}\,.$$

By definition

$$\left|\frac{\partial n}{\partial t}\right|_{\hat{H}^{-1}} = |v|_{L^2} = \left|\frac{\omega^3}{t^3}\,rN\left(\frac{\omega r}{t}\right)\right|_{L^2} = \frac{1}{|t|}\left|\frac{\omega}{t}\left(\frac{\omega r}{t}\,N\left(\frac{\omega r}{t}\right)\right)\right|_{L^2} = \frac{1}{|t|}\,|rN|_{L^2}\,,$$

which concludes the proof of part i). Furthermore

$$\begin{aligned}\left|\nabla\left(\frac{\omega}{t}\,P\left(\frac{x\omega}{t}\right)e^{\frac{i|x|^2}{4t}}\right)\right|_{L^2}^2 &= \left|\left(\frac{\omega^2}{t^2}\,\nabla P\left(\frac{x\omega}{t}\right) + i\,\frac{x}{2t}\,\frac{\omega}{t}\,P\left(\frac{x\omega}{t}\right)\right)e^{\frac{i|x|^2}{4t}}\right|_{L^2}^2 \\ &= \left|\frac{\omega^2}{t^2}\,\nabla P\left(\frac{x\omega}{t}\right)\right|_{L^2}^2 + \left|\frac{x}{2}\,\frac{\omega}{t^2}\,P\left(\frac{x\omega}{t}\right)\right|_{L^2}^2 \\ &= \frac{\omega^2}{t^2}\left|\frac{\omega}{t}\,\nabla P\left(\frac{x\omega}{t}\right)\right|_{L^2}^2 + \frac{1}{4\omega^2}\left|\frac{\omega}{t}\left(\frac{x\omega}{t}\right)P\left(\frac{x\omega}{t}\right)\right|_{L^2}^2 \\ &= \frac{\omega^2}{t^2}\,|\nabla P|_{L^2}^2 + \frac{1}{4\omega^2}\,|xP|_{L^2}^2\,,\end{aligned}$$

that is

$$|\nabla u|_{L^2}^2 = \frac{\omega^2}{t^2}\,|\nabla P|_{L^2}^2 + \frac{1}{4\omega^2}\,|xP|_{L^2}^2\,. \tag{2.21}$$

In addition,

$$|n|_{L^2} = \frac{\omega}{|t|}\left|\frac{\omega}{t}\,N\left(\frac{x\omega}{t}\right)\right|_{L^2} = \frac{\omega}{|t|}\,|N|_{L^2}\,. \tag{2.22}$$

From i),

$$\left|\frac{\partial n}{\partial t}\right|_{\hat{H}^{-1}} = |v|_{L^2} = \frac{1}{|t|}\,|rN|_{L^2}\,.$$

Therefore, we have that $\forall t\in[0,T)$, (u,n) is a solution of (I_{c_0}) in $\hat{H}_1$ and in H_k, $\forall k\geq 1$.

This equation has two invariants $\left(\text{since } \dfrac{\partial n}{\partial t} \in \hat{H}^{-1}\right)$:

Proposition 2.15. $\forall t$,

i) $|u(t)|_{L^2} = |P|_{L^2}$,

ii)
$$\begin{aligned} \mathscr{H}(t) &= \mathscr{H}(u(t), n(t), v(t)) \\ &= \frac{\omega^2}{t^2}\left[\int_{\mathbb{R}^2} |\nabla P(x)|^2\,dx + \int_{\mathbb{R}^2} N(x)\,P^2(x)\,dx + \frac{1}{2}\int_{\mathbb{R}^2}\left(\frac{|x|^2}{c_0^2\omega^2}+1\right)N^2(x)\,dx\right] \\ &\quad + \frac{1}{4\omega^2}\int_{\mathbb{R}^2} |x|^2\,P^2(x)\,dx\,. \end{aligned} \tag{2.23}$$

Corollary 2.16. $\forall t \in [0, T)$,

i)
$$\mathscr{H}(t) = \frac{1}{4\omega^2}\int_{\mathbb{R}^2} |x|^2\,P^2(x)\,dx\,, \tag{2.24}$$

ii)
$$\int_{\mathbb{R}^2} (|\nabla P(x)|^2 + N(x)\,P^2(x))\,dx + \frac{1}{2}\int_{\mathbb{R}^2}\left(\frac{|x|^2}{c_0^2\omega^2}+1\right)N^2(x)\,dx = 0\,. \tag{2.25}$$

Proof of Proposition 2 15.

i)
$$|u(t)|_{L^2} = \left|\frac{\omega}{t}\,e^{i\left(\frac{|x|^2}{4t}-\frac{\omega^2}{t}\right)}P\left(\frac{\omega x}{t}\right)\right|_{L^2} = \left|\frac{\omega}{t}\,P\left(\frac{\omega x}{t}\right)\right|_{L^2} = |P|_{L^2}\,.$$

ii) By definition we have

$$\mathscr{H}(t) = \int_{\mathbb{R}^2}\left(|\nabla u(t,x)|^2 + n(t,x)\,|u(t,x)|^2 + \frac{1}{2c_0^2}\,v^2(t,x) + \frac{1}{2}\,n^2(t,x)\right)dx\,.$$

We have

$$\begin{aligned} \int_{\mathbb{R}^2} n(t,x)\,|u(t,x)|^2\,dx &= \int_{\mathbb{R}^2} \frac{\omega^2}{t^2}\,N\left(\frac{\omega x}{t}\right)\left(\frac{\omega}{t}\right)^2 P^2\left(\frac{\omega}{t}\right)dx \\ &= \frac{\omega^2}{t^2}\int_{\mathbb{R}^2} \frac{\omega^2}{t^2}\,N\left(\frac{\omega x}{t}\right)P^2\left(\frac{\omega}{t}\right)dx \\ &= \frac{\omega^2}{t^2}\int_{\mathbb{R}^2} N(x)\,P^2(x)\,dx\,. \end{aligned} \tag{2.26}$$

Combination of (2.19), (2.21), (2.22), and (2.26) leads to (2.23).

Proof of Corollary 2.16. Since $\mathscr{H}(t)$ is time invariant $\left(\dfrac{\partial n}{\partial t} \in \hat{H}^{-1}\right)$, Proposition 2.15 leads to

– $\forall t$, $\mathscr{H}(t) = \dfrac{1}{4\omega^2}\displaystyle\int_{\mathbb{R}^2} |x|^2\,P^2(x)\,dx$,

– $\displaystyle\int_{\mathbb{R}^2} (|\nabla P(x)|^2 + N(x)\,P^2(x))\,dx + \frac{1}{2}\int_{\mathbb{R}^2}\left(\frac{|x|^2}{c_0^2\omega^2}+1\right)N^2(x)\,dx = 0.$

Remark. Relation (2.25) is a combination of the energy identity (2.13),

$$\int_{\mathbb{R}^2} |\nabla P(x)|^2\,dx + \int_{\mathbb{R}^2} P^2(x)\,dx = \int_{\mathbb{R}^2} -N(x)\,P^2(x)\,dx\,,$$

and the Pohozaev identity obtained in (2.14):

$$\int_{\mathbb{R}^2} P^2(x)\,dx = \frac{1}{2}\int_{\mathbb{R}^2} (\lambda^2|x|^2+1)\,N^2(x)\,dx$$

$\left(\text{here we use the relation } \lambda = \dfrac{1}{c_0\omega}\right)$. Therefore Pohozaev identity can be shown as a consequence of the conservation in time of the Hamiltonian $\mathscr{H}$.

From Proposition 2.16 and from [23], we have

Corollary 2.17. *We have if $P \not\equiv 0$ and $\lambda > 0$,*

$$\int_{\mathbb{R}^2} P^2(x)\,dx > \int_{\mathbb{R}^2} Q^2(x)\,dx\,. \tag{2.27}$$

Proof. Indeed, from Corollary 2.16,

$$\int_{\mathbb{R}^2} (|\nabla P(x)|^2 + N(x)\,P^2(x))\,dx + \frac{1}{2}\int_{\mathbb{R}^2}\left(\frac{|x|^2}{c_0^2\omega^2}+1\right) N^2(x)\,dx = 0\,.$$

Therefore,

$$\int_{\mathbb{R}^2} (|\nabla P(x)|^2 - \frac{1}{2}\,P^4(x))\,dx + \frac{1}{2}\int_{\mathbb{R}^2} (N(x)+P^2(x))^2\,dx + \frac{1}{2}\int_{\mathbb{R}^2} \frac{|x|^2}{c_0^2\omega^2}\,N^2(x)\,dx = 0\,.$$

Thus

$$\mathscr{E}(P) = \int_{\mathbb{R}^2} |\nabla P(x)|^2\,dx - \int_{\mathbb{R}^2} \frac{1}{2}\,P^4(x)\,dx < 0\,.$$

We recall that [23]

$$\frac{1}{2}\int_{\mathbb{R}^2} P^4(x)\,dx \le \frac{\int_{\mathbb{R}^2} P^2(x)\,dx}{\int_{\mathbb{R}^2} Q^2(x)\,dx}\int_{\mathbb{R}^2} |\nabla P(x)|^2\,dx\,. \tag{2.28}$$

Therefore

$$\left(1 - \frac{\int_{\mathbb{R}^2} P^2(x)\,dx}{\int_{\mathbb{R}^2} Q^2(x)\,dx}\right)\int_{\mathbb{R}^2} |\nabla P(x)|^2\,dx < 0\,,$$

which implies

$$\int_{\mathbb{R}^2} P^2(x)\,dx > \int_{\mathbb{R}^2} Q^2(x)\,dx\,.$$

III. Asymptotic Behavior of Radial Solution of (II_λ) as $\lambda \longrightarrow 0$

In this section, we consider radial solutions of system

$$(\mathrm{II}_\varepsilon) \qquad \begin{cases} \Delta P - P = NP\,, & (3.1)\\ \varepsilon^2(r^2 N_{rr} + 6rN^r + 6N) - \Delta N = \Delta|P|^2\,, & (3.2) \end{cases}$$

where $\varepsilon \longrightarrow 0$ which are uniformly bounded in a weak sense. Under some assumptions, we show that their limit points are solutions of equation (II_0).

Proposition 3.1 (Asymptotics Behavior of $(P_\varepsilon, N_\varepsilon)$ as $\varepsilon \longrightarrow 0$). *Let $(P_n, N_n) \not\equiv (0,0)$ radially symmetric solutions of* ($\mathrm{II}_{\varepsilon_n}$) *with $\varepsilon_n \longrightarrow 0$.*
i) *Assume that there is a constant $c > 0$ such that*

$$\forall n\,, \quad |P_n|_{L^2} \le c\,. \tag{3.3}$$

Then there is a subsequence also denoted (P_n, N_n) and a solution $V \not\equiv 0$ of equations

$$(\mathrm{V}) \qquad V = \Delta V + |V|^2 V$$

such that

$$(P_n, N_n) \to (V, -V^2) \quad in \quad H^1 \times L^2\,.$$

ii) *Assume that for a $c > 0$,*

$$\forall n\,, \quad \forall r \ge 0\,, \quad P_n(r) \ge 0 \quad and \quad \forall n\,, \quad |P_n|_{L^2} \le c\,.$$

The full sequence (P_n, N_n) then converges as $n \longrightarrow +\infty$ to $(Q, -Q^2)$ in $H^1 \times L^2$, where Q is the unique radially symmetric positive solution of equation

$$(\mathrm{V}^+) \qquad Q = \Delta Q + |Q|^2 Q\,, \qquad Q > 0\,.$$

Remark. i) Better convergence can be proved (see Part II.B [6]).
ii) From an heuristic point of view, the assumption on $|P_n|_{L^2}$ is crucial since there is an unbounded sequence in L^2 of solutions of equation (V).

