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Solutions with High Dimensional Singular Set, to a Conformally Invariant Elliptic Equation in $\mathbb{R}^4$ and in $\mathbb{R}^6$

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Abstract: We construct positive weak solutions to the equation $-\Delta_0 v = v^{\frac{n+2}{n-2}}$, where $-\Delta_0$ denotes the conformal Laplacian on the n-sphere ($n = 4, 6$), having singular sets of Hausdorff dimension greater than or equal to $\frac{n-2}{2}$.

1. Introduction

In their paper, Schoen and Yau have stated the following conjecture:

Conjecture [8]: All positive weak solutions of $-\Delta_0 v = v^{\frac{n+2}{n-2}}$, with $v \in L^{\frac{n+2}{n-2}}(\mathbb{S}^n)$, have singular set of Hausdorff dimension less than or equal to $(n-2)/2$. Here $-\Delta_0$ denotes the conformal Laplacian for the standard metric on the sphere $\mathbb{S}^n$.

This problem can be formulated in $\mathbb{R}^n$ as follows [9]: We define the measure $d\mu = (1 + |x|^2)^{-n} dx$ on $\mathbb{R}^n$. Assume that $u \in L^{\frac{n+2}{n-2}}(\mathbb{R}^n, d\mu)$ is a weak positive solution of

$$-\Delta u = u^{\frac{n+2}{n-2}} . \tag{1}$$

Then, the Hausdorff dimension of the singular set if u is less than or equal to $(n-2)/2$.

Many attempts have been made to find solutions of (1) with a prescribed singular set. In a very difficult paper [7], Schoen builds solutions of (1) with prescribed isolated singularities. In another paper [8], Schoen and Yau have used the geometrical meaning of Eq. (1) in order to derive, through ideas of conformal geometry, the existence of singular solutions having a singular set whose Hausdorff dimension is less than or equal to $(n-2)/2$. More recently Mazzeo and Smale have proved in [4] the existence of solutions of (1) singular over some manifold which is a small deformation of a sphere $\mathbb{S}^k$, with $k < (n-2)/2$. Their method is based on the study of degenerate operators.

In this paper, we give some counter-examples to the conjecture stated above when $n = 4$ and when $n = 6$. More precisely, we prove the result:

Theorem 1. *Assume that $n = 4$ or $n = 6$, then for any $d \in [(n-2)/2, n]$, there exists a positive weak solution of* (1) *in $L^{\frac{n+2}{n-2}}(\mathbb{R}^n, d\mu)$, having a singular set of finite d-Hausdorff dimension.*

In order to derive the existence of solutions of (1) with prescribed singularities, we use a variational approach already used in [5, 6] in order to prescribe isolated singularities of $-\Delta u = u^{\frac{n}{n-2}}$.

Remark 1. The proof of Theorem 1 by variational techniques does not hold any more in dimension $n \notin \{4, 6\}$.

We must emphasize an important point: In the case where $\delta = (n-2)/2$, we are able to build positive weak solutions of $-\Delta_0 v = v^{\frac{n+2}{n-2}}$ whose singular set Σ is given by a finite union of $(n-2)/2$-dimensional spheres. These solutions allow us to define a complete metric $g = v^{\frac{4}{n-2}} g_0$ on $\mathbb{S}^n \backslash \Sigma$.

2. Quasi-Solutions

In both cases $n = 4$ and $n = 6$, one can write $n = 2(m-1)$. With this notation we notice that

$$\frac{n+2}{n-2} = \frac{m}{m-2}.$$

In this section, we are only interested in solutions of

$$-\Delta u = u^{\frac{m}{m-2}},$$

in a m-dimensional space. We state some results very similar to the results used in [6] in order to prescribe the singularities of solutions of

$$-\Delta u = u^{\frac{m}{m-2}} \quad \text{in} \quad B^m. \tag{2}$$

More precisely, we produce some positive functions which are quasi-solutions of (2) in the sense that $-\Delta u = u^{\frac{m}{n-2}} - f$ in some bounded domain of $\mathbb{R}^m$, where f can be taken as small as we want in a suitable space.

