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Ising Model and $N=2$ Supersymmetric Theories

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Abstract. We establish a direct link between massive Ising model and arbitrary massive $N = 2$ supersymmetric QFT's in two dimensions. This explains why the equations which appear in the computation of spin-correlations in the non-critical Ising model are the same as those describing the geometry of vacua in $N = 2$ theories. The tau-function appearing in the Ising model (i.e., the spin correlation function) is reinterpreted in the $N = 2$ context as a new "index". In special cases this new index is related to the Ray–Singer analytic torsion, and can be viewed as a generalization of that to the loop space of Kähler manifolds.

1. Introduction

The two dimensional Ising model was among the first integrable models to be studied [1]. It is a very rich theory and has played a prominent role in the development of integrable hierarchies. In particular the notion of a tau function arose as a result of studying this theory. This line of development arose by the realization [1] that for the non-critical Ising model, which is equivalent to free massive fermions, spin correlations can be computed by relating them to solutions of certain differential equations. For example, for the two point function the corresponding equation turns out to be a special case of Painlevé III. Formalizing these ideas and generalizing them led the Japanese school [2] to a set of differential equations in n variables, whose solutions could be used to construct the n point spin correlation function for the massive free fermion theory; the first example of a *tau* function. From this point they went on to develop the more general notion of a tau function.

More recently, in a seemingly unrelated development, in studying the geometry of vacua of two dimensional $N = 2$ supersymmetric quantum field theories we encountered a set of equations [3] which captures Berry's curvature for the vacuum subsector of the theory. These equations which were derived using the $N = 2$ algebra and relied heavily on the topological nature of these theories were called *topological-anti-topological* or tt* equations. Some aspects of these computations were recently reinterpreted [4] as a way to define and compute a new supersymmetric index: $\mathrm{Tr}(-1)^F F \exp(-\beta H)$. In the simplest example, with just two vacua, the tt* equations turn out to be equivalent to Painlevé III.

This appearance of the same equation in two seemingly unrelated physical problems raises the question of whether there is any deeper relation between the two. Surprisingly it turns out that, with a suitable choice of coordinates, the tt* equations for an arbitrary $N = 2$ theory with n vacua are identical to the equations written in [2] for the n point spin correlation function for the massive Ising model! It is the aim of this paper to shed some light on this connection. In particular the tau function which is naturally defined in the Ising model, should find an interesting interpretation in the $N = 2$ context. We find that the $N = 2$ analog of the tau function is another "index".

In Sect. 2 we review the work on the Ising model following [2]. In particular we see how to rephrase the spin correlation computation as solving a certain cohomological problem. This cohomological description is very useful in connecting it to $N = 2$ supersymmetric theories as the vacua in these theories are also characterized cohomologically. In Sect. 3 we review the tt* equations, and also present an alternative (less rigorous) derivation for them. We then write these equations in coordinates suitable for making contact with those of the Ising model. In Sect. 4 we establish a more direct link between the Ising model and the $N = 2$ theories, by considering the supersymmetric quantum mechanical Landau–Ginzburg models. This provides a dictionary between the two theories and makes manifest why a priori the tt* equations are the same as the equations appearing for the Ising model. Once this is established we use the Ising model correlators to compute the exact ground state wave function for certain SQM problems. In Sect. 5 we discuss the differential geometrical aspects of the tau function which can be constructed from any solution of the tt* equations. In Sect. 6 we discuss how to obtain the tau function via path integrals which paves the way to interpreting tau-function as a new supersymmetric index. This is done by defining it as an integral of the toroidal partition function (with insertion of F^2) integrated over the moduli of torus. In this way we are led to a generalization of the Ray–Singer torsion in the context of *loop space* of Kähler manifolds. This also turns out to be related to computations of moduli dependence of gravitational and gauge coupling constants for string theory [5].

Finally in Sect. 7 we present our conclusions.

2. Ising Correlations as a Problem in Hodge Theory

The theory of $N = 2$ supersymmetric systems is closely related to that of the $2d$ Ising model [1]. This is particularly evident from the "monodromic" approach advocated by the Japanese school [2]. Solving the Ising model is equivalent to computing some cohomology which is a canonical model for the Q-cohomology of a "generic" $N = 2$ model. To establish the dictionary between the two theories, we briefly review the main results of ref. [2] from a viewpoint appropriate to making contact with $N = 2$ susy. (Experts may wish to skip this section.) This will also make clear the relation of the Ising model to Hodge theory [6]. A more physical link between the two theories will be discussed in Sect. 4.

2.1. The Ising Model. The starting point is the well known fact that, in the scaling limit, the Ising model is just a free massive Majorana field $\Psi(z)$ satisfying the

Euclidean Dirac equation

$$(\not{\partial} - m)\Psi = 0 , \tag{2.1}$$

where

$$\not{\partial} = \begin{pmatrix} 0 & \partial \\ \bar{\partial} & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.$$

If Ψ is a solution to (2.1), $\Psi^\star \equiv -\sigma_1 \Psi^*$ is also a solution. Therefore the space of solutions to (2.1) admits a *real structure*: A solution is said to be *real* if $\Psi = \Psi^\star$. Of course, this is just the (Euclidean) Majorana condition. From this point on we will usually set, by a choice of units, $m = 1$.

In addition to $\Psi(z)$, one considers two other fields [7]: The order operator $\mu(z)$ and the disorder one $\sigma(z)$. Insertion of these operators creates a cut in the plane, around which the Fermi field $\Psi(z)$ picks up a sign as it is transported around it. These fields can be written (non-locally) in terms of the fundamental fermion Ψ [2]. One has $\mu = \exp[\varrho]$ with ϱ a (non-local) bilinear in Ψ. Instead σ has the form $\psi \exp[\varrho]$ with ψ an operator linear in Ψ. Then μ is an even element of the Grassmann algebra generated by Ψ while σ is odd. By definition they satisfy $\langle \mu \rangle = 1$ and $\langle \sigma \rangle = 0$. The precise form of these operators can be extracted from the path-integral representation of the order/disorder correlation functions. These functions are equal [8] to the functional integral over a free fermion Ψ having a prescribed monodromy around the points z_i where the order/disorder operators are inserted. After doubling the model by making Ψ complex, this boundary condition can be implemented by coupling its Fermi current to a flat Abelian connection A_μ whose holonomy equals the given monodromy. In this way one gets e.g.

$$\mu(z) = N \exp\left[i \int d^2w \bar{\Psi}(w) \not{A}(w; z) \Psi(w)\right] , \tag{2.2}$$

where N is some normalization coefficient and $A_\mu(w; z)$ is a flat connection on the punctured plane with

$$\oint_C A = \pi \quad \text{for all paths } C \text{ enclosing } z .$$

The disorder operator $\sigma(z)$ is defined similarly, except that one couples A_μ to the axial current [8] (the procedure is rather subtle since the axial current is not conserved). After the path integral is twisted in this way one (and only one) normalizable zero-mode for Ψ appears.[1] Since the zero-mode absorbs a fermion, $\sigma(z)$ becomes an *odd* element of the field algebra.

The relation of the order/disorder fields to the fundamental fermion is conveniently summarized by their OPE with $\Psi(z)$. The operator expansions can be extracted from the above functional representation [8]. Consider the product $\Psi(z)\sigma(w)$ as $z \to w$. By definition of disorder operator (2.2), this product changes sign as we transport z around w. Moreover, as a function of z, it satisfies the Dirac equation (2.1). Let $\phi_i(z)$ be a basis of solutions with the right monodromy around the origin. Then we can expand the above product in this basis. In particular we can use a basis with definite angular momentum[2]. For each $l \in \mathbf{Z}$ there are two such

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[1] The zero-mode wave functions are discussed in detail below

[2] In the first quantized sense

functions $\varphi_l(z)$ and $\varphi^\star{}_l(z)$, whose explicit expression in terms of modified Bessel functions reads

$$\varphi_l(z) \equiv \begin{pmatrix} \zeta_l(z) \\ \zeta_{l+1}(z) \end{pmatrix} = \begin{pmatrix} \exp[i(l-\frac{1}{2})\theta]\, I_{l-\frac{1}{2}}(2r) \\ \exp[i(l+\frac{1}{2})\theta]\, I_{l+\frac{1}{2}}(2r) \end{pmatrix}, \tag{2.3}$$

where $z = r\exp[i\theta]$. Note that for small z

$$\zeta_l \sim z^{l-\frac{1}{2}}\,.$$

Then, the basic OPE are [2]

$$\begin{aligned} \Psi(z)\sigma(w) &= \frac{i}{2}\mu(w)(\varphi_0(z-w) + \varphi_0^\star(z-w)) \\ &\quad + \frac{1}{2}\sum_{l=1}^{\infty}[\mu_l(w)\varphi_l(z-w) + \mu_{-l}(w)\varphi_l^\star(z-w)]\,, \\ \Psi(z)\mu(w) &= \frac{1}{2}\sigma(w)(\varphi_0(z-w) - \varphi_0^\star(w-z)) \\ &\quad - \frac{1}{2}\sum_{l=1}^{\infty}[\sigma_l(w)\varphi_l(z-w) - \sigma_{-l}(w)\varphi_l^\star(z-w)]\,. \end{aligned} \tag{2.4}$$

Here $\sigma_l(z)$ and $\mu_l(z)$ are "descendants" of the basic order/disorder fields. In particular,

$$\mu_1 = i\partial\mu, \quad \mu_{-1} = -i\bar{\partial}\mu, \quad \sigma_1 = \partial\sigma, \quad \sigma_{-1} = \bar{\partial}\sigma\,,$$

as is easily seen by rewriting (2.4) in terms of free fields.

In the space of solutions we have a natural inner product, corresponding to the norm in the (first quantized) Hilbert space (where we take, for later convenience, a factor of $1/2\pi$ as part of the definition of the integral)

$$\|\Psi\|^2 = \int d^2z(|\psi_+|^2 + |\psi_-|^2)\,. \tag{2.5}$$

The main problem in the theory is to compute the wave function induced by the insertion of one disorder and many order operators [2]

$$\Psi^{(j)}(z) = \frac{\langle\Psi(z)\mu(w_1)\ldots\mu(w_{j-1})\sigma(w_j)\mu(w_{j+1})\ldots\mu(w_n)\rangle}{\langle\mu(w_1)\ldots\mu(w_n)\rangle}\,. \tag{2.6}$$

Once these functions are known, the order/disorder correlations can be computed using (2.4). In fact, as $z \to w_j$ we have

$$\begin{aligned} \Psi^{(j)}(z) &= \frac{i}{2}[\varphi_0(z-w_j) - \varphi_0^\star(z-w_j)] - \frac{i}{2}\varphi_1(z-w_j)\partial_{w_j}\log\langle\mu(w_1)\ldots\mu(w_n)\rangle \\ &\quad + \frac{i}{2}\varphi_1^\star(z-w_j)\partial_{\bar{w}_j}\log\langle\mu(w_1)\ldots\mu(w_n)\rangle + O(|z-w_j|^{3/2}) \end{aligned} \tag{2.7}$$

so the function

$$\tau(w_1,\ldots,w_n) \equiv \langle\mu(w_1)\ldots\mu(w_n)\rangle\,, \tag{2.8}$$

can be recovered by quadrature from $\Psi^{(j)}$. From the path integral point of view, τ is just the determinant

$$\tau = \mathrm{Det}[\not{D} - m]\,,$$

where $\not{D}$ is the Dirac operator coupled to the connection $A_\mu^{(n)}$ induced by the insertion of the order fields.

On the other hand, as $z \to w_k$ $(k \neq j)$ we get

$$\Psi^{(j)}(z) = \frac{1}{2}[\varphi_0(z - w_k) - \varphi_0^\star(z - w_k)]\tau_{jk} + \text{non-singular}\,, \tag{2.9}$$

where

$$\tau_{jk} \equiv \frac{\langle \mu(w_1) \ldots \sigma(w_j) \ldots \sigma(w_k) \ldots \mu(w_n)\rangle}{\langle \mu(w_1) \ldots \mu(w_n)\rangle}\,.$$

Since σ is a Grassmann odd operator τ_{jk} is a real skew-symmetric matrix. This is obvious since in view of (2.4) the correlation function of two σ's in the presence of μ's can be obtained by taking the leading singularity of the 2-point function $\langle \Psi(w_j)\Psi(w_k)\rangle_{\mathrm{A}}$, where $\langle \ldots \rangle_{\mathrm{A}}$ denotes expectation values with the fermions coupled to $A_\mu^{(n)}$. Similarly $\langle \sigma(w_1) \ldots \sigma(w_n)\rangle$ can be computed by applying Wick's theorem. This gives the identity [2]

$$\langle \sigma(w_1) \ldots \sigma(w_n)\rangle = \tau \mathrm{Pf}[\tau_{ij}]\,.$$

To get the wave functions $\Psi^{(j)}(z)$ one looks for the multivalued solutions to (2.1) having singularities at the branching points w_k as predicted by the OPE, and behaving as $O(e^{-2|mz|})$ at infinity. To find such functions is the "cohomological" problem which we will elaborate in the next subsection.

There is another Ising quantity which plays a crucial role in the susy case. Let L be the (first quantized) angular momentum. It is a first order differential operator acting on Ψ. The wave functions $\Psi^{(j)}$ have no definite angular momentum since the insertion of order/disorder operators breaks rotation invariance. Nevertheless, $L\Psi^{(j)}$ is again a solution of (2.1) with the same monodromy properties as $\Psi^{(j)}$ and slightly more singular at the w_k, as is easily seen by acting with this differential operator on the local expansions (2.7) (2.9). We define the matrix[3]

$$Q_{ji} = -\langle \Psi^{(j)}|L|\Psi^{(i)}\rangle\,. \tag{2.10}$$

This Q-matrix will correspond to an "universal" supersymmetric index in the $N = 2$ context.

2.2. The Cohomological Problem. In order to emphasize the geometrical meaning[4] of the Euclidean Dirac equation, let us replace $\Psi(z)$ by the 1-form

$$\varpi = \psi_+(z)dz + \psi_-(z)d\bar{z}\,. \tag{2.11}$$

The map $\Psi \mapsto \varpi$ is very natural. First of all, it is consistent with the Hilbert space structure (2.5)

$$\|\Psi\|^2 = \int \varpi \wedge *\varpi^* \tag{2.12}$$

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[3] Notice that this matrix is independent of the choice of the origin in the z-plane

[4] Our discussion here can be seen as the *massive* analog of the well-known fact that, on a Ricci-flat Kähler manifold, the Kähler–Dolbeault equation is equivalent to the massless Dirac equation [10] Th. 2.2. (See also Sect. 3 of [11]). In our case the relevant Calabi–Yau space is just the flat complex plane. It would be interesting to extend the massive result to higher dimensional CY spaces

(again a factor of $1/2\pi$ is taken as part of the definition of the integral). Moreover, in terms of ϖ, the real structure becomes the standard one: Ψ is real in the Majorana sense if $i\varpi$ is real in the usual sense. Finally, the form $\varpi^{(j)}$ associated to the basic wave-function (2.6) is regular everywhere. More precisely, it is regular when pulled back on the branched cover on which Ψ is univalued. This branched cover is just a hyperelliptic surface which is branched over the points where the spin operators are inserted. In fact, $dz/\sqrt{z-w_k}$ is regular at w_k, as is seen by writing it in terms of the local uniformizing parameter $\zeta^2=(z-w_k)$ of the hyperelliptic surface. This regularity condition encodes the essential part of the OPE (2.4). Let χ be the monodromy defined by picking up a minus sign when going around the branch points w_k. Then $\varpi^{(j)}$ may also be seen as a form taking value in the flat vector bundle V_χ defined by the monodromy χ.

