Commun. Math. Phys. 157, 549–572 (1993)

Communications in Mathematical Physics

Recovering Singularities of a Potential from Singularities of Scattering Data

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Received October 14, 1992; in revised form February 23, 1993

Abstract. In this paper we show that the leading singularities of certain potentials can be determined from the leading singularities of the backscattering (as well as other determined sets of scattering data). The potentials in question are conormal with respect to smooth surfaces of arbitrary dimension; the restrictions on their orders allow for unbounded potentials in all dimension greater than or equal to three.

0. Introduction

Let $q(x)$ be a real-valued, compactly supported potential on $\mathbb{R}^n$, $n \geqq 3$, and $a(\lambda, \theta, \omega)$, $\lambda \in \mathbb{R}$, $\theta, \omega \in S^{n-1}$, the scattering amplitude of $q(x)$. The nonlinear transform $q(x) \rightsquigarrow a(\lambda, \theta, \omega)$ is overdetermined and there has been much interest in the inverse problem of determining $q(x)$ from $a(\lambda, \theta, \omega)$ and the restrictions of a to subsets of $\mathbb{R} \times S^{n-1} \times S^{n-1}$, e.g., [BC, ER, HN, No, N]. In this paper we will be interested in formally determined (n-dimensional) sets of scattering data; moreover, we will work in the time domain, i.e., with the scattering *kernel*,

$$\alpha(s, \theta, \omega) = c_n \int e^{is\lambda} \lambda^{\frac{n-1}{2}} \overline{a(\lambda, \theta, \omega)}\, d\lambda\, .$$

The class of q's considered will be those conormal to a smooth submanifold $S \subset \mathbb{R}^n$ of arbitrary codimension k. The inverse problem solved consists in determining S and the symbol of $q(x)$ from the leading singularities of the scattering data. The strongest singularity of the full scattering kernel $\alpha(s, \theta, \omega)$ is of course the peak scattering; we show that for the class of potentials considered here, $\alpha(s, \theta, \omega)$ is, away from the contribution of the tangential rays, a sum of the peak scattering and a (weaker) lagrangian distribution associated with a reflected lagrangian $\hat{\Lambda}_- \subset T^*(\mathbb{R} \times S^{n-1} \times S^{n-1})$. It is the restriction of this reflected component of

* Partially supported by NSF Grant DMS-9101298 and an Alfred P. Sloan Research Fellowship
** Partially supported by NSF Grant DMS-9100178

$\alpha(s, \theta, \omega)$ to various n-dimensional submanifolds of $\mathbb{R} \times S^{n-1} \times S^{n-1}$ which we show determines S and the symbol of q at S.

A particularly interesting case of our results is that of $q(x)$ having a Heaviside-type singularity across a smooth hypersurface; the location and size of the jump can then be determined from $\alpha|_{\mathbb{B}}$, where $\mathbb{B}$ is the backscattering data,

$$\mathbb{B} = \{(s, \theta, \omega) \colon \theta = -\omega\} .$$

More precisely, we prove the following; a more detailed statement, as well as the extension to other, possibly time-dependent sets of scattering data, can be found in Sect. 4.

Theorem 0.1. *Let $S \subset \mathbb{R}^n$ be smooth of codimension k and $q(x)$ conormal of order μ to S, with*

$$\mu < -\max\left(\frac{(n-2)}{n}k, k-1\right), \quad n \geqq 5 \quad \text{and}$$

$$\mu < -\max\left(\frac{k}{2}, k-1\right), \quad n = 3, 4 .$$

Then S and the principal symbol of $q(x)$ are determined by the singularities of the backscattering $\alpha|_{\mathbb{B}}$.

The restriction that the order of $q(x)$ be less than $-\frac{n-2}{n}k$ or $-k/2$, respectively, insures that the scattering kernel is defined ([P]); the restriction $\mu < 1 - k$ is needed so that the operator $\Box^{-1} M_q$ considered in the proof is slightly smoothing.

Working in the frequency domain (i.e., with $a(\lambda, \theta, \omega)$), Prosser [Pr] gave a formal procedure to determine $q(x)$ from backscattering under a small norm assumption. In Eskin and Ralston [ER], the map from complex q to the backscattering was shown to be generically a local homeomorphism with respect to certain norms. Note that in the theorem, although $q(x)$ belongs to a rather special class, there is no smallness assumption. Only the leading singularities of $q(x)$ are determined by $\alpha|_{\mathbb{B}}$, but only the leading singularities of $\alpha|_{\mathbb{B}}$ are needed to do this. (After the completion of this paper, J. Ralston brought to our attention the related paper of Päivärinta and Somersalo [PäS], which treats the question of recovering the singularities of the potential from the scattering amplitude as a function of all its variables. Their results are from the point of view of the Born approximation, rather than the time-domain approach taken here.)

The method of proof we use is to construct an approximate solution to the direct problem

$$\begin{cases} (\Box + q(x))u(x, t) = 0 & \text{on } \mathbb{R}^{n+1} \\ u(x, t) = \delta(t - x \cdot \omega), & t \ll 0 . \end{cases}$$

It follows from the Lax–Phillips approach to scattering theory [LP, MU2] that $\alpha(s, \theta, \omega)$ can be expressed in terms of $u(x, t)$, cf. (3.34). It is crucial for our approach to incorporate the parameter ω as one of the independent variables. We construct an approximate solution $u \sim u_0 + u_1$, with $u_0 = \delta(t - x \cdot \omega)$. Away from the tangential rays, u_1 is a sum of the product-type lagrangian distributions; for $t \gg 0$, it is a sum of two lagrangian distributions, from which we find that, away from a small bad set,

$$\alpha \in I^{\frac{1}{4}}(\hat{\Lambda}_+) + I^{\mu+\frac{k}{2}-\frac{5}{4}}(\hat{\Lambda}_-)\,,$$

where $\hat{\Lambda}_+$, $\hat{\Lambda}_- \subset T^*(\mathbb{R}\times S^{n-1}\times S^{n-1})$ are the peak and reflected lagrangians, respectively. From this, the solution of the inverse problem follows easily. In Sect. 1 we recall and establish some basic results concerning classical and product-type conormal and lagrangian distributions. The action of operators such as $\Box^{-1}$ on such classes is considered in Sect. 2 under various geometric assumptions. The first two terms of an approximate solution to the direct problem are constructed in Sect. 3; the analysis of the higher terms seems to be considerably more intricate and may only be possible under a strict convexity assumption on S; we hope to return to this point in the future. Finally, in Sect. 4, the approximate solution to the direct problem is used to solve the inverse problem.

Much of this work was completed while the first author was on leave at the University of Washington; he would like to thank that institution for its hospitality and support.

1. Spaces of Lagrangian Distributions

In this section we recall the spaces of conormal distributions and distributions associated with either a single lagrangian or two cleanly intersecting lagrangian manifolds.

Let X be an n-dimensional smooth manifold, and $\Lambda \subset T^*X\backslash 0$ a conic lagrangian manifold. The Hörmander space $I^m(\Lambda)$ of lagrangian distributions on X associated with Λ consists [Hö] of all locally finite sums of distributions of the form

$$u(x) = \int\limits_{\mathbb{R}^N} e^{i\phi(x,\theta)}a(x,\theta)\,d\theta\,,$$

where $\phi(x,\theta)$ is a nondegenerate phase function parametrizing Λ and

$$\begin{aligned} a \in S^{m+\frac{n}{4}-\frac{N}{2}}(X\times\mathbb{R}^N\backslash 0) = \{&a \in C^\infty(X\times(\mathbb{R}^N\backslash 0)):\\ &|\partial_x^\alpha\partial_\theta^\beta a(x,\theta)| \leqq C_{\alpha\beta K}\langle\theta\rangle^{m+\frac{n}{4}-\frac{N}{2}-|\alpha|}\,,\\ &\forall\alpha\in\mathbb{Z}_+^N,\beta\in\mathbb{Z}_+^n, x\in K \Subset X\}\,. \end{aligned}$$

(Here we use the standard notation $\langle\theta\rangle = (1+|\theta|^2)^{\frac{1}{2}}$.) For $u \in I^m(\Lambda)$, the wavefront set $WF(u) \subset \Lambda$.

Now let $S \subset X$ be a smooth submanifold of codimension k. Then the conormal bundle of S,

$$N^*S = \{(x,\xi) \in T^*X\backslash 0\colon x \in S, \xi \perp T_xS\}\,,$$

is a lagrangian submanifold of $T^*X\backslash 0$; the space of distributions on X conormal to S is by definition

$$I^\mu(S) = I^{\mu+\frac{1}{2}-\frac{n}{4}}(N^*S)\,. \tag{1.1}$$

If $h \in C^\infty(X, \mathbb{R}^k)$ is a defining function for S, with rank $(dh) = k$ at S, then $u(x) \in I^\mu(S) \Rightarrow$

$$u(x) = \int_{\mathbb{R}^k} e^{ih(x)\cdot\theta} a(x, \theta)\, d\theta, \quad a \in S^\mu(X \times (\mathbb{R}^k \setminus 0)) . \tag{1.2}$$

For example, if δ_S is a smooth density on S, then $\delta_S \in I^0(S)$, while a distribution on $X \setminus S$ having a Heaviside-type singularity at S belongs to $I^{-k}(S)$. One easily sees that

$$I^\mu(S) \subset L^p_{\text{loc}}(X) \quad \text{if } \mu < -k\left(1 - \frac{1}{p}\right). \tag{1.3}$$

Now, let $\Lambda_0, \Lambda_1 \subset T^*X \setminus 0$ be a cleanly intersecting pair of lagrangians in the sense of [MU1]. Thus, $\Sigma = \Lambda_0 \cap \Lambda_1$ is smooth and

$$T_{\lambda_0}\Sigma = T_{\lambda_0}\Lambda_0 \cap T_{\lambda_0}\Lambda_1, \quad \forall \lambda_0 \in \Sigma .$$

Associated to the pair (Λ_0, Λ_1) is a class of lagrangian distributions, $I^{p,l}(\Lambda_0, \Lambda_1)$, indexed by $p, l \in \mathbb{R}$, which satisfy $WF(u) \subset \Lambda_0 \cup \Lambda_1$ [MU1, GuU]. Microlocally, away from Σ,

$$I^{p,l}(\Lambda_0, \Lambda_1) \subset I^{p+l}(\Lambda_0 \setminus \Lambda_1) \quad \text{and} \quad I^{p,l}(\Lambda_0, \Lambda_1) \subset I^p(\Lambda_1) . \tag{1.4}$$

If $Y_2 \subset Y_1 \subset X$ are smooth submanifolds with $\operatorname{codim}_X(Y_1) = d_1$, $\operatorname{codim}_X(Y_2) = d_1 + d_2$, then N^*Y_1 and N^*Y_2 intersect cleanly in codimension d_2. The space of distributions on X conormal to the pair (Y_1, Y_2) of orders μ, μ' is

$$\begin{aligned} I^{\mu,\mu'}(Y_1, Y_2) &= I^{\mu + \mu' + \frac{d_1 + d_2}{2} - \frac{n}{4}, -\frac{d_2}{2} - \mu'}(N^*Y_1, N^*Y_2) \\ &= I^{\mu + \frac{d_1}{2} - \frac{n}{4}, \mu' + \frac{d_2}{2}}(N^*Y_2, N^*Y_1) . \end{aligned}$$

If one introduces local coordinates $(x_1, \ldots, x_n)$ on X such that

$$Y' = \{x_1 = \cdots = x_{d_1} = 0\} = \{x' = 0\}$$

and

$$Y_2 = \{x_1 = \cdots = x_{d_1 + d_2} = 0\} = \{x' = 0, x'' = 0\} ,$$

then $u(x) \in I^{\mu,\mu'}(Y_1, Y_2)$ iff it can be written locally as

$$u(x) = \int_{\mathbb{R}^{d_1 + d_2}} e^{i(x'\cdot\xi' + x''\cdot\xi'')} a(x; \xi'; \xi'')\, d\xi'\, d\xi'' \tag{1.5}$$

with $a(x; \xi'; \xi'')$ belongs to the product-type symbol class

$$\begin{aligned} S^{\mu,\mu'}(X \times (\mathbb{R}^{d_1} \setminus 0) \times \mathbb{R}^{d_2}) = \{a \in C^\infty : &|\partial_x^\gamma \partial_{\xi''}^\beta \partial_{\xi'}^\alpha a(x, \xi)| \\ &\leqq C_{\alpha\beta\gamma K} \langle \xi', \xi'' \rangle^{\mu - |\alpha|} \langle \xi'' \rangle^{\mu' - |\beta|}\} . \end{aligned} \tag{1.6}$$

We will need the following series of lemmas concerning multiplication of conormal distributions.

