Higher Algebraic Structures and Quantization

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Abstract: We derive (quasi-)quantum groups in $2 + 1$ dimensional topological field theory directly from the classical action and the path integral. Detailed computations are carried out for the Chern-Simons theory with finite gauge group. The principles behind our computations are presumably more general. We extend the classical action in a $d + 1$ dimensional topological theory to manifolds of dimension less than $d + 1$. We then "construct" a generalized path integral which in $d + 1$ dimensions reduces to the standard one and in d dimensions reproduces the quantum Hilbert space. In a $2 + 1$ dimensional topological theory the path integral over the circle is the category of representations of a quasi-quantum group. In this paper we only consider finite theories, in which the generalized path integral reduces to a finite sum. New ideas are needed to extend beyond the finite theories treated here.

Recent work on invariants of low dimensional manifolds utilizes complicated algebraic structures, for both theory and computation. New invariants of 3-manifolds, and of knots and links in 3-manifolds, are constructed from certain types of Hopf algebras [RT] or more generally from special sorts of categories [KR]. These invariants are known to arise from a $2 + 1$ dimensional quantum field theory [W]. In this paper we derive the algebraic structure from the field theory, starting with the classical lagrangian, and so express the relationship between the algebra and the geometry directly. With this understanding the algebra can be put to work to calculate invariants. The guiding principle for us is the *locality* of field theory, as expressed in *gluing laws*. The gluing laws resonate well with cut and paste techniques in topology. They are important tools field theory offers for both theoretical work and computations. We generalize the standard constructs in a $d+1$ dimensional field theory – *classical action* and *path integral* – to spaces of dimension less than $d + 1$, retaining the essential

property of locality. Whereas the classical action is always a finite dimensional integral, the path integral over the space of fields usually involves infinitely many variables. Our focus here is not on the analytical difficulties of path integrals over infinite dimensional spaces; indeed we can only treat path integrals in a "toy model" where they reduce to finite sums. Nevertheless, our generalizations of the classical action and path integral most likely pertain to other topological field theories.

In a $d + 1$ dimensional field theory the classical action of a field Θ on a $(d + 1)$-manifold X is usually a real[1] number $S_X(\Theta)$. Often in topological theories only the exponential $e^{2\pi i S_X(\Theta)}$ is well-defined. The simplest example is the holonomy of a connection: $X = S^1$ is the circle and the field Θ is a connection on a principal circle bundle $P \to S^1$. Notice that the action is not as straightforward if $X = [0, 1]$ has boundary – interpreted as a number the parallel transport of a connection over the interval depends on boundary conditions. Rather, the dependence on boundary conditions is best expressed by regarding the parallel transport as a map $P_0 \to P_1$ from the fiber of the circle bundle over 0 to the fiber over 1. This is the classical action over the interval. Our generalization of the classical action asserts that the classical action of a connection over a point, which is just a principal circle bundle $Q \to pt$, is the fiber Q. The value of that action is a space on which the circle group $\mathbb{T}$ acts simply transitively, a so-called $\mathbb{T}$-*torsor*. Notice that the action of a field (connection) on the interval takes values in the action of the restriction of the field to the boundary. The Chern-Simons invariant in 3 dimensions is similar – the action in 2 dimensions is a $\mathbb{T}$-torsor – and the story continues to lower dimensions [F1, F2].

At the crudest level of structure the classical action in d dimensions is a *set*. (The classical action in $d + 1$ dimensions is a *number*.)

The usual path integral in a $d + 1$ dimensional theory may be written schematically as

$$\int_{\mathscr{E}_X} d\mu_X(\Theta) e^{2\pi i S_X(\Theta)} ,$$

where X is a $(d + 1)$-manifold without boundary, $\mathscr{E}_X$ is the space of fields on X, and $d\mu_X$ is a measure on $\mathscr{E}_X$. Of course, in many examples of interest this is only a formal expression since the measure does not exist, or has not been constructed. This integral is a sum of positive numbers (the measure) times complex numbers (the exponentiated action), so is a complex number. Our generalization to d dimensions is as follows. The action is now a $\mathbb{T}$-torsor, which we extend to a *hermitian line*, i.e., a one dimensional complex inner product space. The original $\mathbb{T}$-torsor is the set of elements of unit norm in the associated hermitian line. The integral is then a sum of positive numbers times hermitian line. If L is a hermitian line and μ a positive number, let $\mu \cdot L$ be the same underlying one dimensional vector space with inner product multiplied by μ. We sum hermitian lines via direct sum; the sum is a hermitian vector space, or Hilbert space. Formally, then, this generalized path integral is the space of L^2 sections of a line bundle over the space of fields. When the space of fields has continuous parameters we can formally reinterpret canonical quantization, or geometric quantization, as the regularization needed to make sense of the integral.

In higher codimensions the classical action and path integral take values in certain generalizations of $\mathbb{T}$-torsors and vector spaces. The next step after a $\mathbb{T}$-torsor is a $\mathbb{T}$-*gerbe* [Gi, Br, BMc] and the next step after a vector space is a *2-vector space* [KV, L]. The underlying structure in both cases is not a set, but rather a *category*. The

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[1] The theories we consider in this paper are unitary

continuation to higher codimensions leads to *multicategories,* and the foundations become rather murky, at least to this author. We attempt an exposition of these "higher algebraic structures" in Sects. 1 and 3. Our treatment has no pretensions of rigor. For this reason throughout this paper we use the term "Assertion" as opposed to "Theorem" or "Proposition", except when dealing with ordinary sets and categories. Since we deal with *unitary* theories our quantum spaces have an inner product, so are Hilbert spaces. In codimension two we therefore obtain *2-inner product spaces* or *2-Hilbert spaces.* The terminology may be confusing: A 2-inner product space is an ordinary category, not a 2-category.

The particular model we treat is gauge theory with finite gauge group. It exists in any dimension. This theory was introduced by Dijkgraaf/Witten [DW] and further developed by many authors [S2, Ko, Q1, Q2, Fg, Y3, FQ]. In some ways this paper is a continuation of [FQ], though it may be read independently. The space of fields (up to equivalence) on a compact manifold is a finite set in this model, hence all path integrals reduce to finite sums. The lagrangian in the $d + 1$ dimensional theory is a singular $(d + 1)$-cocycle, and the generalized classical action is defined as its integral over compact oriented manifolds of dimension less than or equal to $d + 1$. Only the cohomology pairing with the fundamental class of a closed oriented $(d + 1)$-manifold is standardly defined. In the appendix we briefly describe an integration theory which extends this pairing. It is the origin of the torsors, gerbes, etc. that we encounter. We define the generalized classical action in Sect. 2 and the generalized path integral in Sect. 4. Our assertions in these sections are formulated for all codimensions simultaneously, and we suggest that the reader decipher them starting in the top-dimension, where they reduce to the corresponding theorems in [FQ].

In Sect. 5 we explore the structure of the generalized path integral E over a circle in $1 + 1$ dimensional theories and in $2 + 1$ dimensional theories. The treatment here is based on the generalized axioms of topological field theory[2] set out in Assertion 2.5 and Assertion 4.12, not on any particular features of finite gauge theory. In a $1 + 1$ dimensional theory E is an inner product space and we construct a compatible algebra structure and a compatible real structure. The argument here is standard. In a $2 + 1$ dimensional theory E is a 2-inner product space, which in particular is a category. The analogue of the real algebra structure, here derived from the generalized path integral, makes this a *braided monoidal category* with compatible "balancing" and duality. Such categories arise in rational conformal field theory [MS], and have been much discussed in connection with topological invariants and topological field theory. Reconstruction theorems in category theory [DM, Ma1] assert that such a category is the category of representations of a *quasitriangular quasi-Hopf algebra*, or *quasi-quantum group* [Dr].[3] In fact, the reconstruction also requires a special functor from the category E to the category of vector spaces. We remark that a different quasi-Hopf algebra related to field theories was proposed in [Ma3].

We put the abstract theory of Sects. 1–4 to work in Sects. 6–9, where we carry out the computations for the finite gauge theory. We warmup in Sect. 6 by discussing some features of the $1 + 1$ dimensional theory. The remainder of the paper treats the $2 + 1$ dimensional Chern-Simons theory (with finite gauge group). The quasi-Hopf algebras we compute via the generalized path integral are the quasi-Hopf algebras introduced

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[2] These axioms are not meant to be complete, and in any case they must be modified in other examples to allow for central extensions of diffeomorphism groups. See [A, Q2] for a discussion of the general axioms in topological field theory. See [F3] for a discussion of central extensions

[3] I believe that the quasitriangular quasi-Hopf algebras we obtain will always have a "ribbon element" [RT] as well. This certainly holds in the finite gauge theory

by Dijkgraaf/Pasquier/Roche [DPR]. They were further studied by Altschuler/Coste [AC]. The computations are not difficult, but they are nerve-racking! When dealing with categories (and, even worse, multicategories) one must be very careful about equality versus isomorphism, at the next level about equality of isomorphisms versus isomorphisms of isomorphisms, and so on. This sort of algebra seems well-adapted to the geometry of cutting and pasting, but as I said it *is* nerve-racking. We keep close track of the trivializations we need to introduce at various stages of the computation. Some of these trivializations are used to define the functor to the category of vector spaces which we need to reconstruct the quasi-Hopf algebra. Our reconstructions do not follow the procedures in the abstract category theory proofs. Rather, in our examples the algebras are apparent from appropriate descriptions of the braided monoidal category. In Sect. 9 we use more sophisticated gluing arguments to choose special bases of the algebras, and so derive the exact formulas in [DPR]. This involves cutting and pasting manifolds with the simplest kind of corners. We formulate a generalized gluing law for the classical action in Assertion 9.2. Clearly it generalizes to higher codimensional gluing and to the quantum theory. Segal [S1] gives a proof of the "Verlinde diagonalization" [V] using a quantum version of this gluing law. This sort of generalized gluing should be useful in other problems as well. We also briefly describe at the end of Sect. 7 how Segal's *modular functor* [S1] fits in with our approach.

In gauge theory one usually makes special arguments to account for *reducible connections*. In these finite gauge theories every "connections" is reducible, that is, every bundle has nontrivial automorphisms, and all of the constructions must account for the automorphism groups.

We formulate everything in terms of manifolds, whereas others prefer to work more directly with knots and links. The relationship is the following (cf. [W]). Suppose K is a knot in a closed oriented 3-manifold X. Let $X' = X - \nu(K)$ denote the manifold X with an open tubular neighborhood $\nu(K)$ of the knot removed. Then a *framing* of the normal bundle of K in X determines an isotopy class of diffeomorphisms from the standard torus $S^1 \times S^1$ to $\partial X' = -\partial(\nu(X))$. In a $2 + 1$ dimensional topological field theory this induces an isometry between the quantum Hilbert space of $\partial X'$ and the quantum Hilbert space of the standard torus. So the path integral over X' takes values in the Hilbert space of the standard torus. As we explain at the end of Sect. 9 this Hilbert space is the "Grothendieck ring" of the monoidal category discussed above, and it has a distinguished basis consisting of equivalence classes of irreducible representations. These are the "labels" in the theory, and the coefficients of the path integral over X' are the knot invariants for labeled, framed knots. The generalization to links is immediate.

An expository version of some of this material appears in [F3].

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1. Higher Algebra I

Whereas the classical action in a $d + 1$ dimensional field theory typically takes values in the real numbers, often in topological theories only its exponential with values in the circle group

$$\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$

is defined. We remark that nonunitary versions of these theories would replace $\mathbb{T}$ by the group $\mathbb{C}^\times$ of nonzero complex numbers. For the algebra in this section we could replaced $\mathbb{T}$ by any commutative group. The usual action is defined for fields on closed[4] spacetimes of dimension $d + 1$. In Sect. 2 we describe "higher actions" which are defined for fields on manifolds of dimension less than $d + 1$ and take their values in "higher groups." For example, over closed d-manifolds the action takes its values in the *abelian group-like category* of $\mathbb{T}$-*torsors*. On a closed $(d - 1)$-manifold the action takes its values in the *abelian group-like 2-category* of $\mathbb{T}$-*gerbes*. And so on. In this section we briefly describe these "higher groups." We also use the term "higher torsors." As stated in the introduction we only attempt a heuristic treatment, not a rigorous one. Our goal in this section, then, is to explain a hierarchy:

$$\begin{aligned}
\mathscr{T}_0 &= \mathbb{T} \quad \text{circle group}, \\
\mathscr{T}_1 & \quad \text{"group" of } \mathbb{T}\text{-torsors (1-torsors)}, \\
\mathscr{T}_2 & \quad \text{"group" of } \mathbb{T}\text{-gerbes (2-torsors)}, \\
& \quad \text{etc.}
\end{aligned} \tag{1.1}$$

Each of these is an abelian group in the sense that there is a commutative associative composition law, an identity element, and inverses. However, only $\mathscr{T}_0$ is an honest group; in fact, only $\mathscr{T}_0$ is a set! The $\mathbb{T}$-torsors $\mathscr{T}_1$ form a *category*[5], the $\mathbb{T}$-gerbes a *2-category*,[6] etc. So the group structure must be understood in that framework. Although this will not be relevant for us in this paper, we note that $\mathbb{T}$ is a *Lie* group and the higher $\mathscr{T}_n$ also have some smooth structure.

We begin with a definition.

Definition 1.2. A $\mathbb{T}$-torsor T is a manifold with a simply transitive (right) $\mathbb{T}$-action.

So "$\mathbb{T}$-torsor" is a short equivalent to "principal homogeneous $\mathbb{T}$-space." Of course, $\mathbb{T}$ itself is a $\mathbb{T}$-torsor, the *trivial* $\mathbb{T}$-torsor. A nontrivial example, which is of no particular relevance to us, is the nonidentity component of the orthogonal group $O(2)$. An example of more relevance: Let L be any one dimensional complex inner product space. Then the set of elements of unit norm is a $\mathbb{T}$-torsor. Any $\mathbb{T}$-torsor takes this form for some hermitian line L [cf. (3.2)]. Now if T_1, T_2 are $\mathbb{T}$-torsors, then a *morphism* $h : T_1 \to T_2$ is a map which commutes with the $\mathbb{T}$ action: $h(t \cdot \lambda) = h(t) \cdot \lambda$

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[4] Here "closed" means "compact without boundary." There is also a (relative) action on compact manifolds with boundary, which we describe below

[5] We refer to [Mac] for the basics of category theory as well as plenty of examples. Roughly, a category $\mathscr{C}$ is a collection of objects $\mathrm{Obj}(\mathscr{C})$ and for every $A, B \in \mathrm{Obj}(\mathscr{C})$ there is a *set* of morphisms $\mathrm{Mor}(A, B)$. Morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ compose to give a morphism $A \xrightarrow{gf} C$. This composition is associative and there are identity morphisms. Notice that $\mathrm{Obj}(\mathscr{C})$ is not necessarily a set. We often write "$A \in \mathscr{C}$" for "$A \in \mathrm{Obj}(\mathscr{C})$"

[6] A *2-category* $\mathscr{C}$ has a collection of objects $\mathrm{Obj}(\mathscr{C})$ and for each $A, B \in \mathrm{Obj}(\mathscr{C})$ a *category* of morphisms $\mathrm{Mor}(A, B)$. In other words, if $f, g \in \mathrm{Mor}(A, B)$, then there is a set of *2-morphisms* which map from f to g. The composition $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$ is now assumed to be a functor. One obtains different notions depending on whether one assumes that this composition is exactly (strictly) associative or whether one postulates that it is associative up to a given 2-morphism. The former notion generalizes to n-categories. The latter notion was introduced by Benabou [Be] for 2-categories (these are called "bicategories"), and apparently a complete list of axioms for the higher case has not been written down. (See the lists of axioms in [KV] to see the complications involved.) Since for three, $\mathbb{T}$-torsors A, B, C the torsors $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ are strictly speaking different, but isomorphic, the category $\mathscr{T}_1$ does not have a strictly associative tensor product. This propagates through to the higher $\mathscr{T}_n$. Our use of the work "n-category" is in the latter, yet undefined, sense

for all $t \in T_1$, $\lambda \in \mathbb{T}$. The collection of all $\mathbb{T}$-torsors and morphisms forms a category $\mathscr{T}_1$. The group of automorphisms $\mathrm{Aut}(T)$ of any $T \in \mathscr{T}_1$ is naturally isomorphic to $\mathbb{T}$: any $\mu \in \mathbb{T}$ acts as the automorphism $t \mapsto t \cdot \mu$. Also, the set of morphisms $\mathrm{Mor}(T_1, T_2)$ is naturally a $\mathbb{T}$-torsor. Finally, every morphism in $\mathscr{T}_1$ has an inverse.[7]

So far we have only described the category structure on $\mathscr{T}_1$, which is analogous to the set structure on $\mathbb{T}$. The important point is this: Elements of $\mathscr{T}_1$ have automorphisms. We do not identify isomorphic elements which are not equal; the choice of isomorphism matters. In fact, any two elements of $\mathscr{T}_1$ are isomorphic, so all of the information is in the isomorphism. It does make sense to say that two isomorphisms are equal, since $\mathrm{Mor}(T_1, T_2)$ is a set for any $T_1, T_2 \in \mathscr{T}_1$.

To describe the abelian group structure we need to introduce new operations which serve as the group multiplication and group inverse. These are the product of two torsors and the inverse torsor. So if $T_1, T_2 \in \mathscr{T}_1$ are $\mathbb{T}$-torsors, define the product $T_1 \cdot T_2$ as

$$T_1 \cdot T_2 = \{\langle t_1, t_2 \rangle \in T_1 \times T_2\}/\langle t_1 \cdot \lambda, t_2 \rangle \sim \langle t_1, t_2 \cdot \lambda \rangle$$

for all $\lambda \in \mathbb{T}$. The $\mathbb{T}$ action on $T_1 \cdot T_2$ is

$$\langle t_1, t_2 \rangle \cdot \lambda = \langle t_1 \cdot \lambda, t_2 \rangle = \langle t_1, t_2 \cdot \lambda \rangle .$$

The inverse T^{-1} of a torsor T with $\mathbb{T}$ action $\cdot$ has the same underlying set but a new $\mathbb{T}$ action $*$ given by

$$t * \lambda = t \cdot \lambda^{-1} .$$

We denote the element in T^{-1} corresponding to $t \in T$ as $t^{-1} \in T^{-1}$. The trivial torsor $\mathbb{T}$ acts as the identity element under the multiplication. One must remember the maxim that elements in $\mathscr{T}_1$ cannot be declared equal, only isomorphic. So we do not have $T \cdot T^{-1} = \mathbb{T}$, but rather an isomorphism

$$T \cdot T^{-1} \to \mathbb{T}, \qquad \langle t \cdot \lambda, t \rangle \mapsto \lambda .$$

This isomorphism is part of the data describing $\mathscr{T}_1$. All other axioms for an abelian group, such as commutative and associativity, must be similarly modified. For example, now the associative law is not an axiom but a piece of the structure – a system of isomorphisms – and these isomorphisms satisfy a higher-order axiom called the *pentagon diagram*.

We remark that there is a natural identification

$$T_2 \cdot T_1^{-1} \cong \mathrm{Hom}(T_1, T_2) \tag{1.3}$$

for any $T_1, T_2 \in \mathscr{T}_1$.

Starting with the group $\mathbb{T}$ we have outlined the construction of an abelian group-like category $\mathscr{T}_1$. Now we want to repeat the construction replacing $\mathbb{T}$ with $\mathscr{T}_1$. In other words, we consider "$\mathscr{T}_1$-torsors" and then introduce a product law and inverse so as to obtain what is now an abelian group-like *2-category* $\mathscr{T}_2$ of the collection of all "$\mathscr{T}_1$-torsors." The terminology is that a "$\mathscr{T}_1$-torsor" is a $\mathbb{T}$-*gerbe*.

The definitions are analogous to those for $\mathbb{T}$-torsors, so we will be brief and incomplete. A $\mathbb{T}$-gerbe is a category $\mathscr{G}$ equipped with a simply transitive action of $\mathscr{T}_1$. The action is a functor $\mathscr{G} \times \mathscr{T}_1 \to \mathscr{G}$ whose action is denoted $\langle G, T \rangle \mapsto G \cdot T$. The simple transitivity means that the functor

$$\mathscr{G} \times \mathscr{T}_1 \to \mathscr{G} \times \mathscr{G}, \qquad \langle G, T \rangle \mapsto \langle G, G \cdot T \rangle$$

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[7] So $\mathscr{T}_1$ is called a *groupoid*, which is not to be confused with the abelian group-like structure we introduce below

is an equivalence, and we are given an "inverse" functor and equivalences of the composites to the identity. This amounts to the specification of a torsor $T(G_1, G_2)$ for $G_1, G_2 \in \mathscr{G}$ together with natural equivalences $G_2 \cong G_1 \cdot T(G_1, G_2)$ and $T \cong T(G, G \cdot T)$. This definition may be more rigid than the standard definition, but it fits our examples.

Now if $\mathscr{G}_1$ and $\mathscr{G}_2$ are $\mathbb{T}$-gerbes, then a morphism $\mathscr{G}_1 \to \mathscr{G}_2$ is a functor which commutes with the $\mathscr{T}_1$ action. This means that part of the data of the morphism is a natural transformation between the two functors obtained by traveling from northwest to southeast around the square

$$
\begin{array}{ccc}
\mathscr{G}_1 \times \mathscr{T}_1 & \longrightarrow & \mathscr{G}_1 \\
\downarrow & & \downarrow \\
\mathscr{G}_2 \times \mathscr{T}_1 & \longrightarrow & \mathscr{G}_2
\end{array}
$$

It is easy to see that the collection of morphisms $\mathscr{G}_1 \to \mathscr{G}_2$ forms a category and that the morphisms $\mathrm{Mor}(\mathscr{G}_1, \mathscr{G}_2)$ form a $\mathbb{T}$-gerbe. The collection of $\mathbb{T}$-gerbes forms a 2-category $\mathscr{T}_2$. One can introduce an abelian group-like structure on this 2-category by defining the product of two $\mathbb{T}$-gerbes and the inverse of a $\mathbb{T}$-gerbe, which we leave to the reader.

I hope that at this stage it is in principle clear how I mean to define the series of abelian group-like structures listed in (1.1), and that it is clear what their basic properties are, though the detailed definition promises to be a combinatorial mess. We need one more notion, which is a symmetry of such abelian group-like structures. Suppose A is a finite group. To say that A acts on $\mathbb{T}$ by symmetries means that we have a homomorphism $A \to \mathbb{T}$, i.e., a character of A, and then A acts on $\mathbb{T}$ as multiplication by this character. If T is a $\mathbb{T}$-torsor, then since $\mathrm{Aut}(T) \cong \mathbb{T}$, an action of A on T is again given by a character of A. Note that the characters form the cohomology group $H^1(A; \mathbb{T})$. Next, an action of A on $\mathscr{T}_1$ means that we have a "homomorphism" $A \to \mathscr{T}_1$. More precisely, for each $a \in A$ we have a $\mathbb{T}$-torsor T_a and for $a_1, a_2 \in A$ an isomorphism $T_{a_1} \cdot T_{a_2} \cong T_{a_1 a_2}$. These isomorphisms must satisfy an associativity constraint. Such a system of torsors describes a *central extension* $\tilde{A} = \bigcup_{a \in A} T_a$ of A:

$$
1 \to \mathbb{T} \to \tilde{A} \overset{\pi}{\to} A \to 1 . \tag{1.4}
$$

The fiber of π over a is T_a. *Up to isomorphism* the central extension is classified by an element of the cohomology group $H^2(A; \mathbb{T})$. An action of A on a $\mathbb{T}$-gerbe $\mathscr{G}$ also leads to a cohomology class, since different trivializations of $\mathscr{G}$ lead to equivalent extensions of A. The continuation of this discussion to higher $\mathscr{T}_n$ leads to representatives of higher group cohomology (with abelian coefficients).

2. Classical Theory

In this section we describe a classical (gauge) field theory in $d + 1$ dimensions with *finite* gauge group Γ. We generalize the classical theory to higher codimensions, that is, to lower dimensional manifolds. The (exponentiated) action on fields on a $(d+1)$-manifold takes values in $\mathbb{T}$. For fields on a d-manifold the action takes values in $\mathscr{T}_1$, i.e., the value of the action is a $\mathbb{T}$-torsor. More generally, over a $(d+1-n)$-manifold

the action takes values in $\mathscr{T}_n$. We construct the action using the integration theory of the Appendix. Since this is a straightforward generalization of [FQ, Sect. 1], given the algebra in Sect. 1 and the integration theory in the Appendix, we defer to that reference for more details and exposition.

