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On Geometrical Interpretation of the p-Adic Maslov Index

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Abstract: A set of selfdual lattices Λ in a two-dimensional p-adic symplectic space $(\mathscr{V}, \mathscr{B})$ is provided by an integer valued metric d. A realization of the metric space (Λ, d) as a graph Γ is suggested and this graph has been linked to the Bruhat-Tits tree. An action of symplectic group $\mathrm{Sp}(\mathscr{V})$ on a set of cycles of length three of the graph Γ is considered and a geometrical interpretation of the p-adic Maslov index is given in terms of this action.

Introduction

In the paper [Z] a definition of the p-adic Maslov index of a triple of selfdual lattices in a two-dimensional p-adic symplectic space $(\mathscr{V}, \mathscr{B})$ was suggested. In general the construction is as follows. For any selfdual lattice $\mathscr{L}$ in $(\mathscr{V}, \mathscr{B})$ we define an irreducible unitary representation $(H(\mathscr{L}), W_{\mathscr{L}})$ of the Heisenberg group $\tilde{\mathscr{V}}$ of space $(\mathscr{V}, \mathscr{B})$ in a separable Hilbert space $H(\mathscr{L})$. These representations are unitary equivalent and hence for any pair $(H(\mathscr{L}_1), W_{\mathscr{L}_1})$, $(H(\mathscr{L}_2), W_{\mathscr{L}_2})$ of two such representations there exists an intertwining operator $F_{\mathscr{L}_2,\mathscr{L}_1} : H(\mathscr{L}_1) \to H(\mathscr{L}_2)$. Therefore for any triple of such representations the operator $F = F_{\mathscr{L}_1,\mathscr{L}_3} F_{\mathscr{L}_3,\mathscr{L}_2} F_{\mathscr{L}_2,\mathscr{L}_1}$ commutes with all operators $W_{\mathscr{L}_1}(x)$, $x \in \tilde{\mathscr{V}}$. Thus F is proportional to an identity operator Id : $F = \mathfrak{m}(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)\,\mathrm{Id}$. The complex number $\mathfrak{m}(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ is the *p-adic Maslov index* of a triple $(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of selfdual lattices. In the paper [Z] simple properties of this index and explicit formulas for the index are given.

This paper is devoted to a geometrical interpretation of the p-adic Maslov index (we suppose that $p \neq 2$). This interpretation is given in terms of an action of p-adic symplectic group $\mathrm{Sp}(\mathscr{V})$ on a space Λ of selfdual lattices. Section 2 is concerned with the space Λ of selfdual lattices in a two-dimensional symplectic space $(\mathscr{V}, \mathscr{B})$ over the field $\mathbb{Q}_p$ of p-adic numbers. It turns out that the space Λ can be provided with an

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integer valued metric d. Based on this metric the space Λ is realized as a graph Γ. A set of vertices of this graph consists of selfdual lattices, a pair $\mathscr{L}_1, \mathscr{L}_2 \in \Lambda$ forms a link $[\mathscr{L}_1, \mathscr{L}_2]$ of Γ if $d(\mathscr{L}_1, \mathscr{L}_2) = 1$. It is shown that Γ consists of cycles of length three and can be derived from the Bruhat-Tits tree by a transformation "star-triangle."

Symplectic group $\mathrm{Sp}(\mathscr{V})$ acts transitively on sets of vertices and links of the graph Γ. The p-adic Maslov index is invariant under this action and therefore the action of $\mathrm{Sp}(\mathscr{V})$ on the set of cycles of length three is not transitive. The main result of this paper is that the last statement is exact in the following sense: for any two cycles $[\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3]$ and $[\mathscr{L}'_1, \mathscr{L}'_2, \mathscr{L}'_3]$ of length three there is a symplectic transformation $g \in \mathrm{Sp}(\mathscr{V})$ such that $g\mathscr{L}_1 = \mathscr{L}'_1$, $g\mathscr{L}_2 = \mathscr{L}'_2$, $g\mathscr{L}_3 = \mathscr{L}'_3$ if and only if the p-adic Maslov indices of these cycles coincide, that is $\mathfrak{m}(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = \mathfrak{m}(\mathscr{L}'_1, \mathscr{L}'_2, \mathscr{L}'_3)$.