Proof of i). Let us assume that there is a $c > 0$ such that $|P_n|_{L^2} \le c$. The main ingredients are Pohozaev identity (2.14) and identity (2.13). The proof is organized in four steps.

Step 1. Uniform Estimates of (P_n, N_n) in $H^1 \times L^2$. We have from (2.14) and (2.13),

$$\int_{\mathbb{R}^2} P_n^2(x)\,dx = \tfrac{1}{2} \int_{\mathbb{R}^2} (\varepsilon_n^2|x|^2 + 1)\,N_n^2(x)\,dx\,, \tag{3.4}$$

$$\int_{\mathbb{R}^2} |\nabla P_n(x)|^2\,dx + \int_{\mathbb{R}^2} P_n^2(x)\,dx = \int_{\mathbb{R}^2} -N_n(x)\,P_n^2(x)\,dx\,. \tag{3.5}$$

From (3.3) and (3.4) we have that for a $c > 0$,

$$\forall n \ge 0\,, \qquad \int_{\mathbb{R}^2} N_n^2(x)\,dx \le c\,. \tag{3.6}$$

Equations (3.5), (3.6) and Gargliardo Nirenberg identities gives

$$\int_{\mathbb{R}^2} |\nabla P_n(x)|^2\,dx \le \left(\int_{\mathbb{R}^2} N_n^2(x)\,dx\right)^{1/2} \left(\int_{\mathbb{R}^2} P_n^4(x)\,dx\right)^{1/2} + c$$
$$\le c\left[\left(\int_{\mathbb{R}^2} |\nabla P_n(x)|^2\,dx\right)^{1/2} + 1\right],$$

and there is a $c > 0$ such that

$$\int_{\mathbb{R}^2} |\nabla P_n(x)|^2\,dx \le c\,.$$

In conclusion there is a $c > 0$ such that

$$\int_{\mathbb{R}^2} |\nabla P_n(x)|^2\,dx + \int_{\mathbb{R}^2} P_n^2(x)\,dx + \int_{\mathbb{R}^2} N_n^2(x)\,dx \le c\,. \tag{3.7}$$

Step 2. Compactness of (P_n, N_n) in $H^1 \times L^2$ on bounded sets. From Lemma 2.2, (3.7) implies that

$$|P_n|_{L^\infty} \le c \quad \text{and} \quad \forall A\,, \quad |P_n|_{H^2(B(0,A))} \le c_A\,. \tag{3.8}$$

Proof of Lemma 2.4 then implies that

$$\forall A\,, \quad |N_n|_{H^1(B(0,A))} \le c_A\,.$$

Indeed from (2.7) with $r_1 = 0$ and $r_2 = r$,

$$N_n(r) = \frac{N_n(0)}{1-\varepsilon_n^2 r^2} - \frac{1}{(1-\varepsilon_n^2 r^2)^{3/2}} \int_0^r 2P_n(s)\,P_n'(s)\,(1-\varepsilon^2 s^2)^{1/2}\,dy\,. \tag{3.9}$$

$\forall A$, we have for $r \le A$ and n large $\left(\varepsilon_n \le \dfrac{1}{2A}\right)$

$$\left|\frac{1}{(1-\varepsilon_n^2 r^2)^{3/2}} \int_0^r 2P_n(s)\,P_n'(s)\,(1-\varepsilon^2 s^2)^{1/2}\,dy\right|$$
$$\le c\int_0^A |P_n|\,|P_n'| \le c\int_0^A |P_n'|$$
$$\le c\int_0^A \left(|P_n'(A)| + \int_0^A |P_n''|\right) \le c\left(|P_n'(A)| + \int_0^A |P_n''(s)|\,s\,ds\right) \le c\,,$$

since $|P_n|_{H^2(B(0,A))} \le c$.

Therefore

$$|N_n(0)| - c \le |N_n(x)| \le 2|N_n(0)| + c \tag{3.10}$$

on $[0, A]$ for large n. Since $|N_n|_{L^2} \le c$ we have $|N_n(0)| \le c$ and then from (3.10) $|N_n|_{L^\infty(B(0,A))} \le c$.

We then conclude easily. From (2.3) and (3.7),

$$\forall r \leq A\,, \quad |N'(r)| \leq c(|N_n(r)| + 2|P_n(r)P_n'(r)|) \leq c(1 + |P_n'(r)|)$$

and

$$\forall n\,, \quad |N_n|_{H^1(B(0,A))} \leq c_A\,.$$

Using compactness arguments and subtracting a sequence also denoted (P_n, N_n), there is a $(\hat{P}, \hat{N}) \in H^1 \times L^2$ which is radially symmetric such that

$$P_n \rightharpoonup \hat{P} \quad \text{in} \quad H^1\,, \tag{3.11}$$

$$N_n \rightharpoonup \hat{N} \quad \text{in} \quad L^2\,. \tag{3.12}$$

Step 3. Let us prove that $\hat{P} = V$, $\hat{N} = -V^2$ where V is a H^1 solution of (V). Because the functions are radial, a classical compactness theorem implies that

$$P_n \to \hat{P} \quad \text{in} \quad L^r\,, \quad \forall r \in (2, +\infty)\,. \tag{3.13}$$

Therefore, in the distributions sense we have

$$\Delta P_n^2 \to \Delta \hat{P}^2\,,$$

$$N_n P_n \to \hat{N}\hat{P}\,,$$

and passing to the limit as $\varepsilon_n \to 0$, we obtain in the distributions sense that $(\hat{P}, \hat{N}) \in H^1 \times L^2$ is solution of

$$\Delta \hat{P} = \hat{P} + \hat{N}\hat{P}\,, \tag{3.14}$$

$$\Delta(\hat{N} + \hat{P}^2) = 0\,. \tag{3.15}$$

Since $\hat{N}$, $\hat{P}^2$ are in L^2, (3.15) implies that $\hat{N} = -\hat{P}^2$. We thus have that $\hat{P} = V$, $\hat{N} = -V^2$, where V satisfies equation (V) in $\mathbb{R}^2$.

Step 4. Conclusion: Let us show from (3.11)–(3.12), energy (3.5) and Pohozaev identities (3.4) that

$$(P_n, N_n) \to (V, -V^2) \quad \text{in} \quad H^1 \times L^2\,. \tag{3.16}$$

a) $P_n \to V$ in H^1. We have $\forall n$,

$$\int_{\mathbb{R}^2} |\nabla P_n|^2 + \int_{\mathbb{R}^2} P_n^2 = -\int_{\mathbb{R}^2} N_n P_n^2$$

and

$$\int_{\mathbb{R}^2} |\nabla V|^2 + \int_{\mathbb{R}^2} V^2 = \int_{\mathbb{R}^2} V^4\,.$$

We first have from (3.13) that

$$P_n^2 \to V^2 \quad \text{in} \quad L^2\,. \tag{3.17}$$

Indeed,

$$(P_n^2 - V^2)^2 \leq (P_n - V)^2 (P_n + V)^2$$

and

$$|P_n^2 - V^2|_{L^2}^2 \leq c\left(\int_{\mathbb{R}^2} |P_n - V|^4\right)^{1/2} \left(\int_{\mathbb{R}^2} P_n^4 + \int_{\mathbb{R}^2} V^4\right)^{1/2} \to 0\,.$$

From (3.17) and the fact $N_n \rightharpoonup -V^2$ in L^2,

$$-\int_{\mathbb{R}^2} N_n P_n^2 \to \int_{\mathbb{R}^2} V^4 .$$

Therefore

$$\int_{\mathbb{R}^2} |\nabla P_n|^2 + \int_{\mathbb{R}^2} P_n^2 \to \int_{\mathbb{R}^2} V^4 = \int_{\mathbb{R}^2} |\nabla V|^2 + \int_{\mathbb{R}^2} V^2$$

and

$$P_n \rightharpoonup V \quad \text{in} \quad H^1 .$$

This implies that

$$P_n \to V \quad \text{in} \quad H^1 . \tag{3.18}$$

In addition, from Corollary 2.17 $\forall n$, $\int_{\mathbb{R}^2} P_n^2 > \int_{\mathbb{R}^2} Q^2$ we have

$$\int_{\mathbb{R}^2} V^2 \geq \int_{\mathbb{R}^2} Q^2 \neq 0 . \tag{3.19}$$

b) $N_n \to -V^2$ in L^2. We have

$$N_n \rightharpoonup -V^2 \quad \text{in} \quad L^2 . \tag{3.20}$$

From (3.20)

$$\left(\int_{\mathbb{R}^2} V^4\right)^{1/2} \leq \liminf_{n\to+\infty} \left(\int_{\mathbb{R}^2} N_n^2\right)^{1/2} . \tag{3.21}$$

The Pohozaev identity for system ($\mathrm{II}_{\varepsilon_n}$) gives (3.4). Therefore

$$\int_{\mathbb{R}^2} N_n^2 \leq 2 \int_{\mathbb{R}^2} P_n^2 ,$$

and

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^2} N_n^2 \leq 2 \limsup_{n\to+\infty} \int_{\mathbb{R}^2} P_n^2 = 2 \int_{\mathbb{R}^2} V^2 . \tag{3.22}$$

By Pohozaev identity for equation (V) (V is radially symmetric in H^1 thus has exponential decay at infinity – see [3])

$$2 \int_{\mathbb{R}^2} V^2 = \int_{\mathbb{R}^2} V^4 . \tag{3.23}$$

From (3.22), (3.23),

$$\int_{\mathbb{R}^2} (V^2)^2 \geq \limsup_{n\to+\infty} \int_{\mathbb{R}^2} N_n^2 \geq \liminf_{n\to+\infty} \int_{\mathbb{R}^2} N_n^2 \geq \int_{\mathbb{R}^2} (V^2)^2$$

and

$$\int_{\mathbb{R}^2} N_n^2 \to \int_{\mathbb{R}^2} V^4 . \tag{3.24}$$

Finally, from (3.21), (3.22), and (3.24),

$$N_n \to -V^2 \quad \text{in} \quad L^2 \quad \text{as} \quad n \to +\infty .$$

This concludes the proof of Proposition 3.1 i).

Proof of ii) *of Proposition 3.1.*

We now assume

$$P_n \geq 0 \quad \text{and} \quad |P_n|_{L^2} \leq c . \tag{3.25}$$

Let us show that

$$(P_n, N_n) \to (Q, -Q^2) \quad \text{in} \quad H^1 \times L^2 \quad \text{as} \quad n \to +\infty .$$

From part i), we know that there is a subsequence [also denoted (P_n, N_n)] such that

$$(P_n, N_n) \to (V, -V^2) \quad \text{in} \quad H^1 \times L^2 ,$$

where V is a solution of (V) and V is radially symmetric. Letting $n \to +\infty$ in (3.25), we have $V \geq 0$ and from (2.27) $V \not\equiv 0$. From the uniqueness theorem of radial solutions of (V^+) (Kwong [9]), we deduce that $V = Q$.