Definition 1. We will say that $(\bar{u}, f) \in L^{\frac{m}{m-2}}(B^m) \times L^{\frac{2m}{m+2}}(B^m)$ is a quasi-solution if it satisfies

$$-\Delta \bar{u} = \bar{u}^{\frac{m}{m-2}} - f \quad \text{in} \quad B^m.$$

For some $\eta > 0$, we will say that a quasi-solution $(\bar{u}, f) \in L^{\frac{m}{m-2}}(B^m) \times L^{\frac{2m}{m+2}}(B^m)$ satisfies the hypothesis $(\mathscr{H}_\eta)$ if

$$\int_{B^m} \bar{u}^{\frac{m}{m-2}}\, dx < \eta \quad \text{and} \quad \int_{B^m} \left(|\nabla \bar{u}|^{\frac{2m}{m+2}} + f^{\frac{2m}{m+2}} \right) dx < \eta.$$

For all $\eta > 0$, the existence of quasi-solutions satisfying $(\mathscr{H}_\eta)$ is stated in the following proposition:

Proposition 1. *Some $\eta > 0$ and some sequence of points $p_i \in B^m$ being given, for all $N \in \mathbb{N}$, there exists a sequence $(\bar{u}_N, f_N) \in L^{\frac{m}{m-2}}(B^m) \times L^{\frac{2m}{m+2}}(B^m)$ of quasi-solutions satisfying $(\mathscr{H}_\eta)$ and the following properties:*
1. *$\bar{u}_N$ is regular in $B^m \backslash \{p_0, \dots, p_N\}$.*
2. *The supports of u_N and f_N are compact in B^m and included in $\bigcup_{i \in \mathbb{N}} B^m(p_i, 1/4)$.*
3. *The behavior of $\bar{u}_N$ near a point of $\{p_i\}_{i \in \{1, \dots, N\}}$ is given by*

$$\bar{u}_N(x) = \frac{(c_0 + o(1))}{(|x - p_i|^2 \log(1/|x - p_i|))^{\frac{m-2}{2}}},$$

for some constant $c_0 > 0$ which only depends on m.
4. *The sequence $\bar{u}_N$ is increasing and converges strongly in $L^{\frac{m}{m-2}}(B^m)$.*

The proof of this proposition is quite technical and is given in the appendix.

In [2] Aviles has derived the following result concerning the behavior of positive weak solutions of (2) near an isolated singularity.

Theorem 2 [2]. *For $m \geq 3$, assume that u is a positive solution of $-\Delta u = u^{\frac{m}{m-2}}$ in $B^m(1) \subset \mathbb{R}^m$, which is regular in $B^m(1) \backslash \{0\} \subset \mathbb{R}^m$. Then, either there exists some constant $c_0 > 0$ such that the behavior of u in a neighborhood of $0 \in \mathbb{R}^m$ is given by*

$$u(x) = \frac{(c_0 + o(1))}{(|x|^2 \log(1/|x|))^{\frac{m-2}{2}}}$$

or u is regular over $B^m(1)$.

Therefore, all solutions of (2) have the same asymptotic behavior near an isolated singularity. So, it is natural to look for singular solutions, of both (1) and (2), having this type of behavior.

3. A Variational Problem

According to the results of the last section, we can consider that we have built a quasi-solution $(\bar{u}, f) \in L^{\frac{m}{m-2}}(B^m) \times L^{\frac{2m}{m+2}}(B^m)$, with $\bar{u}$ regular, except at the points $\{p_0, \dots, p_N\}$. We will assume that the set of singular points is chosen so that it is disjoint from the hyperplane $x_m = 0$. Namely if $p_i \equiv (x_{i,1}, \dots, x_{i,m})$, we ask that $x_{i,m} > 1/2$.