Consider the operators $\bar{\mathscr{D}}$ and $\mathscr{D}$ acting on forms as

$$\bar{\mathscr{D}}\alpha=\bar{\partial}\alpha+dz\wedge\alpha,\quad \mathscr{D}\alpha=\partial\alpha+d\bar{z}\wedge\alpha\,. \tag{2.13}$$

They satisfy

$$\bar{\mathscr{D}}^2=\mathscr{D}^2=\bar{\mathscr{D}}\mathscr{D}+\mathscr{D}\bar{\mathscr{D}}=0$$

so $\bar{\mathscr{D}}$ (resp. $\mathscr{D}$) is a cohomology operator. They are consistent with the inner product in the sense that, for $\bar{\mathscr{D}}$-closed *one* forms α, and β, the product $\int\alpha\wedge *\beta$ depends only on their classes.

In terms of forms, Eq. (2.1) reads

$$\bar{\mathscr{D}}\varpi=\mathscr{D}\varpi=0\,.$$

Let Λ be the linear operator which vanishes when acting on 0 or 1 forms and maps the 2-form $ia(z)dz\wedge d\bar{z}$ to the zero-form $a(z)$. Then one has the Kähler identity (familiar from the study of Kähler manifold), which can be easily verified

$$\bar{\mathscr{D}}^\dagger=i[\Lambda,\mathscr{D}]\,.$$

Using this identity, we rewrite the Dirac equation as

$$\bar{\mathscr{D}}^\dagger\varpi=\bar{\mathscr{D}}\varpi=0\,. \tag{2.14}$$

The first equation is equivalent to the first component of the Dirac equation and the second equation is equivalent to the second component. Hence the (normalizable) solutions to (2.1) are just the "harmonic" representatives of the $\bar{\mathscr{D}}$-cohomology in a space of forms satisfying suitable boundary conditions.

By standard Hodge theory, these solutions are in one-to-one correspondence with the $\bar{\mathscr{D}}$-cohomology classes in the same space. Complex conjugation interchanges $\bar{\mathscr{D}}$ with $\mathscr{D}$ and thus maps the $\bar{\mathscr{D}}$-cohomology into the $\mathscr{D}$-cohomology (equivalently the $\bar{\mathscr{D}}^\dagger$ one). Notice that a "harmonic" form ϖ can be seen as a $\mathscr{D}$- or a $\bar{\mathscr{D}}$-cohomology class, the two classes being related by the reality structure.

In particular $\varpi^{(j)}$ is a "harmonic" 1-form representing a cohomology class in the space[5] of (regular) forms with coefficients in V_χ. Let Ω_k be a basis of such

[5] A priori one adds also the condition of fast decay at infinity. It can be shown ([12] App. B) that such restriction is immaterial in cohomology, i.e. each cohomology class has a fast decaying representative

"harmonic" forms. Then

$$\varpi^{(j)} = \sum_k A_k^{(j)} \Omega_k \,, \tag{2.15}$$

for some coefficients $A_k^{(j)}$. To get these coefficients, it is enough to compare the two sides of this equation in *cohomology*.

From (2.11) we see that for a "harmonic" one form[6]

$$\varpi = \bar{\mathscr{D}} \psi_+ \,,$$

where ψ_+ is the first component of the corresponding Dirac spinor. Since in cohomology we are free to add to ϖ any $\bar{\mathscr{D}}\xi$ with ξ regular, the $\bar{\mathscr{D}}$-cohomology class of ϖ is completely encoded in the *singularities* of ψ_+. The only allowed singularities of ψ_+ are of the form $a_k/\sqrt{(z - w_k)}$. So, the $\bar{\mathscr{D}}$-class of ϖ is specified by the numbers $(a_1, a_2, \ldots, a_n)$. The *canonical* basis Ω_k is defined by $a_i(\Omega_k) = \delta_{ik}$. In this basis we get

$$\eta_{kj} = \int \Omega_k \wedge *\Omega_j = \delta_{jk} \,. \tag{2.16}$$

This formula can be proven using residue techniques [13]. Alternatively, in the present case we can give an explicit proof of this which will also be useful for some other applications later in this section. Consider functions ψ_a and ψ_b decaying exponentially fast at infinity which, except possibly at w_i, are regular and around w_i pick up a sign. Then we will show that

$$\int \bar{\mathscr{D}} \psi_a \wedge *\bar{\mathscr{D}} \psi_b = \sum_{w_i} \oint_{w_i} dz \psi_a \psi_b \,, \tag{2.17}$$

where $\oint_{w_i}$ is a contour integral taken in an infinitesimal neighborhood of w_i (with a factor $(1/(2\pi i)$ included). The proof is quite simple. We simply use the definition (2.13) and expand the above, and use the definition of $*$ to obtain

$$\int \bar{\mathscr{D}} \psi_a \wedge *\bar{\mathscr{D}} \psi_b = -i \int \bar{\partial}(\psi_a \psi_b) dz \,.$$

Using Stoke's theorem, applied to the complex plane with infinitesimal discs cut out around w_i where ψ's are singular, we end up with Eq. (2.17). Now we can apply this result to $\psi_a = \psi_{k+}$ and $\psi_b = \psi_{j+}$ we obtain the result (2.16).

Now let us consider the natural hermitian metric of the Hilbert space in the canonical basis

$$g_{i\bar{j}} = \int \Omega_i \wedge *\Omega_j^* \,.$$

Comparing with (2.16), we see that $\Omega_j^* = \sum_i g_{i\bar{j}} \Omega_i$, i.e. the matrix g represents the real structure in the canonical basis. Then we must have (by conjugating twice)

$$g^{-1} = g^* = g^T \,.$$

The Ising correlations can be extracted from the metric g as we will now show. We can read the cohomology classes of $\varpi^{(j)}$ from (2.7) and (2.9). Comparing with (2.15), this gives

$$A_k^{(j)} = i \frac{1}{2} \delta_{jk} + \frac{1}{2} \tau_{jk} \,.$$

Using reality of $i\varpi^{(j)}$ we obtain

$$A_k^{(j)*} = -A_k^{(j)} g_{k\bar{l}}$$

[6] In fact, in the present context, only one forms can be harmonic

which leads (using reality of τ_{ij}) to

$$A_k^{(j)} g_{k\bar{l}} = i\frac{1}{2}\delta_{jl} - \frac{1}{2}\tau_{jl} ,$$

which gives $A_k^{(j)}$ and τ_{jk} in terms of g

$$A_k^{(j)} = i(1+g)_{jk}^{-1} , \tag{2.18}$$

$$\tau_{jk} = (A - Ag)_{jk} = i[(1+g)^{-1}(1-g)]_{jk} . \tag{2.19}$$

To complete the computation of the correlators, we need to compute the τ function (2.8). Let $\Omega_k = i\bar{\mathscr{D}}\psi_{k+}$. Near w_j, ψ_{k+} has an expansion of the form

$$\psi_{k+}(z) \simeq \delta_{kj}\zeta_0(z - w_j) + \hat{Z}_{kj}\zeta_1(z - w_j) - \sum_l g_{k\bar{l}}[\delta_{lj}\zeta_1(z - w_j)^* + \tilde{Z}_{lj}^*\zeta_2(z - w_j)^*] + \ldots , \tag{2.20}$$

where we have used the Majorana condition and the fact that complex conjugation of the basis is equivalent to multiplication by g. Comparing with (2.7) we see that $(\partial_j \equiv \partial_{w_j})$

$$(A\tilde{Z})_{jj} = [i(1+g)^{-1}\tilde{Z}]_{jj} = -\frac{i}{2}\partial_j \log\tau . \tag{2.21}$$

Since, as we are going to show, the matrix $\tilde{Z}$ can also be expressed in terms of g, the τ function can be recovered from g by quadratures once we know how to compute g.

It remains to compute g and $\tilde{Z}$ as a function of $w_1, \ldots, w_n$. The idea is to study the differential geometry of the hermitian metric g. In this way we are led to a rather standard problem[7] in "variation of Hodge structures" [6]. There is a natural *metric connection* for g

$$D_i = \partial_i - A_i, \quad \bar{D}_j = \partial_{\bar{j}} - \bar{A}_j ,$$

where the connection is defined by

$$(A_i)_{kl} = \int \partial_i\Omega_k \wedge *\Omega_l ,$$

$$(\bar{A}_i)_{kl} = \int \partial_{\bar{i}}\Omega_k \wedge *\Omega_l . \tag{2.22}$$

It is easy to check with this definition that $D_i g = \bar{D}_i g = 0$. Also, in the basis we have chosen we will now show that the anti-holomorphic components of the connection vanishes,

$$(\bar{A}_i)_{kl} = \int \partial_{\bar{i}}\Omega_k \wedge *\Omega_l \equiv 0 . \tag{2.23}$$

To prove this we apply the result (2.17) with the substitutions $\psi_a = \partial_{\bar{i}}\psi_{k+}$ and $\psi_b = \psi_{l+}$. Since the singularity of ψ_a near w_k is of the form $\delta_{ik}/\sqrt{\bar{z} - \bar{w}_k}$ and that of ψ_b is of the form $\delta_{kl}/\sqrt{z - w_k}$, using (2.17) we end up with

$$(\bar{A}_i)_{kl} = \delta_{ik}\delta_{kl} \oint_{w_k} \frac{dz}{|z - w_k|} = 0 .$$

[7] One can also reduce to a problem of "isomonodromic deformations" as in [2]. The two mathematical theories are closely related [14]. However, in the present language the underlying ring structure is more transparent. In the susy case the ring is the central object of interest

Using this result and using the fact that $D_i g = 0$ we get,

$$A_i = -g\partial_i g^{-1} . \tag{2.24}$$

From (2.20) we have

$$\partial_i \psi_{k+} = -\delta_{kj}\delta_{ji}\zeta_{-1}(z) - \tilde{Z}_{kj}\delta_{ji}\zeta_0(z) + \text{regular as } z \to w_j .$$

Using this, and evaluating the integral (2.22) using (2.17) we get

$$(A_i)_{kl} = \tilde{Z}_{kl}(\delta_{ik} - \delta_{il}) .$$

It is convenient to introduce the matrices $(C_i)_l^k \equiv \delta_{il}\delta_l^k$. Then the connection is rewritten as

$$A_i = [C_i, \tilde{Z}] . \tag{2.25}$$

Comparing with (2.24), we get a formula for off-diagonal part of the matrix $\tilde{Z}$ in terms of the metric g. In particular, $\tilde{Z}^T = \tilde{Z}$. Moreover we have the identity

$$[C_j, A_i] = [C_i, A_j] , \tag{2.26}$$

which follows from (2.25), the Jacobi identity, and the fact that C_i commute with one another.

We still need to compute the diagonal part of $\tilde{Z}$. It is convenient to define the matrix $\bar{C}_{\bar{i}} \equiv gC_i^\dagger g^{-1}$. Then one has the identity

$$\partial_{\bar{i}}\Omega_k + (\bar{C}_{\bar{i}})_k^l \partial_{\bar{z}}\Omega_l = 0 . \tag{2.27}$$

To show this, note that the left-hand side of the above equation is clearly "harmonic" (as $\partial_{\bar{z}}$ and $\partial_{\bar{i}}$ commute with $\bar{\mathscr{D}}$ and $\mathscr{D}$). Moreover using the expansion (2.20), it is easy to see that the left-hand side is a regular "harmonic" form (the singularities cancel between the two terms). Since it is regular (i.e. it has no singularities), it represents the trivial class, and hence vanishes identically. Setting to zero the coefficient of ζ_1, we get

$$\bar{\partial}_i\tilde{Z} = -\bar{C}_{\bar{i}} + C_i^\dagger . \tag{2.28}$$

Let us compute the curvature $R_{i\bar{j}}$ of the above connection. Recalling that $\bar{A} = 0$ one has

$$R_{i\bar{j}} = \partial_{\bar{j}}A_i = [C_i, \partial_{\bar{j}}\tilde{Z}] = -[C_i, \bar{C}_{\bar{j}}] .$$

Using the explicit form of the connection (2.24), this becomes a differential equation for the metric g,

$$\partial_{\bar{j}}(g\partial_i g^{-1}) = [C_i, \bar{C}_j] . \tag{2.29}$$

This is the basic equation governing the correlation functions (if we restored m, there will appear a factor m^2 on the rhs of the above equation).

Let us introduce some short-hand notation,

$$C = \sum_k w_k C_k, \quad \Gamma = \sum_k C_k dw_k, \quad \bar{C} = gC^\dagger g^{-1}, \quad \bar{\Gamma} = g\Gamma^\dagger g^{-1} . \tag{2.30}$$

Then consider the expression

$$(L + C\partial_z - \bar{C}\partial_{\bar{z}})_k^l \Omega_l ,$$

where $L = w_i\partial_i - \bar{w}_i\partial_{\bar{i}} + \frac{1}{2}\sigma_3$ is the angular momentum operator. Then again this is a "harmonic" form which is regular in the sense of the discussion after Eq. (2.12) and hence it is of the form $Q_k^l\Omega_l$ for some matrix Q_k^l. Comparing with (2.10), we see that this matrix is the one defined in (2.10) (note that with the flat metric (2.16) we can lower or raise indices of Q and that Q is antisymmetric). Moreover, using (2.22) and (2.23) we get

$$Q = -\sum_k w_k A_k = [\tilde{Z}, C] \equiv \sum_k w_k g \partial_k g^{-1} . \tag{2.31}$$

From (2.26) one has $[C, g\partial g^{-1}] = [\Gamma, Q]$. This allows to write the connection in terms of Q alone,

$$(g\partial g^{-1})_{kl} = Q_{kl} \frac{d(w_k - w_l)}{w_k - w_l} .$$

This fact can be used to reduce the number of independent equations in (2.29).

From (2.21) we have

$$\partial \log \tau = -2\mathrm{tr}[\Gamma(1+g)^{-1}\tilde{Z}] .$$

Now we use the fact that

$$\mathrm{tr}\, M = \frac{1}{2}(\mathrm{tr}\, M + \mathrm{tr}\, M^t)$$

for any M and use the fact that $\tilde{Z}^t = \tilde{Z}$, $\Gamma^t = \Gamma$ and that $g^t = g^{-1}$ and (2.25) to get

$$\begin{aligned} \partial \log \tau &= \mathrm{tr}(\Gamma(1+g)^{-1}\tilde{Z} + \tilde{Z}(1+g^{-1})^{-1}\Gamma) \\ &= \mathrm{tr}(\Gamma[(1+g)^{-1} + (1+g^{-1})^{-1}]\tilde{Z} + (1+g^{-1})^{-1}[\Gamma, \tilde{Z}]) \\ &= \mathrm{tr}\,\Gamma\tilde{Z} + \mathrm{tr}(1+g^{-1})^{-1}(-g\partial g^{-1}) \\ &= \mathrm{tr}\,\Gamma\tilde{Z} + \mathrm{tr}(1+g)^{-1}\partial g \\ &= \mathrm{tr}\,\Gamma\tilde{Z} + \partial \mathrm{tr} \log(1+g) . \end{aligned} \tag{2.32}$$

We will now show that

$$\tilde{Z} = -\bar{C} + C^\dagger + \frac{1}{2}[Q, \tilde{Z}] + \text{off-diagonal} . \tag{2.33}$$

Let Δ be the difference between the lhs and the rhs of the above equation. Now let us compute $\bar{\partial}_i \Delta$. We will need

$$\bar{\partial}_i Q = [C, \bar{C}_i] \tag{2.34}$$

which follows from (2.29) and

$$\bar{\partial}_i \bar{C} = \bar{C}_i - [Q, \bar{C}_i] . \tag{2.35}$$

To show (2.35) we first use (2.26) to get

$$[Q, C_i] = -[C, g\partial_i g^{-1}] .$$

We then note that $Q^* = -g^{-1}Qg$ which follows from rotational invariance of the theory $Lg = 0$ and $g^* = g^{-1}$. Taking the complex conjugate of the above equation and noting that $C_i = C_i^* = C_i^\dagger$ we get

$$g^{-1}[Q, \bar{C}_i]g = -[g^{-1}\bar{\partial}_i g, C^\dagger] = -g^{-1}(\bar{\partial}_i \bar{C} - \bar{C}_i)g ,$$

which gives Eq. (2.35).

