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The Conformal Factor in the SAS Einstein–Maxwell Field Equations and a Central Extension of a Formal Loop Group

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Abstract: We consider a relation between the conformal factor in the stationary axisymmetric (SAS) Einstein–Maxwell field equations and a central extension of a formal loop group which is described by a group 2-cocycle on the formal loop group. The corresponding 2-cocycle on the Lie algebra of the formal loop group is the one which describes an affine Lie algebra. As a result, we see that the space of formal solutions with conformal factors is a homogeneous space of a central extension of the Hauser group.

0. Introduction

In [HS] we have discussed a σ-model with values in $S(U(1) \times U(2)) \backslash SU(1, 2)$ which is derived from the stationary axisymmetric (SAS) Einstein–Maxwell field equations. We formulated the theory of the σ-model in the category of formal power series by using Takasaki's formal loop group technique [T] and the linearization procedure investigated by Breitenlohner and Maison [BM]. However, we did not incorporate the conformal factor into the theory, neither did we state the homogeneous structure of the space of solutions of the Einstein–Maxwell field equations in stationary axisymmetric space-time.

As to the conformal factor, the second author, in [S], reproduced the results of [BM] in the category of formal power series and obtained an infinite dimensional homogeneous space structure of the space of formal solutions in the case of the Einstein equations.

In the present paper, following [BM, HS, S], we extend the theory of our σ-model to the Einstein–Maxwell field equations with N abelian gauge fields in stationary axisymmetric space-times involving the conformal factor. We prove that there is an elegant relation between the conformal factor and a group 2-cocycle on the formal loop group with values in $SU(1, N+1)$, and show that the trivial central

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extension of the Hauser group acts transitively on the space of formal solutions of the Einstein–Maxwell field equations with N abelian gauge fields. The corresponding 2-cocycle on the Lie algebra of the formal loop group is the one which describes an affine Lie algebra [K]. This relation was first found by [BM].

Now we derive the equations, which are our starting point, from the stationary axisymmetric Einstein–Maxwell field equations with N abelian gauge potentials.

Let $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$ be a metric on $\mathbb{R}^{1+3}$ and $\mathbf{A} = \mathbf{A}_\mu dx^\mu$ an abelian gauge potential with values in $\mathbb{R}^N$. Then the Einstein–Maxwell field equations with N abelian gauge fields are given by

$$R_{\mu\nu} = 8\pi T_{\mu\nu}, \quad \nabla_\kappa \mathbf{F}^{\mu\kappa} = 0 \, (\mu, \nu = 0, 1, 2, 3) \, ,$$

where $R_{\mu\nu}$ is the Ricci curvature and

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu,$$

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\mathbf{F}_{\mu\kappa}{}^t\mathbf{F}_\nu{}^\kappa - \frac{1}{4} g_{\mu\nu} \mathbf{F}_{\kappa\iota}{}^t\mathbf{F}^{\mu\iota} \right) .$$

We adopt the coordinates $(x^0, x^1, x^2, x^3) = (t, \phi, z, \rho)$ with t being time and (ϕ, z, ρ) the cylindrical coordinates of $\mathbb{R}^3$. Stationary axisymmetric space-times amount to the assumption that a metric is of the form

$$g = \begin{pmatrix} h_{00} & h_{01} & & \\ h_{10} & h_{11} & & \\ & & -\lambda & 0 \\ & & 0 & -\lambda \end{pmatrix},$$

$$\det h = -\rho^2 \, ,$$

where $\lambda > 0$, $h_{01} = h_{10}$ and $h = (h_{ij})$. The field λ is called the conformal factor.

For abelian gauge potentials, we fix the gauge so as to $\mathbf{A}_2 = \mathbf{A}_3 = 0$. Since we assume that the fields are stationary and axisymmetric, the functions h_{ij}'s, λ and $\mathbf{A}_i$'s depend only on z and ρ.

There is still a gauge symmetry that remains after setting h and $\mathbf{A}$ as above, i.e.

$$h \to {}^t g h g, \quad \mathbf{A}_i \to \mathbf{A}_i + \mathbf{C}_i$$

for $g \in SL(2, \mathbb{R})$ and $\mathbf{C}_i \in \mathbb{R}^N$ $(i = 0, 1)$. Therefore, we fix the gauge as follows:

$$h|_{(z,\rho)=(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{A}|_{(z,\rho)=(0,0)} = 0 \, . \tag{0.1}$$

Introducing the Ernst potentials $u \in \mathbb{R}$, $v \in \mathbb{C}^N$ constructed from h and $\mathbf{A}$ by the standard method (cf. [DO] [E]), we obtain

Proposition 0.1. *The stationary axisymmetric Einstein–Maxwell field equations with N abelian gauge fields are equivalent to the following equations:*

$$f(d * du + \rho^{-1} d\rho \wedge * du) = (du - 2v^* dv) \wedge * du \, , \tag{0.2}$$

$$f(d * dv + \rho^{-1} d\rho \wedge * dv) = (du - 2v^* dv) \wedge * dv \, , \tag{0.3}$$

$$\frac{\partial_z \lambda}{\lambda} = -\frac{\partial_z f}{2f} + \frac{\rho}{2f^2}(\partial_z f \partial_\rho f)$$

$$-\frac{\rho}{2f^2}(\partial_\rho u - \partial_\rho f - 2v^* \partial_\rho v)(\partial_z u - \partial_z f - 2v^* \partial_z v)$$

$$+\frac{\rho}{f}(\partial_z v^* \partial_\rho v + \partial_z v^* \partial_\rho v)\,, \tag{0.4}$$

$$\frac{\partial_\rho \lambda}{\lambda} = -\frac{\partial_\rho f}{2f} + \frac{\rho}{4f^2}\{(\partial_\rho f)^2 - (\partial_z f)^2\}$$

$$+\frac{\rho}{4f^2}\{(\partial_z u - \partial_z f - 2v^* \partial_z v)^2 - (\partial_\rho u - \partial_\rho f - 2v^* \partial_\rho v)^2\}$$

$$-\frac{\rho}{f}(\partial_z v^* \partial_z v - \partial_\rho v^* \partial_\rho v)\,, \tag{0.5}$$

where $v^* = {}^t\bar{v}$, $|v|^2 = v^* v$, $f = \mathrm{Re}\, u - |v|^2$ *and* $*$ *is the Hodge operator given by* $* dz = d\rho$, $* d\rho = -dz$.

The first two equations are called the Ernst equations.

Corresponding to the gauge fixing (0.1), we shall consider the solutions under the conditions

$$u|_{(z,\rho) = (0,0)} = 1 \quad \text{and} \quad v|_{(z,\rho)=(0,0)} = 0\,. \tag{0.6}$$

It is essential to introduce the function $\tau = f^{1/2}\lambda$ and we shall consider τ, instead of λ, throughout the paper.