Throughout this paper we use a procedure to eliminate the dependence of quantities on extra variables or choices. In [FQ, Sect. 1] we call this the *invariant section construction* after the special case mentioned in the footnote below. Here, following MacLane [Mac] (cf. Quinn [Q1]) we call it an *inverse limit* of a functor. Let $\mathscr{C}$ be a groupoid and $\mathscr{F}: \mathscr{C} \to \mathscr{D}$ a functor to a category (or multicategory) $\mathscr{D}$. We define[8] an element of the inverse limit to be a collection $\{v(C) \in \mathscr{F}(C)\}$ such that $\mathscr{F}(C \to C')v(C) = v(C')$ for all morphisms $C \to C'$. The inverse limit is an object in $\mathscr{D}$. In our applications $\mathscr{D}$ is $\mathscr{T}_n$ for some n or is the multicategory $\mathscr{V}_n$ of higher inner product spaces which we introduce in Sect. 3. Also, in our applications the groupoid $\mathscr{C}$ has only a finite number of components. For $\mathscr{D} = \mathscr{V}_n$ the inverse limit always exists. If $\mathscr{D} = \mathscr{T}_n$ we must also assume that $\mathscr{F}(C \to C)$ is trivial for all automorphisms $C \to C$, i.e., that "$\mathscr{F}$ has no holonomy."

Fix a finite group Γ. For any manifold M we let $\mathscr{C}_M$ denote the category of principal Γ bundles over M. This is the collection of *fields* in the theory. There are symmetries as well: A morphism $f: P' \to P$ is a smooth map which commutes with the Γ action and induces the identity map on M. Notice that every morphism is invertible. Define an equivalence relation by setting $P' \cong P$ if there exists a morphism $P' \to P$. Let $\overline{\mathscr{C}_M}$ denote the space of equivalence classes of fields; it is a finite set if M is compact. If M is connected there is a natural identification

$$\overline{\mathscr{C}_M} \cong \mathrm{Hom}(\pi_1(M, m), \Gamma)/\Gamma$$

for any basepoint $m \in M$. Here Γ acts on a homomorphism by conjugation.

Let $B\Gamma$ be a classifying space for Γ, which we fix together with a universal bundle $E\Gamma \to B\Gamma$. If $P \to M$ is a principal Γ bundle, then there exists a Γ map $P \to E\Gamma$ and any two such *classifying maps* are homotopic through Γ maps.

Fix a singular $(d+1)$-cocycle $\alpha \in C^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$. This is the *lagrangian* of our theory. The action is constructed as follows. Suppose M is a compact oriented manifold of dimension at most $d + 1$. Let $P \in \mathscr{C}_M$. Then if $F: P \to E\Gamma$ is a classifying map for P, with quotient $\bar{F}: M \to B\Gamma$, consider the integral

$$\exp\left(2\pi i \int_M \bar{F}^* \alpha\right),$$

which is defined via the integration theory of the Appendix. We need then to determine the dependence on F and obtain something independent of F. We treat closed manifolds and arbitrary compact manifolds (possibly with boundary) separately, though the second case clearly includes the first.

Suppose first that Y is a closed oriented $(d + 1 - n)$-manifold, $n > 0$, and $Q \in \mathscr{C}_Y$ is a Γ bundle over Y. Define a category $\mathscr{C}_Q$ whose objects are classifying maps $f: Q \to E\Gamma$ and whose morphisms are homotopies $f \xrightarrow{h} f'$. Define a functor

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[8] Think of the following example. Let $\mathscr{C}$ be the category whose objects are the points of a manifold M and whose morphisms are paths on M. Let $\mathscr{D}$ be the category of vector spaces and linear isomorphisms. A vector bundle with connection over M determines a functor $\mathscr{F}: \mathscr{C} \to \mathscr{D}$ (the morphisms act by parallel transport), and the inverse limit is the space of *flat* sections

$\mathscr{F}_{Q;\alpha}:\mathscr{C}_Q \to \mathscr{T}_n$ by

$$\mathscr{F}_{Q;\alpha}(f) = \exp\left(2\pi i \int_Y \bar{f}^*\alpha\right) = I_{Y,\bar{f}^*\alpha}\,,\tag{2.1}$$

where $\bar{f}:Y \to B\Gamma$ is the quotient map determined by $f:Q \to E\Gamma$. For a homotopy $f \xrightarrow{h} f'$, let $\mathscr{F}_{Q;\alpha}(f\xrightarrow{h}f')$ be the morphism

$$\exp\left(2\pi i \int_{[0,1]\times Y} \bar{h}^*\alpha\right) : I_{Y,\bar{f}^*\alpha} \to I_{Y,\bar{f}'^*\alpha}\,.\tag{2.2}$$

Here the homotopy $h:[0,1] \times Q \to E\Gamma$ has quotient map $\bar{h}:[0,1]\times Y \to B\Gamma$. Since $\partial([0,1]\times Y) = \{1\}\times Y \sqcup -\{0\}\times Y$, the isomorphisms (A.6), (A.8), and (1.3) identify the integral (2.2) as a map between the spaces shown. The gluing law (A.10) applied to gluings of cylinders shows that $\mathscr{F}_{Q;\alpha}$ is indeed a functor. An automorphism $f \xrightarrow{h} f$ determines a classifying map $h:S^1 \times Q \to E\Gamma$, by gluing, and so extends to a classifying map $H:D^2\times Q \to E\Gamma$. Then $\bar{h}:S^1\times Y \to E\Gamma$ extends to $\bar{H}:D^2\times Y \to E\Gamma$, and by Stokes' theorem (A.11) the morphism $\mathscr{F}_{Q;\alpha}(f\xrightarrow{h}f)$ acts trivially. So there is an inverse limit of $\mathscr{F}_{Q;\alpha}$ in $\mathscr{T}_n$, which we denote $T_Y^\alpha(Q) = T_Y(Q)$. (We omit the "α" if it is understood from the context.) It should be thought of as the value of the classical action on Q.

Now suppose X is a compact oriented $(d+2-n)$-manifold, possibly with boundary, and $P \in \mathscr{C}_X$ is a Γ bundle over X. Let $\mathscr{C}_P$ be the category of classifying maps $F:P \to E\Gamma$ and homotopies $F\xrightarrow{h}F'$. Restriction to the boundary defines a functor $\mathscr{C}_P \xrightarrow{\partial} \mathscr{C}_{\partial P}$. If $F \in \mathscr{C}_P$ then by integration we obtain

$$\exp\left(2\pi i \int_X \bar{F}^*\alpha\right) \in I_{\partial X, \partial\bar{F}^*a} = \mathscr{F}_{\partial P;\alpha}(\partial F)\,.\tag{2.3}$$

Furthermore, one can check that if $F\xrightarrow{H}F'$ is a homotopy, then (A.11) implies that

$$\mathscr{F}_{\partial P;\alpha}(\partial F \xrightarrow{\partial H} \partial F')\exp\left(2\pi i\int_X \bar{F}^*\alpha\right) = \exp\left(2\pi i\int_X \bar{F}'^*\alpha\right).$$

These equations imply that (2.3) determines an element

$$e^{2\pi i S_X(P)} \in T_{\partial X}(\partial P)\,.\tag{2.4}$$

We state the properties of this action without proof.

Assertion 2.5. *Let Γ be a finite group and $\alpha \in C^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$ a cocycle. Then the assignments* [9]

$$\begin{aligned} Q \mapsto T_Y(Q) \in \mathscr{T}_n\,, \qquad\qquad Q \in \mathscr{C}_Y\,, \\ P \mapsto e^{2\pi i S_X(P)} \in T_{\partial X}(\partial P)\,, \quad P \in \mathscr{C}_X \end{aligned}\tag{2.6}$$

defined above for closed oriented $(d + 1 - n)$-manifolds Y and compact oriented $(d + 2 - n)$-manifolds X satisfy:

---

[9] It is possibly better notation to write $e^{2\pi i S_X(P)} \in e^{2\pi i S_{\partial X}(\partial P)}$ for *any* compact oriented X, or perhaps instead $T_X(P) \in T_{\partial X}(\partial P)$. We will sometimes use the latter notation, especially in Sect. 9

(a) *(Functoriality) If $\psi : Q' \to Q$ is a bundle map covering an orientation preserving diffeomorphism $\bar{\psi} : Y' \to Y$, then there is an induced isomorphism*

$$\psi_* : T_Y(Q') \to T_Y(Q) \tag{2.7}$$

and these compose properly. If $\varphi : P' \to P$ is a bundle map covering an orientation preserving diffeomorphism $\bar{\varphi} : X' \to X$, then there is an induced isomorphism [10]

$$(\partial\varphi)_*(e^{2\pi i S_{X'}(P')}) \to e^{2\pi i S_X(P)}, \tag{2.8}$$

where $\partial\varphi : \partial P' \to \partial P$ is the induced map over the boundary.
(b) *(Orientation) There are natural isomorphisms*

$$T_{-Y}(Q) \cong (T_Y(Q))^{-1}, \tag{2.9}$$

and

$$e^{2\pi i S_{-X}(P)} \cong (e^{2\pi i S_X(P)})^{-1}. \tag{2.10}$$

(c) *(Additivity) If $Y = Y_1 \sqcup Y_2$ is a disjoint union, and Q_i are bundles over Y_i, then there is a natural isomorphism*

$$T_Y(Q_1 \sqcup Q_2) \cong T_Y(Q_1) \cdot T_Y(Q_2).$$

If $X = X_1 \sqcup X_2$ is a disjoint union, and P_i are bundles over X_i, then there is a natural isomorphism

$$e^{2\pi i S_{X_1 \sqcup X_2}(P_1 \sqcup P_2)} \cong e^{2\pi i S_{X_1}(P_1)} \cdot e^{2\pi i S_{X_2}(P_2)} \tag{2.11}$$

(d) *(Gluing) Suppose $Y \hookrightarrow X$ is a closed oriented codimension one submanifold and X^{cut} is the manifold obtained by cutting X along Y. Then $\partial X^{\mathrm{cut}} = \partial X \sqcup Y \sqcup -Y$. Suppose P is a bundle over X, P^{cut} the induced bundle over X^{cut}, and Q the restriction of P to Y. Then there is a natural isomorphism*

$$\mathrm{Tr}_Q(e^{2\pi i S_{X^{\mathrm{cut}}}(P^{\mathrm{cut}})}) \to e^{2\pi i S_X(P)}, \tag{2.12}$$

where Tr_Q is the contraction

$$\mathrm{Tr}_Q : T_{X^{\mathrm{cut}}}(\partial P^{\mathrm{cut}}) \cong T_X(\partial P) \cdot T_Y(Q) \cdot T_Y(Q)^{-1} \to T_X(\partial P).$$

The Functoriality Axiom (a) means in particular that for any $Q \in \mathscr{C}_Y$ there is an action of the finite group $\mathrm{Aut}\, Q$ on $T_Y(Q)$. As explained in Sect. 1 the isomorphism class of this action is an element of $H^n(\mathrm{Aut}\, Q; \mathbb{T})$. For $n = 2$ this action determines a central extension of $\mathrm{Aut}\, Q$ by $\mathbb{T}$. We use an additional property of gluing in Sect. 9: Iterated gluings commute. As always, we must interpret "commute" appropriately in categories.

---

[10] If $n = 1$ then (2.8) is an equality of elements in a $\mathbb{T}$-torsor. Similarly for (2.10), (2.11), and (2.12)

3. Higher Algebra II

The quantum integration process is this: We integrate the classical action over the space of equivalence classes of fields on some manifold. As explained in Sect. 2 the classical action in codimension n takes values in $\mathscr{S}_n$ (or in a $\mathscr{S}_n$-torsor for manifolds *with boundary*). *For example, in the top dimension it takes values in* $\mathbb{T}$. *But we cannot* add elements of $\mathbb{T}$. Rather, to form the quantum path integral we embed $\mathbb{T} \hookrightarrow \mathbb{C}$ and add up the values of the classical action as complex numbers. In higher codimensions we introduce "higher inner product spaces" where we can perform the sum. [11] The collection $\mathscr{V}_n$ of all *complex n-inner product spaces* [12] is an n-category, which is in some sense the trivial complex $(n+1)$-inner product space, and there is an embedding $\mathscr{S}_n \hookrightarrow \mathscr{V}_n$ onto the set of elements of "unit norm." We view the action as taking values in $\mathscr{V}_n$ and then take sums there to perform the path integral. Our goal in this section, then, is to describe this hierarchy:

$$
\begin{aligned}
\mathscr{V}_0 &= \mathbb{C} \quad \text{field of complex numbers}, \\
\mathscr{V}_1 &\quad \text{"ring" of (virtual) finite dimensional complex} \\
&\quad\quad \text{inner product spaces}, \\
\mathscr{V}_2 &\quad \text{"ring" of (virtual) finite dimensional complex} \\
&\quad\quad \text{2-inner product spaces},
\end{aligned}
\tag{3.1}
$$

etc.

The inner product space notions of dual space (or conjugate space), direct sum, and tensor product generalize to $\mathscr{V}_n$, and this gives it a structure analogous to a commutative ring with involution.

The notion of a 2-vector space appears in work of Kapranov and Voevodsky [KV], and also in lectures of Kazhdan and in recent work of Lawrence [L]. We in no way claim to have worked out the category theory in detail, and we feel that this sort of "higher linear algebra" merits further development.

The terminology is confusing: An n-inner product space is an $(n - 1)$-category. Thus a 2-inner product space is an ordinary category.

Recall that an inner product space V is a set with an commutative vector sum $V \times V \to V$, a scalar multiplication $\mathbb{C} \times V \to V$, and an inner product $(\cdot, \cdot) : V \times \bar{V} \to \mathbb{C}$. (The conjugate inner product space $\bar{V}$ is defined below.) We will not review all of the axioms here. There are two trivial examples: the zero inner product space O consisting of one element, and $\mathbb{C}$ with its usual inner product $(z, w) = z \cdot \bar{w}$. If V_1, V_2 are inner product spaces, then a *morphism* is a linear map $V_1 \to V_2$ which preserves the inner product. The collection of inner product spaces and linear maps forms a category $\mathscr{V}_1$.

Suppose $T \in \mathscr{S}_1$ is a $\mathbb{T}$-torsor. From T we form the one dimensional complex inner product space (hermitian line)

$$
\begin{aligned}
L_T &= T \times_{\mathbb{T}} \mathbb{C} \\
&= \{\langle t, z \rangle \in T \times \mathbb{C}\} / \langle t \cdot \lambda, z \rangle \sim \langle t, \lambda \cdot z \rangle
\end{aligned}
\tag{3.2}
$$

---

[11] Since our basic group is $\mathbb{T}$ (as opposed to $\mathbb{C}^\times$) we obtain complex inner product spaces (as opposed to simply complex vector spaces). Presumably one can generalize to other base fields or rings

[12] It is probably better to consider the category of *virtual* complex n-inner product spaces, that is, formal differences of complex n-inner product spaces. This provides additive inverses and is more closely analogous to a ring. However, we will only encounter "positive" elements of this "ring" so do not insist on the inclusion of virtual inner product spaces

for all $\lambda \in \mathbb{T}$. Note that $L_{\mathbb{T}} \cong \mathbb{C}$. The inner product on L_T is

$$(\langle t, z \rangle, \langle t, w \rangle) = z \cdot \bar{w} \,.$$

If $V \in \mathscr{V}_1$ is an inner product space, we form the *dual space* $V^* = \mathrm{Hom}(V, \mathbb{C})$ with its usual inner product. The *conjugate* inner product space $\bar{V}$ has the same underlying abelian group as V but the conjugate scalar multiplication and the transposed inner product. There is a natural isometry $\bar{V} \cong V^*$ given by the inner product. If $V_1, V_2 \in \mathscr{V}_1$ then one can form the *direct sum* $V_1 \oplus V_2$ and the *tensor product* $V_1 \otimes V_2$ with the inner products

$$(v_1 \oplus v_2, w_1 \oplus w_2) = (v_1, w_1) + (v_2, w_2) \,,$$
$$(v_1 \otimes v_2, w_1 \otimes w_2) = (v_1, w_1)(v_2, w_2) \,.$$

Notice that there are natural isomorphisms $O \oplus V \cong V$ and $\mathbb{C} \otimes V \cong V$. Also, if $T_1, T_2 \in \mathscr{V}_1$ then $L_{T^{-1}} \cong L_T^*$ and $L_{T_1 \cdot T_2} \cong L_{T_1} \otimes L_{T_2}$. The direct sum and tensor product give $\mathscr{V}_1$ a *commutative ring-like*[13] *structure with involution*, the involution being the conjugation or duality.

It is useful to observe that for any inner product space V, the induced inner product on $V^* \otimes V$ is

$$(T_1, T_2) = \mathrm{Tr}(T_1 T_2^*) \,, \qquad T_i \in \mathrm{Hom}(V) \,,$$

where we identify $V^* \otimes V \cong \mathrm{Hom}(V)$ via the canonical isomorphism, and T^* is the hermitian adjoint of T.

Finally, we introduce an "inner product"

$$(\cdot, \cdot) \colon \mathscr{V}_1 \times \bar{\mathscr{V}}_1 \to \mathscr{V}_1$$

by

$$(V_1, V_2) = V_1 \otimes \bar{V}_2 \,,$$

and the associated "norm" $|V|^2 = V \otimes \bar{V}$. Notice that the elements of "unit norm," that is of norm $\mathbb{C}$, are precisely the hermitian lines, i.e., the image of the embedding $\mathscr{T}_1 \hookrightarrow \mathscr{V}_1$. The image is closed under tensor product and the embedding is a homomorphism.

Starting with the field $\mathbb{C}$ we have outlined the construction of a commutative ring-like category $\mathscr{V}_1$ (with involution) consisting of inner product spaces over $\mathbb{C}$. Now we iterate and consider inner product spaces over $\mathscr{V}_1$, which we call *complex 2-inner product spaces*.[14] So a complex 2-inner product space $\mathscr{W}$ is a category with an abelian group law $\mathscr{W} \times \mathscr{W} \to \mathscr{W}$, a "scalar multiplication" $\mathscr{V}_1 \times \mathscr{W} \to \mathscr{W}$, and an "inner product" $\mathscr{W} \times \bar{\mathscr{W}} \to \mathscr{V}_1$. There is a zero complex 2-inner product space O. The dual, conjugate, direct sum, and tensor product are defined. The category $\mathscr{V}_1$ is a 2-inner product space which is an identity element for the tensor product. The collection of all (virtual) complex 2-inner product spaces forms a commutative ring-like 2-category $\mathscr{V}_2$ with involution.

Because a 2-inner product space is a category, and not a set, there is an extra layer of structure (natural transformations) and so additional data as part of the definition.

---

[13] As we mentioned above, we should include *virtual* inner product spaces to have additive inverses
[14] Since $\mathscr{V}_1$ is analogous to a ring, not a field, we expect that not all of its modules are free. The ones we consider in this paper are sums of one dimensional cyclic modules, so are free. A formal development of this concept should probably demand freeness in the definition [KV]

We do not claim to have a complete list, but mention some additional structure related to the inner product. Namely, for all $W_1, W_2 \in \mathscr{W}$ there is a specified map

$$(W_2, W_1) \cdot W_1 \to W_2 \,.$$

The "$\cdot$" here is the scalar product. We might further assume that $\mathrm{Mor}(W_1, W_2)$ is isomorphic to the vector space (W_2, W_1); this holds in the examples. In addition, we postulate a preferred isometry

$$(W_1, W_2) \to \overline{(W_2, W_1)}$$

whose "square" is the identity. In particular, (W, W) has a real structure for all $W \in \mathscr{W}$, and we assume the existence of compatible maps

$$\mathbb{C} \to (W, W) \to \mathbb{C} \,. \tag{3.3}$$

The composition is then multiplication by a real number, which we call $\dim W$.

A *linear map* of complex 2-inner product spaces $L \colon \mathscr{W}_1 \to \mathscr{W}_2$ is a functor which preserves the addition and scalar multiplication. The space of all linear maps is the 2-inner product space $\mathrm{Hom}(\mathscr{W}_1, \mathscr{W}_2) \cong \mathscr{W}_2 \otimes \mathscr{W}_1^*$. If we assume some freeness condition on 2-inner product spaces (see previous footnote), then we can clearly generalize other standard notions of linear algebra. For example, we should be able to define linear independence and bases. Then if $P \colon \mathscr{W} \to \mathscr{W}$ is a linear operator on $\mathscr{W}$, a matrix representation relative to a basis of $\mathscr{W}$ is a matrix of inner product spaces $P^i_j \in \mathscr{V}_1$. The *trace* $\mathrm{Tr}(P) = \bigoplus_i P^i_i$ is then an inner product space. The *dimension* of $\mathscr{W}$ is the trace of the identity map, which is $\dim \mathscr{W} = \mathbb{C}^n$ for some n. It makes sense, then, to identify the dimension of $\mathscr{W}$ as n.

If $\mathscr{G}$ is a $\mathbb{T}$-gerbe. Then we form the one dimensional complex 2-inner product space

$$\begin{aligned}
\mathscr{W}_{\mathscr{G}} &= \mathscr{G} \times_{\mathscr{T}_1} \mathscr{V}_1 \\
&= \{\langle G, V \rangle \in \mathscr{G} \times \mathscr{V}_1\} / \langle G \cdot T, V \rangle \sim \langle G, L_T \otimes V \rangle
\end{aligned} \tag{3.4}$$

for all $\mathbb{T}$-torsors T. Note that $\mathscr{W}_{\mathscr{T}_1} \cong \mathscr{V}_1$. If we define the inner product

$$(\mathscr{W}_1, \mathscr{W}_2) = \mathscr{W}_1 \otimes \overline{\mathscr{W}_2}$$

on $\mathscr{V}_2$, then we see that the image of the embedding $\mathscr{T}_2 \hookrightarrow \mathscr{V}_2$ determined by (3.4) consists of complex 2-inner product spaces of "unit norm". The image is closed under tensor product and the embedding is a homomorphism.

Here is a more concrete example of a nontrivial 2-inner product space which is important in what follows. Suppose A is a finite group. Let $(\mathscr{V}_1)^A$ denote the category of finite dimensional unitary representations of A. The morphisms are required to commute with the A action. Then $(\mathscr{V}_1)^A$ is a 2-inner product space as follows. If $W \in (\mathscr{V}_1)^A$ and $V \in \mathscr{V}_1$ then we can "scalar multiply" V by W using the ordinary vector space tensor product. We obtain $V \otimes W$, which is a representation of A. The vector sum in $(\mathscr{V}_1)^A$ is the usual direct sum of representations. The inner product on $(\mathscr{V}_1)^A$ is

$$(W_1, W_2) = (W_1 \otimes \overline{W_2})^A \,, \tag{3.5}$$

where for any representation $W \in (\mathscr{V}_1)^A$ the inner product space $W^A \in \mathscr{V}_1$ is the subspace of *invariants*. Note that if W is an *irreducible* unitary representation of A, then (cf. [FQ, Appendix A])

$$(W, W) = \dim W \cdot \mathbb{C} \,,$$

since dim W is the norm square of the canonical element of $W \otimes \bar{W}$. The composition
(3.3) is dim W in the usual sense. The dimension of $(\mathscr{V}_1)^A$ is the number of
isomorphism classes of irreducible representations of A.