Since the result is true for all subsequences, it is true for the full sequence and Propositon 3.1 ii) is proved.

IV. Existence and Uniqueness Property of Solutions of (II_ε^+) for ε Small

We want to construct solutions of system (II_ε) for $\varepsilon > 0$ and small. From the previous section, up to a subsequence, for $(P_\varepsilon, N_\varepsilon)$ solution of (II_ε), $(P_\varepsilon, N_\varepsilon) \to (V, -V^2)$ in $H^1 \times L^2$ as $\varepsilon \to 0$, where V is a solution of equation (V).

The idea is to construct for a given V solution of equation (V) a branch of solutions $(P_\varepsilon, N_\varepsilon)$ of (II_ε) such that $(P_\varepsilon, N_\varepsilon) \to (V, -V^2)$ as $\varepsilon \to 0$ using fixed point theorem. The fact that these methods are linear implies that V must satisfy in some sense a nondegeneracy condition. For that reason, we assume that

$$V = Q ,$$

where Q is the unique radially symmetric solution in H^1 of (V^+) where nondegeneracy results are known.

The plan of this section is the following:

– in IV.1, we obtain a crucial fixed point formulation of the system (II_ε) near Q

– in IV.2, by a continuity method we show that the branch of solutions $(P_\varepsilon, N_\varepsilon)$ constructed in IV.1 is such that $\forall \varepsilon > 0$, $P_\varepsilon > 0$

– in IV.3, we show uniqueness property of solutions of (II_ε^+) for $\varepsilon > 0$ small from Sect. III and the fixed point formulation near Q.

IV.1 Introduction to the Fixed Point Method near Q

We consider $\lambda = \varepsilon$ small and $P_\varepsilon = Q + h_\varepsilon$, N_ε solution of system (II_ε) or equivalently of ($\mathrm{IV}_\varepsilon^+$), that is

$$\Delta(Q + h_\varepsilon) = (N_\varepsilon + 1)(Q + h_\varepsilon), \tag{4.1}$$

$$N_\varepsilon(r) = \frac{1}{(\varepsilon^2 r^2 - 1)^{3/2}} \int_{1/\varepsilon}^{r} 2(Q(s) + h_\varepsilon(s))(Q'(s) + h'_\varepsilon(s))(\varepsilon^2 s^2 - 1)^{1/2}\, ds. \tag{4.2}$$

Let us define the linear operator

$$\mathcal{N}_\varepsilon(u)(r) = \frac{1}{(\varepsilon^2 r^2 - 1)^{3/2}} \int_{1/\varepsilon}^{r} u'(s)(\varepsilon^2 s^2 - 1)^{1/2}\, ds \tag{4.3}$$

or

$$\mathcal{N}_\varepsilon(u)(r) = \frac{u(r) - u\left(\frac{1}{\varepsilon}\right)}{(\varepsilon^2 r^2 - 1)} - \frac{\varepsilon^2}{(\varepsilon^2 r^2 - 1)^{3/2}} \int_{1/\varepsilon}^{r} \frac{u(s) - u\left(\frac{1}{\varepsilon}\right)}{(\varepsilon^2 s^2 - 1)^{1/2}}\, s\, ds. \tag{4.4}$$

One can write then Eqs. (4.1)–(4.2),

$$\Delta Q + \Delta h_\varepsilon = (\mathcal{N}_\varepsilon((Q + h_\varepsilon)^2) + 1)(Q + h_\varepsilon),$$
$$N_\varepsilon(x) = \mathcal{N}_\varepsilon((Q + h_\varepsilon)^2).$$

From (V^+) and expanding the previous relation with respect to h_ε we obtain

$$\Delta h_\varepsilon + h_\varepsilon + 3Q^2 h_\varepsilon = \mathcal{N}_\varepsilon((Q + h_\varepsilon)^2)(Q + h_\varepsilon) + Q^3 + 3Q^2 h_\varepsilon$$

and

$$\Delta h_\varepsilon - h_\varepsilon + 3Q^2 h_\varepsilon = C_\varepsilon + l_\varepsilon(h_\varepsilon) + q_\varepsilon(h_\varepsilon) + k_\varepsilon(h_\varepsilon) \tag{4.5}$$

with

$$C_\varepsilon = (\mathcal{N}_\varepsilon(Q^2) + Q^2)Q, \tag{4.6}$$
$$l_\varepsilon(h_\varepsilon) = 2(\mathcal{N}_\varepsilon(Qh_\varepsilon) + Qh_\varepsilon)Q + (\mathcal{N}_\varepsilon(Q^2) + Q^2)h_\varepsilon, \tag{4.7}$$
$$q_\varepsilon(h_\varepsilon) = \mathcal{N}_\varepsilon(Qh_\varepsilon)h_\varepsilon + \mathcal{N}_\varepsilon(h_\varepsilon^2)Q, \tag{4 8}$$
$$k_\varepsilon(h_\varepsilon) = \mathcal{N}_\varepsilon(h_\varepsilon^2)h_\varepsilon. \tag{4.9}$$

Thus the problem is to find h_ε in a suitable space such that (4.5) is true.

Define

$$H_r^1 = \{u \in H^1(\mathbb{R}^2), u(x) = u(r), \text{ where } r = |x|\}$$

and $H_r^2 = H^2(\mathbb{R}^2) \cap H_r^1$ with the norm

$$|u|_{H^2}^2 = \int_{\mathbb{R}^2} (|\Delta u|^2 + |\nabla u|^2 + |u|^2).$$

Proposition 4.1 (Nondegeneracy Condition). *For $u \in H_r^1$ there exists a unique $v \in H_r^2$ such that*

$$\Delta v - v + 3Q^2 v = u. \tag{4.10}$$

Define $v = \mathscr{L}u$, *where*

$$\mathscr{L} = (\Delta - \mathrm{Id} + 3Q^2)^{-1}.$$

$\mathscr{L}$ *is a bounded operator of* H^1_r *and there exists a constant* c_1 *such that for* $u \in H^1_r$,

$$|\mathscr{L}(u)|_{H^2} \le c_1 |u|_{L^2}. \tag{4.11}$$

Remark The uniqueness property of v solution of (4.10) is not true for non-radial function. Indeed, the null space in H^1 of the operator $\mathscr{L}^{-1} = \Delta - \mathrm{Id} + 3Q^2\,\mathrm{Id}$ is $\left\langle \frac{\partial Q}{\partial x_1}, \frac{\partial Q}{\partial x_2} \right\rangle$ (see [22] and references therein).

$(Q + h_\varepsilon, \mathscr{N}_\varepsilon((Q + h_\varepsilon)^2))$ is a solution of $(\mathrm{II}_\varepsilon)$ if and only if h_ε is a fixed point of the operator

$$T_\varepsilon(h_\varepsilon) = \mathscr{L}(C_\varepsilon + l_\varepsilon(h_\varepsilon) + q_\varepsilon(h_\varepsilon) + k_\varepsilon(h_\varepsilon)). \tag{4.12}$$

Indeed, a solution of $(\mathrm{II}_\varepsilon)$ is always in the space H^2_r (see Proposition 2.1). We do not consider H^1_r the natural space in order to simplify the control of term on the boundary $|x| = \frac{1}{\lambda}$. Let

$$B = \{u \in H^2_r, |u|_{H^2} \le \delta_0\},$$

where $\delta_0 > 0$ is a fixed constant. We prove that for $\varepsilon > 0$ small enough and $\delta_0 > 0$ small enough, T_ε is contraction mapping of the set B and therefore find solutions of $(\mathrm{II}_\varepsilon)$.

Theorem 4.2 (Fixed Point Theorem for T_ε for ε Small Enough). *There exists* $\varepsilon_0 > 0$ *and a constant* $\delta_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$, T_ε *is a contraction mapping of the set* B. *Therefore there is a unique* $h_\varepsilon \in B$ *such that*

$$h \in B = \{u \in H^2_r, |u|_{H^2} \le \delta_0\} \quad \textit{and} \quad T_\varepsilon(h_\varepsilon) = h_\varepsilon$$

Moreover
– *there exists a constant* c *such that for* $0 < \varepsilon < \varepsilon_0$, $|h_\varepsilon|_{H^2} \le c\varepsilon^2$.
– *the map* $\varepsilon \to h_\varepsilon$ *is continuous in* H^2_r.

Corollary 4.3 (Existence of Non Trivital Solutions of $(\mathrm{II}_\varepsilon)$ for ε Small). *There exists* $\varepsilon_1 > 0$ *such that for* $0 < \varepsilon < \varepsilon_1$ *there is a non-trivial solution* $(P_\varepsilon, Q_\varepsilon)$ *of* $(\mathrm{II}_\varepsilon)$ *continuous in* $H^1 \times L^2$ *with respect to* ε *such that* $(P_\varepsilon, N_\varepsilon) \to (Q, -Q^2)$ *in* $H^1 \times L^2$.

Proof of Corollary 4.3 Let us fix $0 < \varepsilon_1 < \varepsilon_0$ such that for $0 < \varepsilon < \varepsilon_1$, the solution h_ε given in Theorem 4.2 satisfies

$$|h_\varepsilon|_{H^2} < |Q|_{H^2}.$$

Thus $(P_\varepsilon, N_\varepsilon) = (Q + h_\varepsilon, \mathscr{N}_\varepsilon((Q + h_\varepsilon)^2))$ is a solution of $(\mathrm{II}_\varepsilon)$ in $H^1 \times L^2$ such that

$$|P_\varepsilon|_{H^2} > |Q|_{H^2} - |h_\varepsilon|_{H^2} > 0 \quad \text{and} \quad P_\varepsilon \not\equiv 0.$$

Corollary 4.3 follows from the following lemma.

Lemma 4.4. *We have equivalence between the following properties:*
i) *There is a continuous application in* $H^2 : \varepsilon \to h_\varepsilon$, *where* $h_\varepsilon = T_\varepsilon(h_\varepsilon)$ *is such that* $h_0 = 0$.
ii) *There is a continuous application in* $H^1 \times L^2 : \varepsilon \to (P_\varepsilon, N_\varepsilon)$, *where* $P_\varepsilon = Q + h_\varepsilon$ *and* $N_\varepsilon = \mathscr{N}_\varepsilon(P_\varepsilon)$ *is such that* $(P_0, N_0) = (Q, -Q^2)$

Proof. i) $\Rightarrow$ ii). We have to show that $\varepsilon \to N_\varepsilon$ is continuous in L^2 at $\varepsilon' \ge 0$.

Case $\varepsilon' = 0$ It follows from Proposition 3.1 in Sect. III.

We have that $P_\varepsilon \to P_0$ in H^1 as $\varepsilon \to 0$. Let us show that

$$N\varepsilon \to -P_0^2 \quad \text{in} \quad L^2 .$$

Indeed, from Sect. III, up to a subsequence

$$P_\varepsilon \to V \quad \text{in} \quad H^1 \quad \text{and} \quad N_\varepsilon \to -V^2 \quad \text{in} \quad L^2 \quad \text{as} \quad \varepsilon \to 0 .$$

Therefore $V = P_0$ and $N_\varepsilon \to -P_0^2$ in L^2 as $\varepsilon \to 0$.

