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Integrable Operator Representations of $\mathbb{R}_q^2$, $X_{q,\gamma}$ and $SL_q(2,\mathbb{R})$

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Abstract: Let q be a complex number such that $|q| = 1$ and $q^4 \neq 1$. Integrable ("well-behaved") operator representations of the $*$-algebra $SL_q(2,\mathbb{R})$ in Hilbert space are defined and completely classified up to unitary equivalence. In order to do this, the relation $xy - qyx = \gamma(1-q), \gamma \in \mathbb{R}$, for self-adjoint operators x and y is studied in detail. Integrable representations for this relation are defined and classified.

0. Introduction

The study of non-compact quantum groups or more generally of non-compact quantum spaces at the Hilbert space level leads to new features and difficulties which do not occur in the compact case. The main source of these problems is the fact that the generators of the "ring of functions on the non-compact quantum space" do not have (enough) representations by bounded operators in general. On the technical level, we are concerned with (finitely many) *unbounded* operators which satisfy the commutation relations from the definition of the quantum space. The first problem that arises is to select the "well-behaved" representations for this set of relations. Following the terminology commonly used in representation theory of Lie algebras and of general $*$-algebras (see e.g. [J, S1]), we call these representations "integrable." In general, there is no canonical way to define integrability for a given set of commutation relations. The main purpose of this paper is to define and to classify integrable representations for the real quantum vector space $\mathbb{R}_q^2$ (i.e. for the relation $xy = qyx$), for the real quantum hyperboloid $X_{q,\gamma}$ (i.e. for the relation $xy - qyx = \gamma(1-q), \gamma \in \mathbb{R}/\{0\}$) and for the real form $SL_q(2,\mathbb{R})$ of the quantum group $SL_q(2)$, where $x = x^*$, $y = y^*$ and $|q| = 1$, $q^4 \neq 1$. Our main aim was to investigate $SL_q(2,\mathbb{R})$, but it turned out immediately that this requires a very detailed treatment of both $\mathbb{R}_q^2$ and $X_{q,\gamma}$. Knowing the irreducible integrable representations could be a starting point for studying some "analysis" on the quantum group $SL_q(2,\mathbb{R})$. The problem of defining integrability for certain operator relations was touched in [D] and in [W1]. It was studied in [OS1, OS2 and S2]. If the relations are "nice," it may happen that for *irreducible* integrable representations all operators are bounded (like in case of classical matrix groups where they correspond to the group elements) or only one operator is not bounded, cf. [OS1, OS2]. This is not true for $\mathbb{R}_q^2, X_{q,\gamma}$ or $SL_q(2,\mathbb{R})$, where apart from the trivial one-dimensional representations always (at least) two unbounded operators occur.

This paper is organized as follows. In Sects. 2, 3 and 4 we study integrable representations of $\mathbb{R}_q^2$, $X_{q,\gamma}$ and $SL_q(2,\mathbb{R})$, respectively. After giving precise definitions of integrable representations and discussing some motivation, we describe the integrable representations in terms of two models. In cases of $X_{q,\gamma}$ and $SL_q(2,\mathbb{R})$ we also define partial integrability by allowing one operator to be closed and symmetric rather than self-adjoint. As a result of our classification all irreducible integrable representations of $\mathbb{R}_q^2$, $X_{q,\gamma}$ and $SL_q(2,\mathbb{R})$ are listed in Corollaries 2.10, 3.8 and 4.11, respectively. The representation of $SL_q(2,\mathbb{R})$ occurring in the paper [FT] of L.D. Faddeev and L.A. Takhtajan fits into this list. Our classification depends essentially on the self-adjointness of certain operators. The corresponding technical results are contained in the Appendix.

1. Preliminaries

In this paper, q denotes a complex number such that $|q|=1$ and $q^4 \neq 1$. We write $q=e^{-i\varphi}$ with $|\varphi|<\pi$ and we set $q_0=e^{i\frac{\varphi}{2}}$ and $\varphi_n=\varphi-\pi n$ for $n\in\mathbb{Z}$. Throughout, the letters P and Q denote the position operator and the momentum operator, respectively, from quantum mechanics. That is, Q is the multiplication operator by the variable x and P is the differential operator $i\frac{d}{dx}$ acting on the Hilbert space $L^2(\mathbb{R})$. Recall that $FPF^{-1}=-Q$ and $FQF^{-1}=P$, where $(Ff)(x)=(2\pi)^{-1}\int f(t)e^{-itx}dt$ is the Fourier transform of a function f. The operators $e^{\alpha P}$ and $e^{\beta Q}$ with $\alpha,\beta\in\mathbb{R}$ play a crucial role in what follows. The following lemma gives a precise description of the domain $\mathcal{D}(e^{\alpha P})$. We state it only in case $\alpha>0$.

Lemma 1.1. (i) *Suppose that $f(z)$ is a holomorphic function on the strip $I_\alpha := \{z\colon 0<\operatorname{Im} z<\alpha\}$ such that*

$$\sup_{0<y<\alpha} \int |f_y(x)|^2 dx < \infty\ , \tag{1.1}$$

where $f_y(x) := f(x+iy)$.

Then the limits $f := \lim_{y\downarrow 0} f_y$ and $g := \lim_{y\uparrow\alpha} f_y$ exist in $L^2(\mathbb{R})$, $f\in\mathcal{D}(e^{\alpha P})$ and $g=e^{\alpha P}f$. We shall write $f(x+i\alpha) := g(x)$, so that $(e^{\alpha P}f)(x)=f(x+i\alpha)$. Moreover, we have a.e. *on* $\mathbb{R}$,

$$f(x)=\lim_{n\to\infty} f(x+in^{-2}) \quad and \quad f(x+i\alpha)=\lim_{n\to\infty} f(x+i(\alpha-n^{-2}))\ . \tag{1.2}$$

(ii) *For each $f\in\mathcal{D}(e^{\alpha P})$ there exists a unique function $f(z)$ on I_α as in* (i) *such that $f=\lim_{y\downarrow 0} f_y$ in $L^2(\mathbb{R})$.*

Proof. We restrict ourselves to the case $\alpha=2$.
(i): The function $f_1(x)\equiv f(x+i)$ satisfies the assumptions of the classical Paley–Wiener theorem (cf. [K], p. 174), so that $F^{-1}f_1\in\mathcal{D}(e^{|x|})$. Hence $f_1\in\mathcal{D}(e^P)\cap\mathcal{D}(e^{-P})$. Setting $f=e^{-P}f_1$ and $g=e^P f_1$, we have $e^{2P}f=g$. The functions $f(z+i)$ and $(2\pi)^{-1}\int e^{-itz}(F^{-1}f_1)(t)dt$ are both holomorphic for $|\operatorname{Im} z|<1$ and they are equal on the real axis, so they coincide on the whole strip $\{z\colon |\operatorname{Im} z|<1\}$.

This yields $f_y = Fe^{(y-1)Q}F^{-1}f_1 = e^{(y-1)P}f_1$ for $y \in (0,2)$. From $F^{-1}f_1 \in \mathcal{D}(e^{|x|})$ and the dominated convergence theorem we obtain

$$\|f - f_y\| = \|e^{-P}f_1 - e^{(y-1)P}f_1\| = \|(e^{-x} - e^{(y-1)x})F^{-1}f_1\| \to 0 \text{ as } y \downarrow 0\,,$$

and similarly $\|g - f_y\| \to 0$ as $y \uparrow 2$. Set $\zeta_n(x) = f(x + in^{-2})$ and $\eta_n(x) = f(x + i(2 - n^{-2}))$. Since $\sum_n \|\zeta_{n+1} - \zeta_n\| < \infty$ and $\sum_n \|\eta_{n+1} - \eta_n\| < \infty$ because of (1.1), a routine argument from measure theory shows that $\zeta_n \to f$ and $\eta_n \to g$ a.e. on $\mathbb{R}$.
(ii): Since $f \in \mathcal{D}(e^{2P})$, $f_1 := e^P f \in \mathcal{D}(e^P) \cap \mathcal{D}(e^{-P})$ and hence $F^{-1}f_1 \in \mathcal{D}(e^{|x|})$. The assertion follows by a similar reasoning using the converse direction of the Paley–Wiener theorem. ■

Corollary 1.2. (i) *If* $\alpha \in \mathbb{R}$, $f \in \mathcal{D}(e^{\alpha P})$, $\varepsilon > 0$ *and* $\omega \in \mathbb{C}$, *then* $g(x) := e^{-\varepsilon(x+\omega)^2}f(x) \in \mathcal{D}(e^{\alpha P})$ *and* $(e^P g)(x) = e^{-\varepsilon(x+\omega+i\alpha)}f(x + i\alpha)$.
(ii) *If* $\alpha, \beta \in \mathbb{R}$ *and* $f \in \mathcal{D}(e^{\alpha P}e^{\beta Q}) \cap \mathcal{D}(e^{\alpha P})$, *then* $f \in \mathcal{D}(e^{\beta Q}e^{\alpha P})$ *and* $e^{\alpha P}e^{\beta Q}f = e^{i\alpha\beta}e^{\beta Q}e^{\alpha P}f$.

Proof. (i) is clear. We verify (ii) for $\alpha > 0$. If $f(z)$ is the function for $f \in \mathcal{D}(e^{\alpha P})$ as in Lemma 1.1, then obviously $e^{\beta z}f(z)$ is the corresponding function for $e^{\beta Q}f \in \mathcal{D}(e^{\alpha P})$. Formula (1.2) implies that $e^{\alpha P}e^{\beta Q}f = e^{\alpha\beta i}e^{\beta Q}e^{\alpha P}f$. ■

Let us adopt a few notational conventions. If A is an operator on a Hilbert space $\mathcal{G}$ and B is an operator on $L^2(\mathbb{R})$, we write BA for the operator $B \otimes A$ on $L^2(\mathbb{R}) \otimes \mathcal{G}$. If no confusion can arise, an operator and its closure are denoted by the same symbol. If a is a self-adjoint operator, we write $a > 0$ (resp. $a < 0$) if $a \geqq 0$ (resp. $a \leqq 0$) and $\ker a = \{0\}$. A family $\{a_i;\ i \in I\}$ of unbounded operators on a Hilbert space $\mathcal{H}$ is called *irreducible* if a decomposition $a_i = b_i \oplus c_i$ for all $i \in I$ with respect to an orthogonal direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is only possible in the trivial cases $\mathcal{H}_1 = \{0\}$ or $\mathcal{H}_2 = \{0\}$. Symmetric operators are always meant to be densely defined. As usual, a unitary self-adjoint operator is called a *symmetry*. Throughout, we let

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

denote the Pauli matrices acting as operators on Hilbert spaces of the form $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$. We write $s(\alpha)$ for the sign of $\alpha \in \mathbb{R}$.

In what follows, $\mathcal{F}$ will denote a fixed dense linear subspace of $L^2(\mathbb{R})$ such that $\mathcal{F} \subseteq \mathcal{D}(e^{\alpha Q}) \cap \mathcal{D}(e^{\alpha P})$ and $\mathcal{F}$ is invariant under the operators $e^{\alpha Q}$, $e^{\alpha P}$, e^{itQ}, e^{itP} and $e^{-\varepsilon Q^2}$ for α, $t \in \mathbb{R}$ and $\varepsilon > 0$. Note that the invariance under the unitary groups $t \to e^{itQ}$ and $t \to e^{itP}$ implies that $\mathcal{F}$ is a core for all operators $e^{\alpha Q}$ and $e^{\alpha P}$, $\alpha \in \mathbb{R}$ (cf. [S2], Lemma 7.2). For instance, one may take $\mathcal{F} = \mathrm{Lin}\{e^{-\varepsilon x^2 + \gamma x};\ \varepsilon > \varrho,\ \gamma \in \mathbb{C}\}$ for some constant $\varrho > 0$.

2. Integrable Operator Representations of the Real Quantum Plane $\mathbb{R}_q^2$

Recall that $\mathbb{R}_q^2$ is the free $*$-algebra with unit element which is generated by two hermitian elements a and b satisfying the relation

$$ab = qba\,.$$

Throughout this section, let $\{a, b\}$ denote a pair of self-adjoint operators a and b acting on the same Hilbert space $\mathcal{H}$.