2. Space of Selfdual Lattices

2.1. Graph of Selfdual Lattices

Let $\mathscr{V}$ be a two-dimensional vector space over $\mathbb{Q}_p$. A finitely generated $\mathbb{Z}_p$-submodule $\mathscr{L}$ of $\mathscr{V}$ is called a lattice if it contains a basis of $\mathscr{V}$. ($\mathbb{Z}_p$ denotes a ring of integers of $\mathbb{Q}_p$.) Let now $\mathscr{B}$ be a nondegenerated skewsymmetric bilinear form on $\mathscr{V}$. For a lattice $\mathscr{L} \subset \mathscr{V}$ a dual lattice $\mathscr{L}^*$ defines as follows: $\mathscr{L}^* = \{x \in \mathscr{V} : \mathscr{B}(x, y) \in \mathbb{Z}_p \ \forall y \in \mathscr{L}\}$. Notice that $\mathscr{L}^*$ is canonically isomorphic to the module $\mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{L}, \mathbb{Z}_p)$ [MH]. If $\mathscr{L} = \mathscr{L}^*$ then $\mathscr{L}$ is selfdual and a pair $(\mathscr{L}, \mathscr{B})$ forms a space over $\mathbb{Z}_p$ with symplectic inner product. Let Λ denote a set of all selfdual lattices in $(\mathscr{V}, \mathscr{B})$.

Now we define a function $d: \Lambda \times \Lambda \to \mathbb{Z}$ by the formula:

$$d(\mathscr{L}_1, \mathscr{L}_2) = 1/2 \log_p[(\mathscr{L}_1 + \mathscr{L}_2) : (\mathscr{L}_1 \cap \mathscr{L}_2)], \tag{1}$$

where $\mathscr{L}_1, \mathscr{L}_2 \in \Lambda$ and $[(\mathscr{L}_1 + \mathscr{L}_2) : (\mathscr{L}_1 \cap \mathscr{L}_2)]$ denotes order of a group $(\mathscr{L}_1 + \mathscr{L}_2)/(\mathscr{L}_1 \cap \mathscr{L}_2)$.

Proposition 1. *Let $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in \Lambda$. The function d has the following properties.*
(i) $d(\mathscr{L}_1, \mathscr{L}_2) \geq 0$, $d(\mathscr{L}_1, \mathscr{L}_2) = 0 \Leftrightarrow \mathscr{L}_1 = \mathscr{L}_2$;
(ii) $d(\mathscr{L}_1, \mathscr{L}_2) = d(\mathscr{L}_2, \mathscr{L}_1)$,
(iii) $d(\mathscr{L}_1, \mathscr{L}_3) \leq d(\mathscr{L}_1, \mathscr{L}_2) + d(\mathscr{L}_2, \mathscr{L}_3)$.

Properties (i) and (ii) are obvious. For the proof of (iii) we prove the following formula for $\mathscr{L}_1, \mathscr{L}_2 \in \Lambda$:

$$d(\mathscr{L}_1, \mathscr{L}_2) = \log_p[\mathscr{L}_1 : (\mathscr{L}_1 \cap \mathscr{L}_2)] = \log_p[\mathscr{L}_2 : (\mathscr{L}_1 \cap \mathscr{L}_2)]. \tag{2}$$

Notice that from the last relation it follows that the function d does take values in the set of integers $\mathbb{Z}$. Taking into account the relation

$$[(\mathscr{L}_1 + \mathscr{L}_2) : \mathscr{L}_1] = [\mathscr{L}_1^* : (\mathscr{L}_1 + \mathscr{L}_2)^*] = [\mathscr{L}_1 : (\mathscr{L}_1 \cap \mathscr{L}_2)],$$

we get

$$[(\mathscr{L}_1 + \mathscr{L}_2) : (\mathscr{L}_1 \cap \mathscr{L}_2)] = [(\mathscr{L}_1 + \mathscr{L}_2) : \mathscr{L}_1]\,[\mathscr{L}_1 : (\mathscr{L}_1 \cap \mathscr{L}_2)] = [\mathscr{L}_1 : (\mathscr{L}_1 \cap \mathscr{L}_2)]^2.$$

The relations (2) follow directly from the last formula and statement (ii) of Proposition 1.