Finally we are in a position to compute $\bar{\partial}_i \Delta$. We have from (2.28),

$$\bar{\partial}_i \Delta = -(\bar{C}_i - C_i^\dagger) + \bar{\partial}_i \bar{C} - \bar{\partial}_i C^\dagger - \frac{1}{2}[\bar{\partial}_i Q, \tilde{Z}] + \frac{1}{2}[Q, \tilde{C}_i] - \frac{1}{2}[Q, C_i^\dagger] .$$

Now using (2.34) and (2.35) (and noting $\bar{\partial}_i C^\dagger = C_i^\dagger$) we have

$$\bar{\partial}_i \Delta = -\frac{1}{2}[[C, \bar{C}_i], \tilde{Z}] - \frac{1}{2}[Q, \bar{C}_i] - \frac{1}{2}[Q, C_i^\dagger] .$$

Using the Jacobi identity on the first term and using (2.31) we get

$$\bar{\partial}_i \Delta = \frac{1}{2}[[\bar{C}_i, \tilde{Z}], C] - \frac{1}{2}[Q, C_i^\dagger] .$$

The rhs of the above equation has vanishing diagonal elements, since C and $C_i^\dagger$ are diagonal matrices and $[M, D]$ has vanishing diagonal elements for any matrices D and M with D diagonal. So we have now shown that the difference between the diagonal elements of both sides of (2.33) is at most a holomorphic function in w_i's. Since everything falls off exponentially fast as any of $w_i \to \infty$, the diagonal elements of Δ die off at least exponentially fast in this limit, and this implies, since diagonal elements of Δ are holomorphic, that they are identically zero. This concludes showing (2.33).

Now let us define $\tilde{\tau}$ by

$$\log \tilde{\tau} = \log \tau - \log \det(1 + g) ,$$

then using (2.33) and (2.32) and noting that Γ is a diagonal matrix (and using the reality of $\log \tau$) we get

$$d \log \tilde{\tau} = -\mathrm{tr}[\bar{C}\Gamma] + \mathrm{tr}[C^\dagger \Gamma] + \frac{1}{2}\mathrm{tr}(Qg\partial g^{-1}) + \text{c.c.} \tag{2.36}$$

We stress that the rhs of (2.36) is a closed form for all solutions to (2.29). So to each solution we can associate (locally) a τ-function. Of course, only a particular solution of these equations corresponds to the actual correlation functions for the Ising model.

3. The Geometry of $N=2$ Ground States

3.1. The tt* *Equations.* In this section we review the main results of ref. [3]. We consider a general $N = 2$ susy model in two dimensions. A remarkable fact about such models is that they can be "twisted" into a topological field theory (TFT) [15]. This is done by gauging the Fermi number F with the gauge field set equal to one-half the spin-connection. After the twisting, two supercharges Q^+ and $\bar{Q}^+$ become scalars and are consistently interpreted as BRST charges. The observables ϕ_i of the TFT are the operators commuting with these supercharges modulo those that can be written as Q^+ or $\bar{Q}^+$-commutators. In technical terms, the ϕ_i's correspond to the Q^+-cohomology in the space of quantum operators. In the same way the physical states are associated to the Q^+-cohomology in the Hilbert space, i.e.

$$Q^+ |\text{phy}\rangle = \bar{Q}^+ |\text{phy}\rangle = 0, \quad |\text{phy}\rangle \sim |\text{phy}\rangle + Q^+ |\xi\rangle + \bar{Q}^+ |\eta\rangle .$$

The topological observables ϕ_i generate a commutative[8] ring $\mathscr{R}$

$$\phi_i \phi_j = C_{ij}^k \phi_k . \tag{3.1}$$

The ring $\mathscr{R}$ is called the *chiral ring* [16] since, in the original $N = 2$ model, the fields ϕ_i commuting with Q^+ and $\bar{Q}^+$ are just the chiral ones. The two-point function on the sphere serves as a metric for the topological theory

$$\eta_{ij} = \langle \phi_i \phi_j \rangle .$$

If we take the space to be a circle, the physical states are in one-to-one correspondence with the physical operators. Explicitly, the relation reads

$$|\phi_i\rangle = \phi_i |1\rangle ,$$

where $|1\rangle$ is a canonical state whose wave-functional is obtained by filling the circle with a disk and performing the topological path integral over this disk[9]. Then the state $|\phi_i\rangle$ is represented by the path integral with the topological observable ϕ_i inserted at some point on the disk. This shows that

$$\phi_i |\phi_j\rangle = C_{ij}^k |\phi_k\rangle . \tag{3.2}$$

It turns out that all the correlation functions can be computed in terms of C_{jk}^l and η_{jk} [15].

What is the connection of the topological model with the original untwisted $N = 2$ model? Well, if we put our $N = 2$ model on a (say) flat periodic cylinder, the spin-connection vanishes and hence the twisting does not change the functional measure. In this case, the only difference between the two theories is in the definition of the observable and physical states. In the twisted case they are required to be Q^+-closed. But then, if we restrict ourselves to Q^+-closed objects in the $N = 2$ theory the results for the twisted and untwisted models are equal.

Now consider the ground states $|i\rangle$ of the untwisted theory. They satisfy,

$$Q^+|i\rangle = \bar{Q}^+|i\rangle = (Q^+)^\dagger|i\rangle = (\bar{Q}^+)^\dagger|i\rangle = 0 . \tag{3.3}$$

Comparing with the topological theory, we see the vacua are nothing else than the "harmonic" representatives of the BRST-classes. Hence there is a one-to-one correspondence between the $N = 2$ vacua and the TFT physical states $|\phi_i\rangle \mapsto |i\rangle$. For future reference, we describe the path integral realization of this map. Just glue to the boundary of the disk of perimeter β a flat cylinder with the same perimeter and of length T. This does not change the topological state; but from the $N = 2$ viewpoint this is the state $\exp[-HT]|\phi_i\rangle$. As $T \to \infty$, $\exp[-HT]$ becomes a projector on the ground states, and we get the unique vacuum in the $|\phi_i\rangle$ topological class. In particular from (3.2) we have

$$\phi_i |j\rangle = C_{ij}^k |k\rangle \quad \text{mod. positive energy states} .$$

Of course, we could as well have gauged F with *minus* one-half the spin connection. In this case the other two supercharges $Q^- \equiv (Q^+)^\dagger$ and $\bar{Q}^- \equiv (\bar{Q}^+)^\dagger$ are interpreted as BRST charges. This is the so-called *anti-topological* model [3].

[8] If some ϕ_i have odd F, $\mathscr{R}$ is only *graded-commutative*. To keep notations as simple as possible, we assume all ϕ_i to have even F. The extension to the general case is straightforward

[9] For a detailed discussion, see e.g. [3]

The new observable operators $\bar{\phi}_i$ are the CPT conjugates (in the $N = 2$ sense) of the ϕ_i's. Then,

$$\bar{\phi}_i \bar{\phi}_j = C_{ij}^{*k} \bar{\phi}_k .$$

Again there is a natural isomorphism of $\bar{\mathscr{R}}$ with the anti-topological physical states[10] $|\bar{\phi}_i\rangle \equiv \bar{\phi}_i|\bar{1}\rangle$.

From (3.3) we see that the ground states are "harmonic" representatives of both Q^+ and Q^- cohomology. Then the above construction gives us *two* preferred basis in the space of ground states: The topological (or *holomorphic*) one $|i\rangle$, and the anti-topological (or *anti-holomorphic*) one $|\bar{i}\rangle$. The object of primary interest is the hermitian metric[11]

$$g_{i\bar{j}} = \langle \bar{j} | i \rangle , \tag{3.4}$$

which intertwines between the two "topological" bases. It is an highly non-trivial quantity. On the contrary, $\langle i | j \rangle$ is an "elementary" topological object

$$\langle i | j \rangle \equiv \langle \phi_i | \phi_j \rangle = \langle \phi_i \phi_j \rangle_{\text{top.}} = \eta_{ij} . \tag{3.5}$$

Comparing (3.4) and (3.5), we see that the reality structure acts on the vacua as the matrix $g\eta^{-1}$, then CPT implies

$$g\eta^{-1}(g\eta^{-1})^* = 1 . \tag{3.6}$$

Many interesting non-perturbative information about the $N = 2$ model can be extracted from g (see e.g. [3, 4, 17–20]). The main point here is to get a set of differential equations for g. As in Sect. 2, the idea is to study its differential geometry. These equations (known as the tt* equations) were proven with different techniques in refs. [3], [4], and [19]. Here we give a very quick, (and somewhat unrigorous[12]) argument.

The precise framework is the following. We consider a family of $N = 2$ models whose action has the form

$$S = S_0 + \sum_i t_i \int \{Q^-, [\bar{Q}^-, \phi_i]\} + \sum_i \bar{t}_i \int \{Q^+, [\bar{Q}^+, \bar{\phi}_i]\} , \tag{3.7}$$

where t_i are couplings[13] parametrizing the family, and we look for the function $g_{i\bar{j}}(t, \bar{t})$.

As before we introduce the *metric* connection $D_i = \partial_i - A_i$ and $\bar{D}_i = \bar{\partial}_i - \bar{A}_i$, where

$$(A_i)_{ab} = \langle a | \partial_i | b \rangle, \quad (\bar{A}_i)_{ab} = \langle a | \bar{\partial}_i | b \rangle .$$

Clearly g is covariantly constant with respect to these connections.

To get the differential equations for g we need to compute the curvature $R_{i\bar{j}}$ of the connection. Using first order perturbation theory (Eq. (43.6) of [21]), we have

$$D_i |k\rangle = -H^{-1}\mathscr{P}\partial_i H |k\rangle \equiv H^{-1}\mathscr{P}\{Q^-, [\bar{Q}^-, \oint \phi_i]\} |k\rangle ,$$

[10] $|\bar{1}\rangle$ is defined in analogy with $|1\rangle$ but this time with respect to the *anti-topological* path integral

[11] We adhere to the topological conventions. Then the adjoint of $|\phi_i\rangle$ is $|\bar{\phi}_i|$ not $\langle \phi_i|$

[12] People concerned with questions of rigor should refer to [3]

[13] From the TFT viewpoint it may be natural to sum in (3.7) on all topological fields. However from the QFT viewpoint it is safer to sum only over *marginal* and *relevant* operators which lead to renormalizable field theory. This "conservative" assumption is implicit throughout the paper

where $\mathscr{P}$ is the projection on states of *positive* energy and $\oint$ denotes integration over the circle in which we quantize the model. Note that by definition $\langle l|D_i|k\rangle = 0$ which implies that

$$\bar{D}_j(\langle l|D_i|k\rangle) = (\bar{D}_j\langle l|)D_i|k\rangle + \langle l|\bar{D}_j(D_i|k\rangle) = 0$$

which implies that

$$(\bar{D}_j\langle l|)D_i|k\rangle = -\langle l|\bar{D}_j(D_i|k\rangle)\,.$$

Using this and the similar result with D and $\bar{D}$ exchanged we get

$$\begin{aligned}(R_{i\bar{j}})_{kl} &= \langle l|\{Q^-,[\bar{Q}^-,\oint\phi_i]\}\mathscr{P}H^{-2}\{Q^+,[\bar{Q}^+,\oint\bar{\phi}_j]\}|k\rangle\\ &\quad -\langle l|\{Q^+,[\bar{Q}^+,\oint\bar{\phi}_j]\}\mathscr{P}H^{-2}\{Q^-,[\bar{Q}^-,\oint\phi_i]\}|k\rangle\,.\end{aligned}$$

Consider e.g. the first term. Notice that only $P = 0$ intermediate states contribute (there is no momentum flowing in). Now, using the fact that the vacua are annihilated by Q's and commuting Q^- and $\bar{Q}^-$ across the other operators, and using the susy algebra (with $P = 0$), we get $\langle l|\oint\phi_i\mathscr{P}\oint\bar{\phi}_j|k\rangle$. The second term is obtained by $\oint\phi_i\leftrightarrow\oint\bar{\phi}_j$. Given that

$$\mathscr{P} = 1 - |k\rangle\eta^{kl}\langle l|\,,$$

the curvature becomes

$$(R_{i\bar{j}})_{kl} = \langle l|[\oint\phi_i,\oint\bar{\phi}_j]|k\rangle - \beta^2[C_i,\bar{C}_j]_{kl}\,, \tag{3.8}$$

where β is the perimeter of the circle, and the matrices C_i and $\bar{C}_j$ are defined by[14]

$$\phi_i|k\rangle = (C_i)_k^l|l\rangle,\quad \bar{\phi}_j|k\rangle = (\bar{C}_j)_k^l|l\rangle\,.$$

The first term in the rhs of (3.8) vanishes since ϕ_i and $\bar{\phi}_j$ commute at *equal time*. We can now choose the holomorphic gauge which happens to be just the topological basis for the vacua. In this case we have $\bar{A}_i = 0$. Here we used the fact, manifest from (3.7), that a variation of $\bar{t}$ does not change the Q^+-class of a state). Then the connection has the canonical form (2.24), $A_i = -g\partial_i g^{-1}$. On the other hand, comparing with the topological (resp. anti-topological) theory (3.2), we get

$$\begin{aligned}(C_i)_k^l &\equiv C_{ik}^l,\quad \bar{C}_j = gC_j^\dagger g^{-1}\,,\\ &\Rightarrow D_i\bar{C}_{\bar{j}} = \bar{D}_{\bar{i}}C_i = 0\,.\end{aligned} \tag{3.9}$$

Putting everything together, we get the tt* equations for the metric g,

$$\bar{\partial}_j(g\partial_i g^{-1}) = \beta^2[C_i, gC_j^\dagger g^{-1}]\,. \tag{3.10}$$

Notice that the Ising equations (2.29) are just a special instance of the tt* equations, corresponding to a chiral ring $\mathscr{R}$ with

$$C_{ij}^k = \delta_{ij}\delta_j^k,\quad (\beta\leftrightarrow m)\,.$$

Below we shall see that this special case is in fact "generic" and is true for arbitrary $N = 2$ theory (by a choice of coordinates). The same line of argument as in the computation of curvature also shows that

$$D_jC_i = D_iC_j \Rightarrow \partial_iC_j - \partial_jC_i + [g\partial_i g^{-1}, C_j] - [g\partial_j g^{-1}, C_i] = 0 \tag{3.11}$$

[14] These equations are understood to hold up to states of positive energy

which in the Ising case will be equivalent to (2.26). To complete the analogy, we note the equality

$$\tilde{Z} = Z + C^{\dagger} , \tag{3.12}$$

where Z is the matrix defined for an arbitrary $N = 2$ model in Appendix C of [3]. There it is also discussed its connection to the *flat coordinates* of TFT [22].

On the space of couplings there is a special vector $v = v^i \partial_i$ which generates an infinitesimal change of scale. Then the renormalization group (RG) is generated by the differential operator

$$\mathscr{L}_v + \bar{\mathscr{L}}_{\bar{v}} + D\text{-term beta function} ,$$

where $\mathscr{L}_v$ denotes the (covariant) Lie derivative, acting on forms as

$$\mathscr{L}_v = Di(v) + i(v)D, \quad (\text{here } D \equiv dt^i D_i) .$$

The ring $\mathscr{R}$, being topological, is not renormalized. Then the RG group should act by ring automorphisms, i.e. ther exists a matrix Q such that the one-form Γ is defined in (2.30) transforms as

$$\mathscr{L}_v \Gamma = \Gamma + [Q, \Gamma], \quad \bar{\mathscr{L}}_{\bar{v}} \bar{\Gamma} = \bar{\Gamma} - [Q, \bar{\Gamma}] . \tag{3.13}$$

The first term on the right hand side comes from the fact that the only effect of the RG group is to rescale the topological operators by an overall term e^{τ} where τ is the RG time. This is a simple consequence of dimensional analysis of the (F-term part of the) action which remain unrenormalized. Needless to say the first term could be gotten rid of by multiplying Γ with $e^{-\tau}$.