Lemma 1.1. *If $Y, Z \subset X$ are submanifolds with $Y \pitchfork Z$, then*

$$I^{\mu}(Y)\cdot I^{\mu'}(Z) \subset I^{\mu,\mu'}(Y, Y\cap Z) + I^{\mu',\mu}(Z, Y\cap Z)\,. \tag{1.7}$$

If $u \in I^{\mu}(Y)$ satisfies: $\operatorname{supp}(u) \subset Y$, *then*

$$u\cdot I^{\mu'}(Z) \subset I^{\mu,\mu'}(Y, Y\cap Z)\,. \tag{1.8}$$

Proof. If $Y\cap Z = \emptyset$, there is nothing to prove, since $I^{\mu,\mu'}(Y, Y\cap Z) \supset I^{\mu}(Y)$ and $I^{\mu',\mu}(Z, Y\cap Z) \supset I^{\mu'}(Z)$. If $Y\cap Z \neq \emptyset$, let $x^0 \in Y\cap Z$ and introduce local coordinates $(x', x'', x''') \in \mathbb{R}^{d_1}\times\mathbb{R}^{d_2}\times\mathbb{R}^{n-d_1-d_2}$ near x^0 such that a) $x^0 = 0$, b) $Y = \{x' = 0\}$ and c) $Z = \{x'' = 0\}$. If $u(x) \in I^{\mu}(Y)$, u has the local oscillatory representation

$$u(x) = \int_{\mathbb{R}^{d_1}} e^{ix'\cdot\xi'} a(x;\xi')\,d\xi', \quad a \in S^{\mu}(X\times(\mathbb{R}^{d_1}\backslash 0))\,,$$

and $v(x) \in I^{\mu'}(Z)$ has the representation

$$v(x) = \int_{\mathbb{R}^{d_2}} e^{ix''\cdot\xi''} b(x;\xi'')\,d\xi'', \quad b \in S^{\mu'}(X\times(\mathbb{R}^{d_2}\backslash 0))\,,$$

so that

$$(uv)(x) = \int_{\mathbb{R}^{d_1+d_2}} e^{i(x'\cdot\xi' + x''\cdot\xi'')} a(x;\xi')b(x;\xi'')\,d\xi'\,d\xi''\,.$$

Introduce a cutoff function $\chi(t) \in C_0^{\infty}(\mathbb{R})$, $\chi \equiv 1$ for $|t| \leqq \frac{1}{2}$, $\chi \equiv 0$ for $|t| \geqq 1$. Then

$$\chi\left(\frac{\langle\xi''\rangle}{\langle\xi'\rangle}\right) a(x;\xi')b(x;\xi'') \in S^{\mu,\mu'}(X\times(\mathbb{R}^{d_1}\backslash 0)\times\mathbb{R}^{d_2})\,,$$

while

$$(1-\chi)\left(\frac{\langle\xi''\rangle}{\langle\xi'\rangle}\right) a(x;\xi')b(x;\xi'') \in S^{\mu',\mu}(X\times(\mathbb{R}^{d_2}\backslash 0)\times\mathbb{R}^{d_1})\,;$$

making the corresponding decomposition of $u\cdot v$ yield (1.7).

If $\operatorname{supp}(u) \subset Y$, then by [Hö], $WF(uv) \subset N^*Y \cup N^*(Y\cap Z)$, so that in the above decomposition the second term belongs to $I^{\mu+\mu'}(Y\cap Z) \subset I^{\mu,\mu'}(Y, Y\cap Z)$, yielding (1.8). Q.E.D.

We also need the multiplicative properties of conormal distributions associated with a nested pair of submanifolds. Related results for a single submanifold are in [Pi].

Lemma 1.2. *If $Y_1 \supset Y_2$ are submanifolds of X of codimensions d_1, $d_1 + d_2$, respectively, $u_1 \in I^{M_1}(Y_1)$, and $u_2 \in I^{M_2}(Y_2)$ is microlocally supported away from N^*Y_1, then*

$$u_1u_2 \in I^{M',M''}(Y_1, Y_2), \quad M' = (m_1 + d_1)_+ - d_1 + \varepsilon\delta_{m_1,-d_1}\,,$$

$$M'' = m_2 + d_1, \quad \textit{any } \varepsilon > 0\,. \tag{1.9}$$

Proof. Introducing on X local coordinates $x = (x', x'', x''')$ as discussed above (1.5), we have oscillatory representations

$$u_1(x) = \int_{\mathbb{R}^{d_1}} e^{x' \cdot \xi'} a(x; \xi')\, d\xi', \quad a \in S^{M_1}(X \times (\mathbb{R}^{d_1} \backslash 0)) \tag{1.10}$$

and

$$u_2(x) = \int_{\mathbb{R}^{d_1+d_2}} e^{i(x' \cdot \xi' + x'' \cdot \xi'')} b(x; \xi', \xi'')\, d\xi'\, d\xi'', \quad b \in S^{m_2}(X \times (\mathbb{R}^{d_1+d_2} \backslash 0)) \tag{1.11}$$

with $\operatorname{supp}(b) \subset \{|\xi'| \leqq c|\xi''|\}$. Thus,

$$(u_1 u_2)(x) = \int_{\mathbb{R}^{d_1+d_2}} (a *' b)(x; \xi', \xi'')\, d\xi'\, d\xi'' , \tag{1.12}$$

where $a *' b$ is the partial convolution

$$a *' b(x; \xi', \xi'') = \int_{\mathbb{R}^{d_1}} a(x; \eta') b(x; \xi' - \eta', \xi'')\, d\eta' . \tag{1.12}$$

where $a *' b$ is the partial convolution

$$a *' b(x; \xi', \xi'') = \int_{\mathbb{R}^{d_1}} a(x; \eta') b(x; \xi' - \eta', \xi'')\, d\eta' . \tag{1.13}$$

To estimate the size of $a *' b(x; \xi', \xi)$, where $|\xi'| \geqq c|\xi''|$, note that

$$|a *' b(x; \xi', \xi'')| \leqq c\langle \xi' \rangle^{m_1} \int_{|\xi' - \eta'| \leqq c|\xi''|} \langle \xi'' \rangle^{m_2}\, d\eta' \leqq c\langle \xi, \xi' \rangle^{m_1} \langle \xi'' \rangle^{m_2 + d_1} . \tag{1.14}$$

On the other hand, if $|\xi'| \leqq c|\xi''|$,

$$|a *' b(x; \xi', \xi'')| \leqq c\langle \xi'' \rangle^{m_2} \int_{|\eta'| \leqq c|\xi''|} \langle \eta' \rangle^{m_1}\, d\eta' \leqq c\langle \xi'' \rangle^{(m_1 + d_1)_+} + m_2 + \varepsilon\delta_{m_1, -d_1} \tag{1.15}$$

for any $\varepsilon > 0$. Thus, $a *' b$ satisfies the correct size estimate to belong to $S^{M', M''}(X \times (\mathbb{R}^{d_1} \backslash 0) \times \mathbb{R}^{d_2})$, with M', M'' as in (1.9). A derivative in ξ', $\partial^\alpha_{\xi'}(a *' b)$, can be represented as either $(\partial^\alpha_{\xi'} a) *' b$ or $a *' \partial^\alpha_{\xi'} b)$. In the region $|\xi'| \geqq c|\xi''|\}$, we use $\partial^\alpha_{\xi'} a \in S^{m_1 - |\alpha|}$ and (1.14) to get a gain of $\langle \xi' \rangle^{-|\alpha|}$ as long as $|\alpha| \leqq m_1 + d_1$; for $|\alpha| > m_1 + d_1$, we integrate by parts $|\alpha| - m_1 - d_1$ times and then apply (1.14) to obtain the desired gain. On $\{|\xi'| \leqq c|\xi''|\}$, we use $\partial^\alpha_{\xi'} b \in S^{m_2 - |\alpha|}$ and (1.15) to obtain the gain of $\langle \xi'' \rangle^{-|\alpha|} = \langle \xi', \xi'' \rangle^{-|\alpha|}$. A derivative $\partial^\beta_{\xi''}$, however, can only be distributed to $b(x; \xi', \xi'')$, lowering m_2 to $m_2 - |\beta|$ and consequently yielding a gain of only $\langle \xi'' \rangle^{-|\beta|}$. Thus, $a *' b \in S^{M', M''}(X \times (\mathbb{R}^{d_1} \backslash 0) \times \mathbb{R}^{d_2})$ and $u_1 u_2 \in I^{M', M''}(Y_1, Y_2)$ by (1.5). Q.E.D.

If u_2 is microlocally supported near $N^* Y_1$, we have a similar result.

Lemma 1.3. *If Y_1, Y_2 and $u_1(x)$ are as above, and $u_2 \in I^{m_2}(Y_2)$ has amplitude $b(x; \xi', \xi'')$ supported in $\{|\xi''| \leqq c|\xi'|\}$, then*

$$u_1 u_2 \in I^{m_1, m_2 + d_1}(Y_1, Y_2), m_2 < -d_1 , \quad m_1 + m_2 < -d_1 . \tag{1.16}$$

Proof. We repeat the calculations of the previous proof, except that in (1.13), the integral is over $\{|\xi' - \eta'| \geqq c|\xi''|\}$. Thus, on $\{|\xi'| \geqq c|\xi''|\}$,

$$|a *' b(x; \xi', \xi'')| \leqq c\langle\xi'\rangle^{m_1} \int_{|\eta'| \geqq c|\xi''|} \langle\eta', \xi''\rangle^{m_2} d\eta' \leqq \langle\xi'\rangle^{m_1} \langle\xi''\rangle^{m_2+d_1}$$

if $m_2 + d_1 < 0$, while on $\{|\xi'| \leqq |\xi''|\}$,

$$|a *' b(x; \xi', \xi'')| \leqq \int_{|\eta'| \geqq c|\xi''|} \langle\eta'\rangle^{m_1+m_2} d\eta' \tau c \langle\xi''\rangle^{m_1+m_2+d_1}, \; m_1 + m_2 + d_1 < 0 .$$

Q.E.D.

We also will need the action of $I^{m_1}(Y_1)$ on spaces of product-type conormal distributions.

Lemma 1.4. *If Y_1, Y_2 are as above, $u_1 \in I^{m_1}(Y_1)$, and $u_2 \in I^{M',M''}(Y_1, Y_2)$ is supported microlocally near $N^* Y_1$, with $M' < -d_1$, $M' \leqq m_1$, then*

$$u_1 u_2 \in I^{\tilde{M}', M''}(Y_1, Y_2), \; \tilde{M}' = \max((m_1 + d_1)_+ + M' + \varepsilon\delta_{m_1, -d_1}, m_1), \quad \textit{and } \varepsilon > 0 . \tag{1.17}$$

Proof. We again have the oscillatory representation (1.12) of $u_1 u_2$ with $a *' b$ given by (1.13) for $a \in S^{m_1}(X \times (\mathbb{R}^{d_1} \backslash 0))$ but now $b \in S^{M',M''}(X \times (\mathbb{R}^{d_1} \backslash 0) \times \mathbb{R}^{d_2})$; thus, $|a(x; \xi')| \leqq c\langle\xi'\rangle^{m_1}$ and $|b(x; \xi', \xi'')| \leqq c\langle\xi'\rangle^{M'}\langle\xi''\rangle^{M''}$. For $|\xi'| \leqq c|\xi''|$,

$$\begin{aligned} |a *' b(x; \xi', \xi'')| &\leqq c \int_{|\xi' - \eta'| \geqq c|\xi''|} \langle\eta'\rangle^{m_1} \langle\xi' - \eta'\rangle^{M'} \langle\xi''\rangle^{M''} d\eta' \\ &\leqq c\langle\xi''\rangle^{M''} \int_{|\eta'| \geqq c|\xi''|} \langle\eta'\rangle^{m_1+M'} d\eta' \\ &\leqq c\langle\xi''\rangle^{m_1+M'+M''+d_1} \quad \text{if } m_1 + M' < -d_1 . \end{aligned} \tag{1.18}$$