By means of the relation $\mathscr{L}_1 \cap \mathscr{L}_2 \cap \mathscr{L}_3 \subset \mathscr{L}_1 \cap \mathscr{L}_3$ we have

$$\begin{aligned}[\mathscr{L}_1 : (\mathscr{L}_1 \cap \mathscr{L}_3)] &\leq [\mathscr{L}_1 : (\mathscr{L}_1 \cap \mathscr{L}_2 \cap \mathscr{L}_3)] \\ &= [\mathscr{L}_1 : (\mathscr{L}_1 \cap \mathscr{L}_2)]\,[(\mathscr{L}_1 \cap \mathscr{L}_2) : (\mathscr{L}_1 \cap \mathscr{L}_2 \cap \mathscr{L}_3)]\end{aligned}$$

Taking into account the relation $\mathscr{L}/(\mathscr{L} \cap \mathscr{L}') \simeq (\mathscr{L} + \mathscr{L}')/\mathscr{L}'$ [L] we get

$$\begin{aligned}[(\mathscr{L}_1 \cap \mathscr{L}_2) : (\mathscr{L}_1 \cap \mathscr{L}_2 \cap \mathscr{L}_3)] &= [(\mathscr{L}_1 \cap \mathscr{L}_2 \cap \mathscr{L}_3)^* : (\mathscr{L}_1 \cap \mathscr{L}_2)^*] \\ &= [(\mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3) : (\mathscr{L}_1 + \mathscr{L}_2)] \\ &= [\mathscr{L}_3 : (\mathscr{L}_3 \cap (\mathscr{L}_1 + \mathscr{L}_2))] \leq [\mathscr{L}_3 : (\mathscr{L}_3 \cap \mathscr{L}_2)] .\end{aligned}$$

From two last formulas we have

$$[\mathscr{L}_1 : (\mathscr{L}_1 \cap \mathscr{L}_3)] \leq [\mathscr{L}_1 : (\mathscr{L}_1 \cap \mathscr{L}_2)]\,[\mathscr{L}_3 : (\mathscr{L}_2 \cap \mathscr{L}_3)] .$$

Statement (iii) of Proposition 1 directly follows from (2) and the last formula. □

The proved proposition means that the pair (Λ, d) forms a metric space.

Now we realize the space (Λ, d) as a graph Γ. A set of vertices of this graph consists of selfdual lattices, a pair $\mathscr{L}_1, \mathscr{L}_2 \in \Lambda$ forms a link $[\mathscr{L}_1, \mathscr{L}_2]$ of Γ if $d(\mathscr{L}_1, \mathscr{L}_2) = 1$. For understanding of a structure of the graph Γ we recall a construction of the Bruhat-Tits tree (see for example [GP, M, S]).

Let $\mathscr{V}$ be as before a two-dimensional vector space over $\mathbb{Q}_p$. If $s \in \mathbb{Q}_p^*$ and $\mathscr{L}$ is a lattice in $\mathscr{V}$ then $s\mathscr{L}$ is a lattice too and hence $\mathbb{Q}_p^*$ acts on a set of lattices in $\mathscr{V}$. An orbit of this action is called a class of lattice, a set of such classes we denote by X. For a lattice $\mathscr{L}$ from a class $L \in X$ in any class $L' \in X$ there is a unique representative $\mathscr{L}' \in L'$ with the property: $\mathscr{L}' \subset \mathscr{L}$ and the module $\mathscr{L}/\mathscr{L}'$ is cyclic, that is $\mathscr{L}/\mathscr{L}' \simeq \mathbb{Z}_p/p^n\mathbb{Z}_p$ for some nonnegative integer n. The distance $D(L, L')$ between classes L and L' is defined as $D(L, L') = n$ and the map D does define an integer valued metric on the set X. Notice that we have the formula:

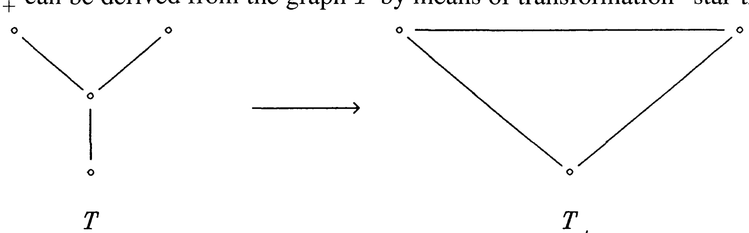
$$D(L, L') = \log_p[\mathscr{L} : \mathscr{L}'] . \tag{3}$$

The space (X, D) can be realized as a graph T in a previous manner: a set of vertices of T consists of classes of lattices, two classes $L, L' \in X$ form a link of T if $D(L, L') = 1$. It turns out that the graph T is a tree. Let us clear up a connection between graphs Γ and T.