1. Ernst Equation

Let θ be Cartan involution of $GL(N+2, \mathbb{C})$ defined by $g \mapsto g^{*-1}$ and G a subgroup of $GL(N+2, \mathbb{C})$ defined by

$$\{g \in GL(N+2, \mathbb{C});\, g^* J g = J, \det g = 1\}\,,$$

where $J = \begin{pmatrix} & & i \\ & 1_N & \\ -i & & \end{pmatrix}$ and 1_N denotes the $N \times N$ identity matrix. Note that G is isomorphic to $SU(1, N+1)$. Let K be the subgroup of G such that each element of K is fixed by θ. Then K is a maximal subgroup of G.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Then $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\,, \tag{1.1}$$

where $\mathfrak{k} = \{x \in \mathfrak{g};\, \theta X = X\}$ and $\mathfrak{p} = \{x \in \mathfrak{g};\, \theta X = -X\}$ with θ the Cartan involution of $\mathfrak{g}$ induced from θ of G.

We fix subgroups A and N of G as follows:

$$A = \left\{ \begin{pmatrix} a & & \\ & 1_N & \\ & & 1/a \end{pmatrix} ; a > 0 \right\} ,$$

$$N = \left\{ \begin{pmatrix} 1 & & \\ v & 1_N & \\ x + i|v|^2/2 & iv^* & 1 \end{pmatrix} ; x \in \mathbb{R},\ v \in \mathbb{C}^N \right\} ,$$

where $|v|^2 = v^*v$. Then we have $G = KAN$ (Iwasawa decomposition).

Let R be a ring of formal power series in z and ρ over $\mathbb{C}$ i.e. $R = \mathbb{C}[[z, \rho]]$. We regard z and ρ as real variables, which means, $\bar{z} = z$ and $\bar{\rho} = \rho$. We denote by * the anti-involution of $\mathfrak{gl}(N + 2, R) = \mathfrak{gl}(N + 2, \mathbb{C}) \otimes_{\mathbb{C}} R$ which is an obvious extension of the canonical anti-involution * of $\mathfrak{gl}(N + 2, \mathbb{C})$ and by θ_R an involution of $GL(N + 2, R)$ defined by $\theta_R(g) = g^{*-1}$ for $g \in GL(N + 2, R)$. Let G_R be a subgroup of $GL(N + 2, R)$ defined by

$$\{ g \in GL(N + 2, R);\ g^*Jg = J, \det g = 1 \} .$$

Then, corresponding to $G = KAN$, G_R decomposes as $G_R = K_R A_R N_R$, where K_R, A_R and N_R denote subgroups of G_R consisting of matrices with values in K, A and N respectively, each of whose components is an element of R.

Now we parametrize an element of $A_R N_R$ as follows:

$$P = \begin{pmatrix} f^{1/2} & 0 & 0 \\ \sqrt{2}v & 1_N & 0 \\ (\psi + i|v|^2)/f^{1/2} & \sqrt{2}iv^*/f^{1/2} & f^{-1/2} \end{pmatrix} , \tag{1.2}$$

where f and v are the same ones as in (0.2) and (0.3), and $\psi = \operatorname{Im} u$.

The following fact is well known.

Proposition 1.1. *Under the parametrization of* (1.2), *we put* $M = \theta_R(P^{-1})P$. *Then the Ernst equations* (0.2) *and* (0.3) *are equivalent to the following equation*:

$$d(\rho * dMM^{-1}) = 0 . \tag{1.3}$$

Moreover the function τ *is a solution of* (0.4) *and* (0.5) *if and only if it is a solution of the following equations*:

$$\tau^{-1}\partial_z \tau = \frac{\rho}{4} \operatorname{tr}(\partial_z MM^{-1} \partial_\rho MM^{-1}) , \tag{1.4}$$

$$\tau^{-1}\partial_\rho \tau = \frac{\rho}{8} \operatorname{tr}((\partial_\rho MM^{-1})^2 - (\partial_z MM^{-1})^2) . \tag{1.5}$$

The integrability of τ follows easily from (1.4) and (1.5). Equation (1.3) is also called the Ernst equation. We shall consider the solutions satisfying

$$P|_{(z,\rho)=(0,0)} = 1 ,$$

which corresponds to the gauge fixing condition (0.6).

We denote by B the real part of the trace form on $\mathfrak{gl}(N+2, R)$:

$$B(X, Y) = \mathrm{Re}\,\mathrm{tr}(XY) \quad \text{for } X, Y \in \mathfrak{gl}(N+2, R)\,.$$

The Lie algebras $\mathfrak{g}_R$ of G_R and $\mathfrak{k}_R$ of K_R can be identified with $\mathfrak{g} \otimes_{\mathbb{R}} R$ and $\mathfrak{k} \otimes_{\mathbb{R}} R$, respectively. Note that $\mathfrak{k}_R$ and $\mathfrak{p}_R = \{X \in \mathfrak{g}_R;\ \theta_R X = -X\} \cong \mathfrak{p} \otimes_{\mathbb{R}} R$ are orthogonal to each other with respect to B, where θ_R also denotes the involution of $\mathfrak{g}_R$ induced from that of G_R.

It is also known that Eq. (1.3) can be rewritten as the integrability condition of a 1-form with values in $\mathfrak{g}$ each of whose component is an element of $\mathbb{C}(z, \rho) \otimes_{\mathbb{C}} \mathbb{C}[[t]]$, where $\mathbb{C}(z, \rho)$ is the quotient field of $R = \mathbb{C}[[z, \rho]]$ and t an indeterminate called "spectral parameter." Namely, let $\mathscr{A}$ and $\mathscr{I}$ be 1-forms defined by

$$\mathscr{A} = \frac{1}{2}(dPP^{-1} + \theta_R(dPP^{-1})), \quad \mathscr{I} = \frac{1}{2}(dPP^{-1} - \theta_R(dPP^{-1}))$$

for any $P \in A_R N_R$, and put

$$\Omega_P = \mathscr{A} + \left(\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} *\right)\mathscr{I}\,,$$

where $*$ is the Hodge operator given by $*dz = d\rho$, $*d\rho = -dz$. We extend the canonical exterior derivative d on $\mathbb{C}(z, \rho)$ to that on $\mathbb{C}(z, \rho) \otimes_{\mathbb{C}} \mathbb{C}[[t]]$ by defining

$$dt = \frac{t}{(1+t^2)\rho}((1-t^2)d\rho + 2t dz)\,. \tag{1.6}$$

Note then that $d^2 t = 0$. Now we have

Proposition 1.2. *Ω_P satisfies the integrability condition*, i.e.,

$$d\Omega_P - \Omega_P \wedge \Omega_P = 0 \tag{1.7}$$

if and only if P is a solution of (1.3).