Since the RG flow preserves the topological metric η,

$$(Q\eta) + (Q\eta)^T = 0 .$$

Given that D is also a metric connection for η, we have $D = D_\eta + T$, where D_η is the η-Cristoffel connection and T is the η-torsion, i.e. the skew-symmetric (with respect to η) part

$$T\eta = \frac{1}{2}[g\partial g^{-1}\eta - (g\partial g^{-1}\eta)^T] .$$

Then $\mathscr{L}_v = \mathscr{L}_v^\eta + i(v)T + Ti(v)$. $\mathscr{L}_v^\eta$ is the topological RG flow. Since this flow is trivial ($\mathscr{L}_v^\eta \Gamma = \Gamma$), we have

$$\mathscr{L}_v \Gamma = \Gamma + i(v)[T, \Gamma] + [T, i(v)\Gamma] = [i(v)T, \Gamma] .$$

Then

$$Q = i(v)T = v^i g \partial_i g^{-1} \Big|_{\eta\text{-skewsymmetric part}} . \tag{3.14}$$

This equation also shows that Q is hermitian with respect to the metric g (thus the second equation in (3.13) follows from the first). Equation (3.14) should be compared with (2.31). It is manifest that the two results agree provided $v = \sum_k w_k \partial_k$. Then we generalize (2.30) by putting $C = i(v)\Gamma$. The matrix C is known as the "superpotential". In Appendix C of [3] is also shown that (cf. (2.25) and (2.31))

$$T = \beta[Z, \Gamma] \Rightarrow Q = \beta[Z, C] . \tag{3.15}$$

As in the Ising case, (3.15) allows writing the connection in terms of Q only, leading to a simplification of the tt* equations. For a more explicit connection with the Ising model see the next subsection, where we consider $N = 2$ LG models.

The matrix Q has also other physical interpretations. Using the general $N = 2$ Ward identity relating dilatations and axial $U(1)$ rotations one shows [3]

$$Q_{kl} = -\frac{1}{2}\langle l|Q^5|k\rangle ,$$

where Q^5 is the (in general non-conserved) axial charge

$$Q^5 = \oint \bar{\psi}\gamma_0\gamma^5\psi .$$

Whenever there is a conserved R-charge, the eigenvalues of the matrix Q_{kl} agree with the values of the conserved charge on the ground states, even if the charge Q^5 itself *is not conserved*[15]. At criticality, there is always a conserved R-charge. The charges q_k of the Ramond ground states are related to the dimension of the primary chiral operators [16] as $h_k = (q_k - q_0)/2$, where q_0 is the smallest q_k. In particular, the central charge is given by $\hat{c} = 2|q_0|$. Then we can see the eigenvalues of Q as a generalization off-criticality of the conformal dimensions h_k and central charge $\hat{c}$. Indeed, these eigenvalues are stationary at a conformal point where they agree with the conformal charges q_k. Thus Q is an alternative to Zamolodchikov's c-function [23] which has the advantage of being computable from the tt* equations[16].

There is another, more interesting, physical interpretation of Q. Indeed Q is a new susy index [4]. To see this, let us quantize the $N = 2$ model on a line instead of a circle. We make the genericity assumption that the theory has a mass-gap and multiple vacua. Then the Hilbert space contains solitonic sectors interpolating between two *distinct* vacuum configurations, k and l, at $x = \pm\infty$ respectively [24]. For concreteness, we define the theory in a segment of length L and take the thermodynamical limit at the end. Then a "modular transformation" shows [4]

$$Q_{kl} = \lim_{L\to\infty} \frac{i\beta}{2L}\,\mathrm{Tr}_{(k,l)}[(-1)^F F e^{-\beta H}] ,$$

where F is the (conserved) Fermi charge and $\mathrm{Tr}_{(k,l)}$ denotes the trace over the solitonic sector specified by the boundary conditions (k, l). Q is an index in the sense that it is independent of the D-term. This is also the reason why it is computable. This "thermodynamical" quantity receives non-trivial contributions from multi-soliton sectors. Then from Q we can extract information about the mass spectrum, the soliton interactions, the Fermi number fractionalization, the RG properties of the model, etc.

3.2. The Canonical Coordinates. Comparing Sects. 2.2 and 3.1 we see that the tt* equations for a family of $N = 2$ models have the same form as the Ising ones (2.29) provided i) all fields ϕ_i have $F = 0$, and ii) we can find new coordinates $w_k(t_i)$ in which the ring coefficients take the form $C_{ij}^k = \delta_{ij}\delta_j^k$. The first condition is

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[15] The analog in the Ising case is that Q is equal to the conserved angular momentum even if we have rotational invariance around a point distinct from the origin

[16] One can also show that for a conformal $N = 2$ family $g/g_{1\bar{1}}$ is equal to the Zamolodchikov metric [3]

automatically satisfied for unorbifoldized LG models. On the other hand, it is a basic geometric consequence of the TFT axioms that such *canonical coordinates* exists under the additional assumption that no element of $\mathcal{R}$ is nilpotent [25, 26]. Physically, this means we have a mass-gap[17].

Since we are primarily interested in the *physics* of this correspondence, we begin by discussing a concrete class of models, i.e. the Landau–Ginzburg models (LG) [27] with a polynomial superpotential $W(X)$. For these theories there is a direct physical identification with the Ising model, explained in Sect. 5 below.

The Family of the A_n-Minimal Model. Without loss of generality, we take a superpotential of the form

$$W(X; t_k) = \frac{1}{n+1} X^{n+1} + t_{n-1} X^{n-1} + \ldots + t_1 X + t_0 \,. \tag{3.16}$$

For generic t_k's, $W(X; t_k)$ is a Morse function[18]. We assume we are in this generic situation. As (local) coordinates in the family we take the critical values $w_i \equiv W(X_i; t_k)$, where X_i are the critical points of $W(X)$ defined by

$$W'(X_i; t_k) = 0 \,.$$

We work in the *canonical* basis for the ground states. This is the "point basis" of [3] but normalized such that $\eta = 1$. An element $|j\rangle$ of this basis belongs to the topological class $|f_j(X)\rangle$, where the $f_j(X)$'s are polynomials such that[19]

$$f_j(X_i) = \delta_i^j \sqrt{W''(X_i)} \,.$$

The canonical basis diagonalizes $\mathcal{R}$

$$g(X) f_j(X) = g(X_j) f_j(X) \quad \text{in } \mathcal{R} \,.$$

Let us compute the action of the observable

$$\mathcal{O}_i \equiv \frac{\partial W(X; t_k)}{\partial w_i} \,,$$

in the canonical basis. Using that $W'(X_i; t_k) = 0$ in $\mathcal{R}$, we get

$$\mathcal{O}_i |j\rangle \equiv (C_i)_j^k |k\rangle = \delta_{ij} \delta_j^k |k\rangle \,, \tag{3.17}$$

i.e. the critical values w_k are just the canonical coordinates we look for. Adding a constant to the superpotential $W(X)$ does not change the physics. Then from $\sum_j \frac{\partial}{\partial w_j} = n \frac{\partial}{\partial t_0}$, it follows that

$$\sum_j \frac{\partial g}{\partial w_j} = 0 \,,$$

$$\sum_j \left(w_j \frac{\partial g}{\partial w_j} - \bar{w}_j \frac{\partial g}{\partial \bar{w}_j} \right) = 0 \,, \tag{3.18}$$

---

[17] However, the presence of a mass-gap may be a slightly stronger requirement than the semi-simplicity of $\mathcal{R}$

[18] I.e. all critical values are non-singular and all critical values are distinct

[19] This condition fixes $f_j(X) \in \mathcal{R}$ uniquely

where the second equation follows from the independence of g from the overall phase of the superpotential[20]. The $N = 2$ Ward Identities (3.18) just express the fact that $g(w_1, \ldots, w_n)$ is Euclidean invariant in the W-plane, i.e. that our tt* metric has the symmetry properties required for the n-point function of a 2d QFT! Then g can be written in terms of the Euclidean invariants of the points w_i. These invariants are objects of direct physical interest. They are the lengths $|w_i - w_j|$ of the segments[21] connecting two points and the angles between these segments. The first invariant is just one-half the mass of the soliton interpolating between the two vacua i and j [24], and the second is related to the S matrix for the scattering of the corresponding solitons [4]. There is a subtle reason for this unexpected symmetry. The tt* theory leads to the identification

$$\left(\frac{1 - g(w_1, \ldots, w_n)}{1 + g(w_1, \ldots, w_n)}\right)_{jk} \leftrightarrow i \frac{\langle \mu(w_1) \ldots \sigma(w_j) \ldots \sigma(w_k) \ldots \mu(w_n)\rangle}{\langle \mu(w_1) \ldots \mu(w_n)\rangle}, \tag{3.19}$$

from which the Euclidean invariance is manifest.

The identification (3.19) should be taken with a grain of salt. In fact we have shown that the equations are the same, but this does not mean that the *solutions* are equal for the two theories. I.e. in general they satisfy different boundary conditions. Selecting the correct boundary condition is a subtler point. The explicit connection between the boundary conditions satisfied by the Ising correlations and the metric g for (3.16) will be discussed in Sect. 4 (for special cases).

The General Case. The general case is very similar. The canonical coordinates w_i always exist. The way to see this is to note that in the sector connecting vacua i and j, the supersymmetry algebra has a central term $\{Q^+, \bar{Q}^+\} = \Delta_{ij}$, satisfying $\Delta_{ij} = -\Delta_{ji}$. Moreover by additivity of the central term, one sees that $\Delta_{ik} = \Delta_{ij} + \Delta_{jk}$, from this one deduces that we can choose w_i for each critical point, which are unique up to an overall shift and such that $\Delta_{ij} = w_i - w_j$. The existence of canonical coordinates w_i can also be shown in a purely topological setting [25, 26]. These coordinates have all the properties needed in the above discussion (in particular in this basis topological metric η is always diagonal). The mass of the soliton connecting the vacua i and j is expressed in terms of the canonical coordinates as

$$m_{ij} = 2|w_i - w_j|\,, \tag{3.20}$$

and the angles between the segments connecting the "points" w_i are still connected with ratios of S-matrix elements. This follows from the IR asymptotics of the solution to the tt* equations and the thermodynamical interpretation of the Q matrix [4]. In fact comparing with the Bogomolnyi bound of [24] we see that the w_j are still the critical values of the superpotential (even for non-LG models). Then the analysis above carries over word for word to the general case.

There is however one major difference. In the general case it is no longer obvious that the solutions to the tt* equations should be regular everywhere in the "W-plane". Physical intuition certainly requires regularity in the subspace corresponding to the relevant or marginal perturbations, but the irrelevant (i.e. non-renormalizable) ones are more tricky and possibly ill-defined.

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[20] Indeed this phase can be absorbed by a redefinition of the Fermi fields

[21] These lengths are positive since $W(X)$ is assumed to be Morse

Fancier Points of View. At a more speculative level one would like to argue the other way around. That is, one starts from the "magical" euclidean invariance (3.18) to construct a $2d$ QFT over coupling space. The correlations of this "target" theory will compute non-trivial quantities for the original $N = 2$ theories.

In principle this can be done using the Euclidean reconstruction theorem [28]. This theorem allows to reconstruct a QFT out of a set of would-be n-point functions provided they satisfy the Osterwalder–Schrader axioms OS1, —, OS5 [28]. Take any physical quantity A for the model (3.16), and consider it as a function of the couplings $(w_1, \ldots, w_n)$. By construction the would-be n points functions $A(w_1, \ldots, w_n)$ satisfy two of the basic axioms, namely OS2 (*Euclidean invariance*) and OS4 (*symmetry*). In fact[22], $A(w_1, \ldots, w_n)$ is invariant under permutations of the w_k's, just because it is a function of the original couplings t_j, which are themselves symmetric functions of the w_k.

Whether the other three axioms are satisfied depend on the quantity A. These should be seen as restrictions on the objects A which can be computed by the reconstruction program. It is easy to construct quantities A which satisfy OS5 (*cluster property*). This requires

$$A(w_1, \ldots, w_r, w'_1, \ldots, w'_s)|_{|w_i - w'_j| \to \infty} \sim A(w_1, \ldots, w_r) A(w'_1, \ldots, w'_s) .$$

Indeed each critical value corresponds to a classial vacuum. As the two groups of critical values are taken apart the potential barrier between the corresponding vacua grows. There is a rigorous bound on the corresponding tunnelling amplitude,

$$\text{tunnelling amplitude} = O(\exp[-2\beta |w_i - w'_j|]) .$$

In the limit $|w_i - w'_j| \to \infty$ the sectors of the Hilbert space built over the two sets of vacua decouple. Thus, say, quantities[23] like $\exp[\langle \mathcal{O} \rangle]$ or $\det[\langle \mathcal{O} \rangle_{ij}]$ satisfy OS5. The axioms OS1 (*regularity*) is not really difficult to realize. In particular the reality condition looks very natural: It requires A to be real whenever the coefficients of W are. The axiom which is non-trivial is OS3 (*reflection positivity*). This is the essential condition on the quantity A needed to compute it by reconstructing a QFT in coupling space.

At least for some $N = 2$ models, our correspondence with the Ising problem can be seeen as a way to reconstruct a "target" QFT (i.e. our old friend the Ising model). In general, it may be necessary to go to a covering of the Euclidean plane in order to realize the "target" QFT, as the fields may have non-abelian anyonic statistics. It may also be that there are other quantities A allowing reconstruction beyond those discussed in the present paper.

4. The Physical Link Between the Ising Model and $N = 2$ Susy

We have seen that the Ising correlations and the g metric for a "generic" class of $N = 2$ models lead to the same abstract mathematical problem described by the tt*

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[22] This discussion is valid if $A(w_1, \ldots, w_n)$ is a univalued function. This needs not to be so, since the change of variables $t_i \leftrightarrow w_j$ can be done only locally. If $A(w_i)$ is multi-valued a standard analysis shows that the would-be fields are anyons rather than bosons. For the LG case the corresponding "statistics" can be fully described in terms of Picard–Leschetz theory [29]

[23] Here $\langle \ldots \rangle_{ij}$ represents the expectation value in the soliton sector conecting the i^{th} vacuum to the j^{th} one. In this sector $W(X) \to w_i$ (resp. w_j) for $X \to -\infty$ (resp. $+\infty$)

geometry. It is natural to ask whether there is a more direct physical link between $N = 2$ susy and the Ising model. Besides its intrinsic interest, this question is very important for practical reasons. In fact, although these two different areas of physics are governed by the same differential equations, the *solutions* are, in general, different for the two cases. I.e. the universality at the level of equations breaks down at the level of the boundary conditions. Having a direct link, we can relate the actual solutions of the two problems. This would be nice since we know the explicit solutions of tt* corresponding to the Ising correlations. To compute, say, the τ function $\langle \mu(w_1) \ldots \mu(w_n) \rangle$, we can insert a complete set of states. Since the theory is free, these are just k fermions of momenta p_i $(i = 1, \ldots, k)$. In this way we can construct all correlation functions in terms of the form factors

$$\langle p_1, \ldots, p_k | \mu | p'_1, \ldots, p'_h \rangle . \tag{4.1}$$

These functions are known [30].