Case $\varepsilon' > 0$.

– We first have from the integral formula

$$\forall r , \quad N_\varepsilon(r) \to N_{\varepsilon'}(r) \quad \text{as} \quad \varepsilon \to \varepsilon' . \tag{4.13}$$

– We claim there is a c such that

$$\forall \varepsilon \in [\varepsilon'/2, 2\varepsilon'] , \qquad |N_\varepsilon(r)| \le \frac{c}{1+r^3} . \tag{4 14}$$

From (4.13) and (4.14), the convergence dominated theorem allows us to conclude that $N_\varepsilon \to N_{\varepsilon'}$ in L^2 as $\varepsilon \to \varepsilon'$.

We will see in Lemma 4.8 that $N_\varepsilon = \mathcal{N}_\varepsilon(P_\varepsilon^2)$ is such that

$$|N_\varepsilon|_{L^\infty} \le c|P_\varepsilon|_{H^2}^2 \le c . \tag{4.15}$$

We then estimate $N_\varepsilon(r)$ for $r \ge 4/\varepsilon'$. From the relation (2.7) with $r_1 = 4/\varepsilon'$ and $r_2 = r$ we have

$$N_\varepsilon(r) = \frac{(\varepsilon^2(4/\varepsilon') - 1)^{3/2}}{(\varepsilon^2 r^2 - 1)^{3/2}} N_\varepsilon(4/\varepsilon') + \frac{1}{(\varepsilon^2 r^2 - 1)^{3/2}} \int_{4/\varepsilon'}^{r} 2P_\varepsilon(s) P_\varepsilon'(s) (\varepsilon^2 s^2 - 1)^{1/2} ds .$$

We first have

$$\left| \int_{4/\varepsilon'}^{r} 2P_\varepsilon(s) P_\varepsilon'(s) (\varepsilon^2 s^2 - 1)^{1/2} ds \right| \le \int_{4/\varepsilon'}^{r} 2P_\varepsilon(s)| \, |P_\varepsilon'(s)| \varepsilon s \, ds \le c\varepsilon .$$

Thus for $r \ge 4/\varepsilon'$ and $\varepsilon \in [\varepsilon'/2, 2\varepsilon']$, $N_\varepsilon(r) \le \frac{c}{r^3}$ and (4.14) follows.

ii) $\Rightarrow$ i). We assume that $(P_\varepsilon, N_\varepsilon) \to (P_{\varepsilon'}, N_{\varepsilon'})$ as $\varepsilon \to \varepsilon'$ in $H^1 \times L^2$, we have to show that $P_\varepsilon \to P_{\varepsilon'}$ in H^2.

It follows directly from Lemma 2.2 and the equation

$$\Delta P_\varepsilon = (N_\varepsilon + 1) P_\varepsilon .$$

Indeed we have $N_{\varepsilon'} \in L^\infty$ and $|P_\varepsilon|_{L^\infty} \le c$ from the regularity theory and Lemma 2.2. Thus

$$\begin{aligned} |\Delta P_\varepsilon - \Delta P_{\varepsilon'}|_{L^2} &= |N_\varepsilon P_\varepsilon - N_{\varepsilon'} P_{\varepsilon'} + P_\varepsilon - P_{\varepsilon'}|_{L^2} \\ &\le |P_\varepsilon - P_{\varepsilon'}|_{L^2} + |(N_\varepsilon - N_{\varepsilon'}) P_\varepsilon|_{L^2} \\ &\quad + |N_{\varepsilon'}(P_\varepsilon - P_{\varepsilon'})|_{L^2} \to 0 \quad \text{as} \quad \varepsilon \to \varepsilon' . \end{aligned}$$

Therefore $P_\varepsilon \to P_{\varepsilon'}$ in H^2 as $\varepsilon \to \varepsilon'$ or equivalenty $h_\varepsilon \to h_{\varepsilon'}$ in H^2 as $\varepsilon \to \varepsilon'$.

This concludes the proof of the lemma and the corollary.

We prove Theorem 4.2 in different steps.

In subsection a), we reduce in IV.2 the estimates in norm H^2 of T_ε to some estimates of $\mathscr{N}_\varepsilon$ in norm L^2 using the regularity of the operator $\mathscr{L}$.

In the two next subsections b) and c), we estimate terms appearing in the different components of the operators T_ε.

d) We conclude the proof of the fixed point theorem.

a) Nondegeneracy condition. In this section, we prove Proposition 4.4 essentially using results of Weinstein [22] on the linearized of the nonlinear Schrödinger equation near Q.

Let

$$(u, v) = \int_{\mathbb{R}^2} u(x)v(x)\,dx \quad \text{and} \quad L = -\Delta - 3Q^2 + 1 .$$

Proposition 4.5. [22].

i) $\forall u \in H^1_r$, $Lu = 0 \Rightarrow u = 0$.

ii) *If* $(u, Q) = 0$, *then* $(Lu, u) \geq 0$.

iii) *Let* $M = \{u \in H^1_r$ *such that* $(u, Q) = (u, Q|x|^2) = 0\}$. *Then there is* $c > 0$ *such that*

$$\forall u \in M , \qquad (Lu, u) \geq c|u|^2_{H^1} .$$

iv) *There are* $\varrho_1 \in H^2$ *and* $\varrho_2 \in H^2$ *such that* $L\varrho_1 = Q$ *and* $L\varrho_2 = |x|^2 Q$.

Proposition 4.5 has been proved in [22] under the assumption that there is a unique radial positive solution of $u = \Delta u + |u|^2 u$. Since this result had been proved by Kwong (see [9]), we have Proposition 4.5.

Let us show now Proposition 4.1.

Proof of Proposition 4.1. We first claim that:

Lemma 4.6. *Let* $f \in H^1_r$. *There is a unique* $u \in H^1_r$ *such that* $Lu = f$.

Proof It follows directly from classical arguments, the theorem of Lax Milgram and Proposition 4.5. Let $f \in H^1_r$. The uniqueness of u is given by Proposition 4.5 i). Let us show the existence of u. Consider

$$f = f_1 + \alpha Q + \beta |x|^2 Q , \quad \text{with} \quad f_1 \in M .$$

We have by direct calculations

$$|f_1|_{L^2} + |\alpha| + |\beta| \leq c|f|_{L^2} . \tag{4.16}$$

By Lax Milgram Theorem applied to $\Pi_M o L : M \to M$, where Π_M is the orthogonal projection on M, there exists u_1 in M such that

$$Lu_1 = f_1 + \alpha' Q + \beta' |x|^2 Q . \tag{4.17}$$

In addition we have

$$c|u_1|^2_{H^1} \leq (Lu_1, u_1) = (f, u_1) \leq |f_1|_{L^2} |u_1|_{L^2}$$

and

$$|u_1|_{H^1} \leq c|f_1|_{L^2} \tag{4.18}$$

Furthermore

$$(LQ, u_1) = \alpha'(Q,Q) + \beta'(|x|^2, Q)\,, \tag{4.19}$$

$$(L|x|^2 Q, u_1) = \alpha'(|x|^2 Q, Q) + \beta'(|x|^2 Q, |x|^2 Q)\,. \tag{4.20}$$

Since $||x^2|Q|_{L^2}|Q|_{L^2} > (Q,|x|^2 Q)^2$ (Cauchy-Schwarz) we have from (4.19)–(4.20):

$$|\alpha'| + |\beta'| \le c|f|_{L^2}\,. \tag{4.21}$$

Let $u = u_1 + (\alpha - \alpha')\varrho_1 + (\beta - \beta')\varrho_2$. Equation (4.17) and the definition of ϱ_1, ϱ_2 give

$$Lu = f_1 + \alpha Q + \beta|x|^2 Q = f\,,$$

and from (4.16), (4.18), and (4.21)

$$|u|_{H^1} \le c|f|_{L^2}\,.$$

The lemma is proved.

Let us conclude now the proof of Proposition 4.1. We claim that $|u|_{H^2} \le c|f|_{L^2}$. Indeed, we have

$$-\Delta u + u - 3Q^2 u = f\,.$$

Thus

$$|\Delta u|_{L^2} \le |f|_{L^2} + |u|_{L^2} + |3Q^2 u|_{L^2} \le c(|f|_{L^2} + |u|_{L^2}) \le c|f|_{L^2}\,.$$

b) Estimates of $|\mathscr{L}(\mathscr{N}_\varepsilon'(uv)w)|_{H^2}$.

Proposition 4.7. *We have the uniform estimate for u and w in H^2_r as $\varepsilon \to 0$:*

$$|\mathscr{L}(\mathscr{N}_\varepsilon'(uv)w)|_{H^2} \le c|u|_{H^2}|v|_{H^2}|w|_{H^2}\,.$$

Proof From (4.11) we have

$$|\mathscr{L}(\mathscr{N}_\varepsilon'(uv)w)|_{H^2} \le c|\mathscr{N}_\varepsilon'(uv)w)|_{L^2} \le c|\mathscr{N}_\varepsilon'(uv)|_{L^\infty}|w|_{L^2}\,. \tag{4.22}$$

Therefore we only need to estimate $|\mathscr{N}_\varepsilon'(uv)|_{L^\infty}$. This is the object of the following lemma:

Lemma 4.8. *We have the following uniform estimates for ε small.*
i) *There is a $c > 0$ such that for u, v, w in H^2_r,*

$$|\mathscr{N}_\varepsilon'(uv)|_{L^\infty} \le c|u|_{H^2}|v|_{H^2}\,.$$

ii) *There is a $\delta > 0$ such that*

$$|\mathscr{N}_\varepsilon'(uv)|_{L^\infty\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)} \le c|u|_{H^2\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}|v|_{H^2\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}\,,$$

$$|\mathscr{N}_\varepsilon'(uQ)|_{L^\infty\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)} \le ce^{-\delta/\varepsilon}|u|_{H^2}\,.$$

Proof of Lemma 4.8. We consider 3 different cases: x near 0, x far from 0 and $1/\varepsilon$, x near $1/\varepsilon$, that is

– $x \in \Omega_{1,\varepsilon} = \left\{|x| < \dfrac{1}{2\varepsilon}\right\}$,

– $x \in \Omega_{2,\varepsilon} = \left\{ |x| > \dfrac{1}{2\varepsilon} \text{ and } \left| |x| - \dfrac{1}{\varepsilon} \right| > 1 \right\}$,

– $x \in \Omega_{3,\varepsilon} = \left\{ \left| |x| - \dfrac{1}{\varepsilon} \right| < 1 \right\}$.