2.A. We repeat the following definition and two propositions from [S2]. Note that some terminology in this paper is different from the one used in [S2].

Definition 2.1. *Suppose that* $a \geqq 0$ *or* $a \leqq 0$. *Then the pair* $\{a, b\}$ *is called an integrable representation of* $\mathbb{R}_q^2$ *if* $\mathcal{H}_0 = \ker a$ *is reducing for* b, i.e. *we have* $b = b_1 \oplus b_0$ *and* $a = a_1 \oplus 0$ *on* $\mathcal{H} = \mathcal{H}_0^{\perp} \oplus \mathcal{H}_0$ *and if there exists a* $k \in \mathbb{Z}$ *such that*

$$|a_1|^{it}\, b_1 = e^{\varphi_{2k} t} b_1 |a_1|^{it} \quad \textit{for all } t \in \mathbb{R}\,.$$

In this case we write $\{a, b\} \in \mathcal{C}_{2k}(q)$.

Proposition 2.2. *Suppose that* $a > 0$ *or* $a < 0$ *and that* $\ker b = \{0\}$. *Consider the following conditions:*

(i) $\{a, b\}$ *is an integrable representation of* $\mathbb{R}_q^2$.
(ii) $\{a, b\}$ *is unitarily equivalent to some pair* $\{\tilde{a} := \varepsilon e^Q,\ \tilde{b} := e^{\varphi_{2k} P} w\}$ *on* $L^2(\mathbb{R}) \otimes \mathcal{K}$, *where* $\varepsilon \in \{1, -1\}, k \in \mathbb{Z}$ *and* w *is a symmetry on some Hilbert space* $\mathcal{K}$.
(iii) *There is an integer* k *such that*

$$|a|^{it} |b|^{is} = e^{i\varphi_{2k} ts} |b|^{is} |a|^{it}, \quad t, s \in \mathbb{R}\,. \tag{2.1}$$

Then (i) $\leftrightarrow$ (ii). *If in addition* $b > 0$ *or* $b < 0$, *then all three conditions are equivalent.*

Proposition 2.3. *Suppose that* $a > 0$ *or* $a < 0$. *Let* $\{a, b\}$ *be an integrable representation of* $\mathbb{R}_q^2$. *Then there exists a dense linear subspace* $\mathcal{D}$ *of* $\mathcal{H}$ *such that* $a\mathcal{D} = \mathcal{D}$, $b\mathcal{D} \subseteq \mathcal{D}$, $\mathcal{D}$ *is a core for* a *and* b *and* $ab\psi = qba\psi$ *for* $\psi \in \mathcal{D}$.

Suppose that $a > 0$ or $a < 0$ and let $n, m \in \mathbb{N}$. If $q^{2nm} \neq \pm 1$ and if $\{a, b\}$ is an integrable representation of $\mathbb{R}_q^2$, then $\{c, d\}$ is an integrable representation of $\mathbb{R}_{q^{nm}}^2$.

2.B. Our next aim is to derive a definition of integrability in the general case. In order to get some motivation, let us assume for a moment that we have already such that a definition and let $\{a, b\}$ be an integrable representation of $\mathbb{R}_q^2$. In view of the above remark, it might be reasonable that $\{a^2, b\}$ is also an integrable representation of $\mathbb{R}_{q^2}^2$. Since $a^2 > 0$, we know already what the latter means. By Definition 2.1, there exists a $k \in \mathbb{Z}$ such that

$$(a^2)^{it} b = e^{(2\varphi - 2\pi k)t} b (a^2)^{it}, \quad t \in \mathbb{R}\,,$$

so that

$$|a|^{is}\, b = e^{(-\varphi - \pi k)s} b\, |a|^{is} \equiv e^{\varphi_k s}\ b\, |a|^{is}, \quad s \in \mathbb{R}\,. \tag{2.2}$$

If k is even, this means that $\{|a|,\, b\}$ is an integrable representation of $\mathbb{R}_q^2$. If k is odd, then (2.2) says that $\{|a|,\, b\}$ is an integrable representation of $\mathbb{R}_{-q}^2$. Set $\varepsilon := (-1)^k$. Further, by Proposition 2.3, we have $|a|\, b\psi = \varepsilon q b\, |a|\, \psi$ for ψ in a common core $\mathcal{D}$ for $|a|$ and b and $|a|\mathcal{D} = \mathcal{D}$. If we require that the operator relation $ab\psi = qba\psi$ is also valid for $\psi \in \mathcal{D}$, it follows from $ab\psi = u_a |a| b\psi = u_a \varepsilon q b |a| \psi$

and $qba\psi = qbu_a|a|\psi$ that $u_q b\eta = \varepsilon\, bu_a\eta$ for $\eta \in |a|\mathcal{D} = \mathcal{D}$. Since $\mathcal{D}$ is a core for b, this yields $u_a b \subseteq \varepsilon bu_a$.

We put the outcome of the preceding discussion into the following

Definition 2.4. *A pair $\{a, b\}$ of self-adjoint operators a and b on a Hilbert space is called an integrable representation of $\mathbb{R}_q^2$ if there is an $\varepsilon \in \{1, -1\}$ such that:*

(D.1) *$\{|a|, b\}$ is an integrable representation of $\mathbb{R}_{\varepsilon q}^2$ and*
(D.2) *$u_a b \subseteq \varepsilon bu_a$.*

If $\{|a|, b\} \in \mathcal{C}_{2k}(q)$, we shall write $\{a, b\} \in \mathcal{C}_{2k}(q)$ for $\varepsilon = 1$ and $\{a, b\} \in \mathcal{C}_{2k+1}(q)$ for $\varepsilon = -1$.

Clearly, if $a \geqq 0$ or $a \leqq 0$, then Definition 2.4 is equivalent to the above Definition 2.1. (Indeed, if $a > 0$ or $a < 0$ then $u_a = I$ or $u_a = -I$, so (D.2) yields $\varepsilon = 1$ if $b \neq 0$.)

Lemma 2.5. *The following three conditions are equivalent:*

(i) *$\{a^2, b\}$ is an integrable representation of $\mathbb{R}_{q^2}^2$.*
(ii) *$\{|a|, b\}$ is an integrable representation of $\mathbb{R}_{\varepsilon q}^2$ for $\varepsilon = 1$ or $\varepsilon = -1$.*
(iii) *$\{|a|, |b|\}$ is an integrable representation of $\mathbb{R}_{\varepsilon q}^2$ for $\varepsilon = 1$ or $\varepsilon = -1$ and $u_b|a| \subseteq |a|u_b$.*

Proof. Without loss of generality we assume that $\ker a = \{0\}$. As already noted above, (i) and (ii) are both equivalent to (2.2) for some $k \in \mathbb{Z}$. Thus (i) $\leftrightarrow$ (ii). Suppose (ii) is valid. From (2.2) we get

$$|a|^{is}|b||a|^{-is} = e^{\varphi_k s}|b|, \quad s \in \mathbb{R}\,, \tag{2.3}$$

which in turn means that $\{|a|,\ |b|\}$ is an integrable representation of $\mathbb{R}_{\varepsilon q}^2$ with $\varepsilon = (-1)^k$. Further, applying first (2.2) and then (2.3), we get

$$\begin{aligned} |a|^{is}u_b|b| = |a|^{is}b = e^{\varphi_k s}b|a|^{is} &= u_b(e^{\varphi_k s}|b||a|^{is}) \\ &= u_b|a|^{is}|b|\,, \end{aligned}$$

so that $|a|^{is}u_b = u_b|a|^{is}$ for $s \in \mathbb{R}$. Consequently, $u_b|a| \subseteq |a|u_b$ and (iii) is proved. Conversely, assume condition (iii). Then we have (2.3) for some $k \in \mathbb{Z}$. Combined with $u_b|a| \subseteq |a|u_b$, this obviously leads to (2.2) and so to (ii). ■

Lemma 2.6. *$u_a b \subseteq \varepsilon bu_a$ if and only if $u_a|b| \subseteq |b|u_a$ and $u_a u_b = \varepsilon u_b u_a$.*

Proof. We can assume that $\ker a = \ker b = \{0\}$. Then u_a and u_b are self-adjoint unitaries. Suppose that $u_a b \subseteq \varepsilon bu_a$. Hence $u_a bu_a = \varepsilon b$ and $u_a|b|u_a = |b|$ which gives $u_a|b| = |b|u_a$. From $u_a u_b = u_a b \subseteq \varepsilon bu_a = \varepsilon u_b|b|u_a = \varepsilon u_b u_a|b|$ and $\ker|b| = \{0\}$ we conclude the $u_a u_b = \varepsilon u_b u_a$. The opposite direction follows similarly. ■

Lemmas 2.5 and 2.6 allow to formulate several equivalent versions of Definition 2.4. We mention one sample stated as

Corollary 2.7. *$\{a, b\}$ is an integrable representation of $\mathbb{R}_q^2$ if and only if we have $u_a|b| \subseteq |b|u_a$, $u_b|a| \subseteq |a|u_b$, $u_a u_b = \varepsilon u_b u_a$ and $\{|a|,\ |b|\}$ is an integrable representation of $\mathbb{R}_{\varepsilon q}^2$ for $\varepsilon = 1$ or for $\varepsilon = -1$.*

One disadvantage of the above Definition 2.4 is that it is not symmetric with respect to a and b. We now remove this defect.

Corollary 2.8. *Consider the following assertions:*

(i) $\{a, b\}$ *is an integrable representation of* $\mathbb{R}^2_q$.
(ii) $\{b, a\}$ *is an integrable representation of* $\mathbb{R}^2_{\bar{q}}$.
(iii) $\{a^{-1}, b\}$ *is an integrable representation of* $\mathbb{R}^2_{\bar{q}}$.
(iv) $\{a^{-1}, b^{-1}\}$ *is an integrable representation of* $\mathbb{R}^2_q$.

We have (i) $\leftrightarrow$ (ii). *If* $\ker a = \{0\}$, *then* (i) $\leftrightarrow$ (iii). *If* $\ker a = \ker b = \{0\}$, *then* (i) $\leftrightarrow$ (iv).

Proof. By Corollary 2.7, it suffices to check that the last condition occurring therein is invariant under the above operations. Hence it is enough to verify Corollary 2.7 in the case where $a > 0$ and $b > 0$. But in this case these assertions follow immediately from formula (2.1) in Proposition 2.2. ■

2.C. In the subsection we describe the structure of the integrable representation up to unitary equivalence by means of the following two models. For this let $\mathcal{K}$ be a Hilbert space and k an integer.

$(\mathcal{M}_1)$: Let w and v be commuting symmetries on $\mathcal{K}$. Define self-adjoint operators $\tilde{a}$ and $\tilde{b}$ on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathcal{K}$ by $\tilde{a} = e^Q w$ and $\tilde{b} = e^{\varphi_{2k} P} v$.
$(\mathcal{M}_{-1})$: Define self-adjoint operators $\tilde{a}$ and $\tilde{b}$ on $\mathcal{H} := L^2(\mathbb{R}) \otimes (\mathcal{K} \oplus \mathcal{K})$ by the operator matrices $\tilde{a} = e^Q \otimes \sigma_0$ and $\tilde{b} = e^{\varphi_{2k+1} P} \otimes \sigma_1$.

Clearly, the couple $\{\tilde{a}, \tilde{b}\}$ of $(\mathcal{M}_1)$ belongs to $\mathcal{C}_{2k}(q)$, while the couple of $(\mathcal{M}_{-1})$ is in $\mathcal{C}_{2k+1}(q)$.

Theorem 2.9. *Suppose that* $\{a, b\}$ *is an integrable representation of* $\mathbb{R}^2_q$ *such that* $\ker a = \ker b = \{0\}$. *Let* $\varepsilon \in \{1, -1\}$ *be as in Definition* 2.4. *Then the pair* $\{a, b\}$ *is unitarily equivalent to a pair* $\{\tilde{a}, \tilde{b}\}$ *described in the model* $(\mathcal{M}_\varepsilon)$.