If $|\xi'| \geqq c|\xi''|$, we use the first inequality in (1.18) and then decompose the integral into three pieces, corresponding to the regions $\mathrm{I} = \{\langle\eta'\rangle \leqq c\langle\xi'\rangle\}$, $\mathrm{II} = \{c\langle\xi''\rangle \leqq \langle\eta'\rangle \leqq c\langle\xi'\rangle\}$ and $\mathrm{III} = \{\langle\xi' - \eta'\rangle \geqq c\langle\xi''\rangle\}$ we have

$$\int_{\mathrm{I}} \leqq c\langle\xi'\rangle^{M'}\langle\xi''\rangle^{M''} \int_{\mathrm{I}} \langle\eta'\rangle^{m_1} d\eta' \leqq c\langle\xi'\rangle^{(m_1+d_1)_+ + M' + \varepsilon\delta}\langle\xi''\rangle^{m} , \tag{1.19}$$

where $\delta = \delta_{m_1, -d_1}$ and $\varepsilon > 0$ is arbitrary,

$$\begin{aligned} \int_{\mathrm{II}} &\leqq c\langle\xi''\rangle^{M''}\left(\langle\xi'\rangle^{M'}\int_{\mathrm{II}} \langle\eta'\rangle^{m_1} d\eta' + \langle\xi'\rangle^{m_1}\int_{\mathrm{II}} \langle\eta'\rangle^{M'} d\eta'\right) \\ &\leqq c\langle\xi''\rangle^{M''}(\langle\xi'\rangle^{M'}\max(\langle\xi''\rangle^{m_1+d_1+\varepsilon_1}, \langle\xi'\rangle^{m_1+d_1+\varepsilon\delta}) \\ &\quad + \langle\xi'\rangle^{m_1}\max(\langle\xi''\rangle^{M''+d_1}, \langle\xi'\rangle^{M'+d_1})) \\ &\leqq \begin{cases} c\langle\xi'\rangle^{m_1}\langle\xi''\rangle^{M'+M''+d_1}, & m_1 + d_1 \geqq 0 \\ c\max(\langle\xi'\rangle^{M'}\langle\xi''\rangle^{m_1+M''+d_1}, \langle\xi'\rangle^{m_1}\langle\xi''\rangle^{M'|M''+d_1}), & m_1 + d_1 < 0 \end{cases} \\ &\leqq c\langle\xi'\rangle\langle\xi''\rangle^{M'+M''+d_1} \quad \text{if } m_1 \geqq M' , \end{aligned} \tag{1.20}$$

and

$$\int_{\mathrm{III}} \leqq c \int_{\mathrm{III}} \langle \eta' \rangle^{m_1 + M'} \langle \xi'' \rangle^{M''} d\eta' \leqq c \langle \xi' \rangle^{m_1 + M' + d_1} \langle \xi'' \rangle^{M''} . \tag{1.21}$$

Since $(m_1 + d_1) + \varepsilon\delta \geqq m_1 + d_1$, the third term is dominated by the first; on the other hand, since $M' + d_1 < 0$, the second term is dominated by $c\langle \xi' \rangle^{m_1} \langle \xi'' \rangle^{M''}$. Thus, on $\{|\xi'| \geqq c|\xi''|\}$,

$$|a *' b(x; \xi', \xi'')| \leqq c \langle \xi' \rangle^{\tilde{M}'} \langle \xi'' \rangle^{M''} . \tag{1.22}$$

The estimates (1.18) and (1.22) together imply the size estimate satisfied by an amplitude belonging to $S^{\tilde{M}', M''}$. As before, $\partial^{\alpha}_{\xi'}$ can be distributed to either a or b, while $\partial^{\beta}_{\xi''}$ must be applied to b, yielding the required estimate

$$|\partial^{\gamma}_{x} \partial^{\beta}_{\xi''} \partial^{\alpha}_{\xi'} (a *' b)| \leqq c \langle \xi', \xi'' \rangle^{\tilde{M}' - |\alpha|} \langle \xi'' \rangle^{M'' - |\beta|} ,$$

so that $u_1 u_2 \in I^{\tilde{M}', M''}(Y_1, Y_2)$. Q.E.D.

Finally, we have

Lemma 1.5. *Let $Y_1 \supset Y_2$ be as above, and $Y_+ \subset X$ such that $Y_1 \pitchfork Y_+$ with $Y_1 \cap Y_+ = Y_2$. Let $u_1 \in I^{m_1}(Y_1)$, and $u_2 \in I^{M', M''}(Y_+, Y_2)$ be supported microlocally near $N^* Y_+$. Then,*

$$u_1 u_2 \in I^{M', \tilde{M}''}(Y_+, Y_2) + I^{\tilde{M}', 0}(Y_+, Y_2) + I^{m_1, m_2}(Y_1, Y_2) , \tag{1.23}$$

where $\tilde{M}'' = M'' + (m_1 + d_1)_+ + \varepsilon\delta_{m_1, -d_1}$, $\tilde{M}' = M' + (m_1 + M'' + d_1)_+ + \varepsilon\delta_{m_1 + M'', -d_1}$, and $m_2 = M' + (M'' + d_1)_+ + \varepsilon\delta_{M'', -d_1}$, any $\varepsilon > 0$.

Proof. We may choose the local coordinates $x = (x', x'', x''')$ so that $Y_1 = \{x' = 0\}$, $Y_+ = \{x'' = 0\}$, *and* $Y_2 = \{x' = 0, x'' = 0\}$. $u_1(x)$ is given by (1.10) while

$$u_2(x) = \int_{\mathbb{R}^{d_1 + d_2}} e^{i(x' \cdot \xi' + x'' \cdot \xi'')} b(x; \xi'', \xi') d\xi' d\xi'', \quad b \in S^{M', M''}(X \times (\mathbb{R}^{d_2} \backslash 0) \times \mathbb{R}) \tag{1.24}$$

with $\operatorname{supp}(b) \subset \{|\xi''| \leqq c|\xi''|\}$. (Note that ξ'' plays the role of the "elliptic" variable.) The product $(u_1, u_2)(x)$ is again represented by (1.12), with

$$a *' b(x; \xi_1, \xi_2) = \int_{\mathbb{R}^{d_1}} a(x; \xi' - \eta') b(x; \xi'', \eta') d\eta' .$$

It suffices to show that

$$a *' b \in S^{M', \tilde{M}''}(X \times (\mathbb{R}^{d_2} \backslash 0) \times \mathbb{R}^{d_1}) + S^{\tilde{M}', 0}(X \times (\mathbb{R}^{d_2} \backslash 0) \times \mathbb{R}^{d_1}) \{|\xi'| \leqq c|\xi''|\} \tag{1.25}$$

while

$$a *' b \in S^{m_1, m_2}(X \times (\mathbb{R}^{d_1} \backslash 0) \times \mathbb{R}^{d_2}) \quad \text{on } \{|\xi''| \leqq c|\xi'|\} . \tag{1.26}$$

Where $|\xi''| \leqq c|\xi'|$, one has

$$|a *' b(x; \xi', \xi'')| \leqq c \int_{|\eta'| \leqq c|\xi''|} \langle \xi' - \eta' \rangle^{m_1} \langle \eta', \xi'' \rangle^{M'} \langle \eta' \rangle^{M''} d\eta'$$

$$\leqq c \langle \xi' \rangle^{m_1} \langle \xi'' \rangle^{M'} \int_{|\eta'| \leqq c|\xi''|} \langle \eta' \rangle^{M''} d\eta'$$

$$\leqq c \langle \xi' \rangle^{m_1} \langle \xi'' \rangle^{M' + (M'' + d_1)_+ + \varepsilon \delta_{M'', -d_1}}, \quad \text{any } \varepsilon > 0, \tag{1.27}$$

which is the correct size estimate for (1.26). Applying $\partial_{\xi'}^{\alpha}$ to the the first factor in $a *' b$, we lower m_1 by $|\alpha|$ and obtain a gain of $\langle \xi' \rangle^{-|\alpha|} \simeq \langle \xi', \xi'' \rangle^{-|\alpha|}$, while $\partial_{\xi''}^{\beta}$ must be applied to the second factor, resulting in a gain of only $\langle \xi'' \rangle^{-|\beta|}$. Thus, (1.26) holds. As for (1.25), on $\{|\xi''| \geqq c|\xi'|\}$, we have

$$|a *' b(x; \xi', \xi'')| = \left| \int_{|\eta'| \leqq c|\xi''|} a(x; \xi' - \eta') b(x; \xi'', \eta') d\eta' \right|$$

$$\leqq c \int_{|\eta'| \leqq c|\xi''|} \langle \eta' \rangle^{m_1 + M''} \langle \xi'' \rangle^{M'} d\eta'$$

$$+ c \int_{|\xi' - \eta'| \leqq \frac{1}{2}|\xi'|} \langle \xi' - \eta' \rangle^{m_1} \langle \xi'' \rangle^{M'} \langle \xi' \rangle^{M''} d\eta'$$

$$\leqq c \langle \xi'' \rangle^{M' + (m_1 + M'' + d_1)_+ + \varepsilon \delta_{m_1 + M'', -d_1}}$$

$$+ c \langle \xi'' \rangle^{M'} \langle \xi' \rangle^{M'' + (m_1 + d_1)_+ + \varepsilon \delta_{m_1, -d_1}}, \tag{1.28}$$

which is the correct size estimate for (1.25). The desired gain of $\langle \xi'' \rangle^{-|\beta|}$ from the application of $\partial_{\xi''}^{\beta}$ follows, since M' is lowered by $|\beta|$. As in the proof of Lemma 1.2, the gain of $\langle \xi' \rangle^{-|\alpha|}$ from $\partial_{\xi'}^{\alpha}$ follows by lowering M'' to $M'' - |\alpha|$ if $m_1 + M'' \geqq -d_1$; otherwise one integrates by parts first. Thus, (1.25) holds and the lemma is proved. Q.E.D.

2. Action of Parametrices on Distribution Spaces

We now consider the mapping properties of a parametrix for a pseudodifferential operator of real principal type, acting on the spaces of distributions associated with one and two lagrangians described in Sect. 1. The intended application in Sect. 3 is to the d'Alembertian on $\mathbb{R}^{n+1} \times S^{n-1}$, but the natural coordinates there do not seem convenient for establishing these results, leading us to formulate and prove them in the generality described below.

Let $P(x, D)$ be an m^{th} order classical ψDO, with real homogeneous principal symbol $p_m(x, \xi)$. Recall that P is of real principal type if a) $dp_m \neq 0$ at $\text{char}(P) = \{(x, \xi) \in T^*X \backslash 0 : p_m(x, \xi) = 0\}$ so that char(P) is smooth, and b) char(P) has no characteristics trapped over a compact set of X. Then $P(x, D)$ is locally solvable and parametrices for $P(x, D)$ were constructed in [DH, MU1]. For $(x, \xi) \in \text{char}(P)$, let $\Xi_{(x,\xi)}$ be the bicharacteristic of $P(x, D)$ (i.e., integral curve of H_{p_m}) through (x, ξ). Then the flowout canonical relation generated by char(P),

$$\Lambda_P = \{(x, \xi; y, \eta) : (x, \xi) \in \text{char}(P), (y, \eta) \in \Xi_{(x,\xi)}\},$$

intersects the diagonal Δ_{T^*X} cleanly in codimension 1. In [MU 1], it was shown that $P(x, D)$ has a parametrix $Q \in I^{\frac{1}{2}-m, -\frac{1}{2}}(\Delta_{T^*X}, \Lambda_P)$.

Proposition 2.1. *Suppose $\Lambda_0 \subset T^*X\backslash 0$ is a conic lagrangian intersecting* char(P) *transversally and such that each bicharacteristics of P intersects Λ_0 a finite number of times. Then, if $T \in I^{p,l}(\Delta_{T^*X}, \Lambda_P)$,*

$$T: I^r(\Lambda_0) \to I^{r+p,l}(\Lambda_0, \Lambda_1), \tag{2.1}$$

where $\Lambda_1 = \Lambda_P \circ \Lambda_0$ is the flowout from Λ on char(P). *Furthermore, for $(x, \xi) \in \Lambda_1 \backslash \Lambda_0$,*

$$\sigma(Tu)(x, \xi) = \sum_j \sigma(T)(x, \xi; y_j, \eta_j)\sigma(u)(y_j, \eta_j), \tag{2.2}$$

where $\{(y_j, \eta_j)\} = \Lambda_0 \cap \Xi_{(x,\xi)}$.