Let

$$\mathscr{N}_\varepsilon(uv)(r) = N_1(r) + N_2(r)\,, \tag{4.23}$$

where

$$N_1(r) = \frac{u(r)v(\varepsilon) - u\left(\frac{1}{\varepsilon}\right) v\left(\frac{1}{\varepsilon}\right)}{(\varepsilon^2 r^2 - 1)}\,,$$

$$N_2(r) = -\frac{\varepsilon^2}{(\varepsilon^2 r^2 - 1)^{3/2}} \int_{1/\varepsilon}^{r} \frac{u(s)v(s) - u\left(\frac{1}{\varepsilon}\right) v\left(\frac{1}{\varepsilon}\right)}{(\varepsilon^2 s^2 - 1)^{1/2}}\, s\, ds\,.$$

We first estimate N_1, then N_2 in L^∞ and conclude the proof.

Step 1 We have the following uniform estimates as $\varepsilon \to 0$:

$$|N_1|_{L^\infty(\Omega_{1,\varepsilon})} \le c|u|_{L^\infty} |v|_{L^\infty}\,, \tag{4.24}$$

$$|N_1|_{L^\infty(\Omega_{2,\varepsilon})} \le c|u|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)} |v|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}\,, \tag{4.25}$$

$$|N_1|_{L^\infty(\Omega_{3,\varepsilon})} \le c|u|_{H^2(\Omega_{3,\varepsilon})} |v|_{H^2(\Omega_{3,\varepsilon})}\,. \tag{4.26}$$

Let us write

$$N_1(r) = \frac{u(r)v(r) - u(r)v\left(\frac{1}{\varepsilon}\right)}{\varepsilon^2 r^2 - 1} + \frac{u(r)v\left(\frac{1}{\varepsilon}\right) - u\left(\frac{1}{\varepsilon}\right) v\left(\frac{1}{\varepsilon}\right)}{\varepsilon^2 r^2 - 1}\,.$$

i) Proof of (4.24). For $0 < r < \dfrac{1}{2\varepsilon}$, we have $(\varepsilon^2 r^2 - 1) \in \left[\frac{1}{2}, \frac{3}{2}\right]$ and we deduce

$$|N_1(r)| \le c|u|_{L^\infty} |v|_{L^\infty}\,.$$

ii) Proof of (4.25). For $\dfrac{1}{2\varepsilon} < r < \dfrac{1}{\varepsilon} - 1$ or $\dfrac{1}{\varepsilon} + 1 < r$, we have $(\varepsilon^2 r^2 - 1)^{-1} \le \dfrac{c}{\varepsilon}$ and from (2.6),

$$\begin{aligned} |N_1(r)| &\le \frac{c}{\varepsilon} |u|_{L^\infty\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)} |v|_{L^\infty\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)} \\ &\le c|u|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)} |v|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}\,. \end{aligned}$$

iii) Proof of (4.26). We write

$$N_1(r) = \frac{1}{\varepsilon} \frac{u(r)}{\varepsilon r + 1} \frac{v(r) - v\left(\frac{1}{\varepsilon}\right)}{r - 1/\varepsilon} + \frac{1}{\varepsilon} \frac{v\left(\frac{1}{\varepsilon}\right)}{\varepsilon r + 1} \frac{u(r) - u\left(\frac{1}{\varepsilon}\right)}{r - 1/\varepsilon}\,.$$

Thus for $\left|r-\frac{1}{\varepsilon}\right|<1$,

$$
\begin{aligned}
|N_1(r)| &\le \frac{c}{\varepsilon}\left(|u|_{L^\infty(\Omega_{3,\varepsilon})}|v'|_{L^\infty(\Omega_{3,\varepsilon})}+|u'|_{L^\infty(\Omega_{3,\varepsilon})}|u|_{L^\infty(\Omega_{3,\varepsilon})}\right)\\
&\le c|u|_{H^2(\Omega_{3,\varepsilon})}|v|_{H^2(\Omega_{3,\varepsilon})}\,.
\end{aligned}
$$

This concludes Step 1.

Step 2 We have the following uniform estimates as $\varepsilon \to 0$:

$$
\begin{aligned}
|N_2|_{L^\infty(\Omega_{1,\varepsilon})} &\le c(\varepsilon^2|u|_{L^2}|v|_{L^2}+|u|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}|v|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}\\
&\quad +|u|_{H^2(\Omega_{3,\varepsilon})}|v|_{H^2(\Omega_{3,\varepsilon})})\,, \qquad (4.27)\\
|N_2|_{L^\infty(\Omega_{2,\varepsilon})} &\le c(+|u|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}|v|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}\\
&\quad +|u|_{H^2(\Omega_{3,\varepsilon})}|v|_{H^2(\Omega_{3,\varepsilon})})\,, \qquad (4.28)\\
|N_2|_{L^\infty(\Omega_{3,\varepsilon})} &\le c|u|_{H^2(\Omega_{3,\varepsilon})}|v|_{H^2(\Omega_{3,\varepsilon})}\,. \qquad (4.29)
\end{aligned}
$$

i) Proof of (4.29). Let us write

$$
\begin{aligned}
N_2(r) &= \frac{-\varepsilon}{(\varepsilon r+1)^{3/2}}\,\frac{1}{(r-1/\varepsilon)^{3/2}}\\
&\quad\times\int_{1/\varepsilon}^{r}\frac{u(s)v(s)-u\left(\frac{1}{\varepsilon}\right)v\left(\frac{1}{\varepsilon}\right)}{\varepsilon^2 s^2-1}(\varepsilon s+1)^{1/2}s\left(s-\frac{1}{\varepsilon}\right)^{1/2}ds\,.
\end{aligned}
$$

That is from (4.26) for $\frac{1}{\varepsilon}-1<r<\frac{1}{\varepsilon}+1$,

$$
\begin{aligned}
|N_2(r)| &\le \frac{c\varepsilon}{(r-1/\varepsilon)^{3/2}}\int_{1/\varepsilon}^{r}|u|_{H^2(\Omega_{3,\varepsilon})}|v|_{H^2(\Omega_{3,\varepsilon})}(1/\varepsilon)\left(s-\frac{1}{\varepsilon}\right)^{1/2}ds\\
&\le c|u|_{H^2(\Omega_{3,\varepsilon})}|v|_{H^2(\Omega_{3,\varepsilon})}\,.
\end{aligned}
$$

ii) Proof of (4.28). From (4.25) and (4.26), for $r>\frac{1}{2\varepsilon}$,

$$
|N_1(r)| \le c(|u|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}|v|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}+|u|_{H^2(\Omega_{3,\varepsilon})}|v|_{H^2(\Omega_{3,\varepsilon})})\,.
$$

Therefore

$$
|N_2(r)| \le \frac{c\varepsilon^2}{(\varepsilon^2 r^2-1)^{3/2}}\int_{1/\varepsilon}^{r}|N_1|_{L^\infty\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}(\varepsilon^2 s^2-1)^{1/2}s\,ds\,,
$$

that is

$$
|N_2|_{L^\infty(\Omega_{2,\varepsilon})} \le c(|u|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}|v|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}+|u|_{H^2(\Omega_{3,\varepsilon})}|v|_{H^2(\Omega_{3,\varepsilon})})\,.
$$

iii) Proof of (4.27). We write

$$N_2(r) = \frac{\varepsilon^2}{(1-\varepsilon^2 r^2)^{3/2}} \int_r^{1/2\varepsilon} \frac{u(s)v(s) - u\left(\frac{1}{\varepsilon}\right) v\left(\frac{1}{\varepsilon}\right)}{(1-\varepsilon^2 s^2)^{1/2}}\, s\,ds + \frac{(3/4)^{3/2}}{(1-\varepsilon^2 r^2)^{3/2}} N_2(1/2\varepsilon)\,.$$

We then have for $r < \dfrac{1}{2\varepsilon}$,

$$|N_2(r)| \le c\varepsilon^2 \int_r^{1/2\varepsilon} |u(s)|\,|v(s)|\,s\,ds + \frac{c\varepsilon^2}{(1-\varepsilon^2 r^2)^{3/2}} \int_r^{1/2\varepsilon} \frac{\left|u\left(\frac{1}{\varepsilon}\right)\right| \left|v\left(\frac{1}{\varepsilon}\right)\right|}{(1-\varepsilon^2 s^2)^{1/2}}\, s\,ds + c|N_2(1/2\varepsilon)|\,,$$

that is

$$|N_2(r)| \le c\left(\varepsilon^2 |u|_{L^2}\,|v|_{L^2} + \left|u\left(\frac{1}{\varepsilon}\right)\right| \left|v\left(\frac{1}{\varepsilon}\right)\right| + |N_2(1/2\varepsilon)|\right),$$

and (4.27) follows from (4.28).

Step 3. Conclusion. The estimates of $|\mathscr{N}_\varepsilon(uv)|_{L^\infty}$ of Lemma 4.8 follows directly from (4.24)–(4.29) and from definition of the norm in H^2_r.

Finally, we remark that if $v = Q$, from the estimate (see [3])

$$|Q^{(k)}(x)| \le c_k\, e^{-2\delta|x|}\,, \tag{4.30}$$

we have

$$|N(uQ)|_{L^\infty(\Omega_{2,\varepsilon}\cup\Omega_{3,\varepsilon})} \le ce^{-\delta/\varepsilon} |u|_{H^2}\,,$$

and this proves Lemma 4.8.

c) Estimates of $|\mathscr{L}((\mathscr{N}_\varepsilon(uQ) + uQ)v)|_{H^2}$.

Proposition 4.9. *We have the uniform estimate as* $\varepsilon \to 0$ *for* $u, v \in H^2_r$,

$$|\mathscr{L}((\mathscr{N}_\varepsilon(Qu) + Qu)v)|_{H^2} \le c\varepsilon^2 |u|_{H^2}\,|v|_{H^2}\,.$$

Proof. Let $u, v \in H^2_r$. We first have from (4.11),

$$\begin{aligned} |\mathscr{L}((\mathscr{N}_\varepsilon(Qu) + Qu)v)|_{H^2} &\le |(\mathscr{N}_\varepsilon(Qu) + Qu)v|_{L^2} \\ &\le |(\mathscr{N}_\varepsilon(Qu) + Qu)v|_{L^2(\Omega_{1,\varepsilon})} + |(\mathscr{N}_\varepsilon(Qu) + Qu)v|_{L^2(\Omega_{2,\varepsilon})} \\ &\quad + |(\mathscr{N}_\varepsilon(Qu) + Qu)v|_{L^2(\Omega_{3,\varepsilon})}\,. \end{aligned} \tag{4.31}$$

By Lemma 4.8 and (4.30) we have for a $\delta > 0$,

$$|(\mathscr{N}_\varepsilon(Qu) + Qu)v|_{L^2(\Omega_{2,\varepsilon})} = O(e^{-\delta/\varepsilon})\,|u|_{H^2}\,|v|_{H^2}\,, \tag{4.32}$$

$$|(\mathscr{N}_\varepsilon(Qu) + Qu)v|_{L^2(\Omega_{3,\varepsilon})} = O(e^{-\delta/\varepsilon})\,|u|_{H^2}\,|v|_{H^2}\,. \tag{4.33}$$

Therefore we only have to estimate $|(\mathcal{N}_\varepsilon(Qu)+Qu)v|_{L^2(\Omega_{1,\varepsilon})}$.