Then a direct link would allow to compute g, the index Q, etc. in terms of the form factors for the Ising model. It may seem that this procedure can work only for very special models, since it is quite rare that the boundary condition satisfied by the ground-state metric is the one corresponding to the Ising correlations. We shall see first of all that there *does exist* an $N = 2$ theory for each n-point correlation of Ising model, which satisfies exactly the same boundary conditions (these correspond to theories where the chiral field lives on a hyperelliptic Riemann surface). Moreover, as we shall see, the class of $N = 2$ systems that can be related to the Ising model is much bigger. The idea is that we can "twist" the Fermi field of the Ising model by inserting in the correlations other operators besides the order/disorder ones. This changes the boundary conditions for the fermion and consequently the boundary conditions for the tt* equations. In this way we can generate families of solutions to tt* in terms of the Ising form factors. It is not clear whether or not in this way we get the full boundary conditions allowed by regularity of tt* equation. At any rate, a more general approach to solving the tt* equations, which leads to a classification program for $N = 2$ quantum field theories in two dimensions will be presented elsewhere [32].

4.1. SQM and the Ising Model. The most direct way to make contact between $N = 2$ Landau–Ginzburg and the Ising model is by looking at the 1d version of the LG model, i.e. by going to SQM.

For a non-degenerate one dimensional LG model the Q^+-cohomology is isomorphic (as a ring) to that of the corresponding $2d$ model, and the ground state metric is the same [12]. Of course, in the SQM case we can also compute the metric from the overlap integrals for vacuum wave function (as we did in Sect. 4.2).

As always, we identify the SQM Hilbert space with the space of square summable forms through the identification [33]

$$\psi^i \mapsto dX^i, \quad \psi^{\bar{i}} \mapsto d\bar{X}^i .$$

The two supercharges having Fermi number $F = +1$, Q^+ and $\bar{Q}^-$, act on forms as [12]

$$Q^+ \alpha = \bar{\partial}\alpha + dW \wedge \alpha, \quad \bar{Q}^- \alpha = \partial\alpha + d\bar{W} \wedge \alpha , \tag{4.2}$$

where $W(X_i)$ is the superpotential. As in Sect. 2, we denote by Λ the contraction with respect to the Kähler form defined by the D-term. Then

$$(Q^+)^\dagger = i[\Lambda, \bar{Q}^-], \quad (\bar{Q}^-)^\dagger = -i[\Lambda, Q^+] .$$

The form ϖ_j associated to the wave-function of a susy ground-state satisfies,

$$Q^+\varpi_j = \bar{Q}^-\varpi_j = \Lambda\varpi_j = 0 .$$

We focus on the case in which we have a LG model with only one superfield X. Then consider the holomorphic change of variables

$$X \mapsto w \equiv W(X) . \tag{4.3}$$

Comparing (2.13) with (4.2) we see that ($m = 1$)

$$Q^+ = W^*\bar{\mathscr{D}}, \quad \bar{Q}^- = W^*\mathscr{D} .$$

Then the map (4.3) transforms the LG model into the Ising model in the W-plane. Stated differently, (4.3) maps the Schroedinger equation for a zero-energy state of the 1d LG model in the X-plane into the Euclidean Dirac equation in the W-plane (2.1). Thus, all we said in Sect. (2.2) for the Ising model applies word-for-word to the 1d LG model. In particular, the wave function viewed on the W-plane, which we denote by $W_*\varpi_j$, can be written as $W_*\varpi_j = i\bar{\mathscr{D}}\psi_{j+}$, where ψ_{j+} is singular only at the branching points w_i, and these singularities encode the Q^+-class of the vacuum ϖ_j.

However on W-plane $W_*\varpi_i$ is not univalued. ϖ_i is required to be univalued only in the X-plane. The branching points of $W_*\varpi_i$ are just the critical values w_k of $W(X)$.

In general not all the pre-images of a given critical value w_i are critical points. If all the pre-images of any critical value are critical points we say that the superpotential $W(X)$ is *nice*. If $W(X)$ is both "nice" and Morse life is particularly easy. In this case the regularity of ϖ_j in X-space requires

$$\psi_{j+}(w) \sim \frac{a_{jk}}{(w - w_k)^{1/2}} + \ldots \quad \text{as } w \to w_k . \tag{4.4}$$

Using standard QM techniques, one easily sees that, as $w \to \infty$, $\psi_{j+} = O(\exp[-2|w|])$. Thus, for $W(X)$ "nice" and Morse, $W_*\varpi_j$ not only satisfies the same equation (2.1) as the Ising wave functions (2.6), but also the same boundary conditions (4.4). Thus the SQM wave-functions are just the correlation functions of the Ising model,

$$W_*\varpi_j(w) = i\bar{\mathscr{D}}\frac{\langle\psi_+(w)\mu(w_1)\ldots\sigma(w_j)\ldots\mu(w_n)\rangle}{\langle\mu(w_1)\ldots\mu(w_n)\rangle} .$$

This explains why the Ising correlations satisfy the tt* equations: The Ising Model *is* an $N = 2$ LG model for some "nice" superpotential!

Conversely, for "nice" Morse superpotentials we can write the explicit solutions to the Schroedinger equation in terms of Ising form factors. The "twisting" procedure mentioned at the beginning allows to extend the space of models which can be solved in terms of Ising form factors. A further extension can be obtained considering more general $\mathbf{Z}_m$ twist-fields for the theory of massive free fermions. This allows to solve simple models without a mass-gap. In the next subsection we present some examples where tt* equations are solved through the correlation functions of Ising model.

4.2. Solving the tt* *Equations by the Ising Map (Elementary Case).* In this subsection some example of models for which the actual g can be written in terms of the Ising form factors is discussed. In particular we discuss the cases of $N = 2$

sine-Gordon model, the Chebyshev superpotentials, andthe hyperelliptic configuration space for chiral fields (which is the exact mirror of Ising model in the $N = 2$ set up).

$N = 2$ sine-Gordon. Let

$$W(X) = \lambda \cos(X) . \tag{4.5}$$

The critical values are $\pm \lambda$ and all points with $W(X) = \pm \lambda$ are critical. So (4.5) is "nice" but it fails to be Morse since the critical values are not all distinct. To get a Morse function one has just to make the identification $X \sim X + 2\pi$, i.e. let X take value on a cylinder $\mathscr{C}$. Then the vacua of the original model correspond to the Q^+-cohomology with coefficients in the flat bundles V_χ, where χ are the unitary representations of $\pi_1(\mathscr{C}) = \mathbf{Z}$. These are just the Block waves, i.e. the θ-vacua, such that

$$T|\theta; a\rangle = e^{i\theta}|\theta; a\rangle, \quad 0 \leqq \theta < 2\pi ,$$

where T is the operator which translates X by 2π. For each θ there are two vacua corresponding to the two critical values $\pm \lambda$.

We start with the simpler case[24], i.e. $\theta = 0$. This corresponds to wave-functions univalued on the cylinder. From the above discussion, we see that – after the change of variables (4.3) – the wave functions for the two $\theta = 0$ vacua are just the Ising functions

$$\varpi_{\pm\lambda}(w; \theta = 0) = i\bar{\mathscr{D}} \frac{\langle \psi_+(w)\sigma(\pm\lambda)\mu(\mp\lambda)\rangle}{\langle \mu(\pm\lambda)\mu(\mp\lambda)\rangle} .$$

So, in this case the boundary condition for the metric g agrees with that for the Ising two-point function. In the canonical basis one has

$$g(\lambda; \theta = 0) = \exp[\sigma_2 u(4|\lambda|)] .$$

In term of $u(z)$, tt* equations for $\theta = 0$ reduce to radial sinh-Gordon (a special case of PIII [34])

$$\frac{\partial}{\partial z}\left(z\frac{\partial u(z)}{\partial z}\right) = \frac{z}{2}\sinh(2u) . \tag{4.6}$$

From the theory of the PIII equation [34] we know that there is one regular solution for each value of the boundary datum r, $|r| \leqq 1$, where

$$u(z) \sim r \log z \quad \text{as } z \to 0 .$$

Recalling (3.14), we see that the eigenvalues of the Q matrix are

$$\pm\frac{1}{2}|\lambda|\frac{\partial u(4|\lambda|)}{d|\lambda|} \sim \pm\frac{1}{2}|r|, \quad \text{as } \lambda \to 0 .$$

In the Ising picture, Q is the mean angular momentum. As $\lambda \to 0$, using the Ising operator expansion, we can replace the order/disorder operators by a single Ψ at the origin. Then in this limit we recover invariance under rotations around the

[24] In fact, $\theta = \pi$ is even simpler. The wave-functions are easily expressed in terms of Bessel functions. Up to normalization, the two $\theta = \pi$ vacua are given by

$$\varpi_{\pm\lambda}(\theta = \pi) = i\bar{\mathscr{D}}\{\zeta_0[2(W(x) \mp \lambda] - \zeta_1[2(W(x) \mp \lambda]\} ,$$

where the functions $\zeta_l(z)$ are as in Eq. (2.3) origin. The angular momentum of Ψ is $\pm 1/2$. Thus, the boundary condition associated to the Ising 2-point function is $r = \pm 1$.

Next, we construct the wave-functions for a general value of θ. When X goes around the cylinder the wave function picks up a phase $\exp(i\theta)$. Since Ψ is now multivalued, we cut the cylinder along one generator, $\mathrm{Re}\, X = a$. When crossing this cut, Ψ gets multiplied by the above phase. The image in the W-plane of this cut is a branch of hyperbola $\mathscr{Y}$. In the Ising language, we mimick this by inserting an operator on this curve which when crossed by the free fermion produces the right phase. However this is not quite correct, since the pre-image of this curve corresponds to two generators of the cylinder, $\mathrm{Re}\, X = a$ and $\mathrm{Re}\, X = a + \pi$. Then it is more convenient to make *two* cuts along these two generators, in such a way that at each cut the wave-function picks up a phase $\exp[i\theta/2]$. The images in the W-plane of these two cuts correspond to the same curve $\mathscr{Y}$, but with opposite orientations.

Since the fermion is free, we make it complex by adding a spectator imaginary part, which does not couple to spins fields σ and μ. Let J_μ be the corresponding $U(1)$ current and insert in the above correlation functions the operator

$$\exp\left[\pm i\frac{\theta}{2}\int_{\mathscr{Y}} \varepsilon_{\mu\nu} J\mu\, dw^\nu\right],$$

where $\pm$ corresponds to the two different sheets. This operator is nothing else than $\exp[-i\theta F/2]$ where F is the "target" Fermi number.

For these "twisted" Ising correlation functions we can repeat the analysis of Sect. 2, getting the same differential equation. Again,

$$g(|\lambda|, \theta) = \exp[\sigma_2 u(4|\lambda|, \theta)],$$

with $u(z, \theta)$ the solution to Eq. (4.6) with

$$\frac{\exp[u(4|\lambda|, \theta)] - 1}{\exp[u(4|\lambda|, \theta)] + 1} = \frac{\langle 0|\sigma(\lambda)\exp[\pm i\theta F/2]\sigma(\lambda)|0\rangle}{\langle 0|\mu(\lambda)\exp[\pm i\theta F/2]\mu(-\lambda)|0\rangle}. \tag{4.7}$$

Now, the crucial fact is that, although the wave-function is multivalued in the W-plane, the correlation functions in the rhs of (4.7) are not. In particular the insertion of the two operators $\exp[\pm i\theta F/2]$ (corresponding to the two sheets) have the same effect (since only the real part of the fermion couples to σ and μ). So (4.7) is unambiguously defined.

Let us compute the corresponding boundary datum r, equal to twice the angular momentum at $\lambda = 0$. Going around the cylinder $\mathscr{C}$ we make a turn around the origin and the wave function picks up a phase $\exp[\pm i\theta]$. Hence the orbital angular momentum m is equal to $\pm\theta/2\pi$ mod. 1. Adding the spin-part, we get

$$r(\theta) = 2l = \pm\left(1 - \frac{\theta}{\pi}\right).$$

Notice that all $|r| \leqq 1$ appear, and hence all regular solutions to PIII can be constructed this way. Since the tt* equations for all the "massive" $N = 2$ model with Witten index 2 can be recast in the form (4.6), this justifies our claim that for all such models the metric g can be written in terms of Ising form factors.

Let us give a more explicit formula for the rhs of Eq. (4.7). Since the Ising model is just a free fermion

$$\langle 0|\sigma(\lambda)e^{i\theta F/2}\sigma(-\lambda)|0\rangle = \sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{2^k}\sum_{e_i=\pm 1}\exp\left[i\frac{1}{2}\sum_{i=1}^{k}e_i\theta\right]\int\prod_{i=1}^{k}\frac{dp_i}{4\pi\sqrt{p_i^2+m^2}}$$
$$\times\langle 0|\sigma(\lambda)|p_1,\ldots,p_k\rangle\langle p_1,\ldots,p_k|\sigma(-\lambda)|0\rangle\,,$$

where we sum over the two possible $U(1)$ charges of each free fermion in the intermediate state. A similar representation holds for the μ 2-point function.

Putting everything together, we get

$$u(z,\theta) = \sum_n [\cos(\theta/2)]^n u_n(z)\,,$$

where $u_n(z)$ is the n-intermediate fermion contribution to the Ising answer; $u_n(z)$ can be extracted from the known form factors. As $\theta \to \theta + 2\pi$, $u(z,\theta)$ changes sign. Then only odd n's contribute. In particular

$$u(z, 2\pi - \theta) = -u(z,\theta)\,.$$

Comparing with [34] we get

$$u_{2n+1}(z) = \frac{2}{2n+1}\int\prod_{i=1}^{2n+1}\frac{e^{-z\cosh\theta_i}}{\cosh\left(\frac{\theta_i-\theta_{i+1}}{2}\right)}\frac{d\theta_i}{4\pi}$$

$$(\text{here } \theta_{2n+1}\equiv\theta_1)\,.$$

Additional information about these solutions can be found in [3] and [4]. Of course, we can also write the θ-vacuum wave-functions in terms of Ising form factors in a similar guise. It is remarkable that we can solve the $N = 2$ sine-Gordon Schroedinger equation by the Ising map.

Chebyshev Superpotentials. We can use the Ising map to compute the ground-state metric for Chebyshev polynomial superpotentials

$$W(X) = \lambda T_n(X), \quad \text{where } T_n(\cos x) = \cos(nx)\,. \tag{4.8}$$

The change of variable f_n: $X \mapsto \cos[Y/n]$ maps this model into the sine-Gordon one. As we saw in Sect. (4.1), the susy charges (and hence the zero-energy Schroedinger equation) transform functorially under such change of coordinates. If we look at Y as a variable taking value in the cylinder $\mathscr{C}$ as above, then the Chebyshev vacua can be seen as solutions to the sine-Gordon belonging to certain representations χ_i $(i = 1, \ldots, n-1)$ of $\pi_1(\mathscr{C})$. Since $Y \to Y + 2\pi n$ acts as the identity on X, these representations should be direct sums of those with $\theta = 2\pi k/n$. The vacua with $\theta = 0, 2\pi$, cannot appear because they would not be singularity free in X space. So $k = 1, \ldots, n-1$. Moreover, the Chebyshev vacua are required to be odd under the symmetry $X \leftrightarrow -X$. These two requirements give $n-1$ vacua. Since this is the Witten index of the model (4.8), they fix completely the vacuum wave functions and hence the metric g.

Then g can be written in terms of $u(z, 2\pi k/n)$. Details can be found in [3].