Proof. Microlocalizing and conjugating by an elliptic Fourier integral operator associated with a canonical transformation, we can assume [MU1] that $X = \mathbb{R}^n$ with coordinates $x = (x_1, x')$, $\Lambda_0 = T_0^*\mathbb{R}^n\backslash 0$, char$(P) = \{(x, \xi): \xi_1 = 0\}$ and thus

$$\Lambda_1 = \{(x_1, 0; 0, \xi'): x_1 \in \mathbb{R}, \xi' \in \mathbb{R}^{n-1}\backslash 0\}.$$

A distribution $u \in I^r(\Lambda_0)$ has the representation

$$u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x; \xi)\, d\xi, \quad a \in S^{r-\frac{n}{4}}(\mathbb{R}^n \times (\mathbb{R}^n\backslash 0)), \tag{2.3}$$

and $T \in I^{p,l}(\Delta_{T^*\mathbb{R}^n}, \Lambda_p)$ has the form

$$Tf(x) = \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\cdot\theta} b(x, y; \theta'; \theta_1) f(y)\, d\theta\, dy,$$

$$b \in S^{p+\frac{1}{2}, l-\frac{1}{2}}(\mathbb{R}^{2n} \times (\mathbb{R}^{n-1}\backslash 0) \times \mathbb{R}). \tag{2.4}$$

Note that on $\Lambda_P \backslash \Delta_{T^*\mathbb{R}^n}$, T is a Fourier integral operator associated with Λ_P,

$$Tf(x) = \int_{\mathbb{R}^{n-1} \times \mathbb{R}^n} e^{i(x'-y')\cdot\theta'} c(x, y; \theta') f(y)\, d\theta'\, dy, \tag{2.5}$$

where, for $x_1 \neq y_1$,

$$c(x, y; \theta') = \int e^{i(x_1-y_1)\theta_1} b(x, y; \theta'; \theta_1)\, d\theta_1 \in S^{p+\frac{1}{2}}(\mathbb{R}^{2n} \times (\mathbb{R}^{n-1}\backslash 0)). \tag{2.6}$$

Now, applying T from (2.4) to $u(y)$, and applying stationary phase in y, ξ_1, θ', we obtain, upon relabelling θ_1 by ξ_1,

$$Tu(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}[a((0, x'); \xi) b(x, (0, x'); \xi'; \xi_1) + \dots]\, d\xi. \tag{2.7}$$

The amplitude in (2.7) is easily seen to belong to $S^{r+p+\frac{1}{2}-\frac{n}{4}, l-\frac{1}{2}}(\mathbb{R}^n \times (\mathbb{R}^{n-1}\backslash 0) \times \mathbb{R})$; by [*GU*, §1], $Tu \in I^{p',l'}(\Lambda_0, \Lambda_1)$, with $p' + l' - \frac{n}{4} = r + p + \frac{1}{2} - \frac{n}{4} + l - \frac{1}{2}$ and $p' + \frac{1}{2} - \frac{n}{4} = r + p + \frac{1}{2} - \frac{n}{4}$, so that $p' = r + p$, $l' = l$.

To calculate the symbol of Tu on $\Lambda_1 \backslash \Lambda_0$, we expand the amplitude a in (2.7) about $\xi_1 = 0$:

$$Tu(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}[a((0,x'); (0,\xi'))b(x,(0,x'); \xi'; \xi_1) + \xi_1 d(x; \xi'; \xi_1)]\,d\xi\,, \quad (2.8)$$

where $d(x; \xi'; \xi_1) \in S^{r+p-\frac{1}{2}-\frac{n}{4}, l-\frac{3}{2}}(\mathbb{R}^n \times (\mathbb{R}^{n-1}\backslash 0) \times \mathbb{R})$. The first term in (2.8) can be written as

$$\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} a((0,x'); (0,\xi'))\left[\int_{\mathbb{R}} e^{ix_1\xi_1} b(x,(0,x');\ \xi'; \xi_1)\,d\xi_1\right] d\xi'\,,$$

which for $x_1 \neq 0$ is an element of $I^{r+p}(\Lambda_1))$ with symbol as in (2.2); the second term is in $I^{r+p-1}(\Lambda_1 \backslash \Lambda_0)$ and thus does not contribute to the principal symbol. Q.E.D.

Finally, we deal with the action of $I^{p,l}(\Delta_{T^*X}, \Lambda_p)$ on the double intersection class $I^{p',l'}(\Lambda_0, \Lambda_1)$.

Proposition 2.2. *Under the same assumptions as Proposition* 2.1,

$$T: I^{p',l'}(\Lambda_0, \Lambda_1) \to I^{p+p'+\frac{1}{2}, l+l'-\frac{1}{2}}(\Lambda_0, \Lambda_1)\,. \quad (2.9)$$

Thus, if Q is a parametrix for $P(x, D)$,

$$Q: I^{p',l'}(\Lambda_0, \Lambda_1) \to I^{p'+1-m, l'-1}(\Lambda_0, \Lambda_1)\,. \quad (2.10)$$

Proof. We argue as in the proof of Proposition 2.1, but now $u \in I^{p',l'}(\Lambda_0, \Lambda_1) \Rightarrow$

$$u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x; \xi'; \xi_1)\,d\xi, \quad a \in S^{p'+\frac{1}{2}-\frac{n}{4}, l'-\frac{1}{2}}(\mathbb{R}^n \times (\mathbb{R}^{n-1}\backslash 0) \times \mathbb{R})\,, \quad (2.11)$$

so that instead of (2.7) we have

$$Tu(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e(x; \xi'; \xi_1)\,d\xi, \quad e \in S^{p+p'+1-\frac{n}{4}, l+l'-1}(\mathbb{R}^n \times (\mathbb{R}^{n-1}\backslash 0) \times \mathbb{R})\,, (2.12)$$

so that $Tu \in I^{p+p'+\frac{1}{2}, l+l'-\frac{1}{2}}(\Lambda_0, \Lambda_1)$. For the parametrix Q, we specialize this to $p = \frac{1}{2} - m$, $l = -\frac{1}{2}$, yielding (2.10). Q.E.D.

Finally, we need

Proposition 2.3. *Suppose $\Lambda_1 \subset T^*X\backslash 0$ is a conic lagrangian which is characteristic for P: $\Lambda_1 \subset \mathrm{char}(P)$. Then, if $T \in I^{p,l}(\Delta_{T^*X}, \Lambda_P)$,*

$$T: I^r(\Lambda_1) \to I^{r+p+\frac{1}{2}}(\Lambda_1)\,, \quad (2.13)$$

and thus

$$Q: I^r(\Lambda_1) \to I^{r+1-m}(\Lambda_1)\,. \quad (2.14)$$

Proof. Microlocalizing, we can find a Λ_0 such that (Λ_0, Λ_1) are as in Proposition 2.2. For each $l' \in \mathbb{R}$, $I^r(\Lambda_1) \subset I^{r,l'}(\Lambda_0, \Lambda_1)$ ([GuU]), which is mapped by T to $I^{r+p+\frac{1}{2}, l'+l-\frac{1}{2}}(\Lambda_0, \Lambda_1)$ by (3.9). Intersecting over all $l' \in \mathbb{R}$, we have T: $I^r(\Lambda_1) \to$

$\bigcap_{l' \in \mathbb{R}} I^{r+p+\frac{1}{2}, l'+l-\frac{1}{2}}(\Lambda_0, \Lambda_1) = I^{r+p+\frac{1}{2}}(\Lambda_1)$, again by the results of [GuU], proving (2.13). Since $Q \in I^{\frac{1}{2}-m, -\frac{1}{2}}(\Delta_{T^*X}, \Lambda_P)$, (2.14) follows. Q.E.D.

3. Plane Wave Ansatz for the Direct Problem

We now analyze the approximate solution, $u_0 + u_1$, to the direct problem described in the Introduction, under the assumption that the potential $q(x)$ is conormal to a smooth codimension k submanifold. Let S be given by a defining function,

$$S = \{x \in \mathbb{R}^n : h(x) = 0\}, \tag{3.1}$$

where $h \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ satisfies $\operatorname{rank}(dh(x)) = k$ for $x \in S$; in addition we assume S has compact closure. Let

$$q(x) \in I^\mu(S), \begin{cases} \mu < -\max\left(\left(1 - \dfrac{2}{n}\right)k, k-1\right), & n \geqq 5 \\ \mu < -\max\left(\dfrac{k}{2}, k-1\right), & n = 3 \text{ or } 4 \end{cases} \tag{3.2}$$

be compactly supported and real-valued. Since, by (1.3), $q \in L^p(\mathbb{R}^n)$, for $p = \frac{n}{2}$ for $n \geqq 5$ and $p > 2$, $n = 3$ or 4, it follows from a theorem of Phillips [P] that the scattering kernel $\alpha(s, \theta, \omega)$ of $q(x)$ exists; furthermore, the representation (3.34) below is valid.

Now define

$$S_1 = \{(x, t, \omega) \in \mathbb{R}^{n-1} \times S^{n-1} : x \in S\}; \tag{3.3}$$

regarding $q(x)$ as a distribution on $\mathbb{R}^{n-1} \times S^{n-1}$ independent of t and ω, one has

$$q \in I^\mu(S_1). \tag{3.4}$$

We wish to find an approximation solution to the problem

$$\begin{cases} (\Box + q(x))u(x, t, \omega) = 0 & \text{on } \mathbb{R}^{n-1} \times S^{n-1} \\ u(x, t, \omega) = \delta(t - x \cdot \omega), & t \ll 0, \end{cases} \tag{3.5}$$

where $\Box = \frac{\partial^2}{\partial t^2} - \Delta_{\mathbb{R}^n}$ is the d'Alembertian on $\mathbb{R}^{n+1}$ acting independently of ω. We look for an approximation

$$u \sim u_0 + u_1 + \cdots + u_j + \cdots,$$

where $u_0(x, t, \omega) = \delta(t - x \cdot \omega)$ and such that the series on the right is (formally) telescoping when $\Box + q$ is applied. Thus $u_{j+1} = -\Box^{-1}(q(x)u_j(x, t, \omega))$, where $\Box^{-1}$ is (say) the forward fundamental solution of $\Box$. For the purposes of this paper, it suffices to consider the first two terms,

$$u_0 + u_1 = \delta(t - x \cdot \omega) - \Box^{-1}(q(x)\delta(t - x \cdot \omega)). \tag{3.6}$$

Now, the leading term in (3.6) is

$$u_0(x, t, \omega) = \delta(t - x \cdot \omega) \in I^0(S_+) , \tag{3.7}$$

where

$$S_+ = \{(x, t, \omega) \in \mathbb{R}^{n+1} \times S^{n-1} : t - x \cdot \omega = 0\} .$$

The submanifolds S_+ and S_1 intersect transversally; let $S_2 = S_+ \cap S_1$ be the resulting codimension $k+1$ submanifold at $\mathbb{R}^{n+1} \times S^{n-1}$. Let $\Lambda_1 = N^*S_1$, $\Lambda_+ = N^*S_+$ and $\Lambda_2 = N^*S_2$ be the respective conormal bundles, which, as described in Sect. 1, are lagrangian submanifolds of $T^*(\mathbb{R}^{n+1} \times S^{n-1})\backslash 0$.

Proposition 3.1. a) *$WF(q) \subset \Lambda_1$ and $WF(u_0) \subset \Lambda_+$.*
b) *Λ and Λ_+ are disjoint.*
c) *Λ_2 intersects Λ_1 and Λ_+ cleanly in codimensions* 1 and k, *respectively, so that (Λ_1, Λ_2) and (Λ_+, Λ_2) are intersecting pairs.*

Proof. a) Follows from (1.1).
b) $N^*_{(x,t,\omega)}S_1 = \{(dh^*_x(\xi), 0, 0) : \xi \in \mathbb{R}^k\backslash 0\}$ while $N^*_{(x,t,\omega)}S_+ = \{(-\sigma\omega, \sigma, -\sigma i^*_\omega x) : \sigma \in \mathbb{R}\backslash 0\}$, where $i_\omega : T_\omega S^{n-1} \subsetneq T_\omega \mathbb{R}^n$.
c) Follows from the fact that $S_2 \subset S_1$ is codimension 1 and $S_2 \subset S_+$ is codimension k. Q.E.D.