This can be proved straightforward, using (1.6).

Moreover, let $\mathscr{I} = \mathscr{I}_z dz + \mathscr{I}_\rho d\rho$. Then Eqs. (1.4) and (1.5) can be written in terms of $\mathscr{I}_z$ and $\mathscr{I}_\rho$ as

$$\tau^{-1}\partial_z \tau = \rho B(\mathscr{I}_z, \mathscr{I}_\rho)\,, \tag{1.8}$$

$$\tau^{-1}\partial_\rho \tau = \frac{\rho}{2}(B(\mathscr{I}_\rho, \mathscr{I}_\rho) - B(\mathscr{I}_z, \mathscr{I}_z))\,. \tag{1.9}$$

It follows from Proposition 1.2 that if P is a solution of the Ernst equation, then there exists a *potential* $p = \sum_{n \geqq 0} p_n t^n$ such that each entry of p_n is an element of $\mathbb{C}(z, \rho)$ and

$$dp = \Omega_P \cdot p \quad \text{and } p_0 = P. \tag{1.10}$$

2. Hauser Group

We introduce formal loop algebras and formal loop groups, following [T].

Put $F_0 = R = \mathbb{C}[[z, \rho]]$ and $F_n = \rho^{|n|}R$ for a nonzero integer n. We introduce a topology in R by declaring that $\{F_n\}_{n \geqq 0}$ forms a fundamental neighborhoods system of 0. Note that $F_m F_n \subset F_{m+n}$ for $m, n \geqq 0$.

Then we define a formal loop algebra $\mathscr{F}\mathfrak{gl}$ by

$$\mathscr{F}\mathfrak{gl} = \left\{ X = \sum_{n \in \mathbb{Z}} X_n t^n;\ X_n \in \mathfrak{gl}(N+2, F_n) \right\}. \tag{2.1}$$

Let * be an anti-involution of $\mathscr{F}\mathfrak{gl}$ defined by

$$X^* = \sum_{n \in \mathbb{Z}} X_n^* (-1/t)^n$$

for $X = \sum_{n \in \mathbb{Z}} X_n t^n$, where the anti-involution * in the right-hand side is the one of $\mathfrak{gl}(N+2, R)$ given in Sect. 1. This is well-defined by the definition of our filtration $\{F_n\}_{n \in \mathbb{Z}}$.

Remark that Ω_P with $P \in A_R N_R$ is not an element of $\mathscr{F}\mathfrak{gl}$, however, we can define Ω_P^* by

$$\Omega_P^* = \mathscr{A}^* + \left(\frac{1-(-1/t)^2}{1+(-1/t)^2} - \frac{2(-1/t)}{1+(-1/t)^2} * \right) \mathscr{I}^*,$$

where * in the right-hand side is the anti-involution of $\mathfrak{gl}(N+2, R)$. Then it follows immediately that

$$\Omega_P^* = -\Omega_P \quad \text{for } P \in A_R N_R. \tag{2.2}$$

We define a formal loop group $\mathscr{F}GL$, following [T], by

$$\mathscr{F}GL = \left\{ g = \sum_{n \in \mathbb{Z}} g_n t^n;\ g_n \in \mathfrak{gl}(N+2, F_n),\ g_0 \text{ is invertible} \right\}. \tag{2.3}$$

Since $\mathscr{F}GL$ is canonically embedded in $\mathscr{F}\mathfrak{gl}$, we can define an involution $\theta^{(\infty)}$ of $\mathscr{F}GL$ by

$$\theta^{(\infty)}(g) = (g^*)^{-1} \quad \text{for } g \in \mathscr{F}GL,$$

which we call Cartan involution of $\mathscr{F}GL$.

Define subgroups of $\mathscr{F}GL$ as follows:

$$\mathscr{F}\mathscr{G} = \left\{ g = \sum_{n \in \mathbb{Z}} g_n t^n \in \mathscr{F}GL;\ g^* J g = J,\ \det g = 1 \right\}, \tag{2.4}$$

$$\mathscr{F}\mathscr{G}_0 = \left\{ g = \sum_{n \in \mathbb{Z}} g_n t^n \in \mathscr{F}\mathscr{G};\ g_0|_{(z,\rho)=(0,0)} = 1 \right\}, \tag{2.5}$$

$$\mathscr{F}\mathscr{K} = \left\{ k = \sum_{n \in \mathbb{Z}} k_n t^n \in \mathscr{F}\mathscr{G};\ \theta^{(\infty)} k = k \right\}, \tag{2.6}$$

$$\mathscr{F}\mathscr{P} = \left\{ p = \sum_{n \in \mathbb{Z}} p_n t^n \in \mathscr{F}\mathscr{G};\ p_0 \in A_R N_R,\ p_n = 0 \text{ if } n < 0 \right\}. \tag{2.7}$$

Then, using the Birkhoff decomposition ((3.17), [T]), we can decompose uniquely an element $g \in \mathscr{F}\mathscr{G}$ as

$$g = kp \quad (k \in \mathscr{F}\mathscr{K},\ p \in \mathscr{F}\mathscr{P}). \tag{2.8}$$

Let s be another indeterminate. Define an infinite dimensional group $\mathscr{G}^{(\infty)}$, which we call Hauser group, by

$$\mathscr{G}^{(\infty)} = \left\{ g = \sum_{n \geqq 0} g_n s^n \in GL(N+2, \mathbb{C}[[s]]);\ g^* J g = J, \det g = 1, g_0 = 1 \right\},$$

where $\mathbb{C}[[s]]$ is a ring of formal power series in s over $\mathbb{C}$ and $g^* = \sum g_n^* s^n$.

Let j be a homomorphism of $GL(N+2, \mathbb{C}[[s]])$ into $\mathscr{F}GL$ given by

$$j\colon g = \sum_{n \geqq 0} g_n s^n \mapsto j(g) = \sum_{n \geqq 0} g_n \left(\rho\left(\frac{1}{t} - t\right) + 2z \right)^n .$$

Then it is easy to see that j is injective and that the image of $\mathscr{G}^{(\infty)}$ by j is in $\mathscr{F}\mathscr{G}_0$. We denote by $\mathscr{F}\mathscr{H}$ the image of $\mathscr{G}^{(\infty)}$ by j. The following equations characterize the elements of $\mathscr{F}\mathscr{H}$ in $\mathscr{F}\mathscr{G}$.

Lemma 2.1. *An element $g \in \mathscr{F}\mathscr{G}$ belongs to $\mathscr{F}\mathscr{H}$ if and only if g satisfies the following equations*:

$$\partial_t g = -\rho\left(\partial_z + \frac{1}{t}\partial_\rho \right) g , \tag{2.9}$$

$$\partial_t g = -\frac{\rho}{2}\left(1 + \frac{1}{t^2} \right) \partial_z g . \tag{2.10}$$

This characterization will play an important role in the proof of our main theorem. For proof, we refer to [S].