Proof. Let $\mathcal{H}_+ := \ker(u_a - I)$ and $\mathcal{H}_- := \ker(u_a + I)$. First suppose that $\varepsilon = 1$. Since $\ker a = \{0\}$, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. From Corollary 2.7 and Lemma 2.6 we conclude that $\mathcal{H}_+$ and $\mathcal{H}_-$ reduce the self-adjoint operator b, i.e. $b = b_1 \oplus b_2$. It is clear that $a = a_+ \oplus a_-$ with $a_+ > 0$ and $a_- < 0$. Applying now Proposition 2.2 to the pairs $\{a_+, b_1\}$ and $\{a_-, b_2\}$, we obtain a pair $\{\tilde{a}, \tilde{b}\}$ as in $(\mathcal{M}_1)$.

Suppose now that $\varepsilon = -1$. Then u_a and u_b are self-adjoint unitaries satisfying the canonical anticommutation relation $u_a u_b + u_b u_a = 0$. Hence we have an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$ such that u_a and u_b act as the Pauli matrices σ_0 and σ_1, respectively. By general properties of the polar decomposition, $\ker(u_a \mp I)$ and $\ker(u_b \mp I)$ are the subspaces of $\mathcal{H}$, where a resp. b are positive and negative, respectively. Let $P_\pm(a)$ and $P_\pm(b)$ denote the orthogonal projections on $\ker(u_a \mp I)$ resp. $\ker(u_b \mp I)$. By $u_a = \sigma_0$, we have $a = a_+ \oplus a_-$ with $a_+ > 0$ and $a_- < 0$. Since $u_b |a| u_b = |a|$ because of Corollary 2.7, we get $a_- = -a_+$. From $u_a b u_a = -b$ by Lemma 2.6, $P_\pm(a) b \subseteq b P_\mp(a)$, so that b and $|b|$ can be written as matrices

$$b = \begin{pmatrix} 0 & c \\ c^* & 0 \end{pmatrix} \quad \text{and} \quad |b| = u_b b = \begin{pmatrix} c^* & 0 \\ 0 & c \end{pmatrix}$$

which in turn yields $c^* = c > 0$. By the last condition in Corollary 2.7, $\{|a|, |b|\}$ is an integrable representation of $\mathbb{R}^2_{-q}$. Therefore, $\{a_+, c\}$ is an integrable representation of $\mathbb{R}^2_{-q}$ on the Hilbert space $\mathcal{H}_1$. Now Proposition 2.2 says that $\{a_+, c\}$ is unitarily equivalent to some pair $\{e^Q, e^{\varphi_{2k+1}P}\}$. (Note that $\varepsilon = 1$ and $w = 1$, since $a_+ > 0$ and $c > 0$.) Hence $\{a, b\}$ is unitarily equivalent to $\{\tilde{a}, \tilde{b}\}$ in $(\mathcal{M}_{-1})$. ∎

We state some consequences of Theorem 2.9.

Corollary 2.10. *Each irreducible integrable representation $\{a, b\}$ of $\mathbb{R}^2_q$ is unitarily equivalent to one of the following list:*

$(\mathrm{I})_{\varepsilon_1,\varepsilon_2,k}$: $a = \varepsilon_1 e^Q$, $b = \varepsilon_2 e^{\varphi_{2k}P}$ *on* $\mathcal{H} = L^2(\mathbb{R})$: $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$, $k \in \mathbb{Z}$.

$(\mathrm{II})_k$: $a = e^Q \otimes \sigma_0$, $b = e^{\varphi_{2k+1}P} \otimes \sigma_1$ *on* $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$: $k \in \mathbb{Z}$.

$(\mathrm{III})_{\alpha,0}$: $a = \alpha$, $b = 0$ *on* $\mathcal{H} = \mathbb{C}$: $\alpha \in \mathbb{R}$.

$(\mathrm{III})_{0,\alpha}$: $a = 0$, $b = \alpha$ *on* $\mathcal{H} = \mathbb{C}$: $\alpha \in \mathbb{R}$.

Corollary 2.11. *Let $\{a, b\}$ be an integrable representation of $\mathbb{R}^2_q$ such that $\ker a = \ker b = \{0\}$. Then there exists a linear subspace $\mathcal{D} \subseteq \mathcal{D}(a) \cap \mathcal{D}(b)$ of $\mathcal{H}$ such that:*

(i) $a\mathcal{D} = \mathcal{D}$, $b\mathcal{D} = \mathcal{D}$, $|a|^{it}\mathcal{D} = \mathcal{D}$ *and* $|b|^{it}\mathcal{D} = \mathcal{D}$ *for* $t \in \mathbb{R}$
(ii) $\mathcal{D}$ *is a core for* a, a^{-1}, b *and* b^{-1}.
(iii) $ab\psi = qba\psi$ *for* $\psi \in \mathcal{D}$.

Proof. The domain $\mathcal{D} := \mathcal{F} \otimes \mathcal{K}$ for $(\mathcal{M}_1)$ resp. $\mathcal{D} := \mathcal{F} \otimes (\mathcal{K} \oplus \mathcal{K})$ for $(\mathcal{M}_{-1})$ has the desired properties. ∎

Corollary 2.12. *If $\{a, b\}$ is an integrable representation of $\mathbb{R}^2_q$ such that $\ker a = \{0\}$, then the operator $(q_0 b + \gamma)a^{-1}$ is symmetric for any real γ.*

Proof. Since it suffices to prove this in case where $\gamma = 0$ and $\ker b = \{0\}$, we can assume that $\{a, b\}$ is as in $(\mathcal{M}_{\pm 1})$. But then the assertion is clear, since the operators $q_0 e^{\varphi_{2k}P} e^{-Q}$ and $iq_0 e^{\varphi_{2k+1}P} e^{-Q}$ are symmetric by Proposition A.1, (v). ∎

3. Integrable Operator Representations of the Real Quantum Hyperboloid $X_{q,\gamma}$

Throughout this section γ is a real non-zero number. By the *real quantum hyperboloid* we mean the free $*$-algebra $X_{q,\gamma}$ with unit element $\mathbb{1}$ which is generated by two hermitian elements x and y satisfying the relation

$$xy - qyx = \gamma(1 - q)\mathbb{1}\,.$$

3.A. In this subsection we want to define x-integrable and integrable representations of $X_{q,\gamma}$. We begin with some simple algebraic manipulations.

Suppose that x and y are hermitian elements of a $*$-algebra $\mathcal{R}$ with unit. Let α be a non-zero complex number such that $\alpha = \bar{\alpha}q$ and put $a := \alpha(yx - \gamma)$. The algebraic relations

$$(xy - qyx - \gamma(1 - q))x = 0$$

and

$$xa = qax \tag{3.2}$$

in $\mathcal{R}$ are obviously equivalent. Thus if x is invertible in $\mathcal{R}$, then (3.1) is equivalent to (3.2) when we set $y = (\alpha^{-1}a + \gamma)x^{-1}$. Moreover, we have

$$a^* - a = \alpha\bar{q}(xy - qyx - \gamma(1-q)),$$

hence a is a hermitian element of $\mathcal{R}$ if and only if (3.1)is valid. That is, on a formal algebraic level we have reduced the real quantum hyperboloid to $\mathbb{R}_q^2$.

Definition 3.1. *Suppose that x is a self-adjoint operator and y is a closed symmetric operator on a Hilbert space $\mathcal{H}$. Let $\alpha \in \mathbb{C}$, $\alpha \neq 0$, be such that $\alpha = \bar{\alpha}q$. We shall say that the couple $\{x, y\}$ is an x-integrable representation of $X_{q,\gamma}$ if there exists a self-adjoint operator a on $\mathcal{H}$ such that $\{x, a\}$ is an integrable representation of $\mathbb{R}_q^2$, $\ker x = \{0\}$ and y is the closure of the operator $(\alpha^{-1}a + \gamma)x^{-1}$.*

Definition 3.2. *A pair $\{x, y\}$ of self-adjoint operators x and y is called an integrable representation of $X_{q,\gamma}$ if $\{x, y\}$ is an x-integrable representation of $X_{q,\gamma}$ and $\{y, x\}$ is a y-integrable representation of $X_{\bar{q},\gamma}$.*

Remarks. (1) Clearly the preceding definitions do not depend on the number α.
(2) Let τ be the $*$-isomorphism of $X_{q,\gamma}$ onto $X_{\bar{q},\gamma}$ defined by $\tau(x) = y$ and $\tau(y) = x$. By the preceding definitions, τ maps x-integrable representations of $X_{q,\gamma}$ into y-integrable representations of $X_{\bar{q},\gamma}$ and vice versa, so τ preserves the integrability. Also the scaling isomorphism $x \to \gamma^{-1}x$, $y \to y$ of $X_{q,\gamma}$ onto $X_{q,1}$ preserves this notion. Therefore, in order to classify the integrable representations of $X_{q,\gamma}$ we could assume without loss of generality that $\varphi > 0$ and $\gamma = 1$.
(3) The reason for allowing only symmetric operators y rather than self-adjoint operators in the above Definition 3.1 will be seen later in Subsect. 3.C: The operators y appearing in our models are not self-adjoint in general.

We shall provide some motivation for our definitions. First note that our assumption $\ker x = \{0\}$ seems to be justified by the following very simple

Lemma 3.3. *Let x and y be a symmetric operators on a Hilbert space and let $\eta \in \mathcal{D}_{xy} := \{\eta \in \mathcal{D}(xy) \cap \mathcal{D}(yx) \colon xy\eta - qyx\eta = \gamma(1-q)\eta\}$. If $x\eta = 0$ or if $y\eta = 0$, then $\eta = 0$.*

Proof. Suppose that $x\eta = 0$. Putting $\zeta := (\gamma(1-q))^{-1}y\eta$, we have $x\zeta = \eta$ and so $0 = \langle x\eta, \zeta\rangle = \langle \eta, x\zeta\rangle = \langle \eta, \eta\rangle$, that is, $\eta = 0$. The proof in case where $y\eta = 0$ is similar. ■

Next we give some arguments justifying the definition of x-integrability in case $x > 0$.

First let us assume that x and y are elements of an algebra $\mathcal{R}$ such that x is invertible in $\mathcal{R}$ and (3.1) is satisfied. A straightforward induction argument shows that

$$f(x)y - y\,f(qx) = \gamma(1-q)(D_q f)(x) \tag{3.3}$$

for any polynomial f, where $D_q f$ denotes the so-called q-derivative

$$(D_q f)(x) = \frac{f(x) - f(qx)}{x - qx}.$$

Suppose now that x and y are self-adjoint operators such that $x > 0$. In order to define integrability for (3.1), it might be natural to require (3.3) for some suitable "nice" functions f such as $f(x) = x^{it}, t \in \mathbb{R}$. (This idea has been used already for

$\mathbb{R}_q^2$ in [S1].) Putting $f(x) = x^{it}$, $t \in \mathbb{R}$, into (3.3), we obtain formally

$$x^{it}y - y(qx)^{it} = \gamma(1-q)\frac{x^{it} - (qx)^{it}}{(1-q)x} .$$

Interpreting $(qx)^{it}$ as $e^{\varphi_{2k}t}x^{it}$ for $k \in \mathbb{Z}$ and multiplying by x from the right, we get

$$x^{it}yx - yxe^{\varphi_{2k}t}x^{it} = \gamma(1 - e^{\varphi_{2k}t})x^{it} ,$$

and hence

$$x^{it}(\alpha(yx-\gamma)) = e^{\varphi_{2k}t}\alpha(yx-\gamma)x^{it}, \quad t \in \mathbb{R} . \tag{3.4}$$

Restricted to the domain $\mathcal{D}_{x,y}$ from Lemma 3.3 the operator $\tilde{a} := \alpha(yx-\gamma)$ is obviously symmetric. Assume now that the closure a of $\tilde{a}|\mathcal{D}_{x,y}$ is a self-adjoint operator. Then, by (3.4),

$$x^{it}\, a = e^{\varphi_{2k}t}\, a\, x^{it} , t \in \mathbb{R} .$$

Since $x > 0$, this means that $\{x, a\}$ is an integrable representation of $\mathbb{R}_q^2$. A similar reasoning with $f(x)$ replaced by $f(x^2)$ can be used in order to justify our definition also in the general case.

3.B. Similarly as in case of $\mathbb{R}_q^2$, we study the x-integrable representations of $X_{q,\gamma}$ in terms of two models.