The second term in (3.6) is

$$u_1 = -\Box^{-1}(q(x, t)\delta(t - x \cdot \omega)) ,$$

where $\Box^{-1}$ acts only in the (x, t) variables. By Lemma (1.1), with $X = \mathbb{R}^{n+1} \times S^{n-1}$, $Y = S_+$ and $Z = S_1$,

$$q(x, t) \cdot \delta(t - x \cdot \omega) \in I^{0,\mu}(S_+, S_2) , \tag{3.8}$$

so that

$$WF(q \cdot \delta) \subset \Lambda_+ \cup \Lambda_2 . \tag{3.9}$$

To obtain $WF(u_1)$, recall that

$$WF(\Box^{-1} v) \subset (\Delta \cap \Lambda_\Box) \circ WF(v), \quad \forall v \in \mathscr{E}'(\mathbb{R}^{n+1} \times S^{n-1}) , \tag{3.10}$$

where Δ is the diagonal of $T^*(\mathbb{R}^{n+1} \times S^{n-1})\backslash 0$ and $\Lambda_\Box$ is the flowout of the characteristic variety

$$\mathrm{char}(\Box) = \{(x, t, \omega; \xi, \tau, \Omega) : |\tau|^2 = |\xi|^2\} \tag{3.11}$$

of $\Box$ (acting on $\mathbb{R}^{n+1} \times S^{n-1}$). In (3.10), $\Delta \cup \Lambda_\Box$ acts as a relation between subsets of $T^*(\mathbb{R}^{n+1} \times S^{n-1})\backslash 0$; if course, Δ acts as the identity. Also, $\Lambda_\Box \circ \Lambda_+ = \Lambda_+$ since Λ_+ is characteristic for $\Box$. Thus

$$WF(u_1) \subset \Lambda_+ \cup \Lambda_2 \cup \Lambda_\Box \circ \Lambda_2 . \tag{3.12}$$

To understand the last term in (3.12), note that Λ_2 is a $(k+1)$-plane over S_2:

$$\Lambda_2 = \{(x, x \cdot \omega, \omega; v - \tau\omega, \tau, -\tau i^*_\omega x) : x \in S, \omega \in S^{n-1}, (v, \tau) \in (N^*_x S \times \mathbb{R})\backslash 0\} , \tag{3.13}$$

where i^*_ω denotes the restriction of an element of $\mathbb{R}^{n^*}$ to $T_\omega S^{n-1}$. The intersection of Λ_2 with char$(\Box)$ in these coordinates is given by

$$\Sigma = \Lambda_2 \cap \mathrm{char}(\Box) = \{v \cdot (v - 2\tau\omega) = 0\} . \tag{3.14}$$

Above a point $(x, x \cdot \omega, \omega) \in S_2$ such that $N^*_x S \subset \omega^\perp$, the fiber of $\Lambda_2 \cap \mathrm{char}(\Box)$ is just $\{(-\tau\omega, \tau, -\tau i^*_\omega x): \tau \in \mathbb{R}\backslash 0\}$, while if $N^*_x S \not\subset \omega^\perp$, the fiber is a smooth k-dimensional cone in $T^*_{(x, x\cdot\omega, \omega)}\mathbb{R}^{n+1} \times S^{n-1}\backslash 0$. Since $N^*_x S \subset \omega^\perp$ iff $\omega \in T_x S$, we see that the degenerate points correspond to the incoming plane wave being tangent to S. Let

$$S_3 = \{(x, x \cdot \omega, \omega) \in S_2 : \omega \in T_x S\} . \tag{3.15}$$

then $S_3 \subset S_2$ is codimension k and away from $\Sigma_3 = \Lambda_2|_{S_3}$, Σ is a smooth hypersurface in Λ_2.

To discuss the geometry further, we first consider the situation when S is a hypersurface ($k = 1$), so that $h(x)$ is scalar-valued with gradient h_x, and (3.13) may be rewritten as

$$\Lambda_2 = \{(x, x \cdot \omega, \omega; \lambda h_x - \tau\omega, \tau, -\tau i^*_\omega x); x \in S, (\lambda, \tau) \in \mathbb{R}^2\backslash 0\} . \tag{3.16}$$

In these coordinates,

$$\Lambda_2 \cap \mathrm{char}(\Box) = \{\lambda(h_x^2 \lambda - 2(\omega \cdot h_x)\tau) = 0\} ,$$

and thus $\Sigma = \Sigma_+ \cup \Sigma_-$, where $\Sigma_+ = \{\lambda = 0\}$ and $\Sigma_- = \{h_x^2 \lambda - 2(\omega \cdot h_x)\tau = 0\}$ are smooth hypersurfaces intersecting transversally over $S_3 = \{\omega \cdot h_x = 0\}$. Note that $\Sigma_+ = \Lambda_+ \cap \Lambda_2$ and thus the flowout $\Lambda_\Box \circ (\Sigma_+ \backslash \Sigma_3)$ of $\Sigma_+ \backslash \Sigma_3$ by the Hamiltonian vector field

$$H_\Box = -\xi \cdot \frac{\partial}{\partial \xi} + \tau \frac{\partial}{\partial \tau}$$

is contained in Λ_+. On the other hand, the flowout of $\Sigma_- \backslash \Sigma_3$ is a new lagrangian, which we denote by Λ_-. In fact, assuming, as we may, that $h_x^2 \equiv 1$,

$$\Sigma_- = \{(y, y \cdot \omega, \omega; \tau v(y, \omega), \tau, -\tau i^*_\omega y): y \in S, \omega \in S^{n-1}, \tau \in \mathbb{R}\backslash 0\} , \tag{3.17}$$

where $v(y, \omega) = 2(\omega \cdot h_y)h_y - \omega$. Note that $v^2 = 1$ and $v(y, \omega) = -\omega$ iff $\omega \cdot h_y = 0$, i.e, only at Σ_3. $H_\Box$ can only be tangent to Σ_- if it arises as the image of a vector $Y \cdot \frac{\partial}{\partial y}$ under the diferential of the parametrization (3.17); but then $Y = -\xi = -\tau v(y, \omega)$ and $\omega \cdot Y = \tau$, which imply that $v(y, \omega) = -\omega$, which only occurs at Σ_3 by the above comment. Thus, $H_\Box \pitchfork | \Sigma_-$ on $\Sigma_- \backslash \Sigma_3$, and $\Lambda_- = \Lambda_\Box \circ (\Sigma_- \backslash \Sigma_3)$ is a smooth lagrangian, intersecting Λ_2 cleanly in codimension 1. Explicitly,

$$\begin{aligned}\Lambda_- = \{&(y - rv(y, \omega), y \cdot \omega + r, \omega; \tau v(y, \omega), \tau, -\tau i^*_\omega y): y \in S, \omega \in S^{n-1}, \omega \cdot h_y \neq 0, \\ &r \in \mathbb{R}, \tau \in \mathbb{R}\backslash 0\} .\end{aligned} \tag{3.18}$$

We remark here that Λ_- is the conormal bundle of a smooth hypersurface, which we denote by S_-. In fact, the differential of the projection π from Λ_- onto the spatial variables is

$$\frac{D(x,t,\omega)}{D(y,r,\omega,\tau)} = \begin{pmatrix} j - rj^*d_yv & -v & -rd_\omega v & 0 \\ j^*\omega & 1 & i_\omega^* x & 0 \\ 0 & 0 & I & 0 \end{pmatrix},$$

where j denotes the differential of the inclusion $S \hookrightarrow \mathbb{R}^n$, from which we see that

$$\operatorname{rank}(d\pi) = n + \operatorname{rank}(j - rj^*d_yv + j^*\omega \otimes v)\,. \tag{3.19}$$

Away from S_3, $v(y,\omega) \neq -\omega$ and the rank of the second term in (3.19) is $n-1$; thus $\operatorname{rank}(d\pi) = 2n-1$ and $\Lambda_- = N^*S_-$, for $S_- \subset \mathbb{R}^{n+1} \times S^{n-1}$ a smooth hypersurface.

Now consider the case when the codimension of S satisfies $1 < k < n$. Then Σ, defined by (3.14), has a conical singularity at Σ_3. We will actually work away from a larger set,

$$\Sigma_2 = \{(y, y\cdot\omega, \omega; v - \tau\omega, \tau, -\tau i_\omega^* y) : (y,v) \in N^*S, \omega \in S^{n-1},$$
$$\tau \in \mathbb{R}\backslash 0, v\cdot\omega = 0\}\,. \tag{3.20}$$

By the same reasoning as for $k=1$, $H_\square \pitchfork \Sigma$ on $\Sigma\backslash\Sigma_2$; thus, Λ_-, which we define in this case to be $\Lambda_\square \circ (\Sigma\backslash\Sigma_2)$, is a smooth lagrangian intersecting Λ_2 cleanly in codimension 1. Furthermore, since $\Lambda_+ \cap \Lambda_2 \subset \Sigma_2$, Λ_+ and Λ_- are disjoint (although $\Lambda_+ \cap \bar{\Lambda}_- \neq \emptyset$.) We can parametrize $\Sigma\backslash\Sigma_2$ and Λ_- by solving $v\cdot(v - 2\tau\omega)$ for τ, which we can do away from Σ_2: $\tau = \tau(y,v,\omega) = \dfrac{v^2}{2(v\cdot\omega)}$. Note that $|v - \tau\omega| = |\tau|$. Thus,

$$\Sigma\backslash\Sigma_2 = \{(y, y\cdot\omega, \omega; v - \tau\omega, \tau, -\tau i_\omega^* y) : (y,v) \in N^*S\backslash 0, \omega \in S^{n-1}, v\cdot\omega \neq 0\} \tag{3.21}$$

and

$$\Lambda_- = \{(y - rw(y,v,\omega), y\cdot\omega, \omega; v - \tau\omega, \tau, -\tau i_\omega^* y) : (y,v) \in N^*S\backslash 0, \omega \in S^{n-1},$$
$$r \in \mathbb{R}, v\cdot\omega \neq 0\}\,, \tag{3.22}$$

where $w(y,v,\omega) = \dfrac{v}{\tau} - \omega$; note that $w^2 = 1$.

When $k = n$, so that S is a finite set of points in $\mathbb{R}^n$, introducing Σ_3 or Σ_2 is unnecessary: Σ is smooth, as is $\Lambda_- = \Lambda_\square \circ \Sigma$. Λ_+ and Λ_- intersect, however, and we will work away from this intersection by using the parametrization (3.22).

Finally, we note that for $1 < k \le n$, Λ_- is contained in the conormal bundle of a submanifold of $\mathbb{R}^{n+1} \times S^{n-1}$ having a conical singularity along S_2. Not needing this fact below, we will not describe its structure further.

Returning now to the second term, $u_1(x,t,\omega)$, of the approximate solution for the direct problem, let $\mathcal{O}' \Subset \mathcal{O} \subset \Lambda_2$ be conic neighborhoods of Σ_3, Σ_2 or $(\Lambda_+ \cap \Lambda_2) \cap \Sigma$ in the cases when $k=1$, $1 < k < n$ or $k = n$, respectively. Let

$L = \Lambda_\square \circ \mathcal{O}$, $L' = \Lambda_\square \circ \mathcal{O}'$ be the conic neighborhoods of $\mathcal{O}$, $\mathcal{O}'$ in $T^*(\mathbb{R}^{n+1} \times S^{n-1})$ invariant under the Hamiltonian flow. Then (3.12) becomes

$$WF(u_1)\backslash L \subset \Lambda_+ \cup \Lambda_2 \cup \Lambda_- \,. \tag{3.23}$$

In fact, microlocalizing u_1 away from L, u_1 belongs to the sum of the spaces of lagrangian distributions associated with the pairs (Λ_2, Λ_+) and (Λ_2, Λ_-). Let $\chi(x, t, \omega, \xi, \tau, \Omega) \in C^\infty(T^*(\mathbb{R}^{n+1} \times S^{n-1})\backslash 0)$ be homogeneous of degree 0 in the fiber variables, with $\chi \equiv 0$ on L' and $\chi \equiv 1$ on L^c. The corresponding pseudodifferential operator $\chi(x, t, \omega, D_x, D_t, D_\omega)$ has the property that $[\square^{-1}, \chi]$ is microlocally supported on the annulus $L \cap L'^c$. Let

$$\tilde{u}_1 = -\square^{-1}(\chi(q \cdot \delta))\,; \tag{3.24}$$

then $\tilde{u}_1 - u_1$ is microlocally smooth on L^c by the above comment. Translating (3.8) from conormal to lagrangian language, we find that

$$q(x) \cdot \delta(t - x \cdot \omega) \in I^{\frac{1-n}{2}, \mu + \frac{k}{2}}(\Lambda_2, \Lambda_+)\,. \tag{3.25}$$

The same is true for $\chi(q \cdot \delta)$, and since, by [GuU], the double intersection spaces microlocalize, one has $\chi(q \cdot \delta) = u^+ + u^-$, where

$$\begin{cases} u^+ \in I^{\frac{1-n}{2}, \mu + \frac{k}{2}}(\Lambda_2\backslash L, \Lambda_+\backslash L) \\ u^- \in I^{\mu + \frac{k+1-n}{2}}(\Lambda_2\backslash L) \end{cases} \tag{3.26}$$

with $u^\pm$ supported microlocally near $\Lambda_2 \cap \Lambda_\pm$. Applying $-\square^{-1} \in I^{-\frac{3}{2}, -\frac{1}{2}}(\Lambda_{T^*(\mathbb{R}^{n+1} \times S^{n-1})}, \Lambda_\square)$ to u^+ and u^-, and using Propositions 2.2 and 2.1, respectively, we find that

$$\begin{cases} -\square^{-1}(u^+) \in I^{-\frac{(n+1)}{2}, \mu + \frac{k}{2} - 1}(\Lambda_2\backslash L, \Lambda_+\backslash L) \\ -\square^{-1}(u^-) \in I^{\mu + \frac{k-2-n}{2}, -\frac{1}{2}}(\Lambda_2\backslash L, \Lambda_-\backslash L)\,. \end{cases} \tag{3.27}$$

Since the variable t is bounded on Λ_2, by (1.4) we have

$$u_1 \in I^{-\frac{(n+1)}{2}}(\Lambda_+\backslash L) + I^{\mu + \frac{k-2-n}{2}}(\Lambda_-\backslash L), \quad t \gg 0\,, \tag{3.28}$$

and thus the approximate solution

$$u_0 + u_1 \in I^{-\frac{(n-1)}{2}}(\Lambda_+\backslash L) + I^{\mu + \frac{k-2-n}{2}}(\Lambda_-\backslash L), \quad t \gg 0\,, \tag{3.29}$$

Furthermore, the analysis shows that $WF(u_1) \subset \Lambda_\square \circ \Lambda_2$.