Once we have this quasi-solution, we can define

$$\tilde{u}(x_1, \dots, x_n) \equiv \bar{u}(x_1, \dots, x_{m-1}, r_m)$$

and

$$\hat{f}(x_1, \dots, x_n) \equiv f(x_1, \dots, x_{m-1}, r_m),$$

where $r_m^2 \equiv x_m^2 + \dots + x_n^2$.

We are looking for a solution of

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in} \quad B^n, \tag{3}$$

with special boundary conditions. Our guess is that this solution is not far away from $\tilde{u}$, so we search $u = \tilde{u} + v$, where v is "more regular" than $\tilde{u}$.

More precisely, we can measure how far is $\tilde{u}$ from a solution of (3) by computing

$$-\Delta \tilde{u} = \tilde{u}^{\frac{n+2}{n-2}} - \hat{f} - \frac{n-m}{r_m} \frac{\partial \tilde{u}}{\partial x_m}.$$

We define $\tilde{f} = \hat{f} + \frac{n-m}{r_m} \frac{\partial \tilde{u}}{\partial x_m}$. In view of the equation above, we need to solve the equation

$$\begin{cases} -\Delta v = (\tilde{u}+v)^{\frac{n+2}{n-2}} - \tilde{u}^{\frac{n+2}{n-2}} + \tilde{f} & \text{in } B^n \\ \tilde{u}+v > 0 & \text{in } B^n \\ (n-2)v = -2\frac{\partial v}{\partial \nu} & \text{on } \partial B^n. \end{cases} \tag{4}$$

The natural functional associated to (4) is the following:

$$E(v) \equiv \int_{B^n} \left(\tfrac{1}{2}|\nabla v|^2 - F_m(\tilde{u}, v) - \tilde{f} v\right) dx + \frac{n-2}{4} \int_{\partial B^n} v^2 \, d\sigma, \tag{5}$$

where

$$F_m(\tilde{u}, v) = \begin{cases} \frac{1}{4}\left(|\tilde{u}+v|^3(\tilde{u}+v) - \tilde{u}^4 - 4\tilde{u}^3 v\right) & \text{if } m=3 \\ \frac{1}{3} v^3 + \tilde{u} v^2 & \text{if } m=4. \end{cases}$$

We define the space

$$H \equiv \{v \in H^1(B^n)/v(x_1, \dots, x_n) = v(x_1, \dots, x_{m-1}, (x_m^2 + \dots + x_n^2)^{1/2}, 0, \dots, 0)\},$$

This is the space of H^1 functions which are invariant by rotation over the $n-m+1$-last variables.

The following proposition is almost standard:

Proposition 2. *The functional $E(.)$ is well defined for $v \in H$ and critical points of this functional are solutions of*

$$\begin{cases} -\Delta v = |\tilde{u}+v|^{\frac{n+2}{n-2}} - \tilde{u}^{\frac{n+2}{n-2}} + \tilde{f} & \textit{in } B^n \\ (n-2)v = -2\frac{\partial v}{\partial \nu} & \textit{on } \partial B^n. \end{cases} \tag{6}$$

Proof. Everything relies on the crucial assumption that the support of $\tilde{f}$ and the supports of the functions $\tilde{u}$ have been prescribed away from the hyperplane $x_m = 0$ (see the assumption $x_{i,m} > 1/2$ and 2 in Proposition 1). Thus, although v only belongs to H^1, we can use the rotational invariance of functions of H in order to prove the following estimates:

$$\left| \int_{B^n} \tilde{f} v dx \right| \leq c_1 \|v\|_H \|\tilde{f}\|_{\frac{2m}{m+2}}$$

and

$$\left| \int_{B^n} F_m(\tilde{u}, v) dx \right| \leq \begin{cases} c_2\left(\|v\|_H^2 \|\tilde{u}\|^2_{\frac{m}{m-2}} + \|v\|_H^4\right) & \text{if } m=3 \\ c_2\left(\|v\|_H^2 \|\tilde{u}\|_{\frac{m}{m-2}} + \|v\|_H^3\right) & \text{if } m=4, \end{cases}$$

for some constant $c_1, c_2 > 0$ depending only on the dimension m. It is now easy to verify the result of the proposition.