More generally, suppose that $\mathscr{G}$ is a $\mathbb{T}$-gerbe with a nontrivial A action, which we
denote by ϱ. For any $G \in \mathscr{G}$ let

$$L_G = \langle G, \mathbb{C} \rangle \in \mathscr{W}_{\mathscr{G}} . \qquad (3.6)$$

[Recall the definition of $\mathscr{W}_{\mathscr{G}}$ in (3.4).] Note that for any line $L \in \mathscr{V}_1$, the element
$\langle G, L \rangle \in \mathscr{W}_{\mathscr{G}}$ is equivalent to $L_{G'}$ for some $G' \in \mathscr{G}$, and so any element of $\mathscr{W}_{\mathscr{G}}$ is
isomorphic to a finite sum $L_{G_1} \oplus \ldots \oplus L_{G_k}$. Let A act on $\mathscr{W}_{\mathscr{G}}$ by

$$a \cdot L_G = L_{a \cdot G} , \qquad (3.7)$$

Finally, set [15]

$$(\mathscr{W}_{\mathscr{G}})^{A, \varrho} = \mathrm{span}\{W = L_{G_1} \oplus \ldots \oplus L_{G_k} : W \text{ is invariant under the } A \text{ action}\} . \quad (3.8)$$

This is our sought-after 2-inner product space. If $\mathscr{G} = \mathscr{T}_1$ is the trivial $\mathbb{T}$-gerbe, then
according to (1.4) the action ϱ determines a central extension $\tilde{A}$ of A by $\mathbb{T}$, and to
each $a \in A$ corresponds a $\mathbb{T}$-torsor T_a which is the fiber of $\tilde{A}$ over a. We can describe
$(\mathscr{W}_{\mathscr{T}_1})^{A, \varrho} = (\mathscr{V}_1)^{A, \varrho}$ as the category of representations of $\tilde{A}$ such that the central $\mathbb{T}$
acts by standard scalar multiplication. For then an element of this category is of the
form $W = L_{T_1} \oplus \ldots \oplus L_{T_k}$, where for each $a \in A$ and each index i there is an
index j with $T_a \cdot T_i = T_j$. Then (3.7) is an isometry $L_{T_a} \otimes L_{T_i} \to L_{T_j}$, and so each
$\tilde{a} \in T_a$ induces an isometry $L_{T_i} \to L_{T_j}$. This describes the $\tilde{A}$ action. The dimension
of $(\mathscr{W}_{\mathscr{T}_1})^{A, \varrho}$ as the number of isomorphism classes of such irreducible representations.
Since any $\mathbb{T}$ gerbe $\mathscr{G}$ is (noncanonically) isomorphic to $\mathscr{T}_1$, this is also the dimension
of $(\mathscr{W}_{\mathscr{G}})^{A, \varrho}$. If ϱ is the trivial A action on $\mathscr{T}_1$, then $(\mathscr{W}_{\mathscr{T}_1})^{A, \varrho} = (\mathscr{V}_1)^{A, \varrho}$ is the 2-inner
product space $(\mathscr{V}_1)^A$ we defined in the previous paragraph.

Think of $(\mathscr{W}_{\mathscr{T}_1})^{A, \varrho}$ as the space of A-invariants in $\mathscr{W}_{\mathscr{G}}$. We can also consider
invariants in the analogous situation "one dimension down." That is, if A acts on a
$\mathbb{T}$-torsor T through a character $\mu : A \to \mathbb{T}$, then A also acts on the hermitian line L_T
through the same character. We define

$$(L_T)^{A, \mu} = \{\ell \in L_t : \ell \text{ is invariant under the } A \text{ action}\} .$$

But this is simple:

$$(L_T)^{A, \mu} = \begin{cases} L_T , & \text{if } \mu \text{ is trivial}; \\ 0, & \text{otherwise}. \end{cases}$$

We remark that whereas $(\mathscr{V}_1)^A$ has a natural *monoidal* structure [16] given by the
tensor product of representations, the category $(\mathscr{V}_1)^{A, \varrho}$ for ϱ nontrivial do not: the
tensor product of representations of $\tilde{A}$ where $\mathbb{T}$ acts as scalar multiplication is a

---

[15] To make good sense of "invariant" we must identify certain *canonically* isomorphic elements.
For example, we need to identify different permutations of the sum $L_{G_1} \oplus \ldots \oplus L_{G_k}$. Also, this
definition is suspicious – the dimension of the invariants is larger than the dimension of $\mathscr{W}_{\mathscr{G}}$
[16] A *monoidal category* is a category equipped with a tensor product and an identity element. In
addition, an "associator" and natural transformations related to the identity element must be specified
explicitly. A monoidal category is the category-theoretic analogue of a monoid, which is a set with
an associative composition law and an identity element

representation of $\tilde{A}$ where $\mathbb{T}$ acts as the *square* of scalar multiplication. Also, if $\mathscr{G}$ is a nontrivial gerbe, then $(\mathscr{W}_{\mathscr{G}})^{A,\varrho}$ is not monoidal in a natural way.

Finally, by forgetting the A action we obtain an "augmentation" linear map

$$(\mathscr{W}_{\mathscr{G}})^{A,\varrho} \to \mathscr{W}_{\mathscr{G}} \, .$$

If $\mathscr{G} = \mathscr{T}_1$ is trivial, it takes values in $\mathscr{W}_{\mathscr{T}_1} = \mathscr{V}_1$.

Clearly these constructions have analogs in the higher complex inner product spaces (3.1).

4. Quantum Theory

Now we are ready to quantize the classical $d + 1$ dimensional classical field theory described in Sect. 2. We carry out the quantization on any compact oriented manifold of dimension less than or equal to $d + 1$ by integrating the classical action over the space of fields. (We first use the constructions in Sect. 3 to convert the values of the classical action from an n-torsor to an n-inner product space.) Since there are symmetries of the fields, we only integrate over *equivalence classes* of fields. The residual symmetry, that is, the automorphism groups of the fields, must also be taken into account. Since the gauge group is finite, the space of equivalence classes of fields on a compact manifold is a *finite set*, so all we need to perform the path integral is a measure on this finite set. We also need to define the product of a positive number μ (the measure) by an element $\mathscr{W} \in \mathscr{V}_n$. This we denote as $\mu \cdot \mathscr{W}$ and interpret it as $\mathscr{W}$ with the inner product multiplied by μ. The rest is a straightforward generalization of [FQ, Sect. 2], given the higher algebra of Sect. 3 and the classical theory of Sect. 2. For a closed oriented $(d + 1 - n)$-manifold Y, $n > 0$, the resulting quantum invariant is a complex n-inner product space $E(Y) \in \mathscr{V}_n$. If $Y = \emptyset$ is the empty manifold, then $E(\emptyset) = \mathscr{V}_{n-1}$ is the trivial space. The quantum invariant of a compact oriented $(d + 2 - n)$-manifold X, possibly with boundary, is an element $Z_X \in E(\partial X)$. For $n = 1$ we recover the quantum invariants of [FQ, Sect. 2], – the ordinary path integral (partition function) and the quantum Hilbert space. For $n = 2$ the quantum invariant of a closed oriented $(d-1)$-manifold S is a 2-inner product space $E(S)$, and the quantum invariant of a compact oriented d-manifold Y is an object Z_Y in the category $E(\partial Y)$. *Et cetera.*

We first introduce a measure μ on the category of principal Γ bundles $\mathscr{C}_M$ over any manifold M. For $P \in \mathscr{C}_M$ set,

$$\mu_P = \frac{1}{\#\operatorname{Aut} P} \, . \tag{4.1}$$

Clearly $\mu_{P'} = \mu_P$ for equivalent bundles $P' \cong P$, so μ determines a measure on the set of equivalence classes $\overline{\mathscr{C}_M}$. This is the assertion that the measure is invariant under the symmetries of the fields.

If M has a boundary, for each $Q \in \mathscr{C}_{\partial M}$ set

$$\mathscr{C}_M(Q) = \{\langle P, \theta \rangle : P \in \mathscr{C}_M, \theta : \partial P \to Q \text{ is an isomorphism}\} \, . \tag{4.2}$$

A morphism $\varphi : \langle P', \theta' \rangle \to \langle P, \theta \rangle$ is an isomorphism $\varphi : P' \to P$ such that $\theta' = \theta \circ \partial \varphi$. The morphisms define an equivalence relation on $\mathscr{C}_M(Q)$, and we denote the set of equivalence classes by $\overline{\mathscr{C}_M}(Q)$. Equation (4.1) determines a measure on $\mathscr{C}_M(Q)$. Note that any automorphism of $\langle P, \theta \rangle \in \mathscr{C}_M(Q)$ is the identity on components of M with

nontrivial boundary. If $\psi : Q' \to Q$ is an isomorphism of Γ bundles over ∂M, then ψ induces a measure-preserving map

$$\psi_* : \overline{\mathscr{C}_M}(Q') \to \overline{\mathscr{C}_M}(Q)$$

by $\psi_*(P, \theta) = \langle P, \psi\theta \rangle$. In particular, for $Q' = Q$ this gives a measure-preserving action of $\operatorname{Aut} Q$ on $\overline{\mathscr{C}_M}(Q)$.

One important property of μ, which is an ingredient in the proof of the gluing law (4.17), is its behavior under cutting and pasting. Suppose $N \hookrightarrow M$ is an oriented codimension one submanifold and M^{cut} the manifold obtained by cutting M along N. For each $Q \in \mathscr{C}_N$, $Q' \in \mathscr{C}_{\partial M}$, we obtain a gluing map

$$g_Q : \overline{\mathscr{C}_{M^{\mathrm{cut}}}}(Q \sqcup Q \sqcup Q') \to \overline{\mathscr{C}_M}(Q')$$

$$\langle P^{\mathrm{cut}}; \theta_1, \theta_2, \theta \rangle \mapsto \langle P^{\mathrm{cut}}/(\theta_1 = \theta_2); \theta \rangle .$$

We refer to [FQ, Sect. 2] for the proof of the following.

Lemma 4.3. *The gluing map g_Q satisfies:*
(a) *g_Q maps onto the set of equivalence classes of bundles over M whose restriction to N is isomorphic to Q.*
(b) *Let $\phi \in \operatorname{Aut} Q$ act on $\langle P^{\mathrm{cut}}; \theta_1, \theta_2, \theta \rangle \in \mathscr{C}_{M^{\mathrm{cut}}}(Q \sqcup Q)$ by*

$$\phi \cdot \langle P^{\mathrm{cut}}; \theta_1, \theta_2, \theta \rangle = \langle P^{\mathrm{cut}}; \phi \circ \theta_1, \phi \circ \theta_2, \theta \rangle .$$

Then the stabilizer of this action at $\langle P^{\mathrm{cut}}; \theta_1, \theta_2, \theta \rangle$ is the image $\operatorname{Aut} P \to \operatorname{Aut} Q$ determined by the θ_i, where $P = g_Q(\langle P^{\mathrm{cut}}; \theta_1, \theta_2, \theta \rangle)$.
(c) *There is an induced action on equivalence classes $\overline{\mathscr{C}_{M^{\mathrm{cut}}}}(Q \sqcup Q)$, and $\operatorname{Aut} Q$ acts transitively on $g_Q^{-1}([P])$ for any $[P] \in \overline{\mathscr{C}_M}$.*
(d) *For any $[P] \in \overline{\mathscr{C}}_M(Q)$ we have*

$$\mu_{[P]} = \operatorname{vol}(g_Q^{-1}([P])) \cdot \mu_Q . \tag{4.4}$$

Now we are ready to carry out the quantization. We treat all codimensions simultaneously, but suggest that the reader first review the top dimensional quantizations in [FQ, Sect. 2]. Again for clarity we first treat closed manifolds and then arbitrary compact manifolds (possibly with boundary), though the second case includes the first.

Suppose first that Y is a closed oriented $(d+1-n)$-manifold, $n > 0$. The classical action defined in Sect. 2 is a map

$$T_Y : \mathscr{C}_Y \to \mathscr{T}_n ,$$

which we can think of as a bundle of "n-torsors" over $\mathscr{C}_Y$. By Assertion 2.5(a) for each $Q \in \mathscr{C}_Y$ there is an action ϱ_Q of $\operatorname{Aut} Q$ on $T_Y(Q)$. Use the construction (3.2), (3.4) to replace each T_Q by the one dimensional n-inner product space

$$\mathscr{W}_Q = \mathscr{W}_{T_Y(Q)} . \tag{4.5}$$

Assertion 2.5(a) also implies that an isomorphism $\psi : Q' \to Q$ induces an isomorphism $\psi_* : \mathscr{W}_{Q'} \to \mathscr{W}_Q$. However, an automorphism $\psi \in \operatorname{Aut} Q$ does not necessarily act trivially on $\mathscr{W}_Q$. Rather, it only acts trivially on the subspace of invariants under the $\operatorname{Aut} Q$ action [cf. (3.8)]. More precisely, we construct a "quotient" complex n-inner product space $\mathscr{W}_{[Q]}$ associated to the *equivalence class* $[Q] \in \overline{\mathscr{C}_Y}$ as an inverse

limit. (The inverse limit picks out the invariants under automorphisms.) Consider the category $\mathscr{C}_{[Q]}$ of bundles Q in the isomorphism class $[Q]$, and let $\mathscr{F}_{[Q]} : \mathscr{C}_{[Q]} \to \mathscr{V}_n$ be the functor whose value at Q is $\mathscr{W}_Q$. Set $\mathscr{W}_{[Q]}$ to be the inverse limit of $\mathscr{F}_{[Q]}$. As $[Q]$ varies we then obtain a map

$$\mathscr{W}_Y : \overline{\mathscr{C}_Y} \to \mathscr{V}_n.$$

The quantum space $E(Y)$ is the integral of $\mathscr{W}_Y$ over $\overline{\mathscr{C}_Y}$, which in this case is a finite sum:

$$E(Y) = \int_{\overline{\mathscr{C}_Y}} d\mu([Q]) \mathscr{W}_Y([Q]) = \bigoplus_{[Q] \in \overline{\mathscr{C}_Y}} \mu[Q] \cdot \mathscr{W}_{[Q]} \in \mathscr{V}_n. \qquad (4.6)$$

If we think of $\mathscr{W}_Y$ as a bundle of n-inner product spaces over $\overline{\mathscr{C}_Y}$, then $E(Y)$ is the space of L^2 sections of that bundle.

Now suppose that X is a compact oriented $(d + 2 - n)$-manifold, possibly with boundary. The classical action on the boundary ∂X is a bundle of n-torsors $T_{\partial X} \to \mathscr{C}_{\partial X}$, and the classical action $e^{2\pi i S_X}$ on X is a section of the pullback $r^* T_{\partial X}$, where r is restriction to the boundary:

$$
\begin{array}{ccc}
r^* T_{\partial X} & \longrightarrow & T_{\partial X} \\
\downarrow & & \downarrow \\
\mathscr{C}_X & \xrightarrow{\ r\ } & \mathscr{C}_{\partial X}
\end{array}
.
$$

By Assertion 2.5(a) the action is invariant under the morphisms in $\mathscr{C}_X$, that is, under symmetries of the fields. Now for each $P \in \mathscr{C}_X$ we use the construction (3.6) to define an element

$$L_X(P) = L_{e^{2\pi i S_X(P)}} \in \mathscr{W}_{\partial P} = \mathscr{W}_{T_{\partial X}(\partial P)}. \qquad (4.7)$$

Now $L_X(P)$ is not necessarily invariant under $\operatorname{Aut} P$; it transforms under $\psi \in \operatorname{Aut} P$ according to the action of the restricted automorphism $\partial \psi \in \operatorname{Aut}(\partial P)$ on $\mathscr{W}_{T_{\partial X}(\partial P)}$. We only obtain invariance after integrating. Thus fix $Q \in \mathscr{C}_{\partial X}$ and consider $\mathscr{C}_X(Q)$ as defined in (4.2). If $\langle P, \theta \rangle \in \mathscr{C}_X(Q)$ then using θ to identify $T_{\partial X}(\partial P) \cong T_{\partial X}(Q)$ we have the action $e^{2\pi i S_X(P,\theta)} \in T_{\partial X}(Q)$ and the associated $L_X(P, \theta) \in \mathscr{W}_Q$, as in (4.7). If $\langle P, \theta \rangle \cong \langle P', \theta' \rangle$ then there is an isomorphism between the values of the actions on these fields as elements of $T_{\partial X}(Q)$. By another inverse limit construction we define $L_X([P, \theta]) \in \mathscr{W}_Q$. Set

$$Z_X(Q) = \int_{\overline{\mathscr{C}_X(Q)}} d\mu([P, \theta]) L_X([P, \theta]) = \bigotimes_{[P,\theta] \in \overline{\mathscr{C}_X(Q)}} \mu_{[P,\theta]} \cdot L_X([P]) \in \mathscr{W}_Q. \qquad (4.8)$$

Now we claim that $Z_X(Q)$ is invariant under the $\operatorname{Aut} Q$ action on $\mathscr{W}_Q$, and so

$$Z_X(Q) \in (\mathscr{W}_{T_{\partial X}(Q)})^{\operatorname{Aut} Q, \varrho_Q}. \qquad (4.9)$$

More generally, we check that for an isomorphism $\psi: Q' \to Q$ we have

$$
\begin{aligned}
\psi_* Z_X(Q') &= \bigoplus_{[\langle P', \theta' \rangle]} \mu_{[P']} \cdot \psi_* L_X([P', \theta']) \\
&\cong \bigoplus_{[\langle P', \theta' \rangle]} \mu_{[P']} \cdot L_X([P', \psi\theta']) \\
&= Z_X(Q),
\end{aligned}
\tag{4.10}
$$

since $\langle P', \psi\theta' \rangle$ runs over a set of equivalence classes in $\mathscr{C}_X(Q)$ as $\langle P', \theta' \rangle$ runs over a set of equivalence classes in $\mathscr{C}_X(Q')$. Using the definition (3.8) of $(\mathscr{W}_{T_{\partial X}(Q)})^{\mathrm{Aut}\,Q, \varrho_Q}$ we deduce (4.9), and furthermore (4.10) shows that $\{Z_X(Q): Q \in [Q]\}$ is a collection of elements in $\{\mathscr{W}_Q: Q \in [Q]\}$ invariant under symmetries. In other words, it is an element of the inverse limit $\mathscr{W}_{[Q]}$:

$$
Z_X([Q]) \in \mathscr{W}_{[Q]}.
$$

Finally, then,

$$
Z_X = \bigoplus_{[Q] \in \overline{\mathscr{C}_{\partial X}}} Z_X([Q]) \in \bigoplus_{[Q] \in \overline{\mathscr{C}_{\partial X}}} \mu_{[Q]} \cdot \mathscr{W}_{[Q]} = E(\partial X)
\tag{4.11}
$$

is the desired quantum invariant.

The basic properties of these quantum invariants, which we might term "higher quantum Hilbert spaces" and "higher path integrals," are listed in the following.

Assertion 4.12. *Let Γ be a finite group and $\alpha \in C^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$ a cocycle. Then the assignments* [17]

$$
\begin{aligned}
Y &\mapsto E(Y) \in \mathscr{V}_n, \\
X &\mapsto Z_X \in E(\partial X),
\end{aligned}
$$

defined above for closed oriented $(d + 1 - n)$-manifolds Y and compact oriented $(d + 2 - n)$-manifolds X satisfy:
(a) (Functoriality) Suppose $f: Y' \to Y$ is an orientation preserving diffeomorphism. Then there is an induced isometry

$$
f_*: E(Y') \to E(Y)
\tag{4.13}
$$

and these compose properly. If $F: X' \to X$ is an orientation preserving diffeomorphism, then there is an induced isometry [18]

$$
(\partial F)_*(Z_{X'}) \to Z_X,
\tag{4.14}
$$

where $\partial F: \partial X' \to \partial X$ is the induced map over the boundary.
(b) (Orientation) There are natural isometries

$$
E(-Y) \cong \overline{E(Y)},
$$

and

$$
Z_{-X} \cong \overline{Z_X}.
\tag{4.15}
$$

---

[17]   Again the notation is awkward, and possibly it is best to use Z_X for all X and write $Z_X \in Z_{\partial X}$
[18]   If $n = 1$ this is an equality, as are (4.15), (4.16), and (4.17)

(c) *(Multiplicativity) If* $Y = Y_1 \sqcup Y_2$ *is a disjoint union, then there is a natural isometry*

$$E(Y_1 \sqcup Y_2) \cong E(Y_1) \otimes E(Y_2).$$

If $X = X_1 \sqcup X_2$ *is a disjoint union, then there is a natural isometry*

$$Z_{X_1 \sqcup X_2} \cong Z_{X_1} \otimes Z_{X_2}.$$ (4.16)

(d) *(Gluing) Suppose* $Y \hookrightarrow X$ *is a closed oriented codimension one submanifold and* X^{cut} *is the manifold obtained by cutting* X *along* Y. *Write* $\partial X^{\mathrm{cut}} = \partial X \sqcup Y \sqcup -Y$. *Then there is a natural isometry*

$$\mathrm{Tr}_Y(Z_{X^{\mathrm{cut}}}) \to Z_X,$$ (4.17)

where Tr_Y *is the contraction*

$$\mathrm{Tr}_Y : E(\partial X^{\mathrm{cut}}) \cong E(\partial X) \otimes E(Y) \otimes \overline{E(Y)} \to E(\partial X)$$ (4.18)

using the inner product on $E(Y)$.

Just as on the classical level, iterated gluings commute.

Proof. We only comment on the gluing law (d). The proof is formally the same as the one in [FQ, Sect. 2], but we repeat it here anyway. Recall that for a field P over a compact oriented $(d + 2 - n)$-manifold X we have the action $e^{2\pi i S_X}(P) \in T_{\partial X}(\partial P)$ which lives in an n-torsor, and the associated $L_X(P) \in \mathscr{W}_{T_{\partial X}(\partial P)}$ which lives in an n-vector space [cf. (2.4) and (4.7)]. Fix a bundle $Q' \to \partial X$. Then for each $Q \to Y$ and each $P^{\mathrm{cut}} \in \mathscr{C}_{X^{\mathrm{cut}}}(Q' \sqcup Q \sqcup Q)$ we have an isometry

$$L_X(g_Q(P^{\mathrm{cut}})) \cong \mathrm{Tr}_Q(L_{X^{\mathrm{cut}}}(P^{\mathrm{cut}}))$$ (4.19)

by (2.12), where now Tr_Q is the contraction

$$\mathrm{Tr}_Q : \mathscr{W}_{T_{\partial X^{\mathrm{cut}}}(\partial P^{\mathrm{cut}})} \cong \mathscr{W}_{T_{\partial X}(\partial P)} \otimes \mathscr{W}_{T_Y(Q)} \overline{\otimes \mathscr{W}_{T_Y(Q)}} \to \mathscr{W}_{T_{\partial X}(\partial P)}$$

using the inner product on $\mathscr{W}_{T_Y(Q)}$, and g_Q is the gluing map

$$g_Q : \overline{\mathscr{C}_{X^{\mathrm{cut}}}}(Q' \sqcup Q \sqcup Q) \to \overline{\mathscr{C}_X}(Q').$$ (4.20)

Fix $[P] \in \overline{\mathscr{C}_X}(Q')$ and consider $g_Q^{-1}([P])$. By Lemma 4.3(c) the group $\mathrm{Aut}\, Q$ acts transitively on $g_Q^{-1}([P])$. This means that the invariants in the representation

$$\bigoplus_{[P^{\mathrm{cut}}] \in g_Q^{-1}([P])} L_{X^{\mathrm{cut}}}([P^{\mathrm{cut}}])$$ (4.21)

of $\mathrm{Aut}\, Q$ by its diagonal action on $\mathscr{W}_{T_Y(Q)} \times \mathscr{W}_{T_{-Y}(Q)}$ via $\varrho_Q \times \varrho_Q$ are the "constant functions" under the isomorphism (4.19). Then the inner product (3.5) in $(\mathscr{W}_{T_Y(Q)})^{\mathrm{Aut}\, Q, \varrho_Q}$ applied to (4.21) gives

$$\left(\bigoplus_{[P^{\mathrm{cut}}] \in g_Q^{-1}([P])} L_{X^{\mathrm{cut}}}([P^{\mathrm{cut}}]) \right)^{\mathrm{Aut}\, Q} \cong \# g_Q^{-1}([P]) \cdot L_X([P]).$$ (4.22)

Fix a set of representatives $\{Q\}$ for $\mathscr{C}_Y$. Let $\overline{\mathscr{C}_X}(Q')_Q$ denote the equivalence classes of bundles over X whose restriction to ∂X is Q' and to Y is Q [with given

isomorphisms as in (4.2)]. Thus using Eq. (4.4) on the measure and the isometry (4.22) we calculate

$$
Z_X(Q') = \int\limits_{\overline{\mathscr{C}}_X(Q')} d\mu([P]) L_X([P])
$$

$$
= \sum_{Q \in \{Q\}} \int\limits_{\overline{\mathscr{C}}_X(Q')_Q} d\mu([P]) L_X([P])
$$

$$
\cong \sum_{Q \in \{Q\}} \mu_Q \cdot \left[\int\limits_{\overline{\mathscr{C}}_X^{\mathrm{cut}}(Q' \sqcup Q \sqcup Q)} d\mu([P^{\mathrm{cut}}]) \, \mathrm{Tr}_Q(L_{X^{\mathrm{cut}}}([P^{\mathrm{cut}}])) \right]^{\mathrm{Aut}\, Q}
$$

$$
= \sum_{Q \in \{Q\}} \mu_Q \cdot \mathrm{Tr}_Q(Z_{X^{\mathrm{cut}}}(Q' \sqcup Q \sqcup Q))^{\mathrm{Aut}\, Q}
$$

$$
= \mathrm{Tr}_Y(Z_{X^{\mathrm{cut}}}(Q')) .
$$

5. Product Structures

Some form of the following assertion holds: In a $d + 1$ dimensional topological quantum field theory the d-inner product space $E(S^1)$ has the structure of a "higher commutative associative algebra with identity and compatible real structure and inner product." In this section we only discuss the cases $d = 1$ and $d = 2$. For $d = 1$ we obtain an ordinary algebra structure on the vector space $E(S^1)$, together with a compatible real structure. The inner product on $E(S^1)$ is compatible with all of these structures. This is a standard argument, which we repeat here as a warmup. For $d = 2$ the quantum space $E(S^1)$ is a 2-inner product space, which in particular is a category. The algebra structure we discuss gives it the structure of a *braided monoidal category* [JS].[19] Here the commutativity and associativity conditions give additional data (rather than being conditions on the multiplication, as in an ordinary algebra), and there is an additional piece of data coming from nontrivial loops of diffeomorphisms of the circle (a *balancing*). All of the arguments in this section proceed directly from the axioms in Assertion 4.12. So they hold for any theory which obeys these axioms, not just for a gauge theory with finite gauge group.