Definition. Let $\mathscr{F}\mathscr{P}$ be as in (2.7). We define $\mathscr{S}\mathscr{P}$ to be a subset of $\mathscr{F}\mathscr{P}$ consisting of elements $p = \sum_{n \geqq 0} p_n t^n$ which satisfy the following conditions:

$$dp = \Omega_{p_0} \cdot p \quad \text{and} \quad p_0|_{(z,\rho)=(0,0)} = 1 . \tag{2.11}$$

We call $\mathscr{S}\mathscr{P}$ the space of potentials.

It follows from (2.11) that p_0 is a solution of the Ernst equation (1.3) for $p = \sum_{n \geqq 0} p_n t^n \in \mathscr{S}\mathscr{P}$. Equation (2.11) is equivalent to the following equations:

$$\partial_z p + \frac{2t^2}{(1+t^2)\rho} \partial_t p = \Omega_1 p \tag{2.12.a}$$

$$\partial_\rho p + \frac{t(1-t^2)}{(1+t^2)\rho} \partial_t p = \Omega_2 p , \tag{2.12.b}$$

where we define Ω_1 and Ω_2 by $\Omega_{p_0} = \Omega_1 dz + \Omega_2 d\rho$.

Put

$$D_1 = \partial_z + \frac{2t^2}{(1+t^2)\rho} \partial_t \quad \text{and} \quad D_2 = \partial_\rho + \frac{t(1-t^2)}{(1+t^2)\rho} \partial_t$$

for brevity.

Theorem 2.2. *An element $p \in \mathscr{S}\mathscr{P}$ satisfies*

$$\partial_t(p^*p) = -\rho\left(\partial_z + \frac{1}{t}\partial_\rho \right)(p^*p) , \tag{2.13.a}$$

$$\partial_t(p^*p) = -\frac{\rho}{2}\left(1 + \frac{1}{t^2} \right) \partial_z(p^*p) . \tag{2.13.b}$$

Conversely, if $p \in \mathscr{F}\mathscr{P} \cap \mathscr{F}\mathscr{G}_0$ *satisfies Eqs.* (2.13.a) *and* (2.13.b), *then* p *belongs to* $\mathscr{S}\mathscr{P}$, *namely, it satisfies Eqs.* (2.12.a) *and* (2.12.b).

Proof. It can be checked by direct calculation that (2.13.a) and (2.13.b) are equivalent to $D_1(p^*p) = 0$ and $D_2(p^*p) = 0$. But then, we have, for $p \in \mathscr{S}\mathscr{P}$,

$$\begin{aligned} D_1(p^*p) &= D_1 p^* p + p^* D_1 p \\ &= (\Omega_1 p)^* p + p^*(\Omega_1 p) \\ &= -p^*\Omega_1 p + p^*\Omega_1 p \\ &= 0\,, \end{aligned}$$

since $\Omega_1^* = -\Omega_1$. Similarly, we can show that $D_2(p^*p) = 0$.

Conversely, let $p \in \mathscr{F}\mathscr{P}$ satisfy (2.13.a) and (2.13.b). Then we have

$$\begin{aligned} &-(\partial_t pp^{-1})^* - \rho\left((\partial_z pp^{-1})^* + \frac{1}{t}(\partial_\rho pp^{-1})^*\right) \\ &\qquad = \partial_t pp^{-1} + \rho\left(\partial_z pp^{-1} + \frac{1}{t}\partial_\rho pp^{-1}\right), \end{aligned} \tag{2.14}$$

$$-(\partial_t pp^{-1})^* - \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)(\partial_z pp^{-1})^* = \partial_t pp^{-1} + \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)\partial_z pp^{-1}\,. \tag{2.15}$$

In (2.14) the left-hand side contains only the terms of t^n ($n \leqq 0$), while the right-hand side those of t^n ($n \geqq -1$), which implies that the coefficients of t^n ($n \leqq -2$) in the l.h.s. vanish. Therefore, we obtain

$$\partial_t pp^{-1} + \rho\left(\partial_z pp^{-1} + \frac{1}{t}\partial_\rho pp^{-1}\right) = \rho\left(\Omega_1 + \frac{1}{t}\Omega_2\right), \tag{2.16}$$

since the coefficient of t^{-1} in the l.h.s. of (2.14) is equal to $\rho\partial_\rho p_0 p_0^{-1}$(= coeff. of t^{-1} in the r.h.s. of (2.14)). Note that, comparing the coefficients of t^{-1} in both sides of (2.14), we obtain

$$\partial_\rho p_0 p_0^{-1} + (\partial_\rho p_0 p_0^{-1})^* = \text{coeff. of } t^{-1} \text{ in } (\partial_z pp^{-1})^*\,. \tag{2.17}$$

Similarly, the l.h.s. of (2.15) contains only the terms of t^n ($n \leqq 0$), while the r.h.s. those of t^n ($n \geqq -2$), which implies that the coefficients of t^n ($n \leqq -3$) in the l.h.s. vanish. Therefore, we obtain

$$\partial_t pp^{-1} + \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)\partial_z pp^{-1} = \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)\Omega_1\,, \tag{2.18}$$

where we used (2.17).

Now it is easy to see that (2.16) and (2.18) are equivalent to (2.12.a) and (2.12.b). ∎

Let $p \in \mathscr{S}\mathscr{P}$ and $g \in \mathscr{G}^{(\infty)}$. By (2.8) there exist $k \in \mathscr{F}\mathscr{K}$ and $p_g \in \mathscr{F}\mathscr{P}$ such that

$$p \cdot j(g) = k^{-1} \cdot p_g\,. \tag{2.19}$$

Then, it follows immediately from Theorem 2.2 that p_g is in $\mathscr{S}\mathscr{P}$. Thus we can define an action of the Hauser group $\mathscr{G}^{(\infty)}$ on $\mathscr{S}\mathscr{P}$ to the right by

$$\mathscr{S}\mathscr{P} \times \mathscr{G}^{(\infty)} \to \mathscr{S}\mathscr{P} \quad (p, g) \mapsto p_g\,, \tag{2.20}$$

where p_g is given by (2.19).

From the fact that an element $g = \sum_{n \geqq 0} g_n s^n \in \mathscr{G}^{(\infty)}$ such that $g^* = g$ and such that g_0 is positive definite decomposes as $g = h^* h$ for some $h \in \mathscr{G}^{(\infty)}$, we have

Corollary 2.3. *The action of* $\mathscr{G}^{(\infty)}$ *on* $\mathscr{S}\mathscr{P}$ *given by* (2.20) *is transitive.*

Remark. As we mentioned in [S], our group $\mathscr{G}^{(\infty)}$ is too small to obtain all solutions of the Ernst equation (1.3) through the action (2.20).