Other Models. The $N = 2$ sine-Gordon is the first model in a family of "nice" superpotentials having Witten indices $n = 2, 3, \ldots$, whose ground-state metric not only satisfies the same equations as the corresponding Ising correlation functions, but also the same boundary conditions. To understand the general case, notice that the important property of W which made the sine-Gordon case work was

$$\left(\frac{dW}{dx}\right)^2 = P_2(W),$$

where $P_2(\cdot)$ is a polynomial of degree 2. From this equation it is obvious that w is a critical value if and only if it is a root of $P_2(\cdot)$. Conversely, all inverse images are critical points. Then the above differential equation guarantees that W is "nice." This argument is easily generalized. The next model satisfies the equation

$$\left(\frac{dW}{dx}\right)^2 = P_3(W),$$

where now $P_3(\cdot)$ is a cubic polynomial. The solution is $W = \wp(X)$, and the argument above shows that this potential is also "nice." To make it Morse, one has just to identify

$$X \sim X + n\omega_1 + m\omega_2\,, \tag{4.9}$$

where ω_i are the periods of the elliptic curve $\mathscr{C}$ defined by $\wp(X)$.

We will now describe this result more invariantly in a more general setup. This turns out to be useful in understanding its relation with the Ising model. Consider a genus g curve (Riemann surface) $\mathscr{C}_g$ as a (branched) double cover of $\mathbf{P}^1$, i.e., a hyperelliptic curve. This cover is described by a degree two meromorphic function

$$W\colon \mathscr{C}_g \to \mathbf{P}^1\,.$$

whose critical values are the branching points $w_1, w_2, \ldots, w_n$ (the other $4g - n$ branching points are set to infinity). Clearly, all inverse images of these points are critical for W. Then we take W as our "nice" superpotential provided we take as field $(P \in \mathscr{C}_g)$

$$X(P) = \int_{P_0}^{P} \omega\,, \tag{4.10}$$

where ω is a holomorphic differential[25] over $\mathscr{C}$. (X is well defined up to periods of ω.) LG models with these kind of superpotentials as well as those associated to more general Riemann surfaces have been introduced and studied in detail by B. Dubrovin, Ref. [35] (see also [25]).

The ground-state metric g for these models satisfy the Ising correlation equations. Indeed, the w_k's are again canonical coordinates [25]. Moreover, they satisfy the same boundary conditions. This can be seen by comparing the corresponding SQM wave functions with the Ising ones. In fact the Ising correlations are

[25] More concretely, if the hyperelliptic curve has equation

$$y^2 = \prod_i (W - e_i)\,,$$

we take $\omega = dW/y$. Then we have $(W')^2 = \prod_i (W - e_i)$, where a prime denotes the derivative with respect the local parameter X

uni-valued when continued on the hyperelliptic surface $\mathscr{C}_g$ just as the wave functions of the LG theory is[26]. The two wave-functions are indeed the same.

Thus the reconstruction program outlined in Sect. 3.2 can be carried out for this class of superpotentials. The target QFT we get this way is just the Ising model itself.

5. Geometry of the N = 2 τ-Function

In Sect. 3 we have seen that the tt* equations for a "massive" $N = 2$ theory take the same form as the equations for the Ising correlators. There we have discussed the $N = 2$ meaning of all Ising quantities but for one: the τ function. One of the purposes of the present paper is to elucidate the physical meaning of the τ function from the $N = 2$ viewpoint. As a preparation to the more substantial physical analysis of Sect. 6, we begin with a general discussion on its geometrical origin in the framework of topological anti-topological fusion (tt*) [3]. For concreteness we fix our attention to the LG case, but our conclusions are fairly general.

5.1. Kähler Metrics in tt*. From many points of view, tt* is a generalization of the so-called "special geometry" (variations of Hodge structure [6]). However, as a generalization of special geometry, tt* is disappointing in one respect: The metric $g_{i\bar{j}}$ is not Kähler in general, and even for a critical theory it is (conformally) Kähler only after restriction to marginal directions.

Luckily in tt* there is another natural (hermitian) metric for $\mathscr{R}$ which *is* Kählerian. It is defined by the following procedure. Recall that the metric $g_{i\bar{j}}$ was defined in [3] by considering an infinitely elongated sphere, in which we consider topological twisting on the left-hemisphere, and anti-topological twisting on the right-hemisphere, and we insert the operator ϕ_i on the left hemisphere and $\bar{\phi}_j$ on the right one. As mentioned in that paper, it is also possible to do a similar thing on an arbitrary genus Riemann surface as long as the regions in which we do the topological twisting and the ones where we do anti-topological twisting are separated by infinitely long tubes. Indeed the computation of the topological OPE, C_{ij}^k and the metrics $g_{i\bar{j}}$ and η_{ij} on the sphere is sufficient to enable us to do an arbitrary such computation on higher genus by the sewing techniques. It turns out the case of interest for us is the metric defined via a torus.

So let us consider a torus with the fields ϕ_i and $\bar{\phi}_j$ inserted on the left and right side of a flat torus respectively which are infinitely separated by two long tubes each of perimeter β (sometimes we set $\beta = 1$ by a choice of units). We denote the resulting correlation by

$$K_{i\bar{j}} = \langle \phi_i \bar{\phi}_j \rangle_{\text{torus}} . \tag{5.1}$$

As in Sect. (3.1), we choose our coordinates in coupling constant space, t^i, such that $\phi_i = \partial_i W$ and $\partial_0 W = 1$. As we shall show below, $ds^2 = K_{i\bar{j}} dt^i d\bar{t}^j$ is a Kähler metric, i.e., (locally) there is a real function K such that

$$K_{i\bar{j}} = \partial_i \partial_{\bar{j}} K .$$

[26] Note that by the above reasoning the two point function in the Ising model can also be related to an $N = 2$ superpotential $W = (X^2 - 1)^{1/2}$ defined on the double cover of the plane

From the previous sections we know that tt* associates a τ function to each $N = 2$ model. (We take Eq. (2.36) as definition of the τ function even for models which have no canonical coordinates, e.g. the critical ones). It turns out that K is simply related to this τ function

$$K = -\log\tilde{\tau} + \mathrm{tr}(CC^\dagger) + f(t_i) + f^*(\bar{t}_i) \tag{5.2}$$

for some holomorphic function f of moduli. This equation is shown in Sect. 5.2 below by showing that $\partial\bar{\partial}$ of both sides of it are equal.

Then the τ function is the analog in toroidal tt* of $\int \varepsilon \wedge \bar{\varepsilon}$ in "special geometry" [36] (ε is the holomorphic $(n, 0)$ form). In fact, $\tilde{\tau}$ is a kind of "partition function". The reason for the additional term $\mathrm{tr}(CC^\dagger)$ in (5.2) is that $\tilde{\tau}$, being translational invariant in W-plane, is independent of t_0, whereas K is not. For if we take $\phi_i = 1$ then $\langle \bar{\phi}_j \rangle_{\mathrm{torus}}$ is not zero and is equal to $\mathrm{tr}\,\bar{C}_j$.[27] The second term in (5.2) restores translational invariance. In this sense then $\log\tilde{\tau}$, which is a minor modification of K, may be a more natural candidate for a Kähler potential, as the perturbation by the identity operator in the superpotential does not change the theory at all, and so $\log\tilde{\tau}$ defines a Kähler metric on the *physical* perturbation space of the theory.

Let us show that the $K_{\bar{i}\bar{j}}$ metric is Kähler. Using the operator representation of the path-integral on the torus, and taking flat metric on the torus we have[28],

$$K_{i\bar{j}} = \lim_{L\to\infty} \mathrm{Tr}[(-1)^F \phi_i e^{-LH} \bar{\phi}_j e^{-LH}] = \mathrm{tr}(C_i \bar{C}_{\bar{j}})\,. \tag{5.3}$$

In particular, $K_{i\bar{j}}$ is known once $g_{i\bar{j}}$ is known and the topological correlations are known.

Using (3.9) and (3.11),

$$\partial_k K_{i\bar{j}} = \partial_k \mathrm{tr}(C_i \bar{C}_{\bar{j}}) = \mathrm{tr}[(D_k C_i)\bar{C}_{\bar{j}}] = \mathrm{tr}[(D_i C_k)\bar{C}_{\bar{j}}] = \partial_i K_{k\bar{j}}\,, \tag{5.4}$$

and so $K_{i\bar{j}}$ is Kähler[29].

Equation (5.4) has the following generalization. Let T be a torus with a flat metric h. Then

$$K_{i\bar{j}}(T, h) = \int_T d^2 z \langle \bar{\phi}_j(z) \phi_i(0) \rangle_{(T,h)}\,, \tag{5.5}$$

has the Kähler property for all T and h. However, for finite periods, $K_{i\bar{j}}(T, h)$ does depend on the D-term, and so is not an index-like quantity in the sense of [4].

5.2. Differential Geometry of τ-Functions. It remains to show Eq. (5.2). Let us introduce some differential forms over coupling constant space

$$\kappa = \mathrm{tr}(\Gamma \wedge \bar{\Gamma}), \quad \bar{\varrho} = i(v)\kappa\,,$$

$$\Omega = \mathrm{tr}(QR), \quad \xi = \frac{1}{2}\mathrm{tr}(QT) \equiv \frac{1}{2}\mathrm{tr}(Qg\partial g^{-1})\,, \tag{5.6}$$

---

[27] From the definition of K we have

$$\frac{\partial}{\partial t_0} K = (\eta^{ij}\partial_i\partial_j F)^*\,,$$

where F is the topological prepotential (the rhs is written in terms of flat coordinates [15])

[28] As mentioned earlier we are taking the chiral field all to have zero fermion number thus no need to insertion of $(-1)^F$ on the rhs of the equation above

[29] If we wish to get a normalized metric, i.e. with $K'_0 = 1$, we have just to divide $K_{i\bar{j}}$ by $K_{0\bar{0}} = \Delta$, the Witten index. Of course, this does not spoil the Kähler property

where $R \equiv -[\Gamma, \bar{\Gamma}]$ is the tt* curvature and T is the torsion (1, 0) form. Notice that κ is just the Kähler form of the metric (5.1). Then from (5.3),

$$\kappa = \partial\bar{\partial}K \,. \tag{5.7}$$

These forms are related by the following identities,

$$\begin{aligned} &\Omega = \kappa - \partial\bar{\varrho}, \quad d\xi = \Omega \,, \\ &\bar{\varrho} = \bar{\partial}\left[\operatorname{tr}(\bar{C}) - \frac{1}{2}\operatorname{tr}(Q^2)\right]. \end{aligned} \tag{5.8}$$

Indeed, we have

$$\begin{aligned} \Omega &= -\operatorname{tr}(Q[\Gamma, \bar{\Gamma}]) = -\operatorname{tr}([Q, \Gamma] \wedge \bar{\Gamma}) \\ &= \operatorname{tr}(\Gamma \wedge \bar{\Gamma}) - \mathscr{L}_v \operatorname{tr}(\Gamma \wedge \bar{\Gamma}) = \kappa - \partial i(v)\kappa \,, \quad \text{using (3.13)} \end{aligned}$$

which shows the first equality in (5.8). In terms of T the tt* equations take the form

$$\begin{aligned} &\bar{\partial}T = -[\Gamma, \bar{\Gamma}] \,, \\ &\Rightarrow \bar{\partial}Q = \bar{\partial}i(v)T = i(v)[\Gamma, \bar{\Gamma}] \,. \end{aligned} \tag{5.9}$$

Then we have

$$\bar{\partial}\operatorname{tr}(QT) = \operatorname{tr}([C, \bar{\Gamma}]T) - \operatorname{tr}(Q[\Gamma, \bar{\Gamma}]) = 2\Omega \,, \tag{5.10}$$

where we used the identity $[Q, \Gamma] = [T, C]$ which follows from (2.26).

On the other hand, we have $DT = T \wedge T$ and $[\Gamma, T] = 0$ (also a consequence of (2.26)). Therefore,

$$\partial\operatorname{tr}(QT) = \operatorname{tr}([\Gamma, \bar{C}]T) + v^i \operatorname{tr}(T_i T_j T_k) dt^j \wedge dt^k = 0 \,. \tag{5.11}$$

Equations (5.10) and (5.11) give the second equality in (5.8).

Finally, (5.9) yields

$$\begin{aligned} \bar{\partial}\operatorname{tr} Q^2 &= 2\operatorname{tr}(Q\bar{\partial}Q) = 2\operatorname{tr}(Q[C, \bar{\Gamma}]) \\ &= 2\operatorname{tr}(C[\bar{\Gamma}, Q]) = -2\operatorname{tr}(C\bar{\Gamma}) + 2\mathscr{L}_{\bar{v}}\operatorname{tr}(C\bar{\Gamma}) \\ &= -2\bar{\varrho} + 2\bar{\partial}\operatorname{tr}(C\bar{C}) \,, \end{aligned}$$

which proves the last equality in (5.8). Comparing (2.36) with (5.6) and using (5.8), we get

$$\begin{aligned} \partial\bar{\partial}\log\tilde{\tau} &= \partial[\bar{\xi} - \bar{\varrho} + \bar{\partial}\operatorname{tr}(CC^\dagger)] \\ &= -\Omega - \partial\bar{\varrho}\operatorname{tr}(CC^\dagger) \\ &= -\kappa + \partial\bar{\partial}\operatorname{tr}(CC^\dagger) \,. \end{aligned} \tag{5.12}$$

Comparing this equation with (5.7) we get (5.2) as was to be shown. This equation is very much reminiscent of the Quillen holomorphic anomaly [37]. This anomaly arises when one studies the determinant of differential operators which depend holomorphically on some parameters. So naively the logarithm of the determinant is expected to be a sum of a holomorphic and its conjugate piece. There is an anomaly which causes a term involving both holomorphic and anti-holomorphic parameters to arise. This mixing is computable, as is the case with all anomalies, and in fact leads to recovering the *full* determinant, i.e. even the purely holomorphic and anti-holomorphic pieces, by the requirement of single-valuedness. Our situation suggests something similar should be happening here, as the mixing of holomorphic and anti-holomorphic pieces are easy to compute for the tau function (5.12), and from that we can fix the other terms by some global single-valuedness criteria. The analogy may be in fact closer than it appears; after all we *are* computing the determinant of a massive Dirac operator. The fact that there is mass implies, however, that it is not a holomorphic operator. But if the mass term is a "soft breaking" of the holomorphicity, then we may expect that the holomorphic anomaly should still be computable as we are finding. It would be interesting to develop this line of thought further.

6. The Tau Function as a Supersymmetric Index

In the context of $N = 2$ supersymmetry a physical quantity A is called an *index* if it is invariant under arbitrary deformations of the D-term. Such a quantity depends at most on n parameters, where n is the dimension of $\mathscr{R}$ (as the F-terms depend only on chiral fields).

The τ function introduced in the previous sections is certainly a quantity of this kind. Indeed it can be computed from the ground-state metric g which is itself invariant under deformations of the D-term. However the most interesting susy indices are those which can be written in the form (α is a super-selection sector of the Hilbert space)

$$\mathrm{Tr}_\alpha[(-1)^F \mathcal{O} e^{-\beta H}]\,,$$

for some "natural" operator $\mathcal{O}$. Only in this case they have a simple path integral representation [38]. For $\mathcal{O} = 1$ we get the original Witten index [39] and for $\mathcal{O} = F$ the index introduced in Ref. [4].

It is natural to ask whether the τ function can be written in this way. In this section we discuss some aspects of this issue.

6.1. τ in the critical case. We begin with the simpler case of a conformal family of $N = 2$ models. In particular the couplings t_i should correspond to exactly marginal deformations. In the conformal case it is more convenient to consider the Kähler metric $K_{i\bar{j}}$ rather than the τ function itself. The two are related by (5.2).