We write

$$\mathcal{N}_\varepsilon(uQ)(x)+Q(x)u(x)=\frac{Q(x)u(x)-Q\left(\frac{1}{\varepsilon}\right)u\left(\frac{1}{\varepsilon}\right)}{\varepsilon^2|x|^2-1}+Q(x)u(x)+N_2(x),$$

where $N_2(x)$ is defined by (4.23). By estimates (4.27) and (4.30),

$$\begin{aligned}|N_2|_{L^\infty(\Omega_{1,\varepsilon})}&\le c(\varepsilon^2|u|_{L^\infty}+|u|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}\\&\quad\times|Q|_{H^1\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}+|u|_{H^2(\Omega_{3,\varepsilon})}|Q|_{H^2(\Omega_{3,\varepsilon})})\\&\le c\varepsilon^2|u|_{H^2}.\end{aligned}$$

Direct computations give

$$\frac{Q(x)u(x)-Q\left(\frac{1}{\varepsilon}\right)u\left(\frac{1}{\varepsilon}\right)}{\varepsilon^2|x|^2-1}+Q(x)u(x)=\varepsilon^2\frac{x^2Q(x)u(x)}{\varepsilon^2|x|^2-1}-\frac{Q\left(\frac{1}{\varepsilon}\right)u\left(\frac{1}{\varepsilon}\right)}{\varepsilon^2|x|^2-1},$$

and

$$\begin{aligned}&|(\mathcal{N}_\varepsilon(Qu)+Qu)v|_{L^2(\Omega_{1,\varepsilon})}\\&\quad\le c\varepsilon^2\left|\frac{|x|^2Q(x)u(x)v(x)}{\varepsilon^2|x|^2-1}\right|_{L^2(\Omega_{1,\varepsilon})}\\&\qquad+c|Q(1/\varepsilon)|\,|u(1/\varepsilon)|\,|v|_{L^2(\Omega_{1,\varepsilon})}+c\varepsilon^2|u|_{H^2}|v|_{L^2(\Omega_{1,\varepsilon})}.\end{aligned}\tag{4.34}$$

In addition,

$$\varepsilon^2\left|\frac{|x|^2Q(x)u(x)v(x)}{\varepsilon^2|x|^2-1}\right|_{L^2(\Omega_{1,\varepsilon})}\le c\varepsilon^2|(|x|^2Q(x))u(x)v(x)|_{L^2}\le c\varepsilon^2|u|_{H^2}|v|_{H^2}$$

and

$$|Q(1/\varepsilon)|\,|u(1/\varepsilon)|\,|v|_{L^2(\Omega_{1,\varepsilon})}\le ce^{-\delta/\varepsilon}|u|_{H^2}|v|_{H^2}\le c\varepsilon^2|u|_{H^2}|v|_{H^2}.$$

Therefore from (4.34), we find

$$|(\mathcal{N}_\varepsilon(Qu)+Qu)v|_{L^2(\Omega_{1,\varepsilon})}\le c\varepsilon^2|u|_{H^2}|v|_{H^2},\tag{4.35}$$

and (4.31)–(4.35) conclude the proof of Proposition 4.9.

d) Proof of the fixed point theorem. Let

$$B=\{u\in H,|u|_{H^2}\le\delta_0\},$$

where $\delta_0>0$ is a constant to be chosen later.

Let us prove T_ε is contraction mapping of the set B for a constant $\delta_0>0$ and for $\varepsilon>0$ small enough. From the definition of T_ε,

$$|T_\varepsilon(u)|_{H^2}\le|C_\varepsilon|_{L^2}+|l_\varepsilon(u)|_{L^2}+|q_\varepsilon(u)|_{L^2}+|k_\varepsilon(u)|_{L^2}$$

and

$$|T_\varepsilon(u)-T_\varepsilon(v)|_{H^2}\le|l_\varepsilon(u)-l_\varepsilon(v)|_{L^2}+|q_\varepsilon(u)-q_\varepsilon(v)|_{L^2}+|k_\varepsilon(u)-k_\varepsilon(v)|_{L^2},$$

where C_ε, l_ε, q_ε, k_ε are defined by (4.6)–(4.9).

1) The constant term: From Proposition 4.9, we have

$$|C_\varepsilon|_{H^2} = |\mathscr{L}((\mathscr{N}_\varepsilon(Q^2) + Q^2)Q)|_{H^2} \le c\varepsilon^2 .$$

2) The linear term: From Proposition 4.9, we have

$$|l_\varepsilon(u)|_{L^2} = |2(\mathscr{N}_\varepsilon(Qu) + Qu)Q + (\mathscr{N}_\varepsilon(Q,Q) + Q^2)u|_{L^2} \le c\varepsilon^2 |u|_{H^2}$$

and

$$|l_\varepsilon(u) - l_\varepsilon(v)|_{L^2} \le c\varepsilon^2 |u - v|_{H^2} .$$

3) The quadratic term: From Proposition 4.7, we have

$$|q_\varepsilon(u)|_{L^2} = |\mathscr{N}_\varepsilon(Qu)u + \mathscr{N}_\varepsilon(u^2)Q|_{L^2} \le c|u|^2_{H^2} ,$$
$$|q_\varepsilon(u) - q_\varepsilon(v)|_{L^2} \le ((|u|_{H^2} + |v|_{H^2})\,|u - v|_{H^2} .$$

4) The cubic term: From Proposition 4.7, we have

$$|k_\varepsilon(u)|_{L^2} = |\mathscr{N}_\varepsilon(u^2)u|_{L^2} \le c|u|^3_{H^2} ,$$
$$|k_\varepsilon(u) - k_\varepsilon(v)|_{L^2} \le ((|u|^2_{H^2} + |v|^2_{H^2})\,|u - v|_{H^2} .$$

Therefore there exist constants $\varepsilon_1 > 0$, $c > 0$ such that for $0 < \varepsilon < \varepsilon_1$ and u, v in H^2_r,

$$|T_\varepsilon(u)|_{H^2} \le c(\varepsilon^2 + |u|_{H^2} + |u|^2_{H^2}) ,$$
$$|T_\varepsilon(u) - T_\varepsilon(v)|_{H^2} \le c(\varepsilon^2 + (|u|_{H^2} + |v|_{H^2} + |u|^2_{H^2} + |v|^2_{H^2}))\,|u - v|_{H^2} .$$

We then deduce that for u, v in B,

$$|T_\varepsilon(u)|_{H^2} \le c(\varepsilon^2 + (\varepsilon^2 + \delta_0 + \delta_0^2)\delta_0) ,$$
$$|T_\varepsilon(u) - T_\varepsilon(v)|_{H^2} \le c(\varepsilon^2 + 2\delta_0 + 2\delta_0^2)\,|u - v|_{H^2} .$$

Let us fix $\delta_0 > 0$ and $0 < \varepsilon_0 < \varepsilon_1$ small enough such that for all $0 < \varepsilon < \varepsilon_0$,

$$c(\varepsilon^2 + 2\delta_0 + 2\delta_0^2) < \tfrac{1}{2} ,$$
$$c(\varepsilon^2 + (\varepsilon^2 + 2\delta_0 + 2\delta_0^2)\delta_0) < \tfrac{1}{2}\,\delta_0 .$$

We then have for $0 < \varepsilon < \varepsilon_0$ and u, v in B,

$$|T_\varepsilon(u)|_{H^2} \le \frac{\delta_0}{2} ,$$
$$|T_\varepsilon(u) - T_\varepsilon(v)|_{H^2} \le \tfrac{1}{2}\,|u - v|_{H^2} .$$

Therefore T_ε is a contraction mapping of the set B for $0 < \varepsilon < \varepsilon_0$ and there is existence and uniqueness of a fixed point $h_\varepsilon \in B$ of the operator T_ε in B.

From the fact that

$$|h_\varepsilon - T_\varepsilon(0)|_{H^2} = |T_\varepsilon(h_\varepsilon) - T_\varepsilon(0)| \le \tfrac{1}{2}\,|h_\varepsilon - 0|_{H^2} ,$$

we deduce that

$$\tfrac{1}{2}\,|h_\varepsilon|_{H^2} \le |T_\varepsilon(0)|_{H^2} \le c\varepsilon^2 .$$

We finally show that h_ε in a continuous function of ε in H^2_r from classical fixed point arguments and the following lemma:

Lemma 4.10. *The function $(\varepsilon, h) \to T_\varepsilon(h)$ is uniformly continuous $\mathbb{R}^+ \times H^2_r \to H^2_r$*

Proof. We recall that from (4.12),

$$T_\varepsilon(h) = \mathscr{L}[\mathscr{N}_\varepsilon((Q+h)^2)(Q+h) + Q^3 + 3Q^2h]\,.$$

The uniform continuity of T_ε follows form standard arguments and the two following properties:

– From Proposition 4.7, for all $\alpha > 0$ there exists a constant $c_\alpha > 0$ such that

$$\forall \varepsilon \in [0,\alpha]\,,\quad \forall u,v,w \in H^2_r\,,\quad |\mathscr{L}(\mathscr{N}_\varepsilon(uv)w)|_{H^2} \le c_\alpha |u|_{H^2}\,|v|_{H^2}\,|w|_{H^2}\,. \tag{4.36}$$

– For fixed $u,v,w \in H^2_r$, the function $\varepsilon \to \mathscr{L}(\mathscr{N}_\varepsilon(uv)w)$ is continuous $\mathbb{R}^+ \to H^2_r$. (4.37)

It is a consequence of the dominated convergence theorem and Lemma 4.8. Indeed, we have from Lemma 4.8 and Proposition 4.7 that for $\alpha > 0$ there exists a constant c_α such that for $0 < \varepsilon < \alpha$, $0 < \varepsilon' < \alpha$, $\forall u,v,w \in H^2_r$,

$$|\mathscr{L}(\mathscr{N}_\varepsilon(uv)w) - \mathscr{L}(\mathscr{N}_{\varepsilon'}(uv)w)|_{H^2} \le c_\alpha |(\mathscr{N}_\varepsilon(uv) - \mathscr{N}_{\varepsilon'}(uv))w|_{L^2}$$

and

$$|\mathscr{N}_\varepsilon(uv)|_{L^\infty} \le c_\alpha |u|_{H^2}\,|u|_{H^2}\,. \tag{4.38}$$

Therefore from (4.38) and the fact that

$$\forall x\,,\quad \mathscr{N}_\varepsilon(uv)(x) \to \mathscr{N}_{\varepsilon'}(uv)(x) \quad \text{as} \quad \varepsilon \to \varepsilon'\,,$$

property (4.37) follows from the dominated convergence theorem. This concludes the proof of Lemma 4.10.

Therefore, h_ε is a continuous function of ε in H^2_r. This completes the proof of Theorem 4.2.

IV.2 Positivity Properties of the Solutions near Q

We prove that the solutions $(P_\varepsilon, N_\varepsilon)$ of $(\mathrm{II}_\varepsilon)$ we find in Part IV.1 are in fact solutions of $(\mathrm{II}^+_\varepsilon)$. That is,

$$\forall \varepsilon\,,\quad \forall r\,,\quad P_\varepsilon(r) > 0\,.$$

Proposition 4.1. *There is a* $\varepsilon_1 > 0$ *such that for* $0 < \varepsilon < \varepsilon_1$, *there exists a solution* $(P_\varepsilon, N_\varepsilon)$ *of* $(\mathrm{II}^+_\varepsilon)$.