3. 2-Cocycle on $\mathscr{F}\mathscr{G}_0$

The formal loop algebra $\mathscr{F}\mathrm{gl}$ becomes a Lie algebra with Lie bracket $[X, Y] = XY - YX$. The map

$$\exp\colon \mathscr{F}\mathrm{gl} \to \mathscr{F}GL$$

given by

$$\exp X = e^X = \sum_{n \geqq 0} \frac{X^n}{n!} \tag{3.1}$$

is called the *formal exponential map.* Note that for any $g \in \mathscr{F}\mathscr{G}_0$ we can find a unique element X in $\mathscr{F}\mathrm{gl}$ such that $g = e^X$, since the *logarithm* given by

$$\log(1 + A) = \sum_{n \geqq 1} \frac{(-1)^{n-1}}{n} A^n \tag{3.2}$$

is well-defined and satisfies

$$e^{\log(1+A)} = 1 + A \tag{3.3}$$

for $A = \sum_{n \in \mathbb{Z}} a_n t^n \in \mathscr{F}\mathrm{gl}$ with $a_0 \in \mathrm{gl}(N + 2, \mathfrak{m})$, where $\mathfrak{m} \subset R$ is the maximal ideal.

For X, Y in $\mathscr{F}\mathrm{gl}$, let $c_n(X, Y)$ $(n = 1, 2, \ldots)$ be the elements in $\mathscr{F}\mathrm{gl}$ which are determined by

$$\exp vX \exp vY = \exp \sum_{n \geqq 0} c_n(X, Y) v^n ,$$

where v is an indeterminate. Furthermore c_n's are uniquely determined by the following recursion formulas (see [V]):

$$c_1(X, Y) = X + Y$$

$$(n + 1)c_{n+1}(X, Y) = \frac{1}{2}[X - Y, c_n(X, Y)]$$

$$+ \sum_{p \geqq 1, 2p \leqq n} K_{2p} \sum_{\substack{k_1, \ldots, k_{2p} > 0 \\ k_1 + \cdots + k_{2p} = n}} [c_{k_1}(X, Y), [\ldots, [c_{k_{2p}}(X, Y), X + Y] \ldots] \ (n \geqq 1),$$

where K_{2p}'s are determined by

$$\frac{x}{1 - e^{-x}} - \frac{1}{2}x = 1 + \sum_{p \geqq 1} K_{2p} x^{2p} .$$

We set $C(X, Y) = \sum_{n \geqq 1} c_n(X, Y)$. Then $C(X, Y)$ is a well-defined element of $\mathscr{F}\mathrm{gl}$ for X, Y such that X_0, $Y_0 \in \mathrm{gl}(N + 2, \mathfrak{m})$.

Lemma 3.1. *For $n \geqq 2$, there exists a $\mathscr{F}\mathfrak{gl}$-valued function $L_n(\cdot\,,\cdot)$ which satisfies*

$$c_n(X, Y) = [X, L_n(X, Y)] + [Y, L_n(-Y, -X)] \tag{3.4}$$

for $X, Y \in \mathscr{F}\mathfrak{gl}$.

Proof. It is easy to see that c_n can be written as

$$c_n(X, Y) = [X, A_n(X, Y)] + [Y, B_n(X, Y)]\,.$$

Let $X, Y \in \mathscr{F}\mathfrak{gl}$. Then, since $c_n(X, Y) = -c_n(-Y, -X)$, we have

$$\begin{aligned} c_n(X, Y) &= \frac{1}{2}(c_n(X, Y) - c_n(-Y, -X)) \\ &= \frac{1}{2}([X, A_n(X, Y)] + [Y, B_n(X, Y)] \\ &\quad - [-Y, A_n(-Y, -X)] - [-X, B_n(-Y, -X)])\,. \end{aligned}$$

Therefore

$$L_n(X, Y) = \frac{1}{2}([X, A_n(X, Y)] + [Y, B_n(X, Y)])$$

satisfies (3.4). ∎

Note that L_n's are not uniquely determined, however, we fix L_n's so that there holds

$$L(X, vY) = \left(\frac{e^{-\operatorname{ad} X} - 1 + \operatorname{ad} X}{\operatorname{ad} X(1 - e^{-\operatorname{ad} X})} - \frac{1}{4}\right) vY + O(v^2)\,, \tag{3.5}$$

where we put $L(X, Y) = \sum_{n \geqq 2} L_n(X, Y)$ for $X, Y \in \mathscr{F}\mathfrak{gl}$ such that $X_0, Y_0 \in \mathfrak{gl}(N + 2, \mathfrak{m})$. Thus, we obtain

$$C(X, Y) = X + Y + [X, L(X, Y)] + [Y, L(-Y, -X)]\,.$$

For a series $f = \sum_{n \in \mathbb{Z}} f_n t^n \in R[[t, t^{-1}]]$, we write

$$\operatorname{Res}_t f = f_{-1} \in R\,.$$

Let $R_0 = \mathbb{R}[[z, \rho]] \subset R$, the formal power series in z and ρ over $\mathbb{R}$. We define a R_0-valued 2-cocycle ω on $\mathscr{F}\mathfrak{gl}$ by

$$\omega(X, Y) = \operatorname{Res}_t B(X, \partial_t Y)$$

for $X, Y \in \mathscr{F}\mathfrak{gl}$. Note that

$$\omega(X^*, Y^*) = -\omega(X, Y) \tag{3.6}$$

for $X, Y \in \mathscr{F}\mathfrak{gl}$.

Now we introduce a group 2-cocycle on $\mathscr{F}\mathscr{G}_0$, following [BM]. Note that, from (3.3), any element $g \in \mathscr{F}\mathscr{G}_0$ can be uniquely written as $g = e^X$ for $X \in \mathscr{F}\mathfrak{gl}$ with $X_0 \in \mathfrak{gl}(N + 2, \mathfrak{m})$.

Definition. Let Ξ be a R_0-valued function on $\mathscr{F}\mathscr{G}_0 \times \mathscr{F}\mathscr{G}_0$ defined by

$$\Xi(e^X, e^Y) = \omega(X, L(X, Y)) + \omega(Y, L(-Y, -X))\,.$$

Then Ξ defines a 2-cocycle on $\mathscr{F}\mathscr{G}_0$, i.e. satisfies the cocycle condition:

$$\Xi(e^X, e^Y) + \Xi(e^X e^Y, e^Z) = \Xi(e^Y, e^Z) + \Xi(e^X, e^Y e^Z) \tag{3.7}$$

for $X, Y, Z \in \mathscr{F}\mathrm{gl}$.