Suppose A is a self-adjoint operator on a Hilbert space $\mathcal{H}_0$ with $\ker A = \{0\}$. Let $\mathcal{K}$ be a Hilbert space and let $k \in \mathbb{Z}$.

$(\mathcal{M}_1^x)$: Suppose that w and v are commuting symmetries on $\mathcal{K}$. Let $\tilde{x}$ and $\tilde{y}$ be the operators on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathcal{K} \oplus \mathcal{H}_0$ defined by

$$\tilde{x} = e^Q w \oplus A^{-1} \quad \text{and} \quad \tilde{y} = (q_0 e^{\varphi_{2k}P}v + \gamma)e^{-Q}w \oplus \gamma A^{-1} . \tag{3.5}$$

$(\mathcal{M}_{-1}^x)$: We define operators $\tilde{x}$ and $\tilde{y}$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes (\mathcal{K} \oplus \mathcal{K}) \oplus \mathcal{H}_0$ by

$$\tilde{x} = e^Q \otimes \sigma_1 \oplus A \quad \text{and} \quad \tilde{y} = (q_0 e^{\varphi_{2k+1}P}e^{-Q} \otimes \sigma_0\sigma_1 + \gamma e^{-Q} \otimes \sigma_1) \oplus \gamma A^{-1} . \tag{3.6}$$

In both cases, $\tilde{x}$ is a self-adjoint operator and $\tilde{y}$ is closed and symmetric (cf. Proposition A.1, (v)). Note that $\tilde{y}$ is not self-adjoint in general, see Proposition A.4.

Proposition 3.4. *The pair $\{\tilde{x},\ \tilde{y}\}$ as defined in $(\mathcal{M}_1^x)$ or in $(\mathcal{M}_{-1}^x)$ is an $\tilde{x}$-integrable representation of $X_{q,\gamma}$. Conversely, each x-integrable representation $\{x, y\}$ is unitarily equivalent to a pair $\{\tilde{x},\ \tilde{y}\}$ of one of the above models $(\mathcal{M}_1^x)$ or $(\mathcal{M}_{-1}^x)$.*

Proof. To verify the first assertion, set $\alpha = \overline{q_0}$ and let $a = e^{\varphi_{2k}P}w \oplus 0$ for $(\mathcal{M}_1^x)$ and $a = e^{\varphi_{2k}P} \otimes \sigma_0 \oplus 0$ for $(\mathcal{M}_{-1}^x)$. In both cases $\{\tilde{x}, a\}$ is an integrable representation of $\mathbb{R}_q^2$ and $\tilde{y} = (q_0 a + \gamma)\tilde{x}^{-1}$, so that $\{\tilde{x},\ \tilde{y}\}$ is an $\tilde{x}$-integrable representation of $X_{q,\gamma}$ by Definition 3.1.

The second assertion follows in a straightforward manner from Definition 2.1 and Theorem 2.9. In case of $(\mathcal{M}_{-1})$ the above form of $\tilde{x}$ and $\tilde{y}$ is obtained after a unitary transformation u, where u denotes the 2×2 matrix (u_{rs}) with $u_{11} = u_{12} = u_{21} = -u_{22} = \frac{1}{\sqrt{2}}$. ■

Corollary 3.5. *Let $\{x,\ y\}$ be an x-integrable representation of $X_{q,\gamma}$. Then there is a domain $\mathcal{D} \subseteq \mathcal{D}(x) \cap \mathcal{D}(y)$ which is invariant under x, x^{-1} and y and a core for these operators such that $xy\eta - qyx\eta = \gamma(1-q)\eta$ for $\eta \in \mathcal{D}$.*

Proof. Set $\mathcal{D} = \mathcal{F} \otimes \mathcal{K} \oplus \mathcal{D}_\infty$ for $(\mathcal{M}_1^x)$ and $\mathcal{D} = \mathcal{F} \otimes (\mathcal{K} \oplus \mathcal{K}) \oplus \mathcal{D}_\infty$ for $(\mathcal{M}_{-1}^x)$, where $\mathcal{D}_\infty := \bigcap_{n \in \mathbb{Z}} \mathcal{D}(A^n)$. That $\mathcal{D}$ is a core for $\tilde{y}$ follows from Proposition A.1, (i). ■

Since the above domain $\mathcal{D}$ is obviously contained in the domain $\mathcal{D}_{x,y}$ from Lemma 3.3, it follows that the operator a in Definition 3.1 is the closure of $\alpha(yx - \gamma)|\mathcal{D}_{x,y}$. In particular, this shows that a is uniquely determined by $\{x, y\}$.

Corollary 3.6. *For arbitrary* $\varepsilon_1, \varepsilon_2 \in \{1, -1\}, k \in \mathbb{Z}$ *and* $\alpha \in \mathbb{R}|\{0\}$, *each couple* $\{x, y\}$ *of the following list is an irreducible x-integrable representation of* $X_{q,\gamma}$. *Any irreducible x-integrable representation of* $X_{q,\gamma}$ *is unitarily equivalent to one of this list:*

$(\mathrm{I})_{\varepsilon_1,\varepsilon_2,k}$: $x = \varepsilon_1 e^Q, y = \varepsilon_1(\varepsilon_2 q_0 e^{\varphi_{2k} P} + \gamma)e^{-Q}$ *on* $\mathcal{H} = L^2(\mathbb{R})$.

$(\mathrm{II})_k$: $x = e^Q \otimes \sigma_1,\ y = q_0 e^{\varphi_{2k+1} P} e^{-Q} \otimes \sigma_0\sigma_1 + \gamma e^{-Q} \otimes \sigma_1$ *on* $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$.

$(\mathrm{III})_\alpha$: $x = \alpha,\ y = \gamma\alpha^{-1}$ *on* $\mathcal{H} = \mathbb{C}$.

3.C. Now we classify the integrable representations of $X_{q,\gamma}$ up to unitary equivalence. The cases where the operator $\tilde{y}$ in our models is self-adjoint are characterized by Proposition A.4. We rewrite our models $(\mathcal{M}_{\pm 1}^x)$ in these cases.

$(\mathcal{M}_1)$: Let $\mathcal{H}$ and $\tilde{x}$ be as in $(\mathcal{M}_1^x)$ and define

$$\tilde{y}_k := \left((-1)^k q_0 e^{\varphi_{2k} P} s(\gamma) + \gamma\right) e^{-Q} w \oplus \gamma A^{-1} \quad \text{for} \quad k \in \{0, s(\varphi)\}\ .$$

$(\mathcal{M}_{-1})$: Let $\mathcal{H}$ and $\tilde{x}$ be as in $(\mathcal{M}_{-1}^x)$ and set

$$\tilde{y} := \begin{pmatrix} 0 & y_+ \\ y_- & 0 \end{pmatrix} \oplus \gamma A^{-1}\ ,$$

where $y_\pm := \left(\pm q_0 e^{(\varphi - s(\varphi)\pi)P} + \gamma\right) e^{-Q}$.

Theorem 3.7. (i) *An x-integrable representation* $\{x, y\}$ *of* $X_{q,\gamma}$ *is integrable if and only if the operator y is self-adjoint.*
(ii) *The couples* $\{\tilde{x}, \tilde{y}\}$ *of* $(\mathcal{M}_1)$ *and* $(\mathcal{M}_{-1})$ *are integrable representations of* $X_{q,\gamma}$. *Conversely, each integrable representation of* $X_{q,\gamma}$ *is unitarily equivalent to one of these couples* $\{\tilde{x}, \tilde{y}\}$.

Proof. (i). For an integrable representation $\{x, y\}$ both operators are self-adjoint by Definition 3.2, so the only if part is trivial. To prove the converse, suppose that $\{x, y\}$ is x-integrable and y is self-adjoint. By Proposition 3.4, we can assume that $\{x, y\}$ is of the form $\{\tilde{x}, \tilde{y}\}$ as in $(\mathcal{M}_1^x)$ or in $(\mathcal{M}_{-1}^x)$. For simplicity, we set $\mathcal{H}_0 = \{0\}$. We treat only the case of $(\mathcal{M}_{-1}^x)$, but the proof for $(\mathcal{M}_1^x)$ is similar. As in the proof of Proposition 3.4, let $a = e^{\varphi_{2k+1} P} \otimes \sigma_0$. Clearly, the unitary group $|a|^{it}$ acts on vectors $f \otimes (\zeta, \eta) \in L^2(\mathbb{R}) \otimes (\mathcal{K} \oplus \mathcal{K})$ as a translation of f by $t\varphi_{2k+1}$. Hence we have $|a|^{it}\tilde{y}\psi = e^{t\varphi_{2k+1}}\tilde{y}|a|^{it}\psi, t \in \mathbb{R}$, for ψ in the domain of $\tilde{y}$ and so for ψ in the domain of its closure which is also denoted by $\tilde{y}$ according to our convention from Sect. 1. Since $\tilde{y}$ is self-adjoint, the preceding means that $\{|a|, \tilde{y}\} \in \mathcal{C}_{2k}(-q)$. Since $u_a = \sigma_0$ and $u_a\tilde{y} \subseteq -\tilde{y}u_a$, we have shown that $\{a, \tilde{y}\}$ is an integrable representation of $\mathbb{R}_q^2$. By Corollary 2.7, $\{\tilde{y}, a\}$ is an integrable representation of $\mathbb{R}_{\bar{q}}^2$. Further, Proposition A.1, (iv), shows that $\ker \tilde{y} = \{0\}$. Next we show that $\tilde{x}$ is the closure of the operator $(q_0 a + \gamma)\tilde{y}^{-1}$. In order to prove this, we first apply the unitary transformation of the Hilbert space induced by the inverse Fourier transform. Then $\tilde{x}$ and $\tilde{y}$ are given by the matrices

$$\tilde{x} = \begin{pmatrix} 0 & e^{-P} \\ e^{-P} & 0 \end{pmatrix} \quad \text{and} \quad \tilde{y} = \begin{pmatrix} 0 & T_+ \\ T_- & 0 \end{pmatrix},$$

where $T_\pm := (\pm q_0 e^{\varphi_{2k+1} Q} + \gamma)e^P$ on $L^2(R) \otimes \mathcal{K}$. Let $\mathcal{D} := \mathcal{F} \otimes (\mathcal{K} \oplus \mathcal{K})$. Since $T_\pm(\mathcal{F} \otimes \mathcal{K})$ is a core for e^{-P} by Proposition A.1, (iii), $\tilde{y}\mathcal{D}$ is a core for $\tilde{x}$. For vectors $\psi \in \mathcal{D}$ we obviously have that $\tilde{x}\tilde{y}\psi = (q_0 a + \gamma)\psi$ by Corollary 1.2, hence $\tilde{x}|\tilde{y}\mathcal{D} = (q_0 a + \gamma)\tilde{y}^{-1}|\tilde{y}\mathcal{D}$. Since $\tilde{x}|\tilde{y}\mathcal{D}$ is essentially self-adjoint and $(q_0 a + \gamma)\tilde{y}^{-1}$ is symmetric by Corollary 2.12, the latter implies that $\tilde{x}$ is the closure of $(q_0 a + \gamma)\tilde{y}^{-1}$. Putting the preceding together, we have proved that $\{\tilde{y}, \tilde{x}\}$ is $\tilde{y}$-integrable representation of $X_{\bar{q},\gamma}$, i.e. the couple $\{\tilde{x}, \tilde{y}\}$ is integrable.
(ii): By (i) and Proposition 3.4, it suffices to determine all pairs $\{\tilde{x}, \tilde{y}\}$ in our models $(\mathcal{M}_1^x)$ and $(\mathcal{M}_{-1}^x)$ for which the operator $\tilde{y}$ is self-adjoint. By Proposition A.4, the latter is true if and only if we are in $(\mathcal{M}_1)$ resp. in $(\mathcal{M}_{-1})$. ∎

Corollary 3.8. *Apart from the one dimensional representations* $(\mathrm{III})_\alpha$ *there are up to unitary equivalence precisely* 5 *irreducible integrable representations of* $X_{q,\gamma}$. *In case* $\gamma > 0, \varphi > 0$ *these are the representations* $(\mathrm{I})_{1,1,0}, (\mathrm{I})_{1,1,0}, (\mathrm{I})_{1,-1,1}$, $(\mathrm{I})_{-1,-1,1}$ *and* $(\mathrm{II})_0$ *from Corollary* 3.6.