We begin with some standard deductions about arbitrary $d+1$ dimensional theories. First, a deduction about the classical theory. Suppose Y is a closed oriented manifold and $Q \in \mathscr{C}_Y$ a Γ bundle. Consider the product $[0,1] \times Q \in \mathscr{C}_{[0,1] \times Y}$, which is a bundle over the "cylinder" $[0,1] \times Y$. The classical action[20] $T_{[0,1] \times Y}([0,1] \times Q)$ is an automorphism of $T_Y(Q)$. Now glue two copies of $[0,1] \times Q$ end to end and apply the gluing law (2.12) to construct an isomorphism

$$
T_{[0,1] \times Y}([0,1] \times Q) \cdot T_{[0,1] \times Y}([0,1] \times Q) \mapsto T_{[0,1] \times Y}([0,1] \times Q) .
$$

This implies that there is a canonical element

$$
t \in T_{[0,1] \times Y}([0,1] \times Q)
$$

---

[19]   In fact, we obtain what some refer to as a *tortile category*. See [Y1, Sect. 1], [Y2] for a precise definition and more thorough discussion. The notion of a tortile category is due to Shum [Sh]

[20]   We use the notation $T_X(P)$ instead of $e^{2\pi i S_X(P)}$, even though $X = [0,1] \times Y$ is not closed

which satisfies $t \cdot t = t$. In other words, the classical action of a product field is trivialized. If $\dim Y = d$ the classical action is the identity map of $T_Y(Q)$. The quantum version of (5.1), obtained from the quantum gluing law (4.17), asserts that

$$Z_{[0,1] \times Y} : E(Y) \to E(Y)$$

is an idempotent. In other words, there is an isometry

$$(Z_{[0,1] \times Y})^2 \to Z_{[0,1] \times Y} \, . \tag{5.4}$$

We may as well assume that $Z_{[0,1] \times Y}$ is isometric to the identity, since in any case we can replace $E(Y)$ by the image of (5.3) to obtain a new theory with this property. Similarly, gluing the ends of $[0,1] \times Y$ together we deduce the existence of an isometry

$$Z_{S^1 \times Y} \cong \dim E(Y) \, . \tag{5.5}$$

Here the dimension of an n-inner product space is an $(n-1)$-inner product space, as discussed in Sect. 3. More generally, if $f : Y \to Y$ is an orientation preserving diffeomorphism, we can glue with a twist by f to form the mapping torus $S^1 \times_f Y$. The axioms now imply the existence of an isometry

$$Z_{S^1 \times_f Y} \cong \mathrm{Tr}_{E(Y)}(f_*) \, ,$$

where $f_* : E(Y) \to E(Y)$ is the isometry (4.13).

Another easily deduced property also relates to the functoriality (4.13). Suppose that $f_0, f_1 : Y' \to Y$ are isotopic orientation preserving diffeomorphisms, and that $f_t : Y' \to Y$ is an isotopy. Form the map

$$F : [0,1] \times Y' \to [0,1] \times Y \, ,$$
$$\langle t, y' \rangle \mapsto \langle t, f_t(y') \rangle \, .$$

(More generally, our considerations apply to *pseudoisotopies* F, that is, to arbitrary diffeomorphisms F which restrict on the ends to f_0 and f_1.) Now apply the functoriality axiom (4.14) as follows. The partition functions $Z_{[0,1] \times Y'}$ and $Z_{[0,1] \times Y}$ are the identity, according to (5.3). The boundary maps f_0 and f_1 induce isometries $(f_i)_* : E(Y') \to E(Y)$. The functoriality axiom asserts that F induces an isometry between $(f_1)_* \circ (f_0)_*^{-1}$ and $\mathrm{id}_{E(Y)}$, or equivalently that

$$F \text{ induces an isometry } F_* : (f_0)_* \to (f_1)_* \, . \tag{5.6}$$

The proper interpretation of (5.6) depends on the dimension of Y. For example, if $\dim Y = d$ then $E(Y)$ is an ordinary inner product space and (5.6) asserts an equality $(f_0)_* = (f_1)_*$. This implies in particular that the action of $\mathrm{Diff}^+(Y)$ on $E(Y)$ factors through an action of isotopy classes of diffeomorphisms $\pi_0 \mathrm{Diff}^+(Y)$ on $E(Y)$. If $\dim Y = d - 1$, then $E(Y)$ is a 2-inner product space, which is a category, and (5.6) asserts that F induces a natural transformation F_* between the functors $(f_0)_*$ and $(f_1)_*$. A further argument shows that isotopic maps F induce the same natural transformation. In the particular case where $f_0 = f_1 = \mathrm{id}$, this shows that $\pi_1 \mathrm{Diff}^+(Y)$ acts on $E(Y)$ by automorphisms of the identity functor.[21] This discussion generalizes to higher codimensions.

---

[21] An automorphism of the identity functor (i.e., a natural transformation from the identity functor to itself) on a category $\mathscr{C}$ is for each object $W \in \mathscr{C}$ a choice of morphism $\theta_W : W \to W$ such that if $W \xrightarrow{f} W'$ is any morphism in $\mathscr{C}$, then

$$f \circ \theta_W = \theta_{W'} \circ f$$

Now fix a $1 + 1$ dimensional theory and denote

$$E = E(S^1) \, .$$

Since any orientation-preserving diffeomorphism of S^1 is isotopic to the identity, (5.6) implies that we can uniquely identify $E(S)$ with E for any connected closed oriented 1-manifold S. Also, any two orientation-reversing diffeomorphisms of S^1 are isotopic, so there is a well-determined isometry

$$c \colon E \to \bar{E} \, .$$

Since the composite of two orientation-reversing diffeomorphisms is orientation-preserving, $\bar{c} c = \mathrm{id}$. Thus c defines a real structure on E:

$$E_{\mathbb{R}} = \{ e \in E \colon c(e) = e \} \, . \tag{5.7}$$

Since c is an isometry, $E_{\mathbb{R}}$ is a real inner product space. The inner product identifies $E_{\mathbb{R}} \cong E_{\mathbb{R}}^*$ as usual. Since any compact oriented 2-manifold has an orientation-reversing diffeomorphism, the generalized partition function of any such manifold is real, by (4.14).

Next, we observe that the generalized partition function of the disk

$$\mathbf{1} = Z_{D^2} \in E_{\mathbb{R}}$$

is a special element of $E_{\mathbb{R}}$.

The partition function of the "pair of pants" P, which is a disk with two smaller disks removed (Fig. 2), is an element

$$Z_P \in E_{\mathbb{R}} \otimes E_{\mathbb{R}} \otimes E_{\mathbb{R}} \, . \tag{5.8}$$

Equation (4.14) applied to diffeomorphisms of P which permute the boundary circles (as in Fig. 5) implies that Z_P lives in the *symmetric* triple tensor product of $E_{\mathbb{R}}$. Identifying $E_{\mathbb{R}} \cong E_{\mathbb{R}}^*$ with the inner product, this defines a commutative multiplication $E_{\mathbb{R}} \otimes E_{\mathbb{R}} \to E_{\mathbb{R}}$. In fact, the trilinear form

$$x \otimes y \otimes z \mapsto (x \cdot y, z)_{E_{\mathbb{R}}} \, , \qquad x, y, z \in E_{\mathbb{R}} \, ,$$

dual to (5.8) is totally symmetric. This symmetry is a compatibility condition between the inner product and the multiplication. For the complex vector space $E = E(S^1)$ we have the analogous statement that

$$x \otimes y \otimes z \mapsto (x \cdot y, c(z))_E \, , \qquad x, y, z \in E \, , \tag{5.9}$$

is totally symmetric. Gluing a disk D^2 onto P any applying (4.17) and (5.3) we deduce that $\mathbf{1}$ acts as the identity map for the multiplication. Finally, a standard gluing argument that we do not repeat here shows that the multiplication is associative.

We summarize this discussion in the following.

Proposition 5.10. *In a $1 + 1$ dimensional topological quantum field theory (which satisfies the axioms of Assertion 4.12) the inner product space $E(S^1)$ has a compatible real algebra structure which is commutative, associative, and has an identity. In addition, the map (5.9) is totally symmetric.*

It is not too hard to see that $E = E(S^1)$ contains no nilpotents. For if $x \neq 0$, then since $(xc(x), \mathbf{1}) = (x, x) \neq 0$, we see that $xc(x) \neq 0$. Iterating we find $x^{2^n} c(x)^{2^n} \neq 0$ and $(x^{2^n} c(x)^{2^n}, \mathbf{1}) = (x^{2^n}, x^{2^n}) \neq 0$ for all n. Standard theorems in algebra imply that E contains a basis of idempotents $e_1, \ldots, e_N$, unique up to permutation, with

$e_i e_j = 0$ for $i \neq j$, and that E is a product of one dimensional algebras.[22] It is easy to express the partition function of a closed oriented surface Σ_g of genus g in terms of the norms $\lambda_i^2 = |e_i|^2$:

$$Z_{\Sigma_g} = \sum_i (\lambda_i^2)^{1-g} \, .$$

Now consider a $2 + 1$ dimensional theory, and as before denote $E = E(S^1)$. Here E is a 2-inner product space, so in particular is a category. If $f \colon S \to S^1$ is an orientation-preserving diffeomorphism, then there is an induced linear isometry $f_* \colon E(S) \to E$. Furthermore, any two such $f_0, f_1 \colon S \to S^1$ are homotopic, and a homotopy $F \colon f_0 \to f_1$ induces an isometry $F_* \colon (f_0)_* \to (f_1)_*$, as in (5.6), but now F_* depends on the choice of F. (In the $1 + 1$ dimensional theory F_* is an equality.) In fact, the positive generator of $\pi_1 \mathrm{Diff}^+(S^1) \cong \mathbb{Z}$ induces an automorphism of the identity functor on E, that is, a morphism

$$\theta_W \colon W \to W \tag{5.11}$$

for each object $W \in \mathrm{Obj}(E)$. So we cannot assert that $E(S)$ and E are uniquely isomorphic.

We do need, however, to identify the spaces $E(S)$ for different circles S to derive the "algebra" structure on E, so we resort to the following device in what follows. We use circles S which lie in $\mathbb{C}$. There is a unique composition of translations and homotheties which maps any such circle S to the standard circle $S^1 = \mathbb{T} \subset \mathbb{C}$. We use this to uniquely identify $E(S) \cong E$ for any such S.

As for the automorphism of the identity θ, we can compute it from the diffeomorphism of the cylinder

$$\tau \colon [0,1] \times S^1 \to [0,1] \times S^1 \, ,$$

$$\langle t, s \rangle \mapsto \langle t, s + t \rangle \, ,$$

where here we write $S^1 = \mathbb{R}/\mathbb{Z}$ additively. This glues to a diffeomorphism of the torus $S^1 \times S^1$ described by the matrix

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} . \tag{5.13}$$

By (5.5) we have an isomorphism

$$E(S^1 \times S^1) \cong \dim E \, ,$$

where $\dim E$ is understood as an inner product space, and in some sense the action of (5.13) on $E(S^1 \times S^1)$ is the action of θ on the identity endomorphism of E.

The reflection $s \mapsto -s$ of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ induces an isometry

$$c \colon E \to \bar{E} \tag{5.14}$$

On the underlying category E determines an involution on the objects. Denote

$$c(W) = W^* \, , \qquad W \in \mathrm{Obj}(E) \, .$$

This is the *definition* of "$*$." As in (5.7) we can consider the invariants $E_{\mathbb{R}}$. For any 2-manifold Y there is an isometry $Z_Y \cong Z_Y^*$ determined by any orientation-reversing

---

[22] We need the *complex* algebra since there exists a nontrivial commutative algebra over $\mathbb{R}$, namely $\mathbb{C}$. Note too that the conjugation $\langle z, w \rangle \mapsto \langle \bar{w}, \bar{z} \rangle$ on $E = \mathbb{C} \times \mathbb{C}$ produces $E_{\mathbb{R}} \cong \mathbb{C}$ as an algebra over $\mathbb{R}$. So it is not true in general that the idempotents belong to $E_{\mathbb{R}}$
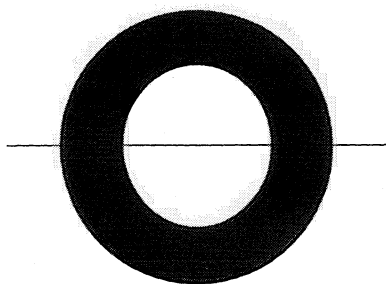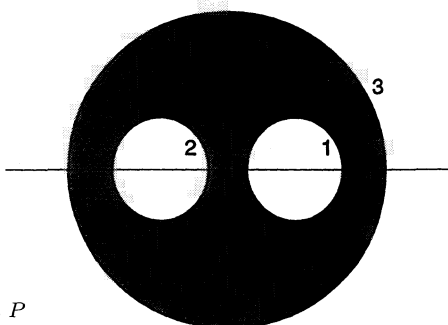
Fig. 1. The cylinder C

Fig. 2. The pair of pants P

diffeomorphism of Y which restricts to r on ∂Y. Of course, this isometry depends on the choice of diffeomorphism, which we will standardize in what follows. Namely, our figures will sit in $\mathbb{C}$, symmetrically about the real axis, and the boundary circles will have centers on that axis. Then reflection about the real axis is our standard orientation-reversing diffeomorphism.

To compute the relationship between c and θ, consider the cylinder C as shown in Fig. 1. The cylinder sits in $\mathbb{C}$, the boundary circles have centers on the real axis, and C is symmetric about the real axis. Now the diffeomorphism (5.12) does not commute with reflection in the real axis, but rather the reflection conjugates it to the diffeomorphism $\langle t, s \rangle \mapsto \langle t, s - t \rangle$. However, since the orientation of the boundary circles are reversed under reflection, this conjugated diffeomorphism represents the positive generator of $\pi_1 \operatorname{Diff}^+ S^1$ for the reflected circle. Thus we conclude that for any $W \in \operatorname{Obj}(E)$,

$$\theta_{W^*} = \theta_W^* , \tag{5.15}$$

Here θ_W^* denotes the image of θ_W under the functor (5.14).

Let D^2 be the unit disk in $\mathbb{C}$. Then

$$\mathbf{1} = Z_{D^2} \in E \tag{5.16}$$

is a distinguished element of E, and reflection in the real axis determines an isometry

$$\mathbf{1} \cong \mathbf{1}^* . \tag{5.17}$$

Fix a standard pair of pants P as shown in Fig. 2. (The ordering of the boundary circles is motivated by Fig. 8.) As with all of our figures it is symmetric about the real axis and the boundary circles have centers on that axis. Any other P' with the same properties is isotopic to P by an isotopy which moves the boundary circles only by translations along the real axis and by homotheties. Furthermore, any two such
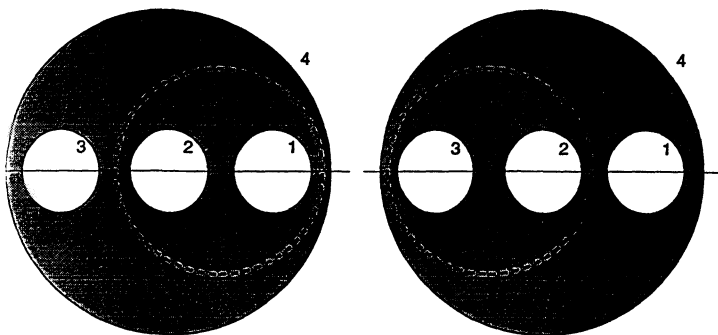
Fig. 3. Associativity

isotopies are isotopic, since any self-diffeomorphism of P which is the identity on ∂P is isotopic to the identity. This means that there is a uniquely defined isotopy $Z_{P'} \cong Z_P$. Now the partition function is

$$Z_P \in E \otimes E \otimes E\,,$$

and reflection about the real axis determines an isometry

$$Z_P \cong Z_P^*\,. \tag{5.18}$$

By duality Z_P determines a map

$$m : E \otimes E \to E\,. \tag{5.19}$$

In particular, m is a functor $E \times E \to E$, but it has linearity properties as well. Denote

$$m(W_1, W_2) = W_1 \odot W_2\,, \qquad W_1, W_2 \in \mathrm{Obj}(E)\,.$$

This is the *definition* of "$\odot$." The isometry (5.18) translates into a natural isometry

$$(W_1 \odot W_2)^* \cong W_1^* \odot W_2^*\,, \qquad W_1, W_2 \in \mathrm{Obj}(E)\,. \tag{5.20}$$

Glue a disk to the inner boundary circles in P to obtain natural isometries

$$\mathbf{1} \odot W \cong W\,, \qquad W \odot \mathbf{1} \cong W\,, \tag{5.21}$$

for all $W \in \mathrm{Obj}(E)$.[23]

It remains to discuss associativity and commutativity. Whereas in the $1 + 1$ dimensional theory these are constraints on the multiplication, here they are new structures which satisfy "higher order" constraints. The associative law is a natural isometry

$$\varphi_{W_1, W_2, W_3} : (W_1 \odot W_2) \odot W_3 \to W_1 \odot (W_2 \odot W_3)\,, \qquad W_1, W_2, W_3 \in \mathrm{Obj}(E)\,, \tag{5.22}$$

obtained from the obvious diffeomorphism indicated in Fig. 3. This figure indicates an isometry between two different contractions of $Z_P \otimes Z_P$, which is equivalent to (5.22). One can think of (5.22) as obtained by gluing and ungluing according to the

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[23] There should also be natural transformations $W \odot W^* \to \mathbf{1}$ and $\mathbf{1} \to W \odot W^*$ which we did not succeed in finding.
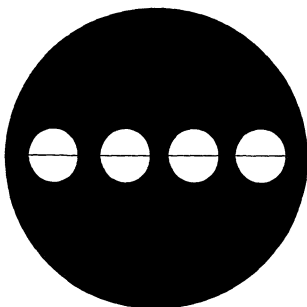**Added in proof:** They are described in [F3]

Fig. 4. Gluings and ungluings of pieces of this surface prove the pentagon

dashed lines in Fig. 3. Performing such gluings and ungluings in Fig. 4 makes obvious the commutativity of the usual pentagon diagram

$$((W_1 \odot W_2) \odot W_3) \odot W_4 \longrightarrow (W_1 \odot W_2) \odot (W_3 \odot W_4)$$

$$W_1 \odot (W_2 \odot (W_3 \odot W_4)) \,.$$

$$(W_1 \odot (W_2 \odot W_3)) \odot W_4 \longrightarrow W_1 \odot ((W_2 \odot W_3) \odot W_4) \qquad (5.23)$$

A similar check shows that

$$(W_1 \odot \mathbf{1}) \odot W_2 \longrightarrow W_1 \odot (\mathbf{1} \odot W_2)$$

$$W_1 \odot W_2 \qquad\qquad (5.24)$$

commutes.

It does not make sense to say that the multiplication (5.19) is commutative. Rather, there is a natural *braiding* isometry

$$R_{W_1, W_2} : W_1 \odot W_2 \to W_2 \odot W_1 \qquad (5.25)$$

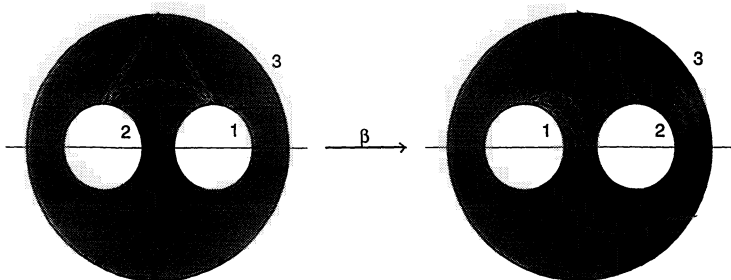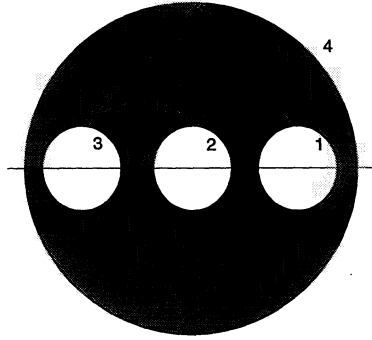**Fig. 5.** The braiding diffeomorphism β

Fig. 6. Surface used to prove hexagon diagrams (5.27) and (5.28)

obtained from the self-diffeomorphism $\beta: P \to P$ indicated in Fig. 5. The auxiliary dashed lines indicate the motion of the boundary circle labeled 2 over that labeled 1. There is a compatibility between the braiding R and the automorphism θ: the diagram

$$
\begin{array}{ccc}
W_1 \odot W_2 & \xrightarrow{\ R_{W_1,W_2}\ } & W_2 \odot W_1 \\[2pt]
{\scriptstyle \theta_{W_1 \odot W_2}}\Big\downarrow & & \Big\downarrow {\scriptstyle \theta_{W_2} \odot \theta_{W_1}} \\[2pt]
W_1 \odot W_2 & \xrightarrow{\ R^{-1}_{W_2,W_1}\ } & W_2 \odot W_1
\end{array}
\qquad (5.26)
$$

commutes for $W_1, W_2 \in \mathrm{Obj}(E)$. (Thus θ is termed "balanced".) This follows from an equation in $\mathrm{Diff}^+(P)$. Namely, let τ_i denote a positive Dehn twist around the boundary labeled i. Then the desired equation is

$$
\tau_2 \tau_1 \beta = \beta^{-1} \tau_3 \,,
$$

which is easily checked using pictures like those in Fig. 5. Similar computations using Fig. 6 show that the hexagon diagrams

$$
\begin{array}{ccccc}
(W_1 \odot W_2) \odot W_3 & \xrightarrow{\ R_{W_1,W_2} \odot \mathrm{id}\ } & (W_2 \odot W_1) \odot W_3 & \xrightarrow{\ \varphi_{W_2,W_1,W_3}\ } & W_2 \odot (W_1 \odot W_3) \\[2pt]
{\scriptstyle \varphi_{W_1,W_2,W_3}}\Big\downarrow & & & & \Big\downarrow {\scriptstyle \mathrm{id} \odot R_{W_1,W_3}} \\[2pt]
W_1 \odot (W_2 \odot W_3) & \xrightarrow{\ R_{W_1,W_2 \odot W_3}\ } & (W_2 \odot W_3) \odot W_1 & \xrightarrow{\ \varphi_{W_2,W_3,W_1}\ } & W_2 \odot (W_3 \odot W_1)
\end{array}
$$

and

$$
\begin{array}{ccccc}
W_1 \odot (W_2 \odot W_3) & \xrightarrow{\ \mathrm{id} \odot R_{W_2,W_3}\ } & W_1 \odot (W_3 \odot W_2) & \xrightarrow{\ \varphi^{-1}_{W_1,W_3,W_2}\ } & (W_1 \odot W_3) \odot W_2 \\[2pt]
{\scriptstyle \varphi^{-1}_{W_1,W_2,W_3}}\Big\downarrow & & & & \Big\downarrow {\scriptstyle R_{W_1,W_3} \odot \mathrm{id}} \\[2pt]
(W_1 \odot W_2) \odot W_3 & \xrightarrow{\ R_{W_1 \odot W_2,W_3}\ } & W_3 \odot (W_1 \odot W_2) & \xrightarrow{\ \varphi^{-1}_{W_3,W_1,W_2}\ } & (W_3 \odot W_1) \odot W_2
\end{array}
$$

commute. Each of (5.27) and (5.28) follows from an equation in the diffeomorphism group of the surface pictured in Fig. 6. The diffeomorphisms are formed from the braiding β shown in Fig. 5. The associators are formed from gluings and ungluings, so do not enter.

We summarize this discussion in the following.