Now we collect some basic properties of Ξ. For details we refer to [BM, S]. It follows from (3.6) that

$$\Xi(\theta^{(\infty)}g_1, \theta^{(\infty)}g_2) = -\Xi(g_1, g_2) \tag{3.8}$$

for $g_1, g_2 \in \mathscr{F}\mathscr{G}_0$. In addition, Ξ satisfies the *anti-symmetric* conditions:

$$\begin{aligned} \Xi(e^X, e^Y) &= -\Xi(e^{-X}, e^X e^Y) \\ &= -\Xi(e^X e^Y, e^{-Y}) \\ &= -\Xi(e^{-Y}, e^{-X}) . \end{aligned} \tag{3.9}$$

Define the mixed form Ξ' of Ξ by

$$\Xi'(e^X, Y) = \frac{d}{dv}\bigg|_{v=0} \Xi(e^X, e^{vY}) .$$

Then, using (3.7), (3.9) and the formula $e^{-X}\partial e^X = \dfrac{1 - e^{-\mathrm{ad}X}}{\mathrm{ad}\,X}\partial X$, we obtain

$$\begin{aligned} \partial\Xi(e^X, e^Y) &= \Xi'(e^{-Y}e^{-X}, \partial e^X e^{-X}) - \Xi'(e^{-X}, \partial e^X e^{-X}) \\ &\quad + \Xi'(e^X e^Y, e^{-Y}\partial e^Y) - \Xi'(e^Y, e^{-Y}\partial e^Y) , \end{aligned} \tag{3.10}$$

where ∂ denotes either ∂_z or ∂_ρ,

Lemma 3.2. *Ξ is trivial on $\mathscr{F}\mathscr{H} \times \mathscr{F}\mathscr{H}$*, i.e.

$$\Xi(g_1, g_2) = 0 \quad \textit{for all } g_1, g_2 \in \mathscr{F}\mathscr{H} .$$

For proof, we refer to [S].

4. Central Extension

For any $p \in \mathscr{S}\mathscr{P}$, we can find an element $g \in \mathscr{F}\mathscr{H}$ which sends the identity element $1 \in \mathscr{S}\mathscr{P}$ to p by Corollary 2.2. Then we have $p = kg$ for some $k \in \mathscr{F}\mathscr{K}$.

Lemma 4.1. *For $p \in \mathscr{S}\mathscr{P}$, let $g \in \mathscr{F}\mathscr{H}$ be as above. If we put*

$$\partial_z p p^{-1} = \sum_{n \geqq 0} q_n t^n$$

and

$$p \cdot g^{-1}\partial_z g \cdot p^{-1} = \sum_{n \in \mathbb{Z}} a_n t^n ,$$

then we have

$$a_0 + a_0^* = q_0 + q_0^* \tag{4.1}$$

and

$$a_n + (-1)^n a_{-n}^* = q_n \tag{4.2}$$

for $n = 1, 2, 3, \ldots$.

Proof. By the choice of g, we have

$$m = g^* g = p^* p ,$$

which imply that

$$\begin{aligned} m^{-1}\partial_z m &= m^{-1}(g^{-1}\partial_z g)^* m + g^{-1}\partial_z g \\ &= m^{-1}(p^{-1}\partial_z p)^* m + p^{-1}\partial_z p . \end{aligned}$$

Therefore we obtain

$$p^{-1}(pg^{-1}\partial_z g p^{-1})^* p + g^{-1}\partial_z g = p^{-1}(\partial_z p p^{-1})^* p + p^{-1}\partial_z p . \tag{4.3}$$

Multiplying (4.3) by p to the left and by p^{-1} to the right, we have

$$(pg^{-1}\partial_z g p^{-1})^* + pg^{-1}\partial_z g p^{-1} = (\partial_z p p^{-1})^* + \partial_z p p^{-1} .$$

Hence

$$\sum_{n \in \mathbb{Z}} \{a_n + (-1)^n a_{-n}^*\} t^n = \sum_{n>0} q_n t^n + q_0 + q_0^* + \sum_{n<0} (-1)^n q_{-n}^* t^n .$$

Comparing the coefficient of t^n in both sides, we obtain (4.1) and (4.2). ∎

Proposition 4.2. *For* $p \in \mathscr{S}\mathscr{P}$, let $g \in \mathscr{F}\mathscr{H}$ *and* $k \in \mathscr{F}\mathscr{K}$ *such that* $p = kg$. *Let* Ξ *be the* 2*-cocycle on* $\mathscr{F}\mathscr{G}_0$ *given in Sect.* 3. *Then we have the following identities*:

$$\begin{aligned} &\partial_z \Xi(kg, g^{-1}) \\ &= \frac{\rho}{2} \operatorname{Res}_t B\left(\frac{1}{2}\left(1 + \frac{1}{t^2}\right)\mathscr{A}_z + \frac{1}{2}\left(-1 + \frac{1}{t^2}\right)\mathscr{I}_z + \frac{1}{t}\mathscr{I}_\rho, pg^{-1}\partial_z g p^{-1}\right) \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\partial_\rho \Xi(kg, g^{-1}) \\ &= \frac{\rho}{2} \operatorname{Res}_t B\left(\frac{1}{2}\left(1 + \frac{1}{t^2}\right)\mathscr{A}_\rho + \frac{1}{2}\left(-1 + \frac{1}{t^2}\right)\mathscr{I}_\rho - \frac{1}{t}\mathscr{I}_z, pg^{-1}\partial_z g p^{-1}\right). \end{aligned} \tag{4.5}$$

Proof. We shall use the following identity:

$$\partial \Xi(kg, g^{-1}) = \frac{1}{2} \operatorname{Res}_t \{B(p^{-1}\partial_t p, g^{-1}\partial g) - B(p^{-1}\partial p, g^{-1}\partial_t g)\} , \tag{4.6}$$

which follows from (3.5) and (3.10), where ∂ denotes either ∂_z or ∂_ρ.