Now let $R\colon \mathscr{E}'(\mathbb{R}^n) \to \mathscr{E}'(\mathbb{R} \times S^{n-1})$ be the Radon transform

$$(Rf)(s, \theta) = \int_{x \cdot \theta = s} f(x)\, d\sigma(x)\,, \tag{3.30}$$

where $d\sigma$ is normalized Lebesgue measure on the hyperplane $\{x \cdot \theta = s\}$. Acting in the x variable, R is defined on those elements of $\mathscr{D}'(\mathbb{R}^n \times \mathbb{R} \times S^{n-1})$ having compact support in x for each t, ω; R is an elliptic Fourier integral operator, $R \in I^{(1-n)/2}(C_R)$, where C_R is the local canonical graph $\subset T^*(\mathbb{R}\times S^{n-1}\times\mathbb{R}\times S^{n-1}\times\mathbb{R}^n\times\mathbb{R}\times S^{n-1})$ given by

$$C_R = \{(x\cdot\theta, \theta, t, \omega, \sigma, -\sigma i_\theta^*(x), \tau, \Omega; x, t, \omega, \sigma\theta, \tau, \Omega): \\ (x, t, \omega) \in \mathbb{R}^{n+1}\times S^{n-1}, \theta\in S^{n-1}, \sigma\in\mathbb{R}\backslash 0, \tau\in\mathbb{R}, \Omega\in T_\omega^* S^{n-1}\}\,. \quad (3.31)$$

The modified (Lax–Phillips) Radon transform [LP], which maps $\mathbb{C}^2$ - to $\mathbb{C}$-valued distributions, is defined by

$$R_{\mathrm{LP}}\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = C_n D_s^{\frac{n-1}{2}}(D_s R v_0 - R v_1), \quad n \text{ odd}\,. \quad (3.32)$$

For n even one replaces in (3.32) $D_s^{\frac{n-1}{2}}$ by $|D_s^{\frac{n-1}{2}}|$. For another discussion of the time dependent approach to scattering theory and also for a more detailed treatment of the odd dimensional case, see [Pe].

The scattering kernel can be expressed in terms of R_{LP} and the solution to the direct problem (3.5) as follows [MU2]: letting

$$w = D_t^{\frac{n-3}{2}}(u(x, t, \omega) - \delta(t - x\cdot\omega)), \quad n \text{ odd}\,, \quad (3.33)$$

one has the relation

$$\alpha(t - s, \theta, \omega) = \delta(t - s)\otimes\delta(\theta - \omega) + R_{\mathrm{LP}}\begin{pmatrix} w \\ D_t\omega \end{pmatrix}, \quad t \gg 0\,. \quad (3.34)$$

For n even one replaces D_t in (3.33) by $|D_t|$.

Now, acting on the argument $u - \delta$, w and $D_t w$ are pseudodifferential operators of orders $\frac{n-3}{2}$ and $\frac{n-1}{2}$, respectively, and thus

$$(\alpha - \delta\otimes\delta)(\tau - s, \theta, \omega) = F(u - \delta)\,, \quad (3.35)$$

where $F\in I^{\frac{n-1}{2}}(C_R)$. Furthermore, if we denote the fiber variables in $T^*(\mathbb{R}\times S^{n-1}\times\mathbb{R}\times S^{n-1})$ and $T^*(\mathbb{R}^{n+1}\times S^{n-1})$ by $(\sigma, \Theta, \tau, \Omega)$ and (ξ, τ', Ω'), respectively, the symbol of F is an elliptic factor times $\sigma - \tau'$. This is elliptic on the region of C_R giving rise to the first components in $(3.40)_1$ and $(3.40)_k$, which are shown below to be those of interest.

To deal with the translation-invariance in (3.35), we introduce, for $t_0 \gg 0$, the mapping $\rho: \mathbb{R}\times S^{n-1}\times S^{n-1}\to\mathbb{R}\times S^{n-1}\times\mathbb{R}\times S^{n-1}$, $\rho(s, \theta, \omega) = (t_0 + s, \theta, t_0, \omega)$, which induces a restriction mapping

$$\rho^*: \mathscr{D}'_\rho(\mathbb{R}\times S^{n-1}\times\mathbb{R}\times S^{n-1})\to\mathscr{D}'(\mathbb{R}\times S^{n-1}\times S^{n-1})\,.$$

Here, $\mathscr{D}'_\rho$ denotes those distributions whose wavefront sets are disjoint from the normals of ρ. ρ^* is a Fourier integral operator, $\rho^*\in I^{\frac{1}{4}}(C_\rho)$, where

$$C_\rho = \{(s, \theta, \omega, \sigma, \Theta, \Omega; t_0 + s; \theta, t_0, \omega, \sigma, \Theta, \eta, \Omega): \\ s\in\mathbb{R}, \theta, \omega\in S^{n-1}, (\sigma, \Theta, \tau, \Omega)\in T^*(\mathbb{R}\times S^{n-1}\times S^{n-1})\backslash 0\}\,. \quad (3.36)$$

Thus, the scattering kernel can be expressed by means of

$$(\alpha - \delta_0\otimes\delta)(s, \theta, \omega) = \rho^* F(u - \delta)\,. \quad (3.37)$$

We first examine the right side of (3.37) applied to the approximate solution $u_0 + u_1$; the comparison with $\rho^* F(u - \delta)$ will be made later. Of course, $\rho^* F(u_0 + u_1 - \delta) = \rho^* F(u_1)$. Since C_R is a local canonical graph, by (3.28) we have

$$F(u_1) \in I^{-1}(C_R \circ \Lambda_+) + I^{\mu + \frac{k-3}{2}}(C_R \circ \Lambda_-) . \tag{3.38}$$

Applying (3.31) to Λ_+, we see that $C_R \circ \Lambda_+$ has two components,

$$\begin{aligned} C_R \circ \Lambda_+ = \{&(y \cdot \omega, \omega, y \cdot \omega, \omega; \sigma, -\sigma i^*_\omega y, -\sigma, \sigma i^*_\omega y) \colon y \in \mathbb{R}^n, \omega \in S^{n-1}, \sigma \in \mathbb{R}\backslash 0\} \\ &\cup [-y \cdot \omega, -\omega, y \cdot \omega, \omega; \sigma, -\sigma i^*_{-\omega} y, \sigma, -\sigma i^*_\omega y) \colon y \in \mathbb{R}^n, \\ &\omega \in S^{n-1}, \sigma \in \mathbb{R}\backslash 0\} . \end{aligned} \tag{3.39}$$

Similarly, applying C_R to Λ_- in the cases $k = 1$ and $1 < k \leqq n$, we obtain

$$\begin{aligned} C_R \circ \Lambda_- = \{&(r - y \cdot v(y, \omega), -v(y, \omega), y \cdot \omega + r, \omega; \sigma, -\sigma i^*_{-v} y, -\sigma, \sigma i^*_\omega y) \colon \\ &y \in S, \omega \in S^{n-1}, r \in \mathbb{R}, \sigma \in \mathbb{R}\backslash 0\} \\ &\cup \{(y \cdot v(y, \omega) - r, v(y, \omega), y \cdot \omega + r, \omega; \sigma, -\sigma i^*_v y, \sigma, -\sigma i^*_\omega y) \colon \\ &y \in S, \omega \in S^{n-1}, r \in \mathbb{R}, \sigma \in \mathbb{R}\backslash 0\} , \end{aligned} \tag{3.40$_1$}$$

$$\begin{aligned} C_R \circ \Lambda_- = \{&(r - y \cdot w(y, v, \omega), -w(y, v, \omega), y \cdot \omega + r, \omega; -\tau(y, v, \omega), \tau i^*_{-w} y, \tau, \\ &-\tau i^*_\omega y) \colon (y, v) \in N^*S\backslash 0, \omega \in S^{n-1}, v \cdot \omega \neq 0, r \in \mathbb{R}\} \\ &\cup \{(y \cdot w(y, v, \omega) - r, w(y, v, \omega), y \cdot \omega + r, \omega; \tau(y, v, \omega), -\tau i^*_\omega y, \tau, \\ &-\tau i^*_\omega y) \colon (y, v) \in N^*S\backslash 0, \omega \in S^{n-1}, v \cdot \omega \neq 0, r \in \mathbb{R}\} , \end{aligned} \tag{3.40$_k$}$$

respectively. We also note that

$$\begin{aligned} C_R \circ \Lambda_1 = \Bigg\{&\left(\pm \frac{v}{|v|} \cdot y, \pm \frac{v}{|v|}, \tau, \omega; \pm |v|, \pm -|v| i^*_{\pm \frac{v}{|v|}} y, 0, 0 \right) \colon \\ &(y, v) \in N^*S\backslash 0, \tau \in \mathbb{R}, \omega \in S^{n-1} \Bigg\} . \end{aligned} \tag{3.41}$$

Since it is impossible for both s and t to be $\gg 0$ on the second component of $C_R \circ \Lambda_+$ in (3.39) and the second component of $C_R \circ \Lambda_-$ in $(3.40)_1$, $(3.40)_k$, the application of C_ρ to these components is empty. Define the *peak lagrangian*

$$\begin{aligned} \hat{\Lambda}_+ = C_\rho \circ C_R \circ \Lambda_+ &= \{(0, \omega, \omega; \sigma, -\sigma i^*_\omega y, \sigma^*_\omega y) \colon \omega \in S^{n-1}, y \in \mathbb{R}^n, \sigma \in \mathbb{R}\backslash 0\} \\ &\subseteq \{(0, \omega, \omega; \sigma, -\Omega, \Omega) \colon (\omega, \Omega) \in T^*S^{n-1}, \sigma \in \mathbb{R}\backslash 0\} \\ &= N^*\{s = 0, \theta = \omega\} , \end{aligned} \tag{3.42}$$

and the *reflected lagrangian*,

$$\begin{aligned} \hat{\Lambda}_- = C_\rho \circ C_R \circ \Lambda_- = \{&(-y \cdot (v(y, \omega) + \omega), v(y, \omega), \omega; \sigma, -\sigma i^*_{-v} y, \sigma i^*_\omega y) \colon \\ &y \in S, \omega \in S^{n-1}, \sigma \in \mathbb{R}\backslash 0\} , \end{aligned} \tag{3.43$_1$}$$

$$\begin{aligned} \hat{\Lambda}_- = \{&(-y \cdot (w(y, v, \omega) + \omega), -w(y, v, \omega), \omega; \tau(y, v, \omega), -\tau i^*_{-w} y, \tau i^*_\omega y) \colon \\ &(y, v) \in N^*S\backslash 0, \omega \in S^{n-1}\} , \end{aligned} \tag{3.43$_k$}$$

for $k = 1$ and $1 < k \leqq n$, respectively. One computes easily that the application of C_ρ to $C_R \circ \Lambda_-$ falls under the transverse intersection calculus. Thus, from (3.38) one obtains

$$\rho_* F(u_1) \in I^{-\frac{3}{4}}(\hat{\Lambda}_+) + I^{\mu+\frac{k}{2}-\frac{5}{4}}(\hat{\Lambda}_-) . \tag{3.44}$$