Let $\mathscr{B}$ be a symplectic form on $\mathscr{V}$, $L \in X$ be a class of a selfdual lattice $\mathscr{L} \in \Lambda$ and X_+ denotes a set of vertices of the graph T placed at even distance D from L:

$$X_+ = \{L' \in X : D(L, L') \equiv 0 \,(\mathrm{mod}\, 2)\} .$$

As before the metric space (X_+, D) can be considered as a graph T_+ with a set of vertices X_+. Vertexes L and L' form a link of T_+ if $D(L, L') = 2$. Notice that the graph T_+ can be derived from the graph T by means of transformation "star-triangle":



Proposition 2. *Graphs Γ and T_+ are isomorphic.*

Let $\mathscr{L}$ be as before a selfdual lattice from a class $L \in X_+$. For $L' \in X_+$ and an arbitrary $\mathscr{L}' \in L'$ there is a symplectic basis $\{e, f\}$ of $(\mathscr{V}, \mathscr{B})$ wherein $\mathscr{L}$ and $\mathscr{L}'$ have the form

$$\mathscr{L} = \mathbb{Z}_p e \oplus \mathbb{Z}_p f ,$$
$$\mathscr{L}' = p^m \mathbb{Z}_p e \oplus p^n \mathbb{Z}_p f$$

for some integers m and n. It is easy to see that $D(L, L') = |m - n|$. As $D(L, L') \equiv 0 \,(\mathrm{mod}\, 2)$ then $p^{-(m+n)/2} \in \mathbb{Q}_p^*$ and $\mathscr{L}'' = p^{-(m+n)/2}\mathscr{L}'$ belongs to the class L'. It is obvious that $\mathscr{L}''$ is selfdual. From the previous discussion it follows that $\mathscr{L}''$ is a unique selfdual lattice in L'. From the formulas (1) and (3) we have

$$D(L, L') = 2d(\mathscr{L}, \mathscr{L}'') , \tag{4}$$

and hence the distance D between classes of selfdual lattices is even. Thus we get a one-to-one correspondence between sets of vertices of graphs Γ and T_+. Formula (4) gives us also the needed correspondence between sets of links of these graphs. □

Notice that unlike T the graph Γ contains cycles of length three and hence Γ is not a tree.

2 2. Action of Sp($\mathscr{V}$) *on* Γ

Let Sp($\mathscr{V}$) denote a symplectic group of the space $(\mathscr{V}, \mathscr{B})$ and Sp($\mathscr{L}$) be a stabilizer of a lattice $\mathscr{L} \in \Lambda$ in Sp($\mathscr{V}$).

As $\mathbb{Z}_p$ is a local ring then there is a symplectic basis $\{e, f\}$ of the space $(\mathscr{V}, \mathscr{B})$ wherein $\mathscr{L}$ has the form $\mathscr{L} = \mathbb{Z}_p e \oplus \mathbb{Z}_p f$ [MH] and therefore the standard left action of Sp($\mathscr{V}$) on Λ is transitive and Λ can be identified with a homogeneous space Sp($\mathscr{V}$)/ Sp($\mathscr{L}$). In other words Sp($\mathscr{V}$) acts transitively on a set of vertices of the graph Γ. As for $\mathscr{L} \in \Lambda$ and $g \in$ Sp($\mathscr{V}$) the modules $\mathscr{L}$ and $g\mathscr{L}$ are isomorphic then this action is isometric.

Moreover, for any two lattices $\mathscr{L}_1$ and $\mathscr{L}_2$ from Λ there is a symplectic basis $\{e, f\}$ of $(\mathscr{V}, \mathscr{B})$ wherein we have

$$\mathscr{L}_1 = \mathbb{Z}_p e \oplus \mathbb{Z}_p f , \qquad \mathscr{L}_2 = p^m \mathbb{Z}_p e \oplus p^{-m} \mathbb{Z}_p f$$

for some nonnegative integer m [W]. Notice that $m = d(\mathscr{L}_1, \mathscr{L}_2)$. From this we have that for any two pairs $\mathscr{L}_1, \mathscr{L}_2$ and $\mathscr{L}_1', \mathscr{L}_2'$ of selfdual lattices such that $d(\mathscr{L}_1, \mathscr{L}_2) = d(\mathscr{L}_1', \mathscr{L}_2')$ there is a symplectic transformation $g \in$ Sp($\mathscr{V}$) such that $g\mathscr{L}_1 = \mathscr{L}_1'$, $g\mathscr{L}_2 = \mathscr{L}_2'$. In particular, the action of Sp($\mathscr{V}$) on the set of links of the graph Γ is transitive.