Since $p \in \mathscr{S}\mathscr{P}$, we have

$$p^{-1}\partial_t p + \frac{\rho(1+t^2)}{t(1-t^2)} p^{-1}\partial_\rho p = \frac{\rho(1+t^2)}{t(1-t^2)} p^{-1}\left\{\mathscr{A}_\rho + \frac{1-t^2}{1+t^2}\mathscr{I}_\rho - \frac{2t}{1+t^2}\mathscr{I}_z\right\} p . \tag{4.7}$$

On the other hand, by Lemma 2.1, we have

$$g^{-1}\partial_t g = -\frac{\rho(1+t^2)}{t(1-t^2)} g^{-1}\partial_\rho g . \tag{4.8}$$

From (4.6) with $\partial = \partial_\rho$, (4.7) and (4.8), we obtain

$$\partial_\rho \Xi(kg, g^{-1}) = \frac{1}{2}\operatorname{Res}_t \frac{\rho(1+t^2)}{t(1-t^2)} B\left(\mathscr{A}_\rho + \frac{1-t^2}{1+t^2}\mathscr{I}_\rho - \frac{2t}{1+t^2}\mathscr{I}_z, pg^{-1}\partial_\rho g p^{-1}\right)$$

$$= \frac{1}{2}\operatorname{Res}_t \frac{\rho(1+t^2)}{t(1-t^2)} B\left(\mathscr{A}_\rho + \frac{1-t^2}{1+t^2}\mathscr{I}_\rho - \frac{2t}{1+t^2}\mathscr{I}_z, \frac{1-t^2}{2t} pg^{-1}\partial_z g p^{-1}\right)$$

$$= (\text{the right-hand side of (4.5)})\,,$$

where we used $\partial_\rho g = \dfrac{1-t^2}{2t}\partial_z g$. Similarly, we can prove (4.4). ∎

By (1.1), each element $X \in \mathfrak{g}$ decomposes uniquely as $X = X_1 + X_2$ with $X_1 \in \mathfrak{k}_R$ and $X_2 \in \mathfrak{p}_R$. We shall denote X_1 and X_2 by $(X)_\mathfrak{k}$ and $(X)_\mathfrak{p}$, respectively.
Now we can prove the following proposition.

Proposition 4.3. *For* $p = \sum_{n \geqq 0} p_n t^n \in \mathscr{SP}$, *let* $g \in \mathscr{FH}$ *and* $k \in \mathscr{FK}$ *be such that* $p = kg$. *Let* τ *be a solution of* (1.8) *and* (1.9) *corresponding to* $P = p_0$. *Then we have the following relations:*

$$\tau^{-1}\partial_z \tau = \partial_z \Xi(kg, g^{-1})\,, \tag{4.9}$$

$$\tau^{-1}\partial_\rho \tau = \partial_\rho \Xi(kg, g^{-1})\,. \tag{4.10}$$

Proof. First, expanding (2.12.a) in a series of t, we have

$$\partial_z p p^{-1} = (\mathscr{A}_z + \mathscr{I}_z) + 2\mathscr{I}_\rho t + O(t^2)$$

$$= \sum_{n \geqq 0} q_n t^n\,.$$

Thus, taking the coefficients of t^0 and t^1, we obtain

$$q_0 = \mathscr{A}_z + \mathscr{I}_z \quad \text{and} \quad q_1 = 2\mathscr{I}_\rho\,. \tag{4.11}$$

In the right-hand side of (4.4), since $\mathfrak{k}_R$ and $\mathfrak{p}_R$ are orthogonal to each other with respect to B, we have

$$\operatorname{Res}_t\left(1 + \frac{1}{t^2}\right) B(\mathscr{A}_z, pg^{-1}\partial_z g p^{-1}) = B(\mathscr{A}_z, (a_1 + a_{-1})_\mathfrak{k})$$

$$\operatorname{Res}_t\left(-1 + \frac{1}{t^2}\right) B(\mathscr{I}_z, pg^{-1}\partial_z g p^{-1}) = B(\mathscr{I}_z, (a_1 - a_{-1})_\mathfrak{p})$$

$$\operatorname{Res}_t \frac{1}{t} B(\mathscr{I}_\rho, pg^{-1}\partial_z g p^{-1}) = B(\mathscr{I}_\rho, (a_0)_\mathfrak{p})\,,$$

where we put $pg^{-1}\partial_z g p^{-1} = \sum_{n\in\mathbb{Z}} a_n t^n$. Then, by Lemma 4.1 and (4.11), we have

$$(a_1 + a_{-1})_{\mathfrak{k}} = \frac{1}{2}(a_1 + a_{-1} - a_1^* - a_{-1}^*)$$

$$= \frac{1}{2}(q_1 - q_1^*) = 0$$

$$(a_1 - a_{-1})_{\mathfrak{p}} = \frac{1}{2}(a_1 - a_{-1} + a_1^* - a_{-1}^*)$$

$$= \frac{1}{2}(q_1 + q_1^*) = 2\mathscr{I}_\rho$$

$$(a_0)_{\mathfrak{p}} = \frac{1}{2}(a_0 + a_0^*)$$

$$= \frac{1}{2}(q_0 + q_0^*) = 2\mathscr{I}_z\,.$$

Therefore, recalling (1.8), we see that (4.9) holds.

Similarly, we can prove (4.10). This completes the proof. ∎

Now we define a central extension of $\mathscr{F}\mathscr{G}_0$ in terms of the cocycle Ξ.

Definition. Let $(\mathscr{F}\mathscr{G}_0)^\sim$ be the set given by

$$(\mathscr{F}\mathscr{G}_0)^\sim = \{(g, e^\mu);\, g \in \mathscr{F}\mathscr{G}_0,\, \mu \in R_0\}\,.$$

Define a product of any two elements of $(\mathscr{F}\mathscr{G}_0)^\sim$ by

$$(g_1, e^{\mu_1})\cdot(g_2, e^{\mu_2}) = (g_1 g_2, e^{\mu_1 + \mu_2 + \Xi(g_1, g_2)}) \tag{4.12}$$

for $(g_1, e^{\mu_1}),\ (g_2, e^{\mu_2}) \in (\mathscr{F}\mathscr{G}_0)^\sim$. Since Ξ satisfies the cocycle condition (3.7), $(\mathscr{F}\mathscr{G}_0)^\sim$ forms a group with group multiplication given by (4.12). Namely, $(\mathscr{F}\mathscr{G}_0)^\sim$ is a central extension of $\mathscr{F}\mathscr{G}_0$.

Let $\tilde{\theta}^{(\infty)}$ be an involution of $(\mathscr{F}\mathscr{G}_0)^\sim$ given by

$$\tilde{\theta}^{(\infty)}(g, e^\mu) = (\theta^{(\infty)}(g), e^{-\mu})\,.$$

If we denote by $(\mathscr{F}\mathscr{K})^\sim$ the subgroup of $(\mathscr{F}\mathscr{G}_0)^\sim$ consisting of elements which are fixed by $\tilde{\theta}^{(\infty)}$, then we have

$$(\mathscr{F}\mathscr{K})^\sim = \{(k, 1) \in (\mathscr{F}\mathscr{G}_0)^\sim;\, k \in \mathscr{F}\mathscr{K}\}\,.$$

Let $(\mathscr{F}\mathscr{P})^\sim$ be a subgroup of $(\mathscr{F}\mathscr{G}_0)^\sim$ given by

$$(\mathscr{F}\mathscr{P})^\sim = \{(p, e^\mu) \in (\mathscr{F}\mathscr{G}_0)^\sim;\, p \in \mathscr{F}\mathscr{P},\, \mu \in R_0\}\,.$$