To compare this with the true scattering kernel given by (3.35), set $\bar{u} = u - (u_0 + u_1)$. Then $\bar{u} \equiv 0$ for $t \gg 0$ and

$$\begin{aligned}(\Box + q)\bar{u} &= (\Box + q)u - (\Box + q)(u_0 + u_1)\\ &= 0 - \Box u_0 - q \cdot u_0 - \Box u_1 - q \cdot u_1\\ &= -q \cdot u_1 . \end{aligned}\tag{3.45}$$

Some care needs to be taken in interpreting the product $q \cdot u_1$. Let M_q denote multiplication by q; we will show that M_q is a "pseudodifferential operator with singular symbol" on $\mathbb{R}^{n+1} \times S^{n-1}$. In fact, the Schwartz kernel of M_q is

$$K_{M_q}((x, \tau, \omega), (x', \tau', \omega')) = q(x)\delta(x - x')\delta(t - t')\delta(\omega - \omega') ,$$

which belongs to $I^{\mu+\frac{k}{2}, -(\mu+\frac{k}{2})}(\Delta'_{T^*(\mathbb{R}^{n-1} \times S^{n-1})}, \Lambda'_S)$, where Λ_S is the flowout of $T^*(\mathbb{R}^{n+1} \times S^{n-1})|_S = \{(x, \tau, \omega, \xi, \tau, \Omega) \colon x \in S\}$,

$$\begin{aligned}\Lambda_S = \{&(x, \tau, \omega, \xi + \eta, \tau, \Omega; x, \tau, \omega, \xi, \tau, \Omega) \colon (x, \tau, \omega, \xi, \tau, \Omega) \in T^*(\mathbb{R}^{n+1} \times S^{n-1})|_S,\\ &\eta \in N_x^* S\} . \end{aligned}\tag{3.46}$$

The resulting operator is defined when acting on distributions $v \in D'(R^{n+1} \times S^{n-1})$ such that $\Lambda_S \circ WF(v) \cap (0) = \phi$, where (0) is the 0-section. From (3.46), we see that $M_q v$ is defined for v such that $WF(v) \cap \Lambda_1 = \phi$. If $K \subset T^*(R^{n+1} \times S^{n-1}) \backslash (0)$ is a closed, conic set disjoint from Λ_1, and H_K^s consists of those elements of the local Sobolev space H_{loc}^s with wave front set contained in K, it follows from the results of [GU] that

$$M_q \colon H_K^s \to H_{\text{loc}}^{s-(\mu+k)_+} . \tag{3.47}$$

To deal with v's with wavefront set near Λ_1, we make use of the results of Sect. 1. Introduce zeroth order pseudodifferential operators $\chi_j(D) = \chi_j(x, t, \omega, D_x, D_t, D_\omega)$, $1 \leqq j \leqq 4$, on $R^{n+1} \times S^{n-1}$, forming a microlocal partition of unity, such that $\chi_1(D)$ is supported on a neighborhood ${}^cK$ of Λ_1, where K is as above; $\chi_2(D)$ is supported near $(\Lambda_+ \cup \Lambda_-) \backslash L$; $\chi_3(D)$ is supported near L; and $\chi_4(D)$ is such that $\sum_{j=1}^4 \chi_j(D) = I$. Now, since $q(x)\delta(t - x \cdot \omega) \in I^{\mu+\frac{k+1-n}{2}}(\Lambda_2)$ away from Λ_+,

$$\chi_1(D)u_1 = -\chi_1(D)\Box^{-1}(q\delta) \in I^{\mu+\frac{k-3-n}{2}}(\Lambda_2) = I^{\mu-2}(S_2)$$

and is supported near Λ_1. By Lemma 1.3,

$$M_q \chi_1 u_1 \in I^{\mu, \mu+k-2}(S_1, S_2) = I^{2\mu+\frac{3k-3-n}{2}, -\mu+\frac{3-k}{2}}(\Lambda_1, \Lambda_2) ,$$

since $\mu - 2 < -k$, $2\mu - 2 < -k$. By (3.27),

$$\begin{aligned}\chi_2(D)u_1 &\in I^{-\frac{n+1}{2}, \mu+\frac{k-2}{2}}(\Lambda_2, \Lambda_+) + I^{\mu+\frac{k-2-n}{2}, -\frac{1}{2}}(\Lambda_2, \Lambda_-)\\ &= I^{-\frac{k+1}{2}, \mu+\frac{k-3}{2}}(S_+, S_2) + I^{\mu-1, -1}(S_-, S_2) .\end{aligned}$$

Applying Lemma 1.5 with $Y_1 = S_1$, $Y_2 = S_2$ and $Y_+ = S_+$ or S_-, we obtain

$$\begin{aligned} M_q\chi_2 u_1 \in\; & I^{-\frac{k+1}{2},\, \mu+\frac{k-3}{2}+(\mu+k)_{+,\varepsilon}}(S_+, S_2) + I^{-\frac{k+1}{2}+\left(2\mu+\frac{3k-3}{2}\right)_{+,\varepsilon},\, 0}(S_+, S_2) \\ & + I^{\mu-1,\, -1+(\mu+k)_{+,\varepsilon}}(S_-, S_2) + I^{\mu-1+(\mu-1+k)_{+,\varepsilon},\, 0}(S_-, S_2) \\ & + I^{\mu,\, -\frac{k+1}{2}+\left(\mu+\frac{3k-3}{2}\right)_{+,\varepsilon}}(S_1, S_2) + I^{\mu,\mu-1+(-1+k)_{+,\varepsilon}}(S_1, S_2)\,, \end{aligned}$$

where $(x+k)_{+,\varepsilon} = (x+k)_+ + \varepsilon\delta_{x,-k}$ for any $\varepsilon > 0$.

Since $\chi_3(D)u_1 \in H_K^{s_0}$ for some $s_0 \in R$, $M_q\chi_2 u_1 \in H_{\text{loc}}^{s_0-(\mu+k)_+}$ by (3.47), and $WF(M_q\chi_2 u_1) \subset L$ by (3.46) and the definition of L. Finally, $M_q\chi_4 u_1 \in I^{-k+(\mu+k)_{+,\varepsilon}}(S_1, S_2)$; this latter space also contains $M_q\chi_1 u_1$, since $-k + (\mu+k)_{+,\varepsilon} \geqq \mu$.

Having described the right side of (3.45), we now apply the forward fundamental solution $\square^{-1}$ to both sides. Since $I^{M,M'}(S_\pm, S_2) = I^{M+\frac{k-n}{2},\, M'+\frac{1}{2}}(\Lambda_2, \Lambda_\pm)$, (2.10) implies that

$$\square^{-1}\colon I^{M,M'}(S_\pm, S_2) \to I^{M-1,M'-1}(S_\pm, S_2)\,.$$

Furthermore, $\square^{-1}$ acts on $I^{M,M'}(S_1, S_2)$ as a pseudodifferential operator of order -2. Finally, $\square^{-1}\colon H^s \to H_{\text{loc}}^{s+1}$. Thus, (3.45) becomes

$$\begin{aligned} (I + \square^{-1}M_q)\bar{u} \in\; & I^{-\frac{k+3}{2},\, \mu+\frac{k-5}{2}+(\mu+k)_{+,\varepsilon}}(S_+, S_2) + I^{-\frac{k+3}{2}+\left(2\mu+\frac{3k-3}{2}\right)_{+,\varepsilon},\, -1}(S_+, S_2) \\ & + I^{\mu-2,\, -2+(\mu+k)_{+,\varepsilon}}(S_-, S_2) + I^{\mu-2+(\mu-1+k)_{+,\varepsilon},\, -1}(S_-, S_2) \\ & + I^{\mu-2,\, -\frac{k+1}{2}+\left(\mu+\frac{3k-3}{2}\right)_{+,\varepsilon}}(S_1, S_2) + I^{\mu-2,\mu-1+(-1+k)_{+,\varepsilon}}(S_1, S_2) \\ & + I^{-k-2+(\mu+k)_{+,\varepsilon},\mu-2+k}(S_1, S_2) + H_{\text{loc}}^{s_0+1-(\mu+k)_+}\,, \end{aligned} \tag{3.48}$$

with the wavefront set of the last term contained in L. Given the $\Lambda_\square$-invariant neighborhood L and any integer $N \geqq 1$, we can find another $\Lambda_\square$ invariant neighborhood, $L_N \Subset L$ such that $WF((\square^{-1}M_q)^N)(L_N) \subset L$. Making all of the above microlocalizations away from L_N, and applying $\sum_{j=0}^{N-1}(-1)^j(\square^{-1}M_q)^j$ to both sides of (3.48), one sees using Lemmas 1.2, 1.3, 1.4 and 1.5 that, since $\mu < 1 - k$, the right side of (3.48) is stable under the application of $(\square^{-1}M_q)^j$ for $0 \leqq j \leqq N-1$. Thus, $(I + (-1)^N(\square^{-1}M_q)^N)\bar{u}$ belongs to the space on the right side of (3.48), and the wavefront set of the Sobolev space term is contained in L. Now apply ρ^*R to both sides; using the mapping properties of R and ρ^*, noting that $C_\rho \circ \Lambda_1 = \emptyset$ for $t \gg 0$ (from (3.36)), and applying the standard Sobolev restriction theorem for hypersurfaces, we find that

$$\rho^*F\bar{u} \in I^{-\frac{7}{4}}(\hat{\Lambda}_+) + I^{\mu+\frac{k}{2}-\frac{9}{4}}(\hat{\Lambda}_-) + H_{\text{loc}}^{r_0+\delta N+n-\frac{3}{2}}\,, \tag{3.49}$$

where $\bar{u} \in H_{\text{loc}}^{r_0}$ and $\delta = 1 - (\mu+k)_+ > 0$.

Taking $N \to +\infty$, we thus have shown

Theorem 3.1. *Microlocally away from $\hat{L}$,*

$$\alpha(s,\theta,\omega) - \rho^*F(u_0 + u_1) \in I^{-\frac{7}{4}}(\hat{\Lambda}_+)_+ + I^{\mu+\frac{k}{2}-\frac{9}{4}}(\hat{\Lambda}_-)\,. \tag{3.50}$$

Thus, $\alpha \in I^{\frac{1}{4}}(\hat{\Lambda}_+ \backslash \hat{L}) + I^{\mu+\frac{k}{2}-\frac{5}{4}}(\hat{\Lambda}_- \backslash \hat{L})$, and hence $\hat{\Lambda}_-$ and the principal symbol $\sigma(\mu_1)|_{\hat{\Lambda}_-}$ are determined by α.