4. Integrable Operator Representations of the $*$-Algebra $SL_q(2, \mathbb{R})$

Recall that $SL_q(2)$ is the free algebra with unit element $\mathbb{1}$ generated by four elements a, b, c, d satisfying the following seven relations:

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb\,, \tag{4.1}$$

$$ad - qbc = \mathbb{1}\,, \tag{4.2}$$

$$ad - da = (q - q^{-1})bc\,. \tag{4.3}$$

In fact, Eqs. (4.1) and (4.3) define the matrix algebra $M_q(2)$ and (4.2) says that the quantum determinant is equal to one, cf. [FRT] or [M]. It is clear that we obtain an equivalent set of relations if (4.3) is replaced by

$$da - q^{-1}bc = \mathbb{1} \tag{4.4}$$

or by

$$ad - q^2 da = (1 - q^2)\mathbb{1}\,, \tag{4.5}$$

or by

$$da - q^{-2}ad = (1 - q^{-2})\mathbb{1}\,. \tag{4.6}$$

(We shall not need the Hopf algebra structure of $SL_q(2)$ in what follows.)

Let $\mathcal{R}$ be an algebra with unit. If a, b, c, and d are elements of $\mathcal{R}$ satisfying the conditions (4.1)–(4.3), then the quadruple $\{a, b, c, d\}$ is called a *representation* of $SL_q(2)$ in $\mathcal{R}$.

Since $|q| = 1$, there in a unique involution on the algebra $SL_q(2)$ such that a, b, c and d are hermitian elements. Endowed with this involution, the algebra $SL_q(2)$ becomes a $*$-algebra which will be denoted by $SL_q(2, \mathbb{R})$.

4.A. The following simple algebraic facts are the key for our integrability definitions given below.

Proposition 4.1. *Let $\mathcal{R}$ be an algebra with unit and let a, b, c, d be elements of $\mathcal{R}$.*

(i) *Suppose that a is invertible in $\mathcal{R}$. Then $\{a, b, c, d\}$ is a representation of $SL_q(2)$ in $\mathcal{R}$ if and only if*

$$ab = qba, \quad ac = qca, \quad bc = cb \tag{4.7}$$

and

$$d = (q^{-1}bc + 1)a^{-1} . \tag{4.8}$$

(ii) *If d is invertible in $\mathcal{R}$, then $\{a, b, c, d\}$ is a representation of $SL_q(2)$ in $\mathcal{R}$ if and only if*

$$db = q^{-1}bd, \quad dc = q^{-1}cd, \quad bc = cb \tag{4.9}$$

and

$$a = (qbc + 1)d^{-1} . \tag{4.10}$$

(iii) *Suppose that $\{a, b, c, d\}$ is a representation of $SL_q(2)$ in $\mathcal{R}$. If b* (resp. c) *is invertible in $\mathcal{R}$, then*

$$z := b^{-1}c \quad \text{(resp. } w := c^{-1}b) \quad \textit{permutes with } a, \ b, \ c \textit{ and } d .$$

(iv) *If b and c are invertible in $\mathcal{R}$, and $z = b^{-1}c$, then (4.7) is equivalent to the three relations*

$$ab = qba, \quad az = za, \quad bz = zb . \tag{4.11}$$

Proof. All assertions follow immediately by straightforward algebraic manipulations. As a sample, we verify that (4.7) and (4.8) imply the relation $cd = qdc$. Indeed, from (4.7) and (4.8) we obtain

$$cd = c(q^{-1}bc + 1)a^{-1} = (q^{-1}bc + 1)ca^{-1} = (q^{-1}bc + 1)qa^{-1}c = qdc. \quad \blacksquare$$

The assertions (i) and (ii) of Proposition 4.1 show that on a formal algebraic level Eqs. (4.1)–(4.3) defining $SL_q(2, \mathbb{R})$ are equivalent to Eqs. (4.7)–(4.8) and also to (4.9)–(4.10). Our integrability definitions for $SL_q(2, \mathbb{R})$ are built on this simple observation. Some justification for the assumption $\ker a = \{0\}$ in Definition 4.3 below is given by

Lemma 4.2. *Let a, b, c, d symmetric operators on a Hilbert space. Let $\psi \in \mathcal{D}(ad) \cap \mathcal{D}(a) \cap \mathcal{D}(bc) \cap \mathcal{D}(cb)$ be a vector such that $ad\psi - qbc\psi = \psi$ and $bc\psi = cb\psi$. If $a\psi = 0$ or if $d\psi = 0$, then $\psi = 0$.*

Proof. We have $0 = \langle d\psi, a\psi \rangle = \langle ad\psi, \psi \rangle = q \langle bc\psi, \psi \rangle + \langle \psi, \psi \rangle$. Since $\langle bc\psi, \psi \rangle = \langle \psi, cb\psi \rangle = \langle \psi, bc\psi \rangle$ is real and q is not real, $\psi = 0$. $\blacksquare$

Definition 4.3. *Let a, b, c be self-adjoint operators and let d be a closed symmetric operator on a Hilbert space $\mathcal{H}$. We shall say that the quadruple $\{a, b, c, d\}$ is an a-integrable representation of $SL_q(2, \mathbb{R})$ if* $\ker a = \{0\}$ *and if the following three conditions are fulfilled:*

(D.1) *There is an integer $n \in \mathbb{Z}$ such that $\{a, b\} \in \mathcal{C}_n(q)$ and $\{a, c\} \in \mathcal{C}_n(q)$.*

(D.2) *The self-adjoint operators b and c strongly commute (i.e. the spectral projections of b and c commute).*

(D.3) *d is the closure of the operator $(q^{-1}bc + 1)a^{-1}$.*

Note that (D.1) and (D.2) imply that the operator $(q^{-1}bc + 1)a^{-1}$ is symmetric by Corollary 2.12.

Definition 4.4. *A quadruple $\{a,b,c,d\}$ of self-adjoint operators on a Hilbert space is called an integrable representation of $SL_q(2,\mathbb{R})$ if $\{a,b,c,d\}$ is an a-integrable representation of $SL_q(2,\mathbb{R})$ and $\{d,b,c,a\}$ is a d-integrable representation of $SL_{\bar{q}}(2,\mathbb{R})$.*

The following slight reformulation of Definition 4.3 is often useful. The proof is straightforward and will be omitted.

Lemma 4.5. *Suppose that a, b, c are self-adjoint operators with* $\ker a = \{0\}$ *and d is a closed symmetric operator on a Hilbert space $\mathcal{H}$. The quadruple $\{a,b,c,d\}$ is an a-integrable representation of $SL_q(2,\mathbb{R})$ if and only if there exist an integer n and a decomposition $a = a_1 \oplus a_2 \oplus a_3$, $b = 0 \oplus b_2 \oplus b_3$, $c = c_1 \oplus 0 \oplus c_3$, $d = d_1 \oplus d_2 \oplus d_3$ with* $\ker b_3 = \ker c_3 = \{0\}$ *with respect to a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ such that:*

(i) $d_1 = a_1^{-1}$ *and* $\{a_1, c_1\} \in \mathcal{C}_n(q)$.

(ii) $d_2 = a_2^{-1}$ *and* $\{a_2, b_2\} \in \mathcal{C}_n(q)$.

(iii) $\{a_3, b_3\} \in \mathcal{C}_n(q)$ *and there exists a self-adjoint operator z_3 on $\mathcal{H}_3$ which commutes strongly with a_3 and b_3 such that c_3 and d_3 are the closure of $b_3 z_3$ and of $(q^{-1} b_3 c_3 + 1) a_3^{-1}$, respectively.*

4.B. Next we study two models of a-integrable representations of $SL_q(2,\mathbb{R})$. They are closely related to the models occurring in the preceding two sections.
Let $\mathcal{K}$ and $\mathcal{H}_0$ be Hilbert spaces and let $k \in \mathbb{Z}$. Suppose that A is a self-adjoint operator on $\mathcal{H}_0$ such that $\ker A = \{0\}$ and that B and C are strongly commuting self-adjoint operators on $\mathcal{K}$.

($\mathcal{M}_1^a$): Suppose w is a symmetry on $\mathcal{K}$ which commutes with B and C. Set $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathcal{K} \oplus \mathcal{H}_0$ and define

$$\begin{aligned}
\tilde{a} &= e^{Q} w \oplus A \,, \\
\tilde{b} &= e^{\varphi_{2k} P} B \oplus 0 \,, \\
\tilde{c} &= e^{\varphi_{2k} P} C \oplus 0 \,, \\
\tilde{d} &= (\bar{q} e^{2\varphi_{2k} P} BC + 1) e^{-Q} w \oplus A^{-1} \,,
\end{aligned}$$

($\mathcal{M}_{-1}^a$): Set $\mathcal{H} = L^2(\mathbb{R}) \otimes (\mathcal{K} \oplus \mathcal{K}) \oplus \mathcal{H}_0$ and

$$\begin{aligned}
\tilde{a} &= e^{Q} \otimes \sigma_0 \oplus A \,, \\
\tilde{b} &= e^{\varphi_{2k+1} P} \otimes B\sigma_1 \oplus 0 \,, \\
\tilde{c} &= e^{\varphi_{2k+1} P} \otimes C\sigma_1 \oplus 0 \,, \\
\tilde{d} &= (\bar{q} e^{2\varphi_{2k+1} P} e^{-Q} \otimes BC\sigma_0 + e^{-Q} \otimes \sigma_0) \oplus A^{-1} \,.
\end{aligned}$$

First note that in both models $\tilde{a}, \tilde{b}, \tilde{c}$ are well-defined self-adjoint operators and $\tilde{d}$ is closed and symmetric (by Proposition A.1, (v)). Further, it is clear that the quadruple $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$ is an a-integrable representation of $SL_q(2,\mathbb{R})$.

Theorem 4.6. *Let $\{a,b,c,d\}$ be an a-integrable representation of $SL_q(2,\mathbb{R})$ and let n be the integer occurring in Definition 4.3. Then $\{a,b,c,d\}$ is unitarily equivalent to a quadruple $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$ of the above model ($\mathcal{M}_\varepsilon^a$), where $\varepsilon = (-1)^n$. For ($\mathcal{M}_1^a$) the self-adjoint operators B and C can be chosen such that the spectrum of B is contained in the set $\{-1, 0, 1\}$ and that the operator $C \mid \ker B$ is a symmetry on* *the reducing subspace* $\ker B$ *for* C*. In case of* $(\mathcal{M}^a_{-1})$ *we can have that* B *is an orthoprojection and that* $C \mid \ker B = I$ *and* $B \mid \ker C = I$.

Proof. First we apply Lemma 4.5. Let $a = a_1 \oplus a_2 \oplus a_3$, $b = 0 \oplus b_2 \oplus b_3$, $c = c_1 \oplus 0 \oplus c_3$, $d = d_1 \oplus d_2 \oplus d_3$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ be the decomposition occurring therein.

Set $\mathcal{H}_0 := \ker c_1 \oplus \ker b_2$. Since $\{a_1, c_1\}$ and $\{a_2, b_2\}$ are integrable representations of $\mathbb{R}^2_q$, $\mathcal{H}_0$ reduces the self-adjoint operator a. Let $A := a \mid \mathcal{H}_0$. Since $\ker a = \{0\}$, A is invertible on $\mathcal{H}_0$. Since $d_1 = a_1^{-1}$ and $d_2 = a_2^{-1}$, $\mathcal{H}_0$ reduces d as well and $d \mid \mathcal{H}_0 = A^{-1}$. By construction, $b \mid \mathcal{H}_0 = c \mid \mathcal{H}_0 = 0$. Thus we have obtained the desired form of the operators a, b, c, d on the space $\mathcal{H}_0$.