2.3. Coordinates on Λ

Proposition 3. *Let $\{e, f\}$ be a symplectic basis of $(\mathscr{V}, \mathscr{B})$ For any lattice $\mathscr{L} \in \Lambda$ there exists a pair (m, μ), $m \in \mathbb{Z}$, $\mu \in \mathbb{Q}_p$ referred to as coordinates of $\mathscr{L}$ in the basis $\{e, f\}$, such that*

$$\mathscr{L} = \mathbb{Z}_p p^m e \otimes \mathbb{Z}_p(\mu p^m e + p^{-m} f) . \tag{5}$$

Two lattices $\mathscr{L}$ and $\mathscr{L}'$ with coordinates (m, μ) and (m', μ') respectively coincide if and only if $m = m'$ and $\mu - \mu' \in \mathbb{Z}_p$.

For the proof see [Z].

As a useful example let us find coordinates of selfdual lattices placed at distance 1 from the reference point. Taking into account Proposition 2 and a structure of the graph T it is easy to calculate the number of such lattices, this number is $p(p+1)$.

Recall that any nonzero p-adic number $x \in \mathbb{Q}_p^*$ can be uniquely represented in the form $x = p^{\mathrm{ord}_p(x)}\varepsilon(x)$, where $\mathrm{ord}_p(x) \in \mathbb{Z}$, $\varepsilon(x) \in \mathbb{Z}_p^*$, and $|x|_p = p^{-\mathrm{ord}_p(x)}$. For the sake of convenience we put $\mathrm{ord}_p(0) = +\infty$.

Proposition 4. *Let $\mathscr{L}_0, \mathscr{L} \in \Lambda$ have coordinates $(0,0)$ and (m, μ) in some basis $\{e, f\}$ respectively Then the following formula is valid.*

$$d(\mathscr{L}_0, \mathscr{L}) = \max\{-m - \mathrm{ord}_p(\mu), |m|\}. \tag{6}$$

It is easy to see that the lattice $\mathscr{L}_0 \cap \mathscr{L}$ consists of elements $\alpha e + \beta f$, where

$$\alpha, \beta \in \mathbb{Z}_p, \quad \alpha p^m + \beta p^m \mu \in \mathbb{Z}_p, \quad p^{-m}\beta \in \mathbb{Z}_p.$$

For the case of $m \geq 0$ the last conditions on α and β are equivalent to the following:

$$\alpha \in \mathbb{Z}_p, \quad \beta \in (p^{-m-\mathrm{ord}_p(\mu)}\mathbb{Z}_p) \cap (p^m\mathbb{Z}_p).$$

Taking into account the last formula and the formula (2) we get (6). For the case of $m < 0$ we choose a new symplectic basis $\{\tilde{e}, \tilde{f}\}: \tilde{e} = p^m e$, $\tilde{f} = p^{-m}f + \mu p^m e$. It is easy to see that in the basis $\{\tilde{e}, \tilde{f}\}$ the lattices $\mathscr{L}_0$ and $\mathscr{L}$ have coordinates $(-m, p^{2m}\mu)$ and $(0,0)$ respectively. Further proof is obvious. □

Corollary. *Coordinates of all lattices from Λ placed at distance 1 from the reference point are given in the following table.*

m	-1	0	1	1	1
μ	0	μ_0/p	0	μ_0/p	$(\mu_0 + \mu_1 p)/p^2$

where $\mu_0 = 1, 2, \ldots, p-1$ *and* $\mu_1 = 0, 1, 2, \ldots, p-1$.

According to Proposition 3 coordinate μ should be considered up to a p-adic integer, for the same reason we consider either $\mu = 0$ or $\mathrm{ord}_p(\mu) < 0$. By virtue of the condition $d(\mathscr{L}_0, \mathscr{L}) = 1$ and the formula (6) the pair $(m, \mathrm{ord}_p(\mu))$ can take values $(-1, +\infty)$, $(0, -1)$, $(1, +\infty)$, $(1, -1)$, and $(1, -2)$. In the above table all possible lattices for which the pair $(m, \mathrm{ord}_p(\mu))$ takes mentioned values are given. It is easy to see that the number of these lattices is equal to $p(p+1)$. □

3. p-Adic Maslov Index

Let $(\mathscr{V}, \mathscr{B})$ be as before a two-dimensional symplectic space over $\mathbb{Q}_p$ $(p \neq 2)$ and $\tilde{\mathscr{V}}$ denotes the Heisenberg group of this space, that is

$$\tilde{\mathscr{V}} = \{(\alpha, x),\ \alpha \in \mathbb{T},\ x \in \mathscr{V}\},$$
$$(\alpha, x)(\beta, y) = (\alpha\beta\chi(1/2\mathscr{B}(x,y)), x + y).$$

Here $\mathbb{T}$ is a unit circle in the field $\mathbb{C}$ of complex numbers and $\chi : \mathbb{Q}_p \to \mathbb{T}$ is a standard additive character of the field $\mathbb{Q}_p$ of rank 0 (that is $\chi(x) = 1 \Leftrightarrow x \in \mathbb{Z}_p$).