Proposition 5.29. *In a* $2 + 1$ *dimensional topological quantum field theory (which satisfies the axioms of Assertion 4.12) the 2-inner product space* $E(S^1)$ *is a braided monoidal category with a compatible balanced automorphism of the identity and compatible duality.*[24]

There is a notion of semisimplicity for such categories [Y2], and it is desirable to prove that E is semisimple using the inner product, as we indicated for the $1 + 1$ dimensional case after Proposition 5.10. Surely one should think of the 2-inner product space structure together with the monoidal structure. In other words, one should think of E as a higher version of the algebra encountered in Proposition 5.10.

There are *reconstruction theorems* in category theory which recover certain algebraic objects from certain types of categories. For example, in [DM] it is shown how to recover a group from its category of representations. The structure in Proposition 5.29 is almost enough to reconstruct a *quasitriangular quasi-Hopf algebra* [Ma1]. (This is often termed a *quasi-quantum group*. Probably there is a ribbon element as well [RT, AC] corresponding to the automorphism of the identity.) Missing is a functor from E to the category of vector spaces, though more abstract reconstructions are possible [Ma2]. We remark that there are examples where no such "fiber functor" exists; the simplest is $\mathscr{V}_1 \times \mathscr{V}_1$. (This arises from a three dimensional σ-model into a space consisting of two points.) But it seems that we can always decompose into a product of spaces where reconstruction is possible. For the finite gauge theory we carry out the reconstruction in Sects. 7–9. There we choose various trivializations to construct a functor from E to the category of vector spaces, and this allows the reconstruction of the quasi-quantum group.

Finally, we remark that we can take products with any closed oriented Y in all of these constructions to obtain a higher algebra structure on $E(S^1 \times Y)$. In particular, the generalized quantum Hilbert space of any torus $S^1 \times \ldots \times S^1$ has a higher algebra structure.

6. The $1 + 1$ Dimensional Theory

We resume our discussion of the finite group gauge theory of Sects. 2 and 4. In this section we examine the $d = 1$ case. We know from Assertion 5.10 that $E(S^1)$ is an algebra, the algebra of central functions $\mathscr{F}_{\text{cent}}(\Gamma)$ under convolution, as was computed in [FQ, Sect. 5]. The new point is to compute $E(pt)$ and $Z_{[0,1]}$. The results are fairly trivial, but they illustrate the definitions and constructions of the previous sections and are a good warmup to the $d = 2$ case we discuss in Sects. 7–9.

Recall that the lagrangian is specified by a cocycle $\alpha \in C^2(B\Gamma; \mathbb{R}/\mathbb{Z})$. We first consider the simplest case (the "untwisted theory") where $\alpha = 0$. Obviously, $\overline{\mathscr{C}_{pt}}$ has a single element, the equivalence class of the trivial bundle $Q_{\text{triv}} = pt \times \mathbb{T}$. The value of the classical action $T_{pt}(Q_{\text{triv}}) \in \mathscr{T}_2$ is the trivial $\mathbb{T}$-gerbe $\mathscr{T}_1$. We identify

---

[24] As mentioned earlier, this is sometimes termed a *tortile category*. Also, there is a gap here in that we did not find the natural transformations mentioned in the footnote following (5.21). **Added in proof:** This gap is filled in [F3]

the automorphism group of Q_{triv} with Γ, acting by *left* multiplication, and it acts trivially on $T_{pt}(Q_{\text{triv}})$. Hence the associated 2-vector space $\mathscr{W}_{Q_{\text{triv}}}$ in (4.5) is $(\mathscr{V}_1)^\Gamma$, the category of representations of Γ. Now $E(pt)$ is computed by the path integral (4.6) as an inverse limit over the category of trivial bundles $Q \to pt$. The automorphism groups $\text{Aut}\,Q$ which enter (4.5) are not *canonically* isomorphic to Γ. Rather, we use the distinguished bundle $Q_{\text{triv}} \to pt$ to trivialize the inverse limit:

$$E(pt) \cong \frac{1}{\#\Gamma} \cdot (\mathscr{V}_1)^\Gamma . \tag{6.1}$$

[Recall that the prefactor is $1/(\#\,\text{Aut}\,Q_{\text{triv}})$.] We use this trivialization in what follows.

According to (5.4) the generalized partition function $Z_{[0,1]}$ is isometric to the identity operator on $E(pt)$. It is instructive to compute this isometry directly from the definition of the path integral (4.8). There is a bijection

$$\overline{\mathscr{C}_{[0,1]}}(Q_{\text{triv}} \sqcup Q_{\text{triv}}) \leftrightarrow \Gamma \tag{6.2}$$

by comparing the trivializations of a bundle $P \to [0,1]$ over the two endpoints of $[0,1]$. More explicitly, fix a basepoint in Q_{triv} and let $p_0 \in P_0$, $p_1 \in P_1$ be the corresponding basepoints in P using the trivializations. Parallel transport along $[0,1]$ is an isomorphism $\psi \colon P_0 \to P_1$. Define $g \in \Gamma$ by $\psi(p_0) = p_1 \cdot g$. Then g is the element of Γ corresponding to P under the correspondence (6.2). The action of $\langle h_0, h_1 \rangle \in \Gamma \times \Gamma \cong \text{Aut}(Q_{\text{triv}}) \times \text{Aut}(Q_{\text{triv}})$ on the left-hand side of (6.2) corresponds to the action

$$\langle h_0, h_1 \rangle \cdot g = h_1 g h_0^{-1}, \qquad g \in \Gamma ,$$

on the right-hand side. The classical action (2.4) is trivial, so in (4.7) we obtain $L_{[0,1]}(g) = \mathbb{C}$ for all g in (6.2). Since $[0,1]$ has nonempty boundary the measure μ in (4.1) is identically equal to 1. Hence the path integral (4.8) gives

$$Z_{[0,1]} = \bigoplus_{g \in \Gamma} \mathbb{C} . \tag{6.3}$$

We identify this as the set of complex-valued functions $\mathscr{F}(\Gamma)$ on Γ, with $\Gamma \times \Gamma$ acting as

$$(\langle h_0, h_1 \rangle \cdot f)(g) = f(h_1 g h_0^{-1}) \qquad f \in \mathscr{F}(\Gamma), \, g \in \Gamma , \tag{6.4}$$

with the standard inner product

$$(f_1, f_2) = \sum_{g \in \Gamma} f_1(g) \overline{f_2(g)} , \qquad f_1, f_2 \in \mathscr{F}(\Gamma) . \tag{6.5}$$

View $\mathscr{F}(\Gamma)$ as an element $\overline{E(pt)} \otimes E(pt)$, or using the inner product on $E(pt)$ as an element in $E(pt)^* \otimes E(pt) \cong \text{Hom}(E(pt))$. Call this endomorphism K. Suppose that $W \in E(pt)$ is a unitary representation of Γ with action $\varrho \colon \Gamma \to \text{Aut}(W)$. According to the inner product (3.5) and the factor $1/\#\Gamma$ in (6.1), the action of K on W is

$$K(W) = \frac{1}{\#\Gamma} \cdot (\mathscr{F}(\Gamma) \otimes W)^\Gamma . $$

Here we take Γ-invariants under the action of $h \in \Gamma$ by $\langle h, 1 \rangle$ on $\mathscr{F}(\Gamma)$ and $\varrho(h)$ on W; then $h \in \Gamma$ acts on $(\mathscr{F}(\Gamma) \otimes W)^\Gamma$ through the action of $\langle 1, h \rangle$ on $\mathscr{F}(\Gamma)$.

Now by (5.4) we can derive from the gluing law an isometry $K^2 \to K$. (The underlying map of categories is a natural transformation.) We compute it by analyzing

the gluing map (4.20) for the gluing of two intervals. We find that the desired isometry is

$$\frac{1}{\#\Gamma} \cdot (\mathscr{F}(\Gamma) \otimes \mathscr{F}(\Gamma))^\Gamma \cong \frac{1}{\#\Gamma} \cdot \mathscr{F}(\Gamma \times \Gamma)^\Gamma \to \mathscr{F}(\Gamma) , \tag{6.6}$$

$$f(\cdot, \cdot) \mapsto f(e, \cdot) .$$

[The Γ invariance in (6.6) refers to the action $(h \cdot f)(g_1, g_2) = f(g_1 h, h^{-1} g_2)$ for $h \in \Gamma$.] This yields the desired isometry $K^2 \to K$ which on $W \in E(pt)$ is

$$K^2(W) = \frac{1}{(\#\Gamma)^2} \cdot (\mathscr{F}(\Gamma \times \Gamma) \otimes W)^{\Gamma \times \Gamma} \to \frac{1}{\#\Gamma} \cdot (\mathscr{F}(\Gamma) \otimes W)^\Gamma = K(W) ,$$

$$f^i \otimes w_i \mapsto (g \mapsto f^i(e, g) w_i) .$$

(These expressions are summed over i.) This is an isometry $K \to \mathrm{id}$ on the image of K, and is compatible with the isometry $K \to \mathrm{id}$ which on $W \in E(pt)$ is

$$\frac{1}{\#\Gamma} \cdot (\mathscr{F}(\Gamma) \otimes W)^\Gamma \to W , \tag{6.7}$$

$$f^i \otimes w_i \mapsto f^i(e) w_i .$$

We can also check the gluing which leads to (5.5). That is to say we can check the gluing law (4.17) when we glue the two ends of $[0, 1]$ together. Now $\overline{\mathscr{C}_{S^1}}$ can be identified with the set of conjugacy classes in Γ, and the gluing map (4.20) with $Q = Q_{\mathrm{triv}}$ sends an element in Γ to its equivalence class. The map Tr_{pt} in (4.18) is $1/\#\Gamma$ times the Γ-invariants under the diagonal action in (6.4), and applied to $Z_{[0,1]} = \mathscr{F}(\Gamma)$ this gives

$$\mathrm{Tr}_{pt}(\mathscr{F}(\Gamma)) = \frac{1}{\#\Gamma} \cdot \mathscr{F}_{\mathrm{cent}}(\Gamma) , \tag{6.8}$$

where $\mathscr{F}_{\mathrm{cent}}(\Gamma)$ is the space of central functions with inner product (6.5). This is $E(S^1)$, as follows easily from (4.6) (cf. [FQ, Sect. 5]).

If the lagrangian $\alpha \in C^2(B\Gamma; \mathbb{R}/\mathbb{Z})$ is nonzero (the "twisted theory"), then the classical action also enters in a nontrivial way. We compute the classical action on the trivial bundle $Q_{\mathrm{triv}} \to pt$. Since there is a unique cycle in $C_0(pt)$ which represents the fundamental class $[pt] \in H_0(pt)$, the integration theory in the appendix gives

$$\exp\left(2\pi i \int\limits_{pt} \bar{f}^* \alpha \right) = \mathscr{T}_1$$

for any $\bar{f} : pt \to B\Gamma$. [This is what we must compute in (2.1).] In other words, we can think of α as defining the trivial $\mathbb{T}$-gerbe bundle over $B\Gamma$, which then lifts to the trivial $\mathbb{T}$-gerbe bundle over $E\Gamma$. The nontrivial part comes from homotopies between classifying maps of Q_{triv}, which we identify with paths in $E\Gamma$. The integral in (2.2) is then a $\mathbb{T}$-torsor. The classical action $T_{pt}^{(\alpha)}(Q_{\mathrm{triv}})$ is a nontrivial $\mathbb{T}$-gerbe computed by an inverse limit over the "path category" of $E\Gamma$. The value of the classical action on a field $P \to [0, 1]$, whose boundary we assume trivialized by an isomorphism $\partial P \cong Q_{\mathrm{triv}} \times Q_{\mathrm{triv}}$, is then an automorphism of $T_{pt}^{(\alpha)}(Q_{\mathrm{triv}})$. By (6.2) we identify the equivalence class of P with an element $g \in \Gamma$, and by (2.8) the classical action is well-defined on the equivalence class. Taking an inverse limit over all such bundles

in the equivalence class we obtain for each $g \in \Gamma$ an automorphism T_g of $T_{pt}^{(\alpha)}(Q_{\text{triv}})$. Furthermore, there are isomorphisms

$$T_{gh} \rightarrow T_g \cdot T_h$$

from the gluing law (2.12) applied to the gluing of intervals.

As in the $\alpha = 0$ case we compute the quantum space $E^{(\alpha)}(pt)$ by taking an inverse limit over the category of all trivial bundles. We use the distinguished object Q_{triv} to trivialize the inverse limit [cf. (4.5) and (3.8)]:

$$E^{(\alpha)}(pt) \cong \frac{1}{\#\Gamma} \cdot (\mathscr{W}_{T_{pt}^{(\alpha)}(Q_{\text{triv}})})^{\Gamma, \varrho} . \tag{6.9}$$

Here ϱ is the action of Γ on $T_{pt}^{(\alpha)}(Q_{\text{triv}})$ via the torsors T_g. If we trivialize the $\mathbb{T}$-gerbe $T_{pt}^{(\alpha)}(Q_{\text{triv}})$, for example by choosing a basepoint in $E\Gamma$, then we obtain an isomorphism $T_{pt}^{(\alpha)}(Q_{\text{triv}}) \cong \mathscr{T}_1$, and so the T_g are identified with $\mathbb{T}$-torsors. As in (1.4) these torsors define a central extension

$$1 \rightarrow \mathbb{T} \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 1 .$$

Incidentally, they are isomorphic to the torsors and central extension which come from the action of $\text{Aut}(Q_{\text{triv}}) \cong \Gamma$ on $T_{pt}^{(\alpha)}(Q_{\text{triv}}) \cong \mathscr{T}_1$. This assertion follows from the fact (5.2) that the classical action of the product bundle $[0, 1] \times Q_{\text{triv}} \rightarrow [0, 1]$ is trivial. With the trivialization of $T_{pt}^{(\alpha)}(Q_{\text{triv}})$ the isometry (6.9) becomes

$$E^{(a)}(pt) \cong \frac{1}{\#\Gamma} \cdot (\mathscr{T}_1)^{\Gamma, \varrho} . \tag{6.10}$$

Recall from the paragraph following (3.8) that $(\mathscr{T}_1)^{\Gamma, \varrho}$ is the category of representations of $\tilde{\Gamma}$, where the central $\mathbb{T}$ acts by scalar multiplication. We emphasize that (6.10) requires two choice of trivialization (of two inverse limits).

Computing with the trivialization (6.10) we find analogous to (6.3) that

$$Z_{[0,1]} \cong \bigoplus_{g \in \Gamma} L_g , \tag{6.11}$$

where L_g is the hermitian line obtained from the torsor T_g as in (3.2). We leave the reader to modify the verification of (6.7) above to show that (6.11) acts isometrically to the identity map. The twisted version of (6.8) is also easy to check.

7. The 2 + 1 Dimensional Chern-Simons Theory and Quasi-Quantum Groups: Untwisted Case

We turn to the $2 + 1$ dimensional case of gauge theory with finite gauge group, which can be considered as a Chern-Simons theory. Our goal is to derive the quasi-Hopf algebras of [DPR] directly from the path integral (4.6). We already investigated several features of this theory in [FQ]. The new point is an investigation of the 2-inner product space $E(S^1)$, which according to Assertion 5.29 is a certain type of braided monoidal category. With suitable trivializations we claim that it is isomorphic to the category

of representations[25] of the quasitriangular quasi-Hopf algebra constructed in [DPR]. We also recover the results of [FQ, Sects. 3–4], including Segal's modular functor [S1], from our approach here. In this section we treat the untwisted case where the lagrangian $\alpha \in C^3(B\Gamma; \mathbb{R}/\mathbb{Z})$ vanishes. In Sects. 8–9 we generalize to the twisted case $\alpha \neq 0$.

The holonomy of a bundle around the circle induces a bijection

$$\overline{\mathscr{C}_{S^1}} \leftrightarrow \text{conjugacy classes in } \Gamma. \tag{7.1}$$

The classical action in the $\alpha = 0$ theory is trivial. Choose a bundle $Q_{[x]} \to S^1$ representing each conjugacy class [x] in Γ under the correspondence (7.1). Then following the same steps as in (6.1), this choice of bundles leads to an isometry

$$E = E(S^1) \cong \bigoplus_{[x]} \frac{1}{\# \operatorname{Aut} Q_{[x]}} \cdot (\mathscr{Y}_1)^{\operatorname{Aut} Q_{[x]}}. \tag{7.2}$$

It is convenient to use a more concrete description of E directly in terms of the group Γ, and this requires a choice of some basepoints. (Compare with the choice of basepoints in [FQ, Sect. 3].) Fix a conjugacy class $[x]$ and consider the fiber $F_{[x]}$ of $Q_{[x]} \to S^1$ over the basepoint $1 \in S^1 = \mathbb{T}$. A point in $F_{[x]}$ determines a particular value of the holonomy of $Q_{[x]}$, which is an element of the conjugacy class $[x]$. Choose a (base)point f_x in the fiber of the holonomy map $F_{[x]} \to [x]$ for each $x \in [x]$. Then f_x induces an isomorphism

$$\operatorname{Aut} Q_{[x]} \to C_x \tag{7.3}$$

by assigning to $\psi \in \operatorname{Aut} Q_{[x]}$ the element $g \in C_x$ in the centralizer of x which satisfies $\psi(f_x) = f_x \cdot g$. Thus if W is a representation of $\operatorname{Aut} Q_{[x]}$, then under this isomorphism W is also a representation of C_x. Let $\mathbf{W}$ denote the trivial vector bundle over $[x]$ whose fiber at each $x \in [x]$ is W. Now Γ acts on $[x]$ on the left by conjugation $(g: x \mapsto gxg^{-1})$, and we want to lift this action to $\mathbf{W}$. For each $x \in [x]$ the stabilizer C_x already acts on the fiber $\mathbf{W}_x = W$. For $x, x' \in [x]$ there is a unique $g_{x,x'} \in \Gamma$ with $f_x = f_{x'} \cdot g_{x,x'}$. Then $x' = g_{x,x'} x g_{x,x'}^{-1}$. Lift $g_{x,x'}: x \mapsto x'$ to the identity map $\operatorname{id}: \mathbf{W}_x \to \mathbf{W}_{x'}$. There is then a unique extension of the C_x action and the action of the $g_{x,x'}$ on W to a Γ action on W which lifts the conjugation action on $[x]$.

Summarizing, the choice of basepoints in the bundles $Q_{[x]}$ leads to an isometry

$$E \cong \frac{1}{\#\Gamma} \cdot \operatorname{Vect}_\Gamma(\Gamma), \tag{7.4}$$

where $\operatorname{Vect}_\Gamma(\Gamma)$ is the 2-inner product space of hermitian vector bundles over Γ with a unitary lift of the left Γ action on Γ by conjugation. We write an element of $\operatorname{Vect}_\Gamma(\Gamma)$ as $W = \bigoplus_{x \in \Gamma} W_x$. If $W_1, W_2 \in \operatorname{Vect}_\Gamma(\Gamma)$, then the inner product is defined as

$$(W_1, W_2)_{\operatorname{Vect}_\Gamma(\Gamma)} = \left(\bigoplus_x (W_1)_x \otimes \overline{(W_2)_x} \right)^\Gamma.$$

It is easy to check that $1/\#\Gamma$ times this inner product is the inner product in (7.2).

---

[25] It is probably more natural to use corepresentations here, but in any case we have enough finiteness to switch back and forth between representations and corepresentations. Also, this will reconstruct the algebras in [DPR] rather than their duals. Our convention here differs from [FQ, Sect. 3], where we use corepresentations. Note also that in [FQ, Sect. 3] we use *right* comodules whereas here we use *left* modules. Thus the groupoid (7.5) is opposite that in [FQ, Sect. 3]

There is another description of E which is useful. Let $\mathscr{G}$ denote the groupoid which is the set $G \times G$ with the composition law

$$\langle x_2, g_2 \rangle \circ \langle x_1, g_1 \rangle = \langle x_1, g_2 g_1 \rangle, \quad \text{if } x_2 = g_1 x_1 g_1^{-1}. \tag{7.5}$$

Composition is not defined if $x_2 \neq g_1 x_1 g_1^{-1}$. Then

$$E \cong \frac{1}{\#\Gamma} \cdot (\mathscr{V}_1)^{\mathscr{G}}, \tag{7.6}$$

where $(\mathscr{V}_1)^{\mathscr{G}}$ is the 2-inner product space of finite dimensional unitary representations of $\mathscr{G}$. What we mean by a representation of the groupoid $\mathscr{G}$ amounts exactly to a Γ-bundle over Γ, so (7.6) is essentially identical to (7.4). More precisely, these are representations (*left* modules) of the "groupoid algebra"

$$\mathbb{C}[\mathscr{G}] = \bigoplus_{x,g} \mathbb{C}\langle x, g \rangle, \tag{7.7}$$

with multiplication

$$\langle x_2, g_2 \rangle \cdot \langle x_1, g_1 \rangle = \begin{cases} \langle x_1, g_2 g_1 \rangle, & x_2 = g_1 x_1 g_1^{-1}; \\ 0, & \text{otherwise}. \end{cases}$$

The unit element is

$$1 = \sum_x \langle x, e \rangle.$$

If $W \in \mathrm{Vect}_\Gamma(\Gamma) = (\mathscr{V}_1)^{\mathscr{G}}$ we use the notation

$$A_g^W = A_g : W_x \to W_{gxg^{-1}}$$

for the action of $\langle \xi, g \rangle \in \mathscr{G}$. In terms of the $\mathscr{G}$ action we have

$$W_x = \langle x, e \rangle \cdot W. \tag{7.8}$$

We use the trivialization (7.4), or equivalently (7.6), in what follows.

An irreducible element $W \in E$ is supported on some equivalence class $[x]$, and the fiber W_x is an irreducible representation ϱ of the centralizer C_x. Since the various C_x, $x \in [x]$ are identified up to inner automorphisms, the equivalence class $[\varrho]$ of the representation is well-defined. Up to isomorphism the irreducible elements of E are labelled by the pair $\langle [x], [\varrho] \rangle$. These labels appear in all treatments of this theory [DVVV, DPR, DW, FQ].

It is convenient to use the isomorphism (7.6) to identify the path integral (4.11) over a compact oriented 2-manifold X, which is an element of $E(\partial X)$, as an element in tensor products of E. Recall our convention stated after (5.11) for identifying ∂X as a disjoint union of copies of the standard circle S^1. For this we restrict to surfaces X which are subsets of $\mathbb{C}$. Under these identifications each component of ∂X has a basepoint corresponding to the standard basepoint $1 \in S^1$. Let $\mathscr{C}'_X$ denote the category of principal Γ bundles $P \to X$ endowed with a basepoint in the fiber over each basepoint in ∂X. Morphisms are required to preserve the basepoints. Let $\overline{\mathscr{C}'_X}$ denote the set of equivalence classes. For the cylinder C the holonomy and parallel transport define a bijection

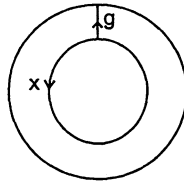$$\overline{\mathscr{C}'_C} \leftrightarrow \mathscr{G} \tag{7.9}$$

Fig. 7. The bundle over C corresponding to $\langle x, g \rangle \in \mathscr{G}$

as illustrated in Fig. 7. [Compare with (6.2).] Now for a surface X we can glue C to any component of ∂X using the basepoints. This induces a $\mathscr{G}$ action on $\overline{\mathscr{C}'_X}$ for each component of ∂X.

Proposition 7.10. *Let* $X \subset \mathbb{C}$ *be a compact oriented 2-manifold.*[26] *Then under the isomorphism (7.6) the path integral over* X *is*

$$Z_X \cong L^2(\overline{\mathscr{C}'_X}) \tag{7.11}$$

with the $\mathscr{G}$ *actions induced by gluing cylinders onto components of* ∂X.

Proof. Let $P \in \mathscr{C}'_X$ and fix a component S of ∂X. The basepoint determines an isomorphism $P|_S \rightarrow Q_{[x]}$ for some $[x]$. If the holonomy around S is x, then the basepoint maps to f_x. Apply this to a pointed bundle $P \in \mathscr{C}'_C$ over the cylinder C which corresponds under (7.9) to an element $\langle x, g \rangle \in \mathscr{G}$. Using parallel transport along the axis of C, this bundle also determines an element of $\operatorname{Aut} Q_{[x]}$. If $g \in C_x$ then the correspondence between the automorphism of $Q_{[x]}$ and g agrees with (7.3). [This follows from (5.2).] Also, the bundle labeled by $\langle x, g_{x,x'} \rangle$ corresponds to the identity in $\operatorname{Aut} Q_{[x]}$ for all $x' \in [x]$. Thus the action of $\mathscr{G} \approx \overline{\mathscr{C}'_C}$ on the quantization (7.11) induced by gluing is the action described in the text leading to (7.4) and (7.6).