Proof. Let $(P_\varepsilon, N_\varepsilon)$ the solution of Corollary 4.3. We then have

$$(P_\varepsilon, N_\varepsilon) \to (Q, -Q^2) \quad \text{in} \quad H^1 \times L^2 \tag{4.39}$$

and from Theorem 4.2,

$$P_\varepsilon \to Q \quad \text{in} \quad H^2\,. \tag{4.40}$$

We first prove some uniform estimates of N_ε with respect to ε at infinity and then in fact that $P_\varepsilon > 0$.

We have the following lemma

Lemma 4.12. *There exists* $\varepsilon_2 > 0$ *and* $r_0 > 0$ *such that for* $0 \le \varepsilon \le \varepsilon_2$,

$$\forall r \le r_0\,,\qquad |N_\varepsilon(r) \le \tfrac{1}{2}\,. \tag{4.41}$$

Proof. We recall that

$$N_\varepsilon(r) = \mathscr{N}_\varepsilon(P^2_\varepsilon)(r)\,,$$

where $\mathscr{N}_\varepsilon$ is defined by (4.3). From equality (2.8) with $r_1 = r$ and $r_3 = \frac{1}{2\varepsilon}$, we have for $r < \frac{1}{2\varepsilon}$,

$$N_\varepsilon(r) = \frac{(3/4)^{3/2}}{(1-\varepsilon^2 r^2)^{3/2}} N_\varepsilon\left(\frac{1}{2\varepsilon}\right) + \frac{1}{(1-\varepsilon^2 r^2)^{3/2}} \int_r^{1/2\varepsilon} 2P_\varepsilon(s)P_\varepsilon'(s)(1-\varepsilon^2 s^2)^{1/2}\,ds\,.$$

That is from (2.6) for $r \leq \frac{1}{2\varepsilon}$,

$$\begin{aligned}|N_\varepsilon(r)| &\leq \left(\left|N_\varepsilon\left(\frac{1}{2\varepsilon}\right)\right| + \int_r^{1/2\varepsilon} 2|P_\varepsilon(s)|\,|P_\varepsilon'(s)|\,ds\right)\\ &\leq c\left(\left|N_\varepsilon\left(\frac{1}{2\varepsilon}\right)\right| + \frac{1}{r}|P_\varepsilon|^2_{H^1}\right). \end{aligned} \tag{4.42}$$

Moreover, Lemma 4.8 gives

$$|N_\varepsilon|_{L^\infty\left(|x|>\frac{1}{2\varepsilon}\right)} \leq c|P_\varepsilon|^2_{H^2\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)}\,. \tag{4.43}$$

Therefore (4.42), (4.43) yield

$$\forall r \geq 0\,,\quad |N_\varepsilon(r)| \leq c\left(|P_\varepsilon|^2_{H^2\left(\left\{|x|>\frac{1}{2\varepsilon}\right\}\right)} + \frac{1}{r}|P_\varepsilon|^2_{H^1}\right). \tag{4.44}$$

Since (4.40), we have

$$\forall r \geq 0\,,\quad |N_\varepsilon(r)| \leq c\left(o(\varepsilon) + \frac{1}{r}\right)$$

and Lemma 4.12 follows.

Let ε_2 and r_0 defined in Lemma 4.12. From (4.40), we have that $P_\varepsilon \to Q$ in $L^\infty(r \leq r_0)$. Since $\forall r \leq r_0$, $Q(r) \geq Q(r_0) > 0$, there is $0 < \varepsilon_3 < \varepsilon_2$ such that for $0 < \varepsilon < \varepsilon_3$,

$$\forall r \leq r_0\,,\quad P_\varepsilon(r) > 0\,.$$

From Lemma 4.12, we have for $0 < \varepsilon < \varepsilon_3$,

$$\forall r \leq r_0\,,\quad (1 + N_\varepsilon(r)) \in \left[\tfrac{1}{2}, \tfrac{3}{2}\right].$$

Since

$$\begin{gathered}\Delta P_\varepsilon - (1+N_\varepsilon)P_\varepsilon = 0\,,\\ P_\varepsilon(r_0) > 0 \quad \text{and} \quad P(+\infty) = 0\,,\end{gathered}$$

we deduce from the maximum principle that $P_\varepsilon(r) > 0$ for $r \geq r_0$.

This concludes the proof of Proposition 4.11.

IV 3 Uniqueness Property of Solutions of $(\mathrm{II}_\varepsilon^+)$ *for* $\varepsilon > 0$ *Small*

We show the uniqueness of the solutions $(P_\varepsilon, N_\varepsilon)$ of $(\mathrm{II}_\varepsilon^+)$ given in Proposition 4.11 under a condition of boundedness.

Proposition 4.13 (Uniqueness of Solutions of $(\mathrm{II}_\varepsilon^+)$ for ε Small). *For all $c > 0$, there exists $\varepsilon_0 = \varepsilon_0(c) > 0$ such that for all $0 \le \varepsilon \le \varepsilon_0$ there exists a unique solution $(P_\varepsilon, N_\varepsilon)$ of $(\mathrm{II}_\varepsilon^+)$ such that $|P_\varepsilon|_{L^2} \le c$*

Proof. It follows from fixed point theorem and from Sect. III. Let $(P_\varepsilon, N_\varepsilon)$ solution of $(\mathrm{II}_\varepsilon^+)$ with $\varepsilon \to 0$ such that $|P_\varepsilon|_{L^2} \le c$. Then $(P_\varepsilon, N_\varepsilon) \to (Q, -Q^2)$ in $H^1 \times L^2$ as $\varepsilon \to 0$. We claim that $P_\varepsilon \to Q$ in H^2.

We consider a sequence (P_n, N_n) of $(\mathrm{II}_{\varepsilon_n}^+)$ with $\varepsilon_n \to 0$. From Sect. III and Lemma 2.2 we have

$$(P_n, N_n) \to (Q, -Q^2) \quad \text{in} \quad H^1 \times L^2 \quad \text{as} \quad n \to +\infty \tag{4.45}$$

and

$$\forall n\,, \quad |P_n|_{H^1} + |N_n|_{L^2} + |P_n|_{L^\infty} \le c\,. \tag{4.46}$$

Since $\Delta P_n = (1 + N_n)P_n$, we deduce that $|P_n|_{H^2} \le c$ and there is a subsequence [also denoted (P_n, N_n)] such that

$$\Delta P_n \rightharpoonup \Delta Q \quad \text{in} \quad L^2 \quad \text{as} \quad n \to +\infty\,. \tag{4.47}$$

From Eq. (4.47), we have

$$\int_{\mathbb{R}^2} |\Delta P_n|^2 = \int_{\mathbb{R}^2} P_n \Delta P_n + \int_{\mathbb{R}^2} (N_n P_n) \Delta P_n\,.$$

From (4.45), (4.46) we have that $P_n \to Q$ in L^2 and $(N_n P_n) \to -Q^3$ in L^2. Indeed,

$$\begin{aligned} |N_n P_n + Q^3|_{L^2} &\le |(N_n + Q^2)P_n|_{L^2} + |-Q^2(P_n - Q)|_{L^2} \\ &\le c(|N_n + Q^2|_{L^2} + |P_n - Q|_{L^2})\,. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^2} |\Delta P_n|^2 \to \int_{\mathbb{R}^2} Q \Delta Q + \int_{\mathbb{R}^2} (-Q^3) \Delta Q = \int_{\mathbb{R}^2} |\Delta Q|^2\,, \tag{4.48}$$

and then from (4.47), $\Delta P_n \to \Delta Q$ in L^2. Thus, $P_n \to Q$ in H^2 as $n \to +\infty$.

Therefore $P_\varepsilon \to Q$ in H^2 as $\varepsilon \to 0$. In particular for $\varepsilon > 0$ small, if $(P_\varepsilon, N_\varepsilon)$ is a solution of $(\mathrm{II}_\varepsilon^+)$ then $k_\varepsilon = P_\varepsilon - Q \in H_r^2$ is such that $T_\varepsilon(k_\varepsilon) = k_\varepsilon$ and $|k_\varepsilon|_{H^2} \le \delta_0$, where δ_0 is defined in Theorem 4.2. Since from the fixed point theorem there is a unique solution $h_\varepsilon \in H_r^2$ such that $T_\varepsilon(h_\varepsilon) = h_\varepsilon$ and $|k_\varepsilon|_{H^2} \le \delta_0$, we have that $k_\varepsilon = h_\varepsilon$ and $P_\varepsilon = Q + h_\varepsilon$. Thus P_ε is unique and N_ε is unique from Propositon 2.6.

This concludes the proof of Proposition 4.13 and also part i) of Theorem 1.

V. Existence of an Unbounded Branch $(\lambda, P_\lambda, N_\lambda)$ of Solutions of (II_λ^+)

In this section, we show using index theory that the branch of $(\lambda, P_\lambda, N_\lambda)$ solutions of (II_λ) for $0 < \lambda < \varepsilon_0$ constructed in Sect. IV can be extended to a connected component $\mathscr{C}_1$ of the set

$$\{(\lambda, P_\lambda, N_\lambda);\ (P_\lambda, N_\lambda) \text{ solution of } (\mathrm{II}_\lambda^+)\}$$

in $\mathbb{R}^+ \times (H_r^1 \times L_r^2)$ which is unbounded. This fact will be a crucial element in the proof of the main theorem of Part II [6].

As in Sect. IV, it is more convenient to consider a different formulation of the problem (II_λ).

Let $\mathscr{C}_1$ be the connected component of $(\lambda, P_\lambda, N_\lambda)$ in $\mathbb{R}^+ \times H^1_r \times L^2_r$ of solutions of (II_λ) containing $(0, Q, -Q^2)$.

Let $\mathscr{C}_2$ be the connected component of (λ, h_λ) solutions of (VI_λ) containing $(0,0)$ in $\mathbb{R}^+ \times H^2_r$, where

$$(\mathrm{VI}_\lambda) \qquad h = T_\lambda(h)$$

and T_λ is defined by (4.12). We have the following relation between $\mathscr{C}_1$ and $\mathscr{C}_2$.

Proposition 5.1.
i) $\mathscr{C}_2 = \{(\lambda, h_\lambda); h_\lambda = P_\lambda - Q$ *with* $(\lambda, P_\lambda, N_\lambda) \in \mathscr{C}_1\}$.
ii) $\mathscr{C}_2$ *is unbounded in* $\mathbb{R}^+ \times H^2$ *if and only if* $\mathscr{C}_1$ *is unbounded in* $\mathbb{R}^+ \times H^1 \times L^2$
iii) *If* $(\lambda, P_\lambda, N_\lambda) \in \mathscr{C}_1$, *then* (P_λ, N_λ) *is a solution of* (II^+_λ): $\forall r$, $P_\lambda(r) > 0$.

Proof i) follows from the fact that (P_λ, N_λ) is a solution of (II_λ) if and only if $h_\lambda = T_\lambda(h_\lambda)$, where $P_\lambda = h_\lambda + Q$, $N_\lambda = \mathscr{N}_\lambda(P^2_\lambda)$, Lemma 4.4 and topological arguments.
ii) $\mathscr{C}_1$ unbounded implies $\mathscr{C}_2$ unbounded. We have to prove that if (P_λ, N_λ) in $H^1 \times L^2$ is unbounded, then h_λ is unbounded in H^2. From Pohozaev identity (2.14), $|N_\lambda|_{L^2} \le |P_\lambda|_{L^2}$ and $|P_\lambda|_{H^1}$ is unbounded. Thus $|h_\lambda|_{H^2} = |P_\lambda - Q|_{H^2}$ is unbounded.