With the help of the last inequalities it is also possible to prove the proposition:

Proposition 3. *For all $\eta > 0$ small enough, there exists $\varrho(\eta) > 0$ and $\theta(\eta) > 0$ such that if $(\bar{u}, f)$ satisfies $(\mathscr{H}_\eta)$ and if $\|v\|_H = \varrho/\eta$, then $E(v) > \theta(\eta)$. In addition, one can assume that $\varrho(\eta) \to 0$ as $\eta \to 0$.*

In order to derive the existence of solutions of (6), we just minimize the functional $E(.)$ over some small ball in H. The minimum exists and is achieved thanks to the result of Proposition 3 and the fact that $E(0) = 0$, if $\varrho(\eta)$ is chosen small enough. The fact that we minimize over the set of functions with small H^1 energy, prevents the lack of compactness phenomenon to occur [3].

4. Proof of the Theorem

We can now prove Theorem 1.

Step 1. We begin the proof for some singular set of finite $(n-2)/2$ Hausdorff dimension. Using Proposition 2 and Proposition 3, we can find a critical point of $E(.)$ by solving the problem

$$\min_{\|v\|_H < \varrho} E(v),$$

as explained in the previous section.

Therefore $u \equiv \tilde{u} + v$ is a solution of (6). Next, we define on all $\mathbb{R}^n$ the function

$$U(x) = \begin{cases} u(x) & \text{for all } |x| \leq 1 \\ \dfrac{1}{|x|^{n-2}}\, u\left(\dfrac{x}{|x|^2}\right) & \text{for all } |x| \geq 1 . \end{cases} \tag{7}$$

This gives us a solution of $-\Delta U = |U|^{\frac{n+2}{n-2}}$ in $\mathbb{R}^n$.

Step 2. Let us denote by Σ_N the union of the set $\{p_0, \ldots, p_N\} \times \mathbb{S}^{n-m}$ and its image by the mapping $x \to x/|x|^2$.

We want to prove that U is positve in $\mathbb{R}^n$. Using the regularity result of H. Brézis and Kato [3], we get that U is regular in $B^n \backslash \Sigma_N$. In particular, using the definition of U outside the unit ball given by (7), we prove that U tends to 0 as x goes to $+\infty$. More precisely $|x|^{n-2} U(x)$ is bounded when $|x|$ goes to $+\infty$. The function U is thus given by the Newtonian potential of $|U|^{\frac{n+2}{n-2}}$ which is positive. It is standard to prove, using Hopf lemma, that $U > 0$ in $\mathbb{R}^n$. So, U is a solution of (4). In addition, using the behavior of U when $|x|$ goes to $+\infty$, we get easily that $U \in L^{\frac{n+2}{n-2}}(\mathbb{R}^n, d\mu)$.

Step 3. We must prove that U has a singular set of correct dimension. Applying the result of Theorem 2 and using the rotational invariance of U, we see that, near a point p of the set Σ_N, if U is regular in a neighborhood of p, then v has a behavior near p given by

$$v(x) = -\frac{(c_0 + o(1))}{(|y|^2 \log(1/|y|))^{\frac{m-2}{2}}},$$

where we have stated $y \equiv (x_1 - x_{p,1}, \ldots, x_{m-1} - x_{p,m-1}, r_m - r_{p,m})$ and, as before, $r_m + x_m^2 + \ldots + x_n^2$, $r_{p,m} = x_{p.m}^2 + \ldots + x_{p,n}^2$. It is easy to verify that v does not belong to $L^{\frac{2n}{n-2}}(B^n)$, which is not possible since $v \in H$. Therefore, we conclude that U must be singular at p. This ends the proof of the theorem in the case of a singular set of finite $(n-2)/2$ Hausdorff dimension.