Now we treat the space $\mathcal{H}_0^\perp$. For notational simplicity, let us assume that $\mathcal{H} = \mathcal{H}_0^\perp$. By the preceding paragraph this implies that $\ker c_1 = \{0\}$ and $\ker b_2 = \{0\}$ in the above decomposition. Also, we have $\ker b_3 = \{0\}$ by Lemma 4.5. Thus Theorem 2.9 applies to the pairs $\{a_1, c_1\}, \{a_2, b_2\}$ and $\{a_3, b_3\}$. Since these three pairs belong to the same class $\mathcal{C}_n(q)$, we obtain the same model $(\mathcal{M}^a_\varepsilon)$ and the same integer k for all three pairs. We treat the case of $(\mathcal{M}^a_1)$. Let $\mathcal{K}_j, w_j$ and $v_j, j = 1, 2, 3$, denote the corresponding Hilbert spaces and symmetries, respectively. Put $w := w_1 \oplus w_2 \oplus w_3$ and $v := 0 \oplus v_2 \oplus v_3$ on $\mathcal{K} := \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}$. Then the operators a and $b = 0 \oplus b_2 \oplus b_3$ have the required structure by Theorem 2.9. Let z_3 be the self-adjoint operator from Lemma 4.5. By condition (iii) therein, z_3 commutes strongly with a_3 and b_3 and hence with $|a_3|$ and $|b_3|$. Since, $|a_3| = e^Q$ and $|b_3| = e^{\varphi_{2k}P}$ on $\mathcal{H}_3 = L^2(\mathbb{R}) \otimes \mathcal{K}_3$, there exist a self-adjoint operator Z_3 on $\mathcal{K}_3$ such that $z_3 = I \otimes Z_3$ (cf. [S2], Lemma 5.3), that is, $z_3 = Z_3$ according to our notational convention. Using once again that z_3 commutes strongly with a_3 and b_3, we conclude that z_3 commutes with w_3 and v_3, so $c_3 := z_3 v_3$ is a well-defined self-adjoint operator on $\mathcal{K}_3$. Setting $c := v_1 \oplus 0 \oplus c_3$, the operator c has the desired form. The above formulae for d follows from the construction and from Lemma 4.5, so our proof is complete in case $(\mathcal{M}^a_1)$.

We sketch the necessary modifications of the proof in case $(\mathcal{M}^a_{-1})$. Then the pairs $\{a_1, c_1\}, \{a_2, b_2\}$ are $\{a_3, b_3\}$ are as in model $(\mathcal{M}_{-1})$ of Sect. 2. Since $|b_3| = e^{\varphi_{2k+1}P}$ on $\mathcal{H}_3$, the same reasoning as in case $(\mathcal{M}^a_1)$ shows that there is a self-adjoint operator Z_3 on $(\mathcal{K}_3 \oplus \mathcal{K}_3)$ such that $z_3 = I \otimes Z_3$. Since $u_{a_3} = \sigma_0$ and $u_{b_3} = \sigma_1$, it follows from $u_{a_3} z_3 \subseteq z_3 u_{a_3}$ and $u_{b_3} z_3 \subseteq z_3 u_{b_3}$ that there is a self-adjoint operator Z on $\mathcal{K}_3$ such that Z_3 is diagonal on $\mathcal{K}_3 \oplus \mathcal{K}_3$ with Z in the main diagonal. To complete the proof, we set $B := 0 \oplus I \oplus I, C := I \oplus 0 \oplus Z$ on $\mathcal{K} := \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3$. ∎

From Theorem 4.6 we easily obtain some important corollaries.

Corollary 4.7. *Each irreducible a-integrable representation of* $SL_q(2, \mathbb{R})$ *is unitarily equivalent to a representation of the following list with* $\alpha \in \mathbb{R}/\{0\}$, $\lambda \in \mathbb{R}, k \in \mathbb{Z}$ *and* $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$:

$$(\mathrm{I})_{\lambda,\varepsilon_1,\varepsilon_2,k} : \quad a = \varepsilon_1 e^Q, b = \varepsilon_2 e^{\varphi_{2k}P}, \quad c = \lambda b \,,$$

$$d = \varepsilon_1(\bar{q}\lambda e^{2\varphi_{2k}P} + 1)e^{-Q} \quad on \ \mathcal{H} = L^2(\mathbb{R}) \,.$$

$$(\mathrm{I})_{\infty,\varepsilon_1,\varepsilon_2,k} : \quad a = \varepsilon_1 e^Q, b = 0, c = \varepsilon_2 e^{\varphi_{2k}P} \,,$$

$$d = \varepsilon_1 e^{-Q} \quad on \ \mathcal{H} = L^2(\mathbb{R}) \,.$$

$(\mathrm{II})_{\lambda,k}$: $a = e^Q \otimes \sigma_0,\ b = e^{\varphi_{2k+1}P} \otimes \sigma_1,\ c = \lambda b\,,$

$d = (\bar{q}\lambda e^{2\varphi_{2k+1}P} + 1)e^{-Q} \otimes \sigma_0$ *on* $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2\,.$

$(\mathrm{II})_{\infty,k}$: $a = e^Q \otimes \sigma_0, b = 0, c = e^{\varphi_{2k+1}P} \otimes \sigma_1\,,$

$d = a^{-1}$ *on* $\mathcal{H} = L^2\mathbb{R} \otimes \mathbb{C}^2\,.$

$(\mathrm{III})_\alpha$: $a = \alpha, b = c = 0, d = \alpha^{-1}$ *on* $\mathcal{H} = \mathbb{C}\,.$

Each quadruple $\{a, b, c, d\}$ *of this list defines an irreducible a-integrable representation of* $SL_q(2, \mathbb{R})$ *and different values of parameters give inequivalent representations.*

Corollary 4.8. *Let* $\{a, b, c, d\}$ *be an a-integrable representation of* $SL_q(2, \mathbb{R})$ *on a Hilbert space* $\mathcal{H}$. *Then there is a dense linear subspace* $\mathcal{D} \subseteq \mathcal{D}(a) \cap \mathcal{D}(b) \cap \mathcal{D}(c) \cap \mathcal{D}(d)$ *of* $\mathcal{H}$ *such that*:

(i) $a\mathcal{D} = \mathcal{D},\ b\mathcal{D} \subseteq \mathcal{D},\ c\mathcal{D} \subseteq \mathcal{D},\ d\mathcal{D} \subseteq \mathcal{D}$ *and* $|a|^{\mathrm{it}}\mathcal{D} = \mathcal{D}$ *for* $t \in \mathbb{R}$.
(ii) *The operator relations* (4.1)–(4.5) *are pointwise satisfied on* $\mathcal{D}$.
(iii) $\mathcal{D}$ *is a core for* a, a^{-1}, b, c *and* d.

If $\ker b = \{0\}$ *or* (resp. *and*) $\ker c = \{0\}$, *we may have also that* $b\mathcal{D} = \mathcal{D}$ *and* $|b|^{\mathrm{it}}\mathcal{D} = \mathcal{D}$ *or* (resp. *and*) $c\mathcal{D} = \mathcal{D}$ *and* $|c|^{\mathrm{it}}\mathcal{D} = \mathcal{D}$, $t \in \mathbb{R}$.

Corollary 4.9. *If* $\{a, b, c, d\}$ *is an a-integrable representation of* $SL_q(2, \mathbb{R})$, *then* $\{a, d\}$ *is an a-integrable representation of* $X_{q^2,1}$.

4.C. Now we characterize the integrable representations of $SL_q(2, \mathbb{R})$.

Theorem 4.10. (i) *An a-integrable representation* $\{a, b, c, d\}$ *of* $SL_q(2, \mathbb{R})$ *is integrable if and only if the operator d is self-adjoint.*
(ii) *The quadruple* $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$ *of the model* $(\mathcal{M}_1^a)$ *is integrable if and only if* $BC > 0$ *and* $k = 0$. *The quadruple* $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$ *of* $(\mathcal{M}_{-1}^a)$ *is integrable if and only if* $BC < 0$ *and* $k = 0$ *if* $\varphi > 0$ resp. $k = 1$ *if* $\varphi < 0$.
(iii) *Each integrable representation of* $SL_q(2, \mathbb{R})$ *is unitarily equivalent to one of the quadruples described by* (ii).

Proof. (i) The only if part is trivial by Definition 4.4. Suppose that $\{a, b, c, d\}$ is an a-integrable representation of $SL_q(2, \mathbb{R})$ for which the operator d is self-adjoint. By Theorem 4.6 we can assume that the quadruple $\{a, b, c, d\}$ is as in one of our models $(\mathcal{M}_{\pm 1}^a)$. We have to show that $\{d, b, c, a\}$ is a d-integrable representation of $SL_{\bar{q}}(2, \mathbb{R})$. We only sketch the proof in case of $(\mathcal{M}_{-1}^a)$. First note that $\ker d = \{0\}$ by Proposition A.1, (iv). To check that $\{d, b\}$ is an integrable representation of $\mathbb{R}_{\bar{q}}^2$, we argue similarly as in the proof of Theorem 3.7. Obviously, $\ker b$ is reducing for d, so we can restrict ourselves to the case where $\ker b = \{0\}$, i.e. $\ker B = \{0\}$ and $\mathcal{H}_0 = \{0\}$. We have $|b|^{it} = e^{it\varphi_{2k+1}P}(e^{it\log|B|} \oplus e^{it\log|B|})$ on $L^2(\mathbb{R}) \otimes (\mathcal{K} \oplus \mathcal{K})$, hence $|b|^{it}d = e^{t\varphi_{2k+1}}d|b|^{it}$ for $t \in \mathbb{R}$. Since obviously $u_b d \subseteq -du_b$, $\{d, b\}$ is an integrable representation of $\mathbb{R}_{\bar{q}}^2$ by Corollary 2.7. The same reasoning works for the pair $\{d, c\}$. Finally, we have to verify that a is the closure of $(qbc + 1)d^{-1}$. For this we first note that $\{a, bc\}$ is obviously an integrable representation of $\mathbb{R}_{q^2}^2$ and d is defined as the closure of $(\bar{q}bc + 1)a^{-1}$. Therefore, by Definition 3.1, $\{a, d\}$ is an a-integrable representation of $X_{q^2,1}$. Since d is self-adjoint, $\{a, d\}$ is integrable by Theorem 3.7 and a is the closure of $(qbc+1)d^{-1}$. This completes the proof of (i).

(ii) follows by combining (i) with Proposition A.4 and (iii) follows from (ii) and Theorem 4.6. ■

Corollary 4.11. *We retain the notation from Corollary 4.7. Then any irreducible integrable representation of $SL_q(2,\mathbb{R})$ is unitarily equivalent to one of the following list:*

$(\mathrm{I})_{\lambda,\varepsilon_1,\varepsilon_2,0}$: $\lambda \in (0,+\infty], \varepsilon_1, \varepsilon_2 \in \{-1,1\}$.

$(\mathrm{II})_{\lambda,s(\varphi)}$: $\lambda \in [-\infty, 0)$.

$(\mathrm{III})_{\alpha}$: $\alpha \in \mathbb{R} \mid \{0\}$.

Each of these representations is an irreducible integrable representation of $SL_q(2, \mathbb{R})$.

Corollary 4.12. *If $\{a, b, c, d\}$ is an integrable representation of $SL_q(2,\mathbb{R})$, then $\{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}$ are integrable representations of $\mathbb{R}_q^2$, $\{a, d\}$ an integrable representation of $X_{q^2,1}$ and $\{d,a\}$ is an integrable representation of $X_{\bar{q}^2,1}$.*

In their study of the quantum Liouville model on the lattice, L.D. Faddeev and L.A. Takhtajan constructed a representation of $SL_q(2, \mathbb{R})$, cf. [FT]. We briefly discuss this in the following

Example 4.13. Set $q = e^i$. For $\lambda \in (0,+\infty)$, we define operators $a := e^{\frac{P}{2}} \sqrt{1+\lambda e^{2Q}}\, e^{\frac{P}{2}}, b := e^Q, c := \lambda e^Q$ and $d := e^{-\frac{P}{2}} \sqrt{1+\lambda e^{2Q}}\, e^{-\frac{P}{2}}$ on $\mathcal{H} = L^2(\mathbb{R})$. The representation from [FT] corresponds to the case $\lambda = 1$.