4. Determination of S and $\sigma(q)$

We now examine how the leading singularity of $q(x)$ is determined by the singularities of $\alpha(s, \theta, \omega)$ and its restriction to various submanifolds of $\mathbb{R} \times S^{n-1} \times S^{n-1}$. We start by showing that $\alpha(s, \theta, \omega)$ determines the surface, S. By Theorem 3.1, it suffices to show that S is determined by $\hat{\Lambda}_-$, where $\hat{\Lambda}_-$ is given by $(3.43)_1$ and $(3.43)_k$ in the cases where the codimension of S is 1 and greater than 1, respectively. For simplicity, consider $k = 1$ first; suppose there are two hypersurfaces, $S, \bar{S}$, such that $\hat{\Lambda}_- = \hat{\bar{\Lambda}}_-$. Then, by $(3.43)_1$, there is a mapping $\bar{y}: S \times S^{n-1} \to \bar{S}$, such that

$$\begin{aligned} &\{(-\bar{y} \cdot (\bar{v} + \omega),\ -\bar{v}, \omega; \sigma,\ -\sigma i^*_{-\bar{v}} \bar{y}, \sigma i^*_\omega \bar{y})\} \\ &\quad = \{(-x \cdot (v + \omega),\ -v, \omega; \sigma,\ -\sigma i^*_{-v} y, \sigma i^*_\omega y)\}\ , \end{aligned} \tag{4.1}$$

where $\bar{v} = \bar{v}(\bar{y}(, \omega), \omega)$. Identifying the second (θ) coordinates in (4.1) yields $2(\bar{h}_{\bar{y}} \cdot \omega)\bar{h}_{\bar{y}} - \omega = 2(h_y \cdot \omega)h_y - \omega \Rightarrow (\bar{h}_{\bar{y}} \cdot \omega)\bar{h}_{\bar{y}} = (h_y \cdot \omega)h_y$; since h_y and $\bar{h}_{\bar{y}}$ are unit vectors, this $\Rightarrow$

$$\bar{h}_{\bar{y}} = \pm h_y\ . \tag{4.2}$$

Identifying the last (Ω) components in (4.1) yields $i^*_\omega \bar{y} = i^*_\omega y \Rightarrow$

$$\bar{y} = y + c_1 \omega, \quad c_1 = c_1(y, \omega)\ . \tag{4.3}$$

Finally, comparing the first (s) components, we have

$$\bar{y} \cdot (\bar{v} + \omega) = y \cdot (v + \omega) \Rightarrow \bar{y} \cdot (\bar{h}_{\bar{y}} \cdot \omega)\bar{h}_y = y \cdot (h_y \cdot \omega) h_y \Rightarrow$$

by (4.2), $\bar{y} \cdot (\pm h_y \cdot \omega)(\pm h_y) = y \cdot (h_y \cdot \omega)h_y \Rightarrow$ by (4.3), $(y + c_1\omega) \cdot (h_y \omega)h_y = y(h_y \cdot \omega)h_y$. Now, $h_y \cdot \omega$ is not identically 0 on $S \times S^{n-1}$, so this $\Rightarrow (y + c_1 \omega) \cdot h_y = y \cdot h_y$ $\Rightarrow c_1 \omega \cdot h_y = 0$. But again, $\omega \cdot h_y \not\equiv 0$, so this $\Rightarrow c_1 = 0 \Rightarrow \bar{y} = y \Rightarrow \bar{S} = S$.

For $1 < k \leq n$, we repeat the above argument, substituting $(3.43)_k$ for $(3.43)_1$. Thus, there is a function $(\bar{y}, \bar{v})$: $(N^*S\backslash 0) \times S^{n-1} \to N^*\bar{S}\backslash 0$ making the identification between $\hat{\Lambda}_-$ and $\hat{\bar{\Lambda}}_-$, and we may assume $\bar{v}(y, \omega)^2 = v^2$. Identifying the θ-coordinates then yields $2(v \cdot \omega)\frac{v}{v^2} = 2(\bar{v} \cdot \omega)\frac{\bar{v}}{\bar{v}^2}$, which implies $v = \pm \bar{v}$, $\tau = \pm \bar{\tau}$. The rest of the proof is the same.

We next consider the restriction of $\alpha(s, \theta, \omega)$ to submanifolds. Consider first the case of backscattering.

Let $\mathbb{B} = \{(s, \theta, \omega): \theta = -\omega\} \subset \mathbb{R} \times S^{n-1} \times S^{n-1}$ be the backscattering surface. Let $j_{\mathbb{B}}: \mathbb{R} \times S^{n-1} \to \mathbb{B}$ be the canonical parametrization, $j_B(s, \omega) = (s,\ -\omega, \omega)$; then the pullback operator,

$$j^*_{\mathbb{B}}: \mathscr{D}'(\mathbb{R} \times S^{n-1} \times S^{n-1}) \to \mathscr{D}'_B(\mathbb{R} \times S^{n-1})\ ,$$

defined on distributions whose wavefronts sets are disjoint from the normals of j_B, is a Fourier integral operator, $j^*_{\mathbb{B}} \in I^{\frac{n-1}{4}}(C_{\mathbb{B}})$, where

$$\begin{aligned} C_{\mathbb{B}} = \{&(s, \omega, \sigma, \Omega; s', \theta, \omega', \sigma', \Theta, \Omega'): s = s', \omega = \omega' = -\theta, \\ &(\sigma, \Omega) = (\sigma', \Theta, \Omega')/N^*_{(s,\ -\omega, \omega)}\mathbb{B}\}\ . \end{aligned}$$

Let $L_{\mathbb{B}} = C_{\mathbb{B}} \circ \hat{L} \subset T^*\mathbb{B}\backslash 0$.

Theorem 4.1. *Away from* $L_{\mathbb{B}}$,

$$\alpha|_{\mathbb{B}} \in I^{\mu+\frac{k-3}{2}+\frac{n}{4}}(\Lambda_{\mathbb{B}}), \tag{4.5}$$

where

$$\Lambda_{\mathbb{B}} = \{(-2y\cdot h_y, h_y; \sigma, \sigma i^*_{h_y} y) : y \in S, \sigma \in \mathbb{R}\backslash 0\}$$
$$\cup \{(2y\cdot h_y, -h_y; \sigma, \sigma i^*_{-h_y} y) : y \in S, \sigma \in \mathbb{R}\backslash 0\} \quad (k=1) \quad (4.6)_1$$

and

$$\Lambda_{\mathbb{B}} = \left\{\left(-2y\cdot\frac{v}{|v|}, \frac{v}{|v|}; |v|, |v| i^*_{\frac{v}{|v|}} y\right) : (y, v) \in N^*S\backslash 0\right\}, \quad 1 < k \leqq n-1\,. \quad (4.6)_k$$

Proof. $\alpha|_{\mathbb{B}} = j^*_{\mathbb{B}}(\alpha) \in I^{\mu+\frac{k}{2}-\frac{5}{4}+\frac{n-1}{4}}(C_{\mathbb{B}} \circ \hat{\Lambda}_-)$ by the transverse intersection calculus, provided a) $\hat{\Lambda}_- \pitchfork T^*(\mathbb{R}\times S^{n-1}\times S^{n-1})|_{\mathbb{B}}$ and b) $\hat{\Lambda}_- \cap N^*\mathbb{B} = \emptyset$; both of these, together with $(4.6)_1$, $(4.6)_k$ follow from $(3.43)_1$, $(3.43)_k$, respectively. Q.E.D.

Corollary 4.2. $\alpha|_{\mathbb{B}}$ *determines* S.

Proof. We deal with the case $k = 1$, the case $1 < k \leqq n$ being handled similarly. By Theorem 4.1, $\alpha|_{\mathbb{B}}$ is, microlocally away from $L_{\mathbb{B}}$, a Fourier integral distribution associated with a lagrangian $\Lambda_B \subset T^*\mathbb{B}\backslash 0$, invariant under the canonical involution induced by $(s, \omega) \to (-s, \omega)$. On either component, we see that

$$\frac{\Omega}{\sigma} - \frac{1}{2} s\omega = i^*_{\pm h_y} y \pm (x\cdot h_y)h_y = y\,.$$

Thus, $S = \left\{\frac{\Omega}{\sigma} - \frac{1}{2} s\omega : (s, \omega, \sigma, \Omega) \in \Lambda_B\right\}$. Q.E.D.

Now consider a more general manifold of scattering data, $\mathbb{D} = \{(s, \theta, \omega) : \theta = \phi(\omega)\}$, where $\phi = S^{n-1} \to S^{n-1}$ is smooth. By $(3.43)_1$, at a point of $\tilde{\Lambda}_- \cap T^*(\mathbb{R}\times S^{n-1}\times S^{n-1})$, we must have

$$\omega - 2(\omega\cdot h_y)h_y = \phi(\omega)$$

or

$$\frac{1}{2}\|\omega - \phi(\omega)\|\left(\frac{\omega - \phi(\omega)}{\|\omega - \phi(\omega)\|}\right) = (\omega\cdot h_y)h_y\,.$$

Letting

$$\varphi(\omega) = \frac{\omega - \phi(\omega)}{\|\omega - \phi(\omega)\|} \tag{4.8}$$

under the assumption $\phi(\omega) \neq \omega$, $\forall \omega \in S^{n-1}$, we thus have

$$\varphi(\omega) = \pm h_y\,.$$

We now assume that φ has a smooth left inverse, at least on the image of the Gaussian map of S. Then the image of $\hat{\Lambda}_-$ under intersection with $T^*(\mathbb{R}\times S^{n-1}\times S^{n-1})|_{\mathbb{D}}$ and modding out by $N^*\mathbb{D}$ is (in $(s, \omega, \sigma, \Omega)$ coordinates),

$$\Lambda_{\mathbb{D}} = \{(-2(\varphi^{-1}(\pm h_y)\cdot h_y)y\cdot h_y), \varphi^{-1}(\pm h_y); \sigma, \sigma i^*_{\varphi^{-1}(\pm h_y)} : y \in S, \sigma \in \mathbb{R}\backslash 0\}\,. \tag{4.9}$$

As in the proof of Corollary 4.2, we can reconstruct $y \in S$ from $\frac{\Omega}{\sigma} = i^*_{\varphi^{-1}(\pm h_y)} y$, the projection of y onto $\varphi^{-1}(\pm h_y)^\perp$, and the dot product of y with any vector *not* in $\varphi^{-1}(\pm h_y)^\perp$; but, if $\varphi^{-1}(v) \cdot v \neq 0$ for all v in the image of the Gaussian map of S, then this is determined by the S component.

We thus have established, for $k = 1$,

Corollary 4.3. *If* $\mathbb{D} = \{(s, \theta, \omega) : \theta = \phi(\omega)\}$ *with*

a) $\phi(\omega) \neq \omega$, $\forall \omega \in S^{n-1}$,

b) $\varphi(\omega) = \dfrac{\omega - \phi(\omega)}{\|\omega = \phi(\omega)\|}$ *a diffeomorphism,*

c) $\varphi^{-1}(v) \cdot v \neq 0$, $v \in$ *Gaussian image of S, then* $\alpha|_{\mathbb{D}}$ *determines S.*

Finally, we note that Corollary 4.3 holds for D of the form $\mathbb{D} = \{(s, \theta, \omega) : \theta = \phi(s, \omega)\}$, satisfying

a′) $\varphi(s, \omega) \neq \omega$, $\forall \omega \in S^{n-1}$.

b′) $\varphi(s, \cdot) = \dfrac{\cdot - \phi(s, \cdot)}{\|\cdot - \phi(s, \cdot)\|}$ is a diffeomorphism $S^{n-1} \to S^{n-1}$ for all $s \in \mathbb{R}$.

c′) $\varphi^{-1}(s, v) \cdot v \neq 0$, $\forall v \in$ Gaussian image, $s \in \mathbb{R}$.

We leave the statement of Corollary 4.3 for $1 < k \leqq n$ to the interested reader.

Finally, we show that for determined sets $\mathbb{D}$ of scattering data as above, the principal symbol $\sigma(q)|_{N^*S}$ is determined by the principal symbol $\sigma(\alpha|_{\mathbb{D}})|_{\Lambda_{\mathbb{D}}}$. For simplicity, we work with $\mathbb{D} = \mathbb{B}$, the backscattering. In fact, by Theorem 3.1, the principal symbol of α on $\hat{\Lambda}_- \backslash \hat{L}$ is the same as that of $\rho^* F(u_1) = -\rho^* F \square_-^{-1}(q \cdot \delta)$. Microlocally near Σ_-, the principal symbol of $q(x) \cdot \delta(t - x \cdot \omega)$ (in $I^{\mu + 1 - \frac{n}{2}}(\Lambda_2)$) is

$$\sigma(q)(x, v) \cdot 1(\tau)$$

in the coordinates of (3.13). By Proposition 2.1, the symbol of $\square^{-1}(q \cdot \delta)$ at a point $(x, \tau, \omega; \xi, \tau, \Omega) \in \hat{\Lambda}_- \backslash \hat{L}$ is proportional (by $\sigma(\square^{-1})$) to the symbol of $q \cdot \delta$ at that point of Σ_- on the same bicharacteristics. Since the Fourier integral operators F, ρ^* and $j^*_{\mathbb{B}}$ are elliptic, the symbol

$$\sigma(\alpha|_{\mathbb{B}})(\pm 2y \cdot h_y, \pm h_y; \sigma, \sigma i^*_{\pm h_y} y) \quad (k = 1) \tag{4.10$_1$}$$

or

$$\sigma(\alpha|_{\mathbb{B}})\left(-2y \cdot \frac{v}{|v|}, \frac{v}{|v|}; |v|, |v| i^*_{\frac{v}{|v|}} y\right) \quad (1 < k \leqq n) \tag{4.10$_k$}$$

is proportional to $\sigma(q)(y, v)$; since $\Sigma_- \backslash \Sigma_2$ is dense in Σ_-, we can recover $\sigma(q)$ on all of $N^*S \backslash 0$.

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Communicated by B. Simon