Step 4. Now we want to build solutions of (1) for all dimensions of the singular set $> (n-2)/2$. This existence result is given by limit argument. We will denote by v the solution obtained in the previous section by the minimizing algorithm. We have the property:

Lemma 1. *There exists $\eta > 0$ such that if $(\bar{u}, f)$ satisfies $(\mathscr{H}_\eta)$ then there exists some constant $c_3 > 0$ depending only on η and n the dimension of the space, such that $\|v\|_H \leq c_3$.*

For all $\delta \in [0, m]$, we can find $\Sigma \subset B^m$ a closed set included in $\{x \in \mathbb{R}^m / x_m > 1/2\}$ of finite δ-Hausdorff dimension. In addition, there exists a sequence of points p_i which is dense in Σ. Now we use the former construction in order to build a sequence of quasi-solutions $(\bar{u}_i, f_i)$ such that the singular set of $\bar{u}_i$, is given by $\{p_0, \ldots, p_i\}$ and $\bar{u}_i$ is an increasing sequence of positive functions which is strongly convergent in $L^{\frac{m}{m-2}}(B^m)$. Denoting by v_i one solution of (4) associated to $\tilde{u}_i$, we get a solution of (1), whose singular set is given by $\{p_0, \ldots, p_i\} \times \mathbb{S}^{n-m}$. By construction, the sequence $\tilde{u}_i$ converges strongly in $L^{\frac{n+2}{n-2}}(B^n)$ and, using Lemma 1, we can always assume that the sequence v_i converges weakly to v in H. In addition

$$-\Delta(\tilde{u}_i + v_i) = (\tilde{u}_i + v_i)^{\frac{n+2}{n-2}} \quad \text{in} \quad B^n .$$

Passing to the limit in this equation gives us the desired result, namely a solution of (1) having as singular set $\Sigma \times \mathbb{S}^2$ in B^n (the fact that the limit is singular on the correct set can be proved like in Step 3). Thanks to the particular extension of the solution in the whole space [see (7)] we get a solution of (1) in $\mathbb{R}^n$ having a singular set of finite $n - m + \delta$-Hausdorff dimension. This ends the proof of Theorem 1.

5. Appendix

In this section we give a proof of Proposition 1.

We begin by recalling some existence result of Aviles [1] in a m-dimensional space:

Theorem 3 [1]. *For $m \geq 3$, there exist radial positive weak solutions of $-\Delta u = u^{\frac{m}{m-2}}$ in $B^m \subset \mathbb{R}^m$ which are regular in $B^m \backslash \{0\}$ and whose behavior in a neighborhood of $0 \in \mathbb{R}^m$ is given by*

$$u(x) = \frac{(c_0 + o(1))}{(-|x|^2 \log|x|)^{\frac{m-2}{2}}},$$

for some positive constant c_0 depending only on m, thanks to Theorem 2.

Remark 2. There exists a positive constant $\bar{R}$ such that

$$\frac{c_0}{2} \le u(x)\left(-|x|^2 \log|x|\right)^{\frac{m-2}{2}} \le 2c_0$$

for all $x \in B^m(\bar{R})$.

We denote by $\underline{u}$ a solution obtained in Theorem 3. Using the equation satisfied by $\underline{u}$, a simple computation shows that the following properties hold:

Lemma 2. *There exists some constants $c_4, c_5 > 0$ such that*

$$|\nabla \underline{u}|(x) = \frac{(c_4 + o(1))}{|x|^{m-1}(-\log|x|)^{\frac{m-2}{2}}} \quad \text{and} \quad |\nabla^2 \underline{u}|(x) = \frac{(c_5 + o(1))}{|x|^{m}(-\log|x|)^{\frac{m-2}{2}}},$$

in a neighborhood of 0.