It follows immediately from the decomposition (2.8) of $\mathscr{F}\mathscr{G}$ that $(\mathscr{F}\mathscr{G}_0)^\sim$ has a unique decomposition:

$$(\mathscr{F}\mathscr{G}_0)^\sim = (\mathscr{F}\mathscr{K})^\sim \cdot (\mathscr{F}\mathscr{P})^\sim\,. \tag{4.13}$$

Furthermore, we put

$$(\mathscr{F}\mathscr{H})^\sim = \{(g, e^\gamma) \in (\mathscr{F}\mathscr{G}_0)^\sim;\, g \in \mathscr{F}\mathscr{H},\, \gamma \in \mathbb{R}\}\,.$$

It follows from Lemma 3.2 that $\mathscr{F}\mathscr{H}$ can be regarded as a subgroup of $(\mathscr{F}\mathscr{H})^\sim$ by

$$\mathscr{F}\mathscr{H} \to (\mathscr{F}\mathscr{H})^\sim, \quad g \mapsto (g, 1) .$$

Let $(\mathscr{S}\mathscr{P})^\sim$ be the subset of $(\mathscr{F}\mathscr{P})^\sim$ given by

$$(\mathscr{S}\mathscr{P})^\sim = \left\{ (p, e^\mu) \in (\mathscr{F}\mathscr{P})^\sim ; \, p = \sum_{n \geqq 0} p_n t^n \in \mathscr{S}\mathscr{P} , \right.$$

$$\left. \tau = e^{-\mu} \text{ satisfies (1.8) and (1.9) with } P = p_0 \right\} . \tag{4.14}$$

We call $(\mathscr{S}\mathscr{P})^\sim$ *the space of potentials with conformal factor.*

Proposition 4.4. *For $p \in \mathscr{S}\mathscr{P}$, let $k \in \mathscr{F}\mathscr{K}$ and $g \in \mathscr{F}\mathscr{H}$ be as above,* i.e. $p = kg$. *Then we have*

$$\Xi(p^*, p) = 2\Xi(kg, g^{-1}) . \tag{4.15}$$

Therefore, any element of $(\mathscr{S}\mathscr{P})^\sim$ can be written as $(p, e^{-\frac{1}{2}\Xi(p^*, p) + \gamma})$ for $p \in \mathscr{S}\mathscr{P}$, $\gamma \in \mathbb{R}$.

Proof. Using the anti-symmetric condition (3.9) and the cocycle condition (3.7), we have

$$\begin{aligned} \Xi(p^*, p) &= -\Xi(p^{-1}, \theta^{(\infty)}p) = -\Xi(p^{-1}, k\theta^{(\infty)}g) \\ &= -\Xi(p^{-1}, k) - \Xi(p^{-1}k, \theta^{(\infty)}g) + \Xi(k, \theta^{(\infty)}g) . \end{aligned} \tag{4.16}$$

For the first and the last terms in the r.h.s. of (4.16), it follows from (3.9) and (3.8) that

$$\Xi(p^{-1}, k) = -\Xi(k, \theta^{(\infty)}g) = \Xi(k, g) .$$

The middle term in the r.h.s. of (4.16) vanishes by Lemma 3.2.

Now using (3.9) again, we obtain

$$\begin{aligned} \Xi(k, g) &= \Xi(pg^{-1}, g) = -\Xi(g^{-1}, gp^{-1}) \\ &= \Xi(g, p^{-1}) = -\Xi(p, g^{-1}) . \end{aligned} \tag{4.17}$$

Thus (4.15) follows.

For any $(p, e^\mu) \in (\mathscr{S}\mathscr{P})^\sim$, put $p = \sum_{n \geqq 0} p_n t^n$. By definition and Proposition 4.3, both $\tau = e^{-\mu}$ and $\tau' = e^{\frac{1}{2}\Xi(p^*, p)}$ satisfies (1.8) and (1.9) with $P = p_0$. Therefore they must be equal to each other, up to constant multiple. This completes the proof of the proposition. ∎

Define an action of $(\mathscr{F}\mathscr{H})^\sim$ on the space of potentials with conformal factor $(\mathscr{S}\mathscr{P})^\sim$ to the right through the decomposition (4.13):

$$(\mathscr{S}\mathscr{P})^\sim \times (\mathscr{F}\mathscr{H})^\sim \to (\mathscr{S}\mathscr{P})^\sim, \; ((p, e^\mu), (g, e^\gamma)) \mapsto (p_g, e^\alpha) . \tag{4.18}$$

Namely, we can find a unique element $(k, 1) \in (\mathscr{F}\mathscr{K})^\sim$ and $(p_g, e^\alpha) \in (\mathscr{F}\mathscr{P})^\sim$ such that

$$(p, e^\mu)(g, e^\gamma) = (k, 1)^{-1}(p_g, e^\alpha) ,$$

where k and p_g are the elements given in (2.19). Since we have

$$\tilde{\theta}^{(\infty)}((p, e^\mu)(g, e^\gamma))^{-1} \cdot (p, e^\mu)(g, e^\gamma) = (g^* p^* p g, e^{2(\mu + \gamma) + \Xi(p^*, p)})$$

and

$$\tilde{\theta}^{(\infty)}(p_g, e^\alpha)^{-1} \cdot (p_g, e^\alpha) = (p_g^* p_g, e^{2\alpha + \Xi(p_g^*, p_g)}) ,$$

we obtain

$$\alpha = \mu + \gamma + \frac{1}{2}(\Xi(p^*, p) - \Xi(p_g^*, p_g))$$

$$= \gamma' - \frac{1}{2}\Xi(p_g^*, p_g)$$

for some $\gamma' \in \mathbb{R}$, where we used Proposition 4.4. Thus (p_g, e^α) belongs to $(\mathscr{SP})^\sim$, i.e. the action (4.18) of $(\mathscr{FH})^\sim$ is well-defined.

Now we state our main theorem:

Theorem 4.5. *The group* $(\mathscr{FH})^\sim$ *acts transitively on the space of potentials with conformal factor* $(\mathscr{SP})^\sim$ *by* (4.18).

Proof. What remains to be proved is transitivity of the action. By Proposition 4.4, any element of $(\mathscr{SP})^\sim$ can be written as $(p, e^{-\frac{1}{2}\Xi(p^*p) + \gamma})$. Moreover, by Corollary 2.3, $p \in \mathscr{SP}$ can be written as $p = kg$ for $k \in \mathscr{FK}$ and $g \in \mathscr{FH}$. Then we have, by (4.17),

$$(k, 1)(g, e^\gamma) = (kg, e^{\gamma + \Xi(k, g)})$$

$$= (p, e^{\gamma - \frac{1}{2}\Xi(p^*, p)}) .$$

This shows that any element of $(\mathscr{SP})^\sim$ is on the oribit of the identity element $(1, 1) \in (\mathscr{SP})^\sim$. ▮

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