In the critical case both right F_R and left F_L Fermi numbers are conserved. Note that the first try at the index, $\mathrm{Tr}(-1)^F F_{L,R}\, q^{H_L}\bar{q}^{H_R}$ on the *torus* vanishes by CPT. So, continuing the logic of [4] it makes sense to consider the following object[30]

$$\mathrm{Tr}[(-1)^F F_R F_L q^{H_L}\bar{q}^{H_R}]\,, \tag{6.1}$$

which is CPT even and generally non-zero[31]. Here the trace is over the periodic (Ramond) sector. We claim that K function defined in previous section is given by

$$K = \int_{\mathscr{F}} \frac{d^2\varrho}{\varrho_2} \mathrm{Tr}[(-1)^F F_R F_L q^{H_L}\bar{q}^{H_R}]\,, \tag{6.2}$$

[30] Conventions are as follows. As usual, $q = \exp[2\pi i\varrho]$, $\bar{q} = \exp[-2\pi i\bar{\varrho}]$, where ϱ is the period of the torus used to represent the functional trace (we call it ϱ to avoid confusion with the τ function) and $\varrho_2 = \mathrm{Im}\,\varrho$. Moreover $2\pi H_L = L_0 - c/24$ and $2\pi H_R = \bar{L}_0 - c/24$

[31] It is easy to show, using arguments similar to the ones in the text that any pure insertion of the form F_L^n or F_R^n is a "simple" index, in that it is completely independent of target moduli

where $\mathscr{F}$ denotes the standard fundamental domain for $SL(2, \mathbf{Z})$. Notice that our definition of the K in the previous section fixes it up to addition of a holomorphic function plus its conjugate. In many interesting cases the holomorphic piece can be determined uniquely by exploiting the *target space* modular invariance [5]. The above definition of K has a fixed holomorphic piece and since it depends only on the moduli of the target theory it is automatically modular invariant.

Remarkably enough integrals similar to those in Eq. (6.2) have already been studied by many authors in a different context [5], namely in computing the stringy one-loop correction to gauge and gravitational couplings. The precise form above has not been encountered. However, it is plausible that provided the central charge $\hat{c}$ has the correct value, (6.2) gives the one-loop correction to the gravitational coupling for a type II super-string "compactified" on the given $N = 2$ superconformal model. In view of the geometric interpretation of $\log \tilde{\tau}$ as a Kähler potential for the *torus* correlations, this is not too surprising.

To show the above claim, let us vary the functional representation of the RHS of (6.2) with respect to the F-term couplings t_i, $\bar{t}_j$. Notice that no D-term perturbation is possible in the critical case. In fact these perturbations are never marginal and so will spoil conformal invariance which is needed to make sense out of (6.2). One obtains[32]

$$\begin{aligned}&\partial_i \partial_{\bar{j}} \mathrm{Tr}[(-1)^F F_R F_L q^{H_L} \bar{q}^{H_R}]\\ &= \varrho_2 \int_{\text{torus}} d^2 z \, \mathrm{Tr}[(-1)^F F_R F_L \{Q^+, [\bar{Q}^+, \bar{\phi}_j(0)]\} \{\bar{Q}^-, [Q^-, \phi_i(z)]\} q^{H_L} \bar{q}^{H_R}] ,\end{aligned} \tag{6.3}$$

where we used translational invariance to fix the position of $\bar{\phi}_j$ at the origin extracting a factor of the area of the torus ϱ_2. Also the propagator $q^{H_L} \bar{q}^{H_R}$ is to be distributed among the terms in the above expression corresponding to where the point z is inserted; but for simplicity of notation we have written it as above and we continue to use this notation throughout this section. Now we use manipulations very similar to the ones used in [4] to simplify (6.3). If we take Q^+ and take it around the trace it will cancel the other term in the anti-commutator, except that we pick two terms: One coming from commutation with F_L and the other one by its anti-commutation with Q^- which leads to $\partial \phi_i$ which vanishes upon integration (note that Q^+ commutes with ϕ_i). Similarly taking $\bar{Q}^+$ around the trace the result is non-vanishing only due to the non-vanishing commutation of $\bar{Q}^+$ with F_R[33]. Since

$$[F_L, Q^\pm] = \pm Q^\pm, \quad [F_R, \bar{Q}^\pm] = \pm \bar{Q}^\pm .$$

So the integrand in (6.3) takes the form

$$\mathrm{Tr}[(-1)^F \bar{Q}^+ Q^+ \bar{\phi}_j(0) \{\bar{Q}^-, [Q^-, \phi_i(z)]\} q^{H_L} \bar{q}^{H_R}] .$$

[32] We have ignored contact terms. For a more careful treatment see [45]

[33] Indeed, if $F_L F_R$ was not present our computation would reduce to the usual argument for the independence of the Witten index from any deformation of the theory

Now taking the Q^- and $\bar{Q}^-$ around the trace the only non-vanishing term comes from the anti-commutators

$$\{Q^-, Q^+\} = H_L, \quad \{\bar{Q}^-, \bar{Q}^+\} = H_R .$$

So finally one has

$$\partial_i \partial_{\bar{j}} \mathrm{Tr}[(-1)^F F_R F_L q^{H_L} \bar{q}^{H_R}] = \varrho_2 \int_{\text{torus}} d^2 z \, \mathrm{Tr}[(-1)^F H_L \bar{\phi}_j(z) H_R \phi_i(0) q^{H_L} \bar{q}^{H_R}] . \tag{6.4}$$

The integrand on the rhs of the above equation looks like a total derivative in toroidal moduli. Indeed, consider the two-form

$$\Omega_{i\bar{j}} = i \mathrm{Tr}[(-1)^F H_L \int d^2 z \, \bar{\phi}_j(z) H_R \phi_i(0) e^{i\varrho H_L} e^{-i\varrho^* H_R}] d\varrho \wedge d\varrho^* ,$$

then Eq. (6.4) is rewritten as

$$\partial_i \partial_{\bar{j}} K = -2 \int_{\mathscr{F}} \Omega_{\bar{i}\bar{j}} , \tag{6.5}$$

and one can write $\Omega_{i\bar{j}}$ as a total derivative,

$$\Omega_{i\bar{j}} = d\omega_{i\bar{j}} , \tag{6.6}$$

where

$$\begin{aligned} \omega_{i\bar{j}} = \int_0^{\varrho_2} dt \, \mathrm{Tr}[(-1)^F \int d^2 z e^{H t} \oint \bar{\phi}_j e^{-H t} H_R \phi_i e^{i\varrho H_L} e^{-i\varrho^* H_R}] d\varrho^* \\ - \frac{1}{2} \mathrm{Tr}[((-1)^F \oint \bar{\phi}_j e^{i\varrho H_L} e^{-i\varrho^* H_R} \phi_i] d\varrho , \end{aligned} \tag{6.7}$$

[as in Sect. 3 we use the convention $\oint \bar{\phi}_j(t) \equiv \int_0^1 dx \, \bar{\phi}_j(x, t)$]. Thus the integral (6.5) reduces to a boundary term

$$\partial_i \partial_{\bar{j}} K = -2 \int_{\partial \mathscr{F}} \omega_{i\bar{j}} .$$

A priori there are four contributions to the LHS: the one coming from infinity, those coming from the two vertical segments $\varrho_1 = \pm \frac{1}{2}$ and finally that of the arc $|\varrho| = 1$. However since $\omega_{i\bar{j}}$ is invariant under $\varrho \to \varrho + 1$, the contributions from the two segments cancel each other. The contribution from the arc cancels as well because the contribution of the two half arcs cancel as they are mapped to one another by the modular transformation $\rho \to -1/\rho$. Thus the only non-vanishing contribution comes from the part of the boundary at infinity (which is what one would expect since the moduli space should be thought of as having only one boundary and that is at infinity).

Let us compute this contribution. One has

$$\begin{aligned} - \int_0^{\varrho_2} dt \, \mathrm{Tr}[(-1)^F \oint \bar{\phi}_j H e^{-H t} \phi_i e^{-H(\varrho_2 - t)}]|_{\varrho_2 \to \infty} \\ + \mathrm{Tr}[(-1)^F \oint \bar{\phi}_j e^{-H \varrho_2} \phi_i]|_{\varrho_2 \to \infty} , \end{aligned} \tag{6.8}$$

The first term in (6.8) can be rewritten as

$$\begin{aligned} \lim_{\varrho_2 \to \infty} \int_0^{\varrho_2} dt \frac{d}{dt} \mathrm{Tr}[(-1)^F \oint \bar{\phi}_j e^{-H t} \phi_i P] \\ = - \mathrm{Tr}[(-1)^F \oint \bar{\phi}_j P' \phi_i P] , \end{aligned}$$

where P is the projector on the vacua and $P' = 1 - P$. Now, since $\bar{\phi}_j$ and ϕ_i *commute at equal time*,

$$\mathrm{Tr}[(-1)^F \bar{\phi}_j P' \phi_i P] = \mathrm{Tr}[(-1)^F \bar{\phi}_j P \phi_i P'] .$$

The second term in (6.8) is

$$\lim_{\varrho_2 \to \infty} \mathrm{Tr}[(-1)^F \oint \bar{\phi}_j e^{-H\varrho_2} \phi_i] = \mathrm{Tr}[(-1)^F \oint \bar{\phi}_j P \phi_i] .$$

Then putting everything together, the $\varrho_2 = \infty$ contribution is

$$\begin{aligned} -\mathrm{Tr}[(-1)^F \oint \bar{\phi}_j P \phi_i (1-P)] + \mathrm{Tr}[(-1)^F \oint \bar{\phi}_j P \phi_i] \\ = \mathrm{Tr}[(-1)^F \oint \bar{\phi}_j P \phi_i P] \\ = \mathrm{tr}[\bar{C}_j C_i] , \end{aligned}$$

which is the formula we wanted to show

$$\partial_i \partial_{\bar{j}} K = \mathrm{tr}[\bar{C}_j C_i] .$$

Let us evaluate K using this relation. In the conformal case the matrix Q is just a constant (in an appropriate basis). Then[34]

$$\partial K = -\mathrm{tr}(Qg\partial g^{-1}) = \sum_k qk\, \partial \log(\det_k[g]) ,$$

where q_k are the eigenvalues of Q, and $\det_k[g]$ means the determinant of the metric restricted to ground states with $U(1)$ charge q_k. In fact, using the constancy of Q, one has

$$\begin{aligned} \partial_{\bar{j}} \mathrm{tr}(Qg\partial_i g^{-1}) &= \mathrm{tr}[Q\partial_{\bar{j}}(g\partial_i g^{-1})] = \mathrm{tr}(Q[C_i, \bar{C}_{\bar{j}}]) \\ &= \mathrm{tr}([Q, C_i]\bar{C}_{\bar{j}}) = -\mathrm{tr}(C_i \bar{C}_{\bar{j}}) , \end{aligned}$$

since, for a marginal deformation, $[Q, C_i] = -C_i$. Then[35]

$$e^{-K} = \prod_k (\det_k[g])^{-q_k} . \tag{6.9}$$

In particular, for the σ-model with target space a torus of period ϱ, we have $\exp[-K] = (\mathrm{Im}\,\varrho)$. Since K should depend only on the target space QFT, i.e., should be a modular invariant object, we can fix the holomorphic plus the antiholomorphic piece of K, by adding $\log(\eta^2 \bar{\eta}^2)$ to the (logarithm of) tau

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[34] Notice that this formula differs for a factor 2 from what we would have guessed from the formula for the massive case. This discrepancy is related to subtleties in taking the limit $\beta \to 0$

[35] Consider a CY 3-fold. Then, this equation reads

$$K = -(m+3)\log\langle \bar{1}|1\rangle - \log\det G + f + f^* ,$$

where G is the Zamolodchikov metric along the marginal (moduli) directions and m is the number of moduli. Acting with $\partial_i \partial_{\bar{j}}$ on the equation and recalling that $-\log\langle \bar{1}|1\rangle$ is the Kähler potential for G, we get

$$\mathrm{tr}(C_i \bar{C}_j) = (m+3)G_{i\bar{j}} - R_{i\bar{j}} ,$$

which gives a relation between the Ricci curvature of G and the ring coefficients. In fact, this relation is a well known fact in special geometry (see e.g. Eq. (38) of [40]). The equation in the text extends this identity to general critical models function. This is the trick which is well known in the literature [5]. This is an example of why Eq. (5.2) may be viewed as a *holomorphic anomaly* equation.

6.2. The Massive Case. Next we try to generalize the above discussion to the massive case. In this case the two chiral charges F_L, F_R are not conserved any longer. Only the vector combination $F = F_L - F_R$ is. So (6.1) makes no sense. However this is not really a problem. Going through the computations for the critical case we see that we can replace $F_L F_R$ by $-F^2/2$ and all the arguments work as before[36].

Then we consider the expression

$$\mathrm{Tr}_\beta[(-1)^F F^2 e^{-tH} e^{ixP}], \tag{6.10}$$

where the index β means that we quantize the theory on a circle of perimeter β. Then (6.10) can be written as a periodic path integral over a torus of periods $(\beta, x + it)$. We write $\varrho = (x + it)/\beta$ for the normalized period. Then all manipulations above hold for the massive case too. In particular

$$i\partial_i \partial_{\bar{j}} \mathrm{Tr}_\beta[(-1)^F F^2 e^{-tH} e^{ixP}] \frac{d\varrho \wedge d\varrho^*}{\varrho_2} = d\omega_{i\bar{j}}, \tag{6.11}$$

where $\omega_{i\bar{j}}$ is as in (6.7). So even in the massive case the second variation is a pure boundary term. The contribution from $t \to \infty$ is computed just as in the critical case and we get

$$\beta^2 \mathrm{tr}[C_i \bar{C}_j].$$

However there is a very important difference with respect to the previous case. In the massless theory we had modular invariance and there was a natural integration region, namely $\mathscr{F}$. In the massive case there is no natural integration region. If we choose to integrate over $\mathscr{F}$, the contributions from the two vertical segments still cancel, since the massive partition function is invariant under $\varrho \to \varrho + 1$ (as a consequence of quantization of momentum in a periodic box). But the arc contribution *will not cancel.* Indeed the argument for cancellation uses explicitly the conformal invariance. A careful evaluation of the arc contribution shows that it is proportional to[37]

$$\beta \frac{\partial}{\partial \beta} \int_{\pi/3}^{2\pi/3} d\theta \langle \phi_i(0) \int d^2 z \bar{\phi}_j(z) \rangle |_{\varrho = e^{i\theta}},$$

i.e. to the variation of $\langle \phi_i \int d^2 z \bar{\phi}_j \rangle$ under an infinitesimal rescaling. This term vanishes only if the theory is conformal and ϕ_i, $\bar{\phi}_j$ are truly marginal deformations. By the same argument, the integral of (6.10) over $\mathscr{F}$ is not invariant under deformations of the D-term. Indeed if we make a change $\delta\kappa$ of the D-term we again

[36] Note that the difference between these two definitions at the critical point is a term involving $F_L^2 + F_R^2$ which is completely independent of target moduli. Or we can just take the integral with F^2 inserted over the doubled fundamental domain which includes the image of the fundamental domain under $\rho \to -1/\rho$ and use the fact that under this modular transformation $F_L - F_R \to i(F_L + F_R)$ to show that the additional pieces cancel out

[37] Here $\langle \cdots \rangle_\varrho$ means the path-integral over a torus whose (normalized) period is ϱ

get a non-vanishing contribution from the arc proportional to

$$\beta \frac{\partial}{\partial \beta} \int_{\pi/3}^{2\pi/3} d\theta \langle \delta\kappa \rangle \bigg|_{\varrho = e^{i\theta}} .$$

From a different point of view, the problem can be seen as due to UV divergences. Formally $\log\tau$ is given by

$$\text{“}\mathrm{Tr}[(-1)^F F^2 \delta(P) \log H]\text{”} . \tag{6.12}$$

One way to see this is to recall that in the Ising case the τ function is the determinant of the twisted Dirac operator. Using the relation between the Dirac operator and the supercharges we get something like (6.12). One has to give a meaning to this purely formal object. We can represent it as the integral of (6.10) over the strip $\varrho_2 \geqq 0$, $|\varrho_1| \leqq \frac{1}{2}$. However this definition is still badly divergent as $\varrho_2 \to 0$. In the critical case we know that this divergence is fake, as it is due to summing over infinite copies of the fundamental domain. To get a finite answer one has just to restrict to a single copy. Unfortunately in the massive case this natural regularization is not available.