Remark 3. Reducing $\bar{R}$, if necessary, we may always assume that for all $x \in B^m(\bar{R})$,

$$|\nabla \underline{u}|(x) \le \frac{2c_4}{|x|^{m-1}(-\log|x|)^{\frac{m-2}{2}}} \quad \text{and} \quad |\nabla^2 \underline{u}|(x) = \frac{2c_5}{|x|^{m}(-\log|x|)^{\frac{m-2}{2}}}.$$

Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^m)$ such that $\chi(x) = 1$ if $x \in B^m(1/2)$ and $\chi(x) = 0$ if $x \in \mathbb{R}^m \backslash B^m(1)$. We denote by $M = \sup_{x \in \mathbb{R}^m} (|\nabla \chi|(x), |\nabla^2 \chi|(x))$. Given some $r > 0$, we define

$$\chi_r(x) = \chi(x/r).$$

Using the solution given in Theorem 3, we define

$$u_\varepsilon(x) = \varepsilon^{m-2} \underline{u}(\varepsilon x).$$

Let us notice that for all $\varepsilon > 0$, u_ε is a solution of $-\Delta u = u^{\frac{m}{m-2}}$.

The proof of Proposition 1 is by induction. The case of zero singularities is straightforward. Therefore, we may assume that we have already built a quasi-solution $(\bar{u}_{N-1}, f_{N-1}) \in L^{\frac{m}{m-2}}(B^m) \times L^{\frac{2}{m+2}}(B^m)$, satisfying all the needed hypotheses. We may always assume that p_N is not a singularity of $\bar{u}_{N-1}$, otherwise the proof is finished. Thus, we can choose $r_N \in \mathbb{R}$ such that the support of $\chi_{r_N}(x - p_N)$ does not contain any $\{p_i\}_{i \in \{1, \dots, N-1\}}$. And finally, we may choose $\varepsilon_N > 0$ such that $\varepsilon_N r_N \le \bar{R}$. We set

$$\bar{u}_N(x) = \bar{u}_{N-1}(x) + u_{\varepsilon_N}(x - p_N)\chi_{r_N}(x - p_N).$$

Let us check that it is always possible to choose r_N and ε_N in order for $\bar{u}_N$ to fulfill all the needed hypotheses.

<u>First Step.</u> We derive an estimate of the norm of $\bar{u}_N$ in $L^{\frac{m}{m-2}}(B^m)$,

$$\left(\int_{B^m} \bar{u}_N^{\frac{m}{m-2}} dx\right)^{\frac{m-2}{m}} \le \left(\int_{B^m} u_{\varepsilon_N}^{\frac{m}{m-2}}(x - p_N)\chi_{r_N}^{\frac{m}{m-2}}(x - p_N)\, dx\right)^{\frac{m-2}{m}} + \left(\int_{B^m} \bar{u}_{N-1}^{\frac{m}{m-2}} dx\right)^{\frac{m-2}{m}}.$$

Now we estimate

$$\int_{B^m} u_{\varepsilon_N}^{\frac{m}{m-2}}(x-p_N)\chi_{r_N}^{\frac{m}{m-2}}(x-p_N)\,dx \le \int_{B^m(\varepsilon_N r_N)} \underline{u}^{\frac{m}{m-2}}\,dx\,.$$

Using the remark following Theorem 3, we get

$$\int_{B^m(r)} \underline{u}^{\frac{m}{m-2}}\,dx \le \frac{c_6}{(-\log r)^{\frac{m-2}{2}}}\,.$$

So, we have

$$\left(\int_{B^m} \bar{u}_N^{\frac{m}{m-2}}\,dx\right)^{\frac{m-2}{m}} \le \left(\int_{B^m} \bar{u}_{N-1}^{\frac{m}{m-2}}\,dx\right)^{\frac{m-2}{m}} + \frac{c_7}{(-\log(\varepsilon_N r_N))^{\frac{(m-2)^2}{2m}}}\,.$$