For any lattice $\mathscr{L} \in \Lambda$ one constructs a unitary irreducible representation of the group $\tilde{\mathscr{V}}$ (so-called $\mathscr{L}$-representation). Let us recall its definition. The space $H(\mathscr{L})$ of the $\mathscr{L}$-representation consists of complex valued functions on $\mathscr{V}$ which satisfies the following properties for all $x \in \mathscr{V}$ and $u \in \mathscr{L}$:

$$f(x+u) = \chi(1/2\mathscr{B}(x,u))f(x)\,, \tag{7}$$

$$\|f\|^2 = \sum_{\alpha \in \mathscr{V}/\mathscr{L}} |f(\alpha)|^2 < \infty\,. \tag{8}$$

The space $H(\mathscr{L})$ is a separable Hilbert space with respect to the scalar product

$$(f,g) = \sum_{\alpha \in \mathscr{V}/\mathscr{L}} f(\alpha)\bar{g}(\alpha)\,. \tag{9}$$

[Taking into account formula (7) it is easy to see that expressions under sum symbols in formulas (8) and (9) don't depend on a choice of an element in a coset $\alpha \in \mathscr{V}/\mathscr{L}$ and in these expressions α denotes an arbitrary representative of a coset α.]

Operators $\tilde{W}_{\mathscr{L}}(\alpha, x)$, $(\alpha, x) \in \tilde{\mathscr{V}}$ of the $\mathscr{L}$-representation are defined as follows:

$$\tilde{W}(\alpha,x)f(u) = \alpha W_{\mathscr{L}}(x)f(u) = \alpha\chi(1/2\mathscr{B}(x,u))f(u-x)\,.$$

$\mathscr{L}$-representation is irreducible and for any two lattices $\mathscr{L}_1, \mathscr{L}_2 \in \Lambda$ $\mathscr{L}_1$- and $\mathscr{L}_2$- representations are unitary equivalent. Therefore there is a unitary intertwining operator $F_{\mathscr{L}_2,\mathscr{L}_1} : H(\mathscr{L}_1) \to H(\mathscr{L}_2)$ which satisfies the properties

$$\begin{aligned} F_{\mathscr{L}_2,\mathscr{L}_1} W_{\mathscr{L}_1}(x) F^{-1}_{\mathscr{L}_2,\mathscr{L}_1} &= W_{\mathscr{L}_2}(x)\,, \\ F^{-1}_{\mathscr{L}_2,\mathscr{L}_1} &= F_{\mathscr{L}_1,\mathscr{L}_2} \end{aligned} \tag{10}$$

for all $x \in \mathscr{V}$. By virtue of (10) for any three lattices $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in \Lambda$ the operator $F = F_{\mathscr{L}_1,\mathscr{L}_3} F_{\mathscr{L}_3,\mathscr{L}_2} F_{\mathscr{L}_2,\mathscr{L}_1}$ commutes with all operators $W_{\mathscr{L}_1}(x)$, $x \in \mathscr{V}$ and therefore it is proportional to an identity operator:

$$F = \mathfrak{m}(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)\,\mathrm{Id}\,.$$

The complex number $\mathfrak{m} = \mathfrak{m}(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \in \mathbb{T}$ is the p-adic Maslov index of a triple of selfdual lattices. The following simple proposition is presented without proof (for the proof see [Z]):

Proposition 5. *Let* $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4 \in \Lambda$ *The following statements are valid.*
(i) $\mathfrak{m}(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = \mathfrak{m}(g\mathscr{L}_1, g\mathscr{L}_2, g\mathscr{L}_3)$ *for all* $g \in \mathrm{Sp}(\mathscr{V})$*;*
(ii) $\mathfrak{m}(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = 1$ *if at least two lattices in the triple coincide,*
(iii) $\mathfrak{m}(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ *remains the same under an even permutation of lattices in the triple and transfers to a conjugate expression under an odd one;*
(iv) *the following cocycle relation holds.*

$$\mathfrak{m}(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)\mathfrak{m}(\mathscr{L}_1, \mathscr{L}_3, \mathscr{L}_4) = \mathfrak{m}(\mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4)\mathfrak{m}(\mathscr{L}_2, \mathscr{L}_4, \mathscr{L}_1)$$