The 2-inner product space E has extra structure determined by the path integral over special surfaces and special diffeomorphisms, as described in Sect. 5.

Proposition 7.12. *The finite gauge theory described in Assertion 4.12 with* $\alpha = 0$ *determines the following structure on* E.
(a) *(Automorphism of the identity (5.11)). For* $W \in E$ *we have*

$$\theta_W|_{W_x} = A_x : W_x \rightarrow W_x. \tag{7.13}$$

(b) *(Involution (5.14)) For* $W \in E$ *the dual* $W^* \in \bar{E}$ *is defined by* $(W^*)_x = W^*_{x^{-1}}$ *and* $A_g^{W^*} = (A_{g^{-1}}^W)^*$.
(c) *(Identity (5.16)) The identity* $\mathbf{1}$ *is*

$$\mathbf{1}_x = \begin{cases} \mathbb{C}, & x = e; \\ 0, & x \neq e, \end{cases} \tag{7.14}$$

with $C_e = \Gamma$ *acting trivially on* $\mathbf{1}_e$.
(d) *(Multiplication (5.19)) The tensor product of* $W_1, W_2 \in E$ *is*

$$(W_1 \odot W_2)_x = \bigoplus_{x_1 x_2 = x} (W_1)_{x_1} \otimes (W_2)_{x_2} \tag{7.15}$$

---

[26] The same arguments apply to arbitrary surfaces with *parametrized* boundary. If the surface has closed components, then we must modify the inner product in (7.11)

with the Γ action

$$A_g^{W_1 \odot W_2} = A_g^{W_1} \otimes A_g^{W_2} \,. \tag{7.16}$$

(e) *(Associated (5.22)) The associator φ is induced from the standard associator of tensor products of vector spaces.*
(f) *(R-matrix (5.25)) For $W_1, W_2 \in E$ we have*

$$R_{W_1, W_2} : (W_1)_{x_1} \otimes (W_2)_{x_2} \to (W_2)_{x_1 x_2 x_1^{-1}} \otimes (W_1)_{x_1} \,,$$
$$w_1 \otimes w_2 \mapsto A_{x_1}^{W_2}(w_2) \otimes w_1 \tag{7.17}$$

and all other components are zero.

A few remarks are in order. First, since x is a central element of C_x, the transformation (7.13) is a scalar on each irreducible component of W_x. (We decompose W_x under the C_x action.) If W is an irreducible element of E labelled by $\langle [x], [\varrho] \rangle$, then the scalar transformation A_x is independent of $x \in [x]$. The *conformal weight* $h_{\langle [x], [\varrho] \rangle}$ is defined up to an integer by the equation

$$A_x = e^{2\pi i h_{\langle [x], [\varrho] \rangle}} \,. \tag{7.18}$$

This agrees with the results of [FQ, Sect. 5], where we calculated the conformal weight from the action of (5.13) on the torus. Notice that θ_W can also be described as the action of

$$v = \sum_x \langle x, x \rangle \tag{7.19}$$

on W, where v is a special element[27] of $\mathbb{C}[\mathcal{G}]$. The identity element $\mathbf{1}$ corresponds to the label $\langle [e], \text{trivial} \rangle$. Another description of the multiplication (7.15), (7.16) is

$$W_1 \odot W_2 = \mu_*(W_1 \boxtimes W_2) \,,$$

where $\mu : \Gamma \times \Gamma \to \Gamma$ is group multiplication and $W_1 \boxtimes W_2 \to \Gamma \times \Gamma$ is the external tensor product. Finally, we invite the reader to verify (5.15), (5.17), (5.20), (5.21), (5.23), (5.24), (5.26), (5.27), and (5.28) directly from the data listed in Proposition 7.12.[28]

Proof. We use Proposition 7.10 to compute the path integrals over the various surfaces.
(a) We compute the action of the diffeomorphism (5.12) on the cylinder C. From (7.9) and (7.11) we obtain an isomorphism

$$Z_C \cong \mathcal{F}(\mathcal{G}) \,. \tag{7.20}$$

An argument similar to that in Sect. 6 [see (6.7)] shows that Z_C acts isometrically to the identity on E via the isometry

$$\frac{1}{\#\Gamma} \cdot (\mathcal{F}(\mathcal{G}) \otimes W)^{\mathcal{G}} \to W$$
$$f^i \otimes w_i \mapsto f^i(\langle \pi(w_i), e \rangle) w_i \,, \tag{7.21}$$

---

[27] This element plays the role of the *inverse* of the ribbon element of Reshetikhin/Turaev [RT]. The quasitriangular quasi-Hopf algebras we encounter have a ribbon structure (cf. [AC])
[28] The natural transformations $W \odot W^* \to \mathbf{1}$ and $\mathbf{1} \to W \odot W^*$ mentioned in the footnote following (5.21) are evidently the duality pairing $\bigoplus_x W_x \otimes W_x^* \to \mathbb{C}$ and its dual

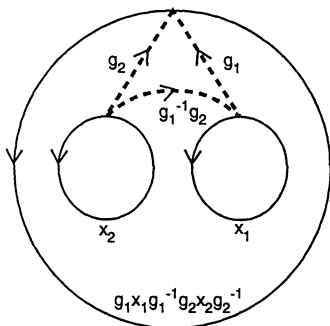**Fig. 8.** The bundle over P corresponding to $\langle x_1, g_1 \rangle \times \langle x_2, g_2 \rangle \in \mathscr{G} \times \mathscr{G}$

where $\pi: W \to \Gamma$ is an element of $\mathrm{Vect}_\Gamma(\Gamma)$. On the left-hand side of (7.21) we take $\mathscr{G}$-invariants under the action $a: f^i(\cdot) \otimes w_i \mapsto f^i(a^{-1}) \otimes a w_i$, and then $a \in \mathscr{G}$ acts on the invariants by $a: f^i(\cdot) \otimes w_i \mapsto f^i(\cdot a) \otimes w_i$. Here "$\cdot$" indicates the argument of the function. Now the diffeomorphism τ in (5.12) induces by pullback the map

$$\tau^* \langle x, g \rangle = \langle x, gx \rangle, \qquad \langle x, g \rangle \in \mathscr{G} \tag{7.22}$$

on fields (7.9), and so the map

$$(\tau^* f)(\langle x, g \rangle) = f(\langle x, gx \rangle), \qquad f \in \mathscr{F}(\mathscr{G}),$$

on the quantization (7.20). In terms of the element $v \in \mathbb{C}[\mathscr{G}]$ in (7.19), this is

$$(\tau_* f)(\cdot) = f(\cdot v).$$

Thus on the left-hand side of (7.21) the diffeomorphism τ induces the action

$$f^i(\cdot) \otimes w_i \mapsto (\tau_* f^i)(\cdot) \otimes w_i = f^i(\cdot v) \otimes w_i,$$

which corresponds to the action $w \mapsto vw$ on the right-hand side of (7.21). This is (7.13).

(b) We first calculate that the reflection of S^1 induces the map $Q_{[x]} \mapsto Q_{[x^{-1}]}$ on fields by pullback. (Actually, this is the map on equivalence classes of fields written using our distinguished representatives.) Since the reflection reverses orientation, this induces a map $\mathscr{T}_1 \mapsto \mathscr{T}_1^{-1}$ on the classical action, and in the quantization leads us to use the dual space. Under the identification (7.4) this gives $(W^*)_x = W^*_{x^{-1}}$. Then the induced representation of $\mathrm{Aut}\, Q_{[x]} \cong \mathrm{Aut}\, Q_{[x^{-1}]}$ is $A_g^{W^*} = (A_{g^{-1}}^W)^*$.

(c) It is easy to see that $\overline{\mathscr{C}'_{D^2}}$ consists of one element, and the restriction of any representative bundle to $\partial D^2 = S^1$ is $Q_{[e]}$. Furthermore, $\mathrm{Aut}\, Q_{[e]} \cong \Gamma$ acts trivially.

(d) For the pair of pants P we identify

$$\overline{\mathscr{C}'_P} \leftrightarrow \mathscr{G} \times \mathscr{G} \tag{7.23}$$

using the parallel transports and holonomies indicated in Fig. 8. This leads to an isometry

$$Z_P \cong \mathscr{F}(\mathscr{G} \times \mathscr{G}). \tag{7.24}$$

The actions of $\langle x, g \rangle \in \mathscr{G}$ corresponding to the two inner components of ∂P are

$$
\begin{aligned}
f(\cdot, \cdot) &\mapsto f(\cdot \langle x, g \rangle^{-1}, \cdot), \\
f(\cdot, \cdot) &\mapsto f(\cdot, \cdot \langle x, g \rangle^{-1}),
\end{aligned}
\tag{7.25}
$$

The action of $\langle x, g \rangle \in \mathscr{G}$ corresponding to the outer component is

$$(\langle x, g \rangle \cdot f)(\langle x_1, g_1 \rangle, \langle x_2, g_2 \rangle) = \begin{cases} f(\langle x_1, gg_1 \rangle, \langle x_2, gg_2 \rangle), & \text{if } x = g_1 x_1 g_1^{-1} g_2 x_2 g_2^{-1}; \\ 0, & \text{otherwise}. \end{cases}$$

(7.26)

Using the inner product (7.6) on E we see that the multiplication (5.19) is the map

$$W_1 \otimes W_2 \mapsto \frac{1}{(\#\Gamma)^2} \cdot (\mathscr{F}(\mathscr{G} \times \mathscr{G}) \otimes W_1 \otimes W_2)^{\mathscr{G} \times \mathscr{G}},$$

where $\mathscr{G} \times \mathscr{G}$ acts on $\mathscr{F}(\mathscr{G} \times \mathscr{G})$ via (7.25). The $\mathscr{G}$ action on the right-hand side is via (7.26). Then a routine check shows that

$$\frac{1}{(\#\Gamma)^2} \cdot (\mathscr{F}(\mathscr{G} \times \mathscr{G}) \otimes W_1 \otimes W_2)^{\mathscr{G} \times \mathscr{G}} \to W_1 \odot W_2$$

(7.27)

$$f^{ij} \otimes w_i^{(1)} \otimes w_j^{(2)} \mapsto f^{ij}(\langle \pi(w_1), e \rangle, \langle \pi(w_2), e \rangle) w_i^{(1)} \otimes w_j^{(2)}$$

is an isometry, where $W_1 \otimes W_2$ is defined by (7.15) and (7.16).
(e) This is immediate from the definition of the associator.
(f) We compute the action of the braiding diffeomorphism β (Fig. 5) on the fields (7.23) by pullback as

$$\langle x_1, g_1 \rangle \times \langle x_2, g_2 \rangle \mapsto \langle x_2, g_1 x_1 g_1^{-1} g_2 \rangle \times \langle x_1, g_1 \rangle.$$

(7.28)

So the action on the quantization (7.24) by pushforward is

$$(\beta_* f)(\langle x_1, g_1 \rangle, \langle x_2, g_2 \rangle) = f(\langle x_2, g_1 x_1 g_1^{-1} g_2 \rangle, \langle x_1, g_1 \rangle).$$

Under the isometry (7.27) this corresponds to (7.17), as desired.

Reconstruction theorems in category theory assert that E is (equivalent to) the category of representations of a Hopf algebra H. In fact, since E is braided H is a *quasitriangular* Hopf algebra [Dr]. We do not need the general arguments from category theory to carry out the reconstruction, as the Hopf algebra H is apparent from our explicit descriptions of E in (7.4) and (7.6), and from the formulas in Proposition 7.12.

Indeed, as an algebra H is the "groupoid algebra" $H = \mathbb{C}[\mathscr{G}]$ defined in (7.7). We have already seen in (7.6) that E is isomorphic to the category of representations of the algebra H. Explicitly, if $\varrho \colon H \to \text{End}(W)$ is a representation of H, set $W_x = \varrho(\langle x, e \rangle)(W)$ as in (7.8) and set $A_g \colon W_x \to W_{gxg^{-1}}$ equal to $\varrho(\langle x, g \rangle)$. The quasitriangular Hopf structure on H is easily deduced from Poposition 7.12. From (7.15) and (7.16) we see that the coproduct $\Delta \colon H \to H \otimes H$ is

$$\Delta(\langle x, g \rangle) = \sum_{x_1 x_2 = x} \langle x_1, g \rangle \otimes \langle x_2, g \rangle.$$

The counit $\varepsilon \colon H \to \mathbb{C}$ is

$$\varepsilon(\langle x, g \rangle) = \begin{cases} 1, & x = e; \\ 0, & \text{otherwise}, \end{cases}$$

as we see from the action of H on $\mathbf{1}$ (7.14). The antipode $S \colon H \to H$ is implemented on the dual (Proposition 7.12(b)), so is

$$S(\langle x, g \rangle) = \langle g x^{-1} g^{-1}, g^{-1} \rangle.$$

The quasitriangular structure is an element $R \in H \otimes H$ such that for every pair of representations (W_1, ϱ_1), (W_2, ϱ_2) of H, we have

$$R_{W_1,W_2} = \tau_{W_1,W_2} \circ (\varrho_1 \otimes \varrho_2)(R),$$

where $\tau_{W_1,W_2} : W_1 \otimes W_2 \to W_2 \otimes W_1$ is the transposition. Hence from (7.17) we deduce

$$R = \sum_{x_1, x_2} \langle x_1, e \rangle \otimes \langle x_2, x_1 \rangle.$$

Since the associator φ is the standard associator on vector spaces (Proposition 7.12(e)), we obtain a Hopf algebra (as opposed to a quasi-Hopf algebra). Finally, we have already observed that the automorphism of the identity θ in (7.13) is implemented by the element v in (7.19):

$$v = \sum_x \langle x, x \rangle.$$

This special element in H is the *inverse* of the ribbon element of Reshetikhin/Turaev [RT]. We interpret it here in terms of the "balancing" of the category of representations.

The quasitriangular Hopf algebra H is identified in [DPR] as the "quantum double" of $\mathscr{F}(\Gamma)$.

Finally, we indicate how to recover the "modular functor" [S1, FQ, Sect. 4]. Once and for all fix a basis $\{W_\lambda\}$ of the 2-inner product space $E = E(S^1)$. Here λ runs over the labeling set Φ mentioned earlier. Now suppose X is a compact oriented 2-manifold with each boundary component parametrized. The parametrizations identify $E(\partial X)$ with a tensor product of copies of E and $\bar{E}$. Thus we can decompose Z_X according to the chosen basis for E:

$$Z_X \cong \bigoplus_\lambda E(X, \lambda) \otimes W_\lambda,$$

where $\lambda = \langle \lambda_1, \ldots, \lambda_k \rangle$ runs over labelings of the boundary components and

$$W_\lambda = W_{\lambda_1}^{\pm 1} \otimes \ldots \otimes W_{\lambda_k}^{\pm 1},$$

the signs chosen according to the orientation. The inner product spaces $E(X, \lambda)$ define the modular functor. The gluing law for the modular functor follows directly from Assertion 4.12(d).

8. The 2 + 1 Dimensional Chern-Simons Theory and Quasi-Quantum Groups: Twisted Case

In this section we extend the results of Sect. 7 to the 2 + 1 dimensional finite gauge theory with nontrivial lagrangian $\alpha \in C^3(B\Gamma; \mathbb{R}/\mathbb{Z})$. The classical theory is nontrivial, and this leads to corresponding modifications of the quantum theory. We must choose additional trivializations (of gerbes) to express the theory in terms of familiar objects, and in particular to construct a quasi-Hopf algebra. (Recall the remarks following Proposition 5.29.) Such trivializations appear more naturally in Sect. 9, where we cut open the circle and make calculations on the interval. We rely here on the exposition in Sect. 7 and only indicate the necessary modifications. The cocycle $\alpha \in C^3(B\Gamma; \mathbb{R}/\mathbb{Z})$ is fixed throughout. We often omit it from the notation.

We use the choices made in Sect. 7 of representative bundles $Q_{[x]} \to S^1$ and basepoints f_x. The classical action $T_{S^1}^{(\alpha)}(Q_{[x]})$ is a $\mathbb{T}$-gerbe, which we denote $\mathscr{G}_{[x]}$. The automorphism group $\mathrm{Aut}\, Q_{[x]}$ acts on this gerbe, and the action is a homomorphism

$$\varrho_{[x]}^{(\alpha)} = \varrho_{[x]} \colon \mathrm{Aut}\, Q_{[x]} \to \mathrm{Aut}(\mathscr{G}_{[x]}) \,. \tag{8.1}$$

Fix a trivializing element

$$G_{[x]} \in \mathscr{G}_{[x]} = T_{S^1}^{(\alpha)}(Q_{[x]}) \,, \tag{8.2}$$

and so an isomorphism $\mathscr{G}_{[x]} \cong \mathscr{T}_1$. This can be done as in [FQ, Sect. 3] by fixing a representative cycle $s \in C_1(S^1)$ for the fundamental class $[S^1] \in H_1(S^1)$, and by fixing classifying maps $Q_{[x]} \to E\Gamma$. With these trivializations the action (8.1) determines a central extension of $\mathrm{Aut}\, Q_{[x]}$ by $\mathbb{T}$, as in (1.4). There is an induced isometry [see (4.6), (6.10)]

$$E^{(\alpha)} = E^{(\alpha)}(S^1) \cong \bigoplus_{[x]} \frac{1}{\#\,\mathrm{Aut}\, Q_{[x]}} \cdot (\mathscr{T}_1)^{\mathrm{Aut}\, Q_{[x]}, \varrho_{[x]}^{(\alpha)}} \,.$$

We want to express this directly in terms of Γ, using the basepoints f_x as in Sect. 7. The central extensions of $\mathrm{Aut}(Q_{[x]})$ lead via the isomorphism (7.3) to central extensions $\tilde{C}_x$ of the centralizer subgroup of any $x \in \Gamma$. That is, for each $g \in C_x$ we have a $\mathbb{T}$-torsor $T(x, g)$ together with appropriate isomorphisms under composition. Note that there are trivializations

$$T(x, e) \cong \mathbb{T} \tag{8.3}$$

since $T(x, e) \cdot T(x, e) \cong T(x, e)$. [This is (5.2).] Extend to a central extension of the groupoid $\mathscr{G}$ in (7.5) as follows. First, for any two elements x, x' in the same conjugacy class let

$$T(x, g_{x,x'}) = \mathbb{T}, \qquad x' \in [x] \,. \tag{8.4}$$

[The element $g_{x,x'} \in \Gamma$ was defined following (7.3).] Then for any $x, g \in \Gamma$ we have $\langle x, g \rangle = \langle x, g_{x,gxg^{-1}} \rangle \circ \langle x, h \rangle$ for some unique $h \in C_x$. Set $T(x, g) = T(x, g_{x,gxg^{-1}}) \cdot T(x, h)$. This determines the desired central extension

$$1 \to \mathbb{T} \to \tilde{\mathscr{G}}^{(\alpha)} \to \mathscr{G} \to 1 \,,$$

where for $\langle x, g \rangle \in \mathscr{G}$ the $\mathbb{T}$-torsor $T(x, g)$ is the preimage of $\langle x, g \rangle$ in $\tilde{\mathscr{G}}^{(\alpha)}$. There are appropriate isomorphisms under composition. Let $L(x, g)$ be the hermitian line corresponding to the $\mathbb{T}$-torsor $T(x, g)$, and

$$l(x, e) \in L(x, e) \tag{8.5}$$

the trivializing element derived from (8.3). We ignore the trivializations (8.4), which are artifacts of our definitions.

With this understood an element of $E^{(\alpha)}$ corresponds to a vector bundle $W = \bigoplus_{x \in \Gamma} W_x$ over Γ with isomorphisms

$$A_g^W = A_g \colon L(x, g) \otimes W_x \to W_{gxg^{-1}}$$

which compose properly. Set

$$H^{(\alpha)} = \bigoplus_{x,g} L(x, g) \,. \tag{8.6}$$

Define an algebra structure[29] using the multiplication in $\tilde{\mathscr{G}}$:

$$L(x_2, g_2) \otimes L(x_1, g_1) \to \begin{cases} L(x_1, g_2 g_1), & x_2 = g_1 x_1 g_1^{-1}, \\ 0, & \text{otherwise}. \end{cases} \tag{8.7}$$

The identity element in $H^{(\alpha)}$ is

$$1 = \sum_x l(x, e). \tag{8.8}$$

We can view $E^{(\alpha)}$ as the 2-inner product space of representations of $H^{(\alpha)}$, with the natural inner product multiplied by $1/\#\Gamma$. Or, by analogy with (7.6), we write

$$E^{(\alpha)} \cong \frac{1}{\#\Gamma} \cdot (\mathscr{G}_1)^{\tilde{\mathscr{G}}}, \tag{8.9}$$

where we only take representations in which the central circles $T(x, e) \cong \mathbb{T}$ act as scalar multiplication.

Now suppose X is a compact oriented surface, either with a given parametrization of the components of ∂X, or with an embedding $X \subset \mathbb{C}$ which induces such parametrizations according to our conventions. Suppose $P \in \mathscr{C}'_X$ is a Γ bundle over X with basepoints on the boundary. Let Y be a component of ∂X and suppose the holonomy of $P|_Y$ is x. Then the basepoint in $P|_Y$ and the parametrization of Y determine an isomorphism $P|_Y \cong Q_{[x]}$, and so an isomorphism $T_{\partial X}^{(\alpha)}(\partial P) \cong T_{S^1}^{(\alpha)}(Q_{[x]}) = \mathscr{G}_{[x]}$. This $\mathbb{T}$-gerbe is trivialized by our choice in (8.2). Hence the classical action (2.6) of P can be identified with a $\mathbb{T}$-torsor $T_X^{(\alpha)}(P)$, using this trivialization. As in (4.7) this $\mathbb{T}$-torsor determines a hermitian line, and by taking an inverse limit we obtain a line $L_X^{(\alpha)}([P])$ depending only on the equivalence class of P. (This line could degenerate to 0 if X has a closed component.) Let

$$L_X^{(\alpha)} \to \overline{\mathscr{C}'_X}$$

denote the resulting line bundle over the finite set $\overline{\mathscr{C}'_X}$. The following generalizes Proposition 7.10.

Proposition 8.10. *Let $X \subset \mathbb{C}$ be a compact oriented 2-manifold.[30] Then under the isomorphism (8.9) the path integral over X is space of L^2 sections*

$$Z_X^{(\alpha)} \cong L^2(\overline{\mathscr{C}'_X}, L_X^{(\alpha)}), \tag{8.11}$$

with the $\tilde{\mathscr{G}}$ action induced by gluing cylinders onto components of ∂X.

Proof. The only new point is an isometry

$$L_C(x, g) \cong L(x, g), \tag{8.12}$$

where $L_C(x, g) = L_C([P_{\langle x, g \rangle}])$ for $P_{\langle x, g \rangle} \to C$ a pointed bundle over the cylinder corresponding to $\langle x, g \rangle \in \mathscr{G}$ under (7.9). Recall the proof of Proposition 7.10, where we show that the basepoints determine an isomorphism $\partial P_{\langle x, g \rangle} \cong Q_{[x]} \sqcup Q_{[x]}$, and so $P_{\langle x, g \rangle}$ determines an element of $\text{Aut} \, Q_{[x]}$. The classical action of $P_{\langle x, g \rangle}$ is then an

---

29 In [FQ] we defined a coalgebra structure instead
30 The same arguments apply to arbitrary surfaces with *parametrized* boundary. If the surface has closed components, then we must modify the inner product in (8.11)

element of $\text{Aut}(\mathscr{G}_{[x]}) \cong \mathscr{T}_1$. But by (5.2) the classical action $T_C([0,1] \times Q_{[x]})$ of a product bundle is trivial, and then the desired isometry (8.12) follows easily.

We adopt the notation

$$l_C(x, e) = l(x, e)$$

for the element in (8.5).

We need a few preliminaries to generalize Proposition 7.12. For any $x \in \Gamma$ there is a trivialization

$$l_C(x, x) \in L_C(x, x) \tag{8.13}$$

as follows. By (7.22) the diffeomorphism $\tau : C \to C$ satisfies $\tau^* \langle x, e \rangle = \langle x, x \rangle$. Notice that τ is the identity on ∂C, so it respects the trivializations (8.2). By the functoriality of the classical action (2.7) the diffeomorphism τ induces an isomorphism $T_C(x, x) \cong T_C(x, e)$, and so an isometry $L_C(x, x) \cong L_C(x, e)$. Then $l_C(x, x)$ corresponds to $l_C(x, e) \in L_C(x, e)$ [cf. (8.5)].