By assumption, we have

$$\int_{B^m} \bar{u}_{N-1}^{\frac{m}{m-2}}\,dx < \eta\,.$$

Therefore, we can always choose ε_N small enough to get

$$\int_{B^m} \bar{u}_N^{\frac{m}{m-2}}\,dx < \eta\,.$$

<u>Second Step.</u> Let us compute $\Delta\bar{u}_N + \bar{u}_N^{\frac{m}{m-2}}$,

$$\Delta\bar{u}_N + \bar{u}_N^{\frac{m}{m-2}} = f_{N-1} + \left(\chi_{r_N}^{\frac{m}{m-2}} - \chi_{r_N}\right)u_{\varepsilon_N}^{\frac{m}{m-2}} + 2\nabla(\chi_{r_N})\nabla(u_{\varepsilon_N}) + \Delta(\chi_{r_N})u_{\varepsilon_N}$$
$$+ (\bar{u}_{N-1} + \chi_{r_N}u_{\varepsilon_N})^{\frac{m}{m-2}} - \bar{u}_{N-1}^{\frac{m}{m-2}} - (\chi_{r_N}u_{\varepsilon_N})^{\frac{m}{m-2}}\,.$$

We denote by

$$g_1 = \left(\chi_{r_N}^{\frac{m}{m-2}} - \chi_{r_N}\right)u_{\varepsilon_N}^{\frac{m}{m-2}}\,,$$
$$g_2 = 2\nabla(\chi_{r_N})\nabla(u_{\varepsilon_N}) + \Delta(\chi_{r_N})u_{\varepsilon_N}\,,$$

and

$$g_3 = (\bar{u}_{N-1} + \chi_{r_N}u_{\varepsilon_N})^{\frac{m}{m-2}} - \bar{u}_{N-1}^{\frac{m}{m-2}} - (\chi_{r_N}u_{\varepsilon_N})^{\frac{m}{m-2}}\,.$$

Notice that g_3 is simply

$$g_3 = \begin{cases} 3\bar{u}_{N-1}^2\chi_{r_N}u_{\varepsilon_N} + 3\bar{u}_{N-1}(\chi_{r_N}u_{\varepsilon_N})^2 & \text{if } m=3 \\ 2\bar{u}_{N-1}\chi_{r_N}u_{\varepsilon_N} & \text{if } m=4\,. \end{cases}$$

Finally, we define $f_N = f_{N-1} + g_1 + g_2 + g_3$. The remark following Theorem 3 allows us to get the estimate

$$\|g_1\|_{L^\infty(B^m)} \le \frac{c_8}{r_N^m(-\log(\varepsilon_N r_N))^{\frac{m}{2}}}\,.$$

Using the estimates of Lemma 2, we get

$$\|g_2\|_{L^\infty(B^m)} \le \frac{c_9}{r_N^m(-\log(\varepsilon_N r_N))^{\frac{m-2}{2}}}.$$

If we choose r_N small enough and if we use the Hölder inequality, we get the estimate:

$$\left(\int_{B^m} g_3^{\frac{2m}{m+2}}\,dx\right)^{\frac{m+2}{2m}} \le c_{10}\left\{(\bar{u}_{N-1}(p_N)+o(1))^{\frac{m}{m-2}}\, r_N^{\frac{m+2}{2}}\right.$$
$$\left. +(\bar{u}_{N-1}(p_N)+o(1))\, r_N^{\frac{m-2}{2}}\left(\int_{B^m(r_N)} (\chi_{r_N} u_{\varepsilon_N})^{\frac{m}{m-2}}\,dx\right)^{\frac{2}{m}}\right\}.$$