Now we present without proof an expression of the p-adic Maslov index in coordinates defined in Sect. 2.3 (for the proof see [Z]). For this according to [VV] we define a function $\lambda_p : \mathbb{Q}_p \to \mathbb{T}$ by the formula

$$\lambda_p(0) = 1\,,$$

$$\lambda_p(x) = \begin{cases} 1, \ \mathrm{ord}_p(x) = 2k, \ k \in \mathbb{Z}\,, \\ \left(\dfrac{\varepsilon(x)}{p}\right), \ \mathrm{ord}_p(x) = 2k+1, \ k \in \mathbb{Z}, \ p \equiv 1 \,(\mathrm{mod}\, 4)\,, \\ i\left(\dfrac{\varepsilon(x)}{p}\right), \ \mathrm{ord}_p(x) = 2k+1, \ k \in \mathbb{Z}, \ p \equiv 3 \,(\mathrm{mod}\, 4)\,, \end{cases}$$

where $\left(\dfrac{\varepsilon(x)}{p}\right)$ is the Legendre symbol of a p-adic unit $\varepsilon(x) \in \mathbb{Z}_p^*$.

Proposition 6. *Let $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in \Lambda$ have in a symplectic basis $\{e, f\}$ coordinates $(0,0)$, (m,μ), and (n,ν) respectively. The following statements are valid*

(i) $\mathfrak{m} = (\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = 1$ *for $\mu, \nu \in \mathbb{Z}_p$ and all $m, n \in \mathbb{Z}$;*

(ii) $$(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = \begin{cases} 1\,, & m \geq 0 \text{ or } \nu \in \mathbb{Z}_p\,, \\ \lambda_p(-\nu)\,, & m < 0, \ 1 < |\nu|_p < p^{-2m}\,, \\ 1\,, & m < 0, \ p^{-2m} \leq |\nu|_p\,, \end{cases}$$

for $\mu \in \mathbb{Z}_p$ and $n = 0$;

(iii) $$\mathfrak{m}(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = \begin{cases} 1\,, & \mu \in \mathbb{Z}_p \text{ or } \nu \in \mathbb{Z}_p \text{ or } \mu - \nu \in \mathbb{Z}_p\,, \\ \lambda_p(\mu\nu(\mu-\nu)) & \textit{in other cases}\,, \end{cases}$$

for $n = m = 0$.

4. Geometrical Interpretation of the p-Adic Maslov Index

As noted above a group $\mathrm{Sp}(\mathscr{V})$ acts transitively on sets of vertices and links of the graph Γ. Let lattices $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in \Lambda$ form a cycle of length three of the graph Γ, that is $d(\mathscr{L}_1, \mathscr{L}_2) = d(\mathscr{L}_2, \mathscr{L}_3) = d(\mathscr{L}_3, \mathscr{L}_1) = 1$ and $[\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3]$ denotes this cycle. (As usual cycle means oriented cycle, that is cycles $[\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3]$ and $[\mathscr{L}_1, \mathscr{L}_3, \mathscr{L}_2]$ are different). Any cycle $[\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3]$ of length three can be provided with the Maslov index $\mathfrak{m}(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ which is called the index of a cycle $[\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3]$. The following theorem gives a connection between the p-adic Maslov index and the action of $\mathrm{Sp}(\mathscr{V})$ on a set of cycles of length three of the graph Γ.

Theorem. *For any two cycles $[\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3]$ and $[\mathscr{L}_1', \mathscr{L}_2', \mathscr{L}_3']$ of length three of the graph Γ there exists a symplectic transformation $g \in \mathrm{Sp}(\mathscr{V})$ which maps one of these cycles to another (that is $g\mathscr{L}_1 = \mathscr{L}_1'$, $g\mathscr{L}_2 = \mathscr{L}_2'$, $g\mathscr{L}_3 = \mathscr{L}_3'$) if and only if the Maslov indices of these cycles coincide: $\mathfrak{m}(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = \mathfrak{m}(\mathscr{L}_1', \mathscr{L}_2', \mathscr{L}_3')$.*

Let $\mathscr{L}$ and $\mathscr{L}'$ have coordinates $(0,0)$ and $(-1,0)$ in some symplectic basis $\{e, f\}$ respectively. It follows from Proposition 4 that these lattices form a link $[\mathscr{L}, \mathscr{L}'] = [(0,0), (-1,0)]$ of the graph Γ. At first we find a stabilizer $\mathrm{Sp}(\mathscr{L}, \mathscr{L}') = \mathrm{Sp}(\mathscr{L}) \cap \mathrm{Sp}(\mathscr{L}')$ of this link in $\mathrm{Sp}(\mathscr{V})$. In the basis $\{e, f\}$ we have the following matrix realizations for $\mathrm{Sp}(\mathscr{L})$ and $\mathrm{Sp}(\mathscr{L}')$:

$$\mathrm{Sp}(\mathscr{L}) \simeq SL(2, \mathbb{Z}_p)\,,$$

$$\mathrm{Sp}(\mathscr{L}') \simeq \begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix} SL(2, \mathbb{Z}_p) \begin{pmatrix} 1/p & 0 \\ 0 & p \end{pmatrix}.$$