The following assertions can be verified by arguing in a similar way as in the proofs of Propositions 1.3 and 1.4: The operators $a \mid \mathcal{F}$ and $d \mid \mathcal{F}$ are essentially self-adjoint. By a slight abuse of notation, we denote their closures again by a and d, respectively. For $\psi \in \mathcal{F}, a\psi = \sqrt{1+\lambda e^{2Q+i}}\ e^P \psi$ and $d\psi = \sqrt{1+\lambda e^{2Q-i}} e^{-P}\psi$. Further, ker $a = \{0\}$ and $a\mathcal{F}$ is a core for d.

From these facts we easily conclude that $\{a, b\} \in \mathcal{C}_0(q), \{a, c\} \in \mathcal{C}_0(q)$ and $d = (\bar{q}bc+1)a^{-1}$, i.e. $\{a, b, c, d\}$ is an integrable representation of $SL_q(2, \mathbb{R})$. Clearly, $\{a, b, c, d\}$ is unitarily equivalent to the representation $(\mathrm{I})_{\lambda,1,1,0}$ from Corollary 4.11.

5. Concluding Remarks

(1) Suppose $\{x, y\}$ is a couple of the class $\mathcal{C}_k(q)$, $k \in \mathbb{Z}$. Then the operator $|x| + |y|$ is essentially self-adjoint if and only if $k \in \{-1, 0, 1, 2s(\varphi)\}$. (This follows at once from Proposition A.5 if we take $\{x, y\}$ as in $(\mathcal{M}_1)$ resp. $(\mathcal{M}_{-1})$ and apply some unitary transformation $e^{i\gamma Q}, \gamma \in \mathbb{R}$.) Further, for the classification of integrable representations of $X_{q,\gamma}$ and $SL_q(2, \mathbb{R})$ only the classes $\mathcal{C}_k(q)$ with $k = -1, 0, 1, 2s(\varphi)$ are needed. For these and other reasons it seems to be sufficient to take only these classes $\mathcal{C}_k(q)$ as integrable representations of $\mathbb{R}_q^2$. Since $x^2 + y^2$ is essentially self-adjoint if and only if $k = 0$ for $|\varphi| < \frac{\pi}{2}$ and $k = s(\varphi)$ for $\frac{\pi}{2} < |\varphi| < \pi$, it might be even justified to consider only the class $\mathcal{C}_k(q)$ with $k = 0$ resp. $k = s(\varphi)$ as integrable representations of $\mathbb{R}_q^2$.

(2) Let A denote one of the $*$-algebras $\mathbb{R}_q^2$, $X_{q,\gamma}$ or $SL_q(2, \mathbb{R})$, and consider an integrable, x-integrable or a-integrable representation of A. Then, by Corollaries

2.11, 3.5 resp. 4.8, the commutation relations defining A are pointwise satisfied for all vectors of a suitable invariant core $\mathcal{D}$ for the corresponding operators $a, b; x, y$ resp. a, b, c, d. Hence such a representation gives really a $*$-representation, say ϱ, of the $*$-algebra A on the domain $\mathcal{D}$ in the sense of unbounded representation theory (see e.g. [S1], Sect 8.1). This justifies to speak about integrable, x-integrable or a-integrable representations of the $*$-algebras A of the previous sections. In case of an integrable representation of A, ϱ^* is a self-adjoint representation of A, but the converse is not true. This suggests the following

Problem. Characterize the self-adjoint representations of the $*$-algebra A which correspond to integrable representations of A.

(3) Knowing the irreducible integrable representations of $\mathbb{R}_q^2$, $X_{q,\gamma}$ resp. $SL_q(2, \mathbb{R})$, the C^*-algebra generated by the corresponding operators can be studied. For this the affiliation notion of [W1] plays a crucial role.

6. Appendix: Some Operator-Theoretic Results

Suppose that A is a self-adjoint operator on a Hilbert space $\mathcal{G}$. Let E_n denote the spectral projection of A corresponding to the interval $(-n, n)$. We study operators $(\omega e^{\alpha Q} A + \gamma) e^P$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathcal{G}$, where $\alpha, \gamma \in \mathbb{R}$, $\omega \in \mathbb{C}$, $\omega \neq 0$. We denote the closure of $(\omega e^{\alpha Q} A + \gamma) e^P$ by T or again by the symbol $(\omega e^{\alpha Q} A + \gamma) e^P$ according to our convention from Sect. 1. Sometimes the operator Q is denoted by the variable x.

Proposition A.1. *Put* $\eta_{\varepsilon,n} := e^{-\varepsilon(x-i)^2} E_n \eta$ *for* $\eta \epsilon \mathcal{H}, \varepsilon > 0$ *and* $n \in \mathbb{N}$. *Suppose that* $\mathcal{D}_0$ *is a core for* A *and set* $\mathcal{D} := \mathcal{F} \otimes \mathcal{D}_0$. *Then:*

(i) *$\mathcal{D}$ is a core for T.*

(ii) *For $\eta \in \mathcal{D}(T^*)$, we have $\eta_{\varepsilon,n} \in \mathcal{D}(T^*) \cap \mathcal{D}(e^P(\bar{\omega} e^{\alpha Q} A + \gamma))$ and $T^* \eta_{\varepsilon,n} = e^P(\bar{\omega} e^{\alpha Q} A + \gamma) \eta_{\varepsilon,n}$, $\varepsilon \in (0, \varepsilon_0)$, $n \in \mathbb{N}$.*

(iii) *$T\mathcal{D}$ is a core for e^{-P}.*

(iv) $\ker T^* = \{0\}$.

(v) *T is symmetric if and only if $\bar{\omega} e^{\alpha i} = \omega$.*

Proof. (i). By taking the closure in the graph norm of A, we can assume that $\mathcal{D}_0 = \mathcal{D}(A)$. Upon writing A as a direct sum of bounded self-adjoint operators it follows that it suffices to prove the assertion in case where A is bounded. Further, it is sufficient to show that $\mathcal{D}((T \mid \mathcal{D})^*) \subseteq \mathcal{D}(T^*)$, since then $(T \mid \mathcal{D})^{**} = T^{**}$. Suppose that $\eta \in \mathcal{D}((T \mid \mathcal{D})^*)$. Then there is a $\zeta \in \mathcal{H}$ such that $\langle T e^{-\varepsilon x^2} \psi, \eta \rangle = \langle e^{-\varepsilon x^2} \psi, \zeta \rangle$ for $\psi \in \mathcal{F} \otimes \mathcal{G}$ and for small $\varepsilon > 0$. Since $(\bar{\omega} e^{\alpha x} A + \gamma) e^{-\varepsilon(x+i)^2}$ is a bounded operator, the latter and Corollary 1.2, (i), yield

$$\langle e^P \psi, (\bar{\omega} e^{\alpha x} A + \gamma) e^{-\varepsilon(x-i)^2} \eta \rangle = \langle \psi, e^{-\varepsilon x^2} \zeta \rangle \tag{6.1}$$

for $\psi \in \mathcal{F} \otimes \mathcal{G}$. Since $\mathcal{F}$ is a core for e^P, (6.1) is valid for all $\psi \in \mathcal{D}(e^P) \otimes \mathcal{G}$. Proceeding in reversed order, we get $\langle e^{-\varepsilon(x-\imath)^2} T \psi, \eta \rangle = \langle e^{-\varepsilon x^2} \psi, \zeta \rangle$ for $\psi \in \mathcal{D}(T)$. Letting $\varepsilon \downarrow 0$, we conclude that $\eta \in \mathcal{D}(T^*)$.

(ii). For $\psi \in \mathcal{D}(T)$ and $\eta \in \mathcal{D}(T^*)$, we have

$$\begin{aligned}\langle Te^{-\varepsilon x^2}E_n\psi, \eta\rangle &= \langle e^{-\varepsilon(x+i)^2}E_nT\psi, \eta\rangle = \langle T\psi, \eta_{\varepsilon,n}\rangle \\ &= \langle e^P\psi, (\bar{\omega}e^{\alpha Q}A+\gamma)\eta_{\varepsilon,n}\rangle \\ &= \langle \psi, e^{-\varepsilon x^2}E_nT^*\eta\rangle\end{aligned}$$

by Corollary 1.2. This yields $(\bar{\omega}e^{\alpha Q}A+\gamma)\eta_{\varepsilon,n} \in \mathcal{D}(e^P)$, $\eta_{\varepsilon,n} \in \mathcal{D}(T^*)$ and $T^*\eta_{\varepsilon,n} = e^P(\bar{\omega}e^{\alpha Q}A+\gamma)\eta_{\varepsilon,n}$.
(iii). First note that $T\mathcal{D} = T(\mathcal{F} \otimes \mathcal{D}_0) \subseteq \mathcal{D}(e^{-P})$ and $e^{-P}T\psi = (\omega e^{-\alpha i}e^{\alpha Q}A+\gamma)\psi$ for $\psi \in \mathcal{D}$. Suppose that $\eta \in \mathcal{D}((e^{-P} \mid T\mathcal{D})^*)$, *i.e* there is a $\zeta \in \mathcal{H}$ such that $\langle e^{-P}T\psi, \eta\rangle = \langle T\psi, \zeta\rangle$, $\psi \in \mathcal{D}$. By closing up in the graph norm of A we can assume that $\mathcal{D}_0 = \mathcal{D}(A)$. Replacing ψ by $e^{-\varepsilon x^2}E_n\psi$, the preceding yields $\langle \psi, (\bar{\omega}e^{\alpha i}e^{\alpha Q}A+\gamma)e^{-\varepsilon x^2}E_n\eta\rangle = \langle T\psi, \zeta_{\varepsilon,n}\rangle$ for $\psi \in \mathcal{D}(T)$, so that $T^*\zeta_{\varepsilon,n} = (\bar{\omega}e^{\alpha i}e^{\alpha Q}A+\gamma)e^{-\varepsilon x^2}E_n\eta$. On the other hand, by (ii) and Corollary 1.2, $T^*\zeta_{\varepsilon,n} = (\bar{\omega}e^{\alpha i}e^{\alpha Q}A+\gamma)e^P\zeta_{\varepsilon,n}$.
A simple operator-theoretic argument shows that the operator $\bar{\omega}e^{\alpha i}e^{\alpha Q}A+\gamma$ has trivial kernel. Therefore, the preceding gives $e^P\zeta_{\varepsilon,n} = e^{-\varepsilon x^2}E_n\eta$. Letting $\varepsilon \downarrow 0$ and $n \to \infty$, we obtain $e^P\zeta = \eta$, i.e. $\eta \in \mathcal{D}(e^{-P})$.
(iv). Let $\eta \in \ker T^*$. Then we have $\eta_{\varepsilon,n} \in \ker T^*$, so $e^P(\bar{\omega}e^{\alpha Q}A+\gamma)\eta_{\epsilon,n} = 0$ by (ii) and hence $\eta_{\varepsilon,n} = 0$ for all $\varepsilon > 0$ and $n \in \mathbb{N}$. Consequently, $\eta = 0$.
(v) follows at once from Corollary 1.2 and (i). ■

Proposition A.2. *Suppose that $A > 0$ on $\mathcal{G}$. Let $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ and $s \in \mathbb{Z}$ be such that $\beta_1 + \beta_2 = \alpha + 2\pi s, \alpha \neq 0$ and $\beta_1 \neq n\pi$ for $j = 1,2$ and for all $n \in \mathbb{Z}$. Define closed operators $T_j, j = 1,2$, on $\mathcal{H}$ by $T_j = (e^{\alpha Q + \beta_j i}A + 1)e^P$. Let $\mathfrak{Z}$ denote the set of real numbers $y_m := \alpha^{-1}((2m+1)\pi + \beta_1), m \in \mathbb{Z}$.*

(i) *If $\mathfrak{Z} \cap (0,1) \neq \phi$, then $T_1^* \neq T_2$,*
(ii) *If $\mathfrak{Z} \cap [0,1] = \phi$, then $T_1^* = T_2$.*

Proof. First let us show that it suffices to prove the assertions (i) and (ii) in the scalar case $\mathcal{G} = \mathbb{C}$. Since we can express A as an orthogonal direct sum of bounded self-adjoint operators with simple spectra, it is enough to verify (i) and (ii) for such operators. But then $\mathcal{G}$ is a separable Hilbert space and we can write A as a direct integral of operators $\lambda \cdot I_\lambda$ over the measurable field $\sigma(A) \ni \lambda \to \mathcal{H}_\lambda := L^2(\mathbb{R})$ of Hilbert spaces. Let $\xi = \xi(\lambda)$ be a cyclic vector for A. Set $T_{j,\lambda} := (e^{\alpha Q+\beta_j i}\lambda + 1)e^P$. Obviously, $\{\psi_{n,m}(\lambda) := x^n e^{-x^2}\lambda^m\xi(\lambda);\ n,m \in \mathbb{N}_0\}$ is a countable family of measurable vector fields such that a.e. $\{\psi_{n,m}(\lambda)\}$ is dense in $\mathcal{D}(T_{j,\lambda})$ with respect to the graph norm. From this we conclude that $\lambda \to T_{j,\lambda}$ is a measurable field of closed operators and T_j is the direct integral of this field (cf. [S1], 12.1). Using standard properties of direct integrals of closed operators (see chapter 12 of [S1] for details), it follows that it suffices to prove the assertions for the operators $T_{j,\lambda}$, $j = 1,2$. For the rest of this proof we assume that $\mathcal{G} = \mathbb{C}$. We set $\delta := \log A$ and $h_j^{\pm}(z) := e^{\alpha z + \delta \pm \beta_j i} + 1, z \in \mathbb{C}$. Then, by definition, $T_j = h_j^+(x)e^P$, where $h_j^+(x)$ is the multiplication operator by the function $h_j^+(x)$ on $\mathcal{H} = L^2(\mathbb{R})$.