(We have used the inequality $(x+y)^{\frac{m}{m-2}} - x^{\frac{m}{m-2}} - y^{\frac{m}{m-2}} \le c_{11}(x^{\frac{m}{m-2}} + xy^{\frac{2}{m-2}})$, for all $x, y \ge 0$, which allows us to have the previous estimate for all $m > 2$.) Thus, it is easy to compute

$$\left(\int_{B^m} (g_1+g_2+g_3)^{\frac{2m}{m+2}}\,dx\right)^{\frac{m+2}{2m}}$$
$$\le c_{12}\left\{(\bar{u}_{N-1}(p_N)+o(1))^{\frac{m}{m-2}}\, r_N^{\frac{m+2}{2}} + (\bar{u}_{N-1}(p_N)+o(1))\, r_N^{\frac{m-2}{2}}\right.$$
$$\left.\times\left(\int_{B^m(r_N)} (\chi_{r_N} u_{\varepsilon_N})^{\frac{m}{m-2}}\,dx\right)^{\frac{2}{m}} + (-r_N\log(\varepsilon_N r_N))^{-\frac{m-2}{2}}\right\}.$$

If $r_N > 0$ and $\varepsilon_N > 0$ are chosen small enough, we can make the last expression as small as we want. Therefore, we conclude by saying that, provided $r_N > 0$ and $\varepsilon_N > 0$ are suitably chosen, we can always get

$$\int_{B^m} \left(\Delta\bar{u}_N + \bar{u}_N^{\frac{m}{m-2}}\right)^{\frac{2m}{m+2}}\,dx < \eta.$$

Third Step. It remains to prove the estimate on $|\nabla u_N|$. As in the previous steps, we only have to prove that the quantity

$$\int_{B^m} |\nabla(u_{\varepsilon_N}(x-p_N)\chi_{r_N}(x-p_N))|^{\frac{2m}{m+2}}\,dx$$

be made as small as we want. We can estimate

$$\int_{B^m} |\nabla(u_{\varepsilon_N}(x-p_N)\chi_{r_N}(x-p_N))|^{\frac{2m}{m+2}}\,dx$$
$$\le c_{13}\int_{B^m(r_N)} \left(|\nabla u_{\varepsilon_N}|^{\frac{2m}{m+2}} + r_N^{-\frac{2m}{m+2}}\, u_{\varepsilon_N}^{\frac{2m}{m+2}}\right)dx.$$

Using the estimates of Lemma 2 we get

$$\int_{B^m} |\nabla(u_{\varepsilon_N}(x-p_N)\chi_{r_N}(x-p_N))|^{\frac{2m}{m+2}}\,dx$$

$$\leq c_{14} \int_{B^m(\varepsilon_N r_N)} \left(|x|^{-2\frac{m(m-1)}{m+2}} + (\varepsilon_N r_N)^{-\frac{2m}{m+2}} |x|^{-2\frac{m(m-2)}{m+2}}\right)(-\log|x|)^{-\frac{m(m-2)}{m+2}}\,dx\,.$$

So finally

$$\int_{B^m} |\nabla(u_{\varepsilon_N}(x-p_N)\chi_{r_N}(x-p_N))|^{\frac{2m}{m+2}}\,dx$$

$$\leq \begin{cases} c_{15}(-\varepsilon_N r_N/\log(\varepsilon_N r_N))^{-\frac{3}{5}} & \text{if } m=3 \\ c_{15}(-\log(\varepsilon_N r_N))^{-\frac{1}{3}} & \text{if } m=4\,, \end{cases}$$

which is a quantity that can be made arbitrarily small if $r_N > 0$ and $\varepsilon_N > 0$ are chosen small enough.

Remark 4. Reducing, if necessary, the values of $r_N > 0$ and $\varepsilon_N > 0$, we can ensure that the sequence u_N converges strongly in $L^{\frac{m}{m-2}}(B^m)$ as N goes to $+\infty$.

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