From the last formula we easily get

$$\mathrm{Sp}(\mathscr{L},\mathscr{L}') = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}_p) : c \equiv 0 \,(\mathrm{mod}\, p^2) \right\}.$$

Notice that from the conditions $c \equiv 0 \,(\mathrm{mod}\, p^2)$ and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$ it follows that $ad \equiv 1 \,(\mathrm{mod}\, p)$.

As $\mathrm{Sp}(\mathscr{V})$ acts transitively on the set of links of the graph Γ then for further proof of the theorem it is sufficient to consider an action of the group $\mathrm{Sp}(\mathscr{L},\mathscr{L}')$ on the set of cycles of length three which contain the link $[\mathscr{L},\mathscr{L}']$. From Proposition 4 we see that in coordates $\{e,f\}$ all these cycles have the form $[(0,0), (-1,0), (0,\mu/p)]$ for $\mu = 1,2,\dots,p-1$. Let $\mathscr{L}(\mu)$ denote the lattice with coordinates $(0,\mu/p)$. For an arbitrary $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \in \mathrm{Sp}(\mathscr{L},\mathscr{L}')$ we have $g\mathscr{L}(\mu) = \mathscr{L}(\tilde{\mu})$ for some $\tilde{\mu} = 1,2,\dots,p-1$, because $\mathrm{Sp}(\mathscr{V})$ acts on Λ isometrically. By virtue of the relation $\mathscr{L}(\mu) = \begin{pmatrix} 1 & \mu/p \\ 0 & 1 \end{pmatrix} \mathscr{L}$ the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \mu/p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \tilde{\mu}/p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{11}$$

is valid for some $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}_p)$. From the relation (11) we get $\mathbb{Z}_p \ni b = \beta + \mu\tilde{\mu}/p^2 c + (\tilde{\mu}d - \mu a)/p$, and therefore $\tilde{\mu}d - \mu a \equiv 0 \,(\mathrm{mod}\, p)$. Taking into account the condition $ad \equiv 1 \,(\mathrm{mod}\, p)$ in the residue class field $\mathbb{F}_p \simeq \mathbb{Z}_p/p\mathbb{Z}_p$, we get the relation $\tilde{\mu} = \mu a_0^2$, where $a_0 \in \mathbb{F}_p^*$ is a class of $a \in \mathbb{Z}_p$ in $\mathbb{F}_p$.

From the above discussion it follows that if there is a symplectic transformation $g \in \mathrm{Sp}(\mathscr{L},\mathscr{L}')$ which transforms $\mathscr{L}(\mu)$ to $\mathscr{L}(\tilde{\mu})$ then μ and $\tilde{\mu}$ are in the same class in $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$.

Let now μ and $\tilde{\mu}$ are in the same class in $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$. By direct calculations it is easy to show that the matrix

$$g = \begin{pmatrix} (\tilde{\mu}/\mu)^{1/2} & 0 \\ 0 & (\mu/\tilde{\mu})^{1/2} \end{pmatrix} \in Sp(\mathscr{L},\mathscr{L}')$$

satisfies the following condition:

$$g \begin{pmatrix} 1 & \mu/p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \tilde{\mu}/p \\ 0 & 1 \end{pmatrix} g,$$

and therefore $g\mathscr{L}(\mu) = \mathscr{L}(\tilde{\mu})$.

From the above discussion we see that for the cycles $[\mathscr{L},\mathscr{L}',\mathscr{L}(\mu)]$ and $[\mathscr{L},\mathscr{L}',\mathscr{L}(\tilde{\mu})]$ there is a symplectic transformation that maps one cycle to another if and only if μ and $\tilde{\mu}$ are in the same class in $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$. From Proposition 6 and properties of the Legendre symbol we see that corresponding Maslov indices have the same properties: $\mathfrak{m}(\mathscr{L},\mathscr{L}',\mathscr{L}(\mu)) = \mathfrak{m}(\mathscr{L},\mathscr{L}',\mathscr{L}(\tilde{\mu}))$ if and only if μ and $\tilde{\mu}$ are in the same class in $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$. This finishes the proof. □

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