$\mathscr{C}_2$ unbounded implies $\mathscr{C}_1$ unbounded. It is sufficient to show that if h_λ is unbounded in H^2 then (P_λ, N_λ) is unbounded in $H^1 \times L^2$. Assume that $|P_\lambda|_{H^1} + |N_\lambda|_{L^2} \le c$, then $|P_\lambda|_{L^\infty} \le c$ (Lemma 2.2) and form the fact that $\Delta P_\lambda = N_\lambda P_\lambda + P_\lambda$, $|P_\lambda|_{H^2} \le c$ which is a contradiction.
iii) Consider

$$\mathscr{C}_1^+ = \{(\lambda, P, N) \in \mathscr{C}_1, P > 0\}.$$

We claim that $\mathscr{C}_1^+ = \mathscr{C}_1$. Indeed, we show that $\mathscr{C}_1^+$ is closed, open and non-empty set in $\mathscr{C}_1$.
– $(0, Q, -Q^2) \in \mathscr{C}_1^+$.
– $\mathscr{C}_1^+$ is closed. Let $(\lambda_n, P_n, N_n) \in \mathscr{C}_1^+$ such that

$$(\lambda_n, P_n, N_n) \to (\lambda, P, N) \quad \text{in} \quad \mathbb{R}^+ \times H^1 \times L^2 \quad \text{as} \quad n \to +\infty \quad \text{and} \quad P_n > 0.$$

We have from Corollary 2.17,

$$|P_n|_{L^2} > |Q|_{L^2}.$$

Thus $P \ge 0$ and $P \not\equiv 0$ from the fact $|P_n|_{L^2} \ge |Q|_{L^2}$. Since

$$\Delta P = (N+1)P,$$

the strong maximum principle yields $P > 0$ and $(\lambda, P, N) \in \mathscr{C}_1^+$.
– $\mathscr{C}_1^+$ is open. It is sufficient to show that, if $(\lambda, P, N) \in \mathscr{C}_1^+$ and $(\lambda_n, P_n, N_n) \in \mathscr{C}_1$, $(\lambda_n, P_n, N_n) \to (\lambda, P, N)$ in $\mathbb{R}^+ \times H^1 \times L^2$, then $(\lambda_n, P_n, N_n) \in \mathscr{C}_1^+$ for n large enough.

Case $\lambda = 0$. Then $(P, N) = (Q, -Q^2)$ (from Kwong [9]) and the result follows from Propositions 4.11 and 4.13.

Case $\lambda > 0$. Let us show that there is a r_0 such that

$$\forall r \ge r_0, \quad |N_n(r)| \le \tfrac{1}{2}.$$

From relation (2.8) with $r_1 = \dfrac{1}{\lambda_n}$ and $r_2 = r$, we have

$$N_n(r) = \frac{1}{(\lambda_n^2 r^2 - 1)^{3/2}} \int_{1/\lambda_n}^{r} 2P_n(s)\,P_n'(s)\,(\lambda^2 s^2 - 1)^{1/2}\,ds\,,$$

that is for $r \geq r_1 = \max\left\{\dfrac{2}{\lambda_n}, n \geq 0\right\}$

$$|N_n(r)| \leq \frac{c}{r^3} \int_{1/\lambda_n}^{r} 2|P_n(s)|\,|P_n'(s)|\,ds \leq \frac{c}{r^3}\,. \tag{5.1}$$

Therefore there exists $r_0 > 0$ such that $\forall r \geq r_0$, $\forall n$, $|N_n(r)| \leq \frac{1}{2}$. In addition, since $P_n \to P$ in H^1, $P_n > 0$ on $[0, r_0]$ for n large. We then can conclude the proof as in Proposition 4.11 using the maximum principle.

We now claim from degree theory that $\mathscr{C}_2$ is unbounded in $\mathbb{R}^+ \times H_r^1$. This will conclude the proof of Proposition 5.1 and the proof of part ii) of Theorem 1.

Theorem 5.2. *$\mathscr{C}_2$ is unbounded in $\mathbb{R}^+ \times H_r^2$.*

It follows from Rabinowitz [17] and following propositions.

Proposition 5.3. *The application*

$$(\lambda, h) \in (0, +\infty) \times H_r^2 \to T_\lambda(h) \in H_r^2\,,$$

where T_λ is defined by (4.12) *is continuous and compact.*

Proposition 5.4. *There exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that*

$$\mathscr{C}_2 \cap [0, \varepsilon_0] \times B = \{(\varepsilon, h_\varepsilon),\ \textit{for}\ 0 \leq \varepsilon \leq \varepsilon_0\}\,,$$

where h_ε is the unique solution in $B = \{h \in H_r^2, |h|_{H^2} \leq \delta_0\}$ of equation $h_\varepsilon = T_\varepsilon(h_\varepsilon)$ and ε_0, δ_0 are defined in Theorem 4.2.

Indeed, assuming Propositions 5.3 and 5.4 and following the method of Rabinowitz in [17], let us prove Theorem 5.2.

Proof of Theorem 5 2. Let us argue by contradiction: assume that the connected component $\mathscr{C}_2$ is bounded. We define

$$\mathscr{T} = \{(\lambda, h_\lambda) \in \mathbb{R}^+ \times H_r^2,\ h_\lambda = T_\lambda(h_\lambda)\}\,.$$

From Propositions 5.3 and 5.4, $\mathscr{C}_2$ is a compact set and there exists a bounded open $\mathscr{O}$ such that

$$\mathscr{C}_2 \subset \mathscr{O} \quad \text{and} \quad \partial\mathscr{O} \cap \mathscr{T} = \emptyset\,.$$

Moreover, we can choose $\mathscr{O}$ such that

$$\mathscr{O} \cap [0, \varepsilon_0] \times H_r^2 = [0, \varepsilon_0] \times \{h \in H_r^2, |h|_{L^2} < \delta_0\}\,. \tag{5.2}$$

We then consider $\mathscr{O}_\lambda = \{h \in H_r^2$ such that $(\lambda, h) \in \mathscr{O}\}$. Therefore the homotopy invariance property of degree yields

$$\deg(\mathrm{Id} - T_\lambda, \mathscr{O}_\lambda) = c\,, \quad \forall \lambda > 0\,. \tag{5.3}$$

– $\mathscr{O}_\lambda = \emptyset$ for λ large since $\mathscr{O}$ is bounded, therefore

$$\deg(\mathrm{Id} - T_\lambda, \mathscr{O}_\lambda) = 0\,. \tag{5.4}$$

– For $0 < \varepsilon \le \varepsilon_0$ small enough, we claim that

$$\deg(\mathrm{Id} - T_\varepsilon, \mathscr{O}_\varepsilon) = 1\,. \tag{5.5}$$

Indeed, since $\mathscr{C}_2$ is bounded in $\mathbb{R}^+ \times H^2$, from (5.2), Proposition 5.4 and Theorem 4.2,

$$\forall \varepsilon \in [0, \varepsilon_0]\,, \qquad \mathscr{T} \cap \mathscr{O}_\varepsilon = \{h_\varepsilon\} \quad \text{and} \quad |h_\varepsilon|_{H^2} \le c\varepsilon^2\,. \tag{5.6}$$

Furthermore the Fréchet differential of $h \to T_\varepsilon(h)$ [defined by (4.12)] is for $h, k \in H^2_r$,

$$dT_\varepsilon(h)\,(k) = \mathscr{L}(l_\varepsilon(k) + dq_\varepsilon(h)\,(k) + dk_\varepsilon(h)\,(k))\,,$$

where

$$dq_\varepsilon(h)\,(k) = \mathscr{N}_\varepsilon(Qh)\,k + \mathscr{N}_\varepsilon(Qk)\,h + 2\mathscr{N}_\varepsilon(kh)\,Q\,,$$
$$dk_\varepsilon(h)\,(k) = \mathscr{N}_\varepsilon(h^2)\,k + 2\mathscr{N}_\varepsilon(hk)\,h\,.$$

Therefore from estimates on $\mathscr{N}_\varepsilon$ and $|h_\varepsilon|_{H^2}$ [Proposition 4.7 and (5.6)], we deduce

$$|dT_\varepsilon(h_\varepsilon)\,(k)|_{H^2_r} \le c\varepsilon^2 |k|_{H^2_r}\,,$$

and for $\varepsilon > 0$ small enough

$$\deg(\mathrm{id} - T_\varepsilon, \mathscr{O}_\varepsilon) = 1\,.$$

Thus, we have a contradiction from (5.3)–(5.5). This concludes the proof of Theorem 5.2 and Part ii) of Theorem 1.

Proof of Proposition 5 3 The continuity follows from Lemma 4.10. Let us prove that the operator

$$(\lambda, h) \to T_\lambda(h)$$

is compact on $(0, +\infty) \times H^2_r$. Let a bounded sequence (λ_n, h_n) of $(0, +\infty) \times H^2_r$ such that

$$\lambda_n \to \lambda > 0 \quad \text{as} \quad n \to +\infty\,.$$

We have to show that the sequence

$$k_n = T_{\lambda_n}(h_n)$$

is relatively compact in H^2_r. Writing $(P_n, N_n) = (Q + h_n, \mathscr{N}_{\lambda_n}((Q + h_n)^2)$, we have from the definition of the operator T_{λ_n} that

$$k_n = \mathscr{L}(P_n N_n + 3Q^2 P_n - 2Q^2)\,.$$

From the property of $\mathscr{L}$ (Proposition 4.1) it is sufficient to show that the sequences $P_n N_n$ and $3Q^2 P_n$ are relatively compact in L^2.

From the estimate $|h_n|_{H^2_r} \le c$ and $h_n = P_n - Q$,

$$|P_n|_{H^2_r} \le c\,, \tag{5.7}$$

and the sequence $3Q^2 P_n$ is relatively compact in L^2.

Furthermore, from Lemma 4.8 we have $|N_n|_{L^\infty} \leq c$ and using estimates (5.1), there exists a constant $c > 0$ such that

$$\forall n, \quad \forall r, \quad |N_n(r)| \leq \frac{c}{1+r^3}. \tag{5.8}$$

From (5.7) and

$$(\lambda_n^2 r^2 - 1) N_n'(r) + 3\lambda^2 r N_n(r) = 2P_n(r) P_n'(r),$$

we deduce that there exists a subsequence of (P_n, N_n) such that

$$P_n \to P \quad \text{in} \quad H^1_{\text{loc}} \quad \text{and} \quad (P_n, N_n) \to (P, N) \quad \text{a.e.} \tag{5.9}$$

From (5.7)–(5.9) we derive that $P_n N_n$ converge to PN in L^2 as $n \to +\infty$. This concludes the proof of Proposition 5.3.

Proof of Proposition 5.4 It follows directly from parts i), ii) of Proposition 5.1 and Proposition 4.13.

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