We prove (i). Put $f_j := h_j^-(x)^{-1}f$ for $f \in L^2(\mathbb{R})$. Note that $f_j \in L^2(\mathbb{R})$, since each function $h_j^{\pm}(x)$ has positive infimum on $\mathbb{R}$ because $\beta_j \neq n\pi$ for $n \in \mathbb{Z}$. Since obviously $\langle T_jf, g_j\rangle = \langle e^Pf, g\rangle = \langle f, e^Pg\rangle$ for $f \in \mathcal{D}(T_j)$ and $g \in \mathcal{D}(e^P)$, we conclude that $g \in \mathcal{D}(T_j^*)$ and $T_j^* g_j = e^P g$. Assume to the contrary that $T_1^* = T_2$. Then we have $\langle T_1^* f_1, g_2 \rangle = \langle T_2 f_1, g_2 \rangle = \langle f_1, T_2^* g_2 \rangle$ and hence $\langle e^P f, g_2 \rangle = \langle h_2^+(x)^{-1} e^P f, g \rangle = \langle f_1, e^P g \rangle$ for all $f, g \in \mathcal{D}(e^P)$. Therefore, $f_1 \in \mathcal{D}(e^P)$. Setting $f(x) = e^{-x^2}$, it follows from Lemma 1.1 that $f_1(x) = h_1^-(x)^{-1} e^{-x^2}$ has a holomorphic extension to the strip $I_1 = \{z : 0 < \operatorname{Im} z < 1\}$. But the set of zeros $z_m = -\delta\alpha^{-1} + iy_m$ of $h_1^-(z)$ intersects I_1, since $\mathfrak{Z} \cap (0,1) \neq \phi$ by assumption. We arrived at a contradiction, so (i) is proven.

Now we prove (ii). Take $\eta \in \mathcal{D}(T_j^*)$ and set $\eta_\varepsilon(x) = e^{-\varepsilon(x-i)^2}\eta(x)$ for $\varepsilon > 0$. Proposition A.1, (ii), says that $\eta_\varepsilon \in \mathcal{D}(T_j^*) \cap \mathcal{D}(e^P h_j^-(x))$ and $T_j^* \eta_\varepsilon = e^P h_j^-(x)\eta_\varepsilon$. (Note that $E_n = I$ and so $\eta_{\varepsilon,n} = \eta_\varepsilon$ if $n > e^\delta$.) By Lemma 1.1, the function $(h_j^- \eta_\varepsilon)(x)$ of $\mathcal{D}(e^P)$ has a holomorphic extension, say $(h_j^- \eta_\varepsilon)(z)$, to the strip I_1. Our assumptions $\mathfrak{Z} \cap [0,1] = \phi$ and $\beta_1 + \beta_2 = \alpha + 2\pi s$ imply that $| h_j^-(z) | \geqq \varrho$ on $\bar{I}_1$ for some constant $\varrho > 0$. Thus $\eta_\epsilon(z) := (h_j^- \eta_\varepsilon)(z) h_j^-(z)^{-1}, z \in I_1$, defines a holomorphic function on I_1 for which

$$\sup_{0<y<1} \int |\eta_\varepsilon(x+iy)|^2 dx \leqq \varrho^{-2} \sup_{0<y<1} \int |(h_j^- \eta_\varepsilon)(x+iy)|^2 dx \,.$$

The latter is finite by Lemma 1.1, since $h_j^- \eta_\varepsilon \in \mathcal{D}(e^P)$. Formula (1.2) shows that $\eta_\epsilon(z)$ has the boundary values $\eta_\varepsilon(x)$ on $\mathbb{R}$, so $\eta_\varepsilon \in \mathcal{D}(e^P)$ by Lemma 1.1. Applying (1.2) once more, we get

$$T_j^* \eta_\varepsilon = e^P (h_j^- \eta_\varepsilon) = h_j^-(x+i)(e^P \eta_\varepsilon)(x) \,. \tag{6.2}$$

In order to complete our proof, take vectors $\eta^1 \in \mathcal{D}(T_1^*)$ and $\eta^2 \in \mathcal{D}(T_2^*)$. We have $T_j^* \eta_\varepsilon^j = e^{-\varepsilon x^2} T_j^* \eta^j$ and $\overline{h_1^-(x)} = h_2(x+i)$. Using these facts and applying (6.2) twice, we get

$$\begin{aligned}
\langle e^{-\varepsilon x^2} T_1^* \eta^1, \eta_\varepsilon^2 \rangle &= \langle T_1^* \eta_\varepsilon^1, \eta_\varepsilon^2 \rangle = \langle e^P (h_1^- \eta_\varepsilon^1), \eta_\varepsilon^2 \rangle \\
&= \langle h_1^- \eta_\varepsilon^1, e^P \eta_\varepsilon^2 \rangle = \langle \eta_\varepsilon^1, h_2^-(x+i) e^P \eta_\varepsilon^2 \rangle \\
&= \langle \eta_\varepsilon^1, T_2^* \eta_\varepsilon^2 \rangle = \langle \eta_\varepsilon^1, e^{-\varepsilon x^2} T_2^* \eta^1 \rangle \,.
\end{aligned}$$

Letting $\varepsilon \downarrow 0$ we obtain $\langle T_1^* \eta^1, \eta^2 \rangle = \langle \eta^1, T_2^* \eta^2 \rangle$. This shows that $T_1^* = T_2^{**} \equiv T_2$. ■

Corollary A.3. *Retain the assumptions and the notation from Proposition* A.2 *and assume that* $\beta_1 = \beta_2$, *so that* $T_1 = T_2 =: T$. *Then the operator* T *is self-adjoint if and only if* s *is even and* $|\alpha| < 2\pi$.

Proof. If s is odd and $|\alpha| > 2\pi$, then $\mathfrak{Z} \cap (0,1) \neq \phi$. If s is even and $|\alpha| < 2\pi$, then $\mathfrak{Z} \cap [0,1] = \phi$. Note that the cases $\alpha = \pm\, 2\pi$ are excluded by assumption. ■

Remarks. (1) The operator T from Corollary A.3 acts as $T\psi = e^{\frac{P}{2}}((-1)^s e^{\alpha Q} A + 1)e^{\frac{P}{2}}$ for $\psi \in \mathcal{F}$. Often this form of T is more convenient. In particular we see that $T \geqq 0$ if and only if s is even.
(2) S.L. Woronowicz [W2] has determined even the deficiency indices of the operator T for $\mathcal{G} = \mathbb{C}$. He showed that for odd s both deficiency indices of T are finite and their difference is one, so T has no self-adjoint extension in $L^2(\mathbb{R})$. If s is even, the deficiency indices of T are equal and finite.

Proposition A.4. (i) *The operator* $\tilde{y}$ *defined by formula* (3.5) *is self-adjoint if and only if* $w = (-1)^k s(\gamma) \cdot I$ *and* $k \in \{0, s(\varphi)\}$.
(ii) *The operator* $\tilde{y}$ *in formula* (3.6) *is self adjoint if and only if* $k = 0$ *and* $\varphi > 0$ *or if* $k = -1$ *and* $\varphi < 0$.

Proof. Without loss of generality we assume that $\mathcal{H}_0 = \{0\}$.
(i). First we study the self-adjointness of the (closed symmetric) operator $S_\lambda := (q_0 e^{\varphi_{2k} P}\lambda + 1)e^{-Q}$ on $L^2(\mathbb{R}) \otimes \mathcal{G}$, where $\lambda \in \mathbb{R}$, $\lambda \neq 0$. A unitary transformation by the inverse Fourier transform yields $F^{-1}S_\lambda F = (q_0 e^{\varphi_{2k} Q}\lambda + 1)e^P$, so Corollary A.3 applies with $A = I, \alpha = \varphi_{2k} = \varphi - 2k\pi$ and $\beta = \frac{\varphi}{2}, s = k$ if $\lambda > 0$ resp. $\beta = \frac{\varphi}{2} + \pi, s = k + 1$ if $\lambda < 0$. Note that $\alpha \neq 0$ and $\beta \neq n\pi$ for $n \in \mathbb{Z}$, because $q \neq 1$. Since $|\varphi| < \pi$ and $\varphi \neq 0$ by assumption, Corollary A.3 shows that S_λ is self-adjoint if and only if $k = 0$ for $\lambda > 0$ or if $k = s(\varphi)$ for $\lambda < 0$.
To Prove the assertion of (i), note that $\gamma^{-1}v\,\tilde{y}$ is the direct sum of two operators S_λ with $\lambda = \gamma^{-1}$, $\mathcal{G} = \ker(w - I)$ and $\lambda = -\gamma^{-1}$, $\mathcal{G} = \ker(w + I)$. By the preceding discussion, both operators are self-adjoint if and only if we are in the above two cases.
(ii). Putting $S_\varepsilon := (\varepsilon q_0 e^{\varphi_{2k+1} P} + \gamma)e^{-Q}$ for $\varepsilon = \pm 1$, the operators $\tilde{y}$ and $\tilde{y}^*$ are given by the matrices

$$\tilde{y} = \begin{pmatrix} 0 & S_1 \\ S_{-1} & 0 \end{pmatrix} \quad \text{and} \quad \tilde{y}^* = \begin{pmatrix} 0 & S_{-1}^* \\ S_1^* & 0 \end{pmatrix}$$

Therefore, $\tilde{y}$ is self-adjoint iff $S_1^* = S_{-1}$. Upon scaling, we can assume that $\gamma = 1$. Then $F^{-1}S_\varepsilon F = (\varepsilon q_0 e^{\varphi_{2k+1} Q} + 1)e^P$ and we are in the situation of Proposition A.2 with $A = I$, $\alpha = \varphi_{2k+1} = \varphi - (2k+1)\pi$, $\beta_1 = \frac{\varphi}{2}, \beta_2 = \frac{\varphi}{2} + \pi$, $s = k + 1$. Using once more that $|\varphi| < \pi$ and $\varphi \neq 0$, one easily verifies that $\mathfrak{Z} \cap (0,1) = \phi$ in the two cases $k = 0, \varphi > 0$ and $k = -1, \varphi < 0$ and that $\mathfrak{Z} \cap [0,1] \neq \phi$ otherwise. By Proposition A.2 these are the only cases where $S_1^* = S_{-1}$ or equivalently where $\tilde{y}$ is self-adjoint. ∎

The next proposition can be derived by using similar arguments.

Proposition A.5. *Suppose that* $\ker A = \{0\}$ *and let* $n \in \mathbb{Z}$. *Then the (closed symmetric) operator* $(\bar{q}e^{2\varphi_n P}A + 1)e^{-Q}$ *is self-adjoint if and only if* $A > 0$ *and* $n = 0$ *or if* $A < 0$ *and* $n = s(\varphi)$.

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