However the UV divergent piece of (6.11) has also the form $\partial_i \partial_{\bar{j}} F$ for some F. Then we can subtract F from the formal definition of τ, getting something finite and with the correct properties[38]. The only drawback of this procedure, is that in general it leads to objects depending on the D-terms. The simplest version of this subtraction is as follows. In Sect. 3 we introduced a family of Kähler potentials $K(\varrho, \beta)$. Then the combination of Kähler potentials

$$K - \beta \frac{\partial}{\partial \beta} \int_{\pi/3}^{2\pi/3} d\theta\, K(e^{i\theta}, \beta) ,$$

has no UV problem and it is in fact given by the integral of (6.10) over the standard domain $\mathscr{F}$. However this combination is not really an index, since it depends on the D-terms.

To get something more "canonical" one has to control the UV structure of the theory in more detail. There is a trick which allows one to control the UV structure whenever the configuration space has $\pi_1 = \mathbf{Z}$. One takes the difference of (6.10) computed in two sectors of the Hilbert space carrying different representations of the fundamental group. The typical instance is the sine-Gordon model, i.e. the LG model with superpotential $W(X) = \lambda \cos(X)$. If one identifies $X \sim X + 2\pi$, the configuration space becomes a circle and $\pi_1 = \mathbf{Z}$. Then we introduce the topological charge (= instanton number)

$$Q = \frac{1}{2\pi} \int dt\, d_t X .$$

The trace over a θ-sector of the Hilbert space are defined as

$$\mathrm{Tr}_\theta[(-1)^F \mathcal{O} e^{-Ht}] = \int [d\Phi] \mathcal{O} \exp\{-S[\Phi] + i\theta Q\} .$$

The problem we had with integrating over the full strip and getting infinity simply because we are adding infinitely many equivalent contributions does not exist anymore, because once we use twisting with the π_1 the modular transforms are all

[38] The same remark applies to the regularization-dependent contact terms

inequivalent. This is a well known trick, already used in the conformal case in [41]. Then we consider the integral over the full strip (and by abbreviating the integral and writing its relevant piece which is the integral over $t = \rho_2$; the integral over ρ_1 simply projects to $P = 0$ subsector) of the difference

$$K(\theta) - K(\theta') = 2\int_0^\infty \frac{dt}{t}\{\mathrm{Tr}_\theta[(-1)^F F^2 e^{-Ht}] - \mathrm{Tr}_{\theta'}[(-1)^F F^2 e^{-Ht}]\}\ . \quad (6.13)$$

This difference is well defined (just as in [41]). Indeed, varying the D-term we get

$$\int_0^\infty dt \frac{d^2}{dt^2}\{\mathrm{Tr}_\theta[(-1)^F \delta\kappa e^{-Ht}] - \mathrm{Tr}_{\theta'}[(-1)^F \delta\kappa e^{-Ht}]\}$$

$$= \frac{d}{dt}\{\mathrm{Tr}_\theta[(-1)^F \delta\kappa e^{-Ht}] - \mathrm{Tr}_{\theta'}[(-1)^F \delta\kappa e^{-Ht}]\}\Big|_{t=0}^{t=\infty}\ .$$

The contribution from $t = \infty$ vanishes since for large times the quantity in brace reduces to a constant (up to exponentially small terms). The contribution from the boundary $t = 0$ also vanishes. Indeed for any operator $\mathcal{O}$

$$\mathrm{Tr}_\theta[(-1)^F \mathcal{O} e^{-Ht}] - \mathrm{Tr}_{\theta'}[(-1)^F \mathcal{O} e^{-Ht}] = O(e^{-c/t}) \quad \text{as } t \to 0\ . \quad (6.14)$$

This equation also shows that the contribution from the boundary at 0 cancels in $\partial_i \partial_{\bar{j}}[K(\theta) - K(\theta')]$. Instead the contribution from $t = \infty$ gives

$$\partial_i \partial_{\bar{j}}[K(\theta) - K(\theta')] = \mathrm{tr}_\theta(C_i \bar{C}_j) - \mathrm{tr}_{\theta'}(C_i \bar{C}_j)\ .$$

To show (6.14) we use the path integral representation of the RHS. We write $\langle \cdots \rangle_\theta$ for $\mathrm{Tr}_\theta[(-1)^F \ldots e^{-Ht}]$. One has $\langle \mathcal{O} \rangle_\theta = \sum_n e^{in\theta} \langle \mathcal{O} \rangle_n$, where $\langle \cdots \rangle_n$ denotes path-integral in the n instanton sector. From this we see that in the difference (6.14) only the non-trivial sectors $n \neq 0$ contribute. Now for small t

$$\langle \mathcal{O} \rangle_n \sim e^{-(2\pi n)^2/t}\ ,$$

because in this sector the action is $\geqq (2\pi n)^2/t$. This is easily seen from the kinetic term (the potential – being positive definite – cannot change the conclusion) where we note that only the $P = 0$ part contributes

$$t\int_0^t ds d_s \bar{X} d_s X \geqq \left|\int_0^t d_s X\, ds\right|^2 = (2\pi n)^2\ .$$

In Eq. (6.13) θ labels the regular solutions to PIII. We fix θ' to have the value π (corresponding to the trivial solution to PIII $u = 0$). Then

$$\log \tilde{\tau}(\theta) = K(\theta) - K(\pi)\ ,$$

corresponds to the standard definition of the τ function for the regular solution of PIII corresponding[39] to θ. So we see that the term which arises from the "UV subtraction" is actually needed to get the correct answer.

It would be very important to generalize this to the case where there is no non-trivial π_1. In the context of SQM this may be possible to do by subtracting

[39] In fact we have

$$\partial_i \partial_{\bar{j}} K(\pi) = \mathrm{tr}(C_i C_j^\dagger)$$

from (6.10) its asymptotic expansion as $t \to 0$ (so that only "exponentially small" terms remain). This would be interesting to study in more detail.

6.3. Relation with the Ray–Singer Torsion. The definition of the function K (6.2) is reminiscent of the Ray–Singer analytic torsion [42]. In particular, in the context of supersymmetric σ-models, interpreting F_L and F_R in terms of holomorphic and anti-holomorphic degree of differential forms and noting that the integral over moduli space is essentially what is needed to give logarithm of the determinant of H acting on the differential forms allows us to build up a dictionary between the two problems. In fact more is true: (A linear combination of) analytic torsion is just $\log \tilde{\tau}$ for a special class of $N = 2$ models. Let us explain this. The 1d susy σ-models satisfy the same tt* equations as the 2d ones [3] (although, in general, with a different, much simpler, $\mathscr{R}$ i.e. the classical cohomology ring, without the instanton corrections which makes the $2d$ case more interesting).

We consider the 1d σ-model with target space a compact Kähler manifold $\mathscr{M}$ having a non-trivial π_1. It is a well known fact that the Hilbert space of this model consists of all (square-summable) (p, q)-forms taking value in the flat bundles V_χ associated to unitary representations χ of $\pi_1(\mathscr{M})$. The existence of these different χ sectors comes from the fact that various classes of maps $S^1 \to \mathscr{M}$ can be weighted with different phases. The susy charges acts on this space as $\bar{\partial}$, ∂ and their adjoints. Then the Hamiltonian is the usual Laplacian,

$$\Delta = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger \bar{\partial}, \quad \text{where } \bar{\partial}^\dagger = - * \partial * . \tag{6.15}$$

The susy vacua are precisely the harmonic (p, q)-forms with coefficients in V_χ. The operators ϕ_i are associated to the corresponding cohomology classes, and act on the Hilbert space by wedge product. Then $\mathscr{R}$ is just the cohomology ring of $\mathscr{M}$ (with coefficients on the flat bundles V_χ). In particular $\mathscr{R}$ is nilpotent and hence has the algebraic structure typical of a critical theory[40]. The conserved charges F and Q^5 act on (p, q) forms as $(p - q)$ and $(p + q - n)$, respectively. The topological metric η_{ij} is just the intersection form in cohomology [15]. Comparing with the Hodge metric, we see that the real structure acts on forms as $\alpha^* = *\alpha^*$. Then, if Ω_k is a "canonical" basis of harmonic forms, we have

$$*\Omega_k^* = g_{l\bar{k}} \Omega_l , \tag{6.16}$$

since, the real structure acts on the vacua as the matrix $g\eta^{-1}$ (3.6).

We consider a family of such models of the form

$$S = S_0 + \left(\sum_k t^k \int ds \{ Q^-, [\bar{Q}^-, \hat{\omega}_k] \} + \text{c.c.} \right),$$

where $\hat{\omega}_k \equiv (\omega_k)_{i\bar{j}} \psi^i \psi^{\bar{j}}$ and the ω_k's give a basis of $H^{1,1}(\mathscr{M})$.

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[40] However, the tt* metric g is a much simpler object in this case. For forms Ω_j of degree $k \leqq n$ having the form $L^r v_j$ with v_j primitive, the ground state metric is

$$g_{j\bar{l}} = (-1)^r (-1)^{k(k-1)/2} \frac{r!}{(n-k+r)!} \int \omega(t, \bar{t})^{n-k+2r} \wedge v_j \wedge C v_l^* ,$$

where $\omega(t, \bar{t})$ is twice the Kähler form (seen as a function of the couplings t), and C is the Weil operator acting on a (p, q)-forms as i^{p-q}. This follows from [43] §I.4. The other entries of g can be obtained from (3.6). This $1d$ result serves also as a boundary condition for the 2d tt* equations, see [17]

For each $p = 1, \ldots, n = \dim \mathscr{M}$, the $\bar{\partial}$-torsion $T_p(\chi)$ is defined as [42]

$$\log T_p(\chi) = \frac{1}{2} \sum_{q=0}^{n} (-1)^q \frac{d}{ds} \zeta_{pq}(s, \chi) \bigg|_{s=0} ,$$

$$\zeta_{pq}(s, \chi) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \, dx \, \mathrm{Tr}_{(p,q)} [\mathscr{P}_\chi (e^{-Hx} - \Pi)] . \tag{6.17}$$

Here $\mathrm{Tr}_{(p,q)}$ is the trace over the (p, q) sector of the Hilbert space, $\mathscr{P}_\chi$ is the projector on the χ representation, and Π is the projector on the ground states. The basic mathematical fact is that the *difference* of the torsions for two χ's depends only on the *class* of the Kähler metric. In physical language this is independence from the D-term. Then this difference is a susy index in the sense of [4].

The relation with the $N = 2 \tau$ function is as follows. Let χ and χ' be real representations. Then

$$\begin{aligned} 2 \sum_p (-1)^p p [\log T_p(\chi) - \log T_p(\chi')] &= \sum (-1)^{p+q} p q \zeta'_{pq}(0, \chi) - (\chi \to \chi') \\ &= -\log \tilde{\tau}(\chi) + \log \tau(\chi') . \end{aligned} \tag{6.18}$$

In fact, let $\mathscr{A}_i$ be the operator which acts on forms as $*^{-1}(\partial_i *)$. Then from (6.15) one gets

$$\partial_i \left\{ \sum_q (-1)^q q \, \mathrm{Tr}_{(p,q)} [\mathscr{P}_\chi (e^{-H_x} - \Pi)] \right\} = \sum_q (-1)^q \frac{d}{dx} \mathrm{Tr}_{(p,q)} [\mathscr{P}_\chi (\mathscr{A}_i e^{-Hx})] .$$

This equation together with (6.17) give [42]

$$\begin{aligned} &\partial_i \sum_p (-1)^p p [\log T_p(\chi) - \log T_p(\chi')] \\ &= \frac{1}{2} \sum_p (-1)^{p+q} p \int_0^\infty dx \frac{d}{dx} \{ \mathrm{Tr}_{(p,q)} [\mathscr{P}_\chi (\mathscr{A}_i e^{-Hx})] - \mathrm{Tr}_{(p,q)} [\mathscr{P}_{\chi'} (\mathscr{A}_i e^{-Hx})] \} \\ &= \frac{1}{2} \sum_p (-1)^{p+q} p \{ \mathrm{Tr}_{(p,q)} [\mathscr{P}_\chi (\mathscr{A}_i \Pi)] - \mathrm{Tr}_{(p,q)} [\mathscr{P}_{\chi'} (\mathscr{A}_i \Pi)] \} , \end{aligned}$$

where we used that, as $x \to 0$,

$$\mathrm{Tr}_{(p,q)} [\mathscr{P}_\chi (\mathscr{A}_i e^{-Hx})] - \mathrm{Tr}_{(p,q)} [\mathscr{P}_{\chi'} (\mathscr{A}_i e^{-Hx})] = O(e^{-c/x}) . \tag{6.19}$$

From (6.16) it follows that

$$\mathscr{A}_i \Pi = -(-1)^F g \partial_i g^{-1} ,$$

and hence we have

$$\begin{aligned} &\sum_p (-1)^{p+q} p \, \mathrm{Tr}_{(p,q)} [\mathscr{P}_\chi (\mathscr{A}_i \Pi)] \\ &= -\frac{1}{2} \mathrm{tr}(\mathscr{P}_\chi Q g \partial_i g^{-1}) - \frac{1}{2} \mathrm{tr}(\mathscr{P}_\chi F g \partial_i g^{-1}) \\ &= -\partial_i \log \tilde{\tau}(\chi) - \frac{1}{2} \mathrm{tr}(\mathscr{P}_\chi F g \partial_i g^{-1}) . \end{aligned}$$

The additional term in the rhs was not present in our definition of the τ function just because in Ising-like models all vacua have $F = 0$; this term should be added to the definition of the τ-function for the general case. Anyhow, for χ real this term vanishes by PCT, and we get Eq. (6.18).

To construct a general theory of susy τ-functions we have to extend this heat kernel argument in two directions: i) to more general rings than those arising from cohomology; and ii) to 2d field theory, i.e. to loop spaces. This discussion suggests that the τ-function which is defined and is computable for an arbitrary 2d susy σ-model (at least with $c_1 > 0$), is the generalization of the analytic torsion to the loop space of Kähler manifolds. This is an exciting mathematical direction to pursue further.

7. Conclusion

We have seen that the Ising model can be viewed as a "target space" description for $N = 2$ QFT's in two dimensions. The equations describing the geometry of $N = 2$ ground states (the tt* equations) are the same equations which characterize the spin correlations for the massive Ising model. In particular the tau function for the Ising model is a new supersymmetric index for $N = 2$ theories. As we have seen this index is related to a generalization of Ray–Singer analytic torsion to the loop space of Kähler manifolds. This index is essentially fixed by its holomorphic anomaly, which can be computed in terms of the metric on the ground states and the chiral ring. Moreover, the tau function can also be interpreted as a "canonical" Kähler potential for the moduli space of $N = 2$ theories which leads to a Kähler metric on this space.

Another direction which would be interesting to explore is to find the corresponding "target theory" for other $N = 2$ theories. Since in some special cases we got the interesting model of massive Ising model, it may be that for other cases we may also get interesting 2d QFT's (perhaps with fields having non-abelian braiding properties).

There are many directions worth pursuing. Probably the most important one is to tame the ultraviolet divergencies in the path-integral formulation for index for the generic massive theory. We were able to do this only for the conformal case (where the fundamental region automatically cuts off UV divergencies) and for massive cases with non-trivial π_1 for the configuration space.

Another possible application of these ideas is to $N = 2$ strings (see [44] and references in it). In this case if the target space is the cotangent of the torus, the one-loop partition function is the same as the tau function defined here! It would be interesting to see if this is true for more general compactifications. Indeed it was conjectured in [44] that the partition function may be characterized by a holomorphic anomaly condition, which is what we have found here to be the case for the tau function.

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