Next, consider the diffeomorphism of the cylinder C

$$\iota : [0, 1] \times S^1 \to [0, 1] \times S^1$$
$$\langle t, s \rangle \mapsto \langle -t, -s \rangle.$$

It is not the identity on ∂C. Rather, $\partial \iota$ swaps the two boundary components, and if we identify them in the obvious way, $\partial \iota$ is the reflection $s \mapsto -s$. By the functoriality (2.7) and the orientation axiom (2.9) this reflection induces an isomorphism $T_{S^1}(Q_{[x]})^{-1} \to T_{S^1}(Q_{[x^{-1}]})$, and so we can compare the trivializations in (8.2). Use this isomorphism to define the $\mathbb{T}$-torsor

$$T_{[x]} = G_{[x]} \cdot G_{[x^{-1}]}. \tag{8.14}$$

Let $L_{[x]}$ be the hermitian line corresponding to the $\mathbb{T}$-torsor $T_{[x]}$. Then since ι induces the map $\iota^* \langle x, g \rangle = \langle g x^{-1} g^{-1}, g^{-1} \rangle$ on fields, the induced isometry on the classical action is

$$\iota_* : L_C(g x^{-1} g^{-1}, g^{-1}) \otimes L_{[x]} \to L_C(x, g) \otimes L_{[g x g^{-1}]}. \tag{8.15}$$

Of course, $L_{[g x g^{-1}]} = L_{[x]}$, so we can cancel these terms from (8.15).

We use (7.23) to identify an equivalence class of pointed bundles over the pair of pants P with an element in $\mathscr{G} \times \mathscr{G}$ (see Fig. 8). Let

$$L_P(x_1 | x_2) = L_P(x_1, e; x_2, e)$$

denote the hermitian line obtained from the classical action on the equivalence class corresponding to $\langle x_1, e \rangle \times \langle x_2, e \rangle$. We claim that for any $x_1, x_2, x_3, g \in \Gamma$ there are isometries

$$\phi_{x_1, x_2, x_3} : L_P(x_1 x_2 | x_3) \otimes L_P(x_1 | x_2) \to L_P(x_1 | x_2 x_3) \otimes L_P(x_2 | x_3), \tag{8.16}$$

$$\sigma_{x_1, x_2} : L_P(x_1 | x_2) \to L_P(x_1 x_2 x_1^{-1} | x_1) \otimes L_C(x_2, x_1), \tag{8.17}$$

and

$$\gamma_{x_1, x_2, g} : L_C(x_1 x_2, g) \otimes L_P(x_1 | x_2)$$
$$\to L_P(g x_1 g^{-1} | g x_2 g^{-1}) \otimes L_C(x_1, g) \otimes L_C(x_2, g). \tag{8.18}$$
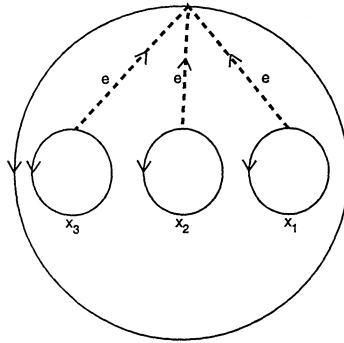
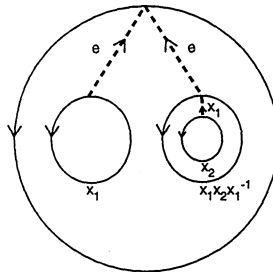**Fig. 9.** Field used in the proof of (8.16)
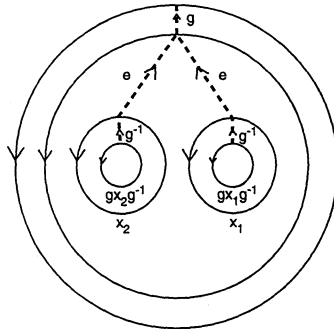

Fig. 10. The isometry (8.19)

Fig. 11. The isometry (8.18)

For (8.16) we use the gluings in Fig. 3 to see that both sides are isomorphic to the bundle $L(x_1, e; x_2, e; x_3, e)$ indicated in Fig. 9. The isometry (8.17) is constructed from the braiding diffeomorphism β, which by (7.28) induces an isometry

$$\beta^* : L_P(x_1, e; x_2, e) \to L_P(x_2, x_1; x_1, e),$$

and from the gluing in Fig. 10, which induces an isometry

$$L_P(x_2, x_1; x_1, e) \to L_P(x_1 x_2 x_1^{-1}, e; x_1, e) \otimes L_C(x_2, x_1). \tag{8.19}$$

The isometry (8.18) is constructed from the gluing in Fig. 11 and the duality

$$L_C(x_i, g) \otimes L_C(x_i, g^{-1}) \to L_C(x_i; e) \cong \mathbb{C}, \tag{8.20}$$

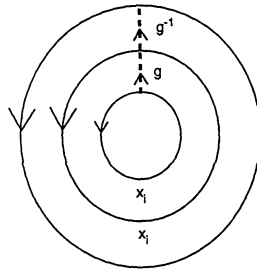**Fig. 12.** The duality (8.20)

which follows from Fig. 12 and (8.5).

Proposition 8.21. *Consider the finite gauge theory described in Assertion 4.12 with lagrangian $\alpha \in C^3(B\Gamma; \mathbb{R}/\mathbb{Z})$. This field theory and the trivializations chosen in (8.2) determine the following structure on $E^{(\alpha)}$.*
(a) *(Automorphism of the identity (5.11)) For $W \in E$ we have*

$$\theta_W|_{W_x} = A_x(l_C(x,x)): W_x \to W_x.$$

(b) *(Involution (5.14)) For $W \in E$ the dual $W^* \in \bar{E}$ is defined by $(W^*)_x = W^*_{x^{-1}} \otimes L^*_{[x]}$ and $A_g^{W^*} = (A_{g^{-1}}^W)^*$.*
(c) *(Identity (5.16) The identity $\mathbf{1}$ is*

$$\mathbf{1}_x = \begin{cases} L_{D^2}([P_{\mathrm{triv}}]), & x = e; \\ 0, & x \neq e, \end{cases} \tag{8.22}$$

with the action of the central extension $\tilde{C}_e$ on $\mathbf{1}_e$ determined by gluing a cylinder C to a disk D^2.
(d) *(Multiplication (5.19)) The tensor product of $W_1, W_2 \in E$ is*

$$(W_1 \odot W_2)_x = \bigoplus_{x_1 x_2 = x} L_P(x_1|x_2) \otimes (W_1)_{x_1} \otimes (W_2)_{x_2} \tag{8.23}$$

with the Γ action

$$A_g^{W_1 \odot W_2} = (\mathrm{id} \otimes A_g^{W_1} \otimes A_g^{W_2}) \circ (\gamma_{x_1, x_2, g} \otimes \mathrm{id}) \tag{8.24}$$

on $L_P(x_1|x_2) \otimes (W_1)_{x_1} \otimes (W_2)_{x_2}$.
(e) *(Associator (5.22)) For $W_1, W_2, W_3 \in E$ the associator is*

$$\varphi_{W_1, W_2, W_3} = \phi_{x_1, x_2, x_3} \otimes \mathrm{id}$$

on $L_P(x_1|x_2) \otimes L_P(x_1 x_2|x_3) \otimes (W_1)_{x_1} \otimes (W_2)_{x_2} \otimes (W_3)_{x_3}$.
(f) *(R-matrix (5.25)) For $W_1, W_2 \in E$ we have*

$$\begin{aligned} R_{W_1, W_2} : L_P(x_1|x_2) &\otimes (W_1)_{x_1} \otimes (W_2)_{x_2} \\ &\to L_P(x_1 x_2 x_1^{-1}|x_1) \otimes (W_2)_{x_1 x_2 x_1^{-1}} \otimes (W_1)_{x_1} \\ l \otimes w_1 &\otimes w_2 \\ &\mapsto (\mathrm{id} \otimes A_{x_1}^{W_2})(\sigma_{x_1, x_2}(l) \otimes (w_2)) \otimes w_1 \end{aligned} \tag{8.25}$$

and all other components are zero.

A few remarks. First, we omitted transposition of ordinary tensor products of vector spaces from the notation in (8.24) and (8.25). Also, the conformal weight is

defined by (7.18) with $A_x(l_C(x,x))$ replacing A_x on the left-hand side. The special (inverse ribbon) element of $H^{(\alpha)}$ replacing (7.19) is

$$v^{(\alpha)} = \sum_x l_C(x,x)\,. \qquad (8.26)$$

In (b) the isometry (8.15) is implicit in the equation $A_g^{W^*} = (A_{g^{-1}}^W)^*$. In (8.22), $[P_{\mathrm{triv}}]$ is the equivalence class of the trivial bundle over the disk, and gluing a cylinder gives isometries

$$L_C(e,g) \otimes L_{D^2}([P_{\mathrm{triv}}]) \to L_{D^2}([P_{\mathrm{triv}}])\,, \qquad (8.27)$$

which is the required action of $\tilde{C}_e$. Of course, (8.27) is equivalent to a linear map

$$\varepsilon\colon \bigoplus_g L_C(e,g) \to \mathbb{C}\,. \qquad (8.28)$$

The verifications of (5.15), (5.17), (5.20), (5.21), (5.23), (5.24), (5.26), (5.27), and (5.28) directly from the data listed in Proposition 8.21 require some additional identities in the classical theory easily derived from simple gluings of the type already considered.

The proof of Proposition 8.21 is a straightforward extension of the proof of Proposition 7.12, so we omit it.

It remains to deduce a quasi-Hopf algebra structure on $H^{(\alpha)}$. For this we need to choose trivializing elements[31]

$$l_P(x_1|x_2) \in L_P(x_1|x_2)\,. \qquad (8.29)$$

Define

$$\omega(x_1,x_2,x_3) = \frac{l_P(x_1|x_2 x_3) \otimes l_P(x_2|x_3)}{\phi_{x_1,x_2,x_3}(l_P(x_1 x_2|x_3) \otimes l_P(x_1|x_2))} \in \mathbb{T}\,. \qquad (8.30)$$

An argument with gluings and ungluings of the four times punctured disk shows that ω satisfies the cocycle identity

$$\frac{\omega(x_1,x_2,x_3)\omega(x_1,x_2 x_3,x_4)\omega(x_2,x_3,x_4)}{\omega(x_1,x_2,x_3 x_4)\omega(x_1 x_2,x_3,x_4)} = 1\,, \qquad x_1,x_2,x_3,x_4 \in \Gamma\,. \qquad (8.31)$$

In a sense this is the classical analog of the pentagon diagram (5.23). So ω defines a class $[\omega] \in H^3(\Gamma; \mathbb{R}\mathbb{Z})$ in group cohomology. The following proposition is analogous to [FQ, Proposition 3.14]. We state it without proof.

Proposition 8.32. *Under the isomorphism $H^\bullet(\Gamma) \cong H^\bullet(B\Gamma)$ the group cohomology class $[\omega]$ corresponds to the singular cohomology class $[\alpha]$.*

Now we write the quasitriangular quasi-Hopf structure on $H^{(\alpha)}$ induced from the data in Proposition 8.21. The coproduct is

$$\Delta^{(\alpha)}(l) = \sum_{x_1 x_2 = x} \frac{\gamma_{x_1,x_2,g}(l \otimes l_P(x_1|x_2))}{l_P(g x_1 g^{-1}|g x_2 g^{-1})}\,, \qquad l \in L_C(x,g)\,. \qquad (8.33)$$

---

[31] From the point of view of the reconstruction theorems, the reason we need to choose these elements is to obtain a functor from E to the category of vector spaces which preserves the tensor product. Hence the line which appears in (8.23) must be trivialized

The counit is the linear map defined in (8.28); it maps $L_C(x,g)$ to 0 if $x \neq 0$. The antipode is computed from Proposition 8.21(b) as the inverse

$$S^{(\alpha)} : L_C(x,g) \to L_C(gx^{-1}g^{-1}, g^{-1}) \tag{8.34}$$

of (8.15). The quasitriangular element $R^{(\alpha)} \in H^{(\alpha)} \otimes H^{(\alpha)}$ is

$$R^{(\alpha)} = \sum_{x_1, x_2} l_C(x_1, e) \otimes \frac{\sigma_{x_1, x_2}(l_P(x_1 | x_2))}{l_P(x_1 x_2 x_1^{-1} | x_1)} . \tag{8.35}$$

Finally, there is an invertible element $\varphi^{(\alpha)} \in H^{(\alpha)} \otimes H^{(\alpha)} \otimes H^{(\alpha)}$ which implements the quasiassociativity condition

$$(\mathrm{id} \otimes \Delta^{(\alpha)}) \Delta^{(\alpha)}(l) = (\varphi^{(\alpha)})(\Delta^{(\alpha)} \otimes \mathrm{id}) \Delta^{(a)}(l) (\varphi^{(a)})^{-1} , \quad l \in H^{(\alpha)} .$$

This is the element

$$\varphi^{(\alpha)} = \sum_{x_1, x_2, x_3} \omega(x_1, x_2, x_3)^{-1} l_C(x_1, e) \otimes l_C(x_2, e) \otimes l_C(x_3, e) . \tag{8.36}$$

A routine check shows that the modular tensor category described in Proposition 8.21 is the category of representations of the quasitriangular quasi-Hopf algebra $H^{(\alpha)}$.

The quasi-Hopf algebra in [DPR, Sect. 3.2] looks similar to $H^{(a)}$, but is expressed in terms of a basis. We will choose this basis geometrically in the next section, and so construct an isomorphism between $H^{(\alpha)}$ and the algebra in [DPR, Sect. 3.2].

9. Higher Gluing and Good Trivializations

In this section we introduce a "higher order gluing law" for gluing manifolds with corners. The corners we use are in codimension two; clearly there are generalizations of this gluing law to higher codimension. Also the gluing law we use here pertains to the classical theory; there are quantum versions as well. While the formulation of this gluing law is rather abstract, the computations which follow should make its meaning clear. We study the classical theory over the interval $[0, 1]$. We choose trivializations (9.4) which replace the trivializations (8.2) we chose in the last section. The procedure here is more natural than that Sect. 8. Furthermore, the trivializations (9.4) induce trivializations of the lines $L_P(x_1 | x_2)$ which we previously chose separately in (8.29), and they also induce trivializations of the lines $L(x, g) \cong L_C(x, g)$. The latter amount to a basis of the algebra $H^{(\alpha)}$ in (8.6). In terms of this basis the quasitriangular quasi-Hopf structure we computed in Sect. 8 is exactly the one constructed in [DPR, Sect. 3.2], as we verify. The reader may wish to consider analogous, but simpler, computations in the $1 + 1$ dimensional theory.

We begin with a statement of the gluing law which should hold in any classical field theory, but for our purposes we consider the classical $d + 1$ dimensional theory of Assertion 2.5. Suppose X is a compact oriented $(d + 2 - n)$-manifold and $Y \hookrightarrow X$
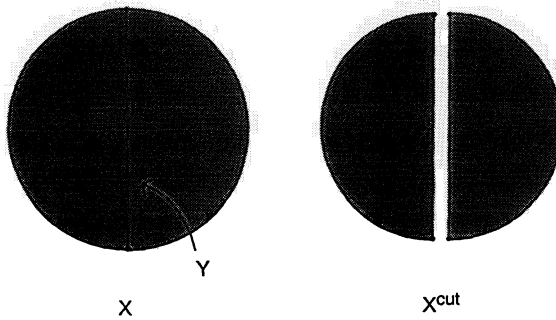
Fig. 13. Gluing manifolds with corners

a *neat* oriented codimension one submanifold (Fig. 13), that is, $\partial Y = Y \cap \partial X$ and Y intersects ∂X transversely. Then $\partial X \hookrightarrow \partial X$ is a closed oriented codimension one submanifold, and

$$\partial X^{\mathrm{cut}} = Y \cup_{\partial Y} (\partial X)^{\mathrm{cut}} \cup_{-\partial X} -Y,$$

$$\partial(\partial X)^{\mathrm{cut}} = -\partial Y \sqcup \partial Y.$$

Suppose $P \to X$ is a Γ bundle and $Q \to Y$ its restriction to Y. Then the usual gluing law Assertion 2.5(d) implies that there is an isomorphism[32]

$$\mathrm{Tr}_{12,34} : T_Y(Q) \cdot T_{(\partial X)^{\mathrm{cut}}}((\partial P)^{\mathrm{cut}}) \cdot T_Y(Q)^{-1} \to T_{\partial X^{\mathrm{cut}}}(\partial P^{\mathrm{cut}}). \tag{9.1}$$

Note that the left-hand side of (9.1) is an element of

$$T_{\partial Y}(\partial Q) \cdot T_{\partial Y}(\partial Q)^{-1} \cdot T_{\partial Y}(\partial Q) \cdot T_{\partial Y}(\partial Q)^{-1}.$$

Also, there is an isomorphism

$$\mathrm{Tr}_{14,23} : T_Y(Q) \cdot T_{(\partial X)^{\mathrm{cut}}}((\partial P)^{\mathrm{cut}}) \cdot T_Y(Q)^{-1} \to T_{\partial X}(\partial P).$$

and so finally an isomorphism

$$\mathrm{Tr}_Q = \mathrm{Tr}_{14,23} \circ \mathrm{Tr}_{12,34}^{-1} : T_{\partial X^{\mathrm{cut}}}(\partial P^{\mathrm{cut}}) \to T_{\partial X}(\partial P).$$

Assertion 9.2. *In the situation described, there is a natural isomorphism*

$$\mathrm{Tr}_Q(e^{2\pi i S_{X^{\mathrm{cut}}}(P^{\mathrm{cut}})}) \to e^{2\pi i S_X(P)}. \tag{9.3}$$

Now we resume our work from Sect. 8, retaining the notations there. As in Sect. 6 fix a trivial bundle $R_{\mathrm{triv}} = pt \times \Gamma$ over a point. Use the correspondence (6.2) to identify equivalence classes of fields over $[0, 1]$ trivialized over the endpoints with elements of Γ. Then the classical action of the equivalence class $[Q_x]$ corresponding to $x \in \Gamma$ is a $\mathbb{T}$-gerbe $\mathscr{G}_x = T_{[0,1]}^{(\alpha)}([Q_x])$. Choose trivializing elements

$$G_x \in \mathscr{G}_x = T_{[0,1]}^{(\alpha)}([Q_x]), \qquad x \in \Gamma. \tag{9.4}$$

Now for $x_1, x_2 \in \Gamma$ we glue $[Q_{x_2}]$ and $[Q_{x_1}]$ to obtain $[Q_{x_1 x_2}]$. Hence the isomorphism (2.12) implies that there is an isomorphism

$$\mathscr{G}_{x_1} \cdot \mathscr{G}_{x_2} \to \mathscr{G}_{x_1 x_2}. \tag{9.5}$$

---

[32] We use the notation $T_Y(Q)$ for the classical action, even though Y is not closed

In particular, (9.5) implies that $\mathscr{G}_e$ has a trivialization compatible with gluing, and we assume that G_e is that trivialization. In other words,

$$G_e \cdot G_e = G_e. \tag{9.6}$$

Define the $\mathbb{T}$-torsor T_{x_1, x_2} by the equation

$$G_{x_1} \cdot G_{x_2} = G_{x_1 x_2} \cdot T_{x_1, x_2}, \qquad x_1, x_2 \in \Gamma, \tag{9.7}$$

where we implicitly use the isomorphism (9.5) to compare the two sides. Equation (9.6) implies that $T_{e,e} = \mathbb{T}$. Three intervals can be glued together in two different ways to obtain a single interval. The behavior of the classical action under iterated gluings, which we did explicitly state in Assertion 2.5(d), implies that for any $x_1, x_2, x_3 \in \Gamma$ the diagram

$$
\begin{array}{ccc}
(\mathscr{G}_{x_1} \cdot \mathscr{G}_{x_2}) \cdot \mathscr{G}_{x_3} & \longrightarrow & \mathscr{G}_{x_1 x_2} \cdot \mathscr{G}_{x_3} \\
& & \searrow \\
& & \qquad \mathscr{G}_{x_1 x_2 x_3} \\
& & \nearrow \\
\mathscr{G}_{x_1} \cdot (\mathscr{G}_{x_2} \cdot \mathscr{G}_{x_3}) & \longrightarrow & \mathscr{G}_{x_1} \cdot \mathscr{G}_{x_2 x_3}
\end{array}
$$

commutes up to a natural transformation. Using (9.7) this natural transformation amounts to an isomorphism

$$T_{x_1, x_2} \cdot T_{x_1 x_2, x_3} \cong T_{x_2, x_3} \cdot T_{x_1, x_2 x_3}, \qquad x_1, x_2, x_3 \in \Gamma. \tag{9.8}$$

In particular, taking two of x_1, x_2, x_3 to be e we deduce isomorphisms

$$T_{x,e} \cong T_{e,x} \cong \mathbb{T}, \qquad x \in \Gamma. \tag{9.9}$$

Now we explain the relationship of the choices (9.4) to the choices (8.2) made in the last section. Fix $x \in \Gamma$ and consider the bundle $Q_{[x]} \to S^1$ with basepoint f_x, as chosen in Sects. 7–8. Cutting the circle at its basepoint, and using the basepoint f_x to identify $\partial Q_{[x]}^{\mathrm{cut}}$ with $R_{\mathrm{triv}} \sqcup R_{\mathrm{triv}}$, we obtain from the gluing law (2.12) an isomorphism

$$\mathscr{G}_x \to \mathscr{G}_{[x]}. \tag{9.10}$$

It is *not* neccessarily true that the trivializations of $\mathscr{G}_{x'}$ in (9.4) for different $x' \in [x]$ lead to the same trivialization of $\mathscr{G}_{[x]}$.[33] Now let X be a compact oriented 2-manifold with parametrized boundary and $P \in \mathscr{C}_X'$ a bundle with basepoints on the boundary. Suppose Y is a component of ∂X and $P|_Y$ has holonomy x. The basepoint and parametrization induce an identification $P|_Y \cong Q_{[x]}$, and so by (9.10) an isomorphism $T_Y^{(\alpha)}(P|_Y) \cong \mathscr{G}_x$. We trivialize this $\mathbb{T}$-gerbe using (9.4). Then as in the argument preceding Proposition 8.10 the classical action of P is a $\mathbb{T}$-torsor. It is not the same $\mathbb{T}$-torsor obtained in Sect. 8, since we use different trivializations. None of the subsequent arguments are affected by this change, and we use these new trivializations in what follows.

---

[33] In this connection notice that whereas $T(x, g_{x,x'})$ was chosen to be $\mathbb{T}$ in (8.4), this torsor is nontrivial with our current set of choices [cf. (9.16)]
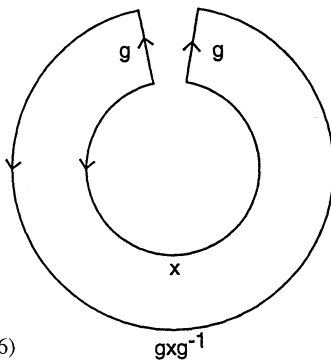
Fig. 14. The isomorphism (9.16)

As a first application of Assertion 9.2 we claim that the classical action of the trivial bundle over the disk is

$$T_{D^2}([P_{\text{triv}}]) = G_e \,. \tag{9.11}$$

This can be deduced from the gluing in Fig. 13 and (9.6).

Next, choose trivializing elements

$$t_{x_1, x_2} \in T_{x_1, x_2} \,, \qquad x_1, x_2 \in \Gamma \,. \tag{9.12}$$

We assume that

$$t_{x,e} = t_{e,x} = 1 \,, \qquad x \in \Gamma \,, \tag{9.13}$$

under the isomorphism (9.9). Define $\omega(x_1, x_2, x_3) \in \mathbb{T}$ by the equation

$$t_{x_1 x_2, x_3} \cdot t_{x_1, x_2} \cdot \omega(x_1, x_2, x_3) = t_{x_1, x_2 x_3} \cdot t_{x_2, x_3} \,, \qquad x_1, x_2, x_3 \in \Gamma \,, \tag{9.14}$$

where the equality refers to the isomorphism (9.8). The behavior of the classical action under iterated gluings of four intervals shows that ω satisfies the cocycle identity (8.31).

Let L_{x_1, x_2} be the hermitian line corresponding to the $\mathbb{T}$-torsor T_{x_1, x_2} and

$$l_{x_1, x_2} \in L_{x_1, x_2} \tag{9.15}$$

the element of unit norm corresponding to t_{x_1, x_2}. We claim that with the choices of trivializations we have made, the higher gluing law (9.3) constructs isometries

$$L_C(x, g) \cong \frac{L_{g,x}}{L_{gxg^{-1}, g}} \,, \qquad x, g \in \Gamma \,, \tag{9.16}$$

$$L_P(x_1 | x_2) \cong L_{x_1, x_2} \,, \qquad x_1, x_2 \in \Gamma \,. \tag{9.17}$$

The isomorphism (9.16) is derived from the gluing in Fig. 14, where we obtain the cylinder C by gluing a disk D^2 along part of its boundary. The usual gluing law (2.12) applied to ∂D^2 yields an isomorphism

$$\mathscr{G}_e \cong \mathscr{G}_{gxg^{-1}} \cdot \mathscr{G}_g \cdot \mathscr{G}_x^{-1} \cdot \mathscr{G}_g^{-1} \,,$$

and a short computation with (9.7) shows that under this isomorphism we have

$$G_e = G_{gxg^{-1}} \cdot G_g \cdot G_x^{-1} \cdot G_g^{-1} \cdot \frac{T_{g,x}}{T_{gxg^{-1}, g}} \,.$$
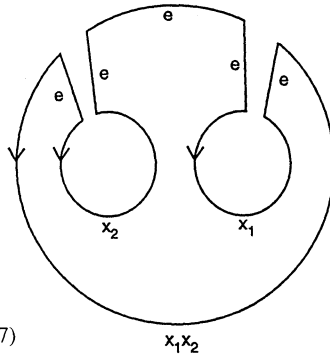
Fig. 15. The isomorphism (9.17)

Now (9.16) follows from (9.11) and the gluing law. The isomorphism (9.17) is derived in a similar manner from Fig. 15. In that figure

$$\mathscr{G}_e \cong \mathscr{G}_{x_1 x_2} \cdot \mathscr{G}_{x_2}^{-1} \cdot \mathscr{G}_{x_1}^{-1},$$

and under this isomorphism

$$G_e = G_{x_1 x_2} \cdot G_{x_2}^{-1} \cdot G_{x_1}^{-1} \cdot T_{x_1, x_2}.$$

The gluing law and (9.11) imply (9.17).

We use (9.15) to trivialize the lines $L_C(x, g)$ and $L_P(x_1 | x_2)$. Namely, set

$$l_C(x, g) = \frac{l_{g,x}}{l_{gxg^{-1}, g}} \tag{9.18}$$

and

$$l_P(x_1 | x_2) = l_{x_1, x_2}. \tag{9.19}$$

The elements in (9.19) replace the arbitrary choice (8.29) we made in Sect. 8. We now define the quasi-Hopf quasitriangular structure on $H^{(\alpha)}$ in terms of the choices (9.19). The elements in (9.18) form a basis of $H^{(\alpha)}$, and our last task is to compute the quasi-Hopf quasitriangular structure in terms of this basis. Observe also that (9.18) agrees with the special trivializations (8.5) and (8.13).

First, we compute the isomorphisms (8.16)–(8.18) in terms of (9.18) and (9.19). We make the obvious computations and leave the justification to the reader. (This involves the compatibility of various gluings and diffeomorphisms.) The isomorphism ϕ_{x_1, x_2, x_3} is still expressed by (8.30), which follows directly from (9.14). For σ_{x_1, x_2} we compute

$$\frac{\sigma_{x_1, x_2}(l_P(x_1 | x_2))}{l_P(x_1 x_2 x_1^{-1} | x_1) \otimes l_C(x_2, x_1)} = \frac{l_{x_1, x_2}}{l_{x_1 x_2 x_1^{-1}, x_1} \otimes \dfrac{l_{x_1, x_2}}{l_{x_1 x_2 x_1^{-1}, x_1}}} = 1.$$

A direct computation yields

$$\frac{\gamma_{x_1, x_2, g}(l_C(x_1 x_2, g) \otimes l_P(x_1 | x_2))}{l_P(g x_1 g^{-1} | g x_2 g^{-1}) \otimes l_C(x_1, g) \otimes l_C(x_2, g)} = \frac{\omega(g, x_1, x_2)\omega(g x_1 g^{-1}, g x_2 g^{-1}, g)}{\omega(g x_1 g^{-1}, g, x_2)}.$$

Now for the structure on $H^{(\alpha)}$. A short computation shows that the multiplication (8.7) is

$$l_C(g_1 x g_1^{-1}, g_2) \cdot l_C(x, g_1) = \frac{\omega(g_2, g_1, x)\omega(g_2 g_1 x g_1^{-1} g_2^{-1}, g_2, g_1)}{\omega(g_2, g_1 x g_1^{-1}, g_1)} l_C(x, g_2 g_1). \quad (9.20)$$

The identity element is (8.8):

$$1 = \sum_x l_C(x, e). \quad (9.21)$$

The coproduct (8.33) is

$$\Delta^{(\alpha)}(l_C(x, g))$$
$$= \sum_{x_1 x_2 = x} \frac{\omega(g, x_1, x_2)\omega(g x_1 g^{-1}, g x_2 g^{-1}, g)}{\omega(g x_1 g^{-1}, g, x_2)} l_C(x_1, g) \otimes l_C(x_2, g). \quad (9.22)$$

The counit (8.28) is

$$\varepsilon(l_C(x, g)) = \begin{cases} 1, & \text{if } x = e; \\ 0, & \text{otherwise}. \end{cases} \quad (9.23)$$

The quasitriangular element (8.35) is

$$R^{(\alpha)} = \sum_{x_1, x_2} l_C(x_1, e) \otimes l_C(x_2, x_1). \quad (9.24)$$

The element $\varphi^{(a)}$ which measures the deviation from coassociativity is (8.36):

$$\varphi^{(\alpha)} = \sum_{x_1, x_2, x_3} \omega(x_1, x_2, x_3)^{-1} l_C(x_1, e) \otimes l_C(x_2, e) \otimes l_C(x_3, e). \quad (9.25)$$

Recall that the antipode (8.34) is the inverse of (8.15). With the trivializations of this section Eq. (8.14) is replaced by the equation

$$G_x \cdot G_{x^{-1}} = G_e \cdot T_{x, x^{-1}},$$

and so (8.15) by a map

$$\iota_* : L_C(g x^{-1} g^{-1}, g^{-1}) \otimes L_{x, x^{-1}} \to L_C(x, g) \otimes L_{g x g^{-1}, g x^{-1} g^{-1}}.$$

The ratio

$$\frac{l_C(x, g) \otimes l_{g x g^{-1}, g x^{-1} g^{-1}}}{\iota_*(l_C(g x^{-1} g^{-1}, g^{-1}) \otimes l_{x, x^{-1}})}$$

is the numerical factor in the expression

$$S^{(\alpha)}(l_C(x, g)) = \frac{\omega(g^{-1}, g x^{-1} g^{-1}, g)\omega(g x g^{-1}, g, x^{-1})}{\omega(g^{-1}, g, x^{-1})\omega(x^{-1}, g^{-1}, g)\omega(g, x, x^{-1})\omega(g x g^{-1}, g x^{-1} g^{-1}, g)}$$
$$\times l_C(g x^{-1} g^{-1}, g^{-1}) \quad (9.26)$$

for the antipode. The inverse ribbon element is (8.26):

$$v^{(\alpha)} = \sum_x l_C(x, x).$$

Equations (9.20)–(9.26) are exactly the equations in [DPR, Sect. 3.2], up to some changes in notation.
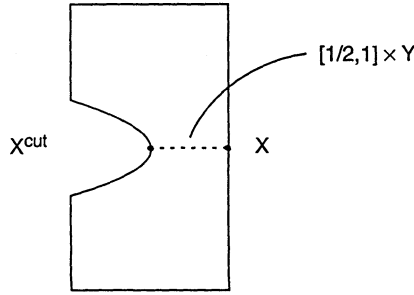
Fig. 16. Gluing along a closed submanifold

Suppose we replace the trivializations t_{x_1,x_2} in (9.12) with $\beta(x_1,x_2)t_{x_1,x_2}$ for some $\beta(x_1,x_2) \in \mathbb{T}$. We assume that $\beta(x,e) = \beta(e,x) = 1$ for all $x \in \Gamma$ so that (9.13) is respected. Then this change of basis has the effect of *twisting* (cf. [Dr]) the formulas (9.20)–(9.26) by the element

$$\sum_{x_1,x_2} \beta(x_1,x_2) l_C(x_1,e) \otimes l_C(x_2,e).$$

We conclude with some brief general remarks about gluing. The first should be valid for arbitrary topological theories in any dimension. Consider $Y \hookrightarrow X$ a closed oriented codimension one submanifold and X^{cut} the cut manifold as in Assertion 2.5(d). Form a new manifold W by identifying the two pieces in the boundary of $[0,1] \times X^{\mathrm{cut}}$ which correspond to $\left[\frac{1}{2}, 1\right] \times Y$, as illustrated in Fig. 16. Then

$$\partial W = X \sqcup -X^{\mathrm{cut}} \sqcup \left[0, \tfrac{1}{2}\right] \times Y \sqcup -\left[0, \tfrac{1}{2}\right] \times Y.$$

In the classical theory we also are given a field P on X and the corresponding P^{cut} on X^{cut}. We claim that the gluing (2.12) of the classical action [resp. the gluing (4.17) of the path integral] is computed by the classical action (resp. path integral) over W. For this we trivialize the classical action (resp. path integral) over $\left[0, \tfrac{1}{2}\right]$ using (5.2) [resp. (5.4)]. Such pictures help compute the gluing isometries.

Figure 16 is a schematic for arbitrary dimensions as well as an exact picture of the gluing of two intervals. The reader may wish to contemplate various gluings of this figure and relate the computations in Sect. 8 to those in Sect. 9.

There should also be refined gluing laws of the following sort. Recall from Proposition 5.29 that in a $2 + 1$ dimensional theory $E(S^1)$ is a "higher commutative associative algebra with compatible real structure" which presumably is semisimple (in a unitary theory). In particular, it is a braided monoidal category, or better a tortile category. For such categories one can define a "Grothendieck ring" $\mathrm{Groth}(E(S^1))$ (see [Y2, Proposition 26]). If $E(S^1)$ is the category of representations of a quasi-Hopf algebra H, then the Grothendieck ring is the ring of equivalence classes of representations, the multiplication given by the tensor product. Equation (5.5) is a gluing law on the level of inner product spaces, and in this case surely there is an extension to an isomorphism

$$E(S^1 \times S^1) \cong \mathrm{Groth}(E(S^1))$$

of algebras. ($E(S^1 \times S^1)$ is an algebra by the remark at the end of Sect. 5. It is commonly called the *Verlinde algebra*.) The Grothendieck ring is the "dimension" of

$E(S^1)$ from the point of view of (5.5). Notice that $\mathrm{Groth}(E(S^1))$ has a distinguished basis of irreducible representations. These are the "labels" mentioned in Sect. 7.

Appendix: Integration of Singular Cocycles Revisited

In [FQ, Appendix B] we describe some elements of an integration theory for singular cocycles with coefficients in $\mathbb{R}/\mathbb{Z}$. Here we describe an extension of that theory to higher codimensions in terms of the higher algebra discussed in Sect. 1. Notice that we do not introduce any basepoints or special choices, as in [FQ, Proposition B.5]. Instead, we extend the integration theory in a more intrinsic manner to all codimensions. The higher algebra of Sect. 1 is a prerequisite to this appendix.

Our goal is to integrate a singular $(d + 1)$-cocycle α over compact oriented manifolds of any dimension less than or equal to $(d + 1)$. In [FQ] we described the integral of α over closed oriented $(d+1)$-manifolds, compact oriented $(d+1)$-manifolds (possibly with boundary), and closed oriented d-manifolds. In the easiest case α is a $(d + 1)$-cocycle on a closed oriented $(d + 1)$-manifold X. Then if $x \in C_{d+1}(X)$ is an oriented cycle which represents the fundamental class $[X] \in H_{d+1}(X)$, we form the pairing $e^{2\pi i \alpha(x)} \in \mathbb{R}/\mathbb{Z}$. If x' is another representative, then $x' - x = \partial w$ for some $w \in C_{d+1}(X)$. Hence $\alpha(x') - \alpha(x) = \alpha(\partial w) = \delta\alpha(w) = 0$ since α is a cocycle. This is the usual argument which shows that the integral

$$\exp\left(2\pi i \int_X \alpha\right) \in \mathscr{T}_0 = \mathbb{T} \tag{A.1}$$

is well-defined. In fact, (A.1) can be viewed as the pairing between the cohomology class $[\alpha] \in H^{d+1}(X; \mathbb{R}/\mathbb{Z})$ and the homology class $[X] \in H_{d+1}(X)$. This is the only one of the integrations we discuss which has cohomological meaning.

Now suppose α is a $(d + 1)$-cocycle on a closed oriented d-manifold Y. Then we claim that there is a well-defined integral

$$I_{Y,\alpha} = \exp\left(2\pi i \int_Y \alpha\right) \in \mathscr{T}_1 \tag{A.2}$$

which is a $\mathbb{T}$-torsor. The following is a slight modification of what appears in [FQ, Appendix B]. The justification for terming this an "integral" are the properties listed in Assertion A.4. Let $\mathscr{C}_Y$ be the category whose objects are oriented cycles $y \in C_d(Y)$ which represent the fundamental class $[Y] \in H_d(Y)$, and with a unique morphism $y \to y'$ for all $y, y' \in \mathscr{C}_Y$. Define a functor $\mathscr{F}_{Y;\alpha} : \mathscr{C}_Y \to \mathscr{T}_1$ by $\mathscr{F}_{Y;\alpha}(y) = \mathbb{T}$ for each y and $\mathscr{F}_{Y;\alpha}(y \to y')$ acts as multiplication by $e^{2\pi i \alpha(x)}$, where x is any $(d+1)$-chain with $y' = y + \partial x$. An easy argument shows that $\alpha(x) = \alpha(x')$ for any two choices of such a chain. Define $I_{Y,\alpha}$ as the *inverse limit* of $\mathscr{F}_{Y;\alpha}$.[34] That is, an element of $I_{Y,\alpha}$ is a function $i(y) \in \mathscr{F}_{Y;\alpha}(y) = \mathbb{T}$ on the objects in $\mathscr{C}_Y$ such that $i(y') = \mathscr{F}_{Y;\alpha}(y \to y')i(y)$ for all morphisms $y \to y'$. It is easy to check that $I_{Y,\alpha}$ exists.

Next, suppose α is a $(d + 1)$-cocycle on a closed oriented $(d - 1)$-manifold S. Then we claim that the integral

$$I_{S,\alpha} = \exp\left(2\pi i \int_S \alpha\right) \in \mathscr{T}_2$$

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[34]   See the beginning of Sect. 2 for a discussion of inverse limits

now makes sense as a $\mathbb{T}$-gerbe. The construction is entirely analogous to the previous one except there is one more layer of argument. So consider the category $\mathscr{C}_S$ whose objects are oriented cycles $s \in C_{d-1}(S)$ which represent the fundamental class $[S] \in H_{d-1}(S)$, and with a unique morphism between any two objects. Now if $s, s' \in \mathscr{C}_S$, construct a category $\mathscr{C}_{s,s'}$ whose objects are d-chains y which satisfy $s' = s + \partial y$, and with a unique morphism between any two objects. Define a functor $\mathscr{F}_{s,s';\alpha} : \mathscr{C}_{s,s'} \to \mathscr{T}_1$ by $\mathscr{F}_{s,s';\alpha}(y) = \mathbb{T}$ for each y and $\mathscr{F}_{s,s';\alpha}(y \to y')$ acts as multiplication by $e^{2\pi i \alpha(x)}$, where x is any $(d+1)$-chain with $y' = y + \partial x$. An easy argument shows that $\alpha(x) = \alpha(x')$ for any two choices of such a chain. Define the $\mathbb{T}$-torsor $I_{s,s';\alpha}$ to be the inverse limit of $\mathscr{F}_{s,s';\alpha}$. Now define a functor $\mathscr{F}_{S;\alpha} : \mathscr{C}_S \to \mathscr{T}_2$ by $\mathscr{F}_{S;\alpha}(s) = \mathscr{T}_1$ for each s and $\mathscr{F}_{S;\alpha}(s \to s')$ acts as multiplication by $I_{s,s';\alpha}$. The $\mathbb{T}$-gerbe $I_{S,\alpha}$ is defined to be the inverse limit of $\mathscr{F}_{S;\alpha}$.

It is clear how to continue to higher codimensions. Now we turn to manifolds with boundary.

If α is a $(d+1)$-cocycle on a compact oriented $(d+1)$-manifold X, then in [FQ, Proposition B.1] we describe the integral

$$\exp\left(2\pi i \int_X \alpha\right) \in I_{\partial X, i^*\alpha},$$

where $i: \partial X \hookrightarrow X$ is the inclusion of the boundary, and $I_{\partial X, i^*\alpha}$ is the $\mathbb{T}$-torsor described previously. We will not review that here, but rather go on to the next case. Namely, suppose that α is a $(d+1)$-cocycle on a compact oriented d-manfiold Y. Then we claim that the integral

$$\exp\left(2\pi i \int_Y \alpha\right) \in I_{\partial Y, i^*\alpha}$$

makes sense, where now $I_{\partial Y, i^*\alpha}$ is the $\mathbb{T}$-gerbe described previously. Call $S = \partial Y$ and let $s \in C_{d-1}(S)$ represent the fundamental class, i.e., $s \in \mathscr{C}_S$. By the definition of $I_{\partial Y, i^*\alpha}$ above we must construct a torsor $I_{Y,s;\alpha} \in \mathscr{T}_1$ and for any $s, s' \in \mathscr{C}_S$ an isomorphism

$$I_{Y,s;\alpha} \otimes I_{s,s';\alpha} \to I_{Y,s';\alpha}. \tag{A.3}$$

To construct $I_{Y,s;\alpha}$ let $\mathscr{C}_{Y,s}$ be the category whose objects are d-chains $y \in C_d(Y)$ such that y represents the fundamental class $[Y, \partial Y] \in H_d(Y, \partial Y)$ and $\partial y = i_* s$. We postulate a unique morphism $y - y'$ between any two objects of $\mathscr{C}_{Y,s}$. Define a functor $\mathscr{F}_{Y,s;\alpha} : \mathscr{C}_{Y,s} \to \mathscr{T}_1$ by $\mathscr{F}_{Y,s;\alpha}(y) = \mathbb{T}$ for each y and $\mathscr{F}_{Y,s;\alpha}(y \to y')$ is multiplication by $e^{2\pi i \alpha(x)}$, where x is any $(d+1)$-chain with $y' = y + \partial x$. As before, this is independent of the choice of x. Set $I_{Y,s;\alpha}$ to be the inverse limit of $\mathscr{F}_{Y,s;\alpha}$. To construct the isomorphism (A.3), suppose that $y \in \mathscr{C}_{Y,s}$ and $a \in \mathscr{C}_{s,s'}$, i.e., $y \in C_d(Y)$ represents $[Y, \partial Y]$ with $\partial y = s$, and $a \in C_d(S)$ with $\partial a = s' - s$. Then $y + a \in \mathscr{C}_{Y,s'}$. The isomorphism (A.3) is defined to be the identity relative to the trivializations of the torsors determined by y, a, and $y + a$.

This discussion indicates the constructions contained in the following assertion, which we boldly state for arbitrary codimension.

Assertion A.4. *Let Y be a closed oriented $(d + 1 - n)$-manifold $(n > 0)$ and $\alpha \in C^{d+1}(Y; \mathbb{R}/\mathbb{Z})$ a singular cocycle. Then there is an element $I_{Y,\alpha} \in \mathscr{T}_n$ defined.*

If X is a compact oriented $(d + 2 - n)$-manifold, $i: \partial X \hookrightarrow X$ the inclusion of the boundary, and $\alpha \in C^{d+1}(X; \mathbb{R}/\mathbb{Z})$ a cocycle, then

$$\exp\left(2\pi i \int_X \alpha\right) \in I_{\partial X, i^* \alpha}$$

is defined. These "higher $\mathbb{T}$-torsors" and integrals satisfy:

(a) (Functoriality) If $f: Y' \to Y$ is an orientation preserving diffeomorphism, then there is an induced isomorphism

$$f_*: I_{Y', f^* \alpha} \to I_{Y, \alpha}$$

and these compose properly. If $F: X' \to X$ is an orientation preserving diffeomorphism, then there is an induced isomorphism [35]

$$(\partial F)_* \left[\exp\left(2\pi i \int_{X'} F^* \alpha\right)\right] \to \exp\left(2\pi i \int_X \alpha\right). \tag{A.5}$$

(b) (Orientation) There are natural isomorphisms

$$I_{-Y, \alpha} \cong (I_{Y, \alpha})^{-1}, \tag{A.6}$$

and

$$\exp\left(2\pi i \int_{-X} \alpha\right) \cong \left[\exp\left(2\pi i \int_X \alpha\right)\right]^{-1}. \tag{A.7}$$

(c) (Additivity) If $Y = Y_1 \sqcup Y_2$ is a disjoint union, then there is a natural isomorphism

$$I_{Y_1 \sqcup Y_2, \alpha_1 \sqcup \alpha_2} \cong I_{Y_1, \alpha_1} \cdot I_{Y_2, \alpha_2}. \tag{A.8}$$

If $X = X_1 \sqcup X_2$ is a disjoint union, then there is a natural isomorphism

$$\exp\left(2\pi i \int_{X_1 \sqcup X_2} \alpha_1 \sqcup \alpha_2\right) \cong \exp\left(2\pi i \int_{X_1} \alpha_1\right) \cdot \exp\left(2\pi i \int_{X_2} \alpha_2\right). \tag{A.9}$$

(d) (Gluing) Suppose $j: Y \hookrightarrow X$ is a closed oriented codimension one submanifold and X^{cut} is the manifold obtained by cutting X along Y. Then $\partial X^{\text{cut}} = \partial X \sqcup Y \sqcup -Y$. Suppose $\alpha \in C^{d+1}(X; \mathbb{R}/\mathbb{Z})$ is a singular $(d + 1)$-cocycle on S, and $\alpha^{\text{cut}} \in C^{d+1}(X^{\text{cut}}; \mathbb{R}/\mathbb{Z})$ the induced cocycle on X^{cut}. Then there is a natural isomorphism

$$\text{Tr}_{Y, j^* \alpha} \left[\exp\left(2\pi i \int_{X^{\text{cut}}} \alpha^{\text{cut}}\right)\right] \to \exp\left(2\pi i \int_X \alpha\right), \tag{A.10}$$

where $\text{Tr}_{Y, j^* \alpha}$ is the contraction

$$\text{Tr}_{Y, j^* \alpha}: I_{\partial X^{\text{cut}}, \alpha^{\text{cut}}} \cong I_{\partial X, i^* \alpha} \otimes I_{X, j^* \alpha} \otimes I_{Y, j^* \alpha}^{-1} \to I_{\partial X, i^* \alpha}.$$

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[35] If $n = 1$ then (A.5) is an *equality* of elements in a $\mathbb{Z}$-torsor. For $n > 1$ it is an *isomorphism* between elements in a "higher $\mathbb{Z}$-torsor." A similar remark hold for (A.7), (A.9), and (A.10)

(e) *(Stokes' Theorem* I*) Let* $\alpha \in C^{d+1}(W; \mathbb{R}/\mathbb{Z})$ *be a singular cocycle on a compact oriented* $(d + 3 - n)$-*manifold* W. *Then there is a natural isomorphism* [36]

$$\exp\left(2\pi i \int_{\partial W} \alpha\right) \cong \mathscr{T}_{n-2}. \tag{A.11}$$

(f) *(Stokes' Theorem* II*) A singular d-cochain* $\beta \in C^d(Y; \mathbb{R}/\mathbb{Z})$ *on a closed oriented* $(d + 1 - n)$-*manifold* Y *determines a trivialization*

$$I_{Y,\delta\beta} \cong \mathscr{T}_{n-1}.$$

A singular d-cochain $\beta \in C^d(X; \mathbb{R}/\mathbb{Z})$ *on a compact oriented* $(d + 2 - n)$-*manifold* X *satisfies*

$$\exp\left(2\pi i \int_X \delta\beta\right) \cong \mathscr{T}_{n-2} \tag{A.12}$$

under this isomorphism.

The assertion in (e) only has real content for $n = 1$. If $n > 1$, then $I_{\partial W, \alpha}$ is trivialized by $\exp\left(2\pi i \int_W \alpha\right)$.

We leave the reader to contemplate higher order gluing laws analogous to [FQ, Proposition B.10] and those discussed in Sect. 9.

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[36] Note that $\mathscr{T}_{n-2}$ is the identity element in $\mathscr{T}_{n-1}$. If $n = 1$, then (A.11) should be interpreted as

$$\exp\left(2\pi i \int_{\partial W} \alpha\right) = 1.$$

A similar remark applies to (